maniflow

A Python library for geometric processing

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Abstract

The purpose of this library is to provide tools for the study of the most beautiful discipline of mathematics: geometry and geometric analysis. In doing so, we restrict ourselves to 2-manifolds. For a mathematician, this may initially seem like a major restriction. However, it allows us, in a relatively simple way, to represent 2-manifolds as ”meshes“ and to develop powerful tools to study them.

The abstraction hardly needs to be restricted at all, because the proposed calculus for meshes makes it possible to develop new geometries with comparatively little effort. Properties of these meshes can then be examined using `maniflow`. For example, we provide tools to break down meshes into their connected components. You can also use `maniflow` to determine the orientability of a mesh. It is also possible to run a geometric flow, such as the mean curvature flow, on a mesh. This means that `maniflow` can also be used to examine meshes with regard to curvature (Gaussian curvature, mean curvature).

`maniflow` also provides the option of creating images of the meshes. This makes it possible, for example, to create animations of geometric flows etc.

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`maniflow` was developed as part of the course M.Mat.0731 ”Advanced practical course in scientific computing“ at Georg-August University Göttingen.

The image on the frontpage is taken from `https://unsplash.com/de/fotos/blaues-und-rotes-licht-digitales-hintergrundbild-wxj729MaPRY`

Getting started

“Be patient, for the world is broad and wide.“

– E. A. ABBOTT, Flatland: A Romance of Many Dimensions

The code of `maniflow` was originally published on

`https://gitlab.gwdg.de/yangshan.xiang/scientific-computing`

To install the libary, simply use

```
pip install dist/maniflow-1.0-py2.py3-none-any.whl
```

To build the wheel file of the library, use

```
python setup.py bdist_wheel -universal
```

Dependencies. The installation and usage of `maniflow` requires the following packages to be installed: `numpy`, `pillow`

Optional dependencies. When using `maniflow.render.SVGPainterRenderer`, one requires the installation of `drawsvg`.

1 Introduction

First, we will look at the basic mathematical concepts that ultimately underpin the whole theory. So let's start with so-called meshes and look at some examples and how these mathematical concepts can be implemented in code using `maniflow`.

Definition 1 (Mesh). *Let V be a vector space over $\mathbb{R}$ of dimension n. Let $\mathcal{V}_M \subset V$ be a set of points in V. We further let $\mathcal{F}_M \subset \mathcal{V}_M^3$. The pair $M = (\mathcal{V}_M, \mathcal{F}_M)$ is then called mesh. The elements of $\mathcal{V}_M$ are called points of M and the elements of $\mathcal{F}_M$ are the faces of the mesh M.*

For a mesh $M = (\mathcal{V}_M, \mathcal{F}_M)$ we will often denote $V_M = |\mathcal{V}_M|$ and $F_M = |\mathcal{F}_M|$.

Remark. Meshes M can be considered as 2-dimensional simplicial complexes. Thus for 2-dimensional manifolds $\tilde{M} \subset V$ we may find a *triangulation* simplicial complex K of $\tilde{M}$. The corresponding mesh will be called *triangulation* mesh of the manifold $\tilde{M}$.

Example 1 (Tetrahedron)**.** *Let*

$$\mathcal{V} = \left\{ \left(\sqrt{\frac{8}{9}}, 0, -\frac{1}{3} \right), \left(-\sqrt{\frac{2}{9}}, \sqrt{\frac{2}{3}}, -\frac{1}{3} \right), \left(-\sqrt{\frac{2}{9}}, -\sqrt{\frac{2}{3}}, -\frac{1}{3} \right), (0,0,1) \right\} \subset \mathbb{R}^3$$

and $\mathcal{F} = \{f \in 2^{\mathcal{V}} : |f| = 3\}$*. The mesh* $T = (\mathcal{V}, \mathcal{F})$ *is the tetrahedron, which is displayed in figure 1. This*

