2 factor stochastic volatility model

$$\frac{dS}{S} = \mu^S dt + \sigma_t^I dW1_t + \beta \sigma_t^M dW2_t \quad \Rightarrow \text{Change in any index component}$$

$$\frac{dS^M}{S^M} = \mu^M dt + \sigma_t^M dW2_t \quad \Rightarrow \text{Change in index}$$

$$d\sigma_t^I = k^I(\bar{\sigma}^I - \sigma_t^I)dt + b^I \sigma_t^I dW3_t \quad \Rightarrow \text{Change in vol of any index component}$$

$$d\sigma_t^M = k^M(\bar{\sigma}^M - \sigma_t^M)dt + b^M \sigma_t^M dW4_t \quad \Rightarrow \text{Change in vol of index}$$

where, $dW1, dW2, dW3, dW4$ are mutually independent BMs

S^M = Market portfolio

β = constant

Vol of stock, $\sigma_t^S = \sqrt{(\sigma_t^I)^2 + (\beta\sigma_t^M)^2}$

Instantaneous correlation of dS & dS^M, $\rho_t = \beta \dfrac{\sigma_t^M}{\sigma_t^S}$

We have a derivative security f who's payoff depends on ρ_t

So, f is $f(S, S^M, \sigma_t^I, \sigma_t^M, t)$. Applying Ito's Lemma, we get

$$\begin{aligned}
dF = {} & \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial S^M}dS^M + \frac{\partial F}{\partial \sigma_t^I}d\sigma_t^I + \frac{\partial F}{\partial \sigma_t^M}d\sigma_t^M + \\
& \frac{1}{2}\frac{\partial^2 F}{\partial S^2}dS^2 + \frac{1}{2}\frac{\partial^2 F}{\partial (S^M)^2}d(S^M)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial (\sigma_t^M)^2}d(\sigma_t^M)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial (\sigma_t^I)^2}d(\sigma_t^I)^2 + \\
& \frac{\partial^2 F}{\partial S \partial S^M}dS\,dS^M + \frac{\partial^2 F}{\partial S \partial \sigma_t^M}dS\,d\sigma_t^M + \frac{\partial^2 F}{\partial S \partial \sigma_t^I}dS\,d\sigma_t^I + \frac{\partial^2 F}{\partial S \partial t}dS\,dt + \\
& \frac{\partial^2 F}{\partial S^M \partial \sigma_t^M}dS^M d\sigma_t^M + \frac{\partial^2 F}{\partial S^M \partial \sigma_t^I}dS^M d\sigma_t^I + \frac{\partial^2 F}{\partial S^M \partial t}dS^M dt + \\
& \frac{\partial^2 F}{\partial \sigma_t^I \partial \sigma_t^M}d\sigma_t^I d\sigma_t^M + \frac{\partial^2 F}{\partial \sigma_t^I \partial t}d\sigma_t^I dt + \frac{\partial^2 F}{\partial \sigma_t^M \partial t}d\sigma_t^M dt
\end{aligned}$$

we can verify the total # of terms by looking at the parameters of $f(S, S^M, \sigma_t^I, \sigma_t^M, t)$

We have a 19 term PDE for f.

We need to substitute all 4 SDE's into the PDE.

We risk neutralize it first.

Step 1: Define market price of risk.

$\theta_1 = (\mu^S - r) / (\sigma_t^I)$

$\theta_2 = (\mu^M - r) / (\sigma_t^M)$

Step 2 : Apply Girsanov's theorem

This is done to change the probability measure from P to Q, which is a risk neutral measure. The Brownian motions under the new measure will be:

$$d\tilde{W}^1_t = dW^{Q^1}_t = dW^1_t - \theta_1 dt$$
$$d\tilde{W}^2_t = dW^{Q^2}_t = dW^2_t - \theta_1 dt$$
$$d\tilde{W}^3_t = dW^{Q^3}_t = dW^3_t$$
$$d\tilde{W}^4_t = dW^{Q^4}_t = dW^4_t$$

Step 3 : Adjust SDE's under the risk neutral measure

We substitute the R.N drifts into the SDE's for $\frac{dS}{S}$ & $\frac{dS^M}{S^M}$.

