# Chapter 2
# (Part 1)

# Relations

- A (binary) relation $R$ from a set $X$ to a set $Y$ is a subset of the Cartesian product $X \times Y$.

- If $(x,y) \in R$, we write

$$x \: R \: y \quad (x \text{ is related to } y)$$

(Binary) relation from $X$ to $Y$, where $x \in X$, $y \in Y$, $(x,y) \in X \times Y$ and $R \subseteq X \times Y$

$$x \: R \: y \: \leftrightarrow \: (x,y) \in R$$

- *A*= { 1, 2, 3, 4 } , *B*= { *p, q, r* }

- *R* = { (1, *q*), (2, *r*), (3, *q*), (4, *p*) }

- *R* ⊆ *A*×*B*

- *R* is the relation from *A* to *B*

  1 *R q*          3 $\not{R}$ *p*

**AxB**

(1,p),(1,q),(1,r),
(2,p),(2,q),(2,r),
(3,p),(3,q),(3,r),
(4,p),(4,q),(4,r)

$aRb \leftrightarrow a-b \in Z^{even}$

- *Finite set: A= { 1, 2 } , B= { 1, 2, 3 }*
  R = { (1, 1), (2, 2), (1, 3) }

- *Infinite set: A=Z and B=Z*
  R = { …(-3, -1), (-2, 2), (1, 3), …. }

(note: Z is set of integers)

- *A* = { New Delhi, Ottawa, London, Paris, Washington }

- *B* = { Canada, England, India, France, United States }

- Let $x \in A$, $y \in B$. Define the relation between $x$ and $y$ by "*x is the capital of y*"

- R = { (New Delhi, India), (Ottawa, Canada),
    (London, England), (Paris, France),
    (Washington, United States) }

- "less than" relation from $A=\{0, 1, 2\}$ to $B=\{1, 2, 3\}$

- Traditional notation:
  $0<1, 0<2, 0<3, 1<2, 1<3, 2<3$

- Set notation
  $A \times B = \{$ (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,1), (2,2), (2,3) $\}$
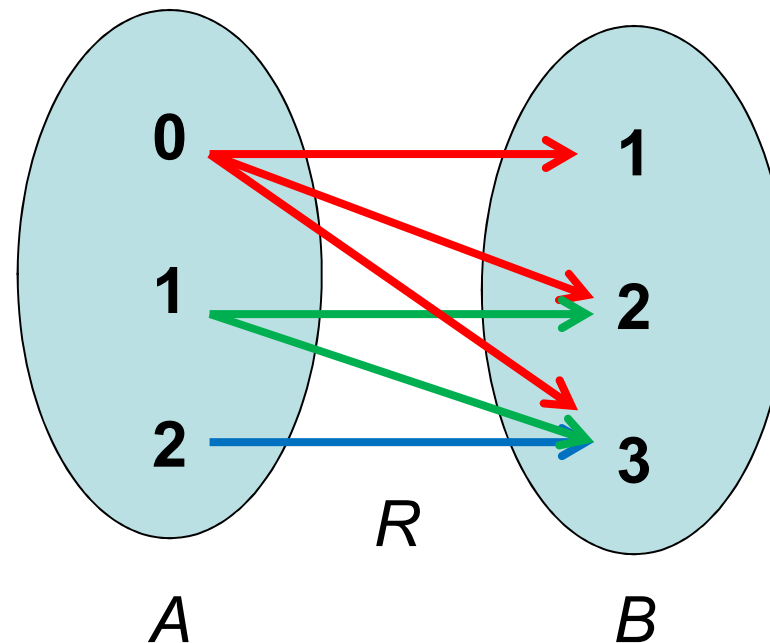  $R = \{$ (0,1), (0,2), (0,3), (1,2), (1,3), (2,3) $\}$

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$R = \{ (0,1), (0,2), (0,3), (1,2), (1,3), (2,3) \}$

Arrow diagrams

Let $R$, a relation from $A$ to $B$.

The set, { $a \in A$ | $(a,b) \in R$ for some $b \in B$ } is called the **domain** of $R$.

The set, { $b \in B$ | $(a,b) \in R$ for some $a \in A$ } is called the **range** of $R$.

In case $A=B$, we call $R$ a(binary) **relation on** $A$.

- Let $R$ be a relation on $X = \{1, 2, 3, 4\}$
  defined by $(x,y) \in R$ if $x \leq y$, and $x, y \in X$.

- Then,
  $R = \{ (1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4) \}$

- The domain and range of $R$ are both equal to $X$.

- Let $X$ = { 2, 3, 4 } and $Y$= { 3, 4, 5, 6, 7 }
  If we define a relation $R$ from $X$ to $Y$ by,
  $(x,y) \in R$ if $x$ divides $y$ (with zero remainder)

- We obtain,
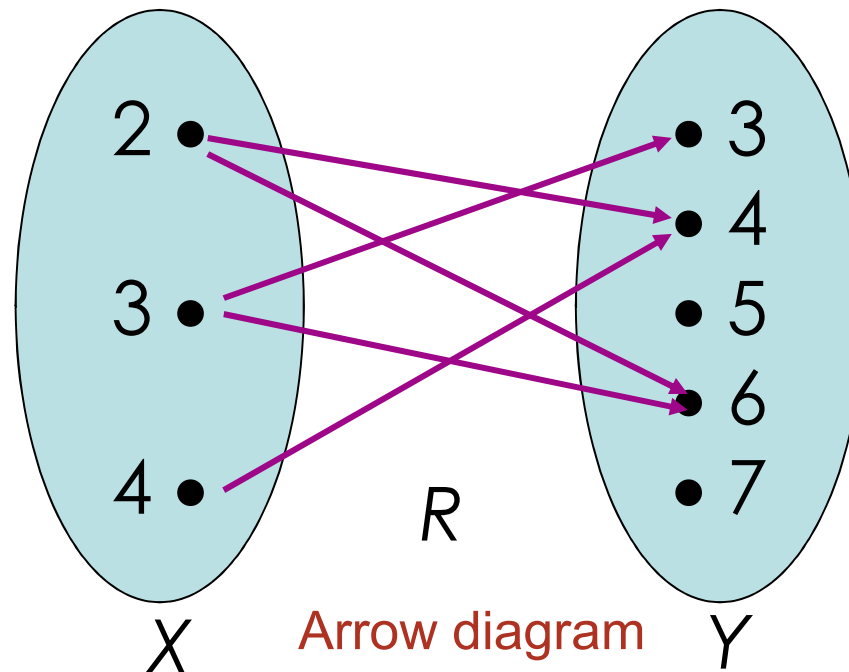  $R$ = { (2,4), (2,6), (3,3), (3,6), (4,4) }

