Lecture 2: Calculus

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Today: Calculus

Much of financial and economic analysis is concerned with marginal analysis, i.e. how a change in the level of one variable x determines a change in the level of another variable y. Calculus provides tools to analyze these changes.

1. Limits
2. Continuous Functions
3. Derivatives
4. Integrals
    - Techniques of Integration
5. Infinite Series
6. Approximation by Polynomial Functions

Limits

The concept of a limit is surely the most important, and probably the most difficult one in all of Calculus.
– Michael Spivak, Calculus

Intuitively, the function f approaches the limit l near a, if we can make $f(x)$ as close as we like to l by requiring that x be sufficiently close, but unequal to, a.

Definition (Limit)
The function f approaches the limit l near a if for every $\epsilon > 0$, there is some $\delta > 0$ such that, for all x, $0 < |x - a| < \delta$, then $|f(x) - l| < \epsilon$

- This definition gives rise to the *epsilon-delta method* of determining whether a limit exists.
- It is useful to think about what is true when the function f does **not** approach a limit l near a.
- This means that there is **some** $\epsilon > 0$, such that **for every** $\delta > 0$, there is **some** x which satisfies $0 < |x - a| < \delta$ but not $|f(x) - l| < \epsilon$.

Let's see an example of this flavor of reasoning.

Limits Are Unique

Theorem (Limits are unique)

A function cannot approach two different limits near a. If f approaches l near a, and f approaches m near a, then $l = m$.

Let us assume that $l \neq m$. We are given that f approaches l near a, so by the definition of a limit, we have that for any $\epsilon > 0$, there is some number $\delta_1 > 0$ such that, for all x,

$$\text{if } 0 < |x - a| < \delta_1, \text{ then } |f(x) - l| < \epsilon$$

Similarly, we know that f approaches m near a, so again have that there is some $\delta_2 > 0$ such that for all x,

$$\text{if } 0 < |x - a| < \delta_2, \text{ then } |f(x) - m| < \epsilon$$

Taking the minimum of δ_1 and δ_2, we then have that there exists δ such that

$$\text{if } 0 < |x - a| < \delta, \text{ then } |f(x) - l| < \epsilon \text{ and } |f(x) - m| < \epsilon$$

Theorem (Limits are unique)

A function cannot approach two different limits near a. If f approaches l near a, and f approaches m near a, then l = m.

$$\text{if } 0 < |x-a| < \delta, \text{ then } |f(x) - l| < \epsilon \text{ and } |f(x) - m| < \epsilon$$

This holds for any ϵ. Since we have assumed that $l \neq m$, we know that $|l - m| > 0$ and so can choose $\epsilon = \dfrac{|l-m|}{2}$. This gives that

$$\text{if } 0 < |x-a| < \delta, \text{ then } |f(x) - l| < \frac{|l-m|}{2} \text{ and } |f(x) - m| < \frac{|l-m|}{2}$$

So for $0 < |x-a| < \delta$, we have

$$|l-m| = |l - f(x) + f(x) - m| \leq |l - f(x)| + |f(x) - m| < |l-m|$$

Contradiction. So, $l = m$.

Theorem (Operations with Limits)

If $\lim_{x\to a} f(x) = l$ *and* $\lim_{x\to a} g(x) = m$ *then:*

1. $\lim_{x\to a}(f+g)(x) = l+m$
2. $\lim_{x\to a}(f \cdot g)(x) = l \cdot m$
3. $\lim_{x\to a} \dfrac{1}{g}(x) = \dfrac{1}{m}$ *if* $m \neq 0$.

Continuous Functions

If f is an arbitrary function, it is not necessarily true that $\lim_{x \to a} f(x) = f(a)$.

- There are many ways in which this can fail to be true.
- Functions for which this condition holds are said to be continuous.
- Intuitively, a function f is continuous if the graph contains no breaks, jumps, or wild oscillations.

Definition (Continuous Function)
The function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$

Theorem (Operations with Continuous Functions)

If f and g are continuous at a then:

1. *$f + g$ is continuous at a*
2. *$f \cdot g$ is continuous at a*
3. *If $g(a) \neq 0$, then $1/g$ is continuous at a.*
4. *If g is continuous at a, and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.*

Continuity and IVT

- We have defined continuity of functions at a single point a.
- However, the concept of continuity is particularly useful to us if we focus our attention on functions that are continuous at all points within some interval.
- These functions are usually regarded to be especially well behaved.

Theorem (Intermediate Value Theorem)

If f is continuous on $[a, b]$ and $f(a) < c < f(b)$, then there is some number x in $[a, b]$ such that $f(x) = c$.

This theorem is very useful in economics e.g. to find an equilibrium market-clearing price.

Continuity, Boundedness, Extreme Values

Theorem (Continuous Functions on a Closed Interval are Bounded)

If f is continuous on $[a, b]$, then f is bounded above on $[a, b]$.

