Math 073 Notes

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§1.1 3-Dimensional Space

Definition 1 (Euclidian Space)**.** This is your standard (x, y, z) coordinate system. Basically just an extension of the 2-dimensional plane into the third dimension.

Definition 2 (Cylindrical Coordinates)**.** Points are represented as (x, y, θ), where x is the horizontal distance from the origin, y is the vertical distance, and θ is the azimuthal angle.

Definition 3 (Spherical Coordinates)**.** Points are represented as (r, θ, γ) where r is the distance from the origin, θ is the azimuthal angle (I think), and γ is the polar angle.

Definition 4 (Plane in 3D Space)**.** A plane in 3D space can be represented by the equation

$$ax + by + cz = k$$

where the vector $\langle a, b, c\rangle$ is the normal vector to the plane and k is a displacement constant.

Definition 5 (Real Spaces)**.** The notation $\mathbb{R}_n$ denotes the space of real numbers in n dimensions.

Example 6

Find the surface bounded by

$$x^2 + y^2 + z^2 = 4y.$$

Solution. We complete the square on y to get that

$$x^2 + (y - 2)^2 + z^2 = 4$$

and see that what results is a spherical shell with radius 2 centered at $(0, 2, 0)$. □

Example 7

Find the volume of the region bounded by the intersection of $x^2 + y^2 + z^2 = 4$ and $(x-2)^2 + (y+1)^2 + (z-2)^2$.

Solution. The distance between the two spheres is 3, and both have a radius of two. So, we can take the volume revolution of one circle centered at the origin from $x = \frac{3}{2}$ to $x = 2$ and double the result. So,

$$\begin{aligned} V &= \pi \int_{\frac{3}{2}}^{2} 4 - x^2 dx \\ &= 4\pi x - \frac{\pi}{3} x^3 \Big|_{x=\frac{3}{2}}^{x=2} \\ &= \frac{11\pi}{24}. \end{aligned}$$

We now double the result to yield a final answer of $\frac{11\pi}{12}$. □

§1.2 Vectors

Definition 8 (Vector)**.** A vector is a direction and magnitude in some n-dimensional space.

Definition 9 (Vector Component Form)**.** For a vector $\vec{v}$ in $\mathbb{R}_n$ space, the component form of $\vec{v}$ is

$$\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle .$$

Definition 10 (Vector Magnitude)**.** A vector $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ in $\mathbb{R}_n$ space has magnitude

$$||\vec{v}|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Remark 11 (Vector and Scalar Similarity) — Vectors act much like scalars in many ways. Their addition and subtraction are commutative and associative but their multiplication is more complicated, given the two types (dot product and cross product). However, multiplication of a vector by a scalar and the vector dot product are commutative, associative, and distributive.

Definition 12 (Vector Addition)**.** Given two vectors $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\vec{u} = \langle b_1, b_2, \ldots, b_n \rangle$ have sum,

$$\vec{v} + \vec{u} = \langle a_1 + b_1, a_2 + b_2, \ldots a_n + b_n \rangle .$$

Note: The two vectors need not have the same dimensionality. All missing dimensions in the vector with lower dimensionality can be treated as zero.

Definition 13 (Vector Subtraction)**.** Vector subtraction is defined analogously to vector addition.

Definition 14 (Vector Multiplication by a Scalar)**.** For a vector $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ and a constant k,

$$k\vec{v} = \langle ka_1, ka_2, \ldots, ka_n \rangle .$$

Corollary 15 (Unit Vector Form)

From the above, it follows that we can represent a vector as the sum of unit vectors. For example, a 3-dimensional vector $\vec{v} = \langle x, y, z \rangle$ can be written as $\vec{v} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ where $\hat{\imath} = \langle 1, 0, 0 \rangle$, $\hat{\jmath} = \langle 0, 1, 0 \rangle$, and $\hat{k} = \langle 0, 0, 1 \rangle$.

