# Bellarmine Math Club Mock AIME Solutions

Henry McNamara

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**Problem 1.** A two-digit integer $\underline{a}\,\underline{b}$ is multiplied by 9. The resulting three-digit integer is of the form $\underline{a}\,\underline{c}\,\underline{b}$ for some digit $c$. Evaluate the sum of all possible $\underline{a}\,\underline{b}$.

*Solution.* Consider the equation

$$90a + 9b = 100a + 10c + b$$

$$4b = 5a + 5c.$$

It follows that $b = 5$ or $b = 0$. If $b = 0$, then $a$ is also zero and there are no solutions. So we substitute 5 for $b$, yielding

$$4 = a + c.$$

We see that there are four possible values for $a$, yielding 15, 25, 35, and 45 as solutions. The requested sum is then 120. □

**Problem 2.** Certain polynomials $p(x) = ax^2 + bx + c$ with integer coefficients satisfy the property that $p(x)$ evenly divides $p(x^2)$. Find the number of such polynomials whose coefficients are positive integers between 1 and 10 inclusive.

*Solution.* So, $p(x^2) = ax^4 + bx^2 + c$. We wish to find when

$$ax^2 + bx + c \mid ax^4 + bx^2 + c$$

or when

$$(ax^2 + bx + c) \cdot q(x) = ax^4 + bx^2 + c$$

for some polynomial $q(x)$. Now, we know that $q(x)$ must be a monic quadratic, so we let $q(x) = x^2 + dx + e$. What we get when we multiply is

$$(ax^2 + bx + c)(x^2 + dx + e) = ax^4 + (ad + b)x^3 + (c + ae + bd)x^2 + (cd + be)x + ce.$$

Observe now that $e = 1$, so the expression simplifies to

$$ax^4 + bx^2 + c = ax^4 + (ad + b)x^3 + (c + a + bd)x^2 + (cd + b)x + c.$$

The conditions we wish to satisfy are

$$ad + b = 0$$

$$a + c + bd = b$$

$$cd + b = 0.$$

From equations one and three, we have $a = c$. Also, notice that $d < 0$. So, $2a > b$. Now,

$$2a + bd = b$$

$$ad + b = 0.$$

For which values of $a$ and $b$ does such a $d$ exist. Well,

$$2a + bd - b = ad + b$$

$$bd - ad = 2b - 2a$$

$$d(b - a)$$

□

**Problem 3.** The value of $x$ which satisfies

$$1 + \log_x(\lfloor x \rfloor) = 2\log_x(\sqrt{3}\{x\})$$

can be written in the form $\frac{a+\sqrt{b}}{c}$, where $a$, $b$, and $c$ are relatively prime integers, and $b$ is not divisible by the square of any prime. Find $a + b + c$.

Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$ and $\{x\}$ denotes the fractional part of $x$.

*Solution.* Begin by simplifying to get rid of the logarithms. What results is

$$x\lfloor x \rfloor = 3\{x\}^2.$$

For simplicity, let $\lfloor x \rfloor = n$ and $\{x\} = p$. We then solve for $p$ in terms of $n$:

$$\begin{aligned} n(n+p) &= 3p^2 \\ n^2 + pn &= 3p^2 \\ 3p^2 - np - n^2 &= 0 \\ p &= \frac{n \pm \sqrt{13}n}{6}. \end{aligned}$$

From the bounds of $\{x\}$, we know that $0 \le p < 1$. This means

$$p = \frac{n + \sqrt{13}n}{6}$$

as the other solution is always negative. Then, for $p$ to remain in these bounds while ensuring the logarithms are still defined, we must have that $n = 1$. So, $x = \frac{1+\sqrt{13}}{6}$. The desired sum is then $1 + 13 + 6 = 20$. □

**Problem 4.** Triangle $ABC$ has side lengths $AB = 4$, $AC = 5$, and $BC = 6$. Points $M$ and $N$ lie on $AB$ and $AC$ respectively so that $MC$ and $NB$ intersect at point $O$. If triangles $MBO$ and $NCO$ both have area 1, evaluate the area of triangle $AMN$.

*Solution.* First, we evalute the area of $ABC$ to be

$$\begin{aligned} [ABC] &= \sqrt{\frac{15}{2}\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)} \\ &= \frac{15\sqrt{7}}{4} \end{aligned}$$

From the two given triangles having equal area, we know that $MNCB$ is a trapezoid, meaning $MN \parallel BC$. So,

$$\frac{[AMN]}{[ABC]} =$$

□

**Problem 5.** Point $P$ is situated inside hexagon $ABCDEF$ with center $O$ such that the feet from $P$ to $AB$, $BC$, $CD$, $DE$, $EF$, and $FA$ respectively are $G$, $H$, $I$, $J$, $K$, and $L$. Given that $PG = \frac{9}{2}$, $PI = 6$, $PK = \frac{15}{2}$ and $PO = \sqrt{3}$, the area of hexagon $GHIJKL$ can be written as $\frac{a\sqrt{b}}{c}$. What is $a + b + c$?

