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Teorema de la convergencia dominada

Teorema 1 (de la convergencia dominada). *Sea* $(f_n)_{n\in\mathbb{N}}$ *una sucesión de fns medibles en un espacio de medida* (X, Σ, μ) *tal que* $\exists \lim f_n =: f \wedge \exists g$ *integrable* $: \forall n \in \mathbb{N} : |f_n| \leq g$[1]

$$\implies \lim_{n\to\infty} \int |f_n - f| \, \mathrm{d}\mu = 0, \quad \textit{en particular,} \qquad \int \lim_{n\to\infty} f_n \, \mathrm{d}\mu = \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu.$$

***Demostración*:** Consideramos $\left(\hat{f}_n\right)_{n\in\mathbb{N}}$ dada por $\forall n \in \mathbb{N} : \hat{f}_n = f_n - f \implies \hat{f}_n \xrightarrow{n\to\infty} 0$.

Definimos $\forall n \in \mathbb{N} : h_n := g - \left|\hat{f}_n\right| \geq 0 \overset{\substack{\text{Fatou}\\ \downarrow}}{\implies} \int \liminf_{n\to\infty} h_n \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int h_n \, \mathrm{d}\mu$.

Ahora, como $\liminf_{n\to\infty} h_n = \lim_{n\to\infty} g - \left|\hat{f}_n\right| = g$ porque $\hat{f}_n \xrightarrow{n\to\infty} 0$, entonces

$$\int g \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int \left(g - \left|\hat{f}_n\right|\right) \mathrm{d}\mu \overset{\substack{\text{linealidad}\\ \downarrow}}{=} \liminf_{n\to\infty} \left(\int g \, \mathrm{d}\mu - \int \left|\hat{f}_n\right| \mathrm{d}mu\right)$$

$$\implies \int g \, \mathrm{d}\mu \leq \liminf_{n\to\infty} \int g \, \mathrm{d}\mu - \limsup_{n\to\infty} \int \left|\hat{f}_n\right| \mathrm{d}\mu$$

Como $\liminf_{n\to\infty} \int g \, \mathrm{d}\mu = \int g \, \mathrm{d}\mu \wedge g \in \mathcal{L}^1(\mu)$, podemos cancelar y nos queda

$$\cancel{\int g \, \mathrm{d}\mu} \leq \cancel{\int g \, \mathrm{d}\mu} - \limsup_{n\to\infty} \int \left|\hat{f}_n\right| \mathrm{d}\mu \implies \limsup_{n\to\infty} \int \left|\hat{f}_n\right| \mathrm{d}\mu \leq 0$$

Lo que implica que $\limsup_{n\to\infty} \int |f_n - f| \, \mathrm{d}\mu = 0 \implies \lim_{n\to\infty} \int |f_n - f| \, \mathrm{d}\mu = 0$.

$$\text{y como } \int |f_n - f| \, \mathrm{d}\mu = \begin{cases} \int (f_n - f) \, \mathrm{d}\mu = \int f_n \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu & \text{si } f_n \geq f \\ \int (f - f_n) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu - \int f_n \, \mathrm{d}\mu & \text{si } f_n < f \end{cases}$$

se tiene $\lim_{n\to\infty} \int |f_n - f| \, \mathrm{d}\mu = 0 \iff \lim_{n\to\infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu = \int \lim_{n\to\infty} f_n \, \mathrm{d}\mu$. ■

[1] Se dice que g "domina" a todas las f_n.