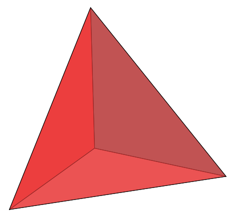


Figure 1: Tetrahedron

can be implemented using **`maniflow`** *by using the* **`Mesh`** *class:*

```
import numpy as np
import itertools
from maniflow.mesh import Mesh, Face

# computing the four vertices of the tetrahedron
v1 = np.array([np.sqrt(8/9), 0, -1/3])
v2 = np.array([-np.sqrt(2/9), np.sqrt(2/3), -1/3])
v3 = np.array([-np.sqrt(2/9), -np.sqrt(2/3), -1/3])
v4 = np.array([0, 0, 1])

tetra = Mesh()
# setting the vertices as the vertices of the new mesh object
tetra.vertices = [v1, v2, v3, v4]
# now we compute the subsets of all the vertices consiting of three vertices
subsets = set(itertools.combinations(list(range(tetra.v)), 3))
# the faces are then set as the faces of tetra
tetra.faces = [Face(tetra, *list(i)) for i in subsets]
```

This way, we obtain the **`Mesh`** *object* **`tetra`** *which represents a tetrahedron.*

1.1 `maniflow.mesh.utils.VertexFunction` – Creating meshes from parameterisations

The way we created a mesh of a tetrahedron in the previous example is very static and absolutely not suitable if you want to study more complicated geometries. `maniflow`, however, provides the option of creating meshes quite easily using parameterisations. For this purpose, `maniflow` provides the wrapper `maniflow.mesh.utils.VertexFunction`, which executes a given function on all vertices of the mesh and has the resulting mesh as output.

Example 2. *For the following example, we assume that we have a* **`Mesh`***-object* **`mesh`** *and we want to shift this mesh by the vector* $\begin{pmatrix}1 & 2 & 3\end{pmatrix}^\intercal \in \mathbb{R}^3$*. For this we make use of a* **`VertexFunction`***:*

```python
# importing the wrapper from maniflow.mesh.utils
from maniflow.mesh.utils import VertexFunction

# implementing the VertexFuntion 'shift'
@VertexFuntion
def shift(vertex):
    return vertex + np.array([1, 2, 3])


# applying 'shift' to 'mesh'
shifted = shift(mesh)
```

The resulting `Mesh`*,* `shifted`*, is* `mesh` *shifted by the vector* $\begin{pmatrix}1 & 2 & 3\end{pmatrix}^\intercal \in \mathbb{R}^3$*.*

Another application of this would be the creation of meshes from parameterisations $\psi\colon \mathbb{R}^2 \supset D \to \mathbb{R}^3$. Oftentimes, the domain D is a cartesian product of two intervals, so $D = I_1 \times I_2$.[1] For this, **`maniflow`** provides the class **`maniflow.mesh.parameterized.Grid`**.

Example 3 (Moebius strip)**.** *We now turn to an example where we want to create a triangulation of a moebius strip. To this end, we will use the parametrisation*

$$\psi\colon [0, 2\pi] \times [-1, 1] \to \mathbb{R}^3,\ (u, v) \mapsto \begin{pmatrix} \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\cos u \\ \left(1 + \frac{v}{2}\cos\frac{u}{2}\right)\sin u \\ \frac{v}{2}\sin\frac{u}{2} \end{pmatrix}.$$

In order to discretise the set $[0, 2\pi] \times [-1, 1]$*, we make use of* `maniflow.mesh.parametrized.Grid` *in order to create a high resolution lattice. Then we can implement a* `VertexFunction` *to capture the parametrisation* ψ *in code and apply it to our lattice.*

```python
from maniflow.mesh.parameterized import Grid
from maniflow.mesh.utils import VertexFunction


# implementing the parametrisation 1:1 in code as a VertexFunction
@VertexFunction
def moebius(vertex):
    x = vertex[0]
    y = vertex[1]
    x0 = np.cos(x) * (1 + (y / 2) * np.cos(x / 2))
    x1 = np.sin(x) * (1 + (y / 2) * np.cos(x / 2))
    x2 = (y / 2) * np.sin(x / 2)
    return np.array([x0, x1, x2])


u = Grid((0, 2 * np.pi), (-1, 1), 30, 10)  # create a high resolution grid
moebiusMesh = moebius(u)  # mapping the vertices from the grid according to the
    parametrisation
coincidingVertices(moebiusMesh) # remove the redundant vertices at the joint after
    making the moebius band
```

[1] Or to be more precise, `maniflow` makes it easy to create meshes from parametrisations, where the domain D is homoeomorphic to a square.