Definition 16 (Unit Vectors)**.** The unit vector $\hat{v}$ in the direction of vector $\vec{v}$ is

$$\hat{v} = \frac{1}{||\vec{v}||}\vec{v}.$$

Definition 17 (Vector Dot Product)**.** The dot product of two vectors $\vec{v} = \langle a_1, a_2, \ldots, a_n \rangle$ and $\vec{u} = \langle b_1, b_2, \ldots, b_n \rangle$ is

$$\vec{v} \cdot \vec{u} = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

This dot product can also be expressed in the form

$$\vec{v} \cdot \vec{u} = ||\vec{v}|| \, ||\vec{u}|| \cos\theta$$

where θ is the angle between the two vectors.

Remark 18 — Note that the dot product will produce a zero result if and only if the two vectors are perpendicular or one of the vectors is a zero-vector.

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§2.1 Cross Products

Aside from the dot product, there is one more product to consider between vectors. This, of course, is the cross product, an operation that results in a vector output and does not follow the typical rules of multiplication that the dot product follows.

Definition 19 (Vector Cross Product)**.** In $\mathbb{R}_3$ space, the cross product of $\vec{u} = \langle a_1, a_2, a_3 \rangle$ and $\vec{v} = \langle b_1, b_2, b_3 \rangle$ is

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}.$$

Remark 20 — Take note that the vector cross product is only generally defined in three dimensions (and seven, but that's beyond the scope of this course).

Example 21

Find the cross product of $\vec{u} = \langle 4, 3, 1 \rangle$ and $\vec{v} = \langle 5, -7, 2 \rangle$.

Solution. Using our above definition, we see that

$$\begin{aligned} \vec{u} \times \vec{v} &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 3 & 1 \\ 5 & -7 & 2 \end{bmatrix} \\ &= \langle 3 \cdot 2 - (-7) \cdot 1, -4 \cdot 2 + 5 \cdot 1, 4 \cdot (-7) - 5 \cdot 3 \rangle \\ &= \langle 13, -3, -43 \rangle . \end{aligned}$$

□

Remark 22 — Another important note is that the cross product always returns a vector that is perpendicular to both input vectors (unless they are parallel, in which the zero-vector is returned).

Example 23

Prove that in general, the cross product returns an orthogonal vector to both input vectors.

Proof. Let vector $\vec{u} = \langle a_1, a_2, a_3 \rangle$ and $\vec{v} = \langle b_1, b_2, b_3 \rangle$. From our original definition of the cross product we derive that

$$\vec{u} \times \vec{v} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle .$$

Now consider $\vec{u} \cdot (\vec{u} \times \vec{v})$, which we wish to show is zero. It follows that

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_1 a_2 b_3 + a_1 a_3 b_2 - a_2 a_3 b_1.$$

Miraculously, all the terms cancel out. Without loss of generality, we apply the same logic to $\vec{v}$ and we are done. □

Going back to the above definition of a cross product, we can arrive at a trigonometric association for the cross product that is similar to the one established for the dot product.

Proposition 24

Two vectors $\vec{u} = \langle a_1, a_2, a_3 \rangle$ and $\vec{v} = \langle b_1, b_2, b_3 \rangle$ satisfy

$$||\vec{u} \times \vec{v}|| = ||\vec{u}|| \, ||\vec{v}|| \sin \theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$.

In the beginning, I note that the cross product has slightly different multiplicative properties than the dot product. Because of this, I list the specific properties below.

- **Anticommutivity:** $\vec{a} \times \vec{b} = -(\vec{a} \times \vec{b})$.
- **Scalar Multiplication:** $(k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$.
- **Distributivity:** $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
- **Non-Associativity:** In general, $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$.

Now, we explore an example that utilizes the trigonometric representation above.

Example 25 (Volume of a Parallelepiped)
Prove that the volume of a parallelepiped is

$$V = \vec{u} \cdot (\vec{v} \times \vec{w})$$

where the vectors represent the three sets of parallel sides.

Proof. Rearranging the above we have

$$\begin{aligned} V &= ||\vec{u}||\,||\vec{v} \times \vec{w}|| \cos\theta \\ &= ||\vec{u}||\,||\vec{v}||\,||\vec{w}|| \cos\theta \sin\alpha \end{aligned}$$

where θ is the angle between $\vec{u}$ and the altitude h between the bases determined by sides in the directions of $\vec{v}$ and $\vec{w}$ and α is the angle between $\vec{v}$ and $\vec{w}$. It is easy to see that $h = ||\vec{u}|| \cos\theta$ and the area of the base $b = ||\vec{v}||\,||\vec{w}|| \sin\alpha$. So we attain $V = bh$, the standard volume formula. □

§2.2 Lines

Lines in 3-dimensional space are not written in the same way that they are in two-dimensional space. In fact, they must be paramaterized or written using symmetric equation (which we get to later).

