Class Book: A Case Study in Multivariable Calculus

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Saturday, November 16, 2013

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Lemma Sheet

Formula 14 - Lemma 1:
If $u : \mathbb{R}^n \to \mathbb{R}^m$ and $v : \mathbb{R}^m \to \mathbb{R}^k$ and $w : \mathbb{R}^k \to \mathbb{R}^l$, then

$$(w \circ v) \circ u = w \circ (v \circ u)$$

Proof of Lemma 1:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = (w \circ v) \circ u$
$g = w \circ (v \circ u)$
$D = \mathbb{R}^n$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}^n}{\forall} ((w \circ v) \circ u)(a) = (w \circ (v \circ u))(a)$$

Now, the computer has:

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = w \circ v$
$g = u$
$x = a$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}^n}{\forall} ((w \circ v) \circ u)(a) = (w \circ v)(u(a))$$

Step 0 (Identify Formula):
Formula 4
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$g = u$
$x = a$
$a^1 = u^1$, $a^2 = u^2$,..., $a^m = u^m$
Step 3 (Replacing Symbols):

$$= (w \circ v)(u^1(a), u^2(a), ..., u^m(a))$$

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = w$
$g = v$
$x = (u^1(a), u^2(a), ..., u^m(a))$
Step 3 (Replacing Symbols):

$$= w(v((u^1(a), u^2(a), ..., u^m(a)))$$

Step 0 (Identify Formula):
Formula 4
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$g = u$
$a^1 = u^1$, $a^2 = u^2$,..., $a^m = u^m$
$x = a$
Step 3 (Replacing Symbols):

$$= w(v((u(a)))$$

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$g = u$
$f = v$
$x = a$
Step 3 (Replacing Symbols):

$$= w((v \circ u)(a))$$

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$g = v \circ u$
$f = w$
$x = a$
Step 3 (Replacing Symbols):

$$= w \circ ((v \circ u)(a))$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f = (w \circ v) \circ u$
$g = w \circ (v \circ u)$
Step 3 (Replacing Symbols):

$$(w \circ v) \circ u = w \circ (v \circ u)$$

End of Proof of Lemma 1

Formula 16 - Lemma 2:

$$p_k \circ p_l = p_{k \cdot l}$$

Proof of Lemma 2:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_k \circ p_l$
$g = p_{k \cdot l}$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}}{\forall} (p_k \circ p_l)(a) = p_{k \cdot l}(a)$$

Now, the computer has:

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side

Step 2 (Identify Symbols):
$f = p_k$
$g = p_l$
$x = a$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}}{\forall} (p_k \circ p_l)(a) = p_k(p_l(a))$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_l$
$x = a$
Step 3 (Replacing Symbols):

$$= p_k(a^l)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_k$
$x = a^l$
Step 3 (Replacing Symbols):

$$= (a^l)^k = a^{k \cdot l}$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$x = a$
$a = k \cdot l$
Step 3 (Replacing Symbols):

$$= p_{k \cdot l}(a)$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f = p_k \circ p_l$
$g = p_{k \cdot l}$
Step 3 (Replacing Symbols):

$$p_k \circ p_l = p_{k \cdot l}$$

End of Proof of Lemma 2

Formula 17 - Lemma 3:

$$\text{If } k > 0 \text{ then } p_k \circ \mathbf{0} = \mathbf{0}$$

Proof of Lemma 3:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_k \circ \mathbf{0}$
$g = \mathbf{0}$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}}{\forall} (p_k \circ \mathbf{0})(a) = \mathbf{0}(a)$$

Now, the computer has:
Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_k$
$g = \mathbf{0}$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}}{\forall} (p_k \circ \mathbf{0})(a) = p_k(\mathbf{0}(a)) = p_k(0)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_k$
$x = 0$
Step 3 (Replacing Symbols):

$$= 0^k = 0 = \mathbf{0}(a)$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f = p_k \circ \mathbf{0}$
$g = \mathbf{0}$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$p_k \circ \mathbf{0} = \mathbf{0}$$

End of Proof of Lemma 3

Formula 18 - Lemma 4:

$$\text{If } f : \mathbb{R} \to \mathbb{R} \text{ then } f \circ p_1 = f$$

Proof of Lemma 4:

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = f \circ p_1$
$g = f$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$\forall_{a \in \mathbb{R}} (f \circ p_l)(a) = f(a)$$

Now, the computer has:

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = f$
$g = p_1$
$x = a$
Step 3 (Replacing Symbols):

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_1$
$x = a$
Step 3 (Replacing Symbols):

$$= f(a^1) = f(a)$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side

Step 2 (Identify Symbols):
$f = f \circ p_1$
$g = f$
Step 3 (Replacing Symbols):

$$f \circ p_1 = f$$

End of Proof of Lemma 4

Formula 19 - Lemma 5:

$$\text{If } f : \mathbb{R} \to \mathbb{R} \text{ and } c \in \mathbb{R} \text{ then } cp_0 \circ f = c$$

Proof of Lemma 5:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = c \circ f$
$g = c$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}}{\forall} (cp_0 \circ f)(a) = c(a)$$

Now, we have:

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = c$
$g = f$
$x = a$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}}{\forall} (cp_0 \circ f)(a) = cp_0(f(a))$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_0$
$x = f(a)$
Step 3 (Replacing Symbols):

$$c(f(a))^0 = c \cdot 1 = c$$

End of Proof of Lemma 5

Formula 20 - Lemma 6:

$$\text{If } f : A \to B \text{ where A and B are any set, then } p_1 \circ f = f$$

Proof of Lemma 6:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = cp_k \circ \mathbf{0}$
$g = \mathbf{0}$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{A}}{\forall} (p_1 \circ f)(a) = f(a)$$

Now, the computer has:
Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_1$
$g = f$
$x = a$ **Step 3 (Replacing Symbols):**

$$\underset{a \in \mathbb{A}}{\forall} (p_1 \circ f)(a) = p_1(f(a))$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_1$
$x = f$
Step 3 (Replacing Symbols):

$$= f(a)$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f = p_1 \circ f$
$g = f$
$D = \mathbb{A}$
Step 3 (Replacing Symbols):

$$p_1 \circ f = f$$

End of Proof of Lemma 6

Formula 21 - Lemma 7:
Let $\mathbf{a} : \mathbb{N} \to \mathbb{R},$ be a sequence in a metric space $(\mathbb{R}, d)$ and $\mathbf{a}(n) = c$ where $c \in \mathbb{R}$ then

$$lim\ \mathbf{a} = c$$

Proof of Lemma 7:

$$\diamondsuit > 0, \heartsuit \in \mathbb{R}$$

$\rho(\mathbf{a}(\heartsuit), c) < \diamondsuit$
$\equiv \rho(c, c) < \diamondsuit$
$\equiv 0 < \diamondsuit$
$\equiv \bar{S}_{\diamondsuit} = (-\infty, \infty) \equiv S_{\diamondsuit} = \bar{S}_{\diamondsuit} \cap \mathbb{N} = \mathbb{N}_k^C \wedge k = 0$
$\equiv lim\ \mathbf{a} = c$
End of Proof of Lemma 7

Formula 22 - Lemma 8:
Let $\mathbf{a}, \mathbf{b} : \mathbb{N} \to \mathbb{R}$ be two sequences that converge. Then $\mathbf{a} + \mathbf{b}$ is also convergent and

$$lim(\mathbf{a} + \mathbf{b}) = lim\ \mathbf{a} + lim\ \mathbf{b}$$

Proof of Lemma 8:

$$\epsilon > 0, \heartsuit \in \mathbb{N}$$
$$lim\ \mathbf{a} = g_1, lim\ \mathbf{b} = g_2, lim(\mathbf{a} + \mathbf{b}) = g$$

$\rho((\mathbf{a} + \mathbf{b})(\heartsuit), g_1 + g_2) < \epsilon$
$\equiv |\mathbf{a}(\heartsuit) + \mathbf{b}(\heartsuit) - g_1 - g_2| \leq |\mathbf{a}(\heartsuit) - g_1 + \mathbf{b}(\heartsuit) - g_2| \leq |\mathbf{a}(\heartsuit) - g_1| + |\mathbf{b}(\heartsuit) - g_2|$
$\forall \dfrac{\epsilon}{2}\ \exists n_\epsilon^1\ \forall \heartsuit > n_\epsilon^1 \implies |\mathbf{a}(\heartsuit) - g_1| < \dfrac{\epsilon}{2}$ and $\exists n_\epsilon^2\ \forall \heartsuit > n_\epsilon^2 \implies |\mathbf{b}(\heartsuit) - g_2| < \dfrac{\epsilon}{2}$
$\forall \epsilon\ \exists n_\epsilon = \max(n_\epsilon^1, n_\epsilon^2)\ \forall \heartsuit > n_\epsilon \implies |(\mathbf{a} + \mathbf{b})(\heartsuit) - (g_1 + g_2)| < \epsilon$
And so $lim(\mathbf{a} + \mathbf{b}) = g_1 + g_2 = lim\ \mathbf{a} + lim\ \mathbf{b}$.
End of Proof of Lemma 8

Formula 23 - Lemma 9:
Let $\mathbf{a}, \mathbf{b} : \mathbb{N} \to \mathbb{R}$ be two sequences that converge. Then $\mathbf{a} \cdot \mathbf{b}$ is also convergent and

$$lim(\mathbf{a} \cdot \mathbf{b}) = lim\ \mathbf{a} \cdot lim\ \mathbf{b}$$

Proof of Lemma 9:

$$\epsilon > 0, \heartsuit \in \mathbb{N}$$
$$lim\ \mathbf{a} = g_1, lim\ \mathbf{b} = g_2, lim(\mathbf{a} \cdot \mathbf{b}) = g$$

$\rho((\mathbf{a}\cdot\mathbf{b})(\heartsuit), g_1\cdot g_2) < \epsilon$
$\equiv |\mathbf{a}(\heartsuit)\cdot\mathbf{b}(\heartsuit) - g_1\cdot g_2| \leq |\mathbf{a}(\heartsuit)\cdot\mathbf{b}(\heartsuit) + g_1\cdot\mathbf{b}(\heartsuit) - g_1\cdot\mathbf{b}(\heartsuit) - g_1\cdot g_2|$
$\leq |\mathbf{b}(\heartsuit)(\mathbf{a}(\heartsuit) - g_1) + g_1(\mathbf{b}(\heartsuit) - g_2)| \leq |\mathbf{b}(\heartsuit)(\mathbf{a}(\heartsuit) - g_1)| + |g_1(\mathbf{b}(\heartsuit) - g_2)|$

Since we know $\mathbf{b}$ is convergent to g_2 then there $\exists n^2$ such that $\forall\heartsuit > n^2\ |b(\heartsuit) - g_2| < 1 \equiv |b| < |g_2| + 1$
So:
$|\mathbf{b}(\heartsuit)(\mathbf{a}(\heartsuit) - g_1)| \leq |\mathbf{b}(\heartsuit)|\,|(\mathbf{a}(\heartsuit) - g_1)| \leq (|g_2| + 1)\,|(\mathbf{a}(\heartsuit) - g_1)| < \frac{\epsilon}{2}$
$\equiv |\mathbf{a}(\heartsuit) - g_1| < \frac{\epsilon}{2\,|g_2| + 2}$

Since $\mathbf{a}$ is convergent to g_1 then,
$\forall\frac{\epsilon}{2}\ \exists n^1_\epsilon\ \forall\heartsuit > n^1_\epsilon \implies |\mathbf{a}(\heartsuit) - g_1| < \frac{\epsilon}{2\,|g_2| + 2} \equiv |\mathbf{b}(\heartsuit)(\mathbf{a}(\heartsuit) - g_1)| < \frac{\epsilon}{2}$
$\forall\frac{\epsilon}{2}\ \exists n^3_\epsilon\ \forall\heartsuit > n^3_\epsilon \implies |\mathbf{b}(\heartsuit) - g_2| < \frac{\epsilon}{2\,|g_1| + 2} \equiv |g_1(\mathbf{b}(\heartsuit) - g_2)| \leq (|g_1|+1)\,|(\mathbf{b}(\heartsuit) - g_2)| < \frac{\epsilon}{2}$

So, $\forall\epsilon\ \exists n_\epsilon = \max(n^1_\epsilon, n^2, n^3_\epsilon)\ \forall\heartsuit > n_\epsilon \implies |(\mathbf{a}\cdot\mathbf{b})(\heartsuit) - (g_1\cdot g_2)| < \epsilon$
And so $lim(\mathbf{a}\cdot\mathbf{b}) = g_1\cdot g_2 = lim\ \mathbf{a}\cdot lim\ \mathbf{b}$.
End of Proof of Lemma 9

Formula 24 - Lemma 10:
Let $\mathbf{a}:\mathbb{N}\to\mathbb{R}$ be a sequences that converges and $lim\ \mathbf{a} \neq 0 \land \forall$ n $\mathbf{a}(n) \neq 0$. Then $\frac{1}{\mathbf{a}}$ is also convergent and

$$lim(\frac{1}{\mathbf{a}}) = \frac{1}{lim\ \mathbf{a}}$$

Proof of Lemma 10:

$$\epsilon > 0, x \in \mathbb{N}$$
$$lim\ \mathbf{a} = g$$

$\rho(\frac{1}{\mathbf{a}(x)}, \frac{1}{g}) < \epsilon$
$\left|\frac{1}{\mathbf{a}(x)} - \frac{1}{g}\right| < \epsilon$
$\left|\frac{1}{\mathbf{a}(x)} - \frac{1}{g}\right| \leq \left|\frac{\mathbf{a}(x) - g}{\mathbf{a}(x)g}\right| \leq \frac{|\mathbf{a}(x) - g|}{|\mathbf{a}(x)|\,|g|}$

$a(x)$ is convergent to g, then
For $\epsilon = \frac{|g|}{2}\ \exists n^1$ such that $\forall x > n^1 \implies |\mathbf{a}(x) - g| < \frac{|g|}{2}$
$\equiv -\frac{|g|}{2} < \mathbf{a}(x) - g < \frac{|g|}{2}$
$\equiv g - \frac{|g|}{2} < \mathbf{a}(x) < g + \frac{|g|}{2}$

If $g > 0$ then, $0 < \frac{|g|}{2} < |\mathbf{a}(x)| < \frac{3\,|g|}{2}$

If $g < 0$ then, $\frac{3g}{2} < \mathbf{a}(x) < \frac{g}{2} < 0 \equiv 0 < -\frac{g}{2} < -\mathbf{a}(x) < -\frac{3g}{2}$
$\equiv 0 < \frac{|g|}{2} < |\mathbf{a}(x)| < \frac{3\,|g|}{2}$

So, $\frac{|\mathbf{a}(x) - g|}{|\mathbf{a}(x)|\,|g|} \leq \frac{|\mathbf{a}(x) - g|}{\frac{|g|}{2}\,|g|} \leq \frac{|\mathbf{a}(x) - g|}{\frac{g^2}{2}} < \epsilon \equiv |\mathbf{a}(x) - g| < \frac{g^2\epsilon}{2}$

Since $a(x)$ converges to g then $\exists n^2\ \forall x > n^2 \implies |\mathbf{a}(x) - g| < \frac{g^2\epsilon}{2}$

Then $\exists n_\epsilon = max(n^1, n^2)$ such that $\forall x > n_\epsilon \implies \left|\frac{1}{\mathbf{a}(x)} - \frac{1}{g}\right| < \epsilon$

We conclude then that: $\lim \frac{1}{\mathbf{a}} = \frac{1}{\lim \mathbf{a}}$

End of Proof of Lemma 10

Formula 25 - Lemma 11:

Let $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \to \mathbb{R}$ have a limit exist at a point $a \in R^n$, then the limit exists at the same point for $\mathbf{f} + \mathbf{g}$ and,

$\underset{a}{lim}(\mathbf{f} + \mathbf{g}) = \underset{a}{lim}\mathbf{f} + \underset{a}{lim}\mathbf{g}$

Proof of Lemma 11:

In this proof bars around each side represent the Euclidean Norm not necessarily the absolute value.

$\underset{a}{lim}\mathbf{f} = L_1, \underset{a}{lim}\mathbf{g} = L_2$

$\forall\epsilon > 0\ \exists\delta$ such that $|x - a| \implies |(\mathbf{f} + \mathbf{g})(x) - (L_1 + L_2)| < \epsilon$ $|(\mathbf{f} + \mathbf{g})(x) - (L_1 + L_2)| \leq |\mathbf{f}(x) - L_1 + g(x) - L_2| \leq |\mathbf{f}(x) - L_1| + |\mathbf{g}(x) - L_2| < \epsilon$

By choosing the ϵ for f and g to be $\frac{\epsilon}{2}$ the final inequality holds and then the δ for $f + g$ would be $\min(\delta_1, \delta_2)$ where the two δs correspond to the functions δs.

End of Proof of Lemma 11

Formula 26 - Lemma 12:

Let $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \to \mathbb{R}$ have a limit exist at a point $a \in R^n$, then

$\underset{a}{lim}(\mathbf{f} \cdot \mathbf{g}) = \underset{a}{lim}\mathbf{f} \cdot \underset{a}{lim}\mathbf{g}$

Proof of Lemma 12:

In this proof bars around each side represent the Euclidean Norm not necessarily the absolute value.

$\underset{a}{lim}\mathbf{f} = L_1, \underset{a}{lim}\mathbf{g} = L_2$

$|x - x_0| < \delta_1 \implies |\mathbf{f}(x) - L_1| < \epsilon$

$|x - x_0| < \delta_2 \implies |\mathbf{g}(x) - L_2| < \epsilon$

$|x - x_0| < min(\delta_1, \delta_2, \delta_3) \implies |\mathbf{f}(x) \cdot \mathbf{g}(x) - L_1 \cdot L_2| < \epsilon$

$|\mathbf{f}(x) \cdot \mathbf{g}(x) - L_1 \cdot L_2| \leq |\mathbf{f}(x) \cdot \mathbf{g}(x) + \mathbf{f}(x) \cdot L_2 - \mathbf{f}(x) \cdot L_2 - L_1 \cdot L_2| \leq |\mathbf{f}(x)(\mathbf{g}(x) - L_2) + L_2(\mathbf{f}(x) - L_1)| \leq$

$|\mathbf{f}(x)(\mathbf{g}(x) - L_2)| + |L_2(\mathbf{f}(x) - L_1)| < \epsilon$

$$|L_2(\mathbf{f}(x) - L_1)| \leq |(L_2 + 1)(\mathbf{f}(x) - L_1)| < \frac{\epsilon}{2} \equiv |(\mathbf{f}(x) - L_1)| < \frac{\epsilon}{2L_2 + 2}$$

We know that $\mathbf{f}$ converges to L_1 so ϵ can be 1 and the corresponding δ will be known as δ_3
$|\mathbf{f}(x) - L_1| < 1 \equiv |\mathbf{f}(x)| < 1 + |L_1|$

$$|\mathbf{f}(x)(\mathbf{g}(x) - L_2)| < \frac{\epsilon}{2}$$

$$|\mathbf{f}(x)(\mathbf{g}(x) - L_2)| \leq |(1 + |L_1|)(\mathbf{g}(x) - L_2)| < \epsilon \equiv |\mathbf{g}(x) - L_2| < \frac{\epsilon}{1 + |L_1|}$$

Now by taking together the two portions that started as $\frac{\epsilon}{2}$ we have that the limit of the product is the product of the limits.

End of Proof of Lemma 12

Formula 27 - Lemma 13:
Let $g : \mathbb{R}^n \to \mathbb{R}$ have a limit exist at a point $a \in R^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\underset{a}{lim} g$, then

$$\underset{a}{lim}(\mathbf{f} \circ \mathbf{g}) = \underset{\underset{a}{lim}\mathbf{g}}{lim}\mathbf{f}$$

Proof of Lemma 13:
$\underset{a}{lim} g = \lambda$
$\underset{\lambda}{lim} f = \sigma$

Since f is continuous at $\lambda f(\lambda) = \underset{\lambda}{lim} f$ So by the limit definition we have:

$$\forall \epsilon_1 > 0 \ \exists \delta_1 > 0 \text{ such that } |y - \lambda < \delta_1| \implies |f(y) - \sigma| < \epsilon_1$$

Which from the continuity at $\lambda \, |f(y) - \sigma| < \epsilon_1 \equiv |f(y) - f(\lambda)| < \epsilon_1 \equiv 0 < \epsilon_1$. The reason it not just approaches 0 but also reaches 0 is because it is continuous there.

Next since the limit exists for g approaching a we have:

$$\forall \epsilon_2 = \delta_1 \exists \delta_2 > 0 \text{ such that } |x - a| \, \delta_2 \implies |g(x) - \lambda| < \epsilon_2$$

Now setting $y = g(x)$ and replacing some values(y for g(x) and an epsilon for a prior delta) we reach $|y - \lambda| < \delta_1$ and swapping out y elsewhere we reach $|f(g(x)) - f(\lambda)| < \epsilon_1$. The last two statements together have proven our lemma.

End of Proof of Lemma 13

Formula 28 - Lemma 14:
Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}$ have a limit exist at a point $a \in R^n$ that limit not equal 0, then

$$\underset{a}{lim}\left(\frac{1}{\mathbf{f}}\right) = \frac{1}{\underset{a}{lim}\mathbf{f}}$$

Proof of Lemma 14:
Similar bars on each side aren't necessarily the absolute value but are the Euclidean Norm.

