The ACPATH Metric: Precise Estimation of the Number of Acyclic Paths in C-like Languages

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Abstract NPATH is a metric introduced by Brian A. Nejmeh in [13] that is aimed at overcoming some important limitations of McCabe's cyclomatic complexity. Despite the fact that the declared NPATH objective is to count the number of acyclic execution paths through a function, the definition given for the C language in [13] fails to do so even for very simple programs. We show that counting the number of acyclic paths in CFG is unfeasible in general. Then we define a new metric for C-like languages, called ACPATH, that allows to quickly compute a very good estimation of the number of acyclic execution paths through the given function. We show that, if the function body does not contain backward gotos and does not contain jumps into a loop from outside the loop, then such estimation is actually exact.

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1 Introduction

Software testing is a process whereby software components or entire systems are executed so as to gather information about their behavior. Although a common expected outcome of software testing is the identification of defects, testing and debugging are two quite different processes [12]: while the latter

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is a development activity, the former is one of the methodologies for software verification and validation.

Despite the increasing adoption of formal methods and static verification techniques, software testing is still the most used verification technique in several industrial sectors costing as much as 50% or even 75% of the total development costs [6].

One of the problems of software testing is the need for adequate test suites, i.e., collections of so-called test cases, each consisting of input and predicted output data. Software testing can only be considered to be acceptable as a verification methodology when the available test suites exercise a significant portion of the code and input space of the component under test. Therefore, the feasibility of meaningfully testing a system and its subsystems with a test suite of manageable size and cost is a qualitative attribute, called testability, of the system/subsystems.

Testing at the unit level (a.k.a. unit testing, i.e., testing individual functions or small groups of functions) is of particular importance as it often allows for the early detection of problems, when the cost of fixing them is much lower than if they are found during integration testing (i.e., when the integration of different units are tested as a whole to assess their ability to work together). Hence, an important reason for limiting the structural complexity of software units is to facilitate unit testing, i.e., to improve testability of the units by limiting the sizes of the unit test suites and the intellectual effort of obtaining them.

The NPATH metric was introduced by Brian A. Nejmeh in [13] in order to automatically quantify the testability of individual procedures or functions, yet addressing the shortcomings of McCabe cyclomatic complexity, another metric meant to quantify testability [7,10]. According to [13], the shortcomings of cyclomatic complexity are:¹

- the number of acyclic paths in the control-flow graph of a procedure varies from a linear to an exponential function of the cyclomatic complexity number [3]; as the fraction of acyclic paths covered by a test suite is an important measure of adequacy of the test suite, it turns out that the cyclomatic complexity number has little correlation with the testing effort;
- cyclomatic complexity does not distinguish between different kinds of controlflow structures (e.g., between conditional and iteration statements) whereas such distinction is important in the assessment of testability;
- cyclomatic complexity does not take into account the way control-flow structures are possibly nested with one another (e.g., two disjoint while loops give rise to the same cyclomatic complexity number as two nested while loops) [2]; again, this distinction is relevant as far as testability is concerned.

While the declared intent of NPATH is to count the number of acyclic paths through a function, the definition given for the C language in [13] fails to do so, as shown by the following example:

¹ Cyclomatic complexity was criticized also by other authors: see, e.g., [4,5,14].

Example 1 Consider the following C function:

```
int f(int a, int b, int c, int d, int e) {
  if (a && b && c)
    return d ? 0 : 1;
  else
    return e ? 0 : 1;
}
```

The algorithm given in [13] gives NPATH = 2 + 2 + 2 = 6, but this is neither the number of possible paths within f() nor an upper bound to it. In fact the number of possible paths is 8, corresponding to the following combinations:

```
1. a && b && c && d
2. a && b && c && !d
3. a && b && !c && e
4. a && b && !c && !e
5. a && !b && e
6. a && !b && !e
7. !a && e
8. !a && !e
```

Section 4 presents a new metric for C-like languages that demonstrably corresponds, under some conditions that are often satisfied, to the number of acyclic paths through the function.

The plan of the paper is as follows: Section 2 introduces preliminary notions and notations; Section 3 recalls the NPATH metric for C-like languages as defined in [13] highlighting the difference between what it is meant to measure and what it really measures; Section 4 presents the new ACPATH metric for C-like languages; Section 5 presents the results of an experimental evaluation that studies, on a number of real-world projects, the relationship between ACPATH and NPATH; Section 6 discusses the contribution of the present paper, some related recent work, and concludes.

2 Preliminaries

In this section we introduce the preliminary notions such as *control flow graphs* and *acyclic paths*, notations including the abstract C syntax used in the paper and some formal definitions needed to prove the theoretical results of Section 4.

2.1 Control Flow Graphs

A control flow graph (CFG) is an abstraction of the computation paths of a procedure.

Definition 1 (Control flow graph.) A control flow graph G is a triple (N, A, s) where (N, A) is a directed graph, hence $A \subseteq N \times N$, and $s \in N$ is called the *entry node of* G. A node $n \in N$ such that n has no successor in A (i.e., for each $(x, y) \in A$ we have $x \neq n$) is called an *exit node of* G.

A node in the graph represents either a basic block of code (i.e., a sequence of statements where control flows from the beginning to the end of the sequence) or a branch point in the procedure. The entry node represents the procedure's entry point and each exit node represents an exit point of the procedure. An arc represents possible flow control.

2.2 Acyclic Paths in a CFG

An acyclic path in a CFG is a path from the entry node to a target node that does not traverse an arc more than once.

Definition 2 (Acyclic path.) An acyclic path in a control flow graph (N, A, s) is any sequence of nodes of the form n_0, \ldots, n_{k-1} such that $n_0 = s$ and, if $M := \{ (n_{i-1}, n_i) \mid i = 1, \ldots, k \}$, then $M \subseteq A$ and |M| = k.

Let G = (N, A, s) be a CFG and $t \in N$ be a target node. The number of acyclic paths in G leading to t, denoted by $\tau(G, t)$, can be computed as follows:

$$\tau(G,t) := \tau(s,A,t),\tag{1}$$

$$\tau(n, A, t) := \begin{cases} 1, & \text{if } n = t; \\ \sum_{(n, m) \in A} \tau(m, A \setminus \{(n, m)\}, t), & \text{otherwise.} \end{cases}$$
 (2)

If we denote by e(G) the set of exit nodes of G, the number of acyclic paths in G, denoted by $\alpha(G)$, is given by

$$\alpha(G) := \sum_{t \in e(G)} \tau(s, A, t). \tag{3}$$

2.3 C Abstract Syntax

The abstract syntax of expressions considered in this paper, which is inspired by the one used in the *Clang* C language family compiler front-end, is approximated by the following grammar in BNF form:

Exp
$$\ni E ::= x \mid k \mid ICE \mid ! E_1 \mid + E_1 \mid - E_1 \mid (E_1) \mid (type) E_1 \mid uop E_1 \mid E_1 \&\& E_2 \mid E_1 \mid | E_2 \mid E_1, E_2 \mid E_1 ? : E_2 \mid E_1 bop E_2 \mid E_1 ? E_2 : E_3$$

where x is a variable, k is an integer literal, 'ICE' is any Integer Constant Expression, that is, a non-literal expression that can be evaluates to a constant at compile time (e.g., 3+4), '(type)' represents (implicit and explicit) cast operators, 'uop' and 'bop' are any unary or binary operators except those already considered. The abstract syntax of commands is approximated by the following grammar:

where stm generates any command except those already considered and L: S is a labeled statement:

```
Lab \ni L ::= case z \mid default \mid id
```

where L is a *label*, z is an ICE and id is a C identifier.

2.4 From C Abstract Syntax To Control Flow Graphs

For any procedure and hence any abstract command Stm that represents it, the actual control flow and the corresponding CFG will depend on the compiler and on the selected optimization level and capabilities. For purposes of this paper, it suffices to define a notion of "reference CFG" and to restrict the possible optimization levels to three:

- 0 no optimization at all,
- 1 branch removal via Boolean interpretation of each integer literal, and
- 2 branch removal via the Boolean interpretation of each ICE.

Note that, if the metrics we are after are meant to measure testability only, there is no difference between a constant literal and an ICE. An alternative point of view is that NPATH's purpose is to evaluate also readability and maintainability and, in this case, an ICE can be considered as an ordinary compound expression. In this paper, we wish to support both views.

The following definition gives, by structural induction on the abstract C syntax of Section 2.3, the reference CFG for any function body. The definition can be skipped unless the reader wishes to check the proofs of the theorems. Appendix B on page 55 provides several examples showing, by means of figures, the definition at work.

Definition 3 (Reference CFG for the C language.) Let CFG denote the set of all CFGs where the nodes are natural numbers and let $i \in \{0, 1, 2\}$ denote the three optimization levels.

To define the reference CFG for expressions we first define the function tv_i : Exp $\rightarrow \{0,1,?\}$ that returns, for each $E \in Exp$, a three-valued Boolean defined as follows, where $E \mapsto b$ means "E evaluates to b":

$$\operatorname{tv}_i(E) := \begin{cases} b, & \text{if } E \mapsto b \text{ and } i = 1 \text{ and } E \text{ is an integer literal;} \\ b, & \text{if } E \mapsto b \text{ and } i = 2 \text{ and } E \text{ is an ICE;} \\ ?, & \text{otherwise.} \end{cases} \tag{4}$$

The function

$$\operatorname{cfg}_{i}[\![\cdot]\!] : \operatorname{Exp} \times \mathbb{N}^{3} \to \operatorname{CFG} \times \mathbb{N}$$

is defined as follows: whenever $\operatorname{cfg}_i[\![E]\!](t,f,m) = \langle G,m' \rangle$, then $G = (N,A,s) \in \operatorname{CFG}$, where the nodes are $N \subseteq \{t,f\} \cup [m,m'-1]$ and t (resp., f) is reached from s if E evaluates to true (resp., false).

Variables:

$$\operatorname{cfg}_{i}[\![x]\!](t,f,m) := \langle (\{m,t,f\}, \{(m,t), (m,f)\}, m), m+1 \rangle. \tag{5}$$

Constants: if E = k or E = ICE,

$$\operatorname{cfg}_{i}\llbracket E \rrbracket(t,f,m) := \begin{cases} \left\langle \left(\{t\},\varnothing,t\right),m\right\rangle, & \text{if } \operatorname{tv}_{i}(E) = 1; \\ \left\langle \left(\{f\},\varnothing,f\right),m\right\rangle, & \text{if } \operatorname{tv}_{i}(E) = 0; \\ \left\langle \left(\{m,t,f\},\{(m,t),(m,f)\},m\right),m+1\right\rangle, & \text{otherwise.} \end{cases}$$

Logical negation:

$$\operatorname{cfg}_{i}[\![! E_{1}]\!](t, f, m) := \operatorname{cfg}_{i}[\![E_{1}]\!](f, t, m),$$
 (7)

where $E = !E_1$.

Unary plus and minus, parentheses and cast expressions:

$$\operatorname{cfg}_{i}[E](t, f, m) := \operatorname{cfg}_{i}[E_{1}](t, f, m),$$
 (8)

where $E \in \{+E_1, -E_1, (E_1), (type)E_1\}.$

Other unary operators:

$$\operatorname{cfg}_{i}[\operatorname{uop} E_{1}](t, f, m) := \langle (N, A, s_{1}), m_{1} \rangle,$$
 (9)

where uop is a unary operator not already considered, $N := N_1 \cup \{m, t, f\}$, $A := A_1 \cup \{(m, t), (m, f)\}$, and $\operatorname{cfg}_i[\![E_1]\!](m, m, m + 1) = \langle (N_1, A_1, s_1), m_1 \rangle$.

Logical conjunction:

$$\operatorname{cfg}_{i}\llbracket E_{1} \&\& E_{2} \rrbracket(t, f, m) := \begin{cases} \left\langle \left(\{f\}, \varnothing, f \right), m \right\rangle, & \text{if } \operatorname{tv}_{i}(E_{1}) = 0, \\ \operatorname{cfg}_{i}\llbracket E_{2} \rrbracket(t, f, m), & \text{if } \operatorname{tv}_{i}(E_{1}) = 1, \\ \left\langle (N, A, s_{1}), m_{2} \right\rangle, & \text{otherwise,} \end{cases}$$
(10)

where $N := N_1 \cup N_2$, $A := A_1 \cup A_2$, $\operatorname{cfg}_i[E_2](t, f, m) = \langle (N_2, A_2, s_2), m_1 \rangle$, and $\operatorname{cfg}_i[E_1](s_2, f, m_1) = \langle (N_1, A_1, s_1), m_2 \rangle$.

Logical disjunction:

$$\operatorname{cfg}_{i}[\![E_{1} \mid\mid E_{2}]\!](t, f, m) := \begin{cases} \langle (\{t\}, \emptyset, t), m \rangle, & \text{if } \operatorname{tv}_{i}(E_{1}) = 1, \\ \operatorname{cfg}_{i}[\![E_{2}]\!](t, f, m), & \text{if } \operatorname{tv}_{i}(E_{1}) = 0, \\ \langle (N, A, s_{1}), m_{2} \rangle, & \text{otherwise,} \end{cases}$$
(11)

where $N := N_1 \cup N_2$, $A := A_1 \cup A_2$, $\operatorname{cfg}_i[\![E_2]\!](t, f, m) = \langle (N_2, A_2, s_2), m_1 \rangle$ and $\operatorname{cfg}_i[\![E_1]\!](t, s_2, m_1) = \langle (N_1, A_1, s_1), m_2 \rangle$.

Comma operator:

$$\operatorname{cfg}_{i}[E_{1}, E_{2}](t, f, m) := \langle (N, A, s_{1}), m_{2} \rangle,$$
 (12)

where $N := N_1 \cup N_2$, $A := A_1 \cup A_2$, $\operatorname{cfg}_i[\![E_2]\!](t, f, m) = \langle (N_2, A_2, s_2), m_1 \rangle$ and $\operatorname{cfg}_i[\![E_1]\!](s_2, s_2, m_1) = \langle (N_1, A_1, s_1), m_2 \rangle$.

Binary conditional operator:

$$\operatorname{cfg}_{i}\llbracket E_{1} ?: E_{2} \rrbracket(t, f, m) := \begin{cases} \langle (\{t\}, \varnothing, t), m \rangle, & \text{if } \operatorname{tv}_{i}(E_{1}) = 1, \\ \operatorname{cfg}_{i}\llbracket E_{2} \rrbracket(t, f, m), & \text{if } \operatorname{tv}_{i}(E_{1}) = 0, \\ \langle (N, A, s_{1}), m_{2} \rangle, & \text{otherwise,} \end{cases}$$
(13)

where $N:=N_1\cup N_2,\,A:=A_1\cup A_2,\,\mathrm{cfg}_i[\![E_2]\!](t,f,m)=\left\langle (N_2,A_2,s_2),m_1\right\rangle$ and $\mathrm{cfg}_i[\![E_1]\!](t,s_2,m_1)=\left\langle (N_1,A_1,s_1),m_2\right\rangle$.

Other binary operators:

$$\operatorname{cfg}_{i}[E_{1} \text{ bop } E_{2}](t, f, m) := \langle (N, A, s_{1}), m_{2} \rangle,$$
 (14)

where $N := N_1 \cup N_2 \cup \{m, t, f\}, A := A_1 \cup A_2 \cup \{(m, t), (m, f)\},$ and

$$\operatorname{cfg}_{i}[E_{2}](m, m, m + 1) = \langle (N_{2}, A_{2}, s_{2}), m_{1} \rangle,$$

$$\operatorname{cfg}_{i}[E_{1}](s_{2}, s_{2}, m_{1}) = \langle (N_{1}, A_{1}, s_{1}), m_{2} \rangle,$$

Conditional operator:

$$\operatorname{cfg}_{i}[\![E_{1} ? E_{2} : E_{3}]\!](t, f, m) := \begin{cases} \operatorname{cfg}_{i}[\![E_{2}]\!](t, f, m), & \text{if } \operatorname{tv}_{i}(E_{1}) = 1, \\ \operatorname{cfg}_{i}[\![E_{3}]\!](t, f, m), & \text{if } \operatorname{tv}_{i}(E_{1}) = 0, \\ \langle (N, A, s_{1}), m_{3} \rangle, & \text{otherwise,} \end{cases}$$
(15)

where $N := N_1 \cup N_2 \cup N_3$, $A := A_1 \cup A_2 \cup A_3$ and

$$\operatorname{cfg}_{i}[E_{2}](t, f, m) = \langle (N_{2}, A_{2}, s_{2}), m_{1} \rangle, \\ \operatorname{cfg}_{i}[E_{3}](t, f, m_{1}) = \langle (N_{3}, A_{3}, s_{3}), m_{2} \rangle, \\ \operatorname{cfg}_{i}[E_{1}](s_{2}, s_{3}, m_{2}) = \langle (N_{1}, A_{1}, s_{1}), m_{3} \rangle.$$

Before defining the reference CFG for statements, we define a special form that will be used for labeled statements:

$$\operatorname{cfg}_{i}[\cdot]: \operatorname{Lab} \times \mathbb{N}^{2} \to \operatorname{CFG} \times \wp((\operatorname{Id} \cup \{\operatorname{cs},\operatorname{df}\}) \times \mathbb{N}) \times \mathbb{N}$$

where $\mathbb{N}_{\perp} := \mathbb{N} \cup \{\perp\}$ and Id denotes the set of identifiers in the C language. If $\operatorname{cfg}_i[\![L]\!](t,m) = \langle G, M^{\operatorname{s}}, m' \rangle$, then:

- $-G = (N, A, s) \in CFG \text{ and } N \subseteq \{t\} \cup [m, m' 1];$
- $-t \in N$ is reached if/when the execution of L terminates;
- M^s is a multimap associating elements of Id ∪ {cs, df} (where cs incorporates all case z, where z is an ICE, and df stands for default, respectively) to nodes, such that at most one occurrence of df is allowed: if (id, n) ∈ M^s , then n is the node in G corresponding to a statement labeled with id; if (cs, n) ∈ M^s , then n is a node in G corresponding to a case-labeled statement; if (df, n) ∈ M^s , then n is the node in G corresponding to a default-labeled statement;
- $-m,m'\in\mathbb{N}$ are, respectively, the lower and the upper bound of the nodes' labels introduced by L

Labels:

$$\operatorname{cfg}_{i}[\![L]\!](t,m) := \begin{cases} \left\langle \left(\{m,t\}, \big\{(m,t)\big\}, m \right), \big\{(\operatorname{cs},m)\big\}, m+1 \right\rangle, & \text{if } L = \operatorname{\mathsf{case}} z, \\ \left\langle \left(\{m,t\}, \big\{(m,t)\big\}, m \right), \big\{(\operatorname{\mathsf{df}},m)\big\}, m+1 \right\rangle, & \text{if } L = \operatorname{\mathsf{default}}, \\ \left\langle \left(\big\{m,t\big\}, \big\{(m,t)\big\}, m \right), \big\{(\operatorname{\mathsf{id}},m)\big\}, m+1 \right\rangle, & \text{if } L = \operatorname{\mathsf{id}}. \end{cases}$$

We can now define the reference CFG for statements

$$\mathrm{cfg}_{i} \llbracket \cdot \rrbracket \colon \mathrm{Stm} \times \mathbb{N} \times \mathbb{N}_{\perp}^{2} \times \mathbb{N} \to \mathrm{CFG} \times \wp \big((\mathrm{Id} \cup \{ \mathrm{cs}, \mathrm{df} \}) \times \mathbb{N} \big) \times \wp (\mathrm{Id} \times \mathbb{N}) \times \mathbb{N}.$$

If $\operatorname{cfg}_{i}[S](t, t_{b}, t_{c}, m) = \langle G, M^{s}, M^{g}, m' \rangle$, then:

- $G = (N, A, s) \in \text{CFG}$ and $N \subseteq \{t, t_{\text{b}}, t_{\text{c}}\} \cup [m, m' 1];$
- $-t \in N$ is reached if/when the execution of S terminates;
- $-t_b \in N$ is reached if/when the execution of S terminates because a break has been executed;

- $-t_c \in N$ is reached if/when the execution of S terminates because a continue has been executed;
- $-M^{s}$ is a multimap for the labels in S, as defined earlier;
- M^{g} is a map associating the identifiers of goto statements in S to their target node in G;
- $-m,m'\in\mathbb{N}$ are, respectively, the lower and the upper bound of the nodes' labels introduced by S.

Expression statement:

$$\operatorname{cfg}_{i}[E;](t, t_{b}, t_{c}, m) := \langle (N, A, s), \varnothing, \varnothing, m_{1} \rangle, \tag{17}$$

where $\operatorname{cfg}_{i}[\![E]\!](t,t,m) = \langle (N,A,s), m_{1} \rangle$.

Sequential composition:

$$\operatorname{cfg}_{i}[[S_{1} \ S_{2}]](t, t_{b}, t_{c}, m) := \langle (N_{1} \cup N_{2}, A_{1} \cup A_{2}, s), M^{s}, M^{g}, m_{2} \rangle, \tag{18}$$

where $M^{s} := M_{1}^{s} \cup M_{2}^{s}$, $M^{g} := M_{1}^{g} \cup M_{2}^{g}$, and

$$\operatorname{cfg}_{i}[[S_{2}]](t, t_{b}, t_{c}, m) = \langle (N_{2}, A_{2}, s_{2}), M_{2}^{s}, M_{2}^{g}, m_{1} \rangle, \operatorname{cfg}_{i}[[S_{1}]](s_{2}, t_{b}, t_{c}, m_{1}) = \langle (N_{1}, A_{1}, s), M_{1}^{s}, M_{2}^{g}, m_{2} \rangle.$$

Return statement:

$$\operatorname{cfg}_{i}[\![\operatorname{return}]\!](t, t_{b}, t_{c}, m) := \langle (\{m\}, \varnothing, m), \varnothing, \varnothing, m + 1 \rangle, \tag{19}$$

Return with expression statement:

$$\operatorname{cfg}_{i} \llbracket \operatorname{return} E \rrbracket (t, t_{b}, t_{c}, m) := \langle (N, A, s), \varnothing, \varnothing, m_{1} \rangle, \tag{20}$$

where $N := N_E \cup \{m\}$ and $\text{cfg}_i[\![E]\!](m, m, m + 1) = \langle (N_E, A, s), m_1 \rangle$.