  The domain of $R$ is {2,3,4}
  The range of $R$ is {3,4,6}

$R = \{ (2,4), (2,6), (3,3), (3,6), (4,4) \}$
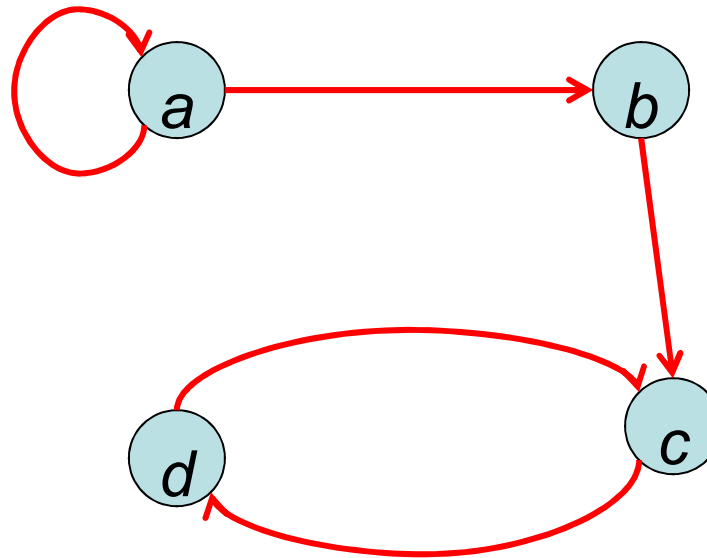


Arrow diagram

- An informative way to picture a relation on a set is to draw its digraph.

- Let $R$ be a relation on a finite set $A$.

- Draw dots (vertices) to represent the elements of $A$.

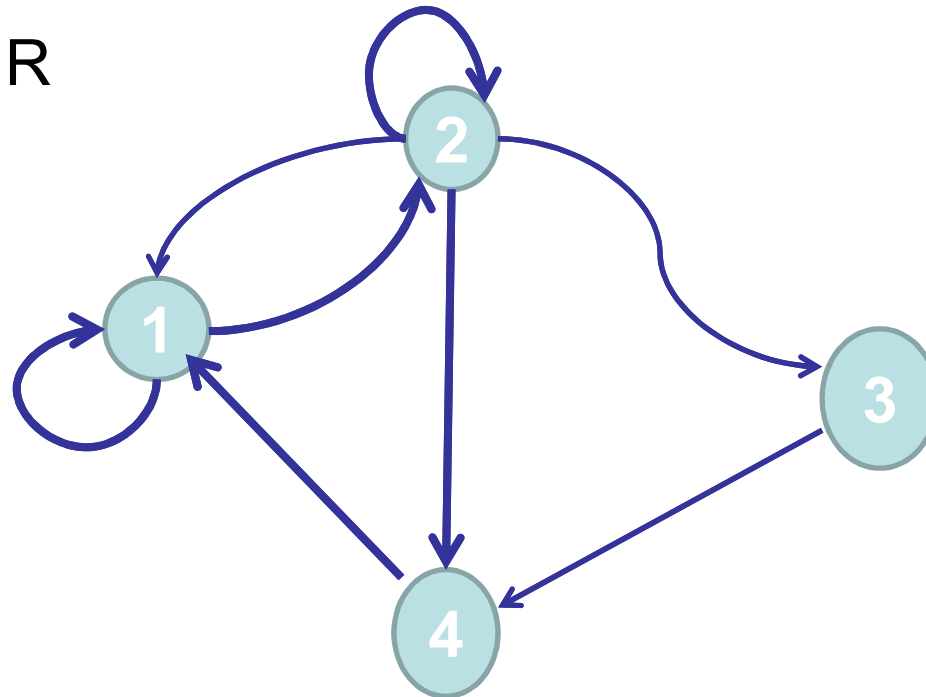- If the element $(a,b) \in R$, draw an arrow (called a directed edge) from a to b .

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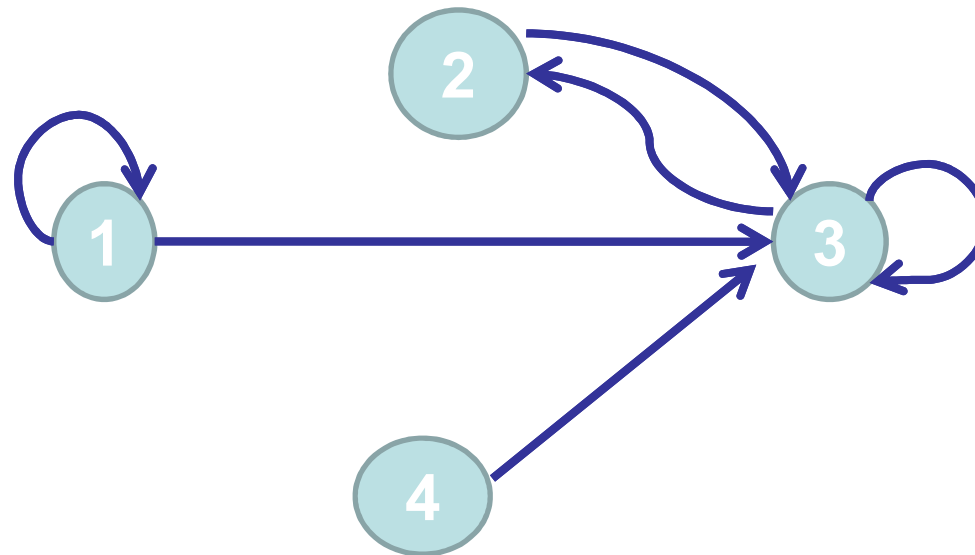The relation $R$ on $A = \{a, b, c, d\}$,

$R = \{(a, a), (a, b), (c, d), (d, c), (b,c)\}$

- Let, A = { 1,2,3,4} and  R = { (1,1), (1,2), (2,1), (2,2), (2,3), (2,4), (3,4) , (4,1)}

- Draw the digraph of R

Find the relation determined by digraph below.