Theorem (Extreme Value Theorem)

If f is continuous on $[a, b]$, then there is a number y in $[a, b]$ such that $f(y) \geq f(x)$ for all x in $[a, b]$. I.e., f attains a **maximum** *(and minimum) on the closed interval.*

Archimedean Property of $\mathbb{N}$

Theorem (Archimedean Property of $\mathbb{N}$)

$\mathbb{N}$ is not bounded above.

Proof.
Suppose that $\mathbb{N}$ were bounded above. Since $\mathbb{N} \neq \emptyset$, we know that there exists a least upper bound α for $\mathbb{N}$. Then, $\alpha \geq n$ for all $n \in \mathbb{N}$. And also, $\alpha \geq n + 1$ for all $n \in \mathbb{N}$.

This is because if $n \in \mathbb{N}$ then $n + 1 \in \mathbb{N}$. However, rearranging, we find that $\alpha - 1 \geq n$ for all $n \in \mathbb{N}$. This means that $\alpha - 1$ must also be an upper bound for $\mathbb{N}$, **contradicting** that α was the last upper bound. Thus $\mathbb{N}$ is not bounded above. $\square$

Theorem (Formulating a Small ϵ)

For any $\epsilon > 0$, there is a natural number $n \in \mathbb{N}$ with $\dfrac{1}{n} < \epsilon$.

Proof.
Suppose not. Then $\dfrac{1}{n} \geq \epsilon$ for all $n \in \mathbb{N}$. Thus $n \leq 1/\epsilon$ for all $n \in \mathbb{N}$. But this would mean that ϵ is a least upper bound for $\mathbb{N}$, which **contradicts** the Archimedean property of $\mathbb{N}$ that it is not bounded above. $\square$

Derivatives

The most useful results for us about functions will be obtained once we restrict our attention even further than simply looking at continuous functions.

- Even continuous functions can have peculiarities such as sharp edges at a point, e.g. $f(x) = |x|$ at 0.
- At such points, it may be that a unique line that is tangent to the graph of the function cannot be drawn.

Definition (Differentiable Function)

The function f is differentiable at a if $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists. In this case the limit is denoted by $f'(a)$ and is called the derivative of f at a. We further say that f is differentiable if f is differentiable at a for every a in the domain of f.

- For any function f, we denote by f' the function whose domain is the set of all numbers a such that f is differentiable at a, and whose value at such a number a is precisely $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.
- The function f' is called the **derivative** of f.

Differentiable if Continuous, Higher Order Derivatives

Theorem (Differentiable if Continuous)

If f is differentiable at a, then f is continuous at a. (The converse is not true, e.g. continuous and nowhere differentiable function).

- Thus, differentiability places more restrictions on functions that continuity.
- Yet, we can actually use the concept of a derivative to formulate even more restrictive conditions on functions.

For any function f, we obtain, by taking the derivative, a new function f' whose domain may be smaller than that of f.

- The notion of differentiability may be applied to the function f', yielding another function f'', called the **second derivative** of f.
- Similarly, we can define the **nth derivative** for any $n \in \mathbb{N}$, collectively referred to as higher order derivatives for f.

Taking a Derivative

Theorem (Derivative of a Constant)

If $f(x) = c$ *for* $c \in \mathbb{R}$ *then* $f'(x) = 0$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{c - c}{h} = 0$$

Theorem (Derivative of a Linear Function)

If $f(x) = mx + b$ *for* $m, b \in \mathbb{R}$*, then* $f'(x) = m$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{m(a+h) + b - (ma+b)}{h} = \lim_{h\to 0} \frac{mh}{h} = m$$

Theorem (Derivative of a Power Function)

If $f(x) = x^n$ *then* $f'(x) = nx^{n-1}$

Proof for $n = 2$:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - a^2}{h} = \lim_{h\to 0} 2a + h = 2a$$

Theorem (Derivative of an Exponential Function)

If $f(x) = e^x$ *then* $f'(x) = e^x$

Theorem (Derivative of a Log Function)

If $f(x) = log(x)$ *then* $f'(x) = 1/x$

Theorem (Derivative of constant times a function)

If $f(x) = a * g(x)$ *then* $f'(x) = a * g'(x)$

Derivative Rules

Theorem (Sum Rule)

If $f(x) = g(x) + h(x)$ *then* $f'(x) = g'(x) + h'(x)$

Theorem (Product Rule)

If $f(x) = g(x)h(x)$ *then* $f'(x) = g'(x)h(x) + g(x)h'(x)$

Theorem (Quotient Rule)

If $f(x) = \frac{g(x)}{h(x)}$ *then* $f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2}$ *for* $h(x) \neq 0$

Theorem (Chain Rule)

$(f \circ g)'(a) = f'(g(a)) * g'(a)$

Theorem (L'Hopital's Rule)

$\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$

L'hopital's rule is particularly useful in obtaining the limit of expressions in which both the numerator and denominator of a function diverge to $+\infty$ or 0. For example,