 $Conditional\ statement:$

$$\begin{split} \operatorname{cfg}_{i} \llbracket \text{if } (E) \ S_{1} \ \text{else} \ S_{2} \rrbracket (t, t_{\operatorname{b}}, t_{\operatorname{c}}, m) \\ &:= \begin{cases} \operatorname{cfg}_{i} \llbracket S_{1} \rrbracket (t, t_{\operatorname{b}}, t_{\operatorname{c}}, m), & \text{if } \operatorname{tv}_{i}(E) = 1 \wedge M_{2}^{\operatorname{s}} = \varnothing, \\ \operatorname{cfg}_{i} \llbracket S_{2} \rrbracket (t, t_{\operatorname{b}}, t_{\operatorname{c}}, m), & \text{if } \operatorname{tv}_{i}(E) = 0 \wedge M_{1}^{\operatorname{s}} = \varnothing, \\ \left\langle (N, A, s), M^{\operatorname{s}}, M^{\operatorname{g}}, m_{3} \right\rangle, & \text{otherwise}, \end{cases} \end{split}$$

where $N:=N_E\cup N_1\cup N_2\cup \left\{m,m_1,t\right\},\, A:=A_E\cup A_1\cup A_2\cup \left\{(m,t),(m_1,t)\right\},\, M^{\mathrm{s}}:=M_1^{\mathrm{s}}\cup M_2^{\mathrm{s}},\, M^{\mathrm{g}}:=M_1^{\mathrm{g}}\cup M_2^{\mathrm{g}},\, \mathrm{and}$

$$cfg_{i}[S_{2}](m, t_{b}, t_{c}, m + 1) = \langle (N_{2}, A_{2}, s_{2}), M_{2}^{s}, M_{2}^{g}, m_{1} \rangle,$$

$$cfg_{i}[S_{1}](m_{1}, t_{b}, t_{c}, m_{1} + 1) = \langle (N_{1}, A_{1}, s_{1}), M_{1}^{s}, M_{1}^{g}, m_{2} \rangle,$$

$$cfg_{i}[E](s_{1}, s_{2}, m_{2}) = \langle (N_{E}, A_{E}, s), m_{3} \rangle.$$

One-armed conditional statement:

$$\operatorname{cfg}_{i}\llbracket\operatorname{if}(E) S_{1}\rrbracket(t, t_{b}, t_{c}, m) = \begin{cases} \operatorname{cfg}_{i}\llbracketS_{1}\rrbracket(t, t_{b}, t_{c}, m), & \text{if } \operatorname{tv}_{i}(E) = 1, \\ \langle (\{t\}, \varnothing, t), \varnothing, \varnothing, m\rangle, & \text{if } \operatorname{tv}_{i}(E) = 0 \wedge M^{s} = \varnothing, \\ \langle (N, A, s), M^{s}, M^{g}, m_{2}\rangle, & \text{otherwise,} \end{cases}$$

$$(22)$$

where
$$N := N_E \cup N_1 \cup \{m, t\}, A := A_E \cup A_1 \cup \{(m, t)\}, \text{ and}$$

$$\operatorname{cfg}_i[S_1](m, t_b, t_c, m + 1) = \langle (N_1, A_1, s_1), M^s, M^g, m_1 \rangle,$$

$$\operatorname{cfg}_i[E](s_1, t, m_1) = \langle (N_E, A_E, s), m_2 \rangle.$$

Switch statement:

$$\begin{aligned} \operatorname{cfg}_{i} \llbracket \operatorname{switch} \left(E \right) S \rrbracket \left(t, t_{\mathrm{b}}, t_{\mathrm{c}}, m \right) \\ &:= \begin{cases} \left\langle (N, A_{1}, s), M^{\mathrm{s}}, M^{\mathrm{g}}, m_{2} \right\rangle, & \text{if } (\operatorname{df}, n) \in M_{1}^{\mathrm{s}}, \\ \left\langle (N, A_{2}, s), M^{\mathrm{s}}, M^{\mathrm{g}}, m_{2} \right\rangle, & \text{otherwise,} \end{cases} \end{aligned}$$
 (23)

where $N := N_E \cup N_S \cup \{m, m_1\}$ and

$$\begin{split} A_1 &:= A_E \cup A_S \cup \{(m,t)\} \\ & \cup \big\{ \, (m_1,n) \; \big| \; \exists l \in \{ \operatorname{cs},\operatorname{df} \} \; . \; (l,n) \in M_1^s \, \big\}, \\ A_2 &:= A_1 \cup \big\{ (m_1,m) \big\}, \\ M^s &:= M_1^s \setminus \big\{ \, (l,n) \; \big| \; l \in \{ \operatorname{cs},\operatorname{df} \} \, \big\}, \\ \operatorname{cfg}_i \llbracket S \rrbracket (m,m,t_c,m+1) &= \big\langle (N_S,A_S,s_S),M_1^s,M^g,m_1 \big\rangle, \\ \operatorname{cfg}_i \llbracket E \rrbracket (m_1,m_1,m_1+1) &= \big\langle (N_E,A_E,s),m_2 \big\rangle. \end{split}$$

While statement:

$$\operatorname{cfg}_{i}[\![\mathsf{while}\ (E)\ S]\!](t, t_{\mathsf{b}}, t_{\mathsf{c}}, m) := \langle (N, A, s_{E}), M^{\mathsf{s}}, M^{\mathsf{g}}, m_{2} \rangle, \tag{24}$$
where $N := N_{E} \cup N_{S} \cup \{m, m_{1}\}, \ A := A_{E} \cup A_{S} \cup \{(m, s_{E}), (m_{1}, s_{S})\}, \text{ and}$

$$\operatorname{cfg}_{i}[\![S]\!](m, t, s_{E}, m + 1) = \langle (N_{S}, A_{S}, s_{S}), M^{\mathsf{s}}, M^{\mathsf{g}}, m_{1} \rangle,$$

$$\operatorname{cfg}_{i}[\![E]\!](m_{1}, t, m_{1} + 1) = \langle (N_{E}, A_{E}, s_{E}), m_{2} \rangle.$$

Do-while statement:

$$cfg_{i} \llbracket \text{do } S \text{ while } (E) \rrbracket (t, t_{b}, t_{c}, m) := \langle (N, A, s_{S}), M^{s}, M^{g}, m_{2} + 1 \rangle$$
 (25)
where $N := N_{E} \cup N_{S} \cup \{m_{1}, m_{2}\}, A := A_{E} \cup A_{S} \cup \{(m_{1}, s_{E}), (m_{2}, s_{S})\},$ and
$$cfg_{i} \llbracket E \rrbracket (m_{2}, t, m) = \langle (N_{E}, A_{E}, s_{E}), m_{1} \rangle,$$

$$cfg_{i} \llbracket S \rrbracket (m_{1}, t, s_{E}, m_{1} + 1) = \langle (N_{S}, A_{S}, s_{S}), M^{s}, M^{g}, m_{2} \rangle.$$

For statement:

$$cfg_{i} \llbracket for (E_{1}; E_{2}; E_{3}) S \rrbracket (t, t_{b}, t_{c}, m) := cfg_{i} \llbracket E_{1}; \text{ while } (E_{2}) \{ S E_{3}; \} \rrbracket (t, t_{b}, t_{c}, m).$$
 (26)

Break statement: assuming $t_b \neq \bot$,

$$\operatorname{cfg}_{i} \llbracket \operatorname{break} \rrbracket (t, t_{\operatorname{b}}, t_{\operatorname{c}}, m) := \big\langle (\{m, t_{\operatorname{b}}\}, \big\{ (m, t_{\operatorname{b}}) \big\}, m), \varnothing, \varnothing, m + 1 \big\rangle. \tag{27}$$

Continue statement: assuming $t_c \neq \bot$,

$$\operatorname{cfg}_{i} \llbracket \operatorname{continue} \rrbracket (t, t_{b}, t_{c}, m) := \langle (\{m, t_{c}\}, \{(m, t_{c})\}, m), \varnothing, \varnothing, m + 1 \rangle. \tag{28}$$

Goto statement:

$$\operatorname{cfg}_{i} [\![\mathsf{goto} \operatorname{id}]\!](t, t_{\mathsf{b}}, t_{\mathsf{c}}, m) := \langle (\{m, t\}, \{(m, t)\}, m), \varnothing, \{(\operatorname{id}, m)\}, m + 1 \rangle.$$
(29)

Labeled statement:

$$\operatorname{cfg}_{i}[L:S](t,t_{b},t_{c},m) := \langle (N,A,m), M^{s}, M^{g}, m_{2} \rangle,$$
 (30)

where $N := N_L \cup N_S$, $A := A_L \cup A_S$, $M^s = M_S^s \cup M_L^s$, and

$$\begin{split} \operatorname{cfg}_i[\![S]\!](t,t_{\operatorname{b}},t_{\operatorname{c}},m) &= \left<(N_S,A_S,s_S),M_S^{\operatorname{s}},M^{\operatorname{g}},m_1\right>,\\ \operatorname{cfg}_i[\![L]\!](s_S,m_1) &= \left<(N_L,A_L,s_L),M_L^{\operatorname{s}},m_2\right>. \end{split}$$

Compound statement:

$$\operatorname{cfg}_{i}[[S]](t, t_{b}, t_{c}, m) := \operatorname{cfg}_{i}[[S]](t, t_{b}, t_{c}, m).$$
 (31)

Other statements:

$$\operatorname{cfg}_{i}[\![\{\operatorname{stm}\}]\!](t, t_{b}, t_{c}, m) := \langle (\{t\}, \emptyset, t), \emptyset, \emptyset, m \rangle. \tag{32}$$

Finally, let $B \in \text{Stm}$ be a full C function body: the CFG constructed for B with respect to optimization level i, denoted by $\operatorname{cfg}_i^b[\![B]\!]$, is given by

$$\operatorname{cfg}_{i}^{b}[\![B]\!] := (N, A, s), \tag{33}$$

where $\operatorname{cfg}_i[\![B]\!](0,\perp,\perp,1) = \langle (N,A_B,s),M^s,M^g,m_1 \rangle$ and A is obtained from A_S by adding the arcs corresponding to goto statements, namely:

$$A := A_B$$

$$\cup \{ (n_1, n_2) \mid \exists s, l . s = (\mathrm{id}_s, n_1) \in M^g \land l = (\mathrm{id}_l, n_2) \in M^s \land \mathrm{id}_s = \mathrm{id}_l \}.$$

In the sequel, we will refer to the overloaded function $\operatorname{cfg}_i^1[\![\cdot]\!] := \pi_1 \circ \operatorname{cfg}_i[\![\cdot]\!]$ where $\pi_1 := \lambda x_1, \dots x_n$. x_1 is the first projection of a variable number $n \geq 1$ of arguments. In other words, for a program phrase $P \in \operatorname{Exp} \cup \operatorname{Stm}$, $\operatorname{cfg}_i^1[\![P]\!](\dots)$ denotes the graph component computed by $\operatorname{cfg}_i[\![P]\!](\dots)$.

3 The NPATH Metric

The definition of the NPATH metric for the C language, extracted from [13] and adapted to the grammar given in Section 2.3, is given in Tables 1 and 2. Note that the syntax used in [13] imposes strong limits on the structure of

Table 1 Inductive definition of function $NP_{\rm E}$

E	$NP_{\rm E}(E)$				
x	0				
c					
$!E_1$					
$+E_1$					
$-E_1$	$\mathrm{NP_E}(E_1)$				
(E_1)					
$(type)E_1$					
$uop E_1$					
$E_1 \&\& E_2$	$NP_{E}(E_{1}) + NP_{E}(E_{2}) + 1$				
$E_1 \mid\mid E_2$	101 E(E1) + 101 E(E2) + 1				
E_1, E_2	$NP_{E}(E_1) + NP_{E}(E_2)$				
$E_1 :: E_2$	$\mathbb{R}^{(E_1)} + \mathbb{R}^{(E_2)}$				
$E_1 \text{ bop } E_2$	$NP_{E}(E_1) + NP_{E}(E_2)$				
$E_1 ? E_2 : E_3$	$NP_{E}(E_{1}) + NP_{E}(E_{2}) + NP_{E}(E_{3}) + 2$				

 ${\bf Table~2}~~{\rm Inductive~definition~of~function~NP_S}$

S	$NP_{\mathrm{S}}(S)$
E;	$NP_{\mathbf{E}}(E)$
$S_1 S_2$	$NP_{S}(S_1)NP_{S}(S_2)$
return	1
$\mathtt{return}\; E$	$max(1, NP_{\rm E}(E))$
if (E) S_1 else S_2	$NP_{E}(E) + NP_{S}(S_1) + NP_{S}(S_2)$
if (E) S_1	$NP_{E}(E) + NP_{S}(S_{1}) + 1$
$\mathtt{switch}\;(E)\;S_{B}$	$NP_{E}(E) + \sum_{i=1}^{k} NP_{S}(S_{i}) + NP_{S}(S_{d})$
while (E) S_1	$NP_{E}(E) + NP_{S}(S_{1}) + 1$
do S_1 while (E)	$\operatorname{F}(D) + \operatorname{F}(S(D)) + 1$
for $(E_1; E_2; E_3)$ S	$NP_{E}(E_{1}) + NP_{E}(E_{2}) + NP_{E}(E_{3}) + NP_{S}(S) + 1$
break	1
continue	1
goto id	1
L: S ₁	$\mathrm{NP}_{\mathrm{S}}(S_1)$
$\{S_1\}$	1VI S(S1)
stm	1

switch statements, hence the definition given in Table 2 is only valid if S_B has the form case $n_1:S_1$ case $n_2:S_2\cdots$ case $n_k:S_k$ default $:S_d.^2$

 $^{^{2}\,}$ The ACPATH metric that we will define in Section 4 has no such limitation.

The introduction of NPATH in [13] is motivated by a convincing argument about the advantages of counting the number of acyclic paths in order to estimate the path complexity of a function. One would assume that the definition of NPATH given in [13] would provide a way of counting the number of acyclic paths but, as we already seen in Example 1 on page 3, this is not the case.

One of the main problems of NPATH is that, as shown by Example 1, in the conditional (resp., loop) statements, the number of acyclic paths in the controlling expressions and in the construct's branches (resp., the body) compound in a multiplicative, not in an additive way. For the conditional, each acyclic path in one branch can be combined with each acyclic path in the controlling expression that directs control flow into that branch.

We now provide further examples where NPATH either underestimates or overestimates the number of acyclic paths in the CFG of a C function.

Example 2 Consider the C function

```
int f(int a, int b, int c, int d) {
  while(a || (b && c && d) ) {
    ... /* no branching statements here */
  }
}
```

We have NPATH = 3, but the possible acyclic paths are 6, corresponding to the following combinations, where the ellipsis separates the values of a, b, c and d before and after the first execution of the while body:

```
1. a ... !a && b && c && !d
2. a ... !a && b && !c
3. a ... !a && !b
4. !a && b && c && !d
5. !a && b && !c
6. !a && !b
```

The problem shown by this example is that NPATH does not consider the backward jump caused by while statement at the end of the execution of the body. In order to correctly compute the number of acyclic paths, in addition to the paths that do not execute the while body, we must consider the paths that first evaluate the guard to true, whereby the body is executed, and then evaluate to false.

Example 3 Consider the C function

```
int f(int a, int b, int c) {
   switch (a) {
    case 1: b ? 0 : 1;
    default: return c ? 0 : 1;
  }
}
```

We have NPATH = 2+2=4, but the possible acyclic paths are 6, corresponding to the following combinations:

```
1. a = 1 && b && c
2. a = 1 && b && !c
3. a = 1 && !b && c
4. a = 1 && !b && !c
5. a <> 1 && c
6. a <> 1 && !c
```

Here the problem is that NPATH does not correctly capture the syntax and semantics of the C switch statement. In the function above, if a is equal to 1, control passed to the case 1 branch and, after the execution of its range, since it does not contain any break statement, the default range is executed. In other words, NPATH does not account for so-called *fall-through* in C switch statements.

Example 4 Consider the following C functions:

```
void f(int a, int b, int c, int d, int e) {
   do {
     if (a)
        break/continue/return;
   if(b)
        ... /* no branching statements here */
   else
        ... /* no branching statements here */
} while (c)
}
```

We have NPATH = 4+1=5, but there are only 3 acyclic paths are instead only 3 paths for the break and the return cases, corresponding to the following:

```
1. a
2. !a, b, !c
3. !a, !b, !c
```

And there are only 2 acyclic paths for the continue case, corresponding to the following:

```
1. !a, b, !c
2. !a, !b, !C
```

In these examples NPATH overstates the number of acyclic paths because it does not distinguish return, continue and break statements from statements that do not affect control flow: in all three cases the while body execution is abandoned if a evaluates to true.

Example 5 Independently from the considerations illustrated by the the previous example, NPATH can overstate the number of acyclic paths for do-while loops. The simplest example is the idiomatic

which is commonly used as a macro body so that macro calls can be terminated with a semicolon without introducing a null statement, while embedding S into a compound statement. If S is a single basic block, we have NPATH ≥ 2 but there is only 1 acyclic path.

Given that NPATH does not count acyclic paths one might think: let us make the compiler build the CFG, and then let us count how many acyclic path it contains from the entry node s to any exit node. Unfortunately, this is unfeasible for general graphs.³

Theorem 1 Consider a directed graph G = (N, A) with entry node s and exit nodes in set T. Counting s - T acyclic paths in G is $\sharp P$ -complete.

Proof First, we can assume that G only has one exit node t. If it has more, then introduce a new node t and place a directed arc from each exit node to t; this does not change the number of paths to an exit node. Form a new directed graph G' = (N', A') with

$$\begin{split} N' &:= \big\{ \left. x_{\mathrm{in}}, x_{\mathrm{out}} \mid x \in N \right. \big\}, \\ A' &:= \big\{ \left. (x_{\mathrm{in}}, x_{\mathrm{out}}) \mid x \in N \right. \big\} \cup \big\{ \left. (x_{\mathrm{out}}, y_{\mathrm{in}}) \mid (x, y) \in A \right. \big\}. \end{split}$$

A path from s to t in G that repeats no nodes corresponds to a path in G' from s_{in} to t_{out} that repeats no arcs. Indeed, every path in G' alternates arcs of the form $(x_{\text{out}}, y_{\text{in}})$ with arcs of the form $(y_{\text{in}}, y_{\text{out}})$. Using node y at most once in G is the same as using arc $(y_{\text{in}}, y_{\text{out}})$ at most once in G'.

This reduces the problem of counting paths with no repeated nodes to that of counting ones with no repeated arcs, and the former is #P-complete [15]. Then the latter is also #P-complete.

Note that the $\sharp P$ -complete problems are at least as difficult as the NP-complete problems. Indeed, the existence of a polynomial-time algorithm for solving a $\sharp P$ -complete problem would imply P = NP.

4 The ACPATH Metric

In this section we present *ACPATH*. This is a new metric for C-like languages that, in contrast to the NPATH metric, corresponds to the exact number of acyclic paths through any function with no backjumps.⁴ Note that, as most coding standards disallow such goto statements, in practice, backjumps are rarely used in critical code. For instance, MISRA C, the most influential C coding standard, in its 2012 edition [11] has an *advisory* rule forbidding all goto statements and a *required* rule forbidding backjumps; so, while a forward goto can be used without justification, a backjump requires a formal deviation.

³ We are grateful to Charles Colbourn for indicating the following reduction to us.

 $^{^4}$ A backjump is a goto statement that jumps to a labeled statement that precedes it.

In Section 4.1, we present algorithms that count the acyclic paths through expressions. In Section 4.2, we deal with the more complex task of counting paths through statements. All the algorithms presented in Sections 4.1 and 4.2, are parametric with respect to an optimization level. As formally stated at the end of the section, all the algorithms are correct for each optimization level.

4.1 Execution Paths Through Expressions

To deal with expressions, we introduce three functions: \mathbf{t}_i , \mathbf{f}_i and \mathbf{p}_i . For each optimization level $i \in \{0, 1, 2\}$ and each $E \in \text{Exp}$ that is evaluated at optimization level i:

- $-\mathbf{t}_i(E)$ counts the number of execution paths through E that may evaluate to true:
- $\mathbf{f}_i(E)$ counts the number of execution paths through E that may evaluate to false;
- $-\mathbf{p}_i(E)$ counts the total number of possible execution paths through E.

It is important to stress that here we are dealing with path counting with respect to a reference CFG and without any semantic inference apart from those encoded in the optimization level. Hence, when we say that a path through E may evaluate to true, we mean that a path exists in the reference CFG for the considered optimization level, and that the same optimization level does not allow concluding that the path evaluates to false.

Definition 4 (\mathbf{t}_i , \mathbf{f}_i , \mathbf{p}_i .) The functions \mathbf{t}_i : Exp $\to \mathbb{N}$, \mathbf{f}_i : Exp $\to \mathbb{N}$ and \mathbf{p}_i : Exp $\to \mathbb{N}$ are inductively defined, for each $i \in \{0, 1, 2\}$ and $E \in \text{Exp}$, as per Table 3.

In order to deal with acyclic paths induced by while loops, we also need functions \mathbf{tt}_i , \mathbf{tf}_i , \mathbf{ff}_i and \mathbf{pp}_i ; for each $i \in \{0, 1, 2\}$ and each $E \in \text{Exp}$:

- $\mathbf{tt}_i(E)$ (resp., $\mathbf{ff}_i(E)$) counts the number of ways in which the expression E can be traversed twice at optimization level i, where both evaluation paths may lead to true (resp., false) and they do not share any arc;
- $\mathbf{tf}_i(E)$ counts the number of ways in which the expression E can be traversed twice at optimization level i, where the two evaluations may lead to different Boolean values (i.e., one to true and the other to false), and the two traversals do not share any arc;
- $-\mathbf{pp}_{i}(E)$ counts the total number of possible ways in which the expression E can be traversed twice at optimization level i, where the two traversals do not share any arc.

Definition 5 (tt_i, tf_i, ff_i, pp_i.) The functions tt_i: Exp $\to \mathbb{N}$, tf_i: Exp $\to \mathbb{N}$, and pp_i: Exp $\to \mathbb{N}$ are inductively defined, for each $i \in \{0, 1, 2\}$ and $E \in \text{Exp}$, as per Tables 4–7.

Table 3 Inductive definition of $\mathbf{t}_i \colon \mathrm{Exp} \to \mathbb{N}$, $\mathbf{f}_i \colon \mathrm{Exp} \to \mathbb{N}$ and $\mathbf{p}_i \colon \mathrm{Exp} \to \mathbb{N}$

$\mathbf{p}_i(E)$	1	П	П	1	$\mathbf{p}_i(E_1)$		B. (E.)	$\mathbf{p}_i(E_1)$		$\mathbf{p}_i(E_1)$	$\mathbf{f}_i(E_1) + \mathbf{t}_i(E_1)\mathbf{p}_i(E_2)$	$\mathbf{t}_i(E_1) + \mathbf{f}_i(E_1)\mathbf{p}_i(E_2)$	$\mathbf{p}_i(E_1)\mathbf{p}_i(E_2)$	$\mathbf{t}_i(E_1) + \mathbf{f}_i(E_1) \mathbf{p}_i(E_2)$	$\mathbf{p}_i(E_1)\mathbf{p}_i(E_2)$	$\mathbf{t}_i(E_1)\mathbf{p}_i(E_2)+\mathbf{f}_i(E_1)\mathbf{p}_i(E_3)$
$\mathbf{f}_i(E)$	1		0	1	$\mathbf{t}_i(E_1)$		f . (F.)	$\mathbf{r}_i(E_1)$		$\mathbf{p}_i(E_1)$	$\mathbf{f}_i(E_1) + \mathbf{t}_i(E_1)\mathbf{f}_i(E_2)$	$\mathbf{f}_i(E_1)\mathbf{f}_i(E_2)$	$\mathbf{p}_i(E_1)\mathbf{f}_i(E_2)$	$\mathbf{f}_i(E_1)\mathbf{f}_i(E_2)$	$\mathbf{p}_i(E_1)\mathbf{p}_i(E_2)$	$\mathbf{t}_i(E_1)\mathbf{f}_i(E_2)+\mathbf{f}_i(E_1)\mathbf{f}_i(E_3)$
$\mathbf{t}_i(E)$	1	1	-	0	$\mathbf{f}_i(E_1)$		+ (E,)	$\mathbf{c}_i(E_1)$		$\mathbf{p}_i(E_1)$	$\mathbf{t}_i(E_1)\mathbf{t}_i(E_2)$	$\mathbf{t}_i(E_1) + \mathbf{f}_i(E_1)\mathbf{t}_i(E_2)$	$\mathbf{p}_i(E_1)\mathbf{t}_i(E_2)$	$\mathbf{t}_i(E_1) + \mathbf{f}_i(E_1)\mathbf{t}_i(E_2)$	$\mathbf{p}_i(E_1)\mathbf{p}_i(E_2)$	$\mathbf{t}_i(E_1)\mathbf{t}_i(E_2) + \mathbf{f}_i(E_1)\mathbf{t}_i(E_3)$
E	x	$c \text{ if } \operatorname{tv}_i(c) = ?$	$c ext{ if } ext{tv}_i(c) = 1$	$c ext{ if } ext{tv}_i(c) = 0$	$:E_1$	$+E_1$	$-E_1$	(E_1)	$(type)E_1$	$\operatorname{lop} E_1$	$E_1 \&\& E_2$	$E_1 \parallel E_2$	E_1, E_2	E_1 ?: E_2	$E_1 \text{ bop } E_2$	$E_1 \mathrel{?} E_2 : E_3$

Table 4 Inductive definition of $\mathbf{tt}_i \colon \text{Exp} \to \mathbb{N}$

E	\mathbf{tt}_i				
\overline{x}	0				
$c ext{ if } ext{tv}_i(c) = ?$	0				
$c \text{ if } \operatorname{tv}_i(c) = 1$	1				
$c \text{ if } \operatorname{tv}_i(c) = 0$	0				
$!E_1$	$\mathbf{ff}_i(E_1)$				
$+E_1$					
$-E_1$	$\mathbf{tt}_i(E_1)$				
(E_1)	$\mathfrak{bb}_i(E_1)$				
$(type)E_1$					
$uop E_1$	0				
$E_1 \&\& E_2$	$\mathbf{tt}_i(E_1)\mathbf{tt}_i(E_2)$				
$E_1 \mid\mid E_2$	$\mathbf{tt}_i(E_1) + 2\mathbf{tf}_i(E_1)\mathbf{t}_i(E_2) + \mathbf{ff}_i(E_1)\mathbf{tt}_i(E_2)$				
E_1, E_2	$\mathbf{pp}_i(E_1)\mathbf{tt}_i(E_2)$				
$E_1 ?: E_2$	$\mathbf{tt}_i(E_1) + 2\mathbf{tf}_i(E_1)\mathbf{t}_i(E_2) + \mathbf{ff}_i(E_1)\mathbf{tt}_i(E_2)$				
$E_1 \text{ bop } E_2$	0				
$E_1 ? E_2 : E_3$	$\mathbf{tt}_i(E_1)\mathbf{tt}_i(E_2) + 2\mathbf{tf}_i(E_1)\mathbf{t}_i(E_2)\mathbf{t}_i(E_3) + \mathbf{ff}_i(E_1)\mathbf{tt}_i(E_3)$				

Table 5 Inductive definition of $\mathbf{tf}_i \colon \mathrm{Exp} \to \mathbb{N}$

E	$\mathbf{tf}_i(E)$
\overline{x}	1
$c ext{ if } ext{tv}_i(c) = ?$	1
$c \text{ if } \operatorname{tv}_i(c) = 1$	0
$c \text{ if } \operatorname{tv}_i(c) = 0$	0
$!E_1$	$\mathbf{tf}_i(E_1)$
$+E_1$	
$-E_1$	$\mathbf{tf}_i(E_1)$
$\overline{}(E_1)$	$\mathbf{G}_{i}(E_{1})$
$(type)E_1$	
$uop E_1$	$\mathbf{pp}_i(E_1)$
$E_1 \&\& E_2$	$\mathbf{tf}_i(E_1)\mathbf{t}_i(E_2) + \mathbf{tt}_i(E_1)\mathbf{tf}_i(E_2)$
$E_1 \parallel E_2$	$\mathbf{tf}_i(E_1)\mathbf{f}_i(E_2) + \mathbf{ff}_i(E_1)\mathbf{tf}_i(E_2)$
E_1, E_2	$\mathbf{pp}_i(E_1)\mathbf{tf}_i(E_2)$
$E_1 : E_2$	$\mathbf{tf}_i(E_1)\mathbf{f}_i(E_2) + \mathbf{ff}_i(E_1)\mathbf{tf}_i(E_2)$
$E_1 \text{ bop } E_2$	$\mathbf{pp}_i(E_1)\mathbf{pp}_i(E_2)$
$E_1 ? E_2 : E_3$	$\mathbf{tt}_i(E_1)\mathbf{tf}_i(E_2) + \mathbf{ff}_i(E_1)\mathbf{tf}_i(E_3)$
$E_1 : E_2 : E_3$	$+\mathbf{tf}_i(E_1)\big(\mathbf{t}_i(E_2)\mathbf{f}_i(E_3)+\mathbf{f}_i(E_2)\mathbf{t}_i(E_3)\big)$

4.2 Execution Paths Through Statements

We first consider a labeled statement L:S: the paths that reach S are those that "fall to L from above" plus those that "switch to L" if L is a case or default label, or "go to L" if L is an identifier label in a goto statement. The (possibly decorated) symbols ft, st and gt will be used as mnemonics for the number of paths that fall through, switch to and go to L, respectively. In the sequel, if Id is the set of identifier labels in a function, gt will be a partial function gt: Id $\rightarrow \mathbb{N}$, mapping any label identifier id \in Id to the cumulative number of paths that reach all the goto id statements in the function that occur before the labeled statement, id: S.