- Since *a R b* if and only if there is an edge from *a* to *b*, so
  R = { (1,1), (1,3), (2,3), (3,2), (3,3), (4,3) }

- A matrix is a convenient way to represent a relation *R* from *A* to *B*.

- Label the rows with the elements of *A* (in some arbitrary order)

- Label the columns with the elements of *B* (in some arbitrary order)

- Let $A=\{a_1, a_2,\ldots, a_n\}$ and $B=\{b_1, b_2,\ldots, b_p\}$ be finite nonempty sets.

- Let $R$ be a relation from $A$ into $B$.

- Let $M_R = [m_{ij}]_{nxp}$ be the Boolean nxp matrix, where

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$$

$$M_R = \begin{bmatrix} m_{11} & m_{12} & .. & .. & m_{1p} \\ m_{21} & m_{22} & .. & .. & m_{2p} \\ \vdots & \vdots & .. & .. & \vdots \\ \vdots & \vdots & .. & .. & \vdots \\ m_{n1} & m_{n2} & .. & .. & m_{np} \end{bmatrix}$$

- Let $A$ = {1, 3, 5} and $B$ = { 1, 2}

- Let $R$ be a relation from $A$ to $B$ and $R$ = {(1,1), (3,2), (5,1)}

- Then the matrix represent $R$ is

$$
\begin{array}{c}
\phantom{0} \\
1 \\
3 \\
5
\end{array}
\begin{array}{cc}
1 & 2 \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}
\end{array}
$$

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The relation,

$R = \{ (1,b),(1,d),(2,c),(3,c),(3,b),(4,a) \}$

from, $X = \{ 1, 2, 3, 4 \}$ to $Y = \{ a, b, c, d \}$

$$\begin{array}{c c c c c} & a & b & c & d \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 \\ 4 & 1 & 0 & 0 & 0 \end{array}$$

**or**

$$\begin{array}{c c c c c} & d & b & a & c \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}$$

The matrix of the relation $R$ from $\{\,2, 3, 4\,\}$ to $\{\,5, 6, 7, 8\,\}$ defined by

$x \, R \, y$   if $x$ divides $y$

$$
\begin{array}{c}
\begin{array}{cccc} 5 & 6 & 7 & 8 \end{array}\\
\begin{array}{c} 2 \\ 3 \\ 4 \end{array}
\begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

- Let A={ *a, b, c, d* }
- Let *R* be a relation on *A*.
- *R* = { (a,a),(b,b),(c,c),(d,d),(b,c),(c,b) }

$$
\begin{array}{c c c c c}
 & a & b & c & d \\
a & 1 & 0 & 0 & 0 \\
b & 0 & 1 & 1 & 0 \\
c & 0 & 1 & 1 & 0 \\
d & 0 & 0 & 0 & 1
\end{array}
$$

- If R is a relation on a set *A* and $a \in A$, then the in-degree of $a$ (relative to relation *R*) is the number of $b \in A$ such that $(b, a) \in R$.

- The out degree of $a$ is the number of $b \in A$ such that $(a, b) \in R$

- Meaning that, in terms of the digraph of R, is that the in-degree of a vertex is

  "the number of edges terminating at the vertex"

- The out-degree of a vertex is

  " the number of edges leaving the vertex"

■ Let $A = \{a, b, c, d\}$, and let $R$ be the relation on $A$ that has the matrix (given below)
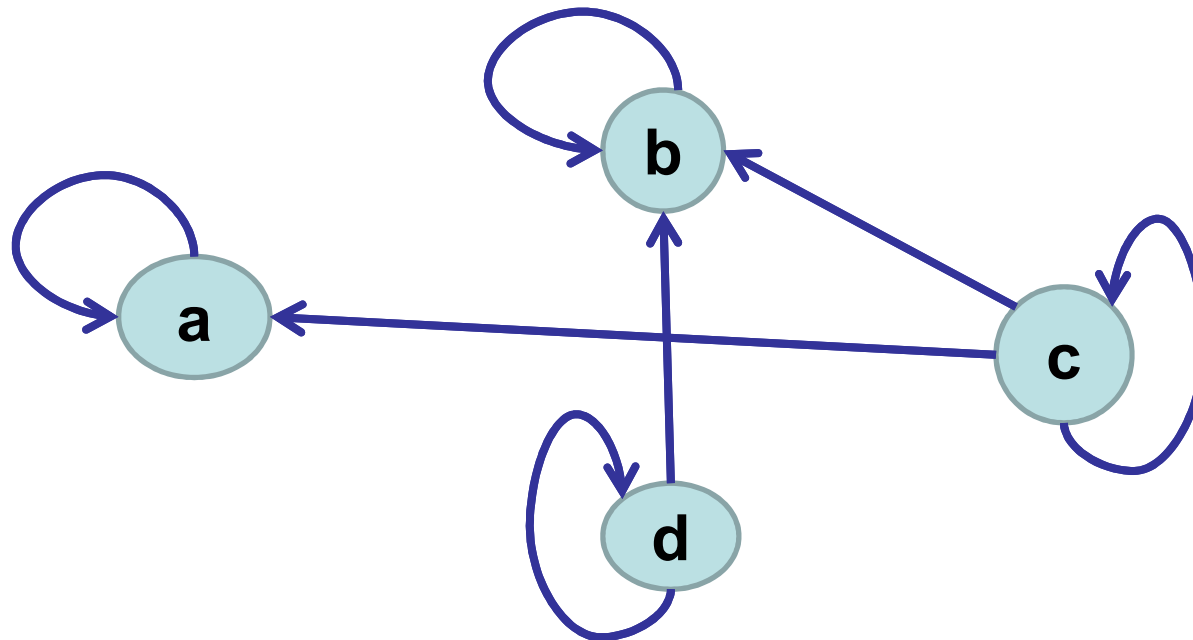
$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