*Solution.* □

**Problem 6.** Let $H$ be the point where the three altitudes of $\triangle ABC$ intersect. If $\angle C = 30^\circ$ and $CH = 625$, the length of $AB$ can be written in the form $\frac{a\sqrt{b}}{c}$ where $a$ and $c$ are relatively prime positive integers and $b$ is not divisible by the square of any prime. Evaluate $a+b+c$.

*Solution.* To solve this problem, we utilize complex numbers. Place the circumcenter of $\triangle ABC$ at the origin and let the circumradius be $r$. Start with $h = a+b+c$. It is known that

$$|h| = |a+b+c| \implies |c-h| = |a+b| = 625.$$

For a given radius $r$ the length of $AB$ is fixed by the inscribed angle theorem. It turns out that $AB$ is a chord from an arc of degree measure $\frac{\pi}{6}$. So, $AB = r$. With this in mind, we wish to find $r$. WLOG, assume that $a = r$ and $b = \frac{1}{2}r + \frac{\sqrt{3}}{2}ri$. So,

$$\begin{aligned}|a+b| &= \sqrt{\left(\frac{3}{2}r\right)^2 + \left(\frac{\sqrt{3}}{2}r\right)^2} \\ &= \sqrt{\frac{9}{4}r^2 + \frac{3}{4}r^2} \\ &= \sqrt{3}r.\end{aligned}$$

This means that with $CH = 625$, $AB = \frac{625\sqrt{3}}{3}$, giving a final answer of 631. □

**Problem 7.** The function $y = x^2$ is graphed in the $xy$-plane. A line from every point on the parabola is drawn to the point $(0, -10, a)$ in three-dimensional space. The locus of points $\mathcal{P}$ where the lines intersect the $xz$-plane forms a closed path with area $\pi$. Given that $a = \frac{p\sqrt{q}}{r}$, evaluate $p+q+r$.

*Solution.* The path formed is an ellipse with top-most point at the horizon line and bottom-most point at the origin. The width of the ellipse is equal to half the width of the parabola at the points whose lines intersect the $xz$-plane halfway above the horizion line. Using the fact that the horizon line is at $z = a$, we have that the width is $\sqrt{10}$. So, the area of the ellipse is $\frac{a}{2} \cdot \frac{\sqrt{10}}{2} \cdot \pi = \frac{a\sqrt{10}}{4} \cdot \pi$. So, $a = \frac{2\sqrt{10}}{5}$ and $p+q+r = 17$. □

**Problem 8.** Triangle $ABC$ with $AB = BC = 22$ has circumcircle $\omega$. The line through $C$ and the midpoint $M$ of $AB$ intersects $\omega$ at point $X \neq C$ and the line through $B$ and the center of $\omega$ intersects $\omega$ at point $Y \neq B$. If $XY$ intersects $AB$ at the foot of the altitude from $C$, then $MX^2$ can be written in the form $\frac{m}{n}$ for relatively prime positive integers $m$ and $n$. Evaluate $m+n$.

*Solution.* First, we establish that $BGDX$ is a cyclic quadrilateral where $G$ is the centroid of $ABC$ and $D$ is the foot of the alitude from $C$ to $AB$. This means that

$$\frac{BM}{XM} = \frac{BG}{XD} = \frac{GM}{DM}$$

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We also have that

$$\begin{aligned}(A, B, ; D, M) &= (A, B; Y, C) \\ \frac{AD}{BD} / \frac{AM}{BM} &= \frac{AY}{BY} / \frac{AC}{BC} \\ \frac{AD}{BD} \cdot \frac{BM}{AM} &= \frac{AY}{BY} \cdot \frac{BC}{AC} \\ \frac{AD}{BD} &= \frac{AY}{BY} \cdot \frac{22}{AC}\end{aligned}$$

Notice that $XY$ is the angle bisector of $\angle AXM$, meaning

$$\frac{AD}{AX} = \frac{MD}{MX}.$$

Combining findings from cyclic quadrilateral $BGDX$ with findings from the above cross ratio, we see that

$$\frac{MD}{MX} = \frac{GM}{BM} = \frac{AD}{AX}$$

We have that $BD = \frac{44}{3}$. That makes $AD = \frac{22}{3}$. Now, going back up to the cross ratio, we have that $\frac{AD}{BD} = \frac{1}{2}$, and

$$\frac{1}{2} = \frac{AY}{BY} \cdot \frac{22}{AC}$$

This also tells us that

$$\frac{BH}{HY} = 2$$

The answer is $363 + 7 = 370$. $\square$

**Problem 9.** The Fibonacci Seqence is defined as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for integers $n \geq 2$. The sum