$\lim\limits_{a} f = g$

$|x-a| < \delta \implies \left|\frac{1}{f} - \frac{1}{g}\right| < \epsilon \; \left|\frac{1}{f} - \frac{1}{g}\right| \leq \left|\frac{f-g}{fg}\right|$

Since f at a is convergent to g, then

For $\epsilon = \frac{|g|}{2}$ $\exists \delta_1$ such that $|x-a| < \delta_1 \implies |\mathbf{f}(x) - g| < \frac{|g|}{2}$

$\equiv -\frac{|g|}{2} < \mathbf{f}(x) - g < \frac{|g|}{2}$

$\equiv g - \frac{|g|}{2} < \mathbf{f}(x) < g + \frac{|g|}{2}$

If $g > 0$ then, $0 < \frac{|g|}{2} < |\mathbf{f}(x)| < \frac{3\,|g|}{2}$

If $g < 0$ then, $\frac{3g}{2} < \mathbf{f}(x) < \frac{g}{2} < 0 \equiv 0 < -\frac{g}{2} < -\mathbf{f}(x) < -\frac{3g}{2}$

$\equiv 0 < \frac{|g|}{2} < |\mathbf{f}(x)| < \frac{3\,|g|}{2}$

So, $\frac{|\mathbf{f}(x) - g|}{|\mathbf{f}(x)|\,|g|} \leq \frac{|\mathbf{f}(x) - g|}{\frac{|g|}{2}\,|g|} \leq \frac{|\mathbf{f}(x) - g|}{\frac{g^2}{2}} < \epsilon \equiv |\mathbf{f}(x) - g| < \frac{g^2\epsilon}{2}$

Since $f(x)$ converges to g then for $\epsilon = \frac{g^2\epsilon}{2} \exists \delta_3$ $|x-a| < \delta_3 \implies |\mathbf{f}(x) - g| < \frac{g^2\epsilon}{2}$

Then $\exists \delta = min(\delta_1, \delta_2)$ such that $|x-a| < \delta \implies \left|\frac{1}{f} - \frac{1}{g}\right| < \epsilon$

We conclude then that: $\lim\limits_{a} \left(\frac{1}{\mathbf{f}}\right) = \frac{1}{\lim\limits_{a} \mathbf{f}}$

End of Proof of Lemma 14

Formula 29 - Lemma 15:

Let $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \to \mathbb{R}$ be continuous at a point $a \in R^n$, then $f + g$ is continuous at a.

Proof of Lemma 15:

By Lemma 11 on limits we have $\lim\limits_{a}(f+g) = \lim\limits_{a} f + \lim\limits_{a} g$ and we also know $(f+g)(a) = f(a) + g(a)$ by Formula 2. Since f is continuous at a then $\lim\limits_{a} f = f(a)$ and similarly we know $\lim\limits_{a} g = g(a)$ so together we know $\lim\limits_{a}(f+g) = (f+g)(a)$ so $f+g$ is continuous at a.

End of Proof of Lemma 15

Formula 30 - Lemma 16:

Let $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \to \mathbb{R}$ be continuous at a point $a \in R^n$, then $f \cdot g$ is continuous at a.

Proof of Lemma 16:

By Lemma 12 on limits we have $\lim\limits_{a}(f \cdot g) = \lim\limits_{a} f \cdot \lim\limits_{a} g$ and we also know $(f \cdot g)(a) = f(a) \cdot g(a)$ by Formula 3. Since f is continuous at a then $\lim\limits_{a} f = f(a)$ and similarly we know $\lim\limits_{a} g = g(a)$ so together we know $\lim\limits_{a}(f \cdot g) = (f \cdot g)(a)$ so $f \cdot g$ is continuous at a.

End of Proof of Lemma 16

Formula 31 - Lemma 17:
Let $g : \mathbb{R}^n \to \mathbb{R}$ be continous at a point $a \in R^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ be continuous at $\underset{a}{lim} g$, then $f \circ g$ is continuous at a.

Proof of Lemma 17:
$(f \circ g)(a) = f(g(a))$ from Formula 5 while we get $\underset{a}{lim}(f \circ g) = \underset{\underset{a}{lim}g}{lim} f$ from Lemma 12 and since the two functions are continuous at the designated places, $\underset{\underset{a}{lim}g}{lim} f = f(\underset{a}{lim} g)$ and we know $\underset{a}{lim} g = g(a)$. Using this we get $f(\underset{a}{lim} g) = f(g(a))$. So with the two parts we know $(f \circ g)(a) = \underset{\underset{a}{lim}g}{lim} f$ and so the lemma is proven.

End of Proof of Lemma 17

Formula 32 - Lemma 18:
Let $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \to \mathbb{R}$ be continuous at a point $a \in R^n$, then $\dfrac{f}{g}$ is continuous at a if and only if $g(a) \neq 0$.

Proof of Lemma 18:
$\dfrac{f}{g} = f \cdot \dfrac{1}{g}$ which by Lemma 14 we know $\underset{a}{lim}\left(\dfrac{1}{g}\right) = \dfrac{1}{\underset{a}{lim} g}$ and by Lemma 12 we know $\underset{a}{lim}(f \cdot \dfrac{1}{g}) = \underset{a}{lim} f \cdot \underset{a}{lim}\left(\dfrac{1}{g}\right)$. Using the two together we have $\underset{a}{lim}(f \cdot \dfrac{1}{g}) = \underset{a}{lim} f \cdot \dfrac{1}{\underset{a}{lim} g}$. Using Formula 3 we know $\dfrac{f}{g}(a) = f(a) \cdot \dfrac{1}{g(a)}$ Now since f and g are continuous at a then, $\underset{a}{lim} f = f(a)$ and $\underset{a}{lim} g = g(a)$. Putting all this together we get $\underset{a}{lim}\left(\dfrac{f}{g}\right) = \dfrac{f}{g}(a)$ and so $\dfrac{f}{g}$ is continuous at a while $g(a) \neq 0$.

Lastly, Lemma 14 requiring that the limit at a of g not be 0 is satisfied by g(a) not being 0 since g is continuous at a.

End of Proof of Lemma 18

Formula 33 - Lemma 19:
If $f : \mathbb{R} \to \mathbb{R}$ and $f = p_1$ then f is continuous.
Proof of Lemma 19:
We'll start with the defintion of the limit at a point:

$$\left(\underset{a}{lim} f = g\right) \equiv \left(\underset{\substack{x:\mathbb{N}\to\mathbb{R}^N \\ limx=a \\ x\neq a}}{\forall} lim y^x = g^x \wedge y^x = f \circ x \wedge \underset{x}{\forall} g^x = g \right)$$

Now our point will be the arbitrary but constant point a. We also won't pick a specific sequence but just say any arbitrary sequence λ.

So $y^x = f \circ \lambda$ which since f is simply the identity function $y^x = \lambda$ based upon Lemma 6.

So the limit of y^x is simply the limit of λ which is simply the value of a based on an earlier condition it had to satisfy. The value of f(a) is also just a. So $\forall \underset{a}{lim} f = f(a)$ which means f is continuous. **End of Proof of Lemma 19**

Formula 34 - Lemma 20:
If $f : \mathbb{R} \to \mathbb{R}$ and $f = p_k$ where $k \in \mathbb{N}$ then f is continuous.

Proof of Lemma 20:
The function f can be rewritten to $f = p_1 \cdot p_1 \cdot ... \cdot p_1$ where p_1 shows up k times. This is the product of the continous function p_1 which we know is continuous from Lemma 19. Using Lemma 16 recursively we can see that the product of any amount of continuous functions is continuous and so f is continuous.
End of Proof of Lemma 20

Formula 35 - Lemma 21:
If $f : \mathbb{R} \to \mathbb{R}$ and $f = sin$ then f is continuous.

Proof of Lemma 21:
We'll start with the defintion of the limit at a point:

$$\left(\underset{a}{lim} f = g\right) \equiv \left(\underset{\substack{x:\mathbb{N}\to\mathbb{R}^N \\ lim x = a \\ x \neq a}}{\forall} lim y^x = g^x \wedge y^x = f \circ x \wedge \underset{x}{\forall} g^x = g \right)$$

Now our point will be the arbritrary but constant point a. We also won't pick a specific sequence but just say any arbritary sequence λ.

So $y^x = f \circ x$ is simply $y^x = sin \circ \lambda$. So the limit at a is simply $\underset{a}{lim} sin = sin(a)$. The value is also just $f(a) = sin(a)$. Since $f(a) = \underset{a}{lim} f$ the function is continuous at a. Since a is arbritary and the result comes about with any a $\forall \underset{a}{lim} f = f(a)$ which means f is continuous.
End of Proof of Lemma 21

Formula 36 - Lemma 22:
If $f : \mathbb{R} \to \mathbb{R}$ and $f = exp$ then f is continuous.

Proof of Lemma 22:
We'll start with the defintion of the limit at a point:

$$\left(\underset{a}{lim} f = g\right) \equiv \left(\underset{\substack{x:\mathbb{N}\to\mathbb{R}^N \\ lim x = a \\ x \neq a}}{\forall} lim y^x = g^x \wedge y^x = f \circ x \wedge \underset{x}{\forall} g^x = g \right)$$

Now our point will be the arbritrary but constant point a. We also won't pick a specific sequence but just say any arbritary sequence λ.

So $y^x = f \circ x$ is simply $y^x = exp \circ \lambda$. So the limit at a is simply $\underset{a}{lim} exp = exp(a)$. The

value is also just $f(a) = exp(a)$. Since $f(a) = \underset{a}{lim} f$ the function is continuous at a. Since a is arbitrary and the result comes about with any a $\forall \underset{a}{lim} f = f(a)$ which means f is continuous.
End of Proof of Lemma 22

Formula 37 - Lemma 23:

$$p_k \cdot p_l = p_{k+l}$$

Proof of Lemma 23:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_k \cdot p_l$
$g = p_{k+l}$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}}{\forall} (p_k \cdot p_l)(a) = p_{k+l}(a)$$

Step 0 (Identify Formula):
Formula 3
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_k$
$g = p_l$
$x = a$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}}{\forall} (p_k \cdot p_l)(a) = p_k(a) \cdot p_l(a)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_k$
$x = a$
Step 3 (Replacing Symbols):

$$= a^k \cdot p_l(a)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$p_a = p_l$
$x = a$
Step 3 (Replacing Symbols):

$$= a^k \cdot a^l = a^{k+l}$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$a = k + l$
$x = a$
Step 3 (Replacing Symbols):

$$= p_{k+l}(a)$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f = p_k \cdot p_l$
$f = p_{k+l}$
Step 3 (Replacing Symbols):

$$p_k \cdot p_l = p_{k+l}$$

End of Proof of Lemma 23

Formula 38 - Lemma 24:
If $f : A \to B$ where A and B represent any set, then $p_1 \circ f \cdot p_{-1} \circ f = 1$
Proof of Lemma 24:
Step 0 (Identify Formula):
Formula 13
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f = p_1$
$g = p_{-1}$
$h = f$
Step 3 (Replacing Symbols):

$$p_1 \circ f \cdot p_{-1} \circ f = (p_1 \cdot p_{-1}) \circ f$$

Step 0 (Identify Formula):
Formula 37
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
$p_l = p_{-1}$
Step 3 (Replacing Symbols):

$$= p_0 \circ f = 1p_0 \circ f$$

Step 0 (Identify Formula):
Formula 19
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$c = 1p_0$
$f = f$
Step 3 (Replacing Symbols):

$$= 1$$

End of Proof of Lemma 24

Formula 39 - Lemma 25:
If $f : \mathbb{R}^n \to \mathbb{R}$ then $(kf)'_l = kf'_l$ where k is any constant relative to what is being derived by (or not a function involving l).
Proof of Lemma 25:
Step 0 (Identify Formula):
Product Rule
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = k$
$g = f$
Step 3 (Replacing Symbols):

$$(kf)'_l = k'_l \cdot f + f'_l \cdot k = k'_l \cdot f + f'_l \cdot k$$

Using Lemma 26 we get $k'_l = 0$ which causes the first term to disappear and leaves what was desired to be proven.
End of Proof of Lemma 25

Formula 40 - Lemma 26:
If $f = k : \mathbb{R}^n \to \mathbb{R}$ then $f_l^{'} = 0$ where k is any constant relative to what is being derived by (or not a function involving l).
Proof of Lemma 26:
The derivative of k'_l can be seen by analogy. Looking at a constant function from $\mathbb{R} \to \mathbb{R}$ we see f(x+h) - f(x) = 0 causing the difference quotient to be $\frac{1}{h} or 0 \cdot \frac{1}{h}$. Taking the limit approaching 0 from any sequence the 0 part won't be affected so when multiplied with all the sequences that could result we simply have 0 left over which the limit of a constant sequence is the constant as has been proven prior.
End of Proof of Lemma 26

FUNCTIONS OF SEVERAL VARIABLES

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PROBLEMS 1

8. Let $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}^2,\ (a,b,c) \mapsto \mathbf{g}\,(a,b,c) = (a^2 + b^3\, e^c, b^3\, \mathbf{sin}\, c + e^a)$.
Express g in terms of Cartesian product function of i^{th}-variable functions.

Solution

$$\mathbf{g}\,(a,b,c) \overset{\mathbf{EF}(1,2,3)}{=} \big(\mathbf{p}_2(a) + \mathbf{p}_3(b)\,\mathbf{exp}(c), \mathbf{p}_3(b)\mathbf{sin}c + \mathbf{exp}(a)\big)$$

$$\begin{pmatrix} \mathbf{RHS} \\ \mathbf{f} = \mathbf{p}_2 \\ i = 1 \\ x^i = a \end{pmatrix} \begin{pmatrix} \mathbf{RHS} \\ \mathbf{f} = \mathbf{p}_3 \\ i = 2 \\ x^i = b \end{pmatrix} \begin{pmatrix} \mathbf{RHS} \\ \mathbf{f} = \mathbf{exp} \\ i = 3 \\ x^i = c \end{pmatrix} \overset{(6)}{=} \begin{pmatrix} \mathbf{RHS} \\ \mathbf{f} = \mathbf{p}_3 \\ i = 2 \\ x^i = b \end{pmatrix} \begin{pmatrix} \mathbf{RHS} \\ 1\mathbf{f} = \mathbf{sin} \\ i = 3 \\ x^i = c \end{pmatrix} \begin{pmatrix} \mathbf{RHS} \\ \mathbf{f} = \mathbf{exp} \\ i = 1 \\ x^i = a \end{pmatrix} \Big(\overset{1}{\mathbf{p}}_2(a,b,c) + \overset{2}{\mathbf{p}}_3(a,b,c)\overset{3}{\mathbf{exp}}(a,b,c), \overset{2}{\mathbf{p}}_3(a,b,c)\overset{3}{\mathbf{sin}}$$

$$\begin{pmatrix} \mathbf{RHS} \\ \mathbf{f} = \overset{2}{\mathbf{p}}_3 \\ \mathbf{g} = \overset{3}{\mathbf{exp}} \\ x = (a,b,c) \end{pmatrix} \overset{(3)}{=} \begin{pmatrix} \mathbf{RHS} \\ \mathbf{f} = \overset{2}{\mathbf{p}}_3 \\ \mathbf{g} = \overset{3}{\mathbf{sin}} \\ x = (a,b,c) \end{pmatrix} \left(\overset{1}{\mathbf{p}}_2(a,b,c) + \overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{exp}}(a,b,c), \overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{sin}}(a,b,c) + \overset{1}{\mathbf{exp}}(a,b,c)\right)$$

$$\begin{pmatrix} \mathbf{RHS} \\ \mathbf{f} = \overset{1}{\mathbf{p}}_2 \\ \mathbf{g} = \overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{exp}} \\ x = (a,b,c) \end{pmatrix} \overset{(2)}{=} \begin{pmatrix} \mathbf{RHS} \\ \mathbf{f} = \overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{sin}} \\ \mathbf{g} = \overset{1}{\mathbf{exp}} \\ x = (a,b,c) \end{pmatrix} \left(\left[\overset{1}{\mathbf{p}}_2 + \left(\overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{exp}}\right)\right](a,b,c), \left[\left(\overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{sin}}\right) + \overset{1}{\mathbf{exp}}\right](a,b,c)\right)$$

$$\overset{(4)}{=} \begin{pmatrix} \mathbf{RHS} \\ \mathbf{a^1} = \overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{exp}} \\ \mathbf{a^2} = \overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{sin}} + \overset{1}{\mathbf{exp}} \\ x = (a,b,c) \end{pmatrix} \left(\left[\overset{1}{\mathbf{p}}_2 + \left(\overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{exp}}\right)\right] \times \left[\left(\overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{sin}}\right) + \overset{1}{\mathbf{exp}}\right]\right)(a,b,c)$$

$$\overset{(1)}{\equiv} \begin{pmatrix} \mathbf{RHS} \\ \mathbf{f} = \mathbf{g} \\ \mathbf{g} = \overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{exp}} \times \overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{sin}} + \overset{1}{\mathbf{exp}} \\ a = (a,b,c) \end{pmatrix} \quad \mathbf{g} = \left[\overset{1}{\mathbf{p}}_2 + \left(\overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{exp}}\right)\right] \times \left[\left(\overset{2}{\mathbf{p}}_3 \cdot \overset{3}{\mathbf{sin}}\right) + \overset{1}{\mathbf{exp}}\right]$$

11. Let $\overset{k}{\mathbf{f}} : \mathbf{R}^n \to \mathbf{R}$ be a k^{th}-variable function generated by any 6-function $\mathbf{f} : \mathbf{R} \to \mathbf{R}$. **Form** the i^{th} quotient function ${}^{i}\mathbf{Q}^{\overset{k}{\mathbf{f}}}_{c}$ for $\overset{k}{\mathbf{f}}$ with respect to a point $c = (c^1, c^2, \ldots, c^n) \in \mathbf{R}^n$ for $i, k = 1, 2, \ldots, n$.

$$ {}^{i}\mathbf{Q}_c^{\overset{k}{\mathbf{f}}} \underset{\begin{pmatrix}\mathbf{LHS}\\ \mathbf{f}=\overset{k}{\mathbf{f}}\\ i=i\\ a=c\end{pmatrix}}{\overset{(\mathbf{14})}{=}} \mathbf{p}_{-1}\cdot\left[\overset{k}{\mathbf{f}}\circ(c^1\mathbf{p}_0\times\cdots\times(c^i\mathbf{p}_0+\mathbf{p}_1)\times\cdots c^n\mathbf{p}_0)-\overset{k}{\mathbf{f}}(c)\mathbf{p}_0\right] $$

Now we consider the two cases of if $k=i$ or if $k\neq i$

If $\underline{k=i}$:

$$ \underset{\begin{pmatrix}\mathbf{LHS}\\ \overset{i}{\mathbf{f}}=\overset{i}{\mathbf{f}}\\ a^1=c^1\mathbf{p}_0\\ \cdots\\ a^i=c^i\mathbf{p}_p+\mathbf{p}_1\\ \cdots\\ a^n=c^n\mathbf{p}_0\end{pmatrix}}{\overset{(\mathbf{10})}{=}} \mathbf{p}_{-1}\cdot\left[\mathbf{f}\circ(c^i\mathbf{p_0}+\mathbf{p_1})-\overset{k}{\mathbf{f}}(c)\mathbf{p}_0\right] \underset{\begin{pmatrix}\mathbf{LHS}\\ \mathbf{f}=\mathbf{f}\\ i=k\\ x=c\end{pmatrix}}{\overset{(6)}{=}} \mathbf{p}_{-1}\cdot[\mathbf{f}\circ(c^i\mathbf{p_0}+\mathbf{p_1})-\mathbf{f}(c^i)\mathbf{p}_0] = $$

$$ \mathbf{Q}_{c^i}^{\mathbf{f}} $$

Thus, when $k=i$, we simply get the *Quotient Function* of $\mathbf{f}$ at c^i.