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = \frac{\infty}{\infty} = ? \text{ is difficult to evaluate directly. Applying L'hopital's,}$$

$$\lim_{x\to\infty} \frac{(\ln x)'}{(\sqrt{x})'} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0$$

Mean Value Theorem

Theorem (Mean Value Theorem)

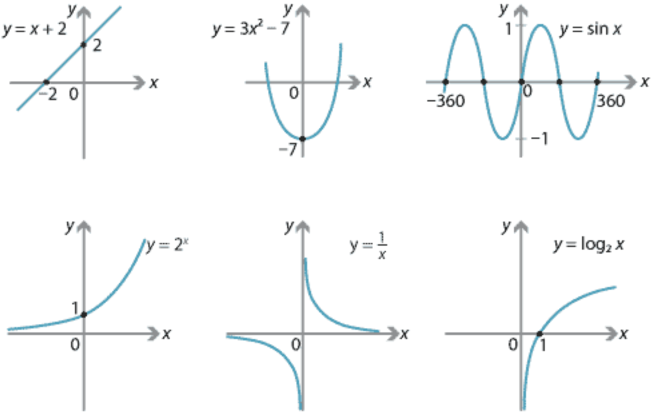
If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number x in (a, b) such that $f'(x) = \dfrac{f(b) - f(a)}{b - a}$

Corollary
If f is defined on an interval and $f'(x) = 0$ for all x in the interval, then f is constant on the interval.

Corollary
If $f'(x) > 0$ for all x in an interval, then f is increasing on the interval. If $f'(x) < 0$ for all x in an interval, then f is decreasing on the interval.

Recap and Intuition: Graphs

We stated that we can draw the graph of a function $(x, f(x))$ on $\mathbb{R}^2$: A function of one variable taking values from the x axis and mapping to the y axis.



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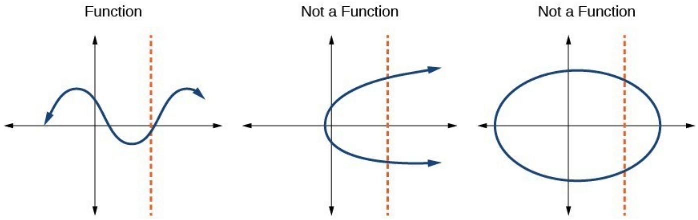
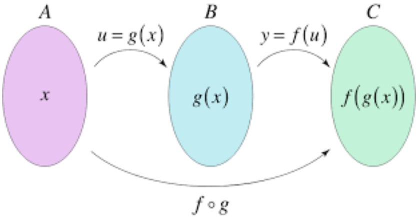


A function maps each value from x axis into a unique element of the y axis!

Definition (Function)
Given two sets A, B a **function** $f : A \mapsto B$ is a map which assigns to every element in A a **unique** element in B.

Recap and Intuition: Composition of Functions



Example:

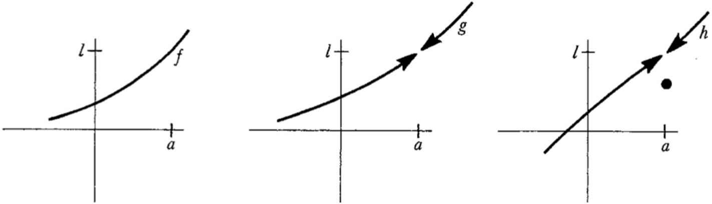
$$
\begin{aligned}
f(x) &= 2x + 3 \\
g(x) &= -x^2 + 5 \\
f(g(x)) &= f(-x^2 + 5) = 2(-x^2 + 5) + 3 = -2x^2 + 13
\end{aligned}
$$

Recap and Intuition: Limits

We begin our study of calculus by trying to find a subset of well-behaved functions.

Definition (Limit)
The function f approaches the limit l near a if for every $\epsilon > 0$, there is some $\delta > 0$ such that, for all x, $0 < |x - a| < \delta$, then $|f(x) - l| < \epsilon$

- The key idea in calculus.
- The limit of $f(x)$ as $x \to a \neq f(a)$ in general!

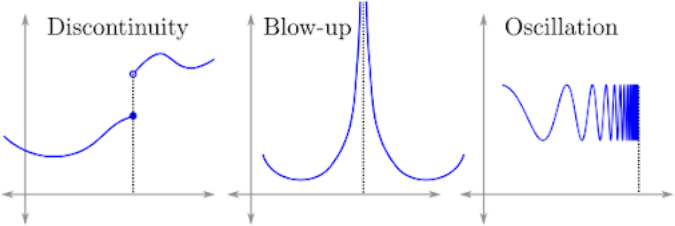


The limit of all 3 functions as $x \to a$ is l, even though g is not defined at a, and $h(a) \neq l$! Perhaps existence of a limit is not a strong enough condition.

Recap and Intuition: Limits

Ways that a limit can fail to exist:

1. The one-sided limits are not equal.
2. The function does not approach a finite value.
3. The function doesn't approach a particular value.