With this we obtain the `Mesh`-object `moebiusMesh`. Using `maniflow.mesh.obj.OBJFile`, we can write this mesh to memory as a `.obj` file

```
from maniflow.mesh.obj import OBJFile

OBJFile.write(moebiusMesh, "examples/moebius.obj")
```

Unsurprisingly, one can then load this file into Blender and create pictures of it etc.[2] *Figure 2 shows a screenshot taken from Blender with the `.obj` file from `examples/moebius.obj`.*

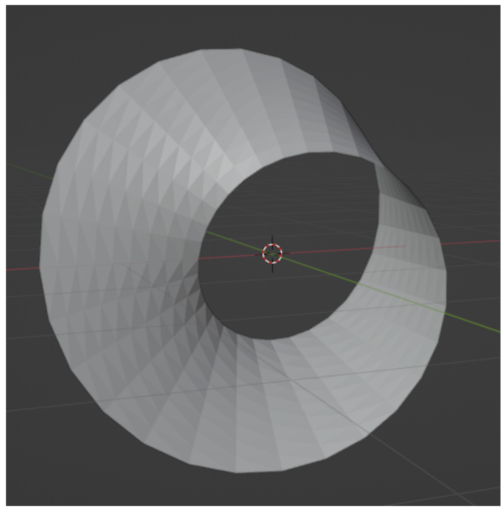


Figure 2: Screenshot of Blender with a moebius strip made with `maniflow`

2 The face graph of a mesh

Definition 2 (Undirected Graph)**.** *Let $\mathcal{V}_G$ be a set and $\mathcal{E}_G \subset \{e \in 2^{\mathcal{V}_G} : |e| = 2\}$ be a set of unordered pairs of elements from $\mathcal{V}_G$. The pair $G = (\mathcal{V}_G, \mathcal{E}_G)$ is then called undirected Graph. The elements from $\mathcal{V}_G$ are called vertices of G and the elements from $\mathcal{E}_G$ are called edges of G.*

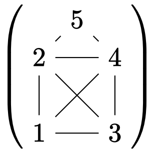
For a Graph $G = (\mathcal{V}_G, \mathcal{E}_G)$ we write

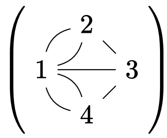
$$x \text{ —— } y$$

if $\{x, y\} \in \mathcal{E}_G$. If we take all edges and points together in this way, we get the picture of a graph with undirected edges.

[2]`maniflow` comes with its own simple renderer. But if you want to do more elaborated computer graphics, you might consider using some other software to render images.

Example 4.

G: 

, H: 

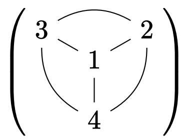
(1)

Definition 3 (Face Graph)**.** *Let $M = (\mathcal{V}_M, \mathcal{F}_M)$ be a mesh and*

$$\mathcal{E} = \left\{(f_1, f_2) \in \mathcal{F}_G^2 : |f_1 \cap f_2| = 2\right\}$$

The face graph of M is the graph $(\mathcal{F}_M, \mathcal{E})$.

Example 5. *The face graph of the tetrahedron is given by*

G: 

(2)

The face graph of a given mesh can be constructed by algorithm 1. Since this algorithm loops over the

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Algorithm 1: Construction of the face graph of a given mesh

Input : A mesh $M = (\mathcal{V}_M, \mathcal{F}_M = \{f_1, f_2, f_3 \ldots\})$
Output: The adjacency matrix of the face graph of the mesh M

$G := 0 \in \mathbb{R}^{F_M \times F_M}$;
for $i = 1$ **to** F_M **do**
    neighbors $:= 0$;
    for $j = 1$ **to** F_M **do**
        if *neighbors* $= 3$ **then**
            break;
        end
        if $|f_i \cap f_j| = 2$ **and** $i \neq j$ **then**
            $G_{ij} \leftarrow 1$;
            neighbors $\leftarrow$ *neighbors* $+ 1$;
        end
    end
end
return G

---

faces of the mesh in a nested way, the complexity of it lies in $O(F_M^2)$. As this runtime complexity has the consequence of the algorithm being very slow at execution for somewhat large meshes, the face graph is computed dynamically by `maniflow.mesh.Mesh.faceGraph`.