If $\underline{k\neq i}$:

$$ \underset{\begin{pmatrix}\mathbf{LHS}\\ \overset{i}{\mathbf{f}}=\overset{k}{\mathbf{f}}\\ \mathbf{a^1}=c^1\mathbf{p}_0\\ \cdots\\ a^i=c^i\mathbf{p}_o+\mathbf{p}_1\\ \cdots\\ a^n=c^n\mathbf{p}_0\end{pmatrix}}{\overset{(\mathbf{10})}{=}} \mathbf{p}_{-1}\cdot\left[\mathbf{f}\circ(c^k\mathbf{p_0})-\overset{k}{\mathbf{f}}(c)\mathbf{p}_0\right] \underset{\begin{pmatrix}\mathbf{LHS}\\ \mathbf{f}=\mathbf{f}\\ i=k\\ x=c\end{pmatrix}}{\overset{(6)}{=}} \mathbf{p}_{-1}\cdot\left[\mathbf{f}(c^k)\mathbf{p_0})-\mathbf{f}(c^k)\mathbf{p}_0\right] = \mathbf{0} $$

10. Find a function $\mathbf{f}:\mathbb{R}\to\mathbb{R}$ satisfying

$$ 3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1+\overset{2}{\mathbf{p}}_1\right) - 2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1-\overset{2}{\mathbf{p}}_1\right) = 3\,\overset{1}{\mathbf{p}}_2 + 30\,\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1 + 3\,\overset{2}{\mathbf{p}}_2 + \mathbf{2} $$

First we compose both sides by the cartesian product of $\mathbf{p}_1$ and $\mathbf{p}_1$

$$\left[3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1+\overset{2}{\mathbf{p}}_1\right) - 2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1-\overset{2}{\mathbf{p}}_1\right)\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) = \left[3\,\overset{1}{\mathbf{p}}_2 + 30\,\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1 + 3\,\overset{2}{\mathbf{p}}_2 + \mathbf{2}\right]\circ(\mathbf{p}_1\times\mathbf{p}_1)$$

$$\left(\begin{array}{l}\textbf{RHS}\\ \mathbf{f}=3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1+\overset{2}{\mathbf{p}}_1\right)\\ g=2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1-\overset{2}{\mathbf{p}}_1\right)\\ h=(\mathbf{p}_1\times\mathbf{p}_1)\end{array}\right)\overset{(12)}{=}\left(\begin{array}{l}\textbf{RHS}\\ \mathbf{f}=3\,\overset{1}{\mathbf{p}}_2+30\,\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1\\ g=3\,\overset{2}{\mathbf{p}}_2+\mathbf{2}\\ h=(\mathbf{p}_1\times\mathbf{p}_1)\end{array}\right)$$

$$\left[3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1+\overset{2}{\mathbf{p}}_1\right)\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) - \left[2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1-\overset{2}{\mathbf{p}}_1\right)\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) = \left[3\,\overset{1}{\mathbf{p}}_2 + 30\,\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) + \left[3\,\overset{2}{\mathbf{p}}_2 + \mathbf{2}\right]\circ(\mathbf{p}_1\times\mathbf{p}_1)$$

$$\left(\begin{array}{l}\textbf{RHS}\\ \mathbf{f}=3\,\overset{1}{\mathbf{p}}_2\\ g=30\,\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1\\ h=(\mathbf{p}_1\times\mathbf{p}_1)\end{array}\right)\overset{(12)}{=}\left(\begin{array}{l}\textbf{RHS}\\ \mathbf{f}=3\,\overset{2}{\mathbf{p}}_2\\ g=2\\ h=(\mathbf{p}_1\times\mathbf{p}_1)\end{array}\right)$$

$$\left[3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1+\overset{2}{\mathbf{p}}_1\right)\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) - \left[2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1-\overset{2}{\mathbf{p}}_1\right)\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) = \left[3\,\overset{1}{\mathbf{p}}_2\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) + \left[30\,\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) + \left[3\,\overset{2}{\mathbf{p}}_2\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) + [\mathbf{2}]\circ(\mathbf{p}_1\times\mathbf{p}_1)$$

$$\overset{(13)}{=}\begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 30\,\overset{1}{\mathbf{p}}_1\\ g = \overset{2}{\mathbf{p}}_1\\ h = (\mathbf{p}_1 \times \mathbf{p}_1)\end{pmatrix}$$

$$\left[3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1\right)\right]\circ(\mathbf{p}_1 \times \mathbf{p}_1) - \left[2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1\right)\right]\circ(\mathbf{p}_1 \times \mathbf{p}_1) = \left[3\,\overset{1}{\mathbf{p}}_2\right]\circ(\mathbf{p}_1 \times \mathbf{p}_1) + \left[30\,\overset{1}{\mathbf{p}}_1\right]\circ(\mathbf{p}_1 \times \mathbf{p}_1)\cdot\left[\overset{2}{\mathbf{p}}_1\right]\circ(\mathbf{p}_1 \times \mathbf{p}_1) + \left[3\,\overset{2}{\mathbf{p}}_2\right]\circ(\mathbf{p}_1 \times \mathbf{p}_1) + [\mathbf{2}]\circ(\mathbf{p}_1 \times \mathbf{p}_1)$$

$$\overset{(10)}{=}\begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1\right)\\ a = \mathbf{p}_1 \times \mathbf{p}_1\\ i = 1\end{pmatrix}\begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1\right)\\ a = \mathbf{p}_1 \times \mathbf{p}_1\\ i = 2\end{pmatrix}\begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1\right)\\ a = \mathbf{p}_1 \times \mathbf{p}_1\\ i = 1\end{pmatrix}\begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1\right)\\ a = \mathbf{p}_1 \times \mathbf{p}_1\\ i = 2\end{pmatrix}\begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 3\,\overset{1}{\mathbf{p}}_2\\ a = \mathbf{p}_1 \times \mathbf{p}_1\\ i = 1\end{pmatrix}\begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 30\,\overset{1}{\mathbf{p}}_1\\ a = \mathbf{p}_1 \times \mathbf{p}_1\\ i = 1\end{pmatrix}\begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 3\,\overset{2}{\mathbf{p}}_2\\ a = \mathbf{p}_1 \times \mathbf{p}_1\\ i = 2\end{pmatrix}\begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 2\\ a = \mathbf{p}_1 \times \mathbf{p}_1\\ i = 0\end{pmatrix}$$

$$3\,\mathbf{f}\circ(\mathbf{p}_1 + \mathbf{p}_1) - 2\,\mathbf{f}\circ(\mathbf{p}_1 - \mathbf{p}_1) = 3\,\mathbf{p}_2 + 30\,\mathbf{p}_1\cdot\mathbf{p}_1 + 3\,\mathbf{p}_2 + \mathbf{2}$$

Simplifying, we get:

$$3\,\mathbf{f}\circ(2\,\mathbf{p}_1) - 2\,\mathbf{f}\circ(\mathbf{0}) = 3\,\mathbf{p}_2 + 30\,\mathbf{p}_2 + 3\,\mathbf{p}_2 + \mathbf{2}$$

Which equals:

$$3\,\mathbf{f}\circ(2\,\mathbf{p}_1) - 2\,\mathbf{f}\,(0)\mathbf{p}_0 = 36\,\mathbf{p}_2 + \mathbf{2}$$

We then find the value of the functions on both sides at 0:

$$\left[3\,\mathbf{f}\circ(2\,\mathbf{p}_1) - 2\,\mathbf{f}\,(0)\mathbf{p}_0\right](0) = \left[36\,\mathbf{p}_2 + \mathbf{2}\right](0)$$

$$\overset{(2)}{=}\begin{pmatrix}\textbf{LHS}\\ \mathbf{f} = 3\,\mathbf{f}\circ(2\,\mathbf{p}_1)\\ g = -2\,\mathbf{f}\,(0)\mathbf{p}_0\\ x = 0\end{pmatrix}\begin{pmatrix}\textbf{LHS}\\ \mathbf{f} = 36\,\mathbf{p}_2\\ g = \mathbf{2}\\ x = 0\end{pmatrix}\quad \left[3\,\mathbf{f}\circ(2\,\mathbf{p}_1)\right](0) - \left[2\,\mathbf{f}\,(0)\mathbf{p}_0\right](0) = \left[36\,\mathbf{p}_2\right](0) + \left[\mathbf{2}\,(0)\right]$$

$$\overset{(5)\,(\mathbf{EF(1)})}{=}\begin{pmatrix}\textbf{LHS}\\ \mathbf{f} = 3\,\mathbf{f}\\ g = 2\,\mathbf{p}_1\\ x = 0\end{pmatrix}\quad 3\,\mathbf{f}(0) - 2\,\mathbf{f}(0) = 0 + 2$$

Simplifying, we get:

$\mathbf{f}(0) = 2$

Substituting this value into the equation, we get:

$3\,\mathbf{f}\circ\big(2\,\mathbf{p}_1\big) - 4 = 36\,\mathbf{p}_2 + \mathbf{2}$

$3\,\mathbf{f}\circ\big(2\,\mathbf{p}_1\big) = 36\,\mathbf{p}_2 + \mathbf{6}$

We divide the whole equation by 3 to get:

$\mathbf{f}\circ\big(2\,\mathbf{p}_1\big) = 12\,\mathbf{p}_2 + \mathbf{2}$

Composing both sides with $1/2\ \mathbf{p}_1$:

$\big[\mathbf{f}\circ\big(2\,\mathbf{p}_1\big)\big]\circ(1/2\mathbf{p}_1) = \big[12\,\mathbf{p}_2 + \mathbf{2}\big]\circ(1/2\mathbf{p}_1)$

$\overset{(12)}{\underset{\begin{pmatrix}\textbf{LHS}\\ \mathbf{f}=12\,\mathbf{p}_2\\ g=\mathbf{2}\\ h=1/2\mathbf{p}_1\end{pmatrix}}{=}}\quad \big[\mathbf{f}\circ\big(2\,\mathbf{p}_1\big)\big]\circ(1/2\mathbf{p}_1) = \big[12\,\mathbf{p}_2\big]\circ(1/2\mathbf{p}_1) + [\mathbf{2}]\circ(1/2\mathbf{p}_1)$

$\overset{(14)}{\underset{\begin{pmatrix}\textbf{LHS}\\ \mathbf{f}=\mathbf{f}\\ g=2\mathbf{p}_1\\ h=1/2\mathbf{p}_1\end{pmatrix}}{=}}\quad \mathbf{f}\circ(2\,(1/2\mathbf{p}_1)) = 12\,\mathbf{p}_2\circ(1/2\mathbf{p}_1) + \mathbf{2}$

where formula 14 is:

$$(f\circ g)\circ h = f\circ(g\circ h)$$

Simplifying, we get:

$\mathbf{f}\circ(\mathbf{p}_1) = 12\,(1/4\mathbf{p}_2) + \mathbf{2}$

Which equals:

$\mathbf{f}\circ(\mathbf{p}_1) = 3\mathbf{p}_2 + \mathbf{2}$

Thus, the function $\mathbf{f} = 3\mathbf{p}_2 + \mathbf{2}$

FUNCTIONS OF SEVERAL VARIABLES

Nathan Yu, Sooraj Boominathan, Steve Zhang, Ben Zhao, and Vincent Po

PROBLEMS 1

8. Let $\mathbf{g}:\mathbb{R}^3\to\mathbb{R}^2,\ (a,b,c)\mapsto \mathbf{g}(a,b,c)=(a^2+b^3e^c, b^3\,\mathbf{sin}\,c+e^a)$.
Express g in terms of Cartesian product function of i^{th}-variable functions.

Solution

$$\mathbf{g}(a,b,c)\overset{\mathbf{EF}(1,2,3)}{=}\left(\mathbf{p}_2(a)+\mathbf{p}_3(b)\,\mathbf{exp}(c),\mathbf{p}_3(b)\mathbf{sin}c+\mathbf{exp}(a)\right)$$

$$\underset{\begin{pmatrix}\mathbf{RHS}\\ \mathbf{f}=\mathbf{p}_2\\ i=1\\ x^i=a\end{pmatrix}\begin{pmatrix}\mathbf{RHS}\\ \mathbf{f}=\mathbf{p}_3\\ i=2\\ x^i=b\end{pmatrix}\begin{pmatrix}\mathbf{RHS}\\ \mathbf{f}=\mathbf{exp}\\ i=3\\ x^i=c\end{pmatrix}\begin{pmatrix}\mathbf{RHS}\\ \mathbf{f}=\mathbf{p}_3\\ i=2\\ x^i=b\end{pmatrix}\begin{pmatrix}\mathbf{RHS}\\ 1\mathbf{f}=\mathbf{sin}\\ i=3\\ x^i=c\end{pmatrix}\begin{pmatrix}\mathbf{RHS}\\ \mathbf{f}=\mathbf{exp}\\ i=1\\ x^i=a\end{pmatrix}}{\overset{(6)}{=}}\left(\overset{1}{\mathbf{p}}_2(a,b,c)+\overset{2}{\mathbf{p}}_3(a,b,c)\overset{3}{\mathbf{exp}}(a,b,c),\overset{2}{\mathbf{p}}_3(a,b,c)\overset{3}{\mathbf{sin}}\right.$$

$$\underset{\begin{pmatrix}\mathbf{RHS}\\ \mathbf{f}=\overset{2}{\mathbf{p}}_3\\ \mathbf{g}=\overset{3}{\mathbf{exp}}\\ x=(a,b,c)\end{pmatrix}\begin{pmatrix}\mathbf{RHS}\\ \mathbf{f}=\overset{2}{\mathbf{p}}_3\\ \mathbf{g}=\overset{3}{\mathbf{sin}}\\ x=(a,b,c)\end{pmatrix}}{\overset{(3)}{=}}\left(\overset{1}{\mathbf{p}}_2(a,b,c)+\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{exp}}(a,b,c),\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{sin}}(a,b,c)+\overset{1}{\mathbf{exp}}(a,b,c)\right)$$

$$\underset{\begin{pmatrix}\mathbf{RHS}\\ \mathbf{f}=\overset{1}{\mathbf{p}}_2\\ \mathbf{g}=\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{exp}}\\ x=(a,b,c)\end{pmatrix}\begin{pmatrix}\mathbf{RHS}\\ \mathbf{f}=\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{sin}}\\ \mathbf{g}=\overset{1}{\mathbf{exp}}\\ x=(a,b,c)\end{pmatrix}}{\overset{(2)}{=}}\left(\left[\overset{1}{\mathbf{p}}_2+\left(\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{exp}}\right)\right](a,b,c),\left[\left(\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{sin}}\right)+\overset{1}{\mathbf{exp}}\right](a,b,c)\right)$$

$$\underset{\begin{pmatrix}\mathbf{RHS}\\ \mathbf{a^1}=\overset{1}{\mathbf{p}}_2+\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{exp}}\\ \mathbf{a^2}=\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{sin}}+\overset{1}{\mathbf{exp}}\\ x=(a,b,c)\end{pmatrix}}{\overset{(4)}{=}}\left(\left[\overset{1}{\mathbf{p}}_2+\left(\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{exp}}\right)\right]\times\left[\left(\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{sin}}\right)+\overset{1}{\mathbf{exp}}\right]\right)(a,b,c)$$

$$\underset{\begin{pmatrix}\mathbf{RHS}\\ \mathbf{f}=\mathbf{g}\\ \mathbf{g}=\overset{1}{\mathbf{p}}_2+\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{exp}}\times\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{sin}}+\overset{1}{\mathbf{exp}}\\ a=(a,b,c)\end{pmatrix}}{\overset{(1)}{\equiv}}\quad \mathbf{g}=\left[\overset{1}{\mathbf{p}}_2+\left(\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{exp}}\right)\right]\times\left[\left(\overset{2}{\mathbf{p}}_3\cdot\overset{3}{\mathbf{sin}}\right)+\overset{1}{\mathbf{exp}}\right]$$

11. Let $\overset{k}{\mathbf{f}}:\mathbf{R}^n\to\mathbf{R}$ be a k^{th}-variable function generated by any 6-function $\mathbf{f}:\mathbf{R}\to\mathbf{R}$. **Form** the i^{th} quotient function ${}^{i}\mathbf{Q}_c^{\overset{k}{\mathbf{f}}}$ for $\overset{k}{\mathbf{f}}$ with respect to a point $c=(c^1,c^2,\ldots,c^n)\in\mathbf{R}^n$ for $i,k=1,2,\ldots,n$.

$${}^{i}\mathbf{Q}_c^{\overset{k}{\mathbf{f}}}\underset{\begin{pmatrix}\mathbf{LHS}\\ \mathbf{f}=\overset{k}{\mathbf{f}}\\ i=i\\ a=c\end{pmatrix}}{\overset{(14)}{=}}\mathbf{p}_{-1}\cdot\left[\overset{k}{\mathbf{f}}\circ(c^1\mathbf{p}_0\times\cdots\times(c^i\mathbf{p}_0+\mathbf{p}_1)\times\cdots c^n\mathbf{p}_0)-\overset{k}{\mathbf{f}}(c)\mathbf{p}_0\right]$$

Now we consider the two cases of if $k=i$ or if $k\neq i$

If $\underline{k = i}$:

$$\underset{\begin{pmatrix}\textbf{LHS}\\ \overset{i}{\mathbf{f}} = \overset{i}{\mathbf{f}}\\ a^1 = c^1\mathbf{p}_0\\ \dots\\ a^i = c^i\mathbf{p}_p + \mathbf{p}_1\\ \dots\\ a^n = c^n\mathbf{p}_0\end{pmatrix}}{\overset{(\mathbf{10})}{=}} \quad \mathbf{p}_{-1}\cdot\left[\mathbf{f}\circ(c^i\mathbf{p}_0 + \mathbf{p}_1) - \overset{k}{\mathbf{f}}(c)\mathbf{p}_0\right] \underset{\begin{pmatrix}\textbf{LHS}\\ \mathbf{f} = \mathbf{f}\\ i = k\\ x = c\end{pmatrix}}{\overset{(6)}{=}} \mathbf{p}_{-1}\cdot[\mathbf{f}\circ(c^i\mathbf{p}_0 + \mathbf{p}_1) - \mathbf{f}(c^i)\mathbf{p}_0] =$$

$$\mathbf{Q}^{\mathbf{f}}_{c^i}$$

Thus, when $k = i$, we simply get the *Quotient Function* of $\mathbf{f}$ at c^i.

If $\underline{k \neq i}$:

$$\underset{\begin{pmatrix}\textbf{LHS}\\ \overset{i}{\mathbf{f}} = \overset{k}{\mathbf{f}}\\ \mathbf{a}^1 = c^1\mathbf{p}_0\\ \dots\\ a^i = c^i\mathbf{p}_o + \mathbf{p}_1\\ \dots\\ a^n = c^n\mathbf{p}_0\end{pmatrix}}{\overset{(\mathbf{10})}{=}} \quad \mathbf{p}_{-1}\cdot\left[\mathbf{f}\circ(c^k\mathbf{p}_0) - \overset{k}{\mathbf{f}}(c)\mathbf{p}_0\right] \underset{\begin{pmatrix}\textbf{LHS}\\ \mathbf{f} = \mathbf{f}\\ i = k\\ x = c\end{pmatrix}}{\overset{(6)}{=}} \mathbf{p}_{-1}\cdot\left[\mathbf{f}(c^k)\mathbf{p}_0) - \mathbf{f}(c^k)\mathbf{p}_0\right] = \mathbf{0}$$

10. Find a function $\mathbf{f}:\mathbb{R}\to\mathbb{R}$ satisfying

$$3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1\right) - 2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1\right) = 3\,\overset{1}{\mathbf{p}}_2 + 30\,\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1 + 3\,\overset{2}{\mathbf{p}}_2 + \mathbf{2}$$

First we compose both sides by the cartesian product of $\mathbf{p}_1$ and $\mathbf{p}_1$

$$\left[3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1\right) - 2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1\right)\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) = \left[3\,\overset{1}{\mathbf{p}}_2 + 30\,\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1 + 3\,\overset{2}{\mathbf{p}}_2 + \mathbf{2}\right]\circ(\mathbf{p}_1\times\mathbf{p}_1)$$

$$\begin{pmatrix}\textbf{RHS}\\ f = 3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1\right)\\ g = 2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1\right)\\ h = (\mathbf{p}_1\times\mathbf{p}_1)\end{pmatrix}\overset{(12)}{=}\begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 3\,\overset{1}{\mathbf{p}}_2 + 30\,\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1\\ g = 3\,\overset{2}{\mathbf{p}}_2 + \mathbf{2}\\ h = (\mathbf{p}_1\times\mathbf{p}_1)\end{pmatrix}$$

$$\left[3\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1\right)\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) - \left[2\,\mathbf{f}\circ\left(\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1\right)\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) = \left[3\,\overset{1}{\mathbf{p}}_2 + 30\,\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1\right]\circ(\mathbf{p}_1\times\mathbf{p}_1) + \left[3\,\overset{2}{\mathbf{p}}_2 + \mathbf{2}\right]\circ(\mathbf{p}_1\times\mathbf{p}_1)$$

$$\begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 3\overset{1}{\mathbf{p}}_2\\ g = 30\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1\\ h = (\mathbf{p}_1 \times \mathbf{p}_1)\end{pmatrix} \overset{(12)}{=} \begin{pmatrix}\textbf{RHS}\\ \mathbf{f} = 3\overset{2}{\mathbf{p}}_2\\ g = 2\\ h = (\mathbf{p}_1 \times \mathbf{p}_1)\end{pmatrix}$$

$$\left[3\,\mathbf{f} \circ \left(\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1\right)\right] \circ (\mathbf{p}_1 \times \mathbf{p}_1) - \left[2\,\mathbf{f} \circ \left(\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1\right)\right] \circ (\mathbf{p}_1 \times \mathbf{p}_1) = \left[3\,\overset{1}{\mathbf{p}}_2\right] \circ (\mathbf{p}_1 \times \mathbf{p}_1) + \left[30\,\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1\right] \circ (\mathbf{p}_1 \times \mathbf{p}_1) + \left[3\,\overset{2}{\mathbf{p}}_2\right] \circ (\mathbf{p}_1 \times \mathbf{p}_1) + [\mathbf{2}] \circ (\mathbf{p}_1 \times \mathbf{p}_1)$$

$$\overset{(13)}{=}$$

$$\begin{pmatrix} \textbf{RHS} \\ \mathbf{f} = 30\,\overset{1}{\mathbf{p}}_1 \\ g = \overset{2}{\mathbf{p}}_1 \\ h = (\mathbf{p}_1 \times \mathbf{p}_1) \end{pmatrix}$$

$$\left[3\,\mathbf{f} \circ \left(\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1\right)\right] \circ (\mathbf{p}_1 \times \mathbf{p}_1) - \left[2\,\mathbf{f} \circ \left(\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1\right)\right] \circ (\mathbf{p}_1 \times \mathbf{p}_1) = \left[3\,\overset{1}{\mathbf{p}}_2\right] \circ (\mathbf{p}_1 \times \mathbf{p}_1) + \left[30\,\overset{1}{\mathbf{p}}_1\right] \circ (\mathbf{p}_1 \times \mathbf{p}_1) \cdot \left[\overset{2}{\mathbf{p}}_1\right] \circ (\mathbf{p}_1 \times \mathbf{p}_1) + \left[3\,\overset{2}{\mathbf{p}}_2\right] \circ (\mathbf{p}_1 \times \mathbf{p}_1) + [\mathbf{2}] \circ (\mathbf{p}_1 \times \mathbf{p}_1)$$

$$\begin{pmatrix} \textbf{RHS} \\ \mathbf{f} = 3\,\mathbf{f} \circ \left(\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1\right) \\ a = \mathbf{p}_1 \times \mathbf{p}_1 \\ i = 1 \end{pmatrix} \begin{pmatrix} \textbf{RHS} \\ \mathbf{f} = 3\,\mathbf{f} \circ \left(\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1\right) \\ a = \mathbf{p}_1 \times \mathbf{p}_1 \\ i = 2 \end{pmatrix} \begin{pmatrix} \textbf{RHS} \\ \mathbf{f} = 2\,\mathbf{f} \circ \left(\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1\right) \\ a = \mathbf{p}_1 \times \mathbf{p}_1 \\ i = 1 \end{pmatrix} \overset{(10)}{=} \begin{pmatrix} \textbf{RHS} \\ \mathbf{f} = 2\,\mathbf{f} \circ \left(\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1\right) \\ a = \mathbf{p}_1 \times \mathbf{p}_1 \\ i = 2 \end{pmatrix} \begin{pmatrix} \textbf{RHS} \\ \mathbf{f} = 3\,\overset{1}{\mathbf{p}}_2 \\ a = \mathbf{p}_1 \times \mathbf{p}_1 \\ i = 1 \end{pmatrix} \begin{pmatrix} \textbf{RHS} \\ \mathbf{f} = 30\,\overset{1}{\mathbf{p}}_1 \\ a = \mathbf{p}_1 \times \mathbf{p}_1 \\ i = 1 \end{pmatrix}$$

$$\begin{pmatrix} \textbf{RHS} \\ \mathbf{f} = 3\,\overset{2}{\mathbf{p}}_2 \\ a = \mathbf{p}_1 \times \mathbf{p}_1 \\ i = 2 \end{pmatrix} \begin{pmatrix} \textbf{RHS} \\ \mathbf{f} = 2 \\ a = \mathbf{p}_1 \times \mathbf{p}_1 \\ i = 0 \end{pmatrix}$$

$$3\,\mathbf{f} \circ \left(\mathbf{p}_1 + \mathbf{p}_1\right) - 2\,\mathbf{f} \circ \left(\mathbf{p}_1 - \mathbf{p}_1\right) = 3\,\mathbf{p}_2 + 30\,\mathbf{p}_1 \cdot \mathbf{p}_1 + 3\,\mathbf{p}_2 + \mathbf{2}$$