However, a limit can exist if there is a **point** discontinuity as in the previous slide!

Recap and Intuition: Limits

As a practical matter, we want to be able to find the limit of a function at some value a. To do so, we can try the following:

1. Directly plug in a: $\lim_{x\to 10} \dfrac{x}{2} = \dfrac{10}{2} = 5$
2. Factor the function:
$\lim_{x\to 1} \dfrac{x^2-1}{x-1} = \lim_{x\to 1} \dfrac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2$
3. Conjugate:
$\lim_{x\to 4} \dfrac{2-\sqrt{x}}{4-x} = \lim_{x\to 4} \dfrac{2-\sqrt{x}}{4-x} * \dfrac{2+\sqrt{x}}{2+\sqrt{x}} = \lim_{x\to 4} \dfrac{4-x}{(4-x)(2+\sqrt{x})}$
$= \lim_{x\to 4} \dfrac{1}{2+\sqrt{x}} = \dfrac{1}{4}$
4. Use L'Hopital's rule:
$\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}} = \lim_{x\to\infty} \dfrac{(\ln x)'}{(\sqrt{x})'} = \lim_{x\to\infty} \dfrac{1/x}{\frac{1}{2}x^{-1/2}} = \lim_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$

Formally, we can prove the limit at a is some value l using the $\epsilon - \delta$ method.

Recap and Intuition: Limits

Definition (Limit)
The function f approaches the limit l near a if for every $\epsilon > 0$, there is some $\delta > 0$ such that, for all x, $0 < |x - a| < \delta$, then $|f(x) - l| < \epsilon$

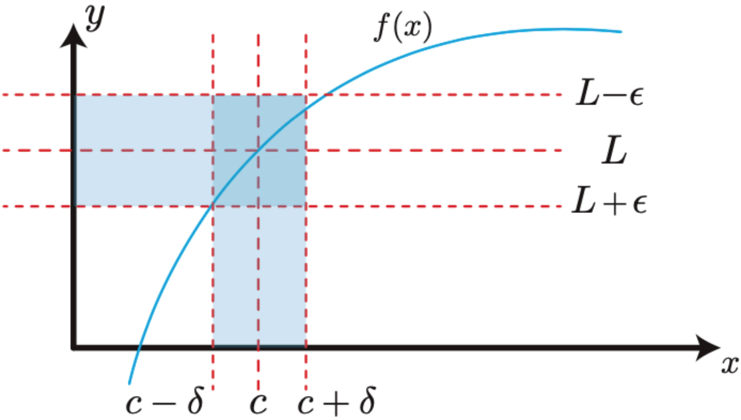


Figure 3: The geometric interpretation of the ϵ and δ "bands".

Recap and Intuition: Limits and Continuity

We would like to exclude functions that exhibit peculiarities which cause a limit to not exist or be equal to the value of the function at a.

Thus, we restrict our attention to continuous functions:

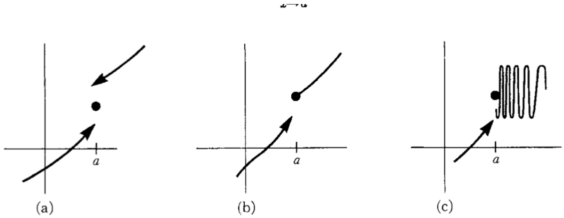


FIGURE 2

Definition (Continuous Function)
The function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$

Recap and Intuition: Continuous Functions

Now that we are looking at continuous functions, we can guarantee the following nice properties:

1. Intermediate Value Theorem: $f(a) < c < f(b)$, then $\exists x$ in $[a, b]$ such that $f(x) = c$.
2. Boundedness on Closed Interval
3. Extreme Value Theorem

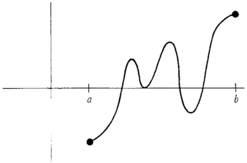


FIGURE 1

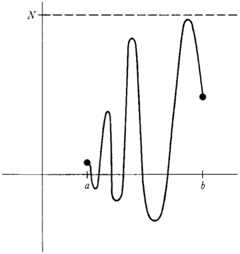


FIGURE 2

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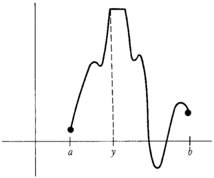


FIGURE 3

Recap and Intuition: Is continuity nice enough?

Even continuous functions sometimes misbehave and are "bent" at certain points, so that a tangent line cannot be drawn, making it difficult to approximate the slope of the graph at that point.