Table 6 Inductive definition of function $\mathbf{ff}_i \colon \mathrm{Exp} \to \mathbb{N}$

E	$ $ $\mathbf{ff}_i(E)$				
\overline{x}	0				
$c ext{ if } ext{tv}_i(c) = ?$	0				
$c \text{ if } \operatorname{tv}_i(c) = 1$	0				
$c \text{ if } \operatorname{tv}_i(c) = 0$	1				
$!E_1$	$\mathbf{tt}_i(E_1)$				
$+E_1$					
$-E_1$	$\mathbf{ff}_i(E_1)$				
$\overline{(E_1)}$	$\mathbf{n}_i(E_1)$				
$(type)E_1$					
$uop E_1$	0				
$E_1 \&\& E_2$	$\mathbf{ff}_i(E_1) + 2 \mathbf{tf}_i(E_1) \mathbf{f}_i(E_2) + \mathbf{tt}_i(E_1) \mathbf{ff}_i(E_2)$				
$E_1 \mid\mid E_2$	$\mathbf{ff}_i(E_1)\mathbf{ff}_i(E_2)$				
E_1, E_2	$\mathbf{pp}_i(E_1)\mathbf{ff}_i(E_2)$				
$E_1 : E_2$	$\mathbf{ff}_i(E_1)\mathbf{ff}_i(E_2)$				
$E_1 \text{ bop } E_2$	0				
$E_1 ? E_2 : E_3$	$\mathbf{tt}_i(E_1)\mathbf{ff}_i(E_2) + 2\mathbf{tf}_i(E_1)\mathbf{f}_i(E_2)\mathbf{f}_i(E_3) + \mathbf{ff}_i(E_1)\mathbf{ff}_i(E_3)$				

Table 7 Inductive definition of function $\mathbf{pp}_i \colon \mathrm{Exp} \to \mathbb{N}$

E	$\mathbf{pp}_i(E)$
\overline{x}	0
$c ext{ if } ext{tv}_i(c) = ?$	0
$c \text{ if } \operatorname{tv}_i(c) = 1$	1
$c \text{ if } \operatorname{tv}_i(c) = 0$	1
$!E_1$	$\mathbf{pp}_i(E_1)$
$+E_1$	
$-E_1$	$\mathbf{pp}_i(E_1)$
$\overline{(E_1)}$	$pp_i(E_1)$
$(type)E_1$	
$uop E_1$	0
$E_1 \&\& E_2$	$\mathbf{ff}_i(E_1) + 2\mathbf{tf}_i(E_1)\mathbf{p}_i(E_2) + \mathbf{tt}_i(E_1)\mathbf{pp}_i(E_2)$
$E_1 \mid\mid E_2$	$\mathbf{tt}_i(E_1) + 2\mathbf{tf}_i(E_1)\mathbf{p}_i(E_2) + \mathbf{ff}_i(E_1)\mathbf{pp}_i(E_2)$
E_1, E_2	$\mathbf{pp}_i(E_1)\mathbf{pp}_i(E_2)$
$E_1 ?: E_2$	$\mathbf{tt}_i(E_1) + 2\mathbf{tf}_i(E_1)\mathbf{p}_i(E_2) + \mathbf{ff}_i(E_1)\mathbf{pp}_i(E_2)$
$E_1 \text{ bop } E_2$	0
$E_1 ? E_2 : E_3$	$\mathbf{tt}_i(E_1)\mathbf{pp}_i(E_2) + 2\mathbf{tf}_i(E_1)\mathbf{p}_i(E_2)\mathbf{p}_i(E_3) + \mathbf{ff}_i(E_1)\mathbf{pp}_i(E_3)$

Definition 6 (apc_i[]: Lab × N × N × (Id \mapsto N) \to N.) Let L be the label for a labeled statement, and ft, st and gt be as defined above. Then apc_i[L](ft, st, gt) is defined as follows:

Case label:

$$\mathrm{apc}_i[\![\mathtt{case}\ n]\!](\mathsf{ft},\mathsf{st},\mathsf{gt}) := \mathsf{ft} + \mathsf{st}. \tag{34}$$

Default label:

$$\operatorname{apc}_i[\![\operatorname{default}]\!](\operatorname{ft},\operatorname{st},\operatorname{gt}) := \operatorname{ft} + \operatorname{st}.$$
 (35)

Identifier label:

$$\mathrm{apc}_{i}[\![\mathrm{id}]\!](\mathsf{ft},\mathsf{st},\mathsf{gt}) := \mathsf{ft} + \mathsf{gt}(\mathrm{id}). \tag{36}$$

We assume that a function terminates with an empty statement ϵ so that each non-empty statement S in a function has a successor S_1 . In order to count the total number of paths that reach S_1 , we introduce the (overloaded) function $\operatorname{apc}_i[\![I]\!]$. If ft, st, gt are described as above, then $\operatorname{apc}_i[\![S]\!]$ (ft, st, gt) computes:

- ft_{out} : the number of acyclic paths that that "fall through S_1 from above";
- bp: the cumulative sum of the acyclic paths that lead to break nodes that terminate the execution of S, i.e., break nodes that are not in switch or loop statements in S;
- cp: the cumulative sum of the acyclic paths that lead to continue nodes that terminate the execution of S, i.e., continue nodes that are not in loop statements in S;
- rp: the cumulative sum of the acyclic paths that lead to return nodes in
 S:
- gt_{out} : a partial function gt : Id $\rightarrowtail \mathbb{N}$, mapping any label identifier id to the cumulative number of paths that reach all the goto id statements that occur before S_1 .

Let $\mathsf{gt}, \mathsf{gt}_1, \mathsf{gt}_2 \colon \mathrm{Id} \rightarrowtail \mathbb{N}$. The function $\mathsf{gt}[n/\mathrm{id}] \colon \mathrm{Id} \rightarrowtail \mathbb{N}$ is given, for each $x \in \mathrm{Id}$, by

$$gt[n/id](x) := \begin{cases} n, & \text{if } x = id, \\ gt(x), & \text{otherwise;} \end{cases}$$

in addition, $(\mathsf{gt}_1 + \mathsf{gt}_2) \colon \mathrm{Id} \rightarrowtail \mathbb{N}$ is given, for each $x \in \mathrm{Id}$, by

$$(\mathsf{gt}_1 + \mathsf{gt}_2)(x) := \mathsf{gt}_1(x) + \mathsf{gt}_2(x).$$

Definition 7 (apc_i[]: Stm $\times \mathbb{N} \times \mathbb{N} \times (\mathrm{Id} \rightarrow \mathbb{N}) \to \mathbb{N}$.) We define the function

$$\operatorname{apc}_{i}[]: \operatorname{Stm} \times \mathbb{N} \times \wp(\operatorname{Id} \times \mathbb{N}) \to \mathbb{N}^{4} \times \wp(\operatorname{Id} \times \mathbb{N})$$

as follows:

Expression statement:

$$\operatorname{apc}_{i}[E;](\mathsf{ft},\mathsf{st},\mathsf{gt}) := (\mathbf{p}_{i}(E)\mathsf{ft},0,0,0,\mathsf{gt}).$$
 (37)

Sequential composition:

$$\operatorname{apc}_{i}[S_{1} S_{2}](\mathsf{ft}, \mathsf{st}, \mathsf{gt}) := (\mathsf{ft}_{2}, \mathsf{bp}, \mathsf{cp}, \mathsf{rp}, \mathsf{gt}_{2}),$$
 (38)

where $bp = bp_1 + bp_2$, $cp = cp_1 + cp_2$, $rp = rp_1 + rp_2$,

$$\begin{aligned} &\text{apc}_i[\![S_1]\!](\mathsf{ft},\mathsf{st},\mathsf{gt}) = (\mathsf{ft}_1,\mathsf{bp}_1,\mathsf{cp}_1,\mathsf{rp}_1,\mathsf{gt}_1), \\ &\text{apc}_i[\![S_2]\!](\mathsf{ft}_1,\mathsf{st},\mathsf{gt}_1) = (\mathsf{ft}_2,\mathsf{bp}_2,\mathsf{cp}_2,\mathsf{rp}_2,\mathsf{gt}_2). \end{aligned}$$

Return statement:

$$apc_i[return](ft, st, gt) := (0, 0, 0, ft, gt).$$
 (39)

Return with expression statement:

$$\operatorname{apc}_{i}[\![\mathsf{return}\ E]\!](\mathsf{ft},\mathsf{st},\mathsf{gt}) := (0,0,0,\mathbf{p}_{i}(E)\mathsf{ft},\mathsf{gt}). \tag{40}$$

Conditional statement:

$$\operatorname{apc}_{i}\llbracket\operatorname{if}(E) S_{1} \text{ else } S_{2}\rrbracket(\operatorname{ft},\operatorname{\mathsf{st}},\operatorname{\mathsf{gt}}) := (\operatorname{\mathsf{ft}}_{out},\operatorname{\mathsf{bp}},\operatorname{\mathsf{cp}},\operatorname{\mathsf{rp}},\operatorname{\mathsf{gt}}_{2}), \tag{41}$$

where we have $ft_{out} = ft_1 + ft_2$, $bp = bp_1 + bp_2$, $cp = cp_1 + cp_2$, $rp = rp_1 + rp_2$,

$$\begin{split} &\operatorname{apc}_i[\![S_1]\!](\mathbf{t}_i(E)\mathsf{ft},\mathsf{st},\mathsf{gt}) = (\mathsf{ft}_1,\mathsf{bp}_1,\mathsf{cp}_1,\mathsf{rp}_1,\mathsf{gt}_1), \\ &\operatorname{apc}_i[\![S_2]\!](\mathbf{f}_i(E)\mathsf{ft},\mathsf{st},\mathsf{gt}_1) = (\mathsf{ft}_2,\mathsf{bp}_2,\mathsf{cp}_2,\mathsf{rp}_2,\mathsf{gt}_2). \end{split}$$

One-armed conditional statement:

$$\operatorname{apc}_i \llbracket \operatorname{if} (E) \ S_1 \rrbracket (\operatorname{ft}, \operatorname{st}, \operatorname{\mathsf{gt}}) := (\operatorname{\mathsf{ft}}_{out}, \operatorname{\mathsf{bp}}_1, \operatorname{\mathsf{cp}}_1, \operatorname{\mathsf{rp}}_1, \operatorname{\mathsf{gt}}_1), \tag{42}$$

where we have

$$\mathsf{ft}_{out} = \mathsf{ft}_1 + \mathbf{f}_i(E), \\ \mathrm{apc}_i \llbracket S_1 \rrbracket (\mathbf{t}_i(E)\mathsf{ft},\mathsf{st},\mathsf{gt}) = (\mathsf{ft}_1,\mathsf{bp}_1,\mathsf{cp}_1,\mathsf{rp}_1,\mathsf{gt}_1).$$

Switch statement:

$$\mathrm{apc}_i \llbracket \mathsf{switch} \ (E) \ S \rrbracket (\mathsf{ft}, \mathsf{st}, \mathsf{gt}) := \begin{cases} (\mathsf{ft}_1, 0, \mathsf{cp}_S, \mathsf{rp}_S, \mathsf{gt}_S), & \text{if } d(S), \\ (\mathsf{ft}_2, 0, \mathsf{cp}_S, \mathsf{rp}_S, \mathsf{gt}_S), & \text{otherwise}, \end{cases} \tag{43}$$

where we have

$$\begin{split} \mathsf{ft}_1 &= \mathsf{ft}_S + \mathsf{bp}_S, \\ \mathsf{ft}_2 &= \mathsf{ft}_S + \mathsf{bp}_S + \mathbf{p}_i(E) \mathsf{ft}, \\ \mathsf{apc}_i \llbracket S \rrbracket (0, \mathbf{p}_i(E) \mathsf{ft}, \mathsf{gt}) &= (\mathsf{ft}_S, \mathsf{bp}_S, \mathsf{cp}_S, \mathsf{rp}_S, \mathsf{gt}_S), \end{split}$$

and d(S) is true if and only if S contains a default label out of all inner switch.

While statement:

$$\operatorname{apc}_{i} \llbracket \mathsf{while} (E) \ S \rrbracket (\mathsf{ft}, \mathsf{st}, \mathsf{gt}) := (\mathsf{ft}_{\mathsf{out}}, 0, 0, \mathsf{rp}_{S}, \mathsf{gt}_{S}) \tag{44}$$

where we have

$$\begin{split} \mathsf{ft}_{\mathsf{out}} &= \mathbf{f}_i(E)\mathsf{ft} + \mathsf{bp}_S \, \mathbf{t}_i(E) + \left(\mathsf{ft}_S + \mathsf{cp}_S\right) \mathbf{t} \mathbf{f}_i(E), \\ \mathsf{apc}_i \llbracket S \rrbracket (\mathsf{ft}, \mathsf{st}, \mathsf{gt}) &= (\mathsf{ft}_S, \mathsf{bp}_S, \mathsf{cp}_S, \mathsf{rp}_S, \mathsf{gt}_S). \end{split}$$

Do-while statement:

$$\mathrm{apc}_i \llbracket \mathsf{do} \ S \ \mathsf{while} \ (E) \rrbracket (\mathsf{ft}, \mathsf{st}, \mathsf{gt}) := (\mathsf{ft}_{out}, 0, 0, \mathsf{rp}_S, \mathsf{gt}_S), \tag{45}$$

where we have

$$\mathsf{ft}_{out} = \mathbf{f}_i(E)\mathsf{ft}_S + \mathsf{bp}_S,$$
$$\mathsf{apc}_i[S](\mathsf{ft},\mathsf{st},\mathsf{gt}) = (\mathsf{ft}_S,\mathsf{bp}_S,\mathsf{cp}_S,\mathsf{rp}_S,\mathsf{gt}_S).$$

For statement:

$$\operatorname{apc}_{i} \llbracket \operatorname{for} (E_{1}; E_{2}; E_{3}) \ S \rrbracket (\operatorname{ft}, \operatorname{st}, \operatorname{gt}) := \operatorname{apc}_{i} \llbracket E_{1} \ \operatorname{while} (E_{2}) \ \{ S \ E_{3}; \} \rrbracket (\operatorname{ft}, \operatorname{st}, \operatorname{gt}). \tag{46}$$

Break statement:

$$apc_i[[break]](ft, st, gt) := (0, ft, 0, 0, gt).$$
 (47)

Continue statement:

$$apc_i[[continue]](ft, st, gt) := (0, 0, ft, 0, gt).$$
 (48)

Goto statement:

$$\operatorname{apc}_{i}[[\operatorname{\mathsf{goto}}\operatorname{id}](\mathsf{ft},\mathsf{st},\mathsf{gt}) := (0,0,0,0,\mathsf{gt}[(\operatorname{\mathsf{gt}}(\operatorname{id}) + \operatorname{\mathsf{ft}})/\operatorname{id}]). \tag{49}$$

Labeled statement:

$$\mathrm{apc}_i[\![L:S]\!](\mathsf{ft},\mathsf{st},\mathsf{gt}) := (\mathsf{ft}_S,\mathsf{bp},\mathsf{cp},\mathsf{rp},\mathsf{gt}_{out}), \tag{50}$$

where we have

$$\begin{split} &\operatorname{apc}_i \llbracket L \rrbracket (\mathsf{ft},\mathsf{st},\mathsf{gt}) = \mathsf{ft}_L, \\ &\operatorname{apc}_i \llbracket S \rrbracket (\mathsf{ft}_L,\mathsf{st},\mathsf{gt}) = (\mathsf{ft}_S,\mathsf{bp},\mathsf{cp},\mathsf{rp},\mathsf{gt}_{out}). \end{split}$$

Compound statement:

$$\operatorname{apc}_{i}[\![\{S\}]\!](\mathsf{ft},\mathsf{st},\mathsf{gt}) := \operatorname{apc}_{i}[\![S]\!](\mathsf{ft},\mathsf{st},\mathsf{gt}). \tag{51}$$

Other statements:

$$\operatorname{apc}_{i}[\operatorname{stm}](\operatorname{ft},\operatorname{st},\operatorname{gt}) := (\operatorname{ft},0,0,0,\operatorname{gt}).$$
 (52)

It is clear from its definition that, for each function body $B \in \text{Stm}$, $\operatorname{apc}_i^b[\![B]\!]$ can be computed with a single traversal of B. It is also easy to prove the following:

Proposition 1 Let G = (N, A, s) be a directed graph with entry node s and exit nodes in set T. Then there exists a C function body $B \in \text{Stm of size } O(A)$ such that, for each $i \in \{0, 1, 2\}$, $\alpha(\text{cfg}_i^b[B]) = \alpha(G)$.

Proof Assume $s = n_1$ and $T = \{t_1, \ldots, t_h\}$. Let $\{\{n_1, \ldots, n_k\}, \{t_1, \ldots, t_h\}\}$ be a partition of N, and let $I = \{\mathrm{id}_{n_1}, \ldots, \mathrm{id}_{n_k}, \mathrm{id}_{t_1}, \ldots, \mathrm{id}_{t_h}\}$ be a set of C identifiers in one-to-one correspondence with N. Then define

$$B := \mathrm{id}_{n_1} : S_{n_1}; \dots; \mathrm{id}_{n_k} : S_{n_k}; \mathrm{id}_{t_1} : S_{t_1}; \dots; \mathrm{id}_{t_k} : S_{t_k}$$

where:

```
- for each i=1,\,\ldots,\,k, S_{n_i}:= \mathtt{switch}\;(x_{n_i})\;S'_{n_i}
```

where x_{n_i} is a variable and, if $\{(n_i, n_{i,1}), \dots, (n_i, n_{i,p_i})\}$ is the subset of A containing all arcs leaving n_i , then

```
S'_{n_i}:={\tt case}\ 1: {\tt goto}\ {\rm id}_{n_{i,1}};\ \cdots\ ;\ {\tt case}\ p_i: {\tt default}: {\tt goto}\ {\rm id}_{n_{i,p_i}}; — for each j=1,\,\ldots,\,h,\,S_{t_j}={\tt return}.
```

Checking that $\operatorname{cfg}_i^b[\![B]\!]$ has the same number of acyclic paths as G is straightforward.

Theorem 1, together with Proposition 1 and the fact that $\operatorname{apc}_i^b[\![B]\!]$ computes the number of acyclic paths in $\operatorname{cfg}_i^b[\![B]\!]$ in time linearly proportional to the size of B, implies that $\operatorname{apc}_i^b[\![B]\!]$ cannot be exact for all function bodies $B \in \operatorname{Stm}$. However, it is correct for a very large class of function bodies, informally characterized as follows (a formal definition is given in Appendix A).

Definition 8 (Controlled function body (informal).) Let $B \in \text{Stm}$ be a full C function body. We call B a controlled function body if it satisfies the following properties:

- it does not contain any backjump;
- if a loop in B can terminate its execution by means of goto, break or return statements, then no goto or switch statement in B will jump or switch into the loop from outside.

The following examples show minimal function bodies that are not controlled.

Example 6 Here we jump into the loop from outside via goto and the loop exits via break:

```
void f(int x) {
   goto l1;
   while (x) {
      break;
   l1: ... /* no branching statements here */
   }
}
```

The value for ACPATH is 1, but there are 2 acyclic paths: the path that jumps into the loop and then evaluates the guard to false, and the path that jumps into the loop, then evaluates the guard to true and exits via the break statement.

Here we jump into the loop from outside via switch and the loop exits via return:

```
void g(int x, int y) {
   switch (x) {
      do {
      return;
   case 0: ... /* no branching statements here */
      } while (y)
   }
}
```

The value of ACPATH is 2, but there are 3 acyclic paths: the path that switches to the end of the function, the path that switches in the loop and evaluates the loop guard false, and the path that switches into the loop, then evaluates the loop guard to true, then exits via return statement.

Definition 9 (apc_i^b[]: Stm $\to \mathbb{N}$.) Let $B \in \text{Stm}$ be a full C function body, and $l(B) \in \wp(\text{Id})$ the set of labels in B; then the number of acyclic paths through B with respect to optimization level i, denoted by apc_i^b[B], is given by

$$\operatorname{apc}_{i}^{b} \llbracket B \rrbracket := \operatorname{ft}_{out} + \operatorname{rp}, \tag{53}$$

where $\mathsf{gt} = \{ (\mathrm{id}, 0) \mid \mathrm{id} \in l(B) \}$ and $\mathrm{apc}_i \llbracket B \rrbracket (1, 0, \mathsf{gt}) = (\mathsf{ft}_{out}, \mathsf{bp}, \mathsf{cp}, \mathsf{rp}, \mathsf{gt}_{out}).$

Theorem 2 Let $B \in \text{Stm}$ be a controlled function body. Then

$$\operatorname{apc}_{i}^{\operatorname{b}} \llbracket B \rrbracket = \alpha (\operatorname{cfg}_{i}^{\operatorname{b}} \llbracket B \rrbracket).$$

The proof of Theorem 2 is in Appendix A.

5 Implementation and Experimental Evaluation

This section reports on a study of the relationship between the metric introduced in this paper, ACPATH, and NPATH [13]. As we have already seen, they are not equivalent from the theoretical point of view: we will now show that they are not equivalent also from the point of view of their practical application.

5.1 Implementation

The ACPATH and NPATH metrics (and many others) have been implemented in ECLAIR, a powerful platform for the automatic analysis, verification, testing and transformation of C, C++ and Java source code, as well as Java bytecode.⁵ In particular, for assessing the complexity of software, ECLAIR provides comprehensive code metrics that can be accumulated over a single function, translation unit, program or even the whole project. The ACPATH algorithm

⁵ http://bugseng.com/products/eclair

has been implemented in ECLAIR as a metric over a complete function body, the optimization level $i \in \{0,1,2\}$ being a parameter of the analysis. Although the ACPATH metric is only fully specified and verified here for C code, the implementation is designed to handle both C and C++ user code, i.e., fully instantiated preprocessed code. The same holds true for the implementation of NPATH. Apart from the generalization to C++, the implementation of both metrics closely follows the definitions given in this paper. The implementation language is a very high-level logical description language that is automatically translated to executable code. The implementation of NPATH is around 300 lines long whereas 550 lines are sufficient to implement ACPATH.

5.2 Sampled Functions

The experimental evaluation was conducted on 61 C projects, for a total of 35284 functions: the majority of such projects involve safety- or mission-critical functionality, mainly from the automotive sector, with projects from other domains (aerospace, railway and medical appliances), some operating system kernels and some, non-critical open-source projects.

The condition about the absence of backjumps is largely satisfied: only 19 C functions (0.05%) have one or more backjumps. We are currently instrumenting ECLAIR in order to count the number of functions that do not satisfy the other conditions of Definition 8. However, we expect the number of functions/methods to which Theorem 2 does not apply to be very small, if not negligible.

5.3 Statistical Analysis

Many studies on software metrics place reliance upon Pearson linear correlation coefficient r, which, however, assumes the variables are approximately normally distributed [14]. This is definitely not our case, as sample skewness of our data is around 200. A common methodology for skew reduction is transformation: in our case sample skew drops to around 50 after taking the logarithm of the metric values. A further drop to around 1 is obtained by adding 1 and taking the logarithm again, that is, by transforming the data with the function $\log(1+\log(x))$: Figure 1 shows the scatter plot after the transformations. The figure shows that there is good, but not absolute correlation, with Pearson correlation coefficient $r \simeq 0.98$. The average error is $\mu \simeq 0$, i.e., NPATH overestimations approximately balance the underestimations, but the standard deviation of the error, $\sigma \simeq 0.12$, confirms, taking into account the $\log(\log(\cdot))$ transformation, that estimation errors can be quite large. To give an idea what this means, let us report this statistics back from the $\log(1 + \log(x))$ to the x scale and let us focus on values of ACPATH in the range [1, 26]. In Figure 2 the red, blue and green lines represent μ , $\mu - \sigma$ and $\mu + \sigma$, respectively. In Figure 3 we report, for each value of ACPATH in the range [1, 26], the computed

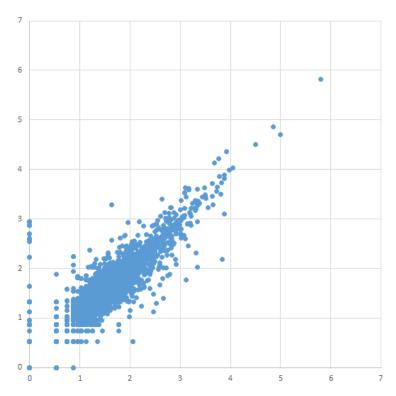


Fig. 1 Scatter plot: $\log(1 + \log(ACPATH))$, x axis, vs $\log(1 + \log(NPATH))$, y axis

values for NPATH, where we discarded the tails, 16% on each side, so that each bar represents 68% of the samples (this matches the range $[\mu - \sigma, \mu + \sigma]$ of Figure 2). It can be seen that, while the error committed by NPATH is low for functions with number of acyclic paths below 10, the error can become rather large even for slightly more complex functions, and their distribution is quite faithfully described by the values of μ and σ obtained as described above.

5.4 Analysis with Respect to Commonly Used Thresholds

The statistical analysis of the previous section does not take into account the fact that organizations that enforce the adoption of sound software engineering principles place strong limitations on the maximum number of acyclic paths that a function may have. Other elements for the comparison can thus be given with reference to the recommended thresholds, under the assumption that such thresholds were actually meant to apply to the number of acyclic paths and not the measure NPATH, that is neither a lower bound nor an upper bound for them even for trivial programs.