Simplifying, we get:

$$3\,\mathbf{f} \circ \left(2\,\mathbf{p}_1\right) - 2\,\mathbf{f} \circ (\mathbf{0}) = 3\,\mathbf{p}_2 + 30\,\mathbf{p}_2 + 3\,\mathbf{p}_2 + \mathbf{2}$$

Which equals:

$$3\,\mathbf{f} \circ \left(2\,\mathbf{p}_1\right) - 2\,\mathbf{f}\,(0)\mathbf{p}_0 = 36\,\mathbf{p}_2 + \mathbf{2}$$

We then find the value of the functions on both sides at 0:

$$\left[3\,\mathbf{f} \circ \left(2\,\mathbf{p}_1\right) - 2\,\mathbf{f}\,(0)\mathbf{p}_0\right]\,(0) = \left[36\,\mathbf{p}_2 + \mathbf{2}\right]\,(0)$$

$$\overset{(2)}{=} \begin{pmatrix} \textbf{LHS} \\ \mathbf{f} = 3\,\mathbf{f} \circ \left(2\,\mathbf{p}_1\right) \\ g = -2\,\mathbf{f}\,(0)\mathbf{p}_0 \\ x = 0 \end{pmatrix} \begin{pmatrix} \textbf{LHS} \\ \mathbf{f} = 36\,\mathbf{p}_2 \\ g = \mathbf{2} \\ x = 0 \end{pmatrix} \left[3\,\mathbf{f} \circ \left(2\,\mathbf{p}_1\right)\right]\,(0) - \left[2\,\mathbf{f}\,(0)\mathbf{p}_0\right]\,(0) = \left[36\,\mathbf{p}_2\right]\,(0) + \left[\mathbf{2}\,(0)\right]$$

$$\overset{(5)\,(\mathbf{EF(1)})}{=} \begin{pmatrix} \textbf{LHS} \\ \mathbf{f} = 3\,\mathbf{f} \\ g = 2\,\mathbf{p}_1 \\ x = 0 \end{pmatrix} 3\,\mathbf{f}(0) - 2\,\mathbf{f}(0) = 0 + 2$$

Simplifying, we get:

$\mathbf{f}(0) = 2$

Substituting this value into the equation, we get:

$3\,\mathbf{f} \circ \left(2\,\mathbf{p}_1\right) - 4 = 36\,\mathbf{p}_2 + \mathbf{2}$

$3\,\mathbf{f} \circ \left(2\,\mathbf{p}_1\right) = 36\,\mathbf{p}_2 + \mathbf{6}$

We divide the whole equation by 3 to get:

$\mathbf{f} \circ \left(2\,\mathbf{p}_1\right) = 12\,\mathbf{p}_2 + \mathbf{2}$

Composing both sides with $1/2\ \mathbf{p}_1$:

$\left[\mathbf{f} \circ \left(2\,\mathbf{p}_1\right)\right] \circ (1/2\mathbf{p}_1) = \left[12\,\mathbf{p}_2 + \mathbf{2}\right] \circ (1/2\mathbf{p}_1)$

$\underset{\begin{pmatrix}\textbf{LHS}\\ \mathbf{f} = 12\,\mathbf{p}_2\\ g = \mathbf{2}\\ h = 1/2\mathbf{p}_1\end{pmatrix}}{\overset{(12)}{=}} \quad \left[\mathbf{f} \circ \left(2\,\mathbf{p}_1\right)\right] \circ (1/2\mathbf{p}_1) = \left[12\,\mathbf{p}_2\right] \circ (1/2\mathbf{p}_1) + [\mathbf{2}] \circ (1/2\mathbf{p}_1)$

$\underset{\begin{pmatrix}\textbf{LHS}\\ \mathbf{f} = \mathbf{f}\\ g = 2\mathbf{p}_1\\ h = 1/2\mathbf{p}_1\end{pmatrix}}{\overset{(14)}{=}} \quad \mathbf{f} \circ (2\,(1/2\mathbf{p}_1)) = 12\,\mathbf{p}_2 \circ (1/2\mathbf{p}_1) + \mathbf{2}$

where formula 14 is:

$$(f \circ g) \circ h = f \circ (g \circ h)$$

Simplifying, we get:

$\mathbf{f} \circ (\mathbf{p}_1) = 12\,(1/4\mathbf{p}_2) + \mathbf{2}$

Which equals:

$\mathbf{f} \circ (\mathbf{p}_1) = 3\mathbf{p}_2 + \mathbf{2}$

Thus, the function $\mathbf{f} = 3\mathbf{p}_2 + \mathbf{2}$

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Formula 14 - Lemma 1:
If $u : \mathbb{R}^n \to \mathbb{R}^m$ and $v : \mathbb{R}^m \to \mathbb{R}^k$ and $w : \mathbb{R}^k \to \mathbb{R}^l$, then

$$(w \circ v) \circ u = w \circ (v \circ u)$$

Proof of Lemma 1:
Step 0 (Identify Formula):
Formula 1

Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = (w \circ v) \circ u$
$g = w \circ (v \circ u)$
$D = \mathbb{R}^n$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}^n}{\forall} ((w \circ v) \circ u)(a) = (w \circ (v \circ u))(a)$$

Now, the computer has:

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = w \circ v$
$g = u$
$x = a$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}^n}{\forall} ((w \circ v) \circ u)(a) = (w \circ v)(u(a))$$

Step 0 (Identify Formula):
Formula 4
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$g = u$
$x = a$
$a^1 = u^1$, $a^2 = u^2$,..., $a^m = u^m$
Step 3 (Replacing Symbols):

$$= (w \circ v)(u^1(a), u^2(a), ..., u^m(a))$$

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = w$
$g = v$
$x = (u^1(a), u^2(a), ..., u^m(a))$
Step 3 (Replacing Symbols):

$$= w(v((u^1(a), u^2(a), ..., u^m(a)))$$

Step 0 (Identify Formula):
Formula 4
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$g = u$
$a^1 = u^1$, $a^2 = u^2$,..., $a^m = u^m$
$x = a$
Step 3 (Replacing Symbols):

$$= w(v((u(a)))$$

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$g = u$
$f = v$
$x = a$
Step 3 (Replacing Symbols):

$$= w((v \circ u)(a))$$

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$g = v \circ u$
$f = w$
$x = a$
Step 3 (Replacing Symbols):

$$= w \circ ((v \circ u)(a))$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f = (w \circ v) \circ u$
$g = w \circ (v \circ u)$
Step 3 (Replacing Symbols):

$$(w \circ v) \circ u = w \circ (v \circ u)$$

End of Proof of Lemma 1

Formula 16 - Lemma 2:

$$p_k \circ p_l = p_{k \cdot l}$$

Proof of Lemma 2:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_k \circ p_l$
$g = p_{k \cdot l}$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}}{\forall} (p_k \circ p_l)(a) = p_{k \cdot l}(a)$$

Now, the computer has:

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_k$
$g = p_l$
$x = a$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}}{\forall} (p_k \circ p_l)(a) = p_k(p_l(a))$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_l$
$x = a$
Step 3 (Replacing Symbols):

$$= p_k(a^l)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):

Left Side
Step 2 (Identify Symbols):
$p_a = p_k$
$x = a^l$
Step 3 (Replacing Symbols):

$$= (a^l)^k = a^{k \cdot l}$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$x = a$
$a = k \cdot l$
Step 3 (Replacing Symbols):

$$= p_{k \cdot l}(a)$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f = p_k \circ p_l$
$g = p_{k \cdot l}$
Step 3 (Replacing Symbols):

$$p_k \circ p_l = p_{k \cdot l}$$

End of Proof of Lemma 2

Formula 17 - Lemma 3:

$$\text{If } k > 0 \text{ then } p_k \circ \mathbf{0} = \mathbf{0}$$

Proof of Lemma 3:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_k \circ \mathbf{0}$
$g = \mathbf{0}$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}}{\forall} (p_k \circ \mathbf{0})(a) = \mathbf{0}(a)$$

Now, the computer has:

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_k$
$g = \mathbf{0}$
Step 3 (Replacing Symbols):

$$\underset{a\in\mathbb{R}}{\forall}(p_k\circ\mathbf{0})(a) = p_k(\mathbf{0}(a)) = p_k(0)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_k$
$x = 0$
Step 3 (Replacing Symbols):

$$= 0^k = 0 = \mathbf{0}(a)$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f = p_k\circ\mathbf{0}$
$g = \mathbf{0}$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$p_k\circ\mathbf{0} = \mathbf{0}$$

End of Proof of Lemma 3

Formula 18 - Lemma 4:

$$\text{If } f : \mathbb{R} \to \mathbb{R} \text{ then } f \circ p_1 = f$$

Proof of Lemma 4:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = f \circ p_1$
$g = f$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$\underset{a\in\mathbb{R}}{\forall}(f\circ p_l)(a)=f(a)$$

Now, the computer has:

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f=f$
$g=p_1$
$x=a$
Step 3 (Replacing Symbols):

$$\underset{a\in\mathbb{R}}{\forall}(f\circ p_l)(a)=f(p_1(a))$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a=p_1$
$x=a$
Step 3 (Replacing Symbols):

$$=f(a^1)=f(a)$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f=f\circ p_1$
$g=f$
Step 3 (Replacing Symbols):

$$f\circ p_1=f$$

End of Proof of Lemma 4

Formula 19 - Lemma 5:

$$\text{If } f:\mathbb{R}\to\mathbb{R} \text{ and } c\in\mathbb{R} \text{ then } c\circ f=c$$

Proof of Lemma 5:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side

Step 2 (Identify Symbols):
$f = c \circ f$
$g = c$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$\underset{a\in\mathbb{R}}{\forall}(c \circ f)(a) = c(a)$$

Now, we have:

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = c$
$g = f$
$x = a$
Step 3 (Replacing Symbols):

$$\underset{a\in\mathbb{R}}{\forall}(c \circ f)(a) = c(f(a)) = c$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f = c \circ p_1$
$g = c$
Step 3 (Replacing Symbols):

$$c \circ p_1 = c$$

End of Proof of Lemma 5

Formula 20 - Lemma 6:

$$\text{If } k > 0 \text{ and } c \in \mathbb{R} \text{ and } k \in \mathbb{R} \text{ then } cp_k \circ \mathbf{0} = \mathbf{0}$$

Proof of Lemma 6:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = cp_k \circ \mathbf{0}$
$g = \mathbf{0}$
$D = \mathbb{R}$
Step 3 (Replacing Symbols):

$$\underset{a\in\mathbb{R}}{\forall}(cp_k\circ\mathbf{0})(a)=\mathbf{0}(a)$$

Now, the computer has:

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f=cp_k$
$g=\mathbf{0}$
Step 3 (Replacing Symbols):

$$\underset{a\in\mathbb{R}}{\forall}(cp_k\circ\mathbf{0})(a)=cp_k(\mathbf{0}(a))=cp_k(0)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a=cp_k$
$x=0$
Step 3 (Replacing Symbols):

$$=c0^k=c\cdot 0=0=\mathbf{0}(a)$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f=cp_k\circ\mathbf{0}$
$g=\mathbf{0}$
$D=\mathbb{R}$
Step 3 (Replacing Symbols):

$$cp_k\circ\mathbf{0}=\mathbf{0}$$

End of Proof of Lemma 6

<u>**Problem 10**</u>
Find a function $\mathbf{f}$: $\mathbb{R}\to\mathbb{R}$ satisfying

$$3f\circ(\overset{1}{p}_1+\overset{2}{p}_1)-2f\circ(\overset{1}{p}_1-\overset{2}{p}_1)=3\overset{1}{p}_2+30\overset{1}{p}_1\cdot\overset{2}{p}_1+3\overset{2}{p}_2+2$$

Consider that:

$$L=3f\circ(\overset{1}{p}_1+\overset{2}{p}_1)-2f\circ(\overset{1}{p}_1-\overset{2}{p}_1)$$
$$R=3\overset{1}{p}_2+30\overset{1}{p}_1\cdot\overset{2}{p}_1+3\overset{2}{p}_2+2$$
$$u=(p_1\times\mathbf{0})$$

Now, the computer has:

$$L = R$$

Then:

$$L \circ u = R \circ u$$

Now:

$$L \circ u = (3f \circ (\overset{1}{p}_1 + \overset{2}{p}_1) - 2f \circ (\overset{1}{p}_1 - \overset{2}{p}_1)) \circ u$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 3f \circ (\overset{1}{p}_1 + \overset{2}{p}_1)$
$g = -(2f \circ (\overset{1}{p}_1 - \overset{2}{p}_1)$
$h = u$
Step 3 (Replacing Symbols):

$$= (3f \circ (\overset{1}{p}_1 + \overset{2}{p}_1)) \circ u - (2f \circ (\overset{1}{p}_1 - \overset{2}{p}_1)) \circ u$$

Step 0 (Identify Formula):
Formula 14
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$w = 3f$
$v = \overset{1}{p}_1 + \overset{2}{p}_1$
$u = u$
Step 3 (Replacing Symbols):

$$= 3f \circ ((\overset{1}{p}_1 + \overset{2}{p}_1) \circ u) - (2f \circ (\overset{1}{p}_1 - \overset{2}{p}_1)) \circ u$$

Step 0 (Identify Formula):
Formula 14
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$w = 2f$
$v = \overset{1}{p}_1 - \overset{2}{p}_1$
$u = u$
Step 3 (Replacing Symbols):

$$= 3f \circ ((\overset{1}{p}_1 + \overset{2}{p}_1) \circ u) - 2f \circ ((\overset{1}{p}_1 - \overset{2}{p}_1) \circ u)$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{1}{p}_1$
$g = \overset{2}{p}_1$
$h = u$
Step 3 (Replacing Symbols):

$$= 3f \circ (\overset{1}{p}_1 \circ u + \overset{2}{p}_1 \circ u) - 2f \circ ((\overset{1}{p}_1 - \overset{2}{p}_1) \circ u)$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{1}{p}_1$
$g = -\overset{2}{p}_1$
$h = u$
Step 3 (Replacing Symbols):

$$= 3f \circ (\overset{1}{p}_1 \circ u + \overset{2}{p}_1 \circ u) - 2f \circ (\overset{1}{p}_1 \circ u - \overset{2}{p}_1 \circ u)$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_1$
$i = 1$
$a^1 = p_1$
Step 3 (Replacing Symbols):

$$= 3f \circ (p_1 \circ p_1 + \overset{2}{p}_1 \circ u) - 2f \circ (p_1 \circ p_1 - \overset{2}{p}_1 \circ u)$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_1$
$i = 2$

$a^2 = textbf0$
Step 3 (Replacing Symbols):

$$= 3f \circ (p_1 \circ p_1 + p_1 \circ textbf0) - 2f \circ (p_1 \circ p_1 - p_1 \circ textbf0)$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
$p_l = p_1$
Step 3 (Replacing Symbols):

$$= 3f \circ (p_1 + p_1 \circ textbf0) - 2f \circ (p_1 - p_1 \circ textbf0)$$

Step 0 (Identify Formula):
Formula 17
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
Step 3 (Replacing Symbols):

$$= 3f \circ (p_1 + 0) - 2f \circ (p_1 - 0) = 3f \circ p_1 - 2f \circ p_1$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):
$f = 3f$
$g = -2f$
$h = p_1$
Step 3 (Replacing Symbols):

$$= (3f - 2f) \circ p_1 = f \circ p_1$$

Step 0 (Identify Formula):
Formula 18
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = f$
$p_1 = p_1$
Step 3 (Replacing Symbols):

$$= f$$

Now:

$$R \circ u = (3\overset{1}{p}_2 + 30\overset{1}{p}_1 \cdot \overset{2}{p}_1 + 3\overset{2}{p}_2 + \mathbf{2}) \circ u$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 3\overset{1}{p}_2$
$g = 30\overset{1}{p}_1 \cdot \overset{2}{p}_1 + 3\overset{2}{p}_2 + \mathbf{2}$
$h = u$
Step 3 (Replacing Symbols):

$$= 3\overset{1}{p}_2 \circ u + (30\overset{1}{p}_1 \cdot \overset{2}{p}_1 + 3\overset{2}{p}_2 + \mathbf{2}) \circ u$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 30\overset{1}{p}_1 \cdot \overset{2}{p}_1$
$g = 3\overset{2}{p}_2 + 2$
$h = u$
Step 3 (Replacing Symbols):

$$= 3\overset{1}{p}_2 \circ u + 30(\overset{1}{p}_1 \cdot \overset{2}{p}_1) \circ u + (3\overset{2}{p}_2 + \mathbf{2}) \circ u$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 3\overset{2}{p}_2$
$g = 2$
$h = u$
Step 3 (Replacing Symbols):

$$= 3\overset{1}{p}_2 \circ u + 30(\overset{1}{p}_1 \cdot \overset{2}{p}_1) \circ u + 3\overset{2}{p}_2 \circ u + \mathbf{2} \circ u$$

Step 0 (Identify Formula):
Formula 13
Step 1 (Identify Side):

Left Side
Step 2 (Identify Symbols):
$f = 30\overset{1}{p}_1$
$g = \overset{2}{p}_1$
$h = u$
Step 3 (Replacing Symbols):

$$= 3\overset{1}{p}_2 \circ u + 30(\overset{1}{p}_1 \circ u) \cdot (\overset{2}{p}_1 \circ u) + 3\overset{2}{p}_2 \circ u + \mathbf{2} \circ u$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 3p_2$
$i = 1$
$a^1 = p_1$
Step 3 (Replacing Symbols):

$$= 3p_2 \circ p_1 + 30(\overset{1}{p}_1 \circ u) \cdot (\overset{2}{p}_1 \circ u) + 3\overset{2}{p}_2 \circ u + \mathbf{2} \circ u$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 30p_1$
$i = 1$
$a^1 = p_1$
Step 3 (Replacing Symbols):

$$= 3p_2 \circ p_1 + 30(p_1 \circ p_1) \cdot (\overset{2}{p}_1 \circ u) + 3\overset{2}{p}_2 \circ u + \mathbf{2} \circ u$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_1$
$i = 2$
$a^1 = textbf0$
Step 3 (Replacing Symbols):

$$= 3p_2 \circ p_1 + 30(p_1 \circ p_1) \cdot (p_1 \circ textbf0) + 3\overset{2}{p}_2 \circ u + \mathbf{2} \circ u$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 3p_2$
$i = 2$
$a^1 = textbf0$
Step 3 (Replacing Symbols):

$$= 3p_2 \circ p_1 + 30(p_1 \circ p_1) \cdot (p_1 \circ 0) + 3p_2 \circ textbf0 + \mathbf{2} \circ u$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_2$
$p_l = p_1$
Step 3 (Replacing Symbols):

$$= 3p_2 + 30(p_1 \circ p_1) \cdot (p_1 \circ 0) + 3p_2 \circ 0 + \mathbf{2} \circ u$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
$p_l = p_1$
Step 3 (Replacing Symbols):

$$= 3p_2 + 30p_1 \cdot (p_1 \circ 0) + 3p_2 \circ 0 + \mathbf{2} \circ u$$

Step 0 (Identify Formula):
Formula 17
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
Step 3 (Replacing Symbols):

$$= 3p_2 + 30p_1 \cdot 0 + 3p_2 \circ 0 + \mathbf{2} \circ u = 3p_2 + 3p_2 \circ 0 + 2 \circ u$$

Step 0 (Identify Formula):
Formula 20

Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$c = 3$
$p_k = p_2$
Step 3 (Replacing Symbols):

$$= 3p_2 + 30p_1 \cdot 0 + 3 * 0 + \mathbf{2} \circ u = 3p_2 + \mathbf{2} \circ u$$

Step 0 (Identify Formula):
Formula 19
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$c = \mathbf{2}$
$f = u$
Step 3 (Replacing Symbols):

$$= 3p_2 + 2$$

In conclusion, the solution for our equation is:

$$f = 3p_2 + 2$$

Functions of Several Variables

Problems 1

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2. $\mathbf{g}\ (a, b) = 5\ \mathbf{exp} \circ (\overset{1}{\mathbf{p}}_1 \bullet \overset{2}{\mathbf{p}}_3)$

$$\begin{pmatrix} LHS \\ f = 5\mathbf{exp}2 \\ a^1 = (\overset{1}{\mathbf{p}}_1 \bullet \overset{2}{\mathbf{p}}_3) \\ x = (a, b) \end{pmatrix}$$
$\overset{(5)}{=} 5\mathbf{exp}[\overset{1}{\mathbf{p}}_1 \bullet \overset{2}{\mathbf{p}}_3](a, b)$

$$\begin{pmatrix} LHS \\ f = \overset{1}{\mathbf{p}}_1 \bullet \overset{2}{\mathbf{p}}_3 \\ x = (a, b) \end{pmatrix}$$
$\overset{(3)}{=} 5\mathbf{exp}[\overset{1}{\mathbf{p}}_1(a, b) \bullet \overset{2}{\mathbf{p}}_3(a, b)]$

$$\begin{pmatrix} LHS \\ f = \overset{1}{\mathbf{p}}_1 \\ g = \overset{2}{\mathbf{p}}_3 \\ x = (a, b) \end{pmatrix}$$