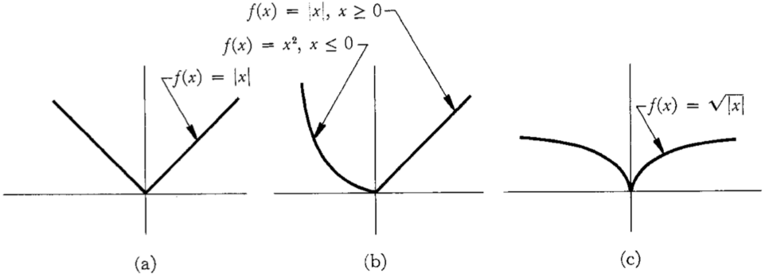
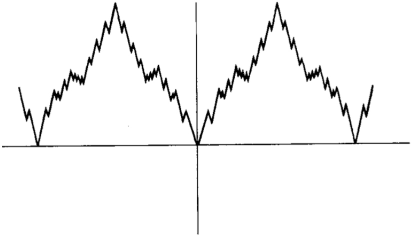


FIGURE 1

We want to restrict our attention further to functions for which we can define a *slope* or *derivative* at each point in its domain.

Recap and Intuition: Is continuity nice enough?

The limit of the function below is continuous everywhere and differentiable nowhere!



(d)

We want to restrict our attention further to functions for which we can define a *slope* or *derivative* at each point in its domain.

Definition (Differentiable Function)

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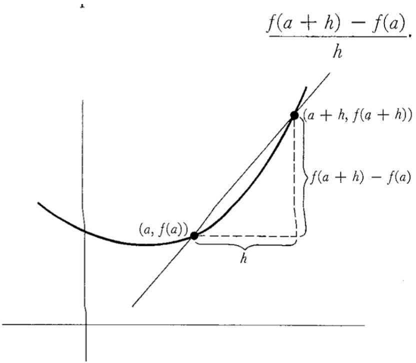

FIGURE 6

Recap and Intuition: Mean Value Theorem

Imposing differentiability allows us to state a strong property:

- Some tangent line is parallel to the line between $(a, f(a))$ and $(b, f(b))$.
- Allows us to use "local" derivative information to say interesting things about "global" function properties.

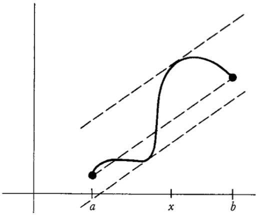

FIGURE 7

Theorem (Mean Value Theorem)

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number x in (a, b) such that $f'(x) = \dfrac{f(b) - f(a)}{b - a}$

Recap and Intuition: Chain Rule Practice

Theorem (Chain Rule)

$(f \circ g)'(a) = f'(g(a)) * g'(a)$

Example 1:

$$\begin{aligned} h(x) &= (6x^2 + 7x)^4 \\ f(x) &= x^4 \\ g(x) &= 6x^2 + 7x \\ h'(x) &= 4(6x^2 + 7x)^3 * (12x + 7) \end{aligned}$$

Theorem (Chain Rule)

$(f \circ g)'(a) = f'(g(a)) * g'(a)$

Example 2:

$$\begin{aligned} h(x) &= \log(1 - 5x^2 + x^3) \\ f(x) &= \log x \\ g(x) &= 1 - 5x^2 + x^3 \\ h'(x) &= \frac{1}{1 - 5x^2 + x^3} * (-10x + 3x^2) \end{aligned}$$

Inverse Functions

Recall that a function $f : A \to B$ is **injective** or **one-to-one** if $f(a) \neq f(b)$ whenever $a \neq b$.

- A simple example of an injective function is the Identity function $f(x) = x$.
- The key defining feature of a function is that it maps every element in its domain A to a unique element in its codomain B.
- If we have an injective function, we can think of a function $g : B \to A$ defined by $g(b) = a$ where $f(a) = b$.

Definition (Inverse Function)

For any function f, the inverse of f, denoted by f^{-1}, is the set of all pairs (a, b) for which the pair (b, a) is in f. f^{-1} is a function if and only if f is injective.

Theorem (Inverse Function Theorem)

Let f be a continuous injective function defined on an interval, and suppose that f is differentiable at $f^{-1}(b)$, with nonzero derivative. Then f^{-1} is differentiable at b, and the derivative of the inverse function at b is the reciprocal of the derivative of f at a:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

Inverse Functions

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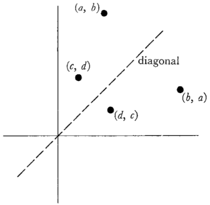


FIGURE 4

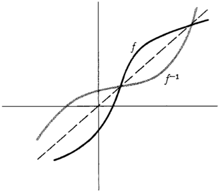


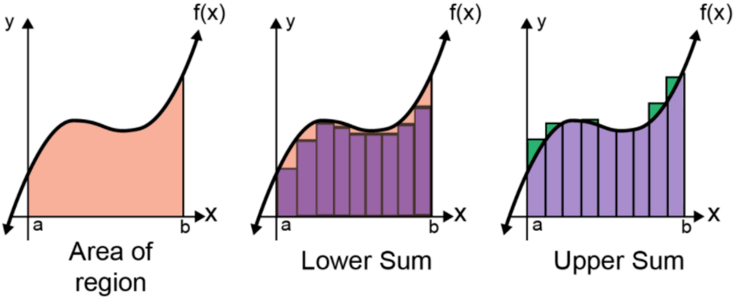
FIGURE 5

The graph of f^{-1} is the reflection of the graph of f over the diagonal $y = x$

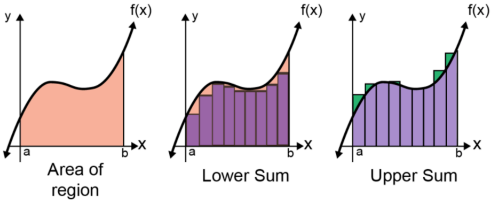
Integrals

Intuitively, an integral of a function gives the area beneath its graph.