In [13] a threshold value of 200 is recommended based on studies conducted at AT&T Bell Laboratories. The HIS⁶ Software Test Working Group, in the document defining the *HIS metrics* [9], which are widespread in the automotive and other industrial sectors, is stricter and recommends not to exceed 80. In our experiment, we found that the ACPATH threshold of 80 is respected by 95% of the C functions, whereas the threshold of 200 is respected by 97% of the C functions. The number of functions that would be miscategorized with respect to violating or complying with the thresholds is rather small:

ACPATH > 80	$NPATH \le 80$	#f = 195
≤ 80	> 80	= 281
> 200	≤ 200	= 152
< 200	> 200	= 182

which also shows that, for "borderline" functions, NPATH errs on the "non-compliant" side more often than it errs on the "compliant" side. The differences can be very large though, here are the worst cases we have found in our experimentation:

ACPATH = 1,	NPATH = 67108864,
ACPATH = 597781,	NPATH = 21,
ACPATH = 2329612972,	NPATH = 130.

The first such worst-case result concerns a function that contains 26 macro invocations that expand to as many do S while (0) statements, where S is a basic block; so while ACPATH gives 1, i.e., the correct number of acyclic paths, NPATH results in $2^{26} = 67108864$.

6 Conclusion

Path complexity is a program complexity measure that can be used to assess the testability of units, that is, functions or methods, depending on the programming language. It is recognized that this is a much better estimator of testability than cyclomatic complexity [7,10], which was found to perform no better that LSLOCs (Logical Source Lines of Code) in this regard [13,14].

As the number of paths in a unit can be unbounded, in [13] it was proposed to use the number of acyclic paths as a proxy. In the same paper, the NPATH metric for C programs was proposed with a claim that it would count the number of acyclic paths for programs without goto statements [13, page 192]:

⁶ HIS, Hersteller initiative Software, is an interest group set up by Audi, BMW, Daimler, Porsche, and Volkswagen in order to join forces on methods of software design and quality assurance for microprocessor based control units.

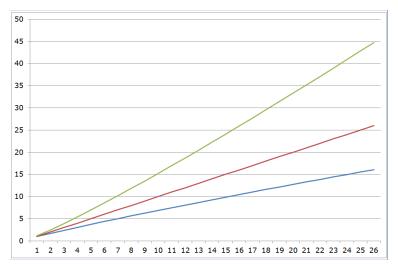
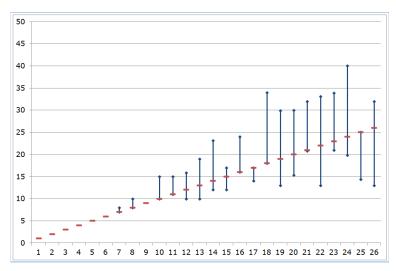


Fig. 2 ACPATH value, x axis, vs NPATH predicted value distribution, y axis



 ${\bf Fig.~3}~$ ACPATH value, x axis, vs NPATH, y axis: each bar contains the central 68% of the samples

NPATH is a measure that is more closely related to the number of acyclic execution paths through a function. In particular, the NPATH measure differs from the actual number of acyclic execution paths by the number of acyclic execution paths resulting from goto statements.

In reality, the syntax for the switch statement is significantly more restricted than what is actually permitted by the definition of the C programming language so, while not acknowledged explicitly in [13, page 192], this is another restriction of the NPATH metric.

The starting point for this paper was the experimental discovery that, in fact, the number of acyclic execution paths through a function can differ from NPATH enormously. Indeed, the difference can be seen even for very simple examples and these small variations can compound in a multiplicative way to very large differences.

We then asked whether or not the number of acyclic paths could be computed exactly by working directly on the control flow graph of the function rather than on the syntax of its body. Unfortunately, we discovered that the problem of counting the acyclic paths in a graph is $\sharp P$ -complete, which leaves little hope for an efficient algorithm. If one wants simply to know whether the number of acyclic paths is below or above a certain (small) threshold, then a possibility is to enumerate all the acyclic paths exhaustively or until the threshold is attained. For a CFG G = (N, A, s) this can be done in time $O(|A| \cdot \Delta(G))$ per enumerated acyclic path, where $\Delta(G)$ is the maximum degree of G, using Johnson's algorithm [8] on the line graph of G. This is probably still not efficient enough and, moreover, such an approach does not allow to differentiate between functions that violate the threshold.

In this paper, we defined a new metric, called ACPATH, for C-like languages: even though we defined it formally only for the C programming language, we have extended it to C++. The metric can be computed very efficiently with a single traversal of the abstract syntax tree of the function. Moreover, we have proved that ACPATH does correspond to the number of acyclic paths under two conditions: (1) absence of backjumps; (2) absence of jumps into a loop whenever there are jumps out of the same loop, terminating the loop in ways unrelated to the evaluation of its guard. We proved that, if condition (1) is removed, then the existence of an efficient algorithm to compute the number of acyclic paths would imply P = NP. The study of what can be done if condition (2) is removed or weakened is one direction for future work.

We thus proposed ACPATH as a natural successor and replacement for NPATH: the former solves most of the problems of the latter while retaining the existence of a very efficient algorithm to compute it, with only a minor increase in the complexity of the definition. Moreover, ACPATH has been proved correct on counting the number of acyclic paths for most programs that are

 $^{^{7}}$ The *degree* of a node is the number of arcs incident to the node. In the CFG of a C function, the maximum degree is essentially given by the maximum number of cases in switch statements.

⁸ The line graph of a directed graph G = (N, A) is the directed graph L(G) whose node set is A and arcs the set of adjacent arcs in A.

written in practice, and the cases where ACPATH is not exact are easy to detect. Exactness of ACPATH on many functions allowed us to conduct experiments, using the ECLAIR software verification platform, on the adequacy of NPATH, both from a statistical and from a more pragmatic point of view.

A different approach to the estimation of execution path complexity has been recently proposed in [1], called asymptotic path complexity. Using techniques from algebraic graph theory and linear algebra, the authors show how to obtain a closed form upper bound to path(n)—the number of (possibly cyclic) paths in the unit of length at most n— of the form n, n^2 , n^3 and so on, or b^n for some exponential base b. The bound is only valid to the limit, i.e., as n goes to infinity.

For future work, we plan to formalize the extension to C++ and to other languages, Java source code and bytecode to start with. A C++ extension is already implemented in ECLAIR but its correctness has not yet been formally proved. The only language feature that requires special care in the generalization to C++ is structured exception handling. We believe that an extension for Java can then easily be derived from the one to C++: additionally, as Java has no goto statements and the syntax of switch is more restrictive than in C++ so that it is not possible to jump into loops, we conjecture ACPATH for Java will always be exact. The generalization to Java bytecode is more problematic, as it depends on the ability to reconstruct loops, which is nontrivial especially if obfuscation techniques have been used.

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A Technical Proofs

We now show that, for each optimization level i and each expression E, the following hold:

- $\mathbf{t}_i(E)$ (resp., $\mathbf{f}_i(E)$) is the number of acyclic paths in $\operatorname{cfg}_i^1[\![E]\!](t,f,m)$ leading to t (resp., f):
- $-\mathbf{p}_{i}(E)$ is the total number of acyclic paths in $\operatorname{cfg}_{i}^{1}[E](t, f, m)$.

Note that in all the proofs, a "path to s" means a path leading to s., "pair of paths" always means a pair of acyclic paths that do not share an arc

Lemma 1 Let $i \in \{0,1,2\}$ be an optimization level, $E \in \text{Exp}$ be an expression, and let $m,t,f \in \mathbb{N}$ be such that $m > \max\{t,f\}$. Then:

$$\mathbf{t}_{i}(E) = \tau \left(\operatorname{cfg}_{i}^{1} \llbracket E \rrbracket(t, f, m), t\right), \tag{54}$$

$$\mathbf{f}_i(E) = \tau \left(\operatorname{cfg}_i^1 \llbracket E \rrbracket(t, f, m), f \right), \tag{55}$$

$$\mathbf{p}_i(E) = \alpha \left(\operatorname{cfg}_i^1 \llbracket E \rrbracket (t, t, m) \right). \tag{56}$$

Proof The proof is by structural induction on E.

Variables: By Definition 4, $\mathbf{t}_i(E) = \mathbf{f}_i(E) = \mathbf{p}_i(E) = 1$. By (5), letting $s \in \{t, f\}$, we have

$$\begin{split} \tau \Big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(t,f,m),s \Big) &= \tau \Big(\big(\{m,t,f\}, \{(m,t),(m,f)\},m \big),s \Big) \\ &= \tau \Big(m, \{(m,t),(m,f)\},s \big) \\ &= \tau \Big(s, \{(m,t),(m,f)\} \setminus \{(m,s)\},s \big) = 1, \\ \alpha \Big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(t,t,m) \Big) &= \alpha \Big(\big(\{m,t\}, \{(m,t)\},m \big) \Big) \\ &= \tau \Big(m, \{(m,t),(m,f)\},t \big) = 1. \end{split}$$

Constants: There are three cases:

 $\operatorname{tv}_i(E) = 1$: by Definition 4, $\mathbf{t}_i(E) = 1$, $\mathbf{f}_i(E) = 0$ and $\mathbf{p}_i(E) = 1$; hence, by (6),

$$\begin{split} &\delta\left(\operatorname{cfg}_{i}^{1}\llbracket E\rrbracket(t,f,m),t\right) = \delta\left(\left(\{t\},\{\},t\right),t\right) = \delta\left(t,\{\},t\right) = 1,\\ &\delta\left(\operatorname{cfg}_{i}^{1}\llbracket E\rrbracket(t,f,m),f\right) = \delta\left(\left(\{t\},\{\},t\right),f\right) = \delta\left(t,\{\},f\right) = 0,\\ &\alpha\left(\operatorname{cfg}_{i}^{1}\llbracket E\rrbracket(t,t,m)\right) = \alpha\left(\left(\{t\},\{\},t\right)\right) = 1. \end{split}$$

 $tv_i(E) = 0$: the proof is similar to the previous case;

 $tv_i(E) = ?$: the proof is similar to the case of variables.

Logical negation: By Definition 4, $\mathbf{t}_i(!E) = \mathbf{f}_i(E)$ and $\mathbf{p}_i(!E) = \mathbf{p}_i(E)$ and, by (7), $\operatorname{cfg}_i^1[\![E]\!](t,f,m) = \operatorname{cfg}_i^1[\![E]\!](f,t,m)$. Hence, by the inductive hypothesis,

$$\begin{split} \mathbf{f}_i(!\,E_1) &= \mathbf{t}_i(E_1) = \tau \left(\mathrm{cfg}_i^1 [\![E_1]\!] (f,t,m), f \right) = \tau \left(\mathrm{cfg}_i^1 [\![!E_1]\!] (t,f,m), f \right), \\ \mathbf{t}_i(!\,E_1) &= \mathbf{f}_i(E_1) = \tau \left(\mathrm{cfg}_i^1 [\![E_1]\!] (f,t,m), t \right) = \tau \left(\mathrm{cfg}_i^1 [\![!E]\!] (t,f,m), t \right), \\ \mathbf{p}_i(!\,E_1) &= \mathbf{p}_i(E_1) = \alpha \left(\mathrm{cfg}_i^1 [\![E_1]\!] (t,t,m) \right) = \alpha \left(\mathrm{cfg}_i^1 [\![!E_1]\!] (t,t,m) \right). \end{split}$$

Unary plus, unary minus, parenthesis and cast operators: By Definition 4,

$$\mathbf{t}_i(\operatorname{uop} E_1) = \mathbf{t}_i(E_1), \quad \mathbf{f}_i(\operatorname{uop} E_1) = \mathbf{f}_i(E_1), \quad \mathbf{p}_i(\operatorname{uop} E_1) = \mathbf{p}_i(E_1)$$

and, by (8),

$$\operatorname{cfg}_{i}^{1} \llbracket \operatorname{uop} E_{1} \rrbracket (t, f, m) = \operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (t, f, m).$$

Hence the proof follows by induction.

Other unary operators: By Definition 4,

$$\mathbf{t}_i(\operatorname{uop} E_1) = \mathbf{f}_i(\operatorname{uop} E_1) = \mathbf{p}_i(\operatorname{uop} E_1) = \mathbf{p}_i(E_1).$$

Moreover, for each of the paths to m in $\operatorname{cfg}_i^1[\![E_1]\!](m,m,m+1)$, there is one path to t passing through the arc (m,t) and one path to f passing through (m,f) in $\operatorname{cfg}_i^1[\![\operatorname{uop} E_1]\!](t,f,m)$, and one path to an exit node in $\operatorname{cfg}_i^1[\![E_1]\!](t,t,m)$. Hence by (9),

$$\begin{split} &\alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket(t,t,m)\right) = \tau \left(\operatorname{cfg}_{i}^{1} \llbracket \operatorname{uop} E_{1} \rrbracket(t,f,m),t\right), \\ &\alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket(t,t,m)\right) = \tau \left(\operatorname{cfg}_{i}^{1} \llbracket \operatorname{uop} E_{1} \rrbracket(t,f,m),f\right), \\ &\alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket(t,t,m)\right) = \alpha \left(\operatorname{cfg}_{i}^{1} \llbracket \operatorname{uop} E_{1} \rrbracket(t,t,m)\right). \end{split}$$

Hence the proof follows by induction.

Logical conjunction: By Definition 4,

$$\mathbf{t}_{i}(E_{1} \&\& E_{2}) = \mathbf{t}_{i}(E_{1}) \,\mathbf{t}_{i}(E_{2}),$$

$$\mathbf{f}_{i}(E_{1} \&\& E_{2}) = \mathbf{f}_{i}(E_{1}) + \mathbf{t}_{i}(E_{1}) \,\mathbf{f}_{i}(E_{2}),$$

$$\mathbf{p}_{i}(E_{1} \&\& E_{2}) = \mathbf{f}_{i}(E_{1}) + \mathbf{t}_{i}(E_{1}) \,\mathbf{p}_{i}(E_{2}).$$
(57)

There are three cases:

 $\operatorname{tv}_i(E_1) = 0$: then by (6), $\mathbf{t}_i(E_1) = 0$ and $\mathbf{f}_i(E_1) = 1$; also by (10), $\operatorname{cfg}_1^1 \llbracket E_1 \&\& E_2 \rrbracket (t, f, m) = (\{f\}, \varnothing, f)$, so there are no paths to t and one path to f. Moreover, in control-flow graph $\operatorname{cfg}_1^1 \llbracket E_1 \&\& E_2 \rrbracket (t, t, m) = (\{t\}, \varnothing, t)$, there is one path to an exit node. Hence, by (57),

$$\tau\left(\operatorname{cfg}_{i}^{1}\llbracket E_{1} \&\& E_{2} \rrbracket(t, f, m), t\right) = 0 = \mathbf{t}_{i}(E_{1} \&\& E_{2}),$$

$$\tau\left(\operatorname{cfg}_{i}^{1}\llbracket E_{1} \&\& E_{2} \rrbracket(t, f, m), f\right) = 1 = \mathbf{f}_{i}(E_{1} \&\& E_{2}),$$

$$\alpha\left(\operatorname{cfg}_{i}^{1}\llbracket E_{1} \&\& E_{2} \rrbracket(t, t, m)\right) = 1 = \mathbf{p}_{i}(E_{1} \&\& E_{2}).$$

 $\text{tv}_i(E_1) = 1$: then, by (6), $\mathbf{t}_i(E_1) = 1$ and $\mathbf{f}_i(E_1) = 0$; also, by (10), $\text{cfg}_i^1 \llbracket E_1 \&\& E_2 \rrbracket(t, f, m) = \text{cfg}_i^1 \llbracket E_2 \rrbracket(t, f, m)$. Hence, by (57) and the inductive hypothesis:

$$\begin{split} \mathbf{t}_{i}(E_{1} \&\& E_{2}) &= \mathbf{t}_{i}(E_{2}) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(t,f,m),t \big) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \&\& E_{2} \rrbracket(t,f,m),t \big), \\ \mathbf{f}_{i}(E_{1} \&\& E_{2}) &= \mathbf{f}_{i}(E_{2}) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(t,f,m),f \big) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \&\& E_{2} \rrbracket(t,f,m),f \big), \\ \mathbf{p}_{i}(E_{1} \&\& E_{2}) &= \mathbf{p}_{i}(E_{2}) \\ &= \alpha \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(t,t,m) \big) \\ &= \alpha \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \&\& E_{2} \rrbracket(t,t,m) \big). \end{split}$$

 $\begin{aligned} \operatorname{tv}_i(E_1) &= ? \colon \text{using the notation in } (10), \text{ since the exit node for true evaluation for graph} \\ \operatorname{cfg}_i^1[\![E_1]\!](s_2,f,m_1) & \text{ is the entry node for graph } \operatorname{cfg}_i^1[\![E_2]\!](t,f,m), \text{ for each path to } s_2 & \text{ in } \\ \operatorname{cfg}_i^1[\![E_1]\!](s_2,f,m_1), \text{ there are } \tau\big(\operatorname{cfg}_i^1[\![E_2]\!](t,f,m),t\big) & \text{ paths to } t \text{ and } \tau\big(\operatorname{cfg}_i^1[\![E_2]\!](t,f,m),f\big) \\ \text{ paths to } f & \text{ in } \operatorname{cfg}_i^1[\![E_1]\!](s_2,f,m), \text{ as well as } \alpha\big(\operatorname{cfg}_i^1[\![E_2]\!](t,t,m)\big) & \text{ paths to an exit } \\ \text{ node in } \operatorname{cfg}_i^1[\![E_1]\!](s_2,f,m_1) & \text{ has } \end{aligned}$

 $\tau(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2},f,m),f)$ paths to f. Hence, by (57), (10) and the inductive hypothesis:

$$\begin{split} \mathbf{t}_{i}(E_{1} \&\& E_{2}) &= \mathbf{t}_{i}(E_{1}) \, \mathbf{t}_{i}(E_{2}) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), s_{2} \big) \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), t \big) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \&\& E_{2} \rrbracket (t, f, m), t \big), \\ \mathbf{f}_{i}(E_{1} \&\& E_{2}) &= \mathbf{f}_{i}(E_{1}) + \mathbf{t}_{i}(E_{1}) \, \mathbf{f}_{i}(E_{2}) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), f \big) \\ &+ \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), s_{2} \big) \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), f \big) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \&\& E_{2} \rrbracket (t, f, m), f \big), \\ \mathbf{p}_{i}(E_{1} \&\& E_{2}) &= \mathbf{f}_{i}(E_{1}) + \mathbf{t}_{i}(E_{1}) \, \mathbf{p}_{i}(E_{2}) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), f \big) \\ &+ \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), s_{2} \big) \alpha \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, t, m) \big) \\ &= \alpha \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \&\& E_{2} \rrbracket (t, t, m) \big). \end{split}$$

Logical disjunction: Dual to the case of logical conjunction.

Comma operator: Using the notation in (12), as $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_2,m_1)$ has both the exit nodes equal to the entry node of $\operatorname{cfg}_i^1[\![E_2]\!](t,f,m)$, for each path leading to s_2 in $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_2,m_1)$ there are $\tau(\operatorname{cfg}_i^1[\![E_2]\!](t,f,m),t)$ paths to t and $\tau(\operatorname{cfg}_i^1[\![E_2]\!](t,f,m),f)$ paths to f in $\operatorname{cfg}_i^1[\![E_1,E_2]\!](t,f,m)$, as well as $\alpha(\operatorname{cfg}_i^1[\![E_2]\!](t,t,m))$ paths to an exit node in $\operatorname{cfg}_i^1[\![E_1,E_2]\!](t,t,m)$. Hence, by Definition 4, (12) and the inductive hypothesis:

$$\begin{split} \mathbf{t}_{i}(E_{1},E_{2}) &= \mathbf{p}_{i}(E_{1})\,\mathbf{t}_{i}(E_{2}) \\ &= \alpha \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket(s_{2},s_{2},m_{1}) \big) \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(t,f,m),t \big) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1},E_{2} \rrbracket(t,f,m),t \big), \\ \mathbf{f}_{i}(E_{1},E_{2}) &= \mathbf{p}_{i}(E_{1})\,\mathbf{f}_{i}(E_{2}) \\ &= \alpha \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket(s_{2},s_{2},m_{1}) \big) \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(t,f,m),f \big) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1},E_{2} \rrbracket(t,f,m),f \big), \\ \mathbf{p}_{i}(E_{1},E_{2}) &= \mathbf{p}_{i}(E_{1})\,\mathbf{p}_{i}(E_{2}) \\ &= \alpha \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket(s_{2},s_{2},m_{1}) \big) \alpha \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(t,t,m) \big) \\ &= \alpha \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1},E_{2} \rrbracket(t,t,m) \big). \end{split}$$

Binary conditional operator: By Definition 4, the functions \mathbf{t}_i , \mathbf{f}_i and \mathbf{p}_i are defined the same for logical disjunction and the binary conditional operator. Moreover, the definitions for logical disjunction in (11) and the binary conditional operator in (13) are the same. Hence the proof of this case is identical to the case of logical disjunction.

Other binary operators: Using the notation in (14), as graph $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_2,m_1)$ has both exit nodes equal to the entry node of graph $\operatorname{cfg}_i^1[\![E_2]\!](m,m,m+1)$, for each path to s_2 in $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_2,m_1)$ there are $\alpha(\operatorname{cfg}_i^1[\![E_2]\!](m,m,m+1))$ paths to m in control-flow graph $\operatorname{cfg}_i^1[\![E_1]\!](t,f,m)$. Moreover, since $(m,t),(m,f)\in A$, each path to m can be extended to paths to t and t in $\operatorname{cfg}_i^1[\![E_1]\!](t,f,m)$ and an exit node in $\operatorname{cfg}_i^1[\![E_1]\!](t,t,m)$. Hence, by Definition 4, (14) and the inductive hypothesis:

$$\begin{aligned} \mathbf{t}_{i}(E_{1} \operatorname{bop} E_{2}) &= \mathbf{p}_{i}(E_{1}) \, \mathbf{p}_{i}(E_{2}) \\ &= \alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{2}, m_{1})\right) \alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (m, m, m+1)\right) \\ &= \tau \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \operatorname{bop} E_{2} \rrbracket (t, f, m), t\right), \\ \mathbf{f}_{i}(E_{1} \operatorname{bop} E_{2}) &= \mathbf{p}_{i}(E_{1}) \, \mathbf{p}_{i}(E_{2}) \\ &= \alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{2}, m_{1})\right) \alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (m, m, m+1)\right) \\ &= \tau \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \operatorname{bop} E_{2} \rrbracket (t, f, m), f\right), \end{aligned}$$

$$\begin{aligned} \mathbf{p}_i(E_1 \text{ bop } E_2) &= \mathbf{p}_i(E_1) \, \mathbf{p}_i(E_2) \\ &= \alpha \left(\operatorname{cfg}_i^1 \llbracket E_1 \rrbracket (s_2, s_2, m_1) \right) \alpha \left(\operatorname{cfg}_i^1 \llbracket E_2 \rrbracket (m, m, m+1) \right) \\ &= \alpha \left(\operatorname{cfg}_i^1 \llbracket E_1 \text{ bop } E_2 \rrbracket (t, t, m) \right). \end{aligned}$$

Conditional operator: By Definition 4,

$$\mathbf{t}_{i}(E_{1} ? E_{2} : E_{3}) = \mathbf{t}_{i}(E_{1}) \mathbf{t}_{i}(E_{2}) + \mathbf{f}_{i}(E_{1}) \mathbf{t}_{i}(E_{3})$$

$$\mathbf{f}_{i}(E_{1} ? E_{2} : E_{3}) = \mathbf{f}_{i}(E_{1}) \mathbf{f}_{i}(E_{2}) + \mathbf{f}_{i}(E_{1}) \mathbf{f}_{i}(E_{3})$$

$$\mathbf{p}_{i}(E_{1} ? E_{2} : E_{3}) = \mathbf{t}_{i}(E_{1}) \mathbf{p}_{i}(E_{2}) + \mathbf{f}_{i}(E_{1}) \mathbf{p}_{i}(E_{3}).$$
(58)

There are three cases:

 $tv_i(E_1) = 1$: by (15),

$$\operatorname{cfg}_{i}^{1} [E_{1} ? E_{2} : E_{3}](t, f, m) = \operatorname{cfg}_{i}^{1} [E_{2}](t, f, m),$$

and by (6), $\mathbf{t}_i(E_1) = 1$, and $\mathbf{f}_i(E_1) = 0$. Hence by (58) and the inductive hypothesis,

$$\begin{split} \mathbf{t}_{i}(E_{1} ? E_{2} : E_{3}) &= \mathbf{t}_{i}(E_{2}) \\ &= \tau (\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), t) \\ &= \tau (\operatorname{cfg}_{i}^{1} \llbracket E_{1} ? E_{2} : E_{3} \rrbracket (t, f, m), t), \\ \mathbf{f}_{i}(E_{1} ? E_{2} : E_{3}) &= \mathbf{f}_{i}(E_{2}) \\ &= \tau (\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), f) \\ &= \tau (\operatorname{cfg}_{i}^{1} \llbracket E_{1} \mid\mid E_{2} \rrbracket (t, f, m), f), \\ \mathbf{p}_{i}(E_{1} ? E_{2} : E_{3}) &= \mathbf{p}_{i}(E_{2}) \\ &= \alpha (\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, t, m)) \\ &= \alpha (\operatorname{cfg}_{i}^{1} \llbracket E_{1} ? E_{2} : E_{3} \rrbracket (t, t, m)). \end{split}$$

 $tv_i(E_1) = 0$: symmetric to the previous case.

 $\operatorname{tv}_i(E_1) = ?$: Using the notation in (15), as $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_3,m_2)$ has the exit node for true evaluation equal to the entry node of $\operatorname{cfg}_i^1[\![E_2]\!](t,f,m)$, for each path leading to s_2 in $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_3,m_2)$ there are $\tau(\operatorname{cfg}_i^1[\![E_2]\!](t,f,m),t)$ paths to t and $\tau(\operatorname{cfg}_i^1[\![E_2]\!](t,f,m),f)$ paths to f in $\operatorname{cfg}_i^1[\![E_1]\!](t,f,m)$, and $\alpha(\operatorname{cfg}_i^1[\![E_2]\!](t,t,m))$ paths to an exit node in $\operatorname{cfg}_i^1[\![E_1]\!](t,f,m)$.