$\overset{(4)}{=} 5\mathbf{exp}[\mathbf{p}_1(a) \bullet \mathbf{p}_3(b)]$

$= 5e^{ab^3}$

$$\begin{pmatrix} LHS \\ f = \mathbf{p}_1 \\ i = 1 \\ x = (a, b) \end{pmatrix}$$

$$\begin{pmatrix} LHS \\ f = \mathbf{p}_3 \\ i = 2 \\ x = (a, b) \end{pmatrix}$$

3. $\mathbf{h}(a, b, c) = 2\mathbf{sin}[\overset{1}{\mathbf{p}}_2 + (3\overset{2}{\mathbf{p}}_1 \bullet \overset{3}{\mathbf{p}}_1)](a, b, c)$

$$\begin{pmatrix} LHS \\ f = 2\mathbf{sin} \\ a^1 = \overset{1}{\mathbf{p}}_2 + (3\overset{2}{\mathbf{p}}_1 \bullet \overset{3}{\mathbf{p}}_1) \\ \lambda = (a, b, c) \end{pmatrix}$$

$\overset{(2)}{=} 2\mathbf{sin}[\overset{1}{\mathbf{p}}_2(a, b, c) + (3\overset{2}{\mathbf{p}}_1 \bullet \overset{3}{\mathbf{p}}_1)(a, b, c)]$

$$\begin{pmatrix} LHS \\ f = \overset{1}{\mathbf{p}}_2 \\ g = 3\overset{2}{\mathbf{p}}_1 \bullet \overset{3}{\mathbf{p}}_1 \\ \lambda = (a, b, c) \end{pmatrix}$$

$\overset{(3)}{=} 2\mathbf{sin}[\overset{1}{\mathbf{p}}_2(a, b, c) + (3\overset{2}{\mathbf{p}}_1(a, b, c) \bullet \overset{3}{\mathbf{p}}_1(a, b, c,))]$

$$\begin{pmatrix} LHS \\ f = \overset{1}{\mathbf{p}}_2 \\ g = 3\overset{2}{\mathbf{p}}_1 \\ h = \overset{3}{\mathbf{p}}_1 \\ \lambda = (a, b, c) \end{pmatrix}$$

$\overset{(4)}{=} 2\mathbf{sin}[\mathbf{p}_2(a) + (3\mathbf{p}_1(b) \bullet \mathbf{p}_1(c))]$

$= 2\mathbf{sin}(a^2 + 3bc)\ \mathbb{R} \to \mathbb{R}^3$

$$\begin{pmatrix} LHS \\ f = \mathbf{p}_2 \\ i = 1 \\ \lambda = (a,b,c) \end{pmatrix}$$
$$\begin{pmatrix} LHS \\ g = 3\mathbf{p}_1 \\ i = 2 \\ \lambda = (a,b,c) \end{pmatrix}$$
$$\begin{pmatrix} LHS \\ h = \mathbf{p}_1 \\ i = 3 \\ \lambda = (a,b,c) \end{pmatrix}$$

6. $\mathbf{f}(a,x) \overset{def}{=} [(\overset{1}{\mathbf{sin}} \bullet \overset{2}{\mathbf{p}}_2 + \overset{1}{\mathbf{p}}_3) \times (\overset{1}{\mathbf{exp}} + \overset{2}{\mathbf{p}}_1) \times (\overset{2}{\mathbf{p}}_4)](a,x)$

$\overset{(4)}{=} [(\overset{1}{\mathbf{sin}} \bullet \overset{2}{\mathbf{p}}_2 + \overset{1}{\mathbf{p}}_3)(a,x), (\overset{1}{\mathbf{exp}} + \overset{2}{\mathbf{p}}_1)(a,x), (\overset{2}{\mathbf{p}}_4)(a,x)]$

$$\begin{pmatrix} LHS \\ a^1 = \overset{1}{\mathbf{sin}} \bullet \overset{2}{\mathbf{p}}_2 + \overset{1}{\mathbf{p}}_3 \\ a^2 = \overset{1}{\mathbf{exp}} + \overset{2}{\mathbf{p}}_1 \\ a^3 = \overset{2}{\mathbf{p}}_4 \\ x = (a,x) \end{pmatrix}$$

$\overset{(2)}{=} [(\overset{1}{\mathbf{sin}} \bullet \overset{2}{\mathbf{p}}_2)(a,x) + \overset{1}{\mathbf{p}}_3(a,x), \overset{1}{\mathbf{exp}}(a,x) + \overset{2}{\mathbf{p}}_1(a,x), \overset{2}{\mathbf{p}}_4(a,x)]$

$$\begin{pmatrix} LHS \\ f = \overset{1}{\mathbf{sin}} \bullet \overset{2}{\mathbf{p}}_2 \\ g = \overset{1}{\mathbf{p}}_3 \\ x = (a,x) \end{pmatrix}$$
$$\begin{pmatrix} LHS \\ f = \overset{1}{\mathbf{exp}} \\ g = \overset{2}{\mathbf{p}}_1 \\ x = (a,x) \end{pmatrix}$$

$\overset{(3)}{=} [\overset{1}{\mathbf{sin}}(a,x)\overset{2}{\mathbf{p}}_2(a,x) + \overset{1}{\mathbf{p}}_3(a,x), \overset{1}{\mathbf{exp}}(a,x) + \overset{2}{\mathbf{p}}_1(a,x), \overset{2}{\mathbf{p}}_4(a,x)]$

$$\begin{pmatrix} LHS \\ f = \overset{1}{\mathbf{sin}} \\ g = \overset{2}{\mathbf{p}}_2 \\ x = (a,x) \end{pmatrix}$$

$\overset{(6)}{=} [\mathbf{sin}(a)\mathbf{p}_2(x) + \mathbf{p}_3(a), \mathbf{exp}(a) + \mathbf{p}_1(x), \mathbf{p}_4(x)]$

$$\begin{pmatrix} LHS \\ f = \mathbf{sin} \\ i = 1 \\ x = (a, x) \end{pmatrix}$$
$$\begin{pmatrix} LHS \\ f = \mathbf{p}_2 \\ i = 2 \\ x = (a, x) \end{pmatrix}$$
$$\begin{pmatrix} LHS \\ f = \mathbf{p}_3 \\ i = 1 \\ x = (a, x) \end{pmatrix}$$
$$\begin{pmatrix} LHS \\ f = \mathbf{exp} \\ i = 1 \\ x = (a, x) \end{pmatrix}$$
$$\begin{pmatrix} LHS \\ f = \mathbf{p}_1 \\ i = 2 \\ x = (a, x) \end{pmatrix}$$
$$\begin{pmatrix} LHS \\ f = \mathbf{p}_4 \\ i = 2 \\ x = (a, x) \end{pmatrix}$$
$$= [(\mathbf{sin}a)x^2 + a^3, e^a + x, x^4] \ \mathbb{R}^2 \to \mathbb{R}^3$$

Functions of Several Variables

Sooraj Boominathan, Vincent Po, Ben Zhao, Nathan Yu, Steve Zhang, Varun Ennamuri

Problems 6

5. Let $m = X \equiv (x, y, z)$ be the line and A the point in the Euclidean space $\mathbb{E}_3$. Find a **vector equation** and a **number equation** for the line l passing through the point A and parallel to the line m.

Since we don't have three points, we can't form two "linearly independent" vectors to create a linear combination of to describe the line. Instead, we have to use the inner product.

The vector equation, following this method, is:

$$\mathbf{E}_3 \supset l = \{X \equiv (x, y, z); [AX] \bullet \mathbf{m} = \pm |AX| \cdot |\mathbf{m}|\}$$

6. Find a **vector equation** and a **number equation** for the plane p that is perpendicular to the vector $\mathbf{n}$ and is passing through the point A.

Similar to problem 5, we do not have 3 points, and cannot form two vectors by which all other position vectors of the plan can be described. We thus again use an inner product:

$$\mathbf{E}_{_3} \supset p = \{X \equiv (x, y, z); [AX] \bullet \mathbf{n} = 0\}$$

7. Find a **vector equation** and a **number equation** for the plane p passing through the points A, B, and C.

Program

Step 1: Form (download) vectors [AB] and [AC], such that the line passing through [AB] is not parallel to the line passing through [AC].
Step 2: Form the vector equation of the plane.
Step 3: Form the number equation of the plane.

Running the program:

Step 1: Form (download) vectors [AB] and [AC], such that the line passing through [AB] is not parallel to the line passing through [AC].

$$[AB] = \bar{b} - \bar{a}$$

$$[AC] = \bar{c} - \bar{a}$$

Vectors AB and AC are "linearly independent", and thus any other vector on the same plane can be described as a linear combination of these two vectors.

Step 2: Form the vector equation of the plane.

$$\mathbf{E}_{_3} \supset p \equiv \{X; \bar{x} = \alpha[AB] + \beta[AC] \wedge \alpha, \beta \in R\}$$

This is the vector equation of the plane.

Step 3: Form the number equation of the plane.

$$\{X \equiv x = (x^1, x^2, x^3); x = \mathbf{f}(\alpha, \beta) \wedge \mathbf{f} = \mathbf{f}_{_1} \times \mathbf{f}_{_2} \times \mathbf{f}_{_3} : \mathbf{R}^2 \to \mathbf{R}^3$$

$$\begin{aligned}
&\wedge \mathbf{f}_{1} = d^1\mathbf{p}_{0} + a^1\mathbf{p}_{1}\\
&\wedge \mathbf{f}_{2} = d^2\mathbf{p}_{0} + a^2\mathbf{p}_{1}\\
&\wedge \mathbf{f}_{3} = d^3\mathbf{p}_{0} + a^3\mathbf{p}_{1} \wedge \alpha, \beta \in \mathbf{R}\}
\end{aligned}$$

This is the number equation.

The other way to do this problem is to use an inner product: The vector equation of this method is given by:

$$\mathbf{E}_{3} \supset l = \{X \equiv (x, y, z); (\bar{x} - \bar{a}) \bullet \bar{n} = 0\}$$

To solve this, we can take x to be the position vectors of our other two points given, B and C. From this, we get two equations:

$$(\bar{c} - \bar{a}) \bullet \bar{n} = 0 \text{ and } (\bar{b} - \bar{a}) \bullet \bar{n} = 0$$

This of course assumes an orthonormal basis. If we solved all of this with the familiar i-j-k system, we would get two equations that look similar to:

$$n^1 + n^2 + n^3 = 0 \text{ with appropriate coefficients in front.}$$

Eventually, if we solved the equations for $n^1, n^2, and n^3$, we could get a number equation similar to:

$$n^1 x + n^2 y + n^3 z + C = 0$$

Which we could rewrite as a function by using the vector equation for a plane:

$$\mathbf{E}_{3} \supset p = \{X \equiv (x, y, z) = a; f(a) = 0 \wedge f = n^1\overset{1}{\mathbf{p}}_{1} + n^2\overset{2}{\mathbf{p}}_{1} + n^3\overset{3}{\mathbf{p}}_{1} + C \wedge f : \mathbf{R}^3 \to \mathbf{R}$$

8. Find a **vector equation** and a **number equation** for the line l passing through the point D and perpendicular to the plane p.

This combines problems 5 and 7. Take [YZ] to be the vector formed by two arbitrary points on the plane p. The vector equation is:

$$\mathbf{E}_{3} \supset l \equiv \{X \equiv (x, y, z); [DX] \bullet [YZ] = 0\}$$

9. Find the point P that is the intersection of p and l from **8.**

Functions of Several Variables

Mehdi Drissi, Rahul Rajala, Cody Poteet, Kenneth Teel, Jerry Ni

Formula 14 - Lemma 1:
If $u : \mathbb{R}^n \to \mathbb{R}^m$ and $v : \mathbb{R}^m \to \mathbb{R}^k$ and $w : \mathbb{R}^k \to \mathbb{R}^l$, then

$$(w \circ v) \circ u = w \circ (v \circ u)$$

Proof of Lemma 1:
Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = (w \circ v) \circ u$$
$$g = w \circ (v \circ u)$$
$$D = \mathbb{R}^n$$

Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}^n}{\forall} ((w \circ v) \circ u)(a) = (w \circ (v \circ u))(a)$$

Now, we have:
Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = w \circ v$$
$$g = u$$
$$x = a$$

Step 3 (Replacing Symbols):

$$\underset{a \in \mathbb{R}^n}{\forall} ((w \circ v) \circ u)(a) = (w \circ v)(u(a))$$

Step 0 (Identify Formula):
Formula 4

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$g = u$

$x = a$

$a^1 = u^1$**,** $a^2 = u^2$**,...,** $a^m = u^m$

Step 3 (Replacing Symbols):

$$= (w \circ v)(u^1(a), u^2(a), ..., u^m(a))$$

Step 0 (Identify Formula):

Formula 5

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$f = w$

$g = v$

$x = (u^1(a), u^2(a), ..., u^m(a))$

Step 3 (Replacing Symbols):

$$= w(v((u^1(a), u^2(a), ..., u^m(a)))$$

Step 0 (Identify Formula):

Formula 4

Step 1 (Identify Side):

Right Side

Step 2 (Identify Symbols):

$g = u$

$a^1 = u^1$**,** $a^2 = u^2$**,...,** $a^m = u^m$

$x = a$

Step 3 (Replacing Symbols):

$$= w(v((u(a)))$$

Step 0 (Identify Formula):

Formula 5

Step 1 (Identify Side):

Right Side

Step 2 (Identify Symbols):

$$g = u$$
$$f = v$$
$$x = a$$

Step 3 (Replacing Symbols):

$$= w((v \circ u)(a))$$

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):

$$g = v \circ u$$
$$f = w$$
$$x = a$$

Step 3 (Replacing Symbols):

$$= w \circ ((v \circ u)(a))$$

Step 0 (Identify Formula):
Formula 1
Step 1 (Identify Side):
Right Side
Step 2 (Identify Symbols):

$$f = (w \circ v) \circ u$$
$$g = w \circ (v \circ u)$$

Step 3 (Replacing Symbols):

$$(w \circ v) \circ u = w \circ (v \circ u)$$

End of Proof of Lemma 1

<u>Problem 12a</u>

Form the quotient functions ${}^{i}Q_c^f$ if $i = 1, 2$ $c = (c^1, c^2) \in \mathbb{R}^2$ and

$$f = \overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \left(p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)\right) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2) \cup \mathbf{0}_{|\{(0,0)\}} : \mathbb{R}^2 \to \mathbb{R}$$

Step 0 (Identify Formula):
Formula 15

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = f$$
$$i = 1$$
$$a = c$$

Step 3 (Replacing Symbols):

$$ {}^{1}Q_c^f = p_{-1} \cdot ((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2)) \circ (c^1 p_0 + p_1 \times c^2 p_0) $$
$$ -((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2)p_0)) $$

Step 0 (Identify Formula):

Formula 13

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = \overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2))$$
$$g = \overset{1}{p}_2 - \overset{2}{p}_2$$
$$h = (c^1 p_0 + p_1 \times c^2 p_0)$$

Step 3 (Replacing Symbols):

$$ = p_{-1} \cdot ((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2))) \circ (c^1 p_0 + p_1 \times c^2 p_0) \cdot ((\overset{1}{p}_2 - \overset{2}{p}_2) \circ (c^1 p_0 + p_1 \times c^2 p_0)) $$
$$ -((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2)p_0)) $$

Step 0 (Identify Formula):

Formula 13

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = \overset{1}{p}_1 \cdot \overset{2}{p}_1$$
$$g = p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)$$
$$h = (c^1 p_0 + p_1 \times c^2 p_0)$$

Step 3 (Replacing Symbols):

$$ = p_{-1} \cdot ((\overset{1}{p}_1 \cdot \overset{2}{p}_1) \circ (c^1 p_0 + p_1 \times c^2 p_0) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2))) \circ (c^1 p_0 + p_1 \times $$
$$ c^2 p_0) \cdot ((\overset{1}{p}_2 - \overset{2}{p}_2) \circ (c^1 p_0 + p_1 \times c^2 p_0)) $$
$$ -((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2)p_0)) $$

Step 0 (Identify Formula):
Formula 13
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = \overset{1}{p}_1$$
$$g = \overset{2}{p}_1$$
$$h = (c^1 p_0 + p_1 \times c^2 p_0)$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((\overset{1}{p}_1 \circ (c^1 p_0 + p_1 \times c^2 p_0)) \cdot (\overset{2}{p}_1 \circ (c^1 p_0 + p_1 \times c^2 p_0)) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \circ (c^1 p_0 + p_1 \times c^2 p_0)) \cdot ((\overset{1}{p}_2 - \overset{2}{p}_2) \circ (c^1 p_0 + p_1 \times c^2 p_0)) - ((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2) p_0))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = p_1$$
$$i = 1$$
$$a^1 = c^1 p_0 + p_1$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((p_1 \circ (c^1 p_0 + p_1)) \cdot (\overset{2}{p}_1 \circ (c^1 p_0 + p_1 \times c^2 p_0)) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \circ (c^1 p_0 + p_1 \times c^2 p_0)) \cdot ((\overset{1}{p}_2 - \overset{2}{p}_2) \circ (c^1 p_0 + p_1 \times c^2 p_0)) - ((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2) p_0))$$

Step 0 (Identify Formula):
Formula 21
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = c^1 p_0 + p_1$$
$$p_k = 1$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (\overset{2}{p}_1 \circ (c^1 p_0 + p_1 \times c^2 p_0)) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \circ (c^1 p_0 + p_1 \times c^2 p_0)) \cdot ((\overset{1}{p}_2 - \overset{2}{p}_2) \circ (c^1 p_0 + p_1 \times c^2 p_0))$$
$$-((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2)p_0))$$

Step 0 (Identify Formula):

Formula 10

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = p_1$$
$$i = 2$$
$$a^2 = c^2 p_0$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (p_1 \circ (c^2 p_0)) \cdot ((p_{-1} \circ (\overset{1}{p}_1 + \overset{2}{p}_1)) \circ (c^1 p_0 + p_1 \times c^2 p_0)) \cdot ((\overset{1}{p}_2 - \overset{2}{p}_2) \circ (c^1 p_0 + p_1 \times c^2 p_0))$$
$$-((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2)p_0))$$

Step 0 (Identify Formula):

Formula 21

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = c^2 p_0$$
$$p_k = p_1$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \circ (c^1 p_0 + p_1 \times c^2 p_0)) \cdot ((\overset{1}{p}_2 - \overset{2}{p}_2) \circ (c^1 p_0 + p_1 \times c^2 p_0))$$
$$-((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2)p_0))$$

Step 0 (Identify Formula):

Formula 12

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = \overset{1}{p}_2$$

$$g = -\overset{2}{p}_2$$
$$h = (c^1p_0 + p_1 \times c^2p_0)$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1p_0 + p_1) \cdot (c^2p_0) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \circ (c^1p_0 + p_1 \times c^2p_0)) \cdot ((\overset{1}{p}_2 \circ (c^1p_0 + p_1 \times c^2p_0)) - (\overset{2}{p}_2 \circ (c^1p_0 + p_1 \times c^2p_0))) - ((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2)p_0))$$

Step 0 (Identify Formula):

Formula 10

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = p_2$$
$$i = 1$$
$$a^1 = c^1p_0 + p_1$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1p_0 + p_1) \cdot (c^2p_0) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \circ (c^1p_0 + p_1 \times c^2p_0)) \cdot ((p_2 \circ (c^1p_0 + p_1)) - (\overset{2}{p}_2 \circ (c^1p_0 + p_1 \times c^2p_0))) - ((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2)p_0))$$

Step 0 (Identify Formula):

Formula 21

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = c^1p_0 + p_1$$
$$p_k = p_2$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1p_0 + p_1) \cdot (c^2p_0) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \circ (c^1p_0 + p_1 \times c^2p_0)) \cdot ((c^1p_0 + p_1)^2 - (\overset{2}{p}_2 \circ (c^1p_0 + p_1 \times c^2p_0))) - ((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2)p_0))$$

Step 0 (Identify Formula):

Formula 10

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = p_2$$
$$i = 2$$
$$a^2 = c^2 p_0$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \circ (c^1 p_0 + p_1 \times c^2 p_0)) \cdot ((c^1 p_0 + p_1)^2 - (p_2 \circ (c^2 p_0)))$$
$$-((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2) p_0))$$

Step 0 (Identify Formula):

Formula 21

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$p_k = p_2$$
$$f = c^2 p_0$$

Step 3 (Replacing Symbols):

$$=$$
$$p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \circ (c^1 p_0 + p_1 \times c^2 p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2) p_0))$$

Step 0 (Identify Formula):

Formula 14

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$w = p_{-1}$$
$$v = \overset{1}{p}_2 + \overset{2}{p}_2$$
$$u = (c^1 p_0 + p_1 \times c^2 p_0)$$

Step 3 (Replacing Symbols):

$$=$$
$$p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((\overset{1}{p}_2 + \overset{2}{p}_2) \circ (c^1 p_0 + p_1 \times c^2 p_0))) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2) p_0))$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = \overset{1}{p}_2$$
$$g = \overset{2}{p}_2$$
$$h = (c^1 p_0 + p_1 \times c^2 p_0)$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ (\overset{1}{p}_2 \circ (c^1 p_0 + p_1 \times c^2 p_0) + \overset{2}{p}_2 \circ (c^1 p_0 + p_1 \times c^2 p_0))) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2) p_0))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = p_2$$
$$i = 1$$
$$a^1 = c^1 p_0 + p_1$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ (p_2 \circ (c^1 p_0 + p_1) + \overset{2}{p}_2 \circ (c^1 p_0 + p_1 \times c^2 p_0))) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)) \cdot (\overset{1}{p}_2 - \overset{2}{p}_2))(c^1, c^2) p_0))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = p_2$$
$$i = 2$$
$$a^2 = c^2 p_0$$