Adjusted SDE's are:

$$\frac{dS}{S} = (\mu - r)dt + \sigma^I_t d\tilde{W}1_t + \beta\sigma^M_t d\tilde{W}2_t$$
$$\frac{dS^M}{S^M} = (\mu^M - r)dt + \sigma^M_t d\tilde{W}2_t$$
$$d\sigma^I_t = \kappa^I(\bar{\sigma}^I - \sigma^I_t)dt + 6^I\sigma^I_t d\tilde{W}3_t$$
$$d\sigma^M_t = \kappa^M(\bar{\sigma}^M - \sigma^M_t)dt + 6^M\sigma^M_t d\tilde{W}4_t$$

Step 4 : Substitute the adjusted SDE's into dF

We know that: $dW^2_t = dt$

$$dt^2 = 0$$
$$dW_t dt = 0$$
$$d\tilde{W}_n d\tilde{W}_m = 0 \quad \because \text{independent}$$

First, we find the square of all 4 SDE's since we will need it.

$$\left(\frac{dS}{S}\right)^2 = \left((\mu-r)dt + \sigma^I_t d\tilde{W}1_t + \beta\sigma^M_t d\tilde{W}2_t\right)^2 = (\sigma^I_t)^2 dt + (\beta\sigma^M_t)^2 dt$$
$$\left(\frac{dS^M}{S^M}\right)^2 = \left((\mu^M-r)dt + \sigma^M_t d\tilde{W}2_t\right)^2 = (\sigma^M_t)^2 dt$$

$$(d\sigma_t^I)^2 = \left(k^I(\bar{\sigma}^I - \sigma_t^I)dt + b^I\sigma_t^I d\tilde{W}3_t\right)^2 = (b^I\sigma_t^I)dt$$

$$(d\sigma_t^M)^2 = \left(\kappa^M(\bar{\sigma}^M - \sigma_t^M)dt + b^M\sigma_t^M d\tilde{W}4_t\right)^2 = (b^M\sigma_t^M)dt$$

We substitute in dF to find expanded equation.

