Lecture 1: Fundamentals

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Goals

I hope that this course will be useful to you in one of two ways:

1. **If you have taken math courses in these topics previously**, these lectures can help remind you of the key takeaways from these courses as you prepare for your masters degree curriculum.
2. **If you have not taken these math courses**,
    - Treat these lectures as a survey overview. I will make you aware of different tools that will be useful for your upcoming courses.
    - Present a "map" of mathematics while emphasizing the connections to finance and economics and the motivations for learning a given mathematical concept for our applied purposes.

At the end of each section lecture notes, I provide resources for deeper learning and practice for topics in which you feel you need additional review.

MS Math Camp Overview

1. Fundamentals
2. Single Variable Calculus
3. Linear Algebra
4. Multivariable Calculus*
5. Optimization
6. Probability and Statistics

Today: Fundamentals

Fundamentals

1. Sets
   - Examples of Sets
   - Set Operations
   - Properties of Point Sets in $\mathbb{R}^n$
2. Functions
   - Examples of Functions
3. Proofs
   - Proof Techniques

Motivation

The tools of mathematics allow economists to make simple assumptions and represent complex phenomena as stylized mathematical relationships in order to make useful deductions about the world.

- The **Black-Scholes Model** derives the prices of options contracts under certain assumptions, including that prices follow a lognormal distribution.
- The **Solow-Swan Model** explains long-run economic growth using a nonlinear system of a single ordinary differential equation to model the evolution of the per-person stock of capital.
- The **Capital Asset Pricing Model (CAPM)** determines the appropriate required rate of return on a given asset under assumptions about how the market functions and how risk can be measured.

Motivation

In order to understand, use, and create new and better economic models, we must be fluent in the mathematics used to formulate them.

To begin our survey of mathematics that are used in economic and financial analysis, let us review:

1. Two essential objects in the language of mathematics: **sets** and **functions**.
2. How the language of mathematics allows logical deductions from a set of assumptions via **proofs**.

Sets

A SET IS A MANY THAT ALLOWS ITSELF TO BE THOUGHT OF AS A ONE – GEORG CANTOR

Definition (Set)
A set B is any collection of items (elements) x thought of as a whole.

- In economics we will usually think of sets with elements in $\mathbb{R}$.
- The principal concept of set theory is that of *belonging*.
- Sets can be defined by enumerating elements $x \in B$, or by stating a property.

  $$B = \{2, 4, 6, 8, 10\} = \{x | x \text{ is an even number between 1 and 11}\}$$

  Then e.g. $2 \in B$ but $3 \notin B$.
- Sets can have infinitely many elements. For example, the set of real numbers $\mathbb{R}$ is an uncountably infinite set.

Definition (Subset)
If all the elements of a set A are also elements of a set B then, A is a subset of B, $A \subseteq B$. Note that if $C \subseteq B$ and $B \subseteq C$, $\implies C = B$.

$$A = \{2, 4\} \implies A \subset B$$

Examples of Sets

Let us further define a few important sets:

Definition (Null Set)
The empty set or the null set is the set with no elements $\emptyset$. $\emptyset \subset S, \forall S$

Definition (Natural numbers)
The natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$

Definition (Integers)
The integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$

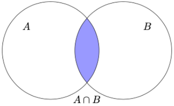
Definition (Rationals)
The rational numbers $\mathbb{Q} = \{\frac{p}{q} \quad | \quad \forall p, q \in \mathbb{Z}, q \neq 0\}$

Definition (Real numbers)
The real numbers $\mathbb{R}$ are the set of numbers that can be represented as a (possibly infinite) decimal expansion, e.g. $\pi = 3.14159...$

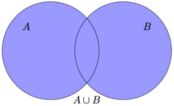
Often the set of real numbers $\mathbb{R}$ is visualized as a straight line, *the Real Line*, on which numbers are *points*. Note that $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are all strict subsets of $\mathbb{R}$.