Step 3 (Replacing Symbols):

$$=$$

$$p_{-1}\cdot((c^1p_0+p_1)\cdot(c^2p_0)\cdot(p_{-1}\circ(p_2\circ(c^1p_0+p_1)+p_2\circ(c^2p_0)))\cdot((c^1p_0+p_1)^2-c^{2^2}p_0)$$
$$-((\overset{1}{p}_1\cdot\overset{2}{p}_1\cdot(p_{-1}\circ(\overset{1}{p}_2+\overset{2}{p}_2))\cdot(\overset{1}{p}_2-\overset{2}{p}_2))(c^1,c^2)p_0))$$

Step 0 (Identify Formula):

Formula 21

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f=c^1p_0+p_1$$
$$p_k=p_2$$

Step 3 (Replacing Symbols):

$$=p_{-1}\cdot((c^1p_0+p_1)\cdot(c^2p_0)\cdot(p_{-1}\circ((c^1p_0+p_1)^2+p_2\circ(c^2p_0)))\cdot((c^1p_0+p_1)^2-c^{2^2}p_0)$$
$$-((\overset{1}{p}_1\cdot\overset{2}{p}_1\cdot(p_{-1}\circ(\overset{1}{p}_2+\overset{2}{p}_2))\cdot(\overset{1}{p}_2-\overset{2}{p}_2))(c^1,c^2)p_0))$$

Step 0 (Identify Formula):

Formula 21

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f=c^2p_0$$
$$p_k=p_2$$

Step 3 (Replacing Symbols):

$$=p_{-1}\cdot((c^1p_0+p_1)\cdot(c^2p_0)\cdot(p_{-1}\circ((c^1p_0+p_1)^2+c^{2^2}p_0))\cdot((c^1p_0+p_1)^2-c^{2^2}p_0)$$
$$-((\overset{1}{p}_1\cdot\overset{2}{p}_1\cdot(p_{-1}\circ(\overset{1}{p}_2+\overset{2}{p}_2))\cdot(\overset{1}{p}_2-\overset{2}{p}_2))(c^1,c^2)p_0)$$

Step 0 (Identify Formula):

Formula 3

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f=\overset{1}{p}_1\cdot\overset{2}{p}_1\cdot(p_{-1}\circ(\overset{1}{p}_2+\overset{2}{p}_2))$$
$$g=\overset{1}{p}_2-\overset{2}{p}_2$$
$$x=c^1,c^2$$

Step 3 (Replacing Symbols):

$$=p_{-1}\cdot((c^1p_0+p_1)\cdot(c^2p_0)\cdot(p_{-1}\circ((c^1p_0+p_1)^2+c^{2^2}p_0))\cdot((c^1p_0+p_1)^2-c^{2^2}p_0)$$

$$-((\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2))(c^1, c^2)) \cdot ((\overset{1}{p}_2 - \overset{2}{p}_2)(c^1, c^2)))p_0)$$

Step 0 (Identify Formula):
Formula 3
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = \overset{1}{p}_1$$
$$g = \overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2))$$
$$x = c^1, c^2$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((c^1 p_0 + p_1)^2 + c^{2^2} p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((\overset{1}{p}_1(c^1, c^2) \cdot (\overset{2}{p}_1 \cdot (p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2))(c^1, c^2)) \cdot ((\overset{1}{p}_2 - \overset{2}{p}_2)(c^1, c^2))p_0)$$

Step 0 (Identify Formula):
Formula 3
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = \overset{2}{p}_1$$
$$g = p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2)$$
$$x = c^1, c^2$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((c^1 p_0 + p_1)^2 + c^{2^2} p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((\overset{1}{p}_1(c^1, c^2) \cdot \overset{2}{p}_1(c^1, c^2) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2))(c^1, c^2)) \cdot ((\overset{1}{p}_2 - \overset{2}{p}_2)(c^1, c^2)))p_0)$$

Step 0 (Identify Formula):
Formula 6
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = p_1$$
$$i = 1$$
$$x^1 = c^1$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((c^1 p_0 + p_1)^2 + c^{2^2} p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((p_1(c^1) \cdot \overset{2}{p}_1(c^1, c^2) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2))(c^1, c^2)) \cdot ((\overset{1}{p}_2 - \overset{2}{p}_2)(c^1, c^2)))p_0)$$

Step 0 (Identify Formula):

Formula 6

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = p_1$$
$$i = 2$$
$$x^1 = c^2$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((c^1 p_0 + p_1)^2 + c^{2^2} p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((p_1(c^1) \cdot p_1(c^2) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2))(c^1, c^2)) \cdot (\overset{1}{p}_2(c^1, c^2) - \overset{2}{p}_2(c^1, c^2)))p_0)$$

Step 0 (Identify Formula):

Formula 2

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = \overset{1}{p}_2$$
$$g = -\overset{2}{p}_2$$
$$x = c^1, c^2$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((c^1 p_0 + p_1)^2 + c^{2^2} p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((p_1(c^1) \cdot p_1(c^2) \cdot ((p_{-1} \circ (\overset{1}{p}_2 + \overset{2}{p}_2))(c^1, c^2)) \cdot (\overset{1}{p}_2(c^1, c^2) - \overset{2}{p}_2(c^1, c^2)))p_0)$$

Step 0 (Identify Formula):

Formula 6

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = p_2$$
$$i = 1$$
$$x^1 = c^1$$

Step 3 (Replacing Symbols):

$$= p_{-1}\cdot((c^1p_0+p_1)\cdot(c^2p_0)\cdot(p_{-1}\circ((c^1p_0+p_1)^2+c^{2^2}p_0))\cdot((c^1p_0+p_1)^2-c^{2^2}p_0)$$
$$-((p_1(c^1)\cdot p_1(c^2)\cdot((p_{-1}\circ(\overset{1}{p}_2+\overset{2}{p}_2))(c^1,c^2))\cdot(p_2(c^1)-\overset{2}{p}_2(c^1,c^2)))p_0)$$

Step 0 (Identify Formula):
Formula 6
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = p_2$$
$$i = 2$$
$$x^2 = c^2$$

Step 3 (Replacing Symbols):

$$= p_{-1}\cdot((c^1p_0+p_1)\cdot(c^2p_0)\cdot(p_{-1}\circ((c^1p_0+p_1)^2+c^{2^2}p_0))\cdot((c^1p_0+p_1)^2-c^{2^2}p_0)$$
$$-((p_1(c^1)\cdot p_1(c^2)\cdot((p_{-1}\circ(\overset{1}{p}_2+\overset{2}{p}_2))(c^1,c^2))\cdot(p_2(c^1)-p_2(c^2)))p_0)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$p_a = p_1$$
$$x = c^1$$

Step 3 (Replacing Symbols):

$$= p_{-1}\cdot((c^1p_0+p_1)\cdot(c^2p_0)\cdot(p_{-1}\circ((c^1p_0+p_1)^2+c^{2^2}p_0))\cdot((c^1p_0+p_1)^2-c^{2^2}p_0)$$
$$-((c^1\cdot p_1(c^2)\cdot((p_{-1}\circ(\overset{1}{p}_2+\overset{2}{p}_2))(c^1,c^2))\cdot(p_2(c^1)-p_2(c^2)))p_0)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$p_a = p_1$$
$$x = c^2$$

Step 3 (Replacing Symbols):

$$= p_{-1}\cdot((c^1p_0+p_1)\cdot(c^2p_0)\cdot(p_{-1}\circ((c^1p_0+p_1)^2+c^{2^2}p_0))\cdot((c^1p_0+p_1)^2-c^{2^2}p_0)$$
$$-((c^1\cdot c^2\cdot((p_{-1}\circ(\overset{1}{p}_2+\overset{2}{p}_2))(c^1,c^2))\cdot(p_2(c^1)-p_2(c^2)))p_0)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$p_a = p_2$$
$$x = c^1$$

Step 3 (Replacing Symbols):

$$= p_{-1}\cdot((c^1p_0+p_1)\cdot(c^2p_0)\cdot(p_{-1}\circ((c^1p_0+p_1)^2+c^{2^2}p_0))\cdot((c^1p_0+p_1)^2-c^{2^2}p_0)$$
$$-((c^1\cdot c^2\cdot((p_{-1}\circ(\overset{1}{p}_2+\overset{2}{p}_2))(c^1,c^2))\cdot(c^{1^2}-p_2(c^2)))p_0)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$p_a = p_2$$
$$x = c^2$$

Step 3 (Replacing Symbols):

$$= p_{-1}\cdot((c^1p_0+p_1)\cdot(c^2p_0)\cdot(p_{-1}\circ((c^1p_0+p_1)^2+c^{2^2}p_0))\cdot((c^1p_0+p_1)^2-c^{2^2}p_0)$$
$$-((c^1\cdot c^2\cdot((p_{-1}\circ(\overset{1}{p}_2+\overset{2}{p}_2))(c^1,c^2))\cdot(c^{1^2}-c^{2^2}))p_0)$$

Step 0 (Identify Formula):
Formula 5
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$f = p_{-1}$$
$$g = \overset{1}{p}_2+\overset{2}{p}_2$$
$$x = c^1, c^2$$

Step 3 (Replacing Symbols):

$$= p_{-1}\cdot((c^1p_0+p_1)\cdot(c^2p_0)\cdot(p_{-1}\circ((c^1p_0+p_1)^2+c^{2^2}p_0))\cdot((c^1p_0+p_1)^2-c^{2^2}p_0)$$
$$-((c^1\cdot c^2\cdot(p_{-1}((\overset{1}{p}_2+\overset{2}{p}_2)(c^1,c^2)))\cdot(c^{1^2}-c^{2^2}))p_0)$$

Step 0 (Identify Formula):
Formula 2

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = \overset{1}{p}_2$$

$$g = \overset{2}{p}_2$$

$$x = c^1, c^2$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((c^1 p_0 + p_1)^2 + c^{2^2} p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((c^1 \cdot c^2 \cdot (p_{-1}(\overset{1}{p}_2(c^1, c^2) + \overset{2}{p}_2(c^1, c^2))) \cdot (c^{1^2} - c^{2^2}))p_0)$$

Step 0 (Identify Formula):

Formula 6

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = p_2$$

$$i = 1$$

$$x^1 = c^1$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((c^1 p_0 + p_1)^2 + c^{2^2} p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((c^1 \cdot c^2 \cdot (p_{-1}(p_2(c^1) + \overset{2}{p}_2(c^1, c^2))) \cdot (c^{1^2} - c^{2^2}))p_0)$$

Step 0 (Identify Formula):

Formula 6

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$$f = p_2$$

$$i = 2$$

$$x^2 = c^2$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((c^1 p_0 + p_1)^2 + c^{2^2} p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0)$$
$$-((c^1 \cdot c^2 \cdot (p_{-1}(p_2(c^1) + p_2(c^2))) \cdot (c^{1^2} - c^{2^2}))p_0)$$

Step 0 (Identify Formula):

Elementary Function 1

Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$p_k = p_2$$
$$x = c^1$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((c^1 p_0 + p_1)^2 + c^{2^2} p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0) \\ -((c^1 \cdot c^2 \cdot (p_{-1}(c^{1^2} + p_2(c^2))) \cdot (c^{1^2} - c^{2^2}))p_0)$$

Step 0 (Identify Formula):
Elementary Function 1
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$$p_k = p_2$$
$$x = c^2$$

Step 3 (Replacing Symbols):

$$= p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((c^1 p_0 + p_1)^2 + c^{2^2} p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0) \\ -((c^1 \cdot c^2 \cdot (p_{-1}(c^{1^2} + c^{2^2})) \cdot (c^{1^2} - c^{2^2}))p_0)$$

$${}^1Q_c^f = p_{-1} \cdot ((c^1 p_0 + p_1) \cdot (c^2 p_0) \cdot (p_{-1} \circ ((c^1 p_0 + p_1)^2 + c^{2^2} p_0)) \cdot ((c^1 p_0 + p_1)^2 - c^{2^2} p_0) \\ -((c^1 \cdot c^2 \cdot (p_{-1}(c^{1^2} + c^{2^2})) \cdot (c^{1^2} - c^{2^2}))p_0) : \mathbb{R} \to \mathbb{R};$$

Now we'll check ${}^1Q_c^f$ at one value u informally.

$${}^1Q_c^f(e) = u^{-1} \cdot (c^1 + u) \cdot c^2 \cdot (c^{1^2} + u^2 + 2c^1 e + c^{2^2})^{-1} \cdot (c^{1^2} + u^2 + 2c^1 u - \\ c^{2^2}) - (c^1 \cdot c^2 \cdot (c^{1^2} + c^{2^2})^{-1}) \cdot (c^{1^2} - c^{2^2})$$

$$= \frac{(c^1 + u) \cdot c^2 \cdot (c^{1^2} + u^2 + 2c^1 u - c^{2^2}) \cdot -c^1 \cdot c^2 \cdot (c^{1^2} - c^{2^2})}{u \cdot (c^{1^2} + u^2 + 2c^1 u + c^{2^2}) \cdot (c^{1^2} + c^{2^2})}$$

Problems 1

Yitong Chen, Preeti Mohan, Caron Song, Divya Velury, Julie Zhu

10. $3\mathbf{f} \circ (\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1) - 2\mathbf{f} \circ (\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1) = 3\overset{1}{\mathbf{p}}_1 + 30\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 + 3\overset{2}{\mathbf{p}}_2 + \mathbf{2}$

We then take the composition of both sides by the Cartesian Product function $(\mathbf{a} \times \mathbf{0})$ to get

$[3\mathbf{f} \circ (\overset{1}{\mathbf{p}}_1 + \overset{2}{\mathbf{p}}_1) - 2\mathbf{f} \circ (\overset{1}{\mathbf{p}}_1 - \overset{2}{\mathbf{p}}_1)] \circ (\mathbf{a} \times \mathbf{0}) = (3\overset{1}{\mathbf{p}}_2 + 30\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 + 3\overset{2}{\mathbf{p}}_2 + \mathbf{2}) \circ (\mathbf{a} \times \mathbf{0})$

Where $\mathbf{a}$ is an arbitrary but fixed constant function and $\mathbf{0}$ is the constant function of the number zero.

Simplifying the left side first, or

$$[3\mathbf{f}\circ(\overset{1}{\mathbf{p}}_1+\overset{2}{\mathbf{p}}_1)-2\mathbf{f}\circ(\overset{1}{\mathbf{p}}_1-\overset{2}{\mathbf{p}}_1)]\circ(\mathbf{a}\times\mathbf{0})$$

$$\underset{\begin{pmatrix} LHS\\ f=3\mathbf{f}\circ(\overset{1}{\mathbf{p}}_1+\overset{2}{\mathbf{p}}_1)\\ g=-2\mathbf{f}\circ(\overset{1}{\mathbf{p}}_1-\overset{2}{\mathbf{p}}_1)\\ h=\mathbf{a}\times\mathbf{0}\end{pmatrix}}{\overset{(12)}{=}} 3\mathbf{f}\circ(\overset{1}{\mathbf{p}}_1+\overset{2}{\mathbf{p}}_1)\circ(\mathbf{a}\times\mathbf{0})-2\mathbf{f}\circ(\overset{1}{\mathbf{p}}_1-\overset{2}{\mathbf{p}}_1)\circ(\mathbf{a}\times\mathbf{0})$$

$$\underset{\begin{pmatrix} LHS\\ f=\overset{1}{\mathbf{p}}_1\\ g=\overset{2}{\mathbf{p}}_1\\ h=\mathbf{a}\times\mathbf{0}\end{pmatrix}\begin{pmatrix} LHS\\ f=\overset{1}{\mathbf{p}}_1\\ g=-\overset{2}{\mathbf{p}}_1\\ h=\mathbf{a}\times\mathbf{0}\end{pmatrix}}{\overset{(12)}{=}} 3\mathbf{f}\circ(\overset{1}{\mathbf{p}}_1\circ(\mathbf{a}\times\mathbf{0})+\overset{2}{\mathbf{p}}_1\circ(\mathbf{a}\times\mathbf{0}))-2\mathbf{f}\circ(\overset{1}{\mathbf{p}}_1\circ(\mathbf{a}\times\mathbf{0})-\overset{2}{\mathbf{p}}_1\circ(\mathbf{a}\times\mathbf{0}))$$

$$\underset{\begin{pmatrix} LHS\\ f=\overset{1}{\mathbf{p}}_1\\ a^i=(\mathbf{a}\times\mathbf{0})\end{pmatrix}\begin{pmatrix} LHS\\ f=\overset{2}{\mathbf{p}}_1\\ a^i=(\mathbf{a}\times\mathbf{0})\end{pmatrix}\begin{pmatrix} LHS\\ f=\overset{1}{\mathbf{p}}_1\\ a^i=(\mathbf{a}\times\mathbf{0})\end{pmatrix}\begin{pmatrix} LHS\\ f=\overset{2}{\mathbf{p}}_1\\ a^i=(\mathbf{a}\times\mathbf{0})\end{pmatrix}}{\overset{(10)}{=}} 3\mathbf{f}\circ(\mathbf{p}_1\circ(\mathbf{a})+\mathbf{p}_1\circ(\mathbf{0}))-2\mathbf{f}\circ(\mathbf{p}_1\circ(\mathbf{a})-\mathbf{p}_1\circ(\mathbf{0}))$$

$$\overset{EF(1)}{=} 3\mathbf{f}\circ(a)-2\mathbf{f}\circ(a)$$

$$\underset{\begin{pmatrix} LHS\\ f=3\mathbf{f}\\ g=-2\mathbf{f}\\ h=\mathbf{a}\end{pmatrix}}{\overset{(12)}{=}} (3\mathbf{f}-2\mathbf{f})\circ\mathbf{a}$$

And since the composition of a function with a constant is just the value of the function at the constant the left hand side equals $\mathbf{f}(a)$

Now simplifying the right side, or

$$(3\overset{1}{\mathbf{p}}_2+30\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1+3\overset{2}{\mathbf{p}}_2+\mathbf{2})\circ(\mathbf{a}\times\mathbf{0})$$

$$\underset{\begin{pmatrix} LHS\\ f=3\overset{1}{\mathbf{p}}_2+30\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1\\ g=-3\overset{2}{\mathbf{p}}_2+\mathbf{2})\circ(\mathbf{a}\times\mathbf{0}\\ h=\mathbf{a}\times\mathbf{0}\end{pmatrix}}{\overset{(12)}{=}} (3\overset{1}{\mathbf{p}}_2+30\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1)\circ(\mathbf{a}\times\mathbf{0})+(3\overset{2}{\mathbf{p}}_2+\mathbf{2})\circ(\mathbf{a}\times\mathbf{0})$$

$$\underset{\begin{pmatrix} LHS\\ f=3\overset{1}{\mathbf{p}}_2\\ g=30\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1\\ h=\mathbf{a}\times\mathbf{0}\end{pmatrix}\begin{pmatrix} LHS\\ f=3\overset{2}{\mathbf{p}}_2\\ g=\mathbf{2}\\ h=\mathbf{a}\times\mathbf{0}\end{pmatrix}}{\overset{(12)}{=}} (3\overset{1}{\mathbf{p}}_2)\circ(\mathbf{a}\times\mathbf{0})+(30\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1)\circ(\mathbf{a}\times\mathbf{0})+(3\overset{2}{\mathbf{p}}_2)\circ(\mathbf{a}\times\mathbf{0})+\mathbf{2}\circ(\mathbf{a}\times\mathbf{0})$$

$$\underset{\begin{pmatrix} LHS\\ f=30\overset{1}{\mathbf{p}}_1\\ g=\overset{2}{\mathbf{p}}_1\\ h=\mathbf{a}\times\mathbf{0}\end{pmatrix}}{\overset{(13)}{=}} (3\overset{1}{\mathbf{p}}_2)\circ(\mathbf{a}\times\mathbf{0})+(30\overset{1}{\mathbf{p}}_1)\circ(\mathbf{a}\times\mathbf{0})\overset{2}{\mathbf{p}}_1\circ(\mathbf{a}\times\mathbf{0})+(3\overset{2}{\mathbf{p}}_2)\circ(\mathbf{a}\times\mathbf{0})+\mathbf{2}\circ(\mathbf{a}\times\mathbf{0})$$

$$\underset{\begin{pmatrix} LHS\\ f=3\overset{1}{\mathbf{p}}_2\\ a^i=(\mathbf{a}\times\mathbf{0})\end{pmatrix}\begin{pmatrix} LHS\\ f=30\overset{1}{\mathbf{p}}_1\\ a^i=(\mathbf{a}\times\mathbf{0})\end{pmatrix}\begin{pmatrix} LHS\\ f=\overset{2}{\mathbf{p}}_1\\ a^i=(\mathbf{a}\times\mathbf{0})\end{pmatrix}\begin{pmatrix} LHS\\ f=3\overset{2}{\mathbf{p}}_2\\ a^i=(\mathbf{a}\times\mathbf{0})\end{pmatrix}}{\overset{(10)}{=}} (3\mathbf{p}_2)\circ(\mathbf{a})+(30\mathbf{p}_1)\circ(\mathbf{a})\mathbf{p}_1\circ(\mathbf{0})+$$

$(3\mathbf{p}_2)\circ(\mathbf{0})+\mathbf{2}$

Note that a constant function's composition is just the constant,

And since the composition of a function with a constant is just the value of the function at the constant, the right hand side equals,

$3\mathbf{p}_2(a)+2 \overset{EF(1)}{=} 3a^2+2$

Setting the left side equal to the right side, or

$$\mathbf{f}(a)=3a^2+2 \underset{\begin{pmatrix} RHS \\ f=\mathbf{p}_2 \\ i=1 \\ x=(a)\end{pmatrix}}{\overset{(6)}{=}} 3\overset{1}{\mathbf{p}}_2(a)+2 \underset{\begin{pmatrix} RHS \\ f=\overset{1}{\mathbf{p}}_2 \\ g=2 \\ x=(a)\end{pmatrix}}{\overset{(2)}{=}} (3\overset{1}{\mathbf{p}}_2+2)(a)$$

$$\underset{\begin{pmatrix} RHS \\ f=\mathbf{f} \\ g=3\overset{1}{\mathbf{p}}_2+2 \\ x=(a)\end{pmatrix}}{\overset{(1)}{\equiv}} \mathbf{f}=3\overset{1}{\mathbf{p}}_2+2$$