2.1 A first application: `maniflow.mesh.utils.connectedComponents`

The method `maniflow.mesh.utils.connectedComponents` decomposes the given mesh into its connected components. Now that we have an algorithm with which to compute the face graph, the connected components of a mesh can now be identified as the connected components of the face graph. These can be determined via the breadth-first traversal of the face graph.

Algorithm 2: Construction of the face graph of a given mesh

Input : A mesh $M = (\mathcal{V}_M, \mathcal{F}_M = \{f_1, f_2, f_3 \ldots\})$
Output: The connected components of the mesh M
Compute the adjacency matrix G using 1;
$start := 1$;
$n := 1$;
while $\mathcal{F}_M \neq \emptyset$ **do**
  Compute a breadth first traversal sequence $T_n \leftarrow \{f_{start}, f_b, f_c, \ldots\} \subseteq \mathcal{F}_M$;
  $n \leftarrow n + 1$;
  $\mathcal{F}_M \leftarrow \mathcal{F}_M \setminus T_n$;
  Set $1 < start \leq F_M$ such that $f_{start} \in \mathcal{F}_M$;
end
return T_1, T_2, $\ldots$

Runtime analysis. The algorithm 1 has a runtime complexity which lies in $O(F_M^2)$. The breadth-first traversal on the face graph has a runtime[3] complexity of $O(F_G + 3 \cdot F_G) = O(F_G)$. The computation of $\mathcal{F}_M \setminus T_n$ has also quadratic complexity $O(|\mathcal{F}_M|^2)$. Thus the overall complexity of algorithm 2 lies in $O(F_G^2)$.

Example 6. *In this example we analyse the connected components of the teapot from* `examples/teapot.obj`*. The teapot is displayed in figure 3.*

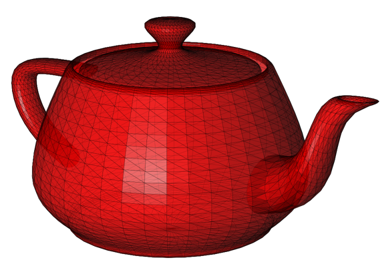


Figure 3: The teapot from `examples/teapot.obj`

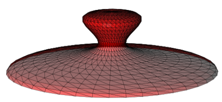
The connected components can be computed using the following code

[3] Since on a graph with the number of vertices being V and the number of edges being E the breadth first search has a complexity of $O(E + V)$. As every face has at most three neighbors we obtain the given runtime complexity.

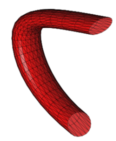
```python
from maniflow.mesh import Mesh
from maniflow.mesh.obj import OBJFile
from maniflow.mesh.utils import connectedComponents, coincidingVertices

teapot = OBJFile.read("examples/teapot.obj")
coincidingVertices(teapot)
components = connectedComponents(teapot)

for i, component_list in enumerate(components):
    component = Mesh.fromFaceList(teapot, *component_list)
    OBJFile.write(component, "teapot" + str(i + 1) + ".obj")
```

2.2 Another application: checking orientability of a `Mesh`

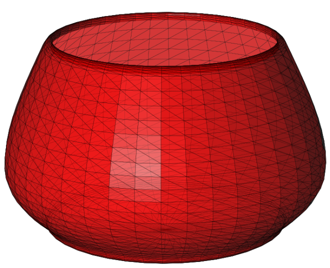
Orientability of a mesh is equivalent to all vertices being aligned



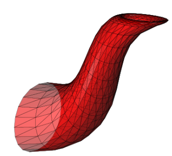
(a) The lid of the teapot



(b) The handle of the teapot

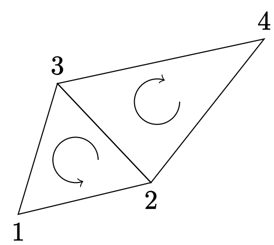


(c) The body of the teapot

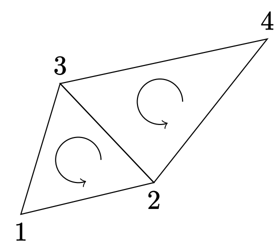


(d) The spout of the teapot

Figure 4: The connected components of the teapot



(a) Non compatible orientation



(b) Compatible orientation

Figure 5: Two triangles with compatible and non compatible orientations