Similarly, as the exit node for false evaluation in graph $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_3,m_2)$ is the entry node of $\operatorname{cfg}_i^1[\![E_3]\!](t,f,m_1)$, for each path to s_3 in $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_3,m_2)$, there are $\tau(\operatorname{cfg}_i^1[\![E_3]\!](t,f,m),t)$ paths to t and $\tau(\operatorname{cfg}_i^1[\![E_3]\!](t,f,m),t)$ paths to f in control-flow graph $\operatorname{cfg}_i^1[\![E_1]\!](E_2)\!](E_3)\!](t,f,m)$; furthermore, there are $\alpha(\operatorname{cfg}_i^1[\![E_3]\!](t,t,m))$ paths to an exit node in $\operatorname{cfg}_i^1[\![E_1]\!](E_2)\!](E_3)\!](E$

$$\begin{split} \mathbf{t}_{i}(E_{1} ? E_{2} : E_{3}) &= \mathbf{t}_{i}(E_{1}) \, \mathbf{t}_{i}(E_{2}) + \mathbf{f}_{i}(E_{1}) \, \mathbf{t}_{i}(E_{3}) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{3}, m_{2}), s_{2} \big) \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), t \big) \\ &+ \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{3}, m_{2}), s_{3} \big) \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{3} \rrbracket (t, f, m), t \big) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{3}, m_{2}), s_{3} \big) \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{3} \rrbracket (t, f, m), t \big) \\ \mathbf{f}_{i}(E_{1} ? E_{2} : E_{3}) &= \mathbf{t}_{i}(E_{1}) \, \mathbf{f}_{i}(E_{2}) + \mathbf{f}_{i}(E_{1}) \, \mathbf{f}_{i}(E_{3}) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{3}, m_{2}), s_{2} \big) \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), f \big) \\ &+ \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{3}, m_{2}), s_{3} \big) \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{3} \rrbracket (t, f, m), f \big) \\ &= \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} ? E_{2} : E_{3} \rrbracket (t, f, m), f \big), \\ \mathbf{p}_{i}(E_{1} ? E_{2} : E_{3}) &= \mathbf{t}_{i}(E_{1}) \, \mathbf{p}_{i}(E_{2}) + \mathbf{f}_{i}(E_{1}) \, \mathbf{p}_{i}(E_{3}) \end{split}$$

$$\begin{split} &=\tau \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{3}, m_{2}), s_{2}\right) \alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, t, m)\right) \\ &+\tau \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{3}, m_{2}), s_{3}\right) \alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{3} \rrbracket (t, t, m)\right) \\ &=\alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} ? E_{2} : E_{3} \rrbracket (t, t, m)\right).\Box \end{split}$$

In order to capture the number of acyclic paths in CFGs containing *while* loops we will need another path-counting function. *While* loops are special in that the subgraphs corresponding to their controlling expressions can be traversed twice: once on a path leading to execution of the body and one leading to exit from the loop; the two traversals should of course not share any arc.

Definition 10 (δ : CFG $\times \mathbb{N}^2 \to \mathbb{N}$.) Let G = (N, A, s) be a CFG and $s_1, s_2 \in N$. Then the number of pairs of acyclic paths in G to s_2 passing by s_1 , denoted by $\delta(G, s_1, s_2)$, is given by:

$$\delta(G, s_1, s_2) := \delta(s, s, A, s_1, s_2),\tag{59}$$

$$\delta(s, n, A, s_1, s_2) := \begin{cases} \tau(s, A, s_2), & \text{if } n = s_1; \\ \sum_{(n, m) \in A} \delta(s, m, A \setminus \{(n, m)\}, s_1, s_2), & \text{otherwise.} \end{cases}$$
(60)

We now show, using Definitions 5 and 10, that for each optimization level i and each expression E, the following hold:

- $\mathbf{tt}_i(E)$ is the number of pairs of acyclic paths in G to true passing by true;
- $-\mathbf{tf}_i(E)$ is the number of pairs of acyclic paths in G to false passing by true;
- $-\mathbf{ff}_i(E)$ is the number of pairs of acyclic paths in G to false passing by false;
- $-\mathbf{pp}_{i}(E)$ is the number of pairs of acyclic paths in G that do not share an arc.

Note that in the following proofs, a "pair of paths" will always means a pair of acyclic paths that do not share an arc.

Lemma 2 Let $i \in \{0,1,2\}$ be an optimization level, $E \in \text{Exp}$ be an expression, and let $m,t,f \in \mathbb{N}$ be such that $m > \max\{t,f\}$. Then:

$$\mathbf{tt}_i(E) = \delta(\operatorname{cfg}_i^1[E](t, f, m), t, t), \tag{61}$$

$$\mathbf{tf}_i(E) = \delta(\mathrm{cfg}_i^1 \llbracket E \rrbracket(t, f, m), t, f) = \delta(\mathrm{cfg}_i^1 \llbracket E \rrbracket(t, f, m), f, t), \tag{62}$$

$$\mathbf{ff}_i(E) = \delta(\mathrm{cfg}_i^1 \llbracket E \rrbracket(t, f, m), f, f), \tag{63}$$

$$\mathbf{pp}_{i}(E) = \delta\left(\mathrm{cfg}_{i}^{1}[E](t, t, m), t, t\right). \tag{64}$$

Proof The proof is by structural induction on E.

Variables: By Definition 5, $\mathbf{tt}_i(E)=0$, $\mathbf{tf}_i(E)=1$, $\mathbf{ff}_i(E)=0$, $\mathbf{pp}_i(E)=0$. By (5) and Definition 10, we have

$$\begin{split} \delta \left(\text{cfg}_i^1 \llbracket E \rrbracket(t,f,m), s_1, s_2 \right) &= \delta \Big(\big(\{m,t,f\}, \{(m,t),(m,f)\}, m \big), s_1, s_2 \big) \\ &= \delta \big(m, m, \{(m,t),(m,f)\}, s_1, s_2 \big) \\ &= \delta \big(m, s_1, \{(m,t),(m,f)\} \setminus \{(m,s_1)\}, s_1, s_2 \big) \\ &= \begin{cases} \delta \big(m, t, \{(m,f)\}, t, s_2 \big) = \tau \big(m, \{(m,f)\}, s_2 \big), & \text{if } s_1 = t, \\ \delta \big(m, f, \{(m,t)\}, f, s_2 \big) = \tau \big(m, \{(m,t)\}, s_2 \big), & \text{if } s_1 = f, \end{cases} \end{split}$$

which evaluates to 0 if $s_1 = s_2$ and 1 otherwise. Similarly,

$$\begin{split} \delta \Big(\mathrm{cfg}_i^1 \big[\![E]\!](t,t,m),t,t \Big) &= \delta \Big(\big(\{m,t\}, \{(m,t)\}, m \big),t,t \Big) \\ &= \delta \Big(m, \{(m,t)\},t,t \Big) \\ &= \tau (m,\varnothing,t) = 0. \end{split}$$

Constants: There are three cases:

 $\operatorname{tv}_i(E) = 1$: By Definition 5, $\operatorname{tt}_i(E) = 1$, $\operatorname{tf}_i(E) = 0$, $\operatorname{ff}_i(E) = 0$, $\operatorname{pp}_i(E) = 1$. By (6), $\operatorname{cfg}_i^1 ||E||(t, f, m) = (\{t\}, \emptyset, t)$ so that

$$\delta\left(\operatorname{cfg}_{i}^{1}[E](t, f, m), s_{1}, s_{2}\right) = \delta\left(\left(\{t\}, \emptyset, t\right), s_{1}, s_{2}\right)$$
$$= \tau(t, \emptyset, s_{2}),$$

which evaluates to 1 if $s_2 = t$ and 0 otherwise. Similarly,

$$\delta(\operatorname{cfg}_{i}^{1}[E](t,t,m),t,t) = \delta((\{t\},\varnothing,t),t,t)$$
$$= \tau(t,\varnothing,t) = 1.$$

 $tv_i(E) = 0$: the proof is similar to the previous case;

 $tv_i(E) = ?$: the proof is similar to the case $tv_i(E) = ?$ for variables.

Logical negation: By (7), $\operatorname{cfg}_{i}^{1}[\![! E]\!](t,f,m) = \operatorname{cfg}_{i}^{1}[\![E]\!](f,t,m)$; hence, by Definition 5 and the inductive hypothesis,

$$\begin{split} \mathbf{ff}_i(!\,E) &= \mathbf{tt}_i(E) \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(f,t,m),f,f \big) \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(f,t,m),f,f \big), \\ \mathbf{tf}_i(!\,E) &= \mathbf{tf}_i(E) \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(f,t,m),t,f \big) \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(f,t,m),t,f \big), \\ \mathbf{tf}_i(!\,E) &= \mathbf{tf}_i(E) \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(f,t,m),f,t \big), \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(f,t,m),f,t \big), \\ \mathbf{tt}_i(!\,E) &= \mathbf{ff}_i(E) \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(f,t,m),t,t \big), \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(f,t,m),t,t \big), \\ \mathbf{pp}_i(!\,E) &= \mathbf{pp}_i(E) \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(t,t,m),t,t \big), \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E \rrbracket(t,t,m),t,t \big), \end{split}$$

Unary plus, unary minus, parentheses and cast operators: By (8), we have $\operatorname{cfg}_i^1[\![E]\!](t,f,m) = \operatorname{cfg}_i^1[\![E_1]\!](t,f,m)$; hence, by Definition 5 and the inductive hypothesis,

$$\begin{aligned} \mathbf{t}\mathbf{t}_{i}(E) &= \mathbf{t}\mathbf{t}_{i}(E_{1}) &= \delta \left(\operatorname{cfg}_{i}^{1}\llbracket E_{1} \rrbracket(t,f,m),t,t\right) &= \delta \left(\operatorname{cfg}_{i}^{1}\llbracket E \rrbracket(t,f,m),t,t\right), \\ \mathbf{t}\mathbf{f}_{i}(E) &= \mathbf{t}\mathbf{f}_{i}(E_{1}) &= \delta \left(\operatorname{cfg}_{i}^{1}\llbracket E_{1} \rrbracket(t,f,m),t,f\right) &= \delta \left(\operatorname{cfg}_{i}^{1}\llbracket E \rrbracket(t,f,m),t,f\right), \\ \mathbf{t}\mathbf{f}_{i}(E) &= \mathbf{t}\mathbf{f}_{i}(E_{1}) &= \delta \left(\operatorname{cfg}_{i}^{1}\llbracket E_{1} \rrbracket(t,f,m),f,t\right) &= \delta \left(\operatorname{cfg}_{i}^{1}\llbracket E \rrbracket(t,f,m),f,t\right), \\ \mathbf{f}\mathbf{f}_{i}(E) &= \mathbf{f}\mathbf{f}_{i}(E_{1}) &= \delta \left(\operatorname{cfg}_{i}^{1}\llbracket E_{1} \rrbracket(t,f,m),f,f\right) &= \delta \left(\operatorname{cfg}_{i}^{1}\llbracket E \rrbracket(t,f,m),f,f\right), \\ \mathbf{p}\mathbf{p}_{i}(E) &= \mathbf{p}\mathbf{p}_{i}(E_{1}) &= \delta \left(\operatorname{cfg}_{i}^{1}\llbracket E_{1} \rrbracket(t,t,m),t,t\right) &= \delta \left(\operatorname{cfg}_{i}^{1}\llbracket E \rrbracket(t,t,m),t,t\right). \end{aligned}$$

Other unary operators: Let $\operatorname{cfg}_i^1[\![\operatorname{uop} E_1]\!](t,f,m)$ be as defined in (8). For each pair of paths to m in $\operatorname{cfg}_i^1[\![E_1]\!](m,m,m+1)$ there is 1 pair of paths, one to t and the other to f, and 0 pairs of paths both to t (resp., f) in $\operatorname{cfg}_i^1[\![\operatorname{uop} E_1]\!](t,f,m)$, and also 0 pairs of paths both to an exit node in $\operatorname{cfg}_i^1[\![E_1]\!](t,t,m)$. Hence, by Definition 5 and the inductive

hypothesis, we have:

$$\begin{split} \mathbf{tt}_{i}(\text{uop}\,E_{1}) &= 0 \\ &= \delta \big(\text{cfg}_{i}^{1} \llbracket \text{uop}\,E_{1} \rrbracket(t,f,m),t,t \big), \\ \mathbf{tf}_{i}(\text{uop}\,E_{1}) &= \mathbf{pp}_{i}(E_{1}) \\ &= \delta \big(\text{cfg}_{i}^{1} \llbracket E_{1} \rrbracket(t,t,m),t,t \big) \\ &= \delta \big(\text{cfg}_{i}^{1} \llbracket \text{uop}\,E_{1} \rrbracket(t,f,m),t,f \big), \\ \mathbf{tf}_{i}(\text{uop}\,E_{1}) &= \mathbf{pp}_{i}(E_{1}) \\ &= \delta \big(\text{cfg}_{i}^{1} \llbracket E_{1} \rrbracket(t,t,m),t,t \big) \\ &= \delta \big(\text{cfg}_{i}^{1} \llbracket \text{uop}\,E_{1} \rrbracket(t,f,m),f,t \big), \\ \mathbf{ff}_{i}(\text{uop}\,E_{1}) &= 0 \\ &= \delta \big(\text{cfg}_{i}^{1} \llbracket \text{uop}\,E_{1} \rrbracket(t,f,m),f,f \big), \\ \mathbf{pp}_{i}(\text{uop}\,E_{1}) &= 0 \\ &= \delta \big(\text{cfg}_{i}^{1} \llbracket \text{uop}\,E_{1} \rrbracket(t,t,m),t,t \big). \end{split}$$

Logical conjunction: There are three cases:

 $\operatorname{tv}_i(E_1) = 0$: then, by (10), $\operatorname{cfg}_i^1[E_1 \&\& E_2](t,f,m) = (\{f\},\varnothing,f)$, so δ in this case is similar to that of the false constant case. By (6), $\operatorname{ff}_i(E_1) = 1$ and $\operatorname{tt}_i(E_1) = \operatorname{tf}_i(E_1) = 0$. Hence, by Definition 5,

$$\begin{split} \mathbf{tt}_i(E_1 \&\& E_2) &= \mathbf{tt}_i(E_1) \, \mathbf{tt}_i(E_2) \\ &= 0 \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E_1 \&\& E_2 \rrbracket (t,f,m),t,t \big), \\ \mathbf{tf}_i(E_1 \&\& E_2) &= \mathbf{tf}_i(E_1) \, \mathbf{t}_i(E_2) + \mathbf{tt}_i(E_1) \, \mathbf{tf}_i(E_2) \\ &= 0 \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E_1 \&\& E_2 \rrbracket (t,f,m),t,f \big), \\ \mathbf{tf}_i(E_1 \&\& E_2) &= \mathbf{tf}_i(E_1) \, \mathbf{t}_i(E_2) + \mathbf{tt}_i(E_1) \, \mathbf{tf}_i(E_2) \\ &= 0 \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E_1 \&\& E_2 \rrbracket (t,f,m),f,t \big), \\ \mathbf{ff}_i(E_1 \&\& E_2) &= \mathbf{ff}_i(E_1) + 2 \, \mathbf{tf}_i(E_1) \, \mathbf{f}_i(E_2) + \mathbf{tt}_i(E_1) \, \mathbf{ff}_i(E_2) \\ &= 1 + 0 + 0 \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E_1 \&\& E_2 \rrbracket (t,f,m),f,f \big), \\ \mathbf{pp}_i(E_1 \&\& E_2) &= \mathbf{ff}_i(E_1) + 2 \, \mathbf{tf}_i(E_1) \, \mathbf{p}_i(E_2) + \mathbf{tt}_i(E_1) \, \mathbf{pp}_i(E_2) \\ &= 1 + 0 + 0 \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E_1 \&\& E_2 \rrbracket (t,t,m),t,t \big). \end{split}$$

 $\operatorname{tv}_i(E_1) = 1 \text{: then } \operatorname{cfg}_1^1[\![E_1 \&\& E_2]\!](t,f,m) = \operatorname{cfg}_1^1[\![E_2]\!](t,f,m) \text{ and, by (6), we have } \mathbf{tt}_i(E_1) = 1 \text{ and } \mathbf{tf}_i(E_1) = \mathbf{ff}_i(E_1) = 0. \text{ Hence, by Definition 5 and the inductive hypothesis:}$

$$\begin{split} \mathbf{tt}_i(E_1 \&\& E_2) &= \mathbf{tt}_i(E_1) \, \mathbf{tt}_i(E_2) \\ &= \mathbf{tt}_i(E_2) \\ &= \delta \big(\mathrm{cfg}_i^1 [\![E_2]\!](t,f,m),t,t \big), \\ &= \delta \big(\mathrm{cfg}_i^1 [\![E_1]\!](t,f,m),t,t \big), \\ \mathbf{tf}_i(E_1 \&\& E_2) &= \mathbf{tf}_i(E_1) \, \mathbf{t}_i(E_2) + \mathbf{tt}_i(E_1) \, \mathbf{tf}_i(E_2) \\ &= \mathbf{tf}_i(E_2) \\ &= \delta \big(\mathrm{cfg}_i^1 [\![E_2]\!](t,f,m),t,f \big), \end{split}$$

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= \delta(\operatorname{cfg}_{i}^{1} [E_{1} \&\& E_{2}](t, f, m), t, f),
 \mathbf{tf}_i(E_1 \&\& E_2) = \mathbf{tf}_i(E_1) \mathbf{t}_i(E_2) + \mathbf{tt}_i(E_1) \mathbf{tf}_i(E_2)
                                 = \mathbf{tf}_i(E_2)
                                 = \delta \left( \operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), f, t \right),
                                 =\delta(\operatorname{cfg}_{i}^{1}[E_{1} \&\& E_{2}](t,f,m),f,t),
  \mathbf{ff}_i(E_1 \&\& E_2) = \mathbf{ff}_i(E_1) + 2 \mathbf{tf}_i(E_1) \mathbf{f}_i(E_2) + \mathbf{tt}_i(E_1) \mathbf{ff}_i(E_2)
                                 =\mathbf{ff}_i(E_2)
                                 =\delta(\operatorname{cfg}_{i}^{1}[E_{2}](t,f,m),f,f),
                                 = \delta(\operatorname{cfg}_{i}^{1} [E_{1} \&\& E_{2}](t, f, m), f, f),
pp_i(E_1 \&\& E_2) = ff_i(E_1) + 2 tf_i(E_1) p_i(E_2) + tt_i(E_1) pp_i(E_2)
                                 = \mathbf{pp}_i(E_2)
                                 = \delta \left( \operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, t, m), t, t \right),
                                 =\delta(\operatorname{cfg}_{i}^{1}[E_{1}\&\&E_{2}](t,t,m),t,t).
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 $\operatorname{tv}_{i}(E_{1}) = ?$: let $\operatorname{cfg}_{i}^{1}[E_{1} \&\& E_{2}](t, f, m)$ be defined as in (10).

Since $\operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, f, m_{1})$ has the exit node for true evaluation equal to the entry node of $\operatorname{cfg}_i^1[\![E_2]\!](t,f,m)$, for each pair of paths in $\operatorname{cfg}_i^1[\![E_1]\!](s_2,f,m_1)$ both to s_2 , there are in $\operatorname{cfg}_i^1[\![E_1]\!](s_2)[\![t,f,m)$: $-\delta(\operatorname{cfg}_i^1[\![E_2]\!](t,f,m),t,t)$ pairs of paths both to t,

- $-\delta(\operatorname{cfg}_{i}^{1}[E_{2}](t,f,m),t,f)$ pairs of paths the first to t and the second to f,
- $-\delta(\operatorname{cfg}_{i}^{1}[E_{2}](t,f,m),f,t)$ pairs of paths the first to f and the second to t,
- $\delta(\operatorname{cfg}_{i}^{1}[E_{2}](t, f, m), f, f)$ pairs of paths both to f,

and in $\operatorname{cfg}_i^1[\![E_1 \&\& E_2]\!](t,t,m)$, $-\delta(\operatorname{cfg}_i^1[\![E_2]\!](t,t,m),t,t)$ pairs of paths both to an exit node.

For each pair of paths, in $\operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, f, m_{1})$, where the first is to s_{2} and the second one to f: there are in $\operatorname{cfg}_{i}^{1}[E_{1} \&\& E_{2}](t, f, m)$:

- $-\tau(\operatorname{cfg}_{i}^{1}[E_{2}](t,f,m),t)$ pairs of paths, the first to t and the second to f,
- $-\tau(\operatorname{cfg}_{i}^{1}[E_{2}](t,f,m),f)$ pairs of paths both to f in $\operatorname{cfg}_{i}^{1}[E_{1}\&\&E_{2}](t,f,m),$ and in $\operatorname{cfg}_{i}^{1}[E_{1} \&\& E_{2}](t, t, m)$,
- $\alpha(\operatorname{cfg}_{i}^{1}\llbracket E_{2} \rrbracket(t,t,m))$ pairs of paths both to an exit node.

For each pair of paths in $\operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, f, m_{1})$, where the first is to f and the second to s_2 , there are in $\operatorname{cfg}_i^1 \llbracket E_1 \&\& E_2 \rrbracket (t, f, m)$:

- $-\tau(\operatorname{cfg}_{i}^{1}\llbracket E_{2}\rrbracket(t,f,m),t)$ pairs of paths, the first to f and the second to t,
- $-\tau(\operatorname{cfg}_{i}^{1}[E_{2}](t,f,m),f)$ pairs of paths both to f in $\operatorname{cfg}_{i}^{1}[E_{1} \&\& E_{2}](t,f,m),$ and in $\operatorname{cfg}_{i}^{1} [E_{1} \&\& E_{2}](t, t, m)$,
- $\alpha(\operatorname{cfg}_{i}^{1}\llbracket E_{2} \rrbracket(t,t,m))$ pairs of paths both to an exit node.

In addition, there are $\delta(\operatorname{cfg}_i^1[\![E_1]\!](s_2,f,m),f,f)$ pairs of paths in $\operatorname{cfg}_i^1[\![E_1]\!](s_2,f,m_1)$ both to f.