■ Construct the digraph of $R$, and list in-degrees and out-degrees of all vertices.

|  | a | b | c | d |
|---|---|---|---|---|
| In-degree | 2 | 3 | 1 | 1 |
| Out-degree | 1 | 1 | 3 | 2 |

■ A relation $R$ on a set $X$ is called reflexive if $(x,x) \in R$ for every $x \in X$.

■ That is, if $xRx$ for all $x \in X$.

($R$ is reflexive if every element $x \in X$ is related to itself)

- The relation $R$ on $X = \{1, 2, 3, 4\}$ defined by $(x,y) \in R$ if $x \leq y, x,y \in X$ is reflexive because for each element $x \in X$, $(x,x) \in R$

- $(1,1), (2,2), (3,3), (4,4)$ are each in $R$.
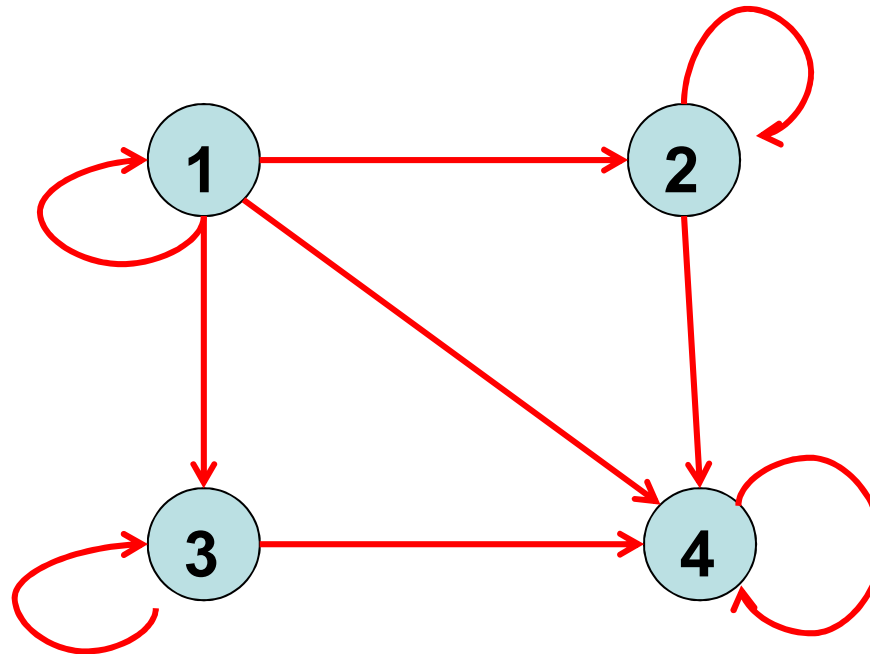
■ The relation,

$$R = \{ (a,a), (b,c), (c,b), (d,d) \}$$

on $X=\{a, b, c, d\}$ is not reflexive.

■ For example,    $b \in X$, but $(b,b) \notin R$

- The digraph of a reflexive relation has a loop at every vertex.

- example

## Irreflexive

- A relation $R$ on a set $A$ is **irreflexive** if $x\not R x$ or $(x,x)$ $R$; $\forall x: x \in X$

## Not Reflexive

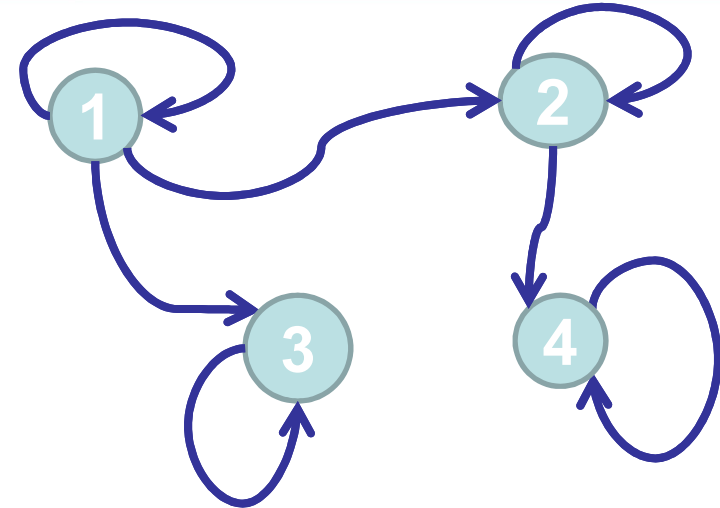- A relation $R$ is **not reflexive** if at least one pair of $(x,x)$ $R$, $\forall x: x \in X$