- This concept is very useful in economics, as it allows us to model the relationship between stocks and flows (e.g. investment in a given year and the accumulation of capital stock) and marginal and total concepts (e.g. the marginal cost of producing an additional unit vs the total cost until that point).



Reimann Integral



- To think about approximating the area under an arbitrary curve, we may find it easier to think about adding up the area of rectangles — a more manageable calculation — that fit under (lower sum) or just over (upper sum) the graph of the function.
- As we make these rectangles smaller and smaller (or “skinnier”), our approximate calculation of the area under the curve approaches the true value.
- This is the intuition behind the **Reimann Integral**.

Loosely speaking, this integral is the limit of this sum as the rectangles get smaller and smaller (i.e. as the *partitions* get finer).

Integrable Function

Definition (Integrable Function)
A function f which is bounded on $[a, b]$ is (Reimann) integrable on $[a, b]$ if the limit of its lower and upper sum are equal. In this case, this common number is called the **integral** of f on $[a, b]$ and is denoted by

$$\int_a^b f(x)dx = \int_a^b f$$

Where a and b are called the lower and upper limits of integration, $f(x)$ is called the integrand, and dx is referred to as “integrating with respect to x”.

Theorem (Integrability over subsets)

Let $a < c < b$. If f is integrable on $[a, b]$, then f is integrable on $[a, c]$ and on $[c, b]$. Conversely, if f is integrable on $[a, c]$ and on $[c, b]$ then f is integrable on $[a, b]$. Further,

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Theorem (Integrability of Sums)

If f and g are integrable on $[a, b]$, then $f + g$ is integrable on $[a, b]$ and:

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g$$

Theorem (Integrability and Scalar Multiplication)

If f is integrable on $[a, b]$, then for any number $c \in \mathbb{R}$, the function cf is integrable on $[a, b]$ and

$$\int_a^b cf = c\int_a^b f$$

Theorem (Bounding the Integral)

Suppose f is integrable on $[a, b]$ and that $m \leq f(x) \leq M$ for all x in $[a, b]$. Then,

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

Fundamental Theorem of Calculus

Note that a function need not be continuous in order to be integrable.

Theorem (Integral is Continuous)

If f is integrable on $[a, b]$ and F is defined on $[a, b]$ by $F(x) = \int_a^x f$, then F is continuous on $[a, b]$.

What if the original function f is continuous? This gives rise to a famous theorem in Calculus:

Theorem (Fundamental Theorem of Calculus)

Let f be integrable on $[a, b]$ and define F on $[a, b]$ by

$$F(x) = \int_a^x f$$

If f is continuous at c on $[a, b]$, then F is differentiable at c, and $F'(c) = f(c)$.

Corollary
If f is continuous on $[a, b]$ and $f = g'$ for some function g, then $\int_a^b f = g(b) - g(a)$. (In fact, this is true even if f is only integrable on $[a, b]$).

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Let f be integrable on $[a, b]$ and define F on $[a, b]$ by

$$F(x) = \int_a^x f$$

If f is continuous at c on $[a, b]$, then F is differentiable at c, and $F'(c) = f(c)$.

Example:

Let $F(x) = \int_{-5}^{x} (t^2 + sint)dt$, then what is $F'(x)$?

By the FTC: $F'(x) = x^2 + sinx$

Fundamental Theorem of Calculus

Corollary
If f is continuous on $[a, b]$ and $f = g'$ for some function g, then $\int_a^b f = g(b) - g(a)$.

Example: Evaluate $\int_0^5 (4x - x^2)dx$. We need to find a function g for which $f = g'$.

- What is $4x$ the derivative of? $2x^2$
- What is $-x^2$ the derivative of? $-\frac{1}{3}x^3$
- By linearity, we have that $g = 2x^2 - \frac{1}{3}x^3$ up to a constant.

Then by the corollary to the FTC, we have that $\int_0^5 (4x - x^2)dx = 2(5)^2 - \frac{1}{3}5^3 - 0 = \frac{32}{3}$.

Techniques of Integration

Nature laughs at the difficulties of integration. – Pierre-Simon Laplace

In general, a function F satisfying $F' = f$ is called a **primitive** of f. Of course, a continuous function f always has a primitive,

$$F(x) = \int_a^x f$$

- Note that in the following discussion, we are interested in the primitive as a **function**, and thus will write integrals without limits to represent this, otherwise known as an indefinite integral.
- In contrast, so far we have been looking at definite integrals, which represents a number when the upper and lower limits are constants.