Definition 26. A line in $\mathbb{R}_3$ space is written in the form $\vec{p} + t\vec{u}$ where $\vec{p}$ represents a position vector, $\vec{u}$ represents a direction vector, and t is a varied parameter.

Remark 27 — Often, $\vec{u}$ in the above is a unit vector, but this is not necessary.

Theorem 28 (Distance between a Point and a Line in 3D)
Given an arbitrary point P in $\mathbb{R}_3$ space not on line ℓ and a point Q on ℓ, let the vector from P to Q be $\vec{u}$ and $\vec{v}$ be a vector parallel to ℓ. The shortest distance from P to ℓ is

$$d = \frac{||\vec{u} \times \vec{v}||}{||\vec{v}||}.$$

Example 29 (Distance between a Point and a Plane in 3D)

Find a general formula for the distance between a point and a plane not containing that point in $\mathbb{R}_3$ space.

Solution. Let our point be P and let Q be an arbitrary point on plane $\mathcal{P}$. Let $\vec{u}$ be the vector from P to Q and let $\vec{v}$ be a vector parallel to the normal vector of $\mathcal{P}$. It follows that the distance is the magnitude of the projection of $\vec{u}$ onto $\vec{v}$, which is defined by

$$d = |\text{proj}_{\vec{v}}\,\vec{u}| = \frac{|\vec{u} \cdot \vec{v}|}{||\vec{v}||}.$$

□

§2.3 The Jacobian

Now that's the end of all the main material that was covered in class, but the professor also brought up the topic of Jacobians, so I include them here. In essence, the Jacobian is a matrix that approximates how a vector-valued function changes at a point. A rigorous definition is below.

Definition 30 (The Jacobian Matrix)**.** A function $f : \mathbb{R}_n \to \mathbb{R}_m$ has Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

While rigorous, the above definition is likely a bit overkill for what we may cover in the class. So, just in case, I include a simpler example.

Example 31 (A Simple Jacobian)

Given a function $f(x, y, z) = xyz + x^2y^2z^2$, find its Jacobian matrix.

Solution. We know that the Jacobian matrix is

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{bmatrix}$$

from our definition above. What's left is to calculate the partial derivatives, yielding an answer of

$$J = \begin{bmatrix} yz + 2xy^2z^2 & xz + 2x^2yz^2 & xy + 2x^2y^2z \end{bmatrix}.$$

□

Remark 32 — Now, we have not yet gone over partial derivatives. However, to evaluate them you only must consider change in one variable. The derivative rules from previous calculus classes still apply, but all other variables are treated as constants.

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§3.1 More Lines

Lines are always defined using a point and a direction. For example, standard $\mathbb{R}_2$ lines are written in the form $y = mx + b$, where $(0, b)$ is a point on the line and m represents the slope, or direction. However, lines in $\mathbb{R}_3$ space and higher spaces cannot be defined with one simple equation only containing scalar inputs. In the last class, we gave the definition of a line $\ell : \vec{p} + t\vec{u}$. However, lines can also be written as a collection of parametric equations as seen in the example below.

Example 33

Line ℓ contains point $\langle 1, 1, 2\rangle$ and has direction $\langle 1, 2, 4\rangle$. Write a collection of parametric equations that represent ℓ.

Solution. To write each parametric equation, we take the corresponding components of a known point and the direction. For example, when solving for x, we have

$$x = p_x + u_x t$$

Doing this for each variable yields an answer of

$$x = 1 + t$$

$$y = 1 + 2t$$

$$z = 2 + 4t.$$

□

If we solve for t in each of the three equations above, we arrive at a symmetric equation of a line.