**Reflexive**

**Irreflexive**
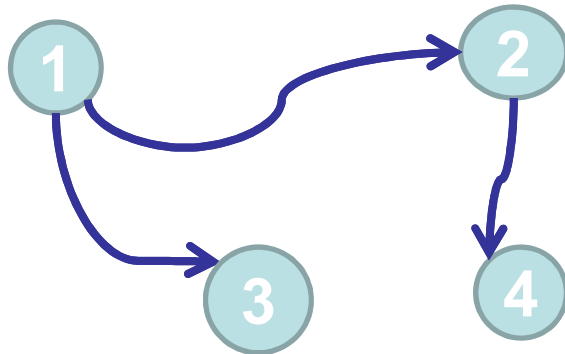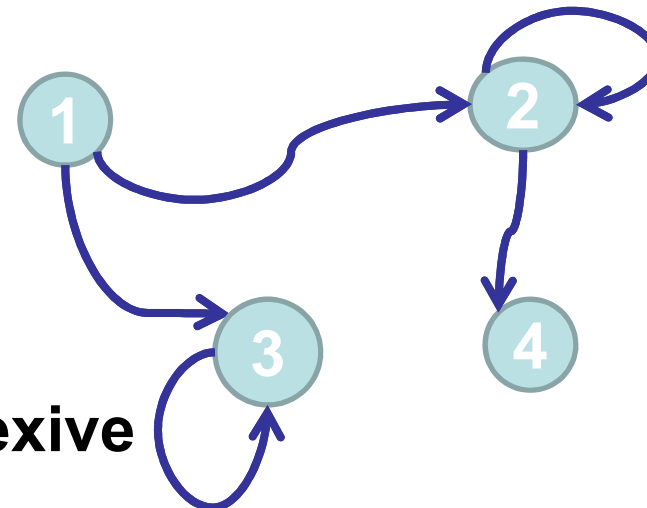
**Not reflexive**

- Consider the following relations on the set {1, 2, 3}

$R_1$ = { (1,1), (1,2), (2,1), (2,2), (3,1), (3,3) }
$R_2$ = { (1,1), (1,3), (2,2), (3,1) }
$R_3$ = { (2,3) }
$R_4$ = { (1,1) }

Which of them are reflexive?

The relation *R* is reflexive if and only if the matrix of relation  has 1's on the main diagonal.

example

$$\begin{array}{c} & a & b & c & d \\ a & 1 & 0 & 0 & 0 \\ b & 0 & 1 & 1 & 0 \\ c & 0 & 1 & 1 & 0 \\ d & 0 & 0 & 0 & 1 \end{array}$$

The relation R is *irreflexive* if and only if the matrix relation have all 0's on its main diagonal

- example

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

- The relation $R$ is **not reflexive**.

$$
\begin{array}{c c c c c}
 & a & b & c & d \\
a & 1 & 0 & 0 & 0 \\
b & 0 & 0 & 1 & 0 \\
c & 0 & 1 & 1 & 0 \\
d & 0 & 0 & 0 & 1
\end{array}
$$

$b \in X$
$(b,b) \notin R$

- A relation $R$ on a set $X$ is called symmetric if for all $x, y \in X$, if $(x,y) \in R$, then $(y,x) \in R$.

$$\forall x,y \in X, (x,y) \in R \rightarrow (y,x) \in R$$

- Let $M$ be the matrix of relation $R$.

  The relation $R$ is symmetric if and only if for all $i$ and $j$, the $ij$th entry of $M$ is equal to the $ji$th entry of $M$.

The matrix of relation $M_R$ is symmetric if $M_R = M_R^T$

Example

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{cccc} a & b & c & d \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} = M_R^T$$

The digraph of a symmetric relation has the property that whenever there is a directed edge from v to w, there is also a directed edge from w to v.

The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$
on $X = \{ a, b, c, d \}$

$(b,c) \in R$
$(c,b) \in R$
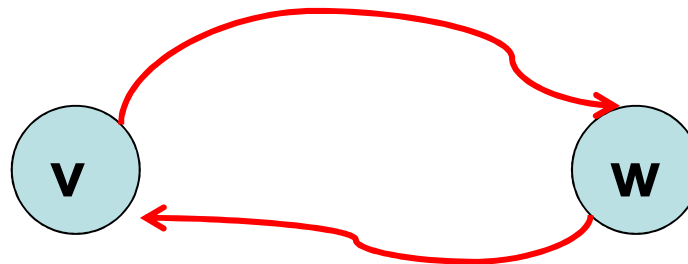
$$\begin{array}{c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

symmetric

The relation $R$ on $X = \{\,1,\,2,\,3\,,4\,\}$, defined by

$(x,y) \in R$ \qquad if $x \leq y,\ x,y \in X$

$$
\begin{array}{c c c c c}
 & 1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 & 1 \\
2 & 0 & 1 & 1 & 1 \\
3 & 0 & 0 & 1 & 1 \\
4 & 0 & 0 & 0 & 1 \\
\end{array}
$$

$(2,3) \in R$

$(3,2) \notin R$

not symmetric

- A relation, $R$ on a set $X$ is called antisymmetric, if for all $x,y \in X$, if $(x,y) \in R$ and $x \neq y$, then $(y,x) \notin R$.

- A relation $R$ on set $X$ is antisymmetric if $x \neq y$, whenever $xRy$, then $y\cancel{R}x$. In other word if whenever $xRy$, then $yRx$ then it implies that $x=y$

$$\forall x,y \in A, (x,y) \in R \wedge x \neq y \to (y, x) \quad R$$

Or

$$\forall x,y \in A, (x,y) \in R \wedge (y, x) \in R \to x = y$$

- Matrix $M_R = [M_{ij}]$ of an antisymmetric relation $R$ satisfies the property that if $i \neq j$, then $m_{ij}=0$ or $m_{ji}=0$.

- If $R$ is antisymmetric relation, then for different vertices $i$ and $j$ there cannot be an edge from vertex $i$ to vertex $j$ and an edge from vertex $j$ to vertex $I$
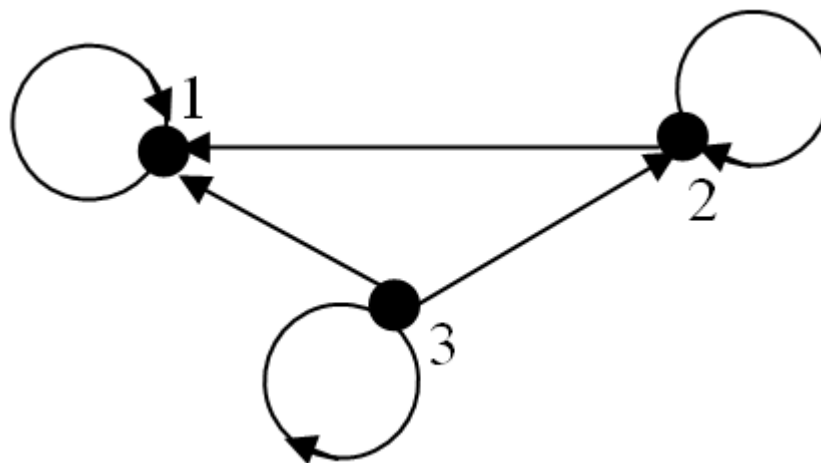
- At least one directed relation and one way