In general, the techniques to evaluate a derivative consists of two parts:

1. Have a good reservoir of known integrals.
2. Be fluent with integration techniques (linearity of integration, substitution, integration by parts).

Some Common Integrals

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Linearity of Integration

Theorem (Sum of Primitives)

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$
$$\int cf(x)dx = c\int f(x)dx$$

As an example, let us evaluate the following integral:

$$\begin{aligned}
&\int (1+x^2)^2 + 9e^x + \frac{\pi}{x}dx \\
&= \int (1+x^2)^2 dx + 9\int e^x dx + \pi\int \frac{1}{x}dx \\
&= \int dx + 2\int x^2 dx + \int x^4 dx + 9\int e^x dx + \pi\int \frac{1}{x}dx \\
&= x + \frac{2}{3}x^3 + \frac{1}{5}x^5 + 9e^x + \pi \ln x
\end{aligned}$$

Integration by Parts

A useful theorem involves viewing a function as a product of a function f with a simple derivative, and a function that is in the form of g':

Theorem (Integration by Parts)

If f' and g' are continuous, then

$$\int fg' = fg - \int f'g$$
$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$
$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$$

Integration by Parts

There are two special tricks which often work with integration by parts. The first is to consider the function g' to be 1, which can always be written in:

$$\int log(x)dx = \int \underbrace{1}_{g'} * \underbrace{log(x)}_{f} dx = xlog(x) - \int x * \frac{1}{x}dx = x(log(x)) - x$$

The second trick is to use integration by parts to find $\int h$ in terms of $\int h$ again, and then solve for $\int h$:

$$\begin{aligned}
\int \underbrace{\frac{1}{x}}_{g'} \underbrace{log(x)}_{f} dx &= log(x)log(x) - \int \frac{1}{x} log(x)dx \\
\implies 2\int \frac{1}{x} log(x)dx &= (log(x))^2 \\
\implies \int \frac{1}{x} log(x) &= \frac{(log(x))^2}{2}
\end{aligned}$$

Substitution

Another important method of integration is a consequence of the chain rule.

Theorem (The Substitution Formula)

If f' and g' are continuous, then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g'$$

$$\int_{g(a)}^{g(b)} f(u) du = \int_a^b f(g(x)) g'(x) dx$$

To use this method, it is useful to use the following procedure:

1. Let $u = g(x)$ and $du = g'(x)dx$ so that only the letter u appears, not x
2. Find a primitive as an expression involving u
3. Substitute $g(x)$ back for u

Theorem (The Substitution Formula)

If f' and g' are continuous, then

$$\int_{g(a)}^{g(b)} f(u)du = \int_a^b f(g(x))g'(x)dx$$

Example:

$$\int \frac{x}{x^2+1}dx \text{ lets rearrange to get in terms of f(g) and g'}$$
$$= \frac{1}{2}\int \frac{2x}{x^2+1}dx$$
$$= \frac{1}{2}\int \frac{1}{u}du \text{ where } u = x^2+1 \text{ and } du = 2xdx$$
$$= \frac{1}{2}\log u = \frac{1}{2}\log(x^2+1)$$

Infinite Sequences

Definition (Infinite Sequence)
An infinite sequence of real numbers $\{a_1, a_2, a_3, ...\}$ is a function $f : \mathbb{N} \to \mathbb{R}$, i.e. whose domain is $\mathbb{N}$.

Definition (Limit of a Sequence)
A sequence $\{a_n\}$ converges to l, $\lim_{n\to\infty} a_n = l$, if for every $\epsilon > 0$, there is a natural number $\mathbb{N}$ such that, for all natural numbers n,

$$n > N \implies |a_n - l| < \epsilon$$

A sequence is said to **converge** if it converges to l for some l, and to **diverge** if it does not converge.

For more details on infinite sequences, refer to lecture notes.

Infinite Series

- We can also consider the "sums" of an infinite sequence now, i.e. $a_1 + a_2 + a_3 + ...$
- This is not fully straightforward, since the sum of infinitely many numbers needs to be defined. What we can easily define are the **partial sums** $s_n = a_1 + ... + a_n$.
- If we hope to compute the infinite sum $a_1 + a_2 + a_3 + ...$, it must be the case that the partial sums s_n represent closer and closer approximations as n grows.

Definition (Sum of Infinite Sequence)
A sequence $\{a_n\}$ is summable if the sequence $\{s_n\}$ converges, where $s_n = a_1 + ... + a_n$. In this case, $\lim_{n\to\infty} s_n$ is denoted by **the infinite series**

$$\sum_{n=1}^{\infty} a_n \text{ or } a_1 + a_2 + a_3 + ...$$

and is called the sum of the sequence $\{a_n\}$.