Hence, by Definition 5 and the inductive hypothesis:

$$\begin{split} \mathbf{tt}_i(E_1 \&\& E_2) &= \mathbf{tt}_i(E_1) \, \mathbf{tt}_i(E_2) \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E_1 \rrbracket (s_2, f, m), s_2, s_2 \big) \delta \big(\mathrm{cfg}_i^1 \llbracket E_2 \rrbracket (t, f, m), t, t \big) \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E_1 \rrbracket (s_2, f, m), t, t \big), \\ \mathbf{tf}_i(E_1 \&\& E_2) &= \mathbf{tf}_i(E_1) \, \mathbf{t}_i(E_2) + \mathbf{tt}_i(E_1) \, \mathbf{tf}_i(E_2) \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E_1 \rrbracket (s_2, f, m), s_2, f \big) \tau \big(\mathrm{cfg}_i^1 \llbracket E_2 \rrbracket (t, f, m), t \big) \\ &+ \delta \big(\mathrm{cfg}_i^1 \llbracket E_1 \rrbracket (s_2, f, m), s_2, s_2 \big) \delta \big(\mathrm{cfg}_i^1 \llbracket E_2 \rrbracket (t, f, m), t, f \big) \\ &= \delta \big(\mathrm{cfg}_i^1 \llbracket E_1 \&\& E_2 \rrbracket (t, f, m), t, f \big), \\ \mathbf{tf}_i(E_1 \&\& E_2) &= \mathbf{tf}_i(E_1) \, \mathbf{t}_i(E_2) + \mathbf{tt}_i(E_1) \, \mathbf{tf}_i(E_2) \end{split}$$

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\begin{split} &= \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), s_{2}, f \right) \tau \left( \mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), t \right) \\ &+ \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), s_{2}, s_{2} \right) \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), f, t \right) \\ &= \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), f, t \right), \\ \mathbf{ff}_{i}(E_{1} \&\& E_{2}) &= \mathbf{ff}_{i}(E_{1}) + 2 \, \mathbf{tf}_{i}(E_{1}) \, \mathbf{f}_{i}(E_{2}) + \mathbf{tt}_{i}(E_{1}) \, \mathbf{ff}_{i}(E_{2}) \\ &= \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), f, f \right) \\ &+ \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), s_{2}, f \right) \tau \left( \mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), t \right) \\ &+ \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), f, s_{2} \right) \tau \left( \mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), f \right) \\ &+ \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), f, s_{2} \right) \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), f, f \right) \\ &= \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), f, f, f \right), \\ \mathbf{pp}_{i}(E_{1} \&\& E_{2}) &= \mathbf{ff}_{i}(E_{1}) + 2 \, \mathbf{tf}_{i}(E_{1}) \, \mathbf{p}_{i}(E_{2}) + \mathbf{tt}_{i}(E_{1}) \, \mathbf{pp}_{i}(E_{2}) \\ &= \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), f, f \right) \\ &+ \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, f, m), f, s_{2} \right) \tau \left( \mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, t, m), t \right) \\ &+ \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{2}, m), s_{2}, s_{2} \right) \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, t, m), t, t \right) \\ &= \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{2}, m), s_{2}, s_{2} \right) \delta \left( \mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, t, m), t, t \right) \end{aligned}
```

Logical disjunction: Dual to the case of logical conjunction.

Comma operator: Let $\operatorname{cfg}_i^1[\![E_1,E_2]\!](t,f,m)$ be defined as in (12). As the exit node for $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_2,m_1)$ and the entry node of $\operatorname{cfg}_i^1[\![E_2]\!](t,f,m)$ must be the same, for each pair of paths to s_2 in $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_2,m_1)$, there are in $\operatorname{cfg}_i^1[\![E_1,E_2]\!](t,f,m)$:

- $\delta(\operatorname{cfg}_{i}^{1} [E_{2}](t, f, m), t, t)$ pairs of paths both to t,
- $-\delta(\operatorname{cfg}_{i}^{1}[E_{2}](t,f,m),t,f)$ pairs of paths, the first to t and the second to f,
- $-\delta(\operatorname{cfg}_{i}^{1}[E_{2}](t,f,m),f,t)$ pairs of paths, the first to f and the second to t,
- $\delta(\operatorname{cfg}_{i}^{1}[E_{2}](t, f, m), f, f)$ pairs of paths both to f,

and in $\operatorname{cfg}_{i}^{1}[E_{1}, E_{2}](t, t, m)$,

 $-\delta(\operatorname{cfg}_{i}^{1}[E_{2}](t,t,m),t,t)$ pairs of paths both to an exit node.

Hence, by Definition 5 and the inductive hypothesis:

$$\begin{split} \mathbf{tt}_{i}(E_{1},E_{2}) &= \mathbf{pp}_{i}(E_{1})\,\mathbf{tt}_{i}(E_{2}) \\ &= \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2},s_{2},m_{1}),s_{2},s_{2} \big) \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t,f,m),t,t \big) \\ &= \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1},E_{2} \rrbracket (t,f,m),t,t \big), \\ \mathbf{tf}_{i}(E_{1},E_{2}) &= \mathbf{pp}_{i}(E_{1})\,\mathbf{tf}_{i}(E_{2}) \\ &= \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2},s_{2},m_{1}),s_{2},s_{2} \big) \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t,f,m),t,f \big) \\ &= \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1},E_{2} \rrbracket (t,f,m),t,f \big), \\ \mathbf{tf}_{i}(E_{1},E_{2}) &= \mathbf{pp}_{i}(E_{1})\,\mathbf{tf}_{i}(E_{2}) \\ &= \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2},s_{2},m_{1}),s_{2},s_{2} \big) \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t,f,m),f,t \big) \\ &= \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1},E_{2} \rrbracket (t,f,m),f,t \big), \\ \mathbf{ff}_{i}(E_{1},E_{2}) &= \mathbf{pp}_{i}(E_{1})\,\mathbf{ff}_{i}(E_{2}) \\ &= \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2},s_{2},m_{1}),s_{2},s_{2} \big) \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t,f,m),f,f \big) \\ &= \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1},E_{2} \rrbracket (t,f,m),f,f \big), \\ \mathbf{pp}_{i}(E_{1},E_{2}) &= \mathbf{pp}_{i}(E_{1})\,\mathbf{pp}_{i}(E_{2}) \\ &= \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2},s_{2},m_{1}),s_{2},s_{2} \big) \tau \big(\mathrm{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t,t,m),t,t \big) \\ &= \delta \big(\mathrm{cfg}_{i}^{1} \llbracket E_{1},E_{2} \rrbracket (t,t,m),t,t \big). \end{split}$$

Binary conditional operator: this case is identical to the case of logical disjunction, as the CFGs are the same and functions \mathbf{t}_i , \mathbf{f}_i and \mathbf{p}_i are defined the same.

Other binary operators: let $\operatorname{cfg}_i^1[\![E_1]\!](t,f,m)$ be defined as in (9). As the exit node for $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_2,m_1)$ and the entry node for $\operatorname{cfg}_i^1[\![E_2]\!](m,m,m+1)$ must be the same, for each pair of paths both to s_2 , there are in $\operatorname{cfg}_i^1[\![E_1]\!](s_2,s_2,m_1)$,

- $\delta(\operatorname{cfg}_{i}^{1}[E_{2}](m, m, m+1), m, m)$ pairs of paths both to m.

Moreover, since m is the second-last node through which each path directed to an exit node passes in $\operatorname{cfg}_i^1[\![E_1]\!]$ top $E_2[\!]$ (t,f,m) and the only arcs that exit from m are $(m,t),(m,f)\in A$, there are in $\operatorname{cfg}_i^1[\![E_1]\!]$ top $E_2[\!]$ (t,f,m):

- 0 pairs of paths both to t,
- -1 pair of paths to t in the first traversal and f in the second one,
- 1 pair of paths to f in the first traversal and t in the second one,
- 0 pairs of paths both to f,

and in $\operatorname{cfg}_{i}^{1}[E_{1} \operatorname{bop} E_{2}](t, t, m)$,

- 0 pairs of paths both to the exit node t.

Hence, by Definition 5 and the inductive hypothesis:

$$\begin{aligned} \mathbf{tt}_{i}(E_{1} \log E_{2}) &= 0 \\ &= \delta \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \log E_{2} \rrbracket(t,f,m),t,t \right), \\ \mathbf{tf}_{i}(E_{1} \log E_{2}) &= \mathbf{pp}_{i}(E_{1}) \, \mathbf{pp}_{i}(E_{2}) \\ &= \delta \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket(s_{2},s_{2},m_{1}),s_{2},s_{2} \right) \delta \left(\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(m,m,m+1),m,m \right) \\ &= \delta \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \log E_{2} \rrbracket(t,f,m),t,f \right), \\ \mathbf{tf}_{i}(E_{1} \log E_{2}) &= \mathbf{pp}_{i}(E_{1}) \, \mathbf{pp}_{i}(E_{2}) \\ &= \delta \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket(s_{2},s_{2},m_{1}),s_{2},s_{2} \right) \delta \left(\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(m,m,m+1),m,m \right) \\ &= \delta \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \log E_{2} \rrbracket(t,f,m),f,t \right), \\ \mathbf{ff}_{i}(E_{1} \log E_{2}) &= 0 \\ &= \delta \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \log E_{2} \rrbracket(t,f,m),f,f \right), \\ \mathbf{pp}_{i}(E_{1} \log E_{2}) &= 0 \\ &= \delta \left(\operatorname{cfg}_{i}^{1} \llbracket E_{1} \log E_{2} \rrbracket(t,t,m),t,t \right). \end{aligned}$$

Conditional operator: Then $E = E_1 ? E_2 : E_3$. There are three cases:

 $tv_i(E_1) = 1$: then, by (15) and (10),

$$\operatorname{cfg}_{i}^{1} [E_{1} ? E_{2} : E_{3}](t, f, m) = \operatorname{cfg}_{i}^{1} [E_{2}](t, f, m) = \operatorname{cfg}_{i}^{1} [E_{1} \&\& E_{2}](t, f, m).$$

Also, by Definition 5,

$$\begin{aligned} \mathbf{tt}_i(E_1 ? E_2 : E_3) &= \mathbf{tt}_i(E_2) = \mathbf{tt}_i(E_1 \&\& E_2), \\ \mathbf{tf}_i(E_1 ? E_2 : E_3) &= \mathbf{tf}_i(E_2) = \mathbf{tf}_i(E_1 \&\& E_2), \\ \mathbf{ff}_i(E_1 ? E_2 : E_3) &= \mathbf{ff}_i(E_2) = \mathbf{ff}_i(E_1 \&\& E_2). \end{aligned}$$

Therefore this case is equivalent to the same case for logical conjunction.

 $\operatorname{tv}_i(E_1) = 0$: symmetric to the previous case.

- $\operatorname{tv}_i(E_1) = ?$: let $\operatorname{cfg}_1^1[\![E]\!](t,f,m) = \operatorname{cfg}_1^1[\![E_1\,?E_2:E_3]\!](t,f,m)$ be as defined in (15). Since $\operatorname{cfg}_1^1[\![E_1]\!](s_2,s_3,m_2)$ has the exit node for true evaluation equal to the entry node of $\operatorname{cfg}_1^1[\![E_2]\!](t,f,m)$, for each pair of paths both to s_2 in $\operatorname{cfg}_1^1[\![E_1]\!](s_2,s_3,m_2)$ there are in $\operatorname{cfg}_1^1[\![E]\!](t,f,m)$:
 - $-\frac{\delta}{\delta}\left(\text{cfg}_i^{-1}[E_2](t,f,m),t,t\right) \text{ pairs of paths both to } t,$
 - $-\delta(\operatorname{cfg}_{i}^{1}[E_{2}](t,f,m),t,f)$ pairs of paths, the first to t and the second to f,
 - $-\delta(\operatorname{cfg}_i^1[E_2](t,f,m),f,t)$ pairs of paths, the first to f and the second to t,
 - $\delta(\operatorname{cfg}_{i}^{1}[E_{2}](t, f, m), f, f)$ pairs of paths both to f;

```
and in \operatorname{cfg}_1^1[\![E]\!](t,t,m), -\delta(\operatorname{cfg}_1^1[\![E_2]\!](t,f,m),t,t) pairs of paths both to an exit node.
Similarly, as \operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, s_{3}, m_{2}) has the exit node for false evaluation equal to the entry
node of \operatorname{cfg}_{i}^{1}[E_{3}](t, f, m), for each pair of paths both to s_{3} in \operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, s_{3}, m_{2}) there
 \begin{aligned} &\text{are in cfg}_i^1 \llbracket E \rrbracket(t,f,m) \colon \\ &- \delta \big( \text{cfg}_i^1 \llbracket E_3 \rrbracket(t,f,m),t,t \big) \text{ pairs of paths both to } t, \end{aligned} 
   -\delta(\operatorname{cfg}_i^1[E_3](t,f,m),t,f) pairs of paths, the first to t and the second to f,
   - \delta(\operatorname{cfg}_{i}^{1}[E_{3}](t, f, m), f, t) pairs of paths, the first to f and the second to t,
   -\delta(\operatorname{cfg}_{i}^{1}[E_{3}](t,f,m),f,f) pairs of paths both to f;
and in \operatorname{cfg}_{i}^{1}\llbracket E \rrbracket(t,t,m),
   - \delta(\operatorname{cfg}_{i}^{1}[E_{3}](t,t,m),t,t) pairs of paths both to an exit node.
In addition, for each of the 2 \cdot \delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, s_{3}, m_{2}), s_{2}, s_{3}) pairs of paths, where the
two paths evaluate to different truth values, there are in \operatorname{cfg}_{i}[E](t, f, m):
   -\tau(\operatorname{cfg}_{i}^{1}[E_{2}](t,f,m),t)\tau(\operatorname{cfg}_{i}^{1}[E_{3}](t,f,m),t) pairs of paths both to t,
   -\tau(\mathrm{cfg}_{i}^{1}[\![E_{2}]\!](t,f,m),t)\tau(\mathrm{cfg}_{i}^{1}[\![E_{3}]\!](t,f,m),f)+\tau(\mathrm{cfg}_{i}^{1}[\![E_{2}]\!](t,f,m),f)\tau(\mathrm{cfg}_{i}^{1}[\![E_{3}]\!](t,f,m),t)
        pairs of paths, the first to t and the second to f,
        \tau\left(\operatorname{cfg}_{i}^{1}\llbracket E_{2}\rrbracket(t,f,m),f\right)\tau\left(\operatorname{cfg}_{i}^{1}\llbracket E_{3}\rrbracket(t,f,m),t\right)+\tau\left(\operatorname{cfg}_{i}^{1}\llbracket E_{2}\rrbracket(t,f,m),t\right)\tau\left(\operatorname{cfg}_{i}^{1}\llbracket E_{3}\rrbracket(t,f,m),f\right)
        pairs of paths, the first to f and the second to t,
        \tau(\operatorname{cfg}_{i}^{1}[E_{2}](t, f, m), f) \cdot \tau(\operatorname{cfg}_{i}^{1}[E_{3}](t, f, m), f) both to f;
and in \operatorname{cfg}_{i}^{1}[E](t, t, m),
        \alpha(\operatorname{cfg}_{i}^{1}[E_{2}](t,f,m))\alpha(\operatorname{cfg}_{i}^{1}[E_{3}](t,f,m)) pairs of paths both to exit node t.
Hence, by Definition 5 and the inductive hypothesis:
             \mathbf{tt}_i(E_1 ? E_2 : E_3) = \mathbf{tt}_i(E_1) \, \mathbf{tt}_i(E_2)
                                                       +2 \mathbf{tf}_{i}(E_{1}) \mathbf{t}_{i}(E_{2}) \mathbf{t}_{i}(E_{3}) + \mathbf{ff}_{i}(E_{1}) \mathbf{tt}_{i}(E_{3})
                                                  = \delta(\operatorname{cfg}_{i}^{1} [E_{1}](s_{2}, s_{3}, m_{2}), s_{2}, s_{2}) \delta(\operatorname{cfg}_{i}^{1} [E_{2}](t, f, m), t, t)
                                                       + \delta \left( \text{cfg}_i^1 \llbracket E_1 \rrbracket (s_2, s_3, m_2), s_2, s_3 \right)
                                                            \cdot \tau (\operatorname{cfg}_{i}^{1} [E_{2}](t, f, m), t) \tau (\operatorname{cfg}_{i}^{1} [E_{3}](t, f, m), t)
                                                       +\delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2},s_{3},m_{2}),s_{3},s_{2})
                                                            \quad \cdot \tau \left(\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(t,f,m),t\right) \tau \left(\operatorname{cfg}_{i}^{1} \llbracket E_{3} \rrbracket(t,f,m),t\right) \\
                                                       + \delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, s_{3}, m_{2}), s_{3}, s_{3})\delta(\operatorname{cfg}_{i}^{1}[E_{3}](t, f, m), t, t)
                                                  = \delta(\operatorname{cfg}_{i}^{1} [E_{1} ? E_{2} : E_{3}](t, f, m), t, t),
            \mathbf{tf}_i(E_1 ? E_2 : E_3) = \mathbf{tt}_i(E_1) \, \mathbf{tf}_i(E_2)
                                                       + \mathbf{tf}_i(E_1)(\mathbf{t}_i(E_2)\mathbf{f}_i(E_3) + \mathbf{f}_i(E_2) + \mathbf{t}_i(E_2))
                                                       +\mathbf{ff}_i(E_1)\mathbf{tf}_i(E_3)
                                                  = \delta \left( \operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{3}, m_{2}), s_{2}, s_{2} \right) \delta \left( \operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), t, f \right)
                                                       +\delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2},s_{3},m_{2}),s_{2},s_{3})
                                                            \cdot \tau (\operatorname{cfg}_{i}^{1} [E_{2}](t, f, m), t) \tau (\operatorname{cfg}_{i}^{1} [E_{3}](t, f, m), f)
                                                       + \delta \left( \text{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{3}, m_{2}), s_{3}, s_{2} \right)
                                                            \cdot \tau (\operatorname{cfg}_{i}^{1} [E_{2}](t, f, m), f) \tau (\operatorname{cfg}_{i}^{1} [E_{3}](t, f, m), t)
                                                       + \delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, s_{3}, m_{2}), s_{3}, s_{3})\delta(\operatorname{cfg}_{i}^{1}[E_{3}](t, f, m), t, f)
                                                  = \delta(\operatorname{cfg}_{i}^{1} [E_{1} ? E_{2} : E_{3}](t, f, m), t, f),
            \mathbf{tf}_i(E_1 ? E_2 : E_3) = \mathbf{tt}_i(E_1) \, \mathbf{tf}_i(E_2)
                                                       + \mathbf{tf}_i(E_1)(\mathbf{t}_i(E_2)\mathbf{f}_i(E_3) + \mathbf{f}_i(E_2) + \mathbf{t}_i(E_2))
```

 $+\delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2},s_{3},m_{2}),s_{2},s_{3})$

 $= \delta(\operatorname{cfg}_{i}^{1} [E_{1}](s_{2}, s_{3}, m_{2}), s_{2}, s_{2}) \delta(\operatorname{cfg}_{i}^{1} [E_{2}](t, f, m), f, t)$

 $+\mathbf{ff}_i(E_1)\mathbf{tf}_i(E_3)$

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\cdot \tau (\operatorname{cfg}_{i}^{1} [E_{2}](t, f, m), f) \tau (\operatorname{cfg}_{i}^{1} [E_{3}](t, f, m), t)
                                                       + \delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, s_{3}, m_{2}), s_{3}, s_{2})
                                                             \cdot \tau (\operatorname{cfg}_{i}^{1} [E_{2}](t, f, m), t) \tau (\operatorname{cfg}_{i}^{1} [E_{3}](t, f, m), f)
                                                       + \delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, s_{3}, m_{2}), s_{3}, s_{3})\delta(\operatorname{cfg}_{i}^{1}[E_{3}](t, f, m), f, t)
                                                 = \delta(\operatorname{cfg}_{i}^{1} [E_{1} ? E_{2} : E_{3}](t, f, m), f, t),
  \mathbf{ff}_i(E_1 ? E_2 : E_3) = \mathbf{tt}_i(E_1) \mathbf{ff}_i(E_2)
                                                       +2 \operatorname{tf}_{i}(E_{1}) \operatorname{f}_{i}(E_{2}) \operatorname{f}_{i}(E_{3}) + \operatorname{ff}_{i}(E_{1}) \operatorname{ff}_{i}(E_{3})
                                                 = \delta \left( \operatorname{cfg}_{i}^{1} \llbracket E_{1} \rrbracket (s_{2}, s_{3}, m_{2}), s_{2}, s_{2} \right) \delta \left( \operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket (t, f, m), f, f \right)
                                                       +\delta\left(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2},s_{3},m_{2}),s_{2},s_{3}\right)
                                                              \cdot \tau (\operatorname{cfg}_{i}^{1} [E_{2}](t, f, m), f) \tau (\operatorname{cfg}_{i}^{1} [E_{3}](t, f, m), f)
                                                       + \delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, s_{3}, m_{2}), s_{3}, s_{2})
                                                             \cdot \tau \left(\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(t,f,m),f\right) \tau \left(\operatorname{cfg}_{i}^{1} \llbracket E_{3} \rrbracket(t,f,m),f\right)
                                                       + \delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, s_{3}, m_{2}), s_{3}, s_{3})\delta(\operatorname{cfg}_{i}^{1}[E_{3}](t, f, m), t, t)
                                                 =\delta(\operatorname{cfg}_{i}^{1}[E_{1}?E_{2}:E_{3}](t,f,m),f,f),
pp_i(E_1 ? E_2 : E_3) = tt_i(E_1) pp_i(E_2)
                                                       +2 \mathbf{tf}_{i}(E_{1}) \mathbf{p}_{i}(E_{2}) \mathbf{p}_{i}(E_{3}) + \mathbf{ff}_{i}(E_{1}) \mathbf{p}_{i}(E_{3})
                                                 = \delta(\operatorname{cfg}_{i}^{1} [E_{1}](s_{2}, s_{3}, m_{2}), s_{2}, s_{2}) \delta(\operatorname{cfg}_{i}^{1} [E_{2}](t, t, m), t, t)
                                                       + \delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2}, s_{3}, m_{2}), s_{2}, s_{3})
                                                              \cdot \alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(t,t,m)\right) \alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{3} \rrbracket(t,t,m)\right)
                                                       +\delta(\operatorname{cfg}_{i}^{1}[E_{1}](s_{2},s_{3},m_{2}),s_{3},s_{2})
                                                             \cdot \alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{2} \rrbracket(t,t,m)\right) \alpha \left(\operatorname{cfg}_{i}^{1} \llbracket E_{3} \rrbracket(t,t,m)\right)
                                                       + \delta(\operatorname{cfg}_{i}^{1} [E_{1}](s_{2}, s_{3}, m_{2}), s_{3}, s_{3}) \delta(\operatorname{cfg}_{i}^{1} [E_{3}](t, t, m), t, t)
                                                 =\delta(\operatorname{cfg}_{i}^{1}[E_{1}?E_{2}:E_{3}](t,f,m),t,t).\Box
```

In the sequel, we will assume that any CFG (N,A,s) constructed by $\operatorname{cfg}_i[\![]\!]$ has $N \subset \mathbb{N}$; and that $B \in \operatorname{Stm}$ denotes a function body and $\operatorname{cfg}_i^b[\![B]\!] = (N_B,A_B,s_B)$. First we provide some additional notation.

- orig $(m) \in \text{Stm}$ denotes a label or statement that generates the node m. Note that, by Definition 3, it follows that, if m > n, then orig(m) occurs before orig(n) in a function.
- For each stm \in {switch, while, do while} in B, we insert two synthetic statements stm enter (resp., stm exit) before (resp., after) the body of stm. Therefore, if $n_1, n_2 \in N_B$ and $\operatorname{orig}(n_1) = \operatorname{stm}$ enter and $\operatorname{orig}(n_2) = \operatorname{stm}$ exit, then $n_1 > n_2$.
- $\operatorname{sb}(t_1, t_2, \operatorname{stm})$, where $t_1, t_2 \in \mathbb{N}$ and $\operatorname{stm} \in \{\operatorname{switch}, \operatorname{while}, \operatorname{do-while}\}$, denotes a predicate that results in true if and only if t_1 and t_2 are the two nodes that enclose the body of a statement stm , namely:

$$sb(t_1, t_2, stm) := \left(orig(t_1) = stm \ enter \land orig(t_2) = stm \ exit \right)$$
$$\land \left| \left\{ c \in [t_2, t_1] \ \middle| \ orig(c) = stm \ exit \right\} \right| = \left| \left\{ c \in [t_2, t_1] \ \middle| \ orig(c) = stm \ exit \right\} \right| \right\}.$$
(65)

Furthermore, for each label L or statement S in B, where $\operatorname{cfg}_i[\![L]\!](t,m)=(N,A,s)$ and $\operatorname{cfg}_i[\![S]\!](t,t_{\operatorname{b}},t_{\operatorname{c}},m)=(N,A,s),\ t,t_{\operatorname{b}},t_{\operatorname{c}},m\in\mathbb{N}$ are such that $t_{\operatorname{b}}\leq m\leq t\leq t_{\operatorname{c}}$, and, finally, $N\subseteq\{t\}\cup[m,m'-1]$:

-l(S) denotes the set of label identifiers contained in S:

$$l(S) := \{ id \in Id \mid \exists m \in N . \operatorname{orig}(m) = id \}.$$
(66)

⁹ sb is a mnemonic for *same body*.

- s(S) denotes the set of case or default nodes contained in S where the controlling switch is external to S:

$$s(S) := \left\{ n \in N \middle| \begin{array}{l} \operatorname{orig}(n) \in \{ \operatorname{case}, \operatorname{default} \} \\ \wedge \not \equiv m_1, m_2 \in N . \operatorname{sb}(m_1, m_2, \operatorname{switch}) \wedge n \in [m_1, m_2] \end{array} \right\}. \tag{67}$$

- t(S,B) denotes the set of switch nodes in B that may pass the control to a case or default node in S:

$$t(S,B) := \left\{ m \in N_B \middle| \begin{array}{l} \operatorname{orig}(m) = \operatorname{switch\ enter} \\ \wedge \exists m_2 \in N_B . \operatorname{sb}(m, m_2, \operatorname{switch}) \\ \wedge \exists n \in N . n \in [m_2, m] \cap [t, s] \end{array} \right\}.$$
 (68)

-b(s,t) denotes the set of break nodes between s and t that are external to any inner switch or loop statements. Formally, if stm \in {switch, while, do - while},

$$b(s,t) := \left\{ m \in [s,t] \middle| \begin{array}{l} \operatorname{orig}(m) = \operatorname{break} \\ \wedge \not\equiv m_1, m_2 \in [s,t] \cdot \left(\operatorname{sb}(m_1, m_2, \operatorname{stm}) \wedge m \in [m_2, m_1] \right) \end{array} \right\}. \tag{69}$$

-c(s,t) denotes the set of continue nodes between s and t that are external to any inner loop statements. Formally, if stm $\in \{\text{while}, \text{do} - \text{while}\}$,

$$c(s,t) := \left\{ m \in [s,t] \middle| \begin{array}{l} \operatorname{orig}(m) = \operatorname{continue} \\ \wedge \not \equiv m_1, m_2 \in [s,t] \cdot \left(\operatorname{sb}(m_1, m_2, \operatorname{stm}) \wedge m \in [m_2, m_1] \right) \end{array} \right\}. \tag{70}$$

-r(s,t) denotes the set of return nodes between s and t. Formally,

$$r(s,t) := \{ m \in [s,t] \mid \operatorname{orig}(m) = \operatorname{return} \}. \tag{71}$$

 $-\sigma(S,B)$ denotes the number of acyclic paths in B to a node in t(S,B):

$$\sigma(S,B) := \sum_{n \in t(S,B)} \tau(s,A,n). \tag{72}$$

 $-g(\mathrm{id},S,B)$ denotes the set of goto id labeled statements in S:

$$g(\mathrm{id}, S, B) := \{ g \in N \mid \mathrm{orig}(g) = \mathsf{goto} \; \mathrm{id} \}. \tag{73}$$

 $-\ \operatorname{pg}(\operatorname{id},S,B)$ denotes the set of $\operatorname{\sf goto}$ id labeled statements in B and before S

$$pg(id, S, B) := \{ g \in N_B, g > s \mid orig(g) = goto id \}.$$

$$(74)$$

 $-\gamma(\mathrm{id},S,B)$ denotes the number of acyclic paths in B to nodes in $g(\mathrm{id},S,B)$:

$$\gamma(\mathrm{id}, S, B) := \sum_{n \in g(\mathrm{id}, S, B)} \tau(s, A, n). \tag{75}$$

 $-\phi(\mathrm{id},S,B)$ denotes the number of acyclic paths in B to nodes in pg(id, S, B):

$$\phi(\mathrm{id}, S, B) := \sum_{n \in \mathrm{pg}(\mathrm{id}, S, B)} \tau(s, A_B, n).$$
(76)

- $\nu(c, S, B)$, where $c \in N$, denotes the number of acyclic paths that "fall through c from above":

$$\nu(c, S, B) := \begin{cases} \tau(s, A_1, c), & \text{if } \operatorname{orig}(c) = \operatorname{id}, \\ \tau(s, A_2, c), & \text{if } \operatorname{orig}(c) \in \{ \operatorname{\mathsf{case}}, \operatorname{\mathsf{default}} \}, \\ \tau(s, A_f, c), & \text{otherwise}, \end{cases}$$
(77)

where $A_1 = \{ (c_1, c) \in A_B \mid c_1 \notin \operatorname{pg}(\operatorname{id}, S, B) \}, A_2 = \{ (c_1, c) \in A_B \mid c_1 \notin t(S, B) \}$ and $A_f = \{ (c_1, c) \in A_B \mid c_1 > c \}.$

 $-\beta(S,B)$ denotes the number of acyclic paths in B to nodes in b(s,t):

$$\beta(S,B) := \sum_{n \in b(s,t)} \tau(s,A,n). \tag{78}$$

 $-\chi(S)$ denotes the number of acyclic paths in B to nodes in c(s,t):

$$\chi(S) := \sum_{n \in c(s,t)} \tau(s, A, n). \tag{79}$$

 $-\rho(S)$ denotes the number of acyclic paths in B to nodes in r(s,t):

$$\rho(S) := \sum_{n \in r(s,t)} \tau(s, A, n). \tag{80}$$

In Section 4, we gave an informal definition of a controlled function body (Definition 8), we now provide a formal version.