- Let $R$ be a relation on $A$ = {1, 2, 3} defined as $(a, b) \in R$ if $a \geq b$, $a, b \in A$ is an antisymmetric relation because for all $a, b \in A$, $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$, for example

$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

$$(3, 3) \in R \text{ and } (3, 3) \in R \text{ implies } a = b$$

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The relation *R* on *X* = { 1, 2, 3 ,4 } defined by,

$(x,y) \in R$     if $x \le y, x,y \in X$

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} &
\left(\begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{array}\right)
\end{array}
$$

$(1,2) \in R$
$(2,1) \notin R$

antisymmetric

The relation $R = \{ (a,a), (b,c), (c,b), (d,d) \}$
on $X = \{ a, b, c, d \}$

$(b,c) \in R$
$(c,b) \in R$

$$
\begin{array}{c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} &
\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
\end{array}
$$

not antisymmetric

- The relation
  $R = \{ (a,a), (b,b), (c,c) \}$
  on $X = \{ a, b, c \}$

- R has no members of the form
  (x,y) with x≠y, then R is
  antisymmetric

$$
\begin{array}{c c c c}
 & a & b & c \\
a & 1 & 0 & 0 \\
b & 0 & 1 & 0 \\
c & 0 & 0 & 1 \\
\end{array}
$$

Antisymmetric

Reflexive

Symmetric

$$
\begin{array}{c} & a & b & c \\ a & \begin{pmatrix} 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{pmatrix} \end{array}
$$

"Antisymmetric" is not the same as "not symmetric"

The digraph of an antisymmetric relation has at most one directed edge between each pair of vertices.

Example
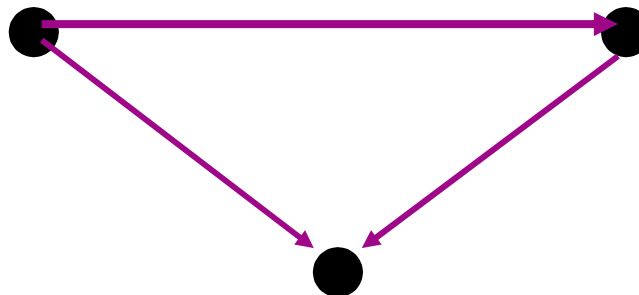
A relation $R$ on set $A$ is asymmetric if whenever $aRb$, then $b\not Ra$.

$$\forall x,y \in X, (x,y) \in R \rightarrow (y,x) \notin R$$

In this sense, a relation is asymmetric if and only if it is both antisymmetric and irreflexive.

- The matrix $M_{R} = [m_{ij}]$ of an asymmetric relation $R$ satisfies the property that

  If $m_{ij} = 1$ then $m_{ji} = 0$

  $m_{ii} = 0$ for all $i$ (the main diagonal of matrix $M_R$ consists entirely of 0's or otherwise)

- If $R$ is asymmetric relation, then the digraph of $R$ cannot simultaneously have an edge from vertex $i$ to vertex $j$ and an edge from vertex $j$ to vertex $i$
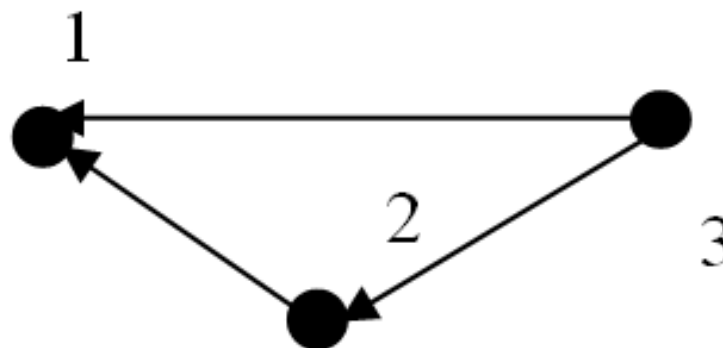
- All edges are "one way street"

- Let $R$ be the relation on A = {1, 2, 3} defined by $(a, b) \in R$ if $a > b$, $a,b \in A$ is an asymmetric relation because,

$$(2, 1) \in R \text{ but } (1, 2) \notin R$$
$$(3, 1) \in R \text{ but } (1, 3) \notin R$$
$$(3, 2) \in R \text{ but } (2, 3) \notin R$$

- Let $R$ be a relation on a set $A$.
- Then $R$ is called **_not symmetric_**, if for all $a, b \in A$, if $(a, b) \in R$, there exist $(b, a)$  $R$.

$$\exists a,b \in A, (a, b) \in R \rightarrow (b, a) \quad R$$

■ Let $R$ be a relation on a set $A$. Then $R$ is called **not symmetric** and **not antisymmetric**, if for all $a, b \in A$, if $(a, b) \in R$, there exist $(b, a)$ $R$ and if $(a, b) \in R$, there exist $(b, a) \in R$.

