Problem 1. A polynomial f with real coefficients satisfies the functional equation

$$f(f(x)+y^2) = f(x+y)f(x-y) + 4f(xy)$$

for all real x, y. What is the sum of all possible values of $|f(1)|$?

Start by setting $y=0$. We get

$$f(f(x)) = f(x)^2 + 4f(0).$$

Now, the left and right side of the equation must have the same degree. Since

$$\deg[f(f(x))] = \deg[f(x)]^2,$$

we must have that

$$\deg[f(x)] \in \{0, 2\}.$$

First, if $f(x) = c$,

$$c = c^2 + 4c \implies f(x) = 0 \text{ or } f(x) = -3.$$

So, the above case contributes $|f(1)| = 3$.

If $f(x) = a(x-p)(x-q)$ for some $p, q \in \mathbb{C}$,

$$a(f(x)-p)(f(x)-q) = f(x)^2 + 4pq$$

$$a(f(x)^2 - (p+q)f(x) + pq) = f(x)^2 + 4pq$$
$$af(x)^2 - a(p+q)f(x) + apq = f(x)^2 + 4pq.$$

Since the above must hold for all $x \in \mathbb{R}$, we must have $a=1$ and $p=q=0$. This yields $f(x) = x^2$ as a solution and contribues $|f(1)| = 1$ to the sum.

Our final answer is thus, $\boxed{4}$.