Corollary 34 (Symmetric Equation of a Line)

Given a line ℓ containing point $\vec{p}$ with direction $\vec{u}$, ℓ can be represented by the symmetric equation

$$\frac{x - p_x}{u_x} = \frac{y - p_y}{u_y} = \frac{z - p_z}{u_z}.$$

Now, if you recall $\mathbb{R}_2$ space, lines are either parallel or intersecting. However, in $\mathbb{R}_3$ space and beyond, lines can also be skew.

Definition 35 (Skew Lines)**.** Two lines in space are skew if they are neither parallel nor intersecting.

For the next two propositions consider distinct lines $\ell_1 : \vec{p}_1 + t\vec{u}_1$ and $\ell_2 : \vec{p}_2 + t\vec{u}_2$.

Theorem 36 (Parallel Lines)
Lines ℓ_1 and ℓ_2 are parallel if and only if $\vec{u}_1 = k\vec{u}_2$ for some scalar constant k.

Theorem 37 (Intersecting Lines)
Lines ℓ_1 and ℓ_2 intersect if and only if there exists t and s such that

$$\vec{p}_1 + t\vec{u}_1 = \vec{p}_2 + s\vec{u}_2.$$

Otherwise, the lines are parallel or skew.

Example 38
Determine if the lines $\ell_1 : \frac{x}{1} = \frac{y-1}{-1} = \frac{z-2}{3}$ and $\ell_2 : \frac{x-2}{2} = \frac{y-3}{2} = \frac{z}{7}$ are parallel, intersecting, or skew.

Solution. It's clear to see that the direction vectors are not multiples of each other, so the lines are not parallel. We now attempt to find a common intersection point. First, we look at x. We know that the x-valeus are the same when

$$t = 2s + 2.$$

Now, looking at y and doing the same, we see that

$$1 - t = 2s + 3.$$

We add the two equations and see that

$$1 = 4s + 5 \implies s = -1, t = 0.$$

So, the lines are skew. □

§3.2 Planes

Now, we move to planes. In previous notes, we deduced the equation of a plane, but let's prove why.

Example 39 (Equation of a Plane)
Find, with proof, the equation of a plane in $\mathbb{R}_3$.

Solution. Let plane $\mathcal{P}$ have a normal vector $\vec{u} = \langle a, b, c \rangle$. Also, define a vector $\vec{v} = \langle x - x_0, y - y_0, z - z_0 \rangle$ from a fixed point in $\mathcal{P}$ to any arbitrary point in $\mathcal{P}$. Since $\vec{u}$ is a normal vector to $\mathcal{P}$, we know that $\vec{u} \perp \vec{v}$. So,

$$\vec{u} \cdot \vec{v} = a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Since $\vec{v}$ represents a vector to any arbitrary point in $\mathcal{P}$, we have the equation of the plane. □

Example 40

Find the equation of the plane containing $X = (2, 3, 5)$, $Y = (4, -1, 6)$, and $Z = (1, 9, -7)$.

Solution. We have $\vec{XY} = \langle 2, -4, 1 \rangle$ and $\vec{YZ} = \langle -3, 10, -13 \rangle$. We already know a fixed point on our plane, so we are left to find a normal vector. To do so, we take the cross product of the two vectors determined above.

$$\begin{aligned} \vec{XY} \times \vec{YZ} &= \det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -4 & 1 \\ -3 & 10 & -13 \end{bmatrix} \\ &= \langle 42, 23, 8 \rangle . \end{aligned}$$

Without loss of generality, we select X as a point within the plane and have the equation

$$42(x - 2) + 23(y - 3) + 8(z - 5) = 0.$$

$\square$

Example 41

Determine if line $\ell : \langle 1 + t, 2 - t, 4 - 3t \rangle$ and plane $\mathcal{P} : 5x + 2y + z = 1$ are parallel.

Solution. We know that $\ell \parallel \mathcal{P}$ if and only if the direction vector of ℓ is orthogonal to a normal vector of $\mathcal{P}$. One such normal vector is $\vec{u} = \langle 5, 2, 1 \rangle$. The direction of ℓ is $\vec{v} = \langle 1, -1, -3 \rangle$. So,

$$\vec{u} \cdot \vec{v} = 5 - 2 - 3 = 0.$$

We thus conclude that $\ell \parallel \mathcal{P}$. $\square$