$$\exists a, b \in A, (a, b) \in R \rightarrow (b, a) \in R$$
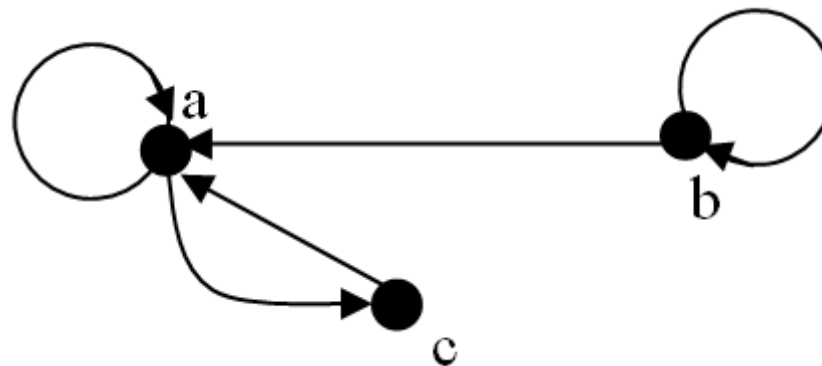$$\text{AND}$$
$$\exists a, b \in A, (a, b) \in R \rightarrow (b, a) \quad R$$

Relation $R = \{(a, c), (b, b), (c, a), (b, a), (a, a)\}$ on $A = \{a, b, c\}$ is not symmetric and not antisymmetric relation because there is,

$(a,c), (c,a) \in R$ and also $(b,a) \in R$ but $(a,b)$   $R$

1. Let $A=Z$, the set of integers and let $R=\{(a,b)\in A\times A|\ a<b\}$. So that $R$ is the relation "less than".

   Is $R$ symmetric, asymmetric or antisymmetric?

2. Let $A=\{1,2,3,4\}$ and let $R = \{(1,2), (2,2), (3,4), (4,1)\}$

   Determine whether $R$ symmetric, asymmetric or antisymmetric.

## Question 1

- **Symmetric** : If *a<b*, then it is not true that *b<a*, so *R* is not symmetric
- **Assymetric** : If *a<b* then *b>a* (*b* is greater than *a*), so *R* is assymetric
- **Antisymmetric** : If *a≠b*, then either *a>b* or *b>a*, so *R* is antisymmetric

## Question 2

- *R* is not symmetric since (1,2)∈*R*, but (2,1)  R
- *R* is not asymmetric , since (2,2)∈*R*
- *R* is antisymmetric, since *a≠b*, either (*a,b*)  R or (*b,a*)  R
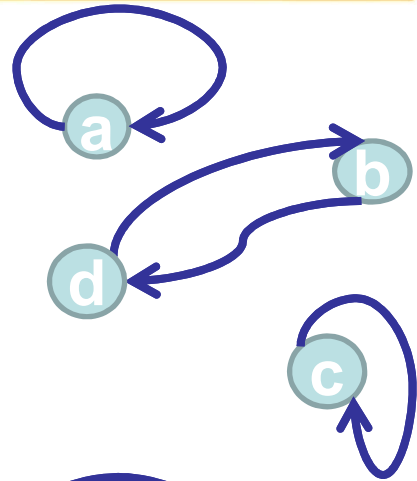
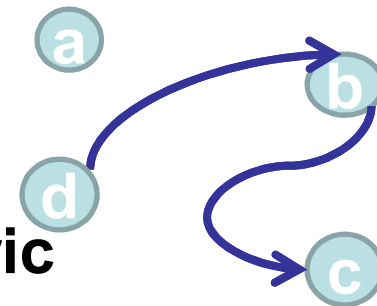**Symmetric**

**Not Symmetric**
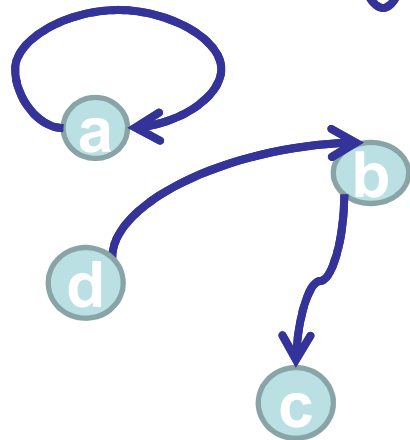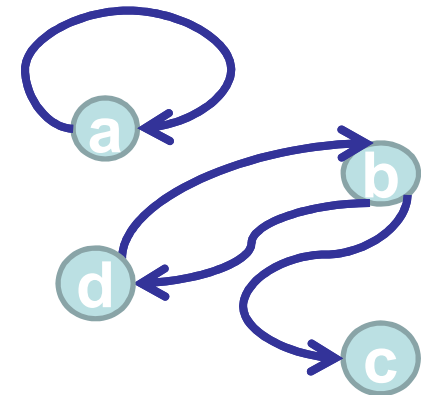
**Antisymmetric**

**Asymmetric**

- A relation $R$ on a set $X$ is called transitive if for all $x,y,z \in X$,

    if $(x,y)$ and $(y,z) \in R$ then $(x,z) \in R$

- It is often convenient to say what it means for a relation to be not transitive.

- A relation $R$ on $X$ is **not transitive** if there exists $x, y$, and $z$ in X so that $xRy$ and $yRz$, but $x\not R z$. If such x,y, and z do not exist, then R is transitive.

- The relation $R$= { $(a,a)$, $(b,c)$, $(c,b)$, $(d,d)$ } on $X$ = { $a, b, c, d$ } is not transitive.

- $(b,c)$ and $(c,b) \in R$, but $(b,b) \notin R$.

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- Let $M_R$ be the matrix of relation R. Let,

$$M_R \otimes M_R = N. \qquad (\otimes \text{ Boolean product})$$

$M_R = [m_{ij}]$ and $N = [n_{ij}]$

The relation $R$ is **transitive** if and only if the following is true:

$$\forall i \forall j, \text{ if } (n_{ij} = 1) \text{ then } (m_{ij} = 1)$$

The relation $R$ is **not transitive** if and only if:

$$\exists i \exists j \ (n_{ij} = 1) \wedge (m_{ij} = 0)$$

| + | 1 | 0 |
|---|---|---|
| 1 | 1 | 1 |
| 0 | 1 | 0 |

| . | 1 | 0 |
|---|---|---|
| 1 | 1 | 0 |
| 0 | 0 | 0 |

Let *R* be a relation on *A={1,2,3}* is defined by *(a,b)∈R* if *a⪯b, a,b∈A*. Find *R*. Is *R* a transitive relation?