Set Operations

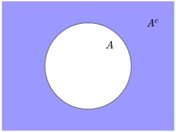
Definition (Intersection)
The intersection C of two sets A and B is the set of elements that are in **both** A and B: $C = A \cap B = \{x|x \in A \wedge x \in B\}$



Definition (Union)
The union D of two sets A and B is the set of elements that are in **either** A or B $D = A \cap B = \{x|x \in A \vee x \in B\}$



Definition (Complement)
The complement A' of a set A is the set of elements (in our "universal" set $U = \mathbb{R}$ or $\mathbb{N}$) that are not in A: $A' = \{x| \notin A\}$



Some Properties of Point Sets in $\mathbb{R}^n$

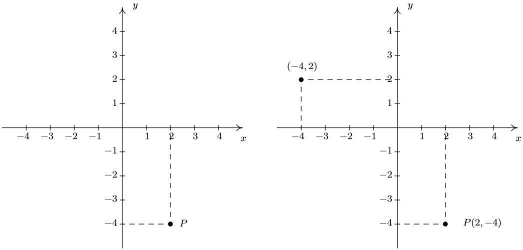
Some important point sets of $\mathbb{R}$ are **intervals**:

Definition (Closed Interval)
$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$ *contains boundary points*

Definition (Half-Open Interval)
$(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$

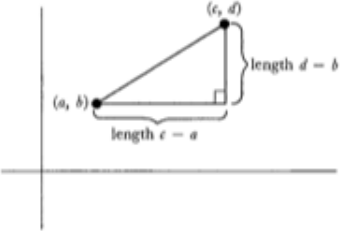
Definition (Open Interval)
$(a, b) = \{x \in \mathbb{R} | a < x < b\}$ *excludes boundary points*

Definition (Cartesian Plane $\mathbb{R}^2$)
$\mathbb{R}^2$ is the set of **ordered** pairs (x, y) formed by the cartesian product of $\mathbb{R} \otimes \mathbb{R}$

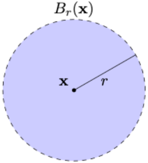


Similarly, we can think about $\mathbb{R}^3$ or $\mathbb{R}^n$ for $n \in \mathbb{N}$. We will use geometric pictures for the purpose of aiding intuition, and define distance as follows so that the Pythagorean theorem is built into our geometry:

Definition (Euclidean Distance)
The Euclidean distance between two elements of $\mathbb{R}^n$ is given by the length of the line segment between them:
$d(a, b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$



Some Properties of Point Sets in $\mathbb{R}^n$

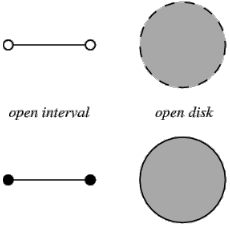


Definition (ϵ-Neighborhood)
An ϵ-neighborhood of a point $x_0 \in \mathbb{R}^n$ is given by the set $N_\epsilon(x_0) = \{x \in \mathbb{R}^n | d(x_0, x) < \epsilon\}$

Of course, when in dimension $n = 1$, this is just an open interval. Similarly, for $n = 2$, this is disc, or circle. In higher dimensions, it can be referred to as a ball or sphere.

Definition (Open Set)
A set $X \subset \mathbb{R}^n$ is open if for every $x \in X$, there exists an $\epsilon > 0$ such that $N_\epsilon(x) \subset X$

Definition (Closed Set)
A set $X \subset \mathbb{R}^n$ is **closed** if its complement X' is an open set.



open interval *open disk*

closed interval *closed disk*

Some Properties of Point Sets in $\mathbb{R}^n$

Definition (Bounded)
A set $X \subset \mathbb{R}^n$ is bounded if for every $x_0 \in X$, there exists $\epsilon < \infty$ such that $X \subset N_\epsilon(x_0)$

- In particular, a set A of real numbers is **bounded above** if there is a number x such that $x \geq a$ for every $a \in A$.
- Such a number x is called an **upper bound** for A.
- The set of real numbers $\mathbb{R}$ and the natural numbers $\mathbb{N}$ are two examples of sets that are **not** bounded above.
- An example of a set that is bounded above is $A = \{x | 0 \leq x < 1\}$. Further, 1 is the **least upper bound** or **supremum** of A, i.e. the smallest such bound. Analogously, we can think of sets **bounded below** and **greatest lower bounds** or **infimums**.

