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Desigualdad maximal de Kolmogorov

Teorema 1 (desigualdad maximal de Kolmogorov). *Sean* $\{X_j\}_{j\in\mathbb{N}_n}$ *variables aleatorias independientes y centradas (i.e.* $\forall j \in \mathbb{N} : \mathbb{E}[X_j] = 0$*) y* $\forall j \in \mathbb{N}_n : X_j \in \mathcal{L}^2(\mathbb{P})$

$$\Longrightarrow \forall t \in \mathbb{R} : \mathbb{P}\left(\max_{1\leq k\leq n} |S_k| > t\right) \leq \frac{\mathbb{V}\left(S_n\right)}{t^2} \qquad \textit{donde} \qquad S_k = \sum_{j=1}^{k} X_j.$$

***Demostración*:** Definimos $\forall k \in \mathbb{N}_n, t \in \mathbb{R}_+$:

$$A_k(t) = \{\omega \in \Omega : |S_k(\omega)| > t \wedge \forall j < k : |S_j(\omega)| \leq t\}.$$

Entonces, $\left\{\omega \in \Omega : \max_{1\leq k\leq n} |S_k(\omega)| > t\right\} = \bigsqcup_{k=1}^{n} A_k(t)$ y, por tanto,

$$\begin{aligned}
\mathbb{V}(S_n) = \mathbb{E}\left[S_n^2\right] = \int_\Omega S_n^2(\omega)\,\mathrm{d}\mathbb{P}(\omega) \geq \int_{\left\{\max_{1\leq k\leq n}|S_n|>t\right\}} S_n^2(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \sum_{k=1}^{n}\int_{A_k(t)} S_n^2(\omega)\,\mathrm{d}\mathbb{P}(\omega) = \\
\geq \sum_{k=1}^{n}\int_{A_k(t)} \left(S_n - S_k + S_k\right)^2\mathrm{d}\mathbb{P} = \sum_{k=1}^{n}\int_{A_k(t)} S_k^2 + 2S_k(S_n - S_k) + (S_n - S_k)^2\,\mathrm{d}\mathbb{P} \\
\geq \sum_{k=1}^{n}\int_{A_k(t)} S_k^2\,\mathrm{d}\mathbb{P} + \cancelto{0\ (*)}{\sum_{k=1}^{n} 2\int_\Omega \mathbb{1}_{A_k(t)}\cdot S_k(S_n - S_k)\,\mathrm{d}\mathbb{P}} \\
\geq \sum_{k=1}^{n} t^2\cdot\mathbb{P}\left(A_k(t)\right) = t^2\cdot\mathbb{P}\left(\max_{1\leq k\leq n} S_k\right)
\end{aligned}$$

$(*)$ Tenemos que $\mathbb{1}_{A_k(t)}\cdot S_k$ es $\sigma\left(X_1,\ldots,X_k\right)$-medible y, además, $S_n - S_k = \sum_{j=k+1}^{n} X_j$ es $\sigma\left(X_{k+1},\ldots,X_n\right)$-medible. Por el teorema Π-λ, $\mathbb{1}_{A_k}S_K$ y $S_n - S_k$ son independientes

$$\Longrightarrow \mathbb{E}\left[\mathbb{1}_{A_k(t)}S_k\cdot(S_n - S_k)\right] = \mathbb{E}\left[\mathbb{1}_{A_k(t)}S_k\right]\cdot\mathbb{E}\left[S_n - S_k\right] = 0.$$

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