Definition 11 (Controlled function body.) Let $B \in \text{Stm}$ be a full C function body. We call B a controlled function body if it satisfies the following properties:

- 1. for each $t_1, t_2 \in N$, if $\operatorname{orig}(t_1) = \operatorname{id}$ and $\operatorname{orig}(t_2) = \operatorname{\mathsf{goto}}$ id, then $t_1 \leq t_2$;
- 2. if stm \in {while, do while}, $w_1, w_2 \in N_B$ are such that $\operatorname{orig}(w_1) = \operatorname{stm}$ enter and $orig(w_2) = stm \ exit$, and one of the following holds:

$$\exists n \in N : n \in [w_2, w_1] \land \operatorname{orig}(n) \in \{\mathtt{break}, \mathtt{return}\} \land n \in b(w_1, w_2), \\ \exists n_1, n_2 \in N : n_1 < w_2 \land n_2 \in [w_2, w_1] \land \operatorname{orig}(n_1) = \operatorname{id} \land \operatorname{orig}(n_2) = \mathtt{goto} \operatorname{id},$$

then neither of the following hold:

```
\exists t_1, t_2 \in N : t_1 \in [w_2, w_1]
                               \land t_2 < w_2 \land \operatorname{orig}(t_2) \in \{ \mathtt{case}, \mathtt{default} \} \land \operatorname{orig}(t_1) = \mathtt{switch} \ \mathtt{enter},
\exists t_1, t_2 \in N : t_1 \in [w_2, w_1]
                               \wedge t_2 > w_1 \wedge \operatorname{orig}(t_1) = \operatorname{id}_w \wedge \operatorname{orig}(t_2) = \operatorname{goto} \operatorname{id}_w.
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Note that it follows from condition 1 of Definition 11 that if $B \in Stm$ is a controlled function body and $\operatorname{cfg}_{i}^{b}[B] = (N_{B}, A_{B}, s_{B})$, for each $(m, n) \in A_{B}$ such that $\operatorname{orig}(m) \notin$ $\{\text{while exit}, \text{do} - \text{while exit}\}$, the statement $\operatorname{orig}(m)$ will be before the statement $\operatorname{orig}(n)$ in B.

Lemma 3 Let:

- $-i \in \{0,1,2\}$ be an optimization level;
- $\begin{array}{ll} & B \in \operatorname{Stm} \ a \ controlled \ function \ body; \\ & \operatorname{cfg}_i^{\, \text{b}}[\![B]\!] = (N_B, A_B, s_B); \end{array}$
- $-t \in N_B$ a target node in $\operatorname{cfg}_i^b[\![B]\!];$
- $\operatorname{stm} \in {\{\text{while}, \text{do} \text{while}\}};$
- $w_1, w_2 \in N_B$ such that $\operatorname{orig}(w_1) = \operatorname{stm}$ enter and $\operatorname{orig}(w_2) = \operatorname{stm}$ exit;
- $-P = s_B, \ldots, a, \ldots, t \text{ an acylic path in } \operatorname{cfg}_i^{\operatorname{b}}[\![B]\!] \text{ where } a \in [w_2, w_1].$

Then P contains a subsequence $a_0, \ldots, a_i = a, \ldots a_k$ where, for each $i \in [1, k-1], a_i \in$ $[w_2, w_1]$ and either $\operatorname{orig}(a_1) = \operatorname{stm}$ enter or $\operatorname{orig}(a_{k-1}) = \operatorname{stm}$ exit.

Proof Suppose $a_0, a_1, \ldots, a_j = a, \ldots, a_k$ is the maximal subsequence of P such that for each $i \in [1, k-1]$, $a_i \in [w_2, w_1]$. Suppose also that $\operatorname{orig}(a_1) \neq \operatorname{stm}$ enter and $\operatorname{orig}(a_{k-1}) \neq \operatorname{stm}$ stm exit. Then we derive a contradiction.

Since $\operatorname{orig}(a_1) \neq \operatorname{stm}$ enter, either $\operatorname{orig}(a_0) = \operatorname{\mathsf{goto}} \operatorname{id}_1$ and $\operatorname{orig}(a_1) = \operatorname{id}_1$ or $\operatorname{orig}(a_0) = \operatorname{\mathsf{goto}} \operatorname{\mathsf{id}}_1$ switch enter and $\operatorname{orig}(a_1) \in \{\operatorname{case}, \operatorname{default}\};$ and also since $\operatorname{orig}(a_{k-1}) \neq \operatorname{stm}$ exit, either $\operatorname{orig}(a_k) = \operatorname{id}_k$ and $\operatorname{orig}(a_{k-1}) = \operatorname{\mathsf{goto}} \operatorname{id}_k$ or $\operatorname{orig}(a_k) \in \{\operatorname{\mathtt{return}},\operatorname{\mathtt{break}}\}$. Therefore, condition 2 of Definition 11 does not hold, contradicting the hypothesis that B is a controlled function body.

Lemma 4 Let: $i \in \{0,1,2\}$ be an optimization level; $B \in \operatorname{Stm}$ a controlled function body; $L \in \operatorname{Lab}$ a label in B; $t,m \in \mathbb{N}$ where t < m; $\operatorname{cfg}_i^b \llbracket B \rrbracket = (N_B,A_B,s_B)$; $\operatorname{cfg}_i^1 \llbracket L \rrbracket (t,m) = (N,A,s)$; $\operatorname{ft} = \nu(s,L,B)$; $\operatorname{st} = \sigma(L,B)$ and $\operatorname{gt} \colon \operatorname{Id} \to \mathbb{N}$ be such that, for each $\operatorname{id} \in l(B)$, $\operatorname{gt}(id) = \phi(\operatorname{id},L,B)$. Then $\operatorname{apc}_i \llbracket L \rrbracket (\operatorname{ft},\operatorname{st},\operatorname{gt}) = \nu(t,L,B)$.

Proof We prove each kind of label separately.

Case label: if L = case z then, by (16), $\text{cfg}_i^1 \llbracket L \rrbracket(t,m) = (\{m,t\}, \{(m,t)\}, m)$ and, by (23), $\exists c \in t(L,B) : (c,m) \in A_B$. By hypothesis,

$$\mathsf{ft} = \tau \big(s_B, A_f \setminus \{(c, m)\}, m \big)$$

and $\operatorname{st} = \tau\left(\operatorname{cfg}_i^{\operatorname{b}}[\![B]\!],c\right)$. The number of paths that reach the entry node of label L without using an arc in A_1 is $\tau(s_B,A_f\setminus \left\{(c,m)\right\},m)$. The number of paths passing through the arc (c,m) is $\tau(\operatorname{cfg}_i^{\operatorname{b}}[\![B]\!],c,m)$. Each of these paths will reach t via the arc (m,t) and will not pass through any arcs in $\left\{(g,t)\mid g\in\operatorname{pg}(\operatorname{id},L,B)\right\}$ when $\operatorname{orig}(t)=\operatorname{id}$ or through an arc $(c_1,t)\in t(L,B)$ if $\operatorname{orig}(t)\in\{\operatorname{case},\operatorname{default}\}$. Concluding,

$$\begin{split} \mathrm{apc}_i \llbracket L \rrbracket (\mathsf{ft},\mathsf{st},\mathsf{gt}) &= \mathsf{ft} + \mathsf{st} \\ &= \tau \big(s_B, A_f \setminus \big\{ (c,s) \big\}, s \big) + \tau \big(\mathrm{cfg}_i^{\,\mathrm{b}} \llbracket B \rrbracket, c \big) \\ &= \nu(t,L,B). \end{split}$$

Default label: the proof is similar to the previous case.

Identifier label: by (16), $\operatorname{cfg}_1^1[[\operatorname{id}](t,m) = (\{m,t\},\{(m,t)\},m)$ and, by hypothesis, $\operatorname{ft} = \tau(N_B,A_f \setminus \{(g,m) \mid g \in \operatorname{pg}(\operatorname{id},L,B)\},m)$. The number of paths that reach the entry node of label L without using any of the arcs in

$$A_g = \{ (g, m) \mid g \in \operatorname{pg}(\operatorname{id}, L, B) \})$$

is $\tau(s_B, A_f \setminus A_g, m)$. The number of paths passing through the arcs in A_g is $\phi(\mathrm{id}, L, B)$. Each of these paths will reach t via (m, t) and will not pass through any arcs in

$$\{(g,t) \mid g \in pg(id, L, B)\}$$

if $\operatorname{orig}(t) = \operatorname{id}$ or through an arc (c,t) . $c \in t(L,B)$ if $\operatorname{orig}(t) \in \{\mathtt{case},\mathtt{default}\}$. Hence, using the hypothesis:

$$\begin{split} \mathrm{apc}_i \llbracket L \rrbracket (\mathsf{ft},\mathsf{st},\mathsf{gt}) &= \mathsf{ft} + \mathsf{gt}(\mathrm{id}) \\ &= \tau \big(N_B, A_f \setminus \big\{ \left. (g,s) \mid g \in \mathrm{pg}(\mathrm{id},L,B) \right. \big\}, s \big) + \phi(\mathrm{id},L,B) \\ &= \nu(t,L,B). \end{split}$$

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Lemma 5 Let: $i \in \{0,1,2\}$ be an optimization level; $B \in \operatorname{Stm}$ be a controlled function body; $S \in \operatorname{Stm}$ be a statement in B; $t, t_b, t_c, c \in \mathbb{N}$ be such that $t_b \leq c < t \leq t_c$; $\operatorname{cfg}_i^b[\![B]\!] = (N_B, A_B, s_B)$; $\operatorname{cfg}_i^1[\![S]\!](t, t_b, t_c, m) = (N, A, s)$; $\operatorname{ft} = \nu(s, S, B)$; $\operatorname{st} = \sigma(S, B)$ and $\operatorname{gt} : \operatorname{Id} \to \mathbb{N}$ be such that, for each $\operatorname{id} \in l(B)$, $\operatorname{gt}(id) = \phi(\operatorname{id}, S, B)$ Then

$$\mathrm{apc}_i \llbracket S \rrbracket (\mathsf{ft},\mathsf{st},\mathsf{gt}) = (\mathsf{ft}_\mathsf{out},\mathsf{bp},\mathsf{cp},\mathsf{rp},\mathsf{gt}_\mathsf{out}),$$

where

$$\begin{split} &\mathsf{ft}_{\mathsf{out}} = \nu(t, S, B) \\ &\mathsf{bp} = \beta(S, B), \\ &\mathsf{cp} = \chi(S, B), \\ &\mathsf{rp} = \rho(S, B), \\ &\mathsf{gt}_{\mathsf{out}} = \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S, B). \end{split}$$

Proof The proof is by induction on S where each kind of statement is considered separately. **Expression statement**: by (17), we have $\operatorname{cfg}_i^1[\![E]\!](t,t_b,t_c,m) = \operatorname{cfg}_i^1[\![E]\!](t,t,m)$, and, by Lemma 1, $\alpha(\operatorname{cfg}_i^1[\![E]\!](t,t,m)) = \mathbf{p}_i(E)$. Moreover, as there are no break, continue, return or goto statements inside S, $\beta(S,B) = \chi(S,B) = \rho(S,B) = 0$ and, for each $\operatorname{id} \in l(B)$, $\gamma(\operatorname{id},S,B) = 0$. Hence, by (37),

$$\begin{split} &\mathsf{ft}_{\mathsf{out}} = \mathsf{ft}\, \mathsf{p}_i(E) \\ &= \tau(s_B, A_f, s) \alpha \big(\mathsf{cfg}_i^1 \llbracket E \rrbracket(t, t, m) \big) \\ &= \nu(t, S, B), \\ &\mathsf{bp} = 0 \\ &= \beta(S, B), \\ &\mathsf{cp} = 0 \\ &= \chi(S, B), \\ &\mathsf{rp} = 0 \\ &= \rho(S, B), \\ &\mathsf{gt}_{\mathsf{out}} = \mathsf{gt} \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot 0 \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S, B). \end{split}$$

Return statement: by (19), $\operatorname{cfg}_i^1[\![\operatorname{return}]\!](t,t_b,t_c,m) = (\{m\},\varnothing,m)$. Hence, for each path in (s_B,A_B,s) to s=m there are 0 paths to t and 1 path to the return node. Moreover, as there are no break, continue or goto statements inside S, $\beta(S,B) = \chi(S,B) = 0$ and, for each id $\in l(B)$, $\gamma(\operatorname{id},S,B) = 0$. Hence, by (39),

$$\begin{split} &\mathsf{ft}_{\mathsf{out}} = 0 \\ &= \tau \big((\{m\}, \varnothing, m), t \big) \\ &= \nu(t, S, B), \\ &\mathsf{bp} = 0 \\ &= \beta(S), \\ &\mathsf{cp} = 0 \\ &= \chi(S), \\ &\mathsf{rp} = \mathsf{ft} \\ &= \tau(s_B, A_B, s) \\ &= \rho(S), \\ &\mathsf{gt}_{\mathsf{out}} = \mathsf{gt} \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot 0 \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S, B). \end{split}$$

Return with expression statement: Let $\operatorname{cfg}_i^1[\![\operatorname{return} E]\!](t,t_{\operatorname{b}},t_{\operatorname{c}},m)$ be defined as in (20). By Lemma 1, $\mathbf{p}_i(E) = \alpha(\operatorname{cfg}_i^1[\![E]\!](m,m,m+1))$. Then, for each path in (s_B,A_B,s) to s, there are 0 paths to t and $\mathbf{p}_i(E)$ paths to a return node. Moreover, as there are no break, continue or goto statements inside S, $\beta(S,B) = \chi(S,B) = 0$ and for each $\operatorname{id} \in l(B)$, $\gamma(\operatorname{id},S,B) = 0$. Hence, by (40), Hence,

$$\begin{aligned} & \text{ft}_{\text{out}} = 0 \\ & = \nu(t, S, B), \\ & \text{bp} = 0 \\ & = \beta(S), \\ & \text{cp} = 0 \end{aligned}$$

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\begin{split} &=\chi(S),\\ \mathsf{rp}&=\mathsf{ft}\,\mathbf{p}_i(E)\\ &=\tau(s_B,A_B,s)\tau\big(\mathsf{cfg}_i^1[\![E]\!](m,m,m+1),m\big)\\ &=\rho(S),\\ \mathsf{gt}_\mathsf{out}&=\mathsf{gt}\\ &=\mathsf{gt}+\lambda\mathrm{id}\in l(B)\cdot 0\\ &=\mathsf{gt}+\lambda\mathrm{id}\in l(B)\cdot \gamma(\mathrm{id},S,B). \end{split}
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Break statement: By (27), $\operatorname{cfg}_1^1[\operatorname{break}](t,t_{\operatorname{b}},t_{\operatorname{c}},m)=(\{m,t_{\operatorname{b}}\},\{(m,t_{\operatorname{b}})\},m)$. Then s=m and, for each path in (s_B,A_B,s) to s there are 0 acyclic paths to t and 1 path to the break node. Moreover, as there are no continue, return or goto statements inside S, $\chi(S,B)=\rho(S,B)=0$ and, for each $\operatorname{id}\in l(B), \gamma(\operatorname{id},S,B)=0$. Hence, by (47),

$$\begin{split} &\mathsf{ft}_\mathsf{out} = 0 \\ &= \nu(t, S, B), \\ &\mathsf{bp} = \mathsf{ft} \\ &= \tau(s_B, A_B, s) \\ &= \beta(S), \\ &\mathsf{cp} = 0 \\ &= \chi(S), \\ &\mathsf{rp} = 0 \\ &= \rho(S), \\ &\mathsf{gt}_\mathsf{out} = \mathsf{gt} \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot 0 \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S, B). \end{split}$$

Continue statement: By (28), $\operatorname{cfg}_i^1[\![\operatorname{continue}]\!](t,t_b,t_c,m) = (\{m,t_b\},\{(m,t_c)\},m)$. Then s=m and, for each path in (s_B,A_B,s) to s, there are 0 paths to t and 1 path to the continue node. Moreover, as there are no break, return or goto statements inside S, $\beta(S,B)=\rho(S,B)=0$ and, for each $\operatorname{id}\in l(B)$, $\gamma(\operatorname{id},S,B)=0$. Hence, by (48),

$$\begin{split} &\mathsf{ft}_{\mathsf{out}} = 0 \\ &= \nu(t, S, B) \\ &\mathsf{bp} = 0 \\ &= \beta(S), \\ &\mathsf{cp} = \mathsf{ft} \\ &= \tau(s_B, A_B, s) \\ &= \chi(S), \\ &\mathsf{rp} = 0 \\ &= \rho(S), \\ &\mathsf{gt}_{\mathsf{out}} = \mathsf{gt} \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot 0 \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S, B). \end{split}$$

Goto statement: Let cfg! [goto id] (t,t_b,t_c,m) be defined as in (29). Then, for each path in (s_B,A_B,s) to s, there are 0 paths to t and 1 path to m where $\operatorname{orig}(m)=\operatorname{id}$; hence $\gamma(\operatorname{id},S,B)=\tau(s_B,A_B,s)$. Moreover, as there are no break, continue or return statements inside S, $\beta(S,B)=\chi(S,B)=\rho(S,B)=0$ and also for each $\operatorname{id}_1\in l(B)\neq\operatorname{id}$, $\gamma(\operatorname{id}_1,S)=0$.

Hence, by (49),

$$\begin{split} &\mathsf{ft}_\mathsf{out} = 0 \\ &= \nu(t, S, B), \\ &\mathsf{bp} = 0 \\ &= \beta(S) \mathsf{ft}, \\ &\mathsf{cp} = 0 \\ &= \chi(S) \mathsf{ft}, \\ &\mathsf{rp} = 0 \\ &= \rho(S) \mathsf{ft}, \\ &\mathsf{gt}_\mathsf{out} = \mathsf{gt} \big[\big(\mathsf{gt}(\mathsf{id}) + \mathsf{ft} \big) / \mathsf{id} \big] \\ &= \mathsf{gt} + \lambda \mathsf{id}_1 \in l(B) \cdot \begin{cases} \mathsf{ft}, & \text{if } \mathsf{id}_1 = \mathsf{id}, \\ 0, & \text{otherwise}; \\ &= \mathsf{gt} + \lambda \mathsf{id} \in l(B) \cdot \gamma(\mathsf{id}, S, B). \end{cases} \end{split}$$

Other statements: By (32), $\operatorname{cfg}_1^1[\operatorname{Im}](t,t_{\operatorname{b}},t_{\operatorname{c}},m)=(\{t\},\varnothing,t)$ and s=t. Moreover, as there are no break, continue, return or goto statements inside $S,\ \beta(S,B)=\chi(S,B)=\rho(S,B)=0$ and, for each $\operatorname{id}\in l(B),\ \gamma(\operatorname{id},S)=0$. Hence, by (52),

$$\begin{split} & \mathsf{ft}_{\mathsf{out}} = \mathsf{ft} \\ &= \nu(s, S, B) \\ &= \nu(t, S, B), \\ & \mathsf{bp} = 0 \\ &= \beta(S), \\ & \mathsf{cp} = 0 \\ &= \chi(S), \\ & \mathsf{rp} = 0 \\ &= \rho(S), \\ & \mathsf{gt}_{\mathsf{out}} = \mathsf{gt} \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot 0 \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S, B). \end{split}$$

Sequential composition: Let $\operatorname{cfg}_i^1 \llbracket S_1 \ S_2 \rrbracket (t,t_b,t_c,m) = (N,A,s)$ be as defined in (18). Since $\operatorname{\mathsf{gt}}_1 = \operatorname{\mathsf{gt}} + \lambda \operatorname{id} \in l(B)$. $\gamma(\operatorname{id},S_1,B) = \lambda \operatorname{id} \in l(B)$. $\phi(\operatorname{id},S_2,B)$, using the inductive hypothesis on S_1 , we can use the inductive hypothesis on S_2 . Moreover, because we have $b(s_1,m_1) \cap b(s_2,m) = c(s_1,m_1) \cap c(s_2,m) = r(s_1,m_1) \cap r(s_2,m) = \varnothing$ the paths to break, continue or return statements are, respectively, $\beta(S_1 \ S_2,B) = \beta(S_1) + \beta(S_2)$, $\chi(S_1 \ S_2,B) = \chi(S_1) + \chi(S_2)$, $\rho(S_1 \ S_2,B) = \rho(S_1) + \rho(S_2)$ and, for each $\operatorname{id} \in l(B)$, since $g(\operatorname{id},S_1,B) \cap g(\operatorname{id},S_2,B) = \varnothing$, $\gamma(\operatorname{id},S_1 \ S_2,B) = \gamma(\operatorname{id},S_1,B) + \gamma(\operatorname{id},S_2,B)$. Hence, by (38),

$$\begin{split} \mathsf{ft}_{\mathsf{out}} &= \mathsf{ft}_2 \\ &= \nu(t, S_2, B) \\ &= \nu(t, S_1 \ S_2, B), \\ \mathsf{bp} &= \mathsf{bp}_1 + \mathsf{bp}_2 \\ &= \beta(S_1) + \beta(S_2) \\ &= \beta(S_1 \ S_2), \\ \mathsf{cp} &= \mathsf{cp}_1 + \mathsf{cp}_2 \\ &= \chi(S_1) \mathsf{ft} + \chi(S_2) \mathsf{ft}_1 \\ &= \chi(S_1 \ S_2), \end{split}$$

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\begin{split} \mathsf{bp} &= \mathsf{bp}_1 + \mathsf{bp}_2 \\ &= \rho(S_1)\mathsf{ft} + \rho(S_2)\mathsf{ft}_1 \\ &= \rho(S_1 \ S_2), \\ \mathsf{gt}_\mathsf{out} &= \mathsf{gt}_2 \\ &= \mathsf{gt}_1 + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S_2, B) \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S_1, B) + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S_2, B) \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S_1 \ S_2, B). \end{split}
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Conditional statement: There are three cases:

$$\operatorname{tv}_i(E) = 1 \land M_2^{\operatorname{s}} = \emptyset$$
: then $\mathbf{t}_i(E) = 1$ and $\mathbf{f}_i(E) = 0$. By (21), $\operatorname{cfg}_i^1[\![\mathbf{if}(E) \ S_1 \ \text{else} \ S_2]\!](t, t_{\operatorname{b}}, t_{\operatorname{c}}, m) = \operatorname{cfg}_i^1[\![S_1]\!](t, t_{\operatorname{b}}, t_{\operatorname{c}}, m).$

Hence $\mathbf{f}_i(E)$ ft = 0 so that $\mathsf{ft}_2 = 0$. Since S_2 does not contain any labeled statements, it is not possible to jump into S_2 so that $\mathsf{bp}_2 = \mathsf{cp}_2 = \mathsf{rp}_2 = 0$ and, also for each $\mathsf{id} \in l(B)$, $\gamma(\mathsf{id}, S_2, B) = 0$. Hence, by (41) and the inductive hypothesis on S_1 ,