Solution:

*R={(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)}*

*R* is a transitive relation because

(1,2) and (2,2)∈*R*, (1,2)∈*R*
(1,2) and (2,3)∈*R*, (1,3)∈*R*
(1,3) and (3,3)∈*R*, (1,3)∈*R*
(2,2) and (2,3)∈*R*, (2,3)∈*R*
(2,3) and (3,3)∈*R*, (2,3)∈*R*

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The matrix of relation $M_R$,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\forall i \forall j$, if $(n_{ij} = 1)$
then $(m_{ij} = 1)$

The product of boolean,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that,$(1,2)$ and $(2,3) \in R$, $(1,3) \in R$

Consider the following relations on the set {1, 2, 3}

R1 = { (1,1), (1,2), (2,3) }
R2 = { (1,2), (2,3), (1,3) }

Which of them is transitive?

R1 = { (1,1), (1,2), (2,3) }

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$\forall i \forall j$, if $(n_{ij} = 1)$ then $(m_{ij} = 1)$

$(n_{13} = 1) \wedge (m_{13} = 0)$

(1,2) and (2,3)$\in R$, (1,3)  $R$

**Not transitive**

R2 = { (1,2), (2,3), (1,3) }

$$\begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} \otimes \begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$$

$\forall i \forall j$, if ($n_{ij}$ = 1) then ($m_{ij}$ = 1)

($n_{13}$ = 1) then ($m_{13}$ = 1)

(1,2) and (2,3)$\in R$, (1,3)$\in R$

*R* is a transitive relation

The relation $R$ on $A=\{a,b,c,d\}$ is $R=\{(a,a), (b,b), (c,c), (d,d), (a,c), (c,b)\}$ is not transitive. The matrix of relation $M_R$,

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{cccc} a & b & c & d \\ \begin{bmatrix} 1 & \boxed{0} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$(n_{12} = 1) \wedge (m_{12} = 0)$

The product of boolean,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{cccc} a & b & c & d \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \otimes \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{cccc} a & b & c & d \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{cccc} a & b & c & d \\ \begin{bmatrix} 1 & \boxed{1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

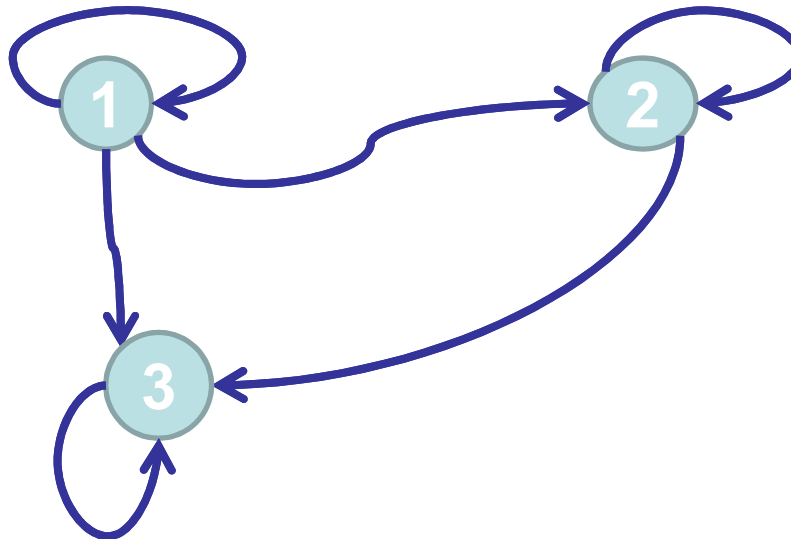Note that,(a,c) and (c,b)$\in R$, (a,b) $R$

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In the digraph of *R*, *R* is a transitive relation if and only if there is a directed edge from one vertex *a* to another vertex *b*, an if there exits a directed edge from vertex *b* to vertex *c*, then there must exists a directed edge from *a* to *c*

$R$={(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)}

The diagraph:

A relation $R$ that is reflexive, symmetric and transitive on a set $X$ is called an equivalence relation on $X$.

Let $R$={(1,1), (1,3), (2,2), (3,1), (3,3)} on {1,2,3}, the matrix of the relation $M_R$,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

All the main diagonal matrix elements are 1 and the matrix is reflexive

Let $R$={(1,1), (1,3), (2,2), (3,1), (3,3)} on {1,2,3}, the matrix of the relation $M_R$,

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

All the main diagonal matrix elements are 1 and the matrix is reflexive

The transpose matrix $M_R$, $M_R^T$ is equal to $M_R$, so $R$ is symmetric

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \end{array} \qquad M_R^T = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \end{array}$$

The product of Boolean show that the matrix is transitive.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- The relation, *R*={ (1,1), (1,3), (1,5), (2,2), (2,4), (3,1), (3,3), (3,5), (4,2), (4,4), (5,1), (5,3), (5,5) }
  on { 1,2,3,4,5 }

- Reflexive?
- Symmetric?
- Transitive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Reflexive?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Reflexive √

Symmetric?

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Symmetric √

Transitive?

$$
\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}
$$

- Reflexive
- Symmetric
- Transitive

Equivalence relation

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

- The relation $R$ on $X=\{ 1, 2, 3, 4 \}$, defined by
  $(x,y) \in R$ if $x \leq y$, $x,y \in X$

- Not symmetric
  - $(2,3) \in R$ but $(3,2) \notin R$

- $R$ is not equivalence relation on $X$.

A relation, *R* on a set *X* is called a partial order if *R* is reflexive, antisymmetric, and transitive.

Let $R$ be a relation on a set $A=\{1,2,3\}$ defined by $(a,b) \in R$ if $a \preceq b$, $a,b \in R$.

$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

$R$ is reflexive, antisymmetric and transitive.

So $R$ is a partial order relation.

The relation $R$ defined on the positive integers by

$(x,y) \in R$ if $x$ divides $y$ (evenly)

is reflexive, antisymmetric and transitive

R is a partial order.