$$
\begin{aligned}
dF = {} & \frac{\partial F}{\partial t}dt \; + \\
& \frac{\partial F}{\partial S}\left(S((\mu-r)dt + \sigma_t^I d\tilde{W}1_t + \beta\sigma_t^M d\tilde{W}2_t)\right) \; + \\
& \frac{\partial F}{\partial S^M}\left(S^M((\mu^M-r)dt + \sigma_t^M d\tilde{W}2_t)\right) \; + \\
& \frac{\partial F}{\partial \sigma_t^I}\left(k^I(\bar{\sigma}^I - \sigma_t^I)dt + b^I\sigma_t^I d\tilde{W}3_t\right) \; + \\
& \frac{\partial F}{\partial \sigma_t^M}\left(\kappa^M(\bar{\sigma}^M - \sigma_t^M)dt + b^M\sigma_t^M d\tilde{W}4_t\right) \; + \\
& \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(\sigma_t^I)^2 dt + \beta(\sigma_t^M)^2 dt \; + \\
& \frac{1}{2}\frac{\partial^2 F}{\partial (S^M)^2}(\sigma_t^M)^2 dt \; + \\
& \frac{1}{2}\frac{\partial^2 F}{\partial (\sigma_t^M)^2}(b^M\sigma_t^M)^2 dt \; + \\
& \frac{1}{2}\frac{\partial^2 F}{\partial (\sigma_t^I)^2}(b^I\sigma_t^I)^2 dt \; + \\
& \frac{\partial^2 F}{\partial S\partial S^M}\left(S((\mu-r)dt + \sigma_t^I d\tilde{W}1_t + \beta\sigma_t^M d\tilde{W}2_t)\right)\left(S^M((\mu^M-r)dt + \sigma_t^M d\tilde{W}2_t)\right) + \\
& \frac{\partial^2 F}{\partial S\partial \sigma_t^M}\left(S((\mu-r)dt + \sigma_t^I d\tilde{W}1_t + \beta\sigma_t^M d\tilde{W}2_t)\right)\left(\kappa^M(\bar{\sigma}^M - \sigma_t^M)dt + b^M\sigma_t^M d\tilde{W}4_t\right) + \\
& \frac{\partial^2 F}{\partial S\partial \sigma_t^I}\left(S((\mu-r)dt + \sigma_t^I d\tilde{W}1_t + \beta\sigma_t^M d\tilde{W}2_t)\right)\left(k^I(\bar{\sigma}^I - \sigma_t^I)dt + b^I\sigma_t^I d\tilde{W}3_t\right) + \\
& \frac{\partial^2 F}{\partial S\partial t}\left(S((\mu-r)dt + \sigma_t^I d\tilde{W}1_t + \beta\sigma_t^M d\tilde{W}2_t)\right)dt \; + \\
& \frac{\partial^2 F}{\partial S^M\partial \sigma_t^M}\left(S^M((\mu^M-r)dt + \sigma_t^M d\tilde{W}2_t)\right)\left(\kappa^M(\bar{\sigma}^M - \sigma_t^M)dt + b^M\sigma_t^M d\tilde{W}4_t\right) + \\
& \frac{\partial^2 F}{\partial S^M\partial \sigma_t^I}\left(S^M((\mu^M-r)dt + \sigma_t^M d\tilde{W}2_t)\right)\left(k^I(\bar{\sigma}^I - \sigma_t^I)dt + b^I\sigma_t^I d\tilde{W}3_t\right) + \\
& \frac{\partial^2 F}{\partial S^M\partial t}\left(S^M((\mu^M-r)dt + \sigma_t^M d\tilde{W}2_t)\right)dt \; + \\
& \frac{\partial^2 F}{\partial \sigma_t^I\partial \sigma_t^M}\left(k^I(\bar{\sigma}^I - \sigma_t^I)dt + b^I\sigma_t^I d\tilde{W}3_t\right)\left(\kappa^M(\bar{\sigma}^M - \sigma_t^M)dt + b^M\sigma_t^M d\tilde{W}4_t\right) + \\
& \frac{\partial^2 F}{\partial \sigma_t^I\partial t}\left(k^I(\bar{\sigma}^I - \sigma_t^I)dt + b^I\sigma_t^I d\tilde{W}3_t\right)dt \; + \\
& \frac{\partial^2 F}{\partial \sigma_t^M\partial t}\left(\kappa^M(\bar{\sigma}^M - \sigma_t^M)dt + b^M\sigma_t^M d\tilde{W}4_t\right)dt
\end{aligned}
$$

dF still contains some terms which are zero based on our rules.

Afer cancelling zero terms, the terms that are left are

$$
\begin{aligned}
\partial F = \; & \frac{\partial F}{\partial t} dt \; + \\
& \frac{\partial F}{\partial S}\left(S\left((\mu - r)dt + \sigma^I_t d\tilde{W}1_t + \beta \sigma^M_t d\tilde{W}2_t\right)\right) \; + \\
& \frac{\partial F}{\partial S^M}\left(S^M\left((\mu^M - r)dt + \sigma^M_t d\tilde{W}2_t\right)\right) \; + \\
& \frac{\partial F}{\partial \sigma^I_t}\left(k^I\left(\bar{\sigma}^I - \sigma^I_t\right)dt + b^I \sigma^I_t d\tilde{W}3_t\right) \; + \\
& \frac{\partial F}{\partial \sigma^M_t}\left(K^M\left(\bar{\sigma}^M - \sigma^M_t\right)dt + b^M \sigma^M_t d\tilde{W}4_t\right) \; + \\
& \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\left(\sigma^I_t\right)^2 dt + \beta\left(\sigma^M_t\right)^2 dt \; + \\
& \frac{1}{2}\frac{\partial^2 F}{\partial (S^M)^2}\left(\sigma^M_t\right)^2 dt \; + \\
& \frac{1}{2}\frac{\partial^2 F}{\partial (\sigma^M_t)^2}\left(b^M \sigma^M_t\right)^2 dt \; + \\
& \frac{1}{2}\frac{\partial^2 F}{\partial (\sigma^I_t)^2}\left(b^I \sigma^I_t\right)^2 dt \; + \\
& \frac{\partial^2 F}{\partial S \partial S^M}\left(\sigma^M_t\right)^2 dt
\end{aligned}
$$

This is the PDE that f's price would have to satisfy in a complete market with no arbitrage.