$$\begin{split} &\text{ft}_{\text{out}} = \text{ft}_1 + \text{ft}_2 \\ &= \text{ft}_1 \\ &= \nu(t, S_1, B), \\ &\text{bp} = \text{bp}_1 + \text{bp}_2 \\ &= \text{bp}_1 \\ &= \beta(S_1) \\ &= \beta\big(\text{if }(E) \, S_1 \, \text{else} \, S_2\big), \\ &\text{cp} = \text{cp}_1 + \text{cp}_2 \\ &= \text{cp}_1 \\ &= \chi(S_1) \, \mathbf{t}_i(E) \\ &= \chi\big(\text{if }(E) \, S_1 \, \text{else} \, S_2\big), \\ &\text{rp} = \text{rp}_1 + \text{rp}_2 \\ &= \text{rp}_1 \\ &= \rho\big(\text{if }(E) \, S_1 \, \text{else} \, S_2\big), \\ &\text{gt}_{\text{out}} = \text{gt}_2 \\ &= \text{gt}_1 + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_2, B) \\ &= \text{gt}_1 \\ &= \text{gt}_1 + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_1, B) \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_1, B), \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_1, B), \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_1, B), \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_1, B), \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_1, B), \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_1, B). \end{split}$$

 $\operatorname{tv}_i(E) = 0 \wedge M_1^s = \varnothing$: the proof is similar to the previous case;

otherwise: Let $\mathrm{cfg}_1^1[\![\mathrm{if}\ (E)\ S_1\ \mathrm{else}\ S_2]\!](t,t_b,t_c,m)$ be as defined in (21). For each path to m or m_1 there is 1 path to t, using the arcs (m,t) or (m_1,t) . Furthermore, since $\mathbf{t}_i(E)$ ft = $\nu(s_1,S_1,B)$, we can use the inductive hypothesis on S_1 , and since $\mathbf{f}_i(E)$ ft = $\nu(s_2,S_2,B)$ and $\mathsf{gt}_1=\mathsf{gt}+\lambda\mathrm{id}\in l(B)$. $\gamma(\mathrm{id},S_1,B)=\lambda\mathrm{id}\in l(B)$. $\phi(\mathrm{id},S_2,B)$ we can also use the inductive hypothesis on S_2 . Moreover, as $b(s_1,m_1)\cap b(s_2,m)=c(s_1,m_1)\cap c(s_2,m)=r(s_1,m_1)\cap r(s_2,m)=\varnothing$ the paths to break, continue or return statements are $\beta(S_1,S_2,B)=\beta(S_1)+\beta(S_2)$, $\chi(S_1,S_2,B)=\chi(S_1)+\chi(S_2)$, $\rho(S_1,S_2,B)=\rho(S_1)+\rho(S_2)$ and for each id $\in l(B)$, since $g(\mathrm{id},S_1,B)\cap g(\mathrm{id},S_2,B)=\varnothing$, $\gamma(\mathrm{id},S_1,S_2)=$

 $\gamma(\mathrm{id}, S_1, B) + \gamma(\mathrm{id}, S_2, B)$. Hence, by (41) and the inductive hypothesis on S_1 ,

$$\begin{split} &\text{ft}_{\text{out}} = \text{ft}_1 + \text{ft}_2 \\ &= \nu(m, S_1, B) + \nu(m_1, S_2, B) \\ &= \tau(N_B, A_f, m) + \tau(N_B, A_f, m_1) \\ &= \nu(t, B), \\ &\text{bp} = \text{bp}_1 + \text{bp}_2 \\ &= \beta(S_1) + \beta(S_2) \\ &= \beta\left(\text{if }(E) \ S_1 \ \text{else} \ S_2\right), \\ &\text{cp} = \text{cp}_1 + \text{cp}_2 \\ &= \chi(S_1) + \chi(S_2) \\ &= \chi\left(\text{if }(E) \ S_1 \ \text{else} \ S_2\right), \\ &\text{bp} = \text{bp}_1 + \text{bp}_2 \\ &= \rho(S_1) + \rho(S_2) \\ &= \rho\left(\text{if }(E) \ S_1 \ \text{else} \ S_2\right), \\ &\text{gt}_{\text{out}} = \text{gt}_2 \\ &= \text{gt}_1 + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_2, B) \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_1, B) + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_2, B) \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, \text{if }(E) \ S_1 \ \text{else} \ S_2, B). \end{split}$$

 ${\bf One\text{-}armed\ conditional\ statement:}\ {\bf There\ are\ three\ cases:}$

$$\begin{split} \operatorname{tv}_i(E) = 1 \colon & \text{ then } \mathbf{t}_i(E) = 1 \text{ and } \mathbf{f}_i(E) = 0. \text{ By (22),} \\ & \operatorname{cfg}_i^1 \llbracket \operatorname{if} (E) \, S_1 \rrbracket(t, t_{\operatorname{b}}, t_{\operatorname{c}}, m) = \operatorname{cfg}_i^1 \llbracket S_1 \rrbracket(t, t_{\operatorname{b}}, t_{\operatorname{c}}, m). \end{split}$$

For each path in (N_B, A_B, m) to m there is 1 path to t using the arc (m, t). Hence, by (42) and the inductive hypothesis on S_1 ,

$$\begin{split} &\mathsf{ft}_{\mathsf{out}} = \mathsf{ft}_1 + \mathsf{f}_i(E)\mathsf{ft} \\ &= \mathsf{ft}_1 \\ &= \nu(t, S, B), \\ &\mathsf{bp} = \mathsf{bp}_1 \\ &= \beta(S_1) \\ &= \beta(\mathsf{if}\ (E)\ S_1), \\ &\mathsf{cp} = \mathsf{cp}_1 \\ &= \chi(S_1) \\ &= \chi(\mathsf{if}\ (E)\ S_1), \\ &\mathsf{rp} = \mathsf{rp}_1 \\ &= \rho(\mathsf{s}_1) \\ &= \rho(\mathsf{if}\ (E)\ S_1), \\ &\mathsf{gt}_{\mathsf{out}} = \mathsf{gt}_1 \\ &= \mathsf{gt} + \lambda \mathsf{id} \in l(B)\ .\ \gamma(\mathsf{id}, S_1, B) \\ &= \mathsf{gt} + \lambda \mathsf{id} \in l(B)\ .\ \gamma(\mathsf{id}, S, B). \end{split}$$

$$\operatorname{tv}_i(E) = 0 \wedge M_1^{\operatorname{s}} = \varnothing \colon \text{ Then } \mathbf{t}_i(E) = 0 \text{ and } \mathbf{f}_i(E) = 1. \text{ By } (22),$$

$$\operatorname{cfg}_{i}^{1}[if(E) S_{1}](t, t_{b}, t_{c}, m) = (\{t\}, \emptyset, t).$$

Hence $\mathsf{ft}_1 = 0$. Since S_1 does not contain any labeled statements, it is not possible to jump into S_1 so that $\mathsf{bp}_1 = \mathsf{cp}_1 = \mathsf{rp}_1 = 0$ and, also, for each $\mathsf{id} \in l(B)$, $\gamma(\mathsf{id}, S_1, B) = 0$. Hence, by (42),

$$\begin{split} &\mathsf{ft}_{\mathsf{out}} = \mathsf{ft}_1 + \mathbf{f}_i(E) \mathsf{ft} \\ &= \mathsf{ft} \\ &= \nu(t, S, B), \\ &\mathsf{bp} = \mathsf{bp}_1 \\ &= \beta(S_1) \\ &= 0 \\ &= \beta \big(\mathsf{if} \ (E) \ S_1 \big), \\ &\mathsf{cp} = \mathsf{cp}_1 \\ &= \chi(S_1) \\ &= 0 \\ &= \chi \big(\mathsf{if} \ (E) \ S_1 \big), \\ &\mathsf{rp} = \mathsf{rp}_1 \\ &= \rho(S_1) \\ &= 0 \\ &= \rho \big(\mathsf{if} \ (E) \ S_1 \big), \\ &\mathsf{gt}_{\mathsf{out}} = \mathsf{gt}_1 \\ &= \mathsf{gt} + \lambda \mathsf{id} \in l(B) \ . \ \gamma(\mathsf{id}, S_1, B) \\ &= \mathsf{gt} + \lambda \mathsf{id} \in l(B) \ . \ 0 \\ &= \mathsf{gt} + \lambda \mathsf{id} \in l(B) \ . \ \gamma(\mathsf{id}, S, B). \end{split}$$

otherwise: Let $\operatorname{cfg}_i^1[\![if(E)S_1]\!](t,t_{\operatorname{b}},t_{\operatorname{c}},m)$ be as defined in (22). By Lemma 1, $\mathbf{t}_i(E) = \tau(\operatorname{cfg}_i^1[\![E]\!](s_1,t,m),s_1)$ and $\mathbf{f}_i(E) = \tau(\operatorname{cfg}_i^1[\![E]\!](s_1,t,m),t)$. Moreover, for each path from s to s_1 in $(\operatorname{cfg}_i^1[\![E]\!](s_1,t,m)$, there are $\tau(\operatorname{cfg}_i^1[\![S_1]\!](m,t_{\operatorname{b}},t_{\operatorname{c}},m+1),m)$ paths to t. Since $\mathbf{t}_i(E)$ ft = $\nu(s_1,B)$, we can use the inductive hypothesis on S_1 . Hence, by (42),

$$\begin{split} & \text{ft}_{\text{out}} = \text{ft}_1 + \mathbf{f}_i(E) \text{ft} \\ &= \nu(m, S_1, B) + \mathbf{f}_i(E) \nu(s, B) \\ &= \nu(t, S, B), \\ & \text{bp} = \text{bp}_1 \\ &= \beta(S_1) \\ &= \beta(\text{if } (E) \, S_1), \\ & \text{cp} = \text{cp}_1 \\ &= \chi(S_1) \\ &= \chi(\text{if } (E) \, S_1), \\ & \text{rp} = \text{rp}_1 \\ &= \rho(S_1) \\ &= \rho(\text{if } (E) \, S_1), \\ & \text{gt}_{\text{out}} = \text{gt}_1 \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_1, B) \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma(\text{id}, S_1, B). \end{split}$$

 ${\bf Switch\ statement} :$ There are two cases:

 $(\mathrm{df},n)\in M_1^s$: Let $\mathrm{cfg}_i^1[\![\mathtt{switch}\ (E)\ S_1]\!](t,t_b,t_c,m)$ be as defined in (23). By Lemma 1, there are $\mathbf{p}_i(E)$ paths from s to m_1 . Moreover, since arc $(m_1,s_S)\in A$ only if $\mathrm{orig}(s_S)\in \{\mathtt{case},\mathtt{default}\},\ \nu(s_S,B)=0$ so that $m_1\in t(S,B)$. In addition to the $\nu(m,B)$ paths that fall through to m, there are $\beta(S_1)$ paths that exit from switch via break nodes. Furthermore, by Definition (69), $b(s,t)=\varnothing$. Also $\mathbf{p}_i(E)$ ft = $\tau(N_B,A_f,m_1)=\sigma(S_1,B)$. Hence, by (43) and applying the inductive hypothesis to S_1 ,

$$\begin{split} &\mathsf{ft}_\mathsf{out} = \mathsf{ft}_S + \mathsf{bp}_S \\ &= \nu(m, S, B) + \beta(S_1, B) \\ &= \nu(t, S, B), \\ &\mathsf{bp} = 0 \\ &= \beta \big(\mathsf{switch} \ (E) \ S_1 \big), \\ &\mathsf{cp} = \mathsf{cp}_S \\ &= \chi(S_1) \\ &= \chi \big(\mathsf{switch} \ (E) \ S_1 \big), \\ &\mathsf{rp} = \mathsf{rp}_S \\ &= \rho(S_1) \\ &= \rho \big(\mathsf{switch} \ (E) \ S_1 \big), \\ &\mathsf{gt}_\mathsf{out} = \mathsf{gt}_S \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S_1, B) \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, \mathsf{switch} \ (E) \ S_1, B). \end{split}$$

 $(\mathrm{df},n) \notin M_1^s$: Let $\mathrm{cfg}_1^1[\![\mathtt{switch}\,(E)\,S_1]\![(t,t_b,t_c,m)]$ be as defined in (23). Then there is 1 path from m_1 to m using the arc (m_1,m) . By Lemma 1, there are $\mathbf{p}_i(E)$ paths to m_1 . Moreover, since arc $(m_1,s_S) \in A$ only if $\mathrm{orig}(s_S) \in \{\mathtt{case},\mathtt{default}\}, \, \nu(s_S,B)=0$. In addition to the $\nu(m,B)$ paths that fall through to m, there are $\beta(S_1)$ nodes that exit from switch via break nodes and $\mathbf{p}_i(E)$ ft paths that reach m via (m_1,m) . Furthermore, by Definition (69), $b(s,t)=\varnothing$. Also, since $0=\nu(s_S,B)$, $\mathbf{p}_i(E)$ ft = $\tau(N_B,A_f,m_1)=\sigma(S_1,B)$. Hence, by (43) and applying the inductive hypothesis to S_1 ,

$$\begin{split} & \text{ft}_{\text{out}} = \text{ft}_S + \text{bp}_S + \text{p}_i(E) \text{ft} \\ &= \nu(m, S, B) + \beta(S_1) + \text{p}_i(E) \text{ft} \\ &= \nu(t, S, B), \\ & \text{bp} = 0 \\ &= \beta \big(\text{switch} \ (E) \ S_1 \big), \\ & \text{cp} = \text{cp}_1 \\ &= \chi(S_1) \\ &= \chi \big(\text{switch} \ (E) \ S_1 \big), \\ & \text{rp} = \text{rp}_1 \\ &= \rho \big(\text{switch} \ (E) \ S_1 \big), \\ & \text{gt}_{\text{out}} = \text{gt}_1 \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma \big(\text{id}, S_1, B \big) \\ &= \text{gt} + \lambda \text{id} \in l(B) \cdot \gamma \big(\text{id}, \text{switch} \ (E) \ S_1, B \big). \end{split}$$

While statement: let $\operatorname{cfg}_i^1[\![\operatorname{while}\ (E)\ S_1]\!](t,t_{\operatorname{b}},t_{\operatorname{c}},m)=(N,A,s)$ be as defined in (24). Then $\operatorname{orig}(m_3)=$ while enter and $\operatorname{orig}(m)=$ while exit so that $b(s,t)=c(s,t)=\varnothing$. As B is a controlled function body, by Lemma 3, each acyclic path in (N,A,s) to t will include the node m_3 or m. Therefore the number of paths to t is the sum of:

- paths that go to t directly from E evaluating 0;
- paths that fall to an entry node for $\operatorname{cfg}(\llbracket i \rrbracket)(E)(t,f,m)$, E evaluating 1, and then exit from S_1 via a break node;
- paths that fall to S to an entry node for $\operatorname{cfg}_{[[i]]}(E)(t,f,m)$, E evaluating 1 and exit via m, (possibly via a continue) node) and then E evaluating 0;
- paths that jump into S from a goto id or switch node, exit via m, (possibly via a continue node) and then E evaluating 0.

Since $\mathbf{t}_i(E)$ ft = $\nu(m_1, B)$, we can use the inductive hypothesis on S_1 . Hence, by (44), letting $T := \mathbf{tf}_i(E)/\mathbf{t}_i(E)$ if $\mathbf{t}_i(E) \neq 0$ and T = 0 otherwise:

$$\begin{split} &\mathsf{ft}_{\mathsf{out}} = \mathbf{f}_i(E)\mathsf{ft} + \mathsf{bp}_S \, \mathbf{t}_i(E) + (\mathsf{ft}_S + \mathsf{cp}_S) \, \mathbf{tf}_i(E) \\ &= \mathbf{f}_i(E)\mathsf{ft} + \beta(S_1) \, \mathbf{t}_i(E) + \big(\nu(m_1, S, B) + \chi(S_1)\big) \mathbf{tf}_i(E) \\ &= \mathbf{f}_i(E)\mathsf{ft} + \beta(S_1) \, \mathbf{t}_i(E) \\ &\quad + \big(\nu(m_1, S_1, B) + \mathsf{ft} \sum_{n \in c(s, t)} \tau(m_1, A, n)\big) \, \mathbf{tf}_i(E) \\ &= \mathbf{f}_i(E)\mathsf{ft} + \beta(S_1) \, \mathbf{t}_i(E) \\ &\quad + \big(\nu(m_1, S_1, B) + \mathsf{ft} \sum_{n \in c(s, t)} \tau(m_1, A, n)\big) \mathbf{tf}_i(E) \\ &= \nu(t, S, B), \\ &\mathsf{bp} = 0 \\ &= \beta\big(\mathsf{while}\,(E) \, S_1\big), \\ &\mathsf{cp} = 0 \\ &= \chi\big(\mathsf{while}\,(E) \, S_1\big), \\ &\mathsf{rp} = \mathsf{rp}_1 \\ &= \rho(S_1) \\ &= \rho\big(\mathsf{while}\,(E) \, S_1\big), \\ &\mathsf{gt}_{\mathsf{out}} = \mathsf{gt}_1 \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, S_1, B) \\ &= \mathsf{gt} + \lambda \mathrm{id} \in l(B) \cdot \gamma(\mathrm{id}, \mathsf{while}\,(E) \, S_1\big). \end{split}$$

Do while statement: let $\operatorname{cfg}_i^1[\![\operatorname{do} S_1 \text{ while } (E)]\!](t,t_b,t_c,m)$ be as defined in (25). Then $\operatorname{orig}(m_1)=\operatorname{do}$ — while enter and $\operatorname{orig}(m)=\operatorname{do}$ — while exit so that $b(s,t)=c(s,t)=\varnothing$. As B is a controlled function body, by Lemma 3, each acyclic path in (N,A,s) to t will include the nodes m or m_1 . Therefore the only paths to t through S are:

- those that fall to m_1 and go to t via a break node; and
- those that fall through to m, (possibly via a continue node) and then E evaluating 0.

Since $\mathsf{ft} = \nu(m_1, S, B)$, we can use the inductive hypothesis on S_1 . Hence, by (45),

$$\begin{split} & \text{ft}_{\text{out}} = \mathbf{f}_i(E) \text{ft}_S + \text{bp}_S \\ &= \nu(m, S, B) \, \mathbf{f}_i(E) + \beta(S_1, B) \\ &= \nu(t, S, B), \\ & \text{bp} = 0 \\ &= \beta \big(\text{do } S_1 \text{ while } (E) \big), \\ & \text{cp} = 0 \\ &= \chi \big(\text{do } S_1 \text{ while } (E) \big), \\ & \text{rp} = \text{rp}_1 \\ &= \rho(S_1) \end{split}$$

$$\begin{split} &= \rho \big(\text{do } S_1 \text{ while } (E) \big), \\ \text{gt}_{\text{out}} &= \text{gt}_1 \\ &= \text{gt} + \lambda \text{id} \in l(B) \ . \ \gamma (\text{id}, S_1, B) \\ &= \text{gt} + \lambda \text{id} \in l(B) \ . \ \gamma \big(\text{id}, \text{do } S_1 \text{ while } (E) \big). \end{split}$$

For statement: the lemma is true by inductive hypothesis for sequential composition and while statement.

Labeled statement: let $\operatorname{cfg}_{i}^{1}[L:S_{1}](t,t_{b},t_{c},m)=(N,A,s)$ be as defined in (30). By Lemma 4, $\operatorname{ft}_{L}=\nu(s_{S},S,B)$ so that, by (50), lemma is true by the inductive hypothesis on S_{1} .

Compound statement: by (31), $\operatorname{cfg}_i[[S_1]](t, t_b, t_c, m) = \operatorname{cfg}_i[[S_1]](t, t_b, t_c, m)$ so that, by (51), the lemma is true by inductive hypothesis on S_1 .

 $\begin{aligned} \textbf{Corollary 1} & \ Let \ B \in \operatorname{Stm} \ be \ a \ full \ C \ controlled \ function \ body, \ if \ \operatorname{apc}_i[\![B]\!](1,0,\lambda \mathrm{id} \in l(B) \ . \\ 0) & = (\mathsf{ft}_{\mathsf{out}},\mathsf{bp},\mathsf{cp},\mathsf{rp},\mathsf{gt}_{\mathsf{out}}) \ \ then \ \alpha(\operatorname{cfg}_i^b[\![B]\!]) = \mathsf{ft}_{\mathsf{out}} + \mathsf{rp} \end{aligned}$

Proof Let $\operatorname{cfg}_i^b[\![B]\!] = (N,A,s)$; then $\nu(s,B,B) = \tau(s,A,s) = 1$, $t(B,B) = \varnothing$, and, for all id $\in l(B)$, $\phi(\operatorname{id},B,B) = 0$. Concluding, by Lemma 5, $\operatorname{ft_{out}} = \nu(0,B,B) = \alpha(N,A,0)$ as $\operatorname{orig}(0) \notin \{\operatorname{case},\operatorname{default},\operatorname{goto}\operatorname{id}\}$ and $\operatorname{rp} = \rho(B,B)$. Then the number of paths leading to an exit node are $\operatorname{ft_{out}} + \operatorname{rp}$.

Theorem 2 Let $B \in \text{Stm}$ be a controlled function body. Then

$$\operatorname{apc}_{i}^{\operatorname{b}}[\![B]\!] = \alpha \left(\operatorname{cfg}_{i}^{\operatorname{b}}[\![B]\!]\right).$$

Proof By Definition 9 and Corollary 1:

$$\operatorname{apc}_{i}^{b}[\![B]\!] = \operatorname{ft_{out}} + \operatorname{rp}$$

= $\alpha(\operatorname{cfg}_{i}^{b}[\![B]\!]).$

B Example Reference CFGs

We now illustrate the CFGs built according to Definition 3 by means of examples. In the examples, all expressions represented by "are assumed to be non-constants and to have a trivial control flow. Similarly, all statements represented by '-' are assumed to result into a basic block, i.e., a single node in the CFG. Moreover, the CFGs have been simplified by removing all nodes and arcs that are unreachable from the entry node, which is represented by a diamond-shaped box. Exit nodes are emphasized by being enclosed into double circles. All the drawings have been obtained automatically from an executable version of Definition 3. Figure 4 shows the CFG generated by a command containing return statements and branching expressions. Figure 5 shows the effect of break and continue in a while statement. Figures 6 and 7 show the difference between while and do - while statements: for do - while the backward arc that generates the loop cannot be crossed in an acyclic path, whereas this is allowed in while statements as the guard expression can be evaluated twice also in acyclic paths. Figures 8 and 9 illustrates CFGs generated from switch statements: the former shows the effect of break statements in switches, the latter is a reduced version of Duff's device. Figure 10 shows the CFG genereted for a program containing a nasty use of goto statements, which is perfectly legal in C: jumping from one of the branches to the other in if-then-else statements.

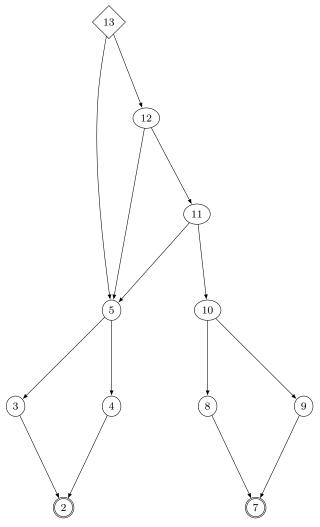
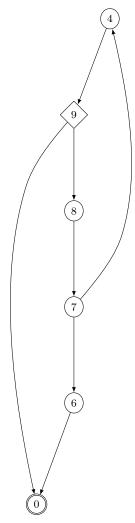


Fig. 4 CFG for { if (\cdot && \cdot && \cdot) return (\cdot ? 0 : 1); else return (\cdot ? 0 : 1); }: 8 acyclic paths



 $\textbf{Fig. 5} \ \ \text{CFG for } \big\{ \, \texttt{while} \, (\cdot) \, \, \texttt{if} \, (\cdot) \, \, \texttt{break}; \, \, \texttt{else continue}; \, \big\} \\ : \, 3 \, \, \texttt{acyclic paths}$

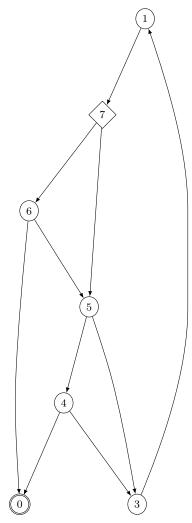
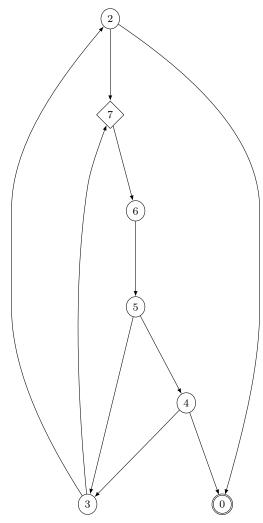
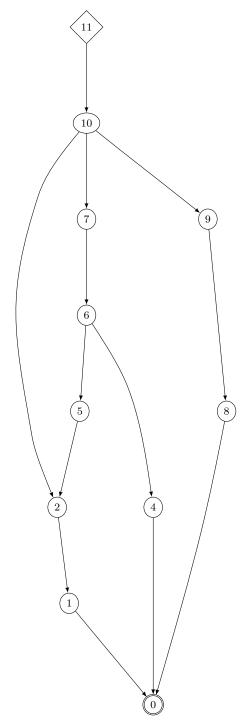


Fig. 6 CFG for $\{ \text{ while } ((\cdot \mid| \cdot) \&\& (\cdot \mid| \cdot)) - \} : 7 \text{ acyclic paths}$



 $\mathbf{Fig.~7}~\mathrm{CFG~for~}\big\{\,\mathtt{do-while}~((\cdot\mid\mid\cdot)\,\&\&~(\cdot\mid\mid\cdot))\,\big\}{:}~3~\mathrm{acyclic~paths}$



 $\begin{aligned} &\textbf{Fig. 8} \ \text{CFG for } \big\{ \, \texttt{switch} \, (\cdot) \, \big\{ \, \texttt{case} \, 1 \colon \big\{ - \, \texttt{break}; \, \big\} \, \texttt{case} \, 2 \colon \texttt{if} \, (\cdot) \, - \, \texttt{else} \, \big\{ - \, \texttt{break}; \, \big\} \, \texttt{default:} \, - \, \big\} \, \big\} \colon \\ &4 \, \text{acyclic paths} \end{aligned}$

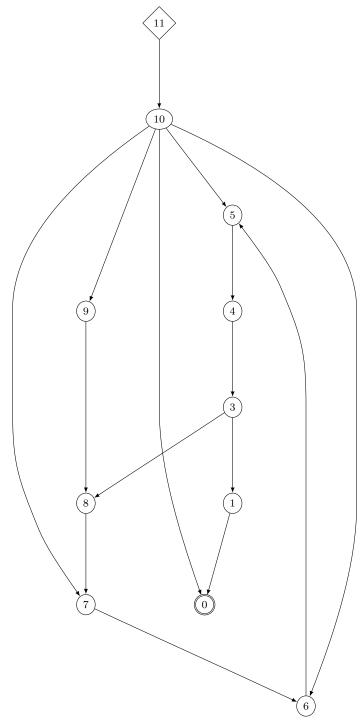


Fig. 9 CFG for $\{$ switch (\cdot) case 0: do $\{$ - case 1: - case 2: - case 3: - $\}$ while (\cdot) $\}$: 5 acyclic paths



 $\textbf{Fig. 10} \ \ \text{CFG for } \big\{ \ \text{if } (\cdot) \ \text{goto 11}; \ \text{else 11: goto 12}; \ \text{while } (\cdot)-12:-\big\} : \ 2 \ \text{acyclic paths} \\$