Theorem (Least Upper Bound Property or The Completeness Axiom)

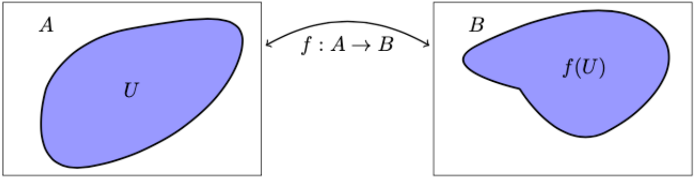
If A is a set of real numbers, $A \neq \emptyset$, and A is bounded above, then A has a least upper bound.

Functions

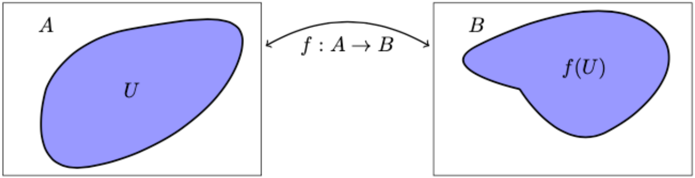
> UNDOUBTEDLY THE MOST IMPORTANT CONCEPT IN ALL OF MATHEMATICS IS THAT OF A FUNCTION – IN ALMOST EVERY BRANCH OF MODERN MATHEMATICS FUNCTIONS TURN OUT TO BE THE CENTRAL OBJECTS OF INVESTIGATION.
> – MICHAEL SPIVAK, CALCULUS

Functions are an object of great generality but for applications to finance and economics, we can restrict our attention to a small subset of functions.

Definition (Function)
Given two sets A, B a **function** $f : A \mapsto B$ is a map which assigns to every element in A a **unique** element in B.



Functions



If $a \in A$, we usually denote the corresponding element of B by $f(a)$. A is the **domain** and B is the **codomain** of function f.

Definition (Domain)

The **domain** of function f is the set $\{a \in A | \exists! b \in B \text{ s.t. } b = f(a)\}$.

Definition (Range)

The **range** of function f is the set $\{b \in B | \exists a \in A \text{ s.t. } b = f(a)\}$.

Definition (Injective Function)

A function f is **injective** or **one-to-one** if $f(x) = f(y) \implies x = y$

Definition (Surjective Function)

A function f is **surjective** if the range of f is B.

Examples of Functions

Let us further define a few important types of functions:

Definition (Linear Function)
$f : \mathbb{R} \mapsto \mathbb{R}$: $f(x) = ax + b = y$ for $a, b \in \mathbb{R}$ where a is the slope and b is the intercept

Definition (Quadratic Function)
$f : \mathbb{R} \mapsto \mathbb{R}$: $f(x) = ax^2 + bx + c = y$ for $a, b, c \in \mathbb{R}$

The simplest quadratic function is the **parabola**, $f(x) = x^2$.

Definition (Polynomial Function)
$f : \mathbb{R} \mapsto \mathbb{R}$: $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$ for $a_i \in \mathbb{R}$

The simplest polynomial function is the **power function**, $f(x) = x^n$ for $n \in \mathbb{N}$.

Definition (Graph)
We can draw the graph of a function $(x, f(x))$ on the Cartesian Plane $\mathbb{R}^2$

In contrast, some of the simplest and most important subsets of the Cartesian Plane are not the graphs of functions. These include the circle, or ϵ-neighborhood defined above. Similarly, ellipses and hyperbolas are not the graphs of a function. *Why?*

Proofs

> Pure mathematics is, in its way, the poetry of logical ideas. –Albert Einstein

- A proof is a mathematical argument intended to convince us that a result is correct.
- A proof of a theorem is thus a series of logical deductions using the assumptions of the theorem, the definitions of the terms involved, and previous results that have been proven.
- In mathematics, a statement (or proposed theorem, etc.) is an assertion that is either **true** or **false**.

Proofs

Many mathematical theorems can be expressed symbolically in the form $P \implies Q$, i.e. P *implies* Q.

- The statement P is the *assumption* of the theorem, and statement Q is the *conclusion*.
- Equivalence theorems can be expressed as $P \iff Q$, which means the same thing as $P \implies Q$ and $Q \implies P$.

Similarly, economic analysis is mainly concerned with deductive statements that require a logical proof, such as *if the money supply increases, then the price level will rise*. Or, even stronger statements such as *the price level rises* **if and only if** *the money supply increases*.