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1 The Problem for $c=(c^1,c^2)$

Form the ${}^{i}\mathbf{Q}^{f}_{c}$ if $i=1,\ 2$ $c=(c^1,c^2)\in\mathbb{R}^2$ and

$$\mathbf{f}=\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1\cup\mathbf{0}_{|(0,0)}:\mathbb{R}^2\longrightarrow\mathbb{R},$$

$$(x,\ y)\longmapsto\mathbf{f}(x,\ y)=\begin{cases} xy & if\ \ (x,y)\neq(0,0)\\ 0 & if\ \ (x,y)=(0,0)\end{cases}$$

If i = 1

$$ {}^{1}\mathbf{Q}^{\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1}_{(c^1,c^2)} \underset{\substack{LHS \\ i=1 \\ a=(c^1,c^2) \\ f=\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1}}{\overset{(14)}{=}} \mathbf{p}_{-1}\cdot[(\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1)\circ((c^1\mathbf{p}_0+\mathbf{p}_1)\times c^2\mathbf{p}_0)-(\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1)(c^1,c^2)\mathbf{p}_0]: $$

$$\mathbb{R}-\{0\}\longrightarrow\mathbb{R}$$

$$\underset{\substack{LHS \\ f=\overset{1}{\mathbf{p}}_1 \\ g=\overset{2}{\mathbf{p}}_1 \\ x=(c^1,c^2)}}{\overset{(3)}{=}} \mathbf{p}_{-1}\cdot[(\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1)\circ((c^1\mathbf{p}_0+\mathbf{p}_1)\times c^2\mathbf{p}_0)-\overset{1}{\mathbf{p}}_1(c^1,c^2)\cdot\overset{2}{\mathbf{p}}_1(c^1,c^2)\mathbf{p}_0]$$

$$\underset{\substack{LHS\\ f=\mathbf{p}_1\\ i=1\\ x^1=c^1\\ x^2=c^2}\ \substack{LHS\\ f=\mathbf{p}_1\\ i=2\\ x^1=c^1\\ x^2=c^2}}{\overset{(6)}{=}} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ ((c^1\mathbf{p}_0 + \mathbf{p}_1) \times c^2\mathbf{p}_0) - \mathbf{p}_1(c^1) \cdot \mathbf{p}_1(c^2)\mathbf{p}_0]$$

$$\overset{(EF1)}{=} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ ((c^1\mathbf{p}_0 + \mathbf{p}_1) \times c^2\mathbf{p}_0) - c^1 \cdot c^2\mathbf{p}_0]$$

$$\underset{\substack{LHS\\ f=\overset{1}{\mathbf{p}}_1\\ g=\overset{2}{\mathbf{p}}_1\\ h=(c^1\mathbf{p}_0+\mathbf{p}_1)\times c^2\mathbf{p}_0}}{\overset{(13)}{=}} \mathbf{p}_{-1} \cdot [\overset{1}{\mathbf{p}}_1 \circ [(c^1\mathbf{p}_0 + \mathbf{p}_1) \times c^2\mathbf{p}_0] \cdot \overset{2}{\mathbf{p}}_1 \circ [(c^1\mathbf{p}_0+\mathbf{p}_1)\times c^2\mathbf{p}_0] - c^1 \cdot c^2\mathbf{p}_0]$$

$$\underset{\substack{LHS\\ f=\mathbf{p}_1\\ i=1\\ a^1=c^1\mathbf{p}_0+\mathbf{p}_1\\ a^2=c^2\mathbf{p}_0}\ \substack{LHS\\ f=\mathbf{p}_1\\ i=2\\ a^1=c^1\mathbf{p}_0+\mathbf{p}_1\\ a^2=c^2\mathbf{p}_0}}{\overset{(10)}{=}} \mathbf{p}_{-1} \cdot [\mathbf{p}_1 \circ (c^1\mathbf{p}_0 + \mathbf{p}_1) \cdot \mathbf{p}_1 \circ (c^2\mathbf{p}_0) - c^1 \cdot c^2\mathbf{p}_0]$$

$$\overset{(EF6)}{=} \mathbf{p}_{-1} \cdot [\mathbf{p}_1(c^1\mathbf{p}_0 + \mathbf{p}_1) \cdot \mathbf{p}_1(c^2\mathbf{p}_0) - c^1 \cdot c^2\mathbf{p}_0]$$

$$\overset{(EF1)}{=} \mathbf{p}_{-1} \cdot [(c^1\mathbf{p}_0 + \mathbf{p}_1) \cdot c^2\mathbf{p}_0 - c^1 \cdot c^2\mathbf{p}_0]$$

$$= \mathbf{p}_{-1} \cdot [\mathbf{p}_1 \cdot (c^2\mathbf{p}_0)]$$

$$= c^2\mathbf{p}_0$$

If i = 2

$${}^2\mathbf{Q}^{\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1}_{(c^1,c^2)} \underset{\substack{LHS\\ i=2\\ a=(c^1,c^2)\\ f=\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1}}{\overset{(14)}{=}} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ (c^1\mathbf{p}_0 \times (c^2\mathbf{p}_0 + \mathbf{p}_1)) - (\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1)(c^1, c^2)\mathbf{p}_0] : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$$

$$\underset{\substack{LHS\\ f=\overset{1}{\mathbf{p}}_1\\ g=\overset{2}{\mathbf{p}}_1\\ x=(c^1,c^2)}}{\overset{(3)}{=}} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ (c^1\mathbf{p}_0 \times (c^2\mathbf{p}_0+\mathbf{p}_1)) - \overset{1}{\mathbf{p}}_1(c^1, c^2) \cdot \overset{2}{\mathbf{p}}_1(c^1, c^2)\mathbf{p}_0]$$

$$\underset{\substack{LHS\\ f=\mathbf{p}_1\\ i=1\\ x^1=c^1\\ x^2=c^2}\ \substack{LHS\\ f=\mathbf{p}_1\\ i=2\\ x^1=c^1\\ x^2=c^2}}{\overset{(6)}{=}} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ (c^1\mathbf{p}_0 \times (c^2\mathbf{p}_0 + \mathbf{p}_1)) - \overset{1}{\mathbf{p}}_1(c^1) \cdot \overset{2}{\mathbf{p}}_1(c^2)\mathbf{p}_0]$$

$$\overset{(EF1)}{=} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ (c^1\mathbf{p}_0 \times (c^2\mathbf{p}_0 + \mathbf{p}_1)) - c^1 \cdot c^2\mathbf{p}_0]$$

$$\overset{(13)}{\underset{\substack{LHS\\ f=\overset{1}{\mathbf{p}}_1\\ g=\overset{2}{\mathbf{p}}_1\\ h=c^1\mathbf{p}_0\times(c^2\mathbf{p}_0+\mathbf{p}_1)}}{=}} \quad \mathbf{p}_{-1}\cdot[\overset{1}{\mathbf{p}}_1\circ[c^1\mathbf{p}_0\times(c^2\mathbf{p}_0+\mathbf{p}_1)]\cdot\overset{2}{\mathbf{p}}_1\circ[c^1\mathbf{p}_0\times(c^2\mathbf{p}_0+\mathbf{p}_1)]-c^1\cdot c^2\mathbf{p}_0]$$

$$\overset{(10)}{\underset{\substack{LHS\\ f=\mathbf{p}_1\\ i=1\\ a^1=c^1\mathbf{p}_0\\ a^2=c^2\mathbf{p}_0+\mathbf{p}_1}\ \substack{LHS\\ f=\mathbf{p}_1\\ i=2\\ a^1=c^1\mathbf{p}_0\\ a^2=c^2\mathbf{p}_0+\mathbf{p}_1}}{=}} \quad \mathbf{p}_{-1}\cdot[\mathbf{p}_1\circ(c^1\mathbf{p}_0)\cdot\mathbf{p}_1\circ(c^2\mathbf{p}_0+\mathbf{p}_1)-c^1\cdot c^2\mathbf{p}_0]$$

$$\overset{(EF6)}{=} \mathbf{p}_{-1}\cdot[\mathbf{p}_1(c^1\mathbf{p}_0)\cdot\mathbf{p}_1(c^2\mathbf{p}_0+\mathbf{p}_1)-c^1\cdot c^2\mathbf{p}_0]$$

$$\overset{(EF1)}{=} \mathbf{p}_{-1}\cdot[c^1\mathbf{p}_0\cdot(c^2\mathbf{p}_0+\mathbf{p}_1)-c^1\cdot c^2\mathbf{p}_0]$$

$$= c^1\mathbf{p}_0$$

2 The Problem for c = (0,0)

Form the ${}^i\mathbf{Q}_c^f$ if $i = 1,\ 2$ $c = (0{,}0) \in \mathbf{R}^2$ and

$$\mathbf{f} = \overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1 \cup \mathbf{0}_{|(0,0)} : \mathbb{R}^2 \longrightarrow \mathbb{R},$$

$$(x,\ y) \longmapsto \mathbf{f}(x,\ y) = \begin{cases} xy & if\ \ (x,y)\neq(0,0)\\ 0 & if\ \ (x,y)=(0,0)\end{cases}$$

If i = 1

$${}^1\mathbf{Q}_{(0,0)}^{\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1} \overset{(14)}{\underset{\substack{LHS\\ i=1\\ a=(0,0)\\ f=\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1}}{=}} \mathbf{p}_{-1}\cdot[(\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1)\circ((0\mathbf{p}_0+\mathbf{p}_1)\times 0\mathbf{p}_0)-(\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1)(0,0)\mathbf{p}_0] : \mathbb{R}-\{0\}\longrightarrow\mathbb{R}$$

$$=\mathbf{p}_{-1}\cdot[(\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1)\circ(\mathbf{p}_1\times\mathbf{0})-(\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1)(0,0)\mathbf{p}_0]$$

$$\overset{(3)}{\underset{\substack{LHS\\ f=\overset{1}{\mathbf{p}}_1\\ g=\overset{2}{\mathbf{p}}_1\\ x=(0,0)}}{=}} \mathbf{p}_{-1}\cdot[(\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1)\circ(\mathbf{p}_1\times\mathbf{0})-\overset{1}{\mathbf{p}}_1(0,0)\cdot\overset{2}{\mathbf{p}}_1(0,0)\mathbf{p}_0]$$

$$\underset{\substack{LHS\ LHS \\ f=\mathbf{p}_1\ f=\mathbf{p}_1 \\ i=1\ \ i=2 \\ x^1=0\ x^1=0 \\ x^2=0\ x^2=0}}{\overset{(6)}{=}} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ (\mathbf{p}_1 \times \mathbf{0}) - \mathbf{p}_1(0) \cdot \mathbf{p}_1(0)\mathbf{p}_0]$$

$$\overset{(EF1)}{=} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ (\mathbf{p}_1 \times \mathbf{0}) - \mathbf{0}]$$

$$\underset{\substack{LHS \\ f=\overset{1}{\mathbf{p}}_1 \\ g=\overset{2}{\mathbf{p}}_1 \\ h=\mathbf{p}_1\times\mathbf{0}}}{\overset{(13)}{=}} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \circ (\mathbf{p}_1 \times \mathbf{0})) \cdot (\overset{2}{\mathbf{p}}_1 \circ (\mathbf{p}_1 \times \mathbf{0}))]$$

$$\underset{\substack{LHS\ \ LHS \\ f=\mathbf{p}_1\ f=\mathbf{p}_1 \\ i=1\ \ i=2 \\ a^1=\mathbf{p}_1\ a^1=\mathbf{p}_1 \\ a^2=\mathbf{0}\ \ a^2=\mathbf{0}}}{\overset{(10)}{=}} \mathbf{p}_{-1} \cdot [\mathbf{p}_1 \circ (\mathbf{p}_1) \cdot \mathbf{p}_1 \circ (\mathbf{0})]$$

$$\overset{(EF6),(EF1)}{=} \mathbf{p}_{-1} \cdot [\mathbf{p}_1 \cdot \mathbf{0}]$$

$$= \mathbf{0}$$

<u>**If i = 2**</u>

$${}^2\mathbf{Q}^{\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1}_{(0,0)} \underset{\substack{LHS \\ i=2 \\ a=(0,0) \\ f=\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_1}}{\overset{(14)}{=}} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ (0\mathbf{p}_0 \times (0\mathbf{p}_0 + \mathbf{p}_1)) - (\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1)(0,0)\mathbf{p}_0] :$$

$$\mathbb{R} - \{0\} \longrightarrow \mathbb{R}$$

$$= \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ (\mathbf{0} \times \mathbf{p}_1) - (\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1)(0,0)\mathbf{p}_0]$$

$$\underset{\substack{LHS \\ f=\overset{1}{\mathbf{p}}_1 \\ g=\overset{2}{\mathbf{p}}_1 \\ x=(0,0)}}{\overset{(3)}{=}} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ (\mathbf{0} \times \mathbf{p}_1) - \overset{1}{\mathbf{p}}_1(0,0) \cdot \overset{2}{\mathbf{p}}_1(0,0)\mathbf{p}_0]$$

$$\underset{\substack{LHS\ LHS \\ f=\mathbf{p}_1\ f=\mathbf{p}_1 \\ i=1\ \ i=2 \\ x^1=0\ x^1=0 \\ x^2=0\ x^2=0}}{\overset{(6)}{=}} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ (\mathbf{0} \times \mathbf{p}_1) - \mathbf{p}_1(0) \cdot \mathbf{p}_1(0)\mathbf{p}_0]$$

$$\overset{(EF1)}{=} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1) \circ (\mathbf{0} \times \mathbf{p}_1) - \mathbf{0}]$$

$$\underset{\substack{LHS\\ f=\overset{1}{\mathbf{p}}_1\\ g=\overset{2}{\mathbf{p}}_1\\ h=\mathbf{0}\times\mathbf{p}_1}}{\overset{(13)}{=}} \mathbf{p}_{-1} \cdot [(\overset{1}{\mathbf{p}}_1 \circ (\mathbf{0} \times \mathbf{p}_1)) \cdot (\overset{2}{\mathbf{p}}_1 \circ (\mathbf{0} \times \mathbf{p}_1))]$$

$$\underset{\substack{LHS\;\;LHS\\ f=\mathbf{p}_1\; f=\mathbf{p}_1\\ i=1\;\; i=2\\ a^1=\mathbf{0}\;\; a^1=\mathbf{0}\\ a^2=\mathbf{p}_1 a^2=\mathbf{p}_1}}{\overset{(10)}{=}} \mathbf{p}_{-1} \cdot [(\mathbf{p}_1 \circ (\mathbf{0})) \cdot (\mathbf{p}_1 \circ (\mathbf{p}_1))]$$

$$\overset{(EF6,EF1)}{=} \mathbf{p}_{-1} \cdot [\mathbf{0} \cdot \mathbf{p}_1]$$

$$= \mathbf{0}$$

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Formula 22 - Lemma 8:

Let $a : \mathbb{N} \to \mathbb{R},$ be a sequence in a metric space $(\mathbb{R}, d)$ and $a = c$ where $c \in \mathbb{R}$ then

$$lim\ a = c$$

Proof of Lemma 8:

$$\diamondsuit > 0, \heartsuit \in \mathbb{R}$$

$\rho(a(\heartsuit), c) < \diamondsuit$
$\equiv \rho(c, c) < \diamondsuit$
$\equiv 0 < \diamondsuit$
$\equiv \bar{S}_{\diamondsuit} = (-\infty, \infty) \equiv S_{\diamondsuit} = \bar{S}_{\diamondsuit} \cap \mathbb{N} = \mathbb{N}_k^C \wedge k = 0$
$\equiv lim\ a = c$
End of Proof of Lemma 8

Problem 1

Use the sequential definition to show that the given limits do not exist.

(a) $\underset{(0,0)}{lim} f$ if

$f = (p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1)$

First the definition of the limit of a function at a point is:

$$\left(\lim_a f = g\right) \equiv \left(\forall_{\substack{x:\mathbb{N}\to\mathbb{R}^N \\ lim x = a \\ x \neq a}} lim y^x = g^x \wedge y^x = f \circ x \wedge \forall_x g^x = g\right)$$

Which implies if the negation of the second half is true than the limit does not exist. The negation is:

$$\left(\lim_a f = g\right) \equiv \left(\exists_{\substack{x:\mathbb{N}\to\mathbb{R}^N \\ lim x = a \\ x \neq a}} lim y^x \neq g^x \vee y^x \neq f \circ x \vee \exists_x g^x \neq g\right)$$

So because of the meaning of or if we can show that any one of the three proposition is true then the entire proposition is true. The simplest one to show true is the last proposition which can be satisfied by finding two sequences x that cause the lim of y^x to be different. This is what the remainder of the proof will focus on doing.

Now let's start by using for formula 5 on sheet 2. $f = f$ and $a = (0, 0)$. Now we must choose a sequence x. The first sequence chosen will be:

$x = (p_{-1} \times -p_{-1})$

Thus:

$y^x = ((p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1)) \circ (p_{-1}, -p_{-1})$

Now we swap to a prior program.

Step 0 (Identify Formula):

Formula 13

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$f = p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)$

$g = \overset{1}{p}_2 - 4\overset{2}{p}_1$

$h = (p_{-1} \times -p_{-1})$

Step 3 (Replacing Symbols):

$$(p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)) \circ (p_{-1} \times -p_{-1}) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times -p_{-1})$$

Step 0 (Identify Formula):
Formula 14
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$w = p_{-1}$
$v = \overset{2}{p}_2 + 4\overset{1}{p}_1$
$u = (p_{-1} \times -p_{-1})$
Step 3 (Replacing Symbols):

$$p_{-1} \circ ((\overset{2}{p}_2 + 4\overset{1}{p}_1) \circ (p_{-1} \times -p_{-1})) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times -p_{-1})$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{2}{p}_2$
$g = 4\overset{1}{p}_1$
$h = (p_{-1} \times -p_{-1})$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (\overset{2}{p}_2 \circ (p_{-1} \times -p_{-1}) + 4\overset{1}{p}_1 \circ (p_{-1} \times -p_{-1})) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times -p_{-1})$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{1}{p}_2$
$g = -4\overset{2}{p}_1$
$h = (p_{-1} \times -p_{-1})$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (\overset{2}{p}_2 \circ (p_{-1} \times -p_{-1}) + 4\overset{1}{p}_1 \circ (p_{-1} \times -p_{-1})) \cdot (\overset{1}{p}_2 \circ (p_{-1} \times -p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times -p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_2$
$i = 2$
$a^2 = -p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1}\circ(p_2\circ(-p_{-1})+4\overset{1}{p}_1\circ(p_{-1}\times -p_{-1}))\cdot(\overset{1}{p}_2\circ(p_{-1}\times -p_{-1})-4\overset{2}{p}_1\circ(p_{-1}\times -p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 4p_1$
$i = 1$
$a^1 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1}\circ(p_2\circ(-p_{-1})+4p_1\circ(p_{-1}))\cdot(\overset{1}{p}_2\circ(p_{-1}\times -p_{-1})-4\overset{2}{p}_1\circ(p_{-1}\times -p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_2$
$i = 1$
$a^1 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1}\circ(p_2\circ(-p_{-1})+4p_1\circ(p_{-1}))\cdot(p_2\circ(p_{-1})-4\overset{2}{p}_1\circ(p_{-1}\times -p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):

Left Side
Step 2 (Identify Symbols):
$f = 4p_1$
$i = 2$
$a^1 = -p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (-p_{-1}) + 4p_1 \circ (p_{-1})) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (-p_{-1}))$$

Step 0 (Identify Formula):
Formula 21
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_2$
$f = -p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_1 \circ (p_{-1})) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (-p_{-1}))$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
$p_l = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (-p_{-1}))$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_2$
$p_l = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_{-2} - 4p_1 \circ (-p_{-1}))$$

Step 0 (Identify Formula):
Formula 21
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
$f = -p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_{-2} + 4p_{-1}) = 1$$

Now that we have simplified y^x we need to find its limit. Since y^x is equal to a constant sequence the limit by Lemma 22 is equal to that constant. So the limit of this sequence is 1.

The second sequence chosen will be:
$x = (p_{-1} \times p_{-1})$

Thus:
$y^x = ((p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1)) \circ (p_{-1}, p_{-1})$
Now we again swap to a prior program.