Geometric Series

Definition (Geometric Series)
The most important of all infinite series are the geometric series.

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + ...$$

Only the cases $|r| < 1$ are interesting, since the individual terms do not approach 0 otherwise. These series can be managed because the partial sums can be evaluated in simple terms:

$$s_n = 1 + r + ... + r^n \implies rs_n = r + r^2 + ...r^{n+1}$$

$$\implies s_n(1-r) = 1 - r^{n+1} \implies s_n = \frac{1-r^{n-1}}{1-r}$$

Then it follows that $\displaystyle\sum_{n=0}^{\infty} r^n = \lim_{n\to\infty} \frac{1-r^{n-1}}{1-r} = \frac{1}{1+r}$ for $|r| < 1$

For example,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - 1 = \frac{1}{1-\frac{1}{2}} - 1 = 1$$

Approximation by Polynomials

It is useful to be able to reduce the calculation of a function f to the evaluation of a polynomial function. The method depends on finding polynomial functions which are close approximations to f.

Consider a polynomial function $p(x)$:

$$p(x) = a_0 + a_1 x + ... + a_n x^n$$

It is interesting to note that the coefficients a_i can be expressed in terms of the value of p and its various derivatives evaluated at 0.

$$\begin{aligned} p'(x) &= a_1 + 2a_2 x + ... + n a_n x^{n-1} \\ p'(0) &= a_1 \end{aligned}$$

We can further differentiate again,

$$\begin{aligned} p''(x) &= 2a_2 + 3 * 2a_3 x + ... + n(n-1) a_n x^{n-2} \\ p''(0) &= 2a_2 \end{aligned}$$

Approximation by Polynomials

In general, we arrive at the following expression for a given coefficient on a polynomial function:

$$a_k = \frac{p^{(k)}(0)}{k!}$$

We can also consider a polynomial around a point a rather than 0, to write:

$$p(x) = a_0 + a_1(x - a) + ... + a_n(x - a)^n$$

$$a_k = \frac{p^{(k)}(a)}{k!}$$

Definition (Taylor Polynomial of degree n for f at a)
Suppose that f is a function such that its first n derivatives $f^{(1)}(a), ..., f^{(n)}(a)$ at a all exist. Let

$$a_k = \frac{f^{(k)}(a)}{k!} \quad , 0 \leq k \leq n$$

And define

$$P_{n,a}(x) = a_0 + a_1(x - a) + ... + a_n(x - a)^n$$

Approximation by Polynomials

Then $P_{n,a}(x)$ is called the **Taylor polynomial of degree** n **for** f **at** a, and has been defined in such a way so that

$$P_{n,a}^{(k)}(a) = f^{(k)}(a) \text{ for } 0 \leq k \leq n$$

- Thus we have found a polynomial that has the same first n derivatives as f at a.
- We argue that this is a good approximation for f at a.
- In order to justify this, we need to look at the error term $r_{n,a}(x) = f(x) - p_{n,a}(x)$.
- In particular, in order for $p_{n,a}(x)$ to represent a good k-th order approximation to f, we should require that the remainder tends to 0 faster than k-th order, i.e.

$$\lim_{x \to a} \frac{r_{n,a}(x)}{(x-a)^k} = 0$$

Theorem (Taylor's Theorem)

Suppose that f is $n+1$ times differentiable on the interval I with $a \in I$. For each $x \in I$, there is a point c between a and x such that

$$r_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Corollary
Suppose that f is $n+1$ times differentiable on the interval I with $a \in I$. Then,

$$\lim_{x \to a} \frac{r_{n,a}(x)}{|x-a|^n} = 0$$

- This corollary implies that the Taylor polynomial is a good approximation, since the error vanishes faster than order n.
- The Taylor formula opens up the way for most of the calculations of applied analysis, and is extremely important from a practical point of view

Notice further that we can make a Taylor polynomial of as high a degree as we'd like.

- The **Taylor Series** is just the Taylor polynomial with infinite degree.

Let us consider an example, and find the Taylor series for $f(x) = e^x$ at $a = 1$.

- Since all derivatives of $f(x)$ are e^x, we have that $f^{(n)}(1) = e$ for all $n \geq 0$.
- Thus, its Taylor series at 1 is:

$$\sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$$

PV of a Stream of Payments

In many economic settings, we need to compute the present value of a series of future cash flows.

- Consider an individual who makes annual payments at the end of each year in amount V for T years, with an interest rate of r.
- Then, we can calculate the present value of this stream of payments as:

$$PV = \frac{V}{(1+r)^1} + \frac{V}{(1+r)^2} + ... + \frac{V}{(1+r)^T} = \sum_{t=1}^{T} \frac{V}{(1+r)^t}$$

Today: Calculus

1. Limits
2. Continuous Functions
3. Derivatives
4. Integrals
    - Techniques of Integration
5. Infinite Series
6. Approximation by Polynomial Functions

In lecture notes: inverse functions, more on infinite sequences and series.