Proof Techniques

Often to prove a statement, we will use one of the standard proof techniques:

1. **Direct Proof:** the conclusion is established by logically combining the axioms, definitions, and earlier theorems.
2. **Contrapositive:** $P \implies Q$ is proved by showing the logically equivalent contrapositive statement, $\neg Q \implies \neg P$.
3. **Proof by Contradiction** or *reductio ad absurdum*: it is shown that if some statement is assumed true, a logical contradiction occurs, and hence the statement must be false.
4. **Proof by Induction:** A base case is provided, and an induction rule is proved that establishes that any arbitrary case implies the next case. Since the induction rule can be applied repeatedly, it follows that all cases are provable.

Direct Proof

Direct Proof: the conclusion is established by logically combining the axioms, definitions, and earlier theorems.

Theorem (Example of Direct Proof)

The sum of two even integers is always even, where even is defined to be an integer that has a factor of 2.

Proof.

- Consider two even integers x, y.
- Then we know that they can be written as $x = 2a$ and $y = 2b$ respectively, for $a, b \in \mathbb{Z}$.
- Then $x + y = 2(a + b)$ has 2 as a factor, and thus by definition is even.

$\square$

Contrapositive Proof

Contrapositive Proof Method: $P \implies Q$ is proved by showing the logically equivalent contrapositive statement, $\neg Q \implies \neg P$.

Theorem (Example of Contrapositive Proof)

Given an integer x, if x^2 is even, then x is even:

Proof.

- Suppose x is not even.
- Then x is odd.
- The product of two odd numbers is odd, thus $x^2 = x * x$ is odd.
- Thus x^2 is not even.
- We have shown the contrapositive, which implies that if x^2 is even it must be that x is even.

$\square$

Proof by Contradiction

Proof by Contradiction: it is shown that if some statement is assumed true, a logical contradiction occurs, and hence the statement must be false.

Theorem (Example of Proof by Contradiction)

$\sqrt{2}$ *is irrational*

Proof.

- Suppose $\sqrt{2} \in \mathbb{Q}$. Then it could be written in lowest terms as $\sqrt{2} = \dfrac{a}{b}$ where $a, b \in \mathbb{Z} \setminus 0$ with no common factor.
- Thus $b\sqrt{2} = a \implies 2b^2 = a^2$. Since this equality shows that a^2 is even, it must be that a is even by the result in the example above of the contrapositive proof, so that 2 is a factor of a. Thus we can write $a = 2c$ for $c \in \mathbb{Z}$.
- Substituting this in, we get $2b^2 = (2c)^2 \implies b^2 = 2c^2$ and so b must also be even, so that 2 is a factor of b.
- This **contradicts** that a, b had no common factor. Thus it must be that $\sqrt{2} \notin \mathbb{Q}$, $\sqrt{2}$ is irrational.

$\square$

Proof by Induction

In proof by mathematical induction, a base case is provided, and an induction rule is proved that establishes that any arbitrary case implies the next case. Since the induction rule can be applied repeatedly, it follows that all cases are provable.

Theorem (Example of Proof by Induction)

All positive integers in the form of $2n - 1$ *are odd. Let* $P(n)$ *denote the statement "*$2n - 1$ *is odd".*

Proof.

1. Base Case: For $n = 1$, $2n - 1 = 2(1) - 1 = 1$ and is odd, since it does not have a factor of 2. Thus $P(1)$ is true.
2. Induction: For any n, if $2n - 1$ is odd such that $P(n)$ is true, then $(2n - 1) + 2$ must also be odd, because adding 2 to an odd number results in an odd number. But $(2n - 1) + 2 = 2n + 1 = 2(n + 1) - 1$ which is $P(n + 1)$. Thus, $P(n) \implies P(n + 1)$.

Thus, $2n - 1$ is odd, for all positive integers n. $\square$

Today: Fundamentals

Fundamentals

1. Sets
   - Examples of Sets
   - Set Operations
   - Properties of Point Sets in $\mathbb{R}^n$
2. Functions
   - Examples of Functions
3. Proofs
   - Proof Techniques

In lecture notes: more set properties.