$$S = \sum_{n=0}^{\infty} \frac{F_n^2}{9^n}$$

can be written as $\frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

*Solution.* We use Binet's formula to represent each term as

$$\begin{aligned}F_n^2 &= \frac{(\phi^n - (1-\phi)^n)^2}{5} \\ &= \frac{\phi^{2n} - 2\phi^n(1-\phi)^n + (1-\phi)^{2n}}{5} \\ &= \frac{\phi^{2n} - 2(\phi - \phi^2)^n + (1-\phi)^{2n}}{5}\end{aligned}$$

which makes the sum equal to

$$\begin{aligned}S &= \frac{1}{5} \sum_{n=0}^{\infty} \frac{\phi^{2n} - 2(\phi - \phi^2)^n + (1-\phi)^{2n}}{9^n} \\ &= \frac{1}{5} \sum_{n=0}^{\infty} \frac{\phi^{2n}}{9^n} - \frac{2}{5} \sum_{n=0}^{\infty} \frac{(\phi - \phi^2)^n}{9^n} + \frac{1}{5} \sum_{n=0}^{\infty} \frac{(1-\phi)^{2n}}{9^n}\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{5}\cdot\frac{1}{1-\frac{\phi^2}{9}} - \frac{2}{5}\cdot\frac{1}{1-\frac{\phi-\phi^2}{9}} + \frac{1}{5}\cdot\frac{1}{1-\frac{(1-\phi)^2}{9}}\\
&= \frac{1}{5}\cdot\frac{1}{1-\frac{3+\sqrt{5}}{18}} - \frac{2}{5}\cdot\frac{1}{1+\frac{1}{9}} + \frac{1}{5}\cdot\frac{1}{1-\frac{3-\sqrt{5}}{18}}\\
&= \frac{1}{5}\cdot\frac{1}{\frac{15-\sqrt{5}}{18}} - \frac{2}{5}\cdot\frac{9}{10} + \frac{1}{5}\cdot\frac{1}{\frac{15+\sqrt{5}}{18}}\\
&= \frac{\frac{15+\sqrt{5}}{18}}{\frac{275}{81}} - \frac{9}{25} + \frac{\frac{15-\sqrt{5}}{18}}{\frac{275}{81}}\\
&= \frac{30\cdot 81}{275\cdot 18} - \frac{9}{25}\\
&= \frac{54}{110} - \frac{9}{25}\\
&= \frac{36}{275}.
\end{aligned}$$

So, $m+n=311$. □

**Problem 10.** For a sequence $s=(s_1,s_2,\dots,s_n)$, define

$$F(s)=\sum_{i=1}^{n-1}(-1)^{i+1}(s_i-s_{i+1})^2.$$

Consider the sequence $S=(2^1,2^2,\dots,2^{1000})$. Let $R$ be the sum of all $F(m)$ for all non-empty *subsequences* $m$ of $S$. Find the remainder when $R$ is divided by 1000.

*Note: A subsequence is a sequence that can be obtained from another sequence by deleting some non-negative number of values without changing the order.*

*Solution.* □

**Problem 11.** One face of a tetrahedron has sides of length 3, 4, and 5. The tetrahedron's volume is 24 and surface area is $n$. If $n=a\sqrt{b}+c$, where $a$, $b$, and $c$ are integers and $b$ is not divisible by the square of any prime, evaluate $a+b+c$.

*Solution.* Let the base triangle be $ABC$ so that $AB=3$, $AC=4$, and $BC=5$. Let $D$ be the final vertex of the tetrahedron and $E$ be the foot from $D$ to the plane defined by $ABC$. Finally, let the distance from $E$ to $AB$, $AC$, and $BC$ be $x$, $y$, and $z$ respectively and the altitudes from $D$ to $AB$, $AC$, and $BC$ be $p$, $q$, and $r$. □

**Problem 12.** A function $f$ defined across the real numbers satisfies

$$f(x+y)=f(x)+f(y)+f(xy).$$

Find $f(2024)$.

*Solution.* The answer would be 0. □

**Problem 13.** A polynomial $p$ defined across the real numbers satisfies $p(1)=1$ and

$$p(x+y)=p(x)+p(y)+xy.$$

Find $p(2024)$.

*Solution.* If $y = 0$, we see that

$$f(x) = f(x) + f(0) \implies f(0) = 0.$$

In general,

$$f(2x) = 2f(x) + x^2$$

If $y = -x$ we see that

$$f(x) + f(-x) = x^2.$$

Now, if $f$ has even powers of $x$, then they will appear in the above sum. So, $f$ only has $\frac{1}{2}x^2$ as an even power. The rest are odd. Let $g(x) = f(x) - \frac{1}{2}x^2$. We have that

$$g(x) + g(-x) = 0$$

and

$$g(x+y) + \frac{1}{2}(x+y)^2 = g(x) + g(y) + xy + \frac{1}{2}(x^2+y^2)$$
$$g(x+y) = g(x) + g(y).$$

Question has already been asked. $\square$