Step 0 (Identify Formula):
Formula 13
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)$
$g = \overset{1}{p}_2 - 4\overset{2}{p}_1$
$h = (p_{-1} \times p_{-1})$
Step 3 (Replacing Symbols):

$$(p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)) \circ (p_{-1} \times p_{-1}) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times p_{-1})$$

Step 0 (Identify Formula):
Formula 14
Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$w = p_{-1}$

$v = \overset{2}{p}_2 + 4\overset{1}{p}_1$

$u = (p_{-1} \times p_{-1})$

Step 3 (Replacing Symbols):

$$p_{-1} \circ ((\overset{2}{p}_2 + 4\overset{1}{p}_1) \circ (p_{-1} \times p_{-1})) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times p_{-1})$$

Step 0 (Identify Formula):

Formula 12

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$f = \overset{2}{p}_2$

$g = 4\overset{1}{p}_1$

$h = (p_{-1} \times p_{-1})$

Step 3 (Replacing Symbols):

$$p_{-1} \circ (\overset{2}{p}_2 \circ (p_{-1} \times p_{-1}) + 4\overset{1}{p}_1 \circ (p_{-1} \times p_{-1})) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times p_{-1})$$

Step 0 (Identify Formula):

Formula 12

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$f = \overset{1}{p}_2$

$g = -4\overset{2}{p}_1$

$h = (p_{-1} \times -p_{-1})$

Step 3 (Replacing Symbols):

$$p_{-1} \circ (\overset{2}{p}_2 \circ (p_{-1} \times p_{-1}) + 4\overset{1}{p}_1 \circ (p_{-1} \times p_{-1})) \cdot (\overset{1}{p}_2 \circ (p_{-1} \times p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times p_{-1}))$$

Step 0 (Identify Formula):

Formula 10

Step 1 (Identify Side):

Left Side

Step 2 (Identify Symbols):

$f = p_2$
$i = 2$
$a^2 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (p_{-1}) + 4\overset{1}{p}_1 \circ (p_{-1} \times p_{-1})) \cdot (\overset{1}{p}_2 \circ (p_{-1} \times p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 4p_1$
$i = 1$
$a^1 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (p_{-1}) + 4p_1 \circ (p_{-1})) \cdot (\overset{1}{p}_2 \circ (p_{-1} \times p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_2$
$i = 1$
$a^1 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (p_{-1}) + 4p_1 \circ (p_{-1})) \cdot (p_2 \circ (p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 4p_1$
$i = 2$
$a^1 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (p_{-1}) + 4p_1 \circ (p_{-1})) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (p_{-1}))$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_2$
$p_l = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_1 \circ (p_{-1})) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (p_{-1}))$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
$p_l = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (p_{-1}))$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_2$
$p_l = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_{-2} - 4p_1 \circ (p_{-1}))$$

Step 0 (Identify Formula):
Formula 21
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$p_k = p_1$
$f = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_{-2} - 4p_{-1}) = p_{-1} \circ (-4p_1 + 1p_0) \cdot (4p_1 + 1p_0)$$

Functions of Several Variables

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Problems 2

1b). Use the sequential definition to show that the given limit does not exist:

$\lim\limits_{(0,2)} \mathbf{f}$ if

$$\mathbf{f} = \left[\mathbf{p}_{-1} \circ \left(\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2 - 4\overset{2}{\mathbf{p}}_1 + \mathbf{4}\right)\right] \cdot \left(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 - 2\overset{1}{\mathbf{p}}_1\right) : \mathbb{R}^2 - \{(0,2)\} \to \mathbb{R}$$

Choose $x_1 = (\mathbf{p}_{-1} \times 2\mathbf{p}_0 + \mathbf{p}_{-1})$

$$y^{x_1} = \left(\left[\mathbf{p}_{-1} \circ \left(\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2 - 4\overset{2}{\mathbf{p}}_1 + \mathbf{4}\right)\right] \cdot \left(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 - 2\overset{1}{\mathbf{p}}_1\right)\right) \circ (\mathbf{p}_{-1} \times 2\mathbf{p}_0 + \mathbf{p}_{-1})$$

We run Program 1 (How to use Formulas) multiple times, omitting many of the details because we know how to do them:

$$= \mathbf{p}_{-1} \circ (\mathbf{p}_{-2} + \mathbf{p}_2 \circ (2\mathbf{p}_0 + \mathbf{p}_{-1}) - 4(2\mathbf{p}_0 + \mathbf{p}_{-1}) + \mathbf{4}) \cdot \left[\mathbf{p}_{-1} \cdot (2\mathbf{p}_0 + \mathbf{p}_{-1}) - 2\mathbf{p}_{-1}\right]$$

$$= \mathbf{p}_{-1} \circ (\mathbf{p}_{-2} + 4\mathbf{p}_0 + 4\mathbf{p}_{-1} + \mathbf{p}_{-2} - 8\mathbf{p}_0 - 4\mathbf{p}_{-1} + \mathbf{4}) \cdot \left(2\mathbf{p}_{-1} + \mathbf{p}_{-2} - 2\mathbf{p}_{-1}\right)$$

$$= \mathbf{p}_{-1} \circ (2\mathbf{p}_{-2}) \cdot (\mathbf{p}_{-2}) = \frac{\mathbf{p}_{-2}}{2\mathbf{p}_{-2}} \Rightarrow lim\ \ y^{x_1} = \frac{1}{2}$$

Choose $x_2 = (2\mathbf{p}_{-1} \times 2\mathbf{p}_0 + \mathbf{p}_{-1})$

$$y^{x_2} = \left(\left[\mathbf{p}_{-1} \circ \left(\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2 - 4\overset{2}{\mathbf{p}}_1 + \mathbf{4}\right)\right] \cdot \left(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 - 2\overset{1}{\mathbf{p}}_1\right)\right) \circ (2\mathbf{p}_{-1} \times 2\mathbf{p}_0 + \mathbf{p}_{-1})$$

Again, we run Program 1:

$$= \mathbf{p}_{-1} \circ (4\mathbf{p}_{-2} + 4\mathbf{p}_0 + 4\mathbf{p}_{-1} + \mathbf{p}_{-2} - 8\mathbf{p}_0 - 4\mathbf{p}_{-1} + \mathbf{4}) \cdot \left(2\mathbf{p}_{-1} \cdot (2\mathbf{p}_0 + \mathbf{p}_{-1}) - 4\mathbf{p}_{-1}\right)$$

$$= \mathbf{p}_{-1} \circ (5\mathbf{p}_{-2}) \cdot (2\mathbf{p}_{-2}) = \frac{2\mathbf{p}_{-2}}{5\mathbf{p}_{-2}} \Rightarrow lim\ \ y^{x_2} = \frac{2}{5}$$

$lim\, y^{x_1} \neq lim\, y^{x_2}. \Rightarrow \lim\limits_{(0,2)} \mathbf{f}\ \ DNE.$

2a) Use the $\delta - \varepsilon$ - definition to verify the limits
$\lim\limits_{(1,2)} \mathbf{f} = 1$ if

$$\mathbf{f} = 7\overset{1}{\mathbf{p}}_1 - 3\overset{2}{\mathbf{p}}_1 : \mathbb{R}^2 \to \mathbb{R}$$

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta_\varepsilon>0}{\exists}\left(\underset{(x,y)}{\forall}\ \boldsymbol{\rho}\left((x,y),(1,2)\right) < \delta_\varepsilon\right) \Rightarrow\ \left|\mathbf{f}\left(x,y\right) - 1\right| < \varepsilon$$

(SIDE WORK) $|x-1| < \rho\left((x,y),(1,2)\right)$
(SIDE WORK) $|y-2| < \rho\left((x,y),(1,2)\right)$
(SIDE WORK) $|(7x-3y)-1| = |7(x-1)+3(2-y)| < 7|x-1|+3|2-y| = 7|x-1|+3|y-2| < 7\delta_\varepsilon + 3\delta_\varepsilon = 10\delta_\varepsilon$

$$|\mathbf{f}(x,y)-1| < 10\delta_\varepsilon \Rightarrow |\mathbf{f}(x,y)-1| < 10\rho\left((x,y),(1,2,)\right) < \varepsilon$$

Let $\delta_\varepsilon = \dfrac{1}{10}\varepsilon$. Then $\underset{\varepsilon>0}{\forall}\ \underset{\delta_\varepsilon>0}{\exists}\left(\underset{(x,y)}{\forall}\ \boldsymbol{\rho}\left((x,y),(1,2)\right) < \dfrac{1}{10}\varepsilon\right) \Rightarrow\ \left|\mathbf{f}\left(x,y\right) - 1\right| < \varepsilon$

3. Verify if the given functions are continuous $\mathbf{f} = \left[\mathbf{p}_{-1} \circ \left(\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2\right)\right] \cdot \left(\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1\right) \cup \mathbf{0}_{|\{(0,0)\}} : \mathbb{R}^2 \to \mathbb{R}$,$

First, we will verify that the function is continuous at (0,0). To find $\underset{(0,0)}{lim} f$:

Choose $x_1 = (\mathbf{p}_{-1} \times \mathbf{p}_{-1})$

$$y^{x_1} = f \circ x_1 = \left(\left[\mathbf{p}_{-1} \circ (\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2)\right] \cdot \left(\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1\right)\right) \circ (\mathbf{p}_{-1} \times \mathbf{p}_{-1})$$

We run Program 1, "How to use Formulas",multiple times to get:

$$\left[\mathbf{p}_{-1} \circ (\mathbf{p}_{-2} + \mathbf{p}_{-2})\right] \cdot (\mathbf{p}_{-3} \cdot \mathbf{p}_{-1}) = \left[\mathbf{p}_{-1} \circ (2\mathbf{p}_{-2})\right] \cdot (\mathbf{p}_{-4}) = \frac{\mathbf{p}_{-4}}{2\mathbf{p}_{-2}} = \frac{\mathbf{p}_{-2}}{2}, lim\ \ y^{x_1} = 0$$

Choose $x_2 = (\mathbf{p}_{-1} \times 2\,\mathbf{p}_{-1})$

$$y^{x_2} = f \circ x_2 = \left(\left[\mathbf{p}_{-1} \circ (\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2)\right] \cdot \left(\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1\right)\right) \circ (\mathbf{p}_{-1} \times 2\,\mathbf{p}_{-1})$$

Again we run Program 1 "How to use Formulas" multiple times to get:

$$\left[\mathbf{p}_{-1} \circ (\mathbf{p}_{-2} + 4\,\mathbf{p}_{-2})\right] \cdot (\mathbf{p}_{-3} \cdot 2\,\mathbf{p}_{-1}) = \left[\mathbf{p}_{-1} \circ (5\,\mathbf{p}_{-2})\right] \cdot (2\,\mathbf{p}_{-4}) = \frac{2\,\mathbf{p}_{-4}}{5\,\mathbf{p}_{-2}} = \frac{2\,\mathbf{p}_{-2}}{5}, lim\ \ y^{x_2} = 0$$

We have now established that $\underset{0}{lim}\, f$ could be 0. We use the $\delta - \varepsilon$ definition to verify that 0 is indeed the limit.

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta_\varepsilon>0}{\exists} \left(\underset{(x,y)}{\forall}\ \boldsymbol{\rho}\ ((x,y),(0,0)) < \delta_\varepsilon\right) \Rightarrow\ |\mathbf{f}\,(x,y) - 0| < \varepsilon$$

(SIDE WORK)$|x - 0| < \rho\,((x,y),(0,0))$
(SIDE WORK)$|y - 0| < \rho\,((x,y),(0,0))$
(SIDE WORK)$|f(x,y) - 0| < \varepsilon \equiv |\dfrac{x^3 \cdot y}{x^2 + y^2}| < \varepsilon$
(SIDE WORK)$|\dfrac{x^3 \cdot y}{x^2 + y^2}| = |x \cdot y||\dfrac{x^2}{x^2 + y^2}| < |x \cdot y| = |x| \cdot |y| < (\sqrt{x^2 + y^2}) \cdot (\sqrt{x^2 + y^2}) = x^2 + y^2 =$
$\mathbf{p}_2 \circ \rho((x,y),(0,0)) < \varepsilon$

$$\text{Let } \delta_\varepsilon = \sqrt{\varepsilon} : \underset{\varepsilon>0}{\forall}\ \underset{\delta_\varepsilon>0}{\exists} \left(\underset{(x,y)}{\forall}\ \boldsymbol{\rho}\ ((x,y),(0,0)) < \sqrt{\varepsilon}\right) \Rightarrow\ |\mathbf{f}\,(x,y) - 0| < \varepsilon$$

This proves that $\underset{(0,0)}{lim}\, \mathbf{f} = 0$, and since $\underset{(0,0)}{lim}\, \mathbf{f} = 0$ and $\mathbf{f} = 0$, the function is continuous at the point (0,0)

In order to verify that the function is continuous at all points within its domain, we choose an ABF point, (a_1, a_2).

To find $\underset{(a_1,a_2)}{lim}\ \mathbf{f}$: Choose $x_1 = (a_1\mathbf{p}_0 + \mathbf{p}_{-1} \times a_2\mathbf{p}_0 + \mathbf{p}_{-1})$

$$y^{x_1} = f \circ x_1 = \left(\left[\mathbf{p}_{-1} \circ (\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2)\right] \cdot \left(\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1\right)\right) \circ (a_1\mathbf{p}_0 + \mathbf{p}_{-1} \times a_2\mathbf{p}_0 + \mathbf{p}_{-1})$$

We run Program 1 multiple times to get:

$$\begin{gathered}\left[\mathbf{p}_{-1} \circ \left((a_1)^2\mathbf{p}_0 + 2\,a_1\mathbf{p}_{-1} + \mathbf{p}_{-2} + (a_2)^2\mathbf{p}_0 + 2\,a_2\mathbf{p}_{-1} + \mathbf{p}_{-2}\right)\right] \cdot \\ \left[\left((a_1)^3\mathbf{p}_0 + 3\,(a_1)^2\mathbf{p}_{-1} + 3(a_1)\mathbf{p}_{-2} + \mathbf{p}_{-3}\right) \cdot (a_2\mathbf{p}_0 + \mathbf{p}_{-1})\right]\end{gathered}$$

$$= \frac{\left(((a_1)^3\mathbf{p}_0 + 3\,(a_1)^2\mathbf{p}_{-1} + 3(a_1)\mathbf{p}_{-2} + \mathbf{p}_{-3}) \cdot (a_2\mathbf{p}_0 + \mathbf{p}_{-1})\right)}{\left((a_1)^2\mathbf{p}_0 + 2\,a_1\mathbf{p}_{-1} + \mathbf{p}_{-2} + (a_2)^2\mathbf{p}_0 + 2\,a_2\mathbf{p}_{-1} + \mathbf{p}_{-2}\right)}$$

After some simplifying, we get that

$$lim\, y^{x_1} = \frac{(a_1)^3\mathbf{p}_0 \cdot a_2\mathbf{p}_0}{(a_1)^2\mathbf{p}_0 + (a_2)^2\mathbf{p}_0} = \frac{(a_1)^3 \cdot a_2}{(a_1)^2 + (a_2)^2}$$

Choose $x_2 = (a_1\mathbf{p}_0 + \mathbf{p}_{-1} \times a_2\mathbf{p}_0 + 2\,\mathbf{p}_{-1})$

$$y^{x_2} = f \circ x_2 = \left(\left[\mathbf{p}_{-1} \circ (\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2)\right] \cdot \left(\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1\right)\right) \circ (a_1\mathbf{p}_0 + \mathbf{p}_{-1} \times a_2\mathbf{p}_0 + 2\,\mathbf{p}_{-1})$$

We run Program 1 multiple times to get:

$$\left[\mathbf{p}_{-1} \circ \left((a_1)^2\mathbf{p}_0 + 2\,a_1\mathbf{p}_{-1} + \mathbf{p}_{-2} + (a_2)^2\mathbf{p}_0 + 4\,a_2\mathbf{p}_{-1} + 4\mathbf{p}_{-2}\right)\right] \cdot \left(((a_1)^3\mathbf{p}_0 + 3\,(a_1)^2\mathbf{p}_{-1} + 3(a_1)\mathbf{p}_{-2} + \mathbf{p}_{-3}\right.$$

$$= \frac{\left(((a_1)^3\mathbf{p}_0 + 3\,(a_1)^2\mathbf{p}_{-1} + 3(a_1)\mathbf{p}_{-2} + \mathbf{p}_{-3}) \cdot (a_2\mathbf{p}_0 + 2\,\mathbf{p}_{-1})\right)}{\left((a_1)^2\mathbf{p}_0 + 2\,a_1\mathbf{p}_{-1} + \mathbf{p}_{-2} + (a_2)^2\mathbf{p}_0 + 4\,a_2\mathbf{p}_{-1} + 4\,\mathbf{p}_{-2}\right)}$$

After some simplifying, we get that

$$lim\, y^{x_2} = \frac{(a_1)^3\mathbf{p}_0 \cdot a_2\mathbf{p}_0}{(a_1)^2\mathbf{p}_0 + (a_2)^2\mathbf{p}_0} = \frac{(a_1)^3 \cdot a_2}{(a_1)^2 + (a_2)^2}$$

We have now established that $\dfrac{(a_1)^3 \cdot a_2}{(a_1)^2 + (a_2)^2}$ is a possible limit of the function $\mathbf{f}at(a_1, a_2)$. We now use the $\delta - \varepsilon$ definition to verify this:

$$\underset{\varepsilon > 0}{\forall}\ \underset{\delta_\varepsilon > 0}{\exists} \left(\underset{(a_1, a_2)}{\forall}\ \boldsymbol{\rho}\ ((x,y),(a_1,a_2)) < \delta_\varepsilon\right) \Rightarrow\ |\mathbf{f}\,(x,y) - \left(\frac{(a_1)^3 \cdot a_2}{(a_1)^2 + (a_2)^2}\right)| < \varepsilon$$

If you continue with the tedious algebraic manipulation, applying many "tricks" of absolute value functions, you can eventually show with the definition that the function is continuous at every point (a_1, a_2) in its domain.

For now, we may simply say that the function is continuous across its entire domain by virtue of the continuity of the Six-Functions from which it is made of. We say that the sum, product, and composition of any of these six functions is thus continuous. This can be easily proven elsewhere, but here, we shall just take it as truth and end at that.

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Formula 22 - Lemma 8:
Let $a : \mathbb{N} \to \mathbb{R},$ be a sequence in a metric space $(\mathbb{R}, d)$ and $a = c$ where $c \in \mathbb{R}$ then

$$lim\ a = c$$

Proof of Lemma 8:

$$\diamondsuit > 0, \heartsuit \in \mathbb{R}$$

$\rho(a(\heartsuit), c) < \diamondsuit$
$\equiv \rho(c, c) < \diamondsuit$
$\equiv 0 < \diamondsuit$
$\equiv \bar{S}_{\diamondsuit} = (-\infty, \infty) \equiv S_{\diamondsuit} = \bar{S}_{\diamondsuit} \cap \mathbb{N} = \mathbb{N}_k^C \wedge k = 0$
$\equiv lim\ a = c$
End of Proof of Lemma 8

<u>**Problem 1**</u>
Use the sequential definition to show that the given limits do not exist.
(a) $\underset{(0,0)}{lim} f$ if

$f = (p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1)$
First the definition of the limit of a function at a point is:

$$\left(\underset{a}{lim} f = g\right) \equiv \left(\underset{\substack{x:\mathbb{N}\to\mathbb{R}^N \\ lim x = a \\ x \neq a}}{\forall} lim y^x = g^x \wedge y^x = f \circ x \wedge \underset{x}{\forall} g^x = g\right)$$

Which implies if the negation of the second half is true than the limit does not exist. The negation is:

$$\left(\underset{a}{lim} f = g\right) \equiv \left(\underset{\substack{x:\mathbb{N}\to\mathbb{R}^N \\ lim x = a \\ x \neq a}}{\exists} lim y^x \neq g^x \vee y^x \neq f \circ x \vee \underset{x}{\exists} g^x \neq g\right)$$

So because of the meaning of or if we can show that any one of the three proposition is true then the entire proposition is true. The simplest one to show true is the last proposition which can be satisfied by finding two sequences x that cause the lim of y^x to be different. This is what the remainder of the proof will focus on doing.

Now let's start by using for formula 5 on sheet 2. $f = f$ and $a = (0, 0)$. Now we must choose a sequence x. The first sequence chosen will be:
$x = (p_{-1} \times -p_{-1})$

Thus:
$y^x = ((p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1)) \circ (p_{-1}, -p_{-1})$
Now we swap to a prior program.

Step 0 (Identify Formula):
Formula 13
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)$
$g = \overset{1}{p}_2 - 4\overset{2}{p}_1$
$h = (p_{-1} \times -p_{-1})$
Step 3 (Replacing Symbols):

$$(p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)) \circ (p_{-1} \times -p_{-1}) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times -p_{-1})$$

Step 0 (Identify Formula):
Formula 14
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$w = p_{-1}$
$v = \overset{2}{p}_2 + 4\overset{1}{p}_1$
$u = (p_{-1} \times -p_{-1})$
Step 3 (Replacing Symbols):

$$p_{-1} \circ ((\overset{2}{p}_2 + 4\overset{1}{p}_1) \circ (p_{-1} \times -p_{-1})) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times -p_{-1})$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{2}{p}_2$
$g = 4\overset{1}{p}_1$
$h = (p_{-1} \times -p_{-1})$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (\overset{2}{p}_2 \circ (p_{-1} \times -p_{-1}) + 4\overset{1}{p}_1 \circ (p_{-1} \times -p_{-1})) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times -p_{-1})$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{1}{p}_2$
$g = -4\overset{2}{p}_1$
$h = (p_{-1} \times -p_{-1})$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (\overset{2}{p}_2 \circ (p_{-1} \times -p_{-1}) + 4\overset{1}{p}_1 \circ (p_{-1} \times -p_{-1})) \cdot (\overset{1}{p}_2 \circ (p_{-1} \times -p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times -p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_2$
$i = 2$
$a^2 = -p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (-p_{-1}) + 4\overset{1}{p}_1 \circ (p_{-1} \times -p_{-1})) \cdot (\overset{1}{p}_2 \circ (p_{-1} \times -p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times -p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 4p_1$
$i = 1$
$a^1 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (-p_{-1}) + 4p_1 \circ (p_{-1})) \cdot (\overset{1}{p}_2 \circ (p_{-1} \times -p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times -p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_2$
$i = 1$
$a^1 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (-p_{-1}) + 4p_1 \circ (p_{-1})) \cdot (p_2 \circ (p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times -p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 4p_1$
$i = 2$
$a^1 = -p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (-p_{-1}) + 4p_1 \circ (p_{-1})) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (-p_{-1}))$$

Step 0 (Identify Formula):
Formula 21
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_2$
$f = -p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_1 \circ (p_{-1})) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (-p_{-1}))$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
$p_l = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (-p_{-1}))$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_2$
$p_l = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_{-2} - 4p_1 \circ (-p_{-1}))$$

Step 0 (Identify Formula):
Formula 21
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
$f = -p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_{-2} + 4p_{-1}) = 1$$

Now that we have simplified y^x we need to find its limit. Since y^x is equal to a constant sequence the limit by Lemma 22 is equal to that constant. So the limit of this sequence is 1.

The second sequence chosen will be:
$x = (p_{-1} \times p_{-1})$

Thus:
$y^x = ((p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1)) \circ (p_{-1}, p_{-1})$
Now we again swap to a prior program.

Step 0 (Identify Formula):
Formula 13
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)$
$g = \overset{1}{p}_2 - 4\overset{2}{p}_1$
$h = (p_{-1} \times p_{-1})$
Step 3 (Replacing Symbols):

$$(p_{-1} \circ (\overset{2}{p}_2 + 4\overset{1}{p}_1)) \circ (p_{-1} \times p_{-1}) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times p_{-1})$$

Step 0 (Identify Formula):
Formula 14
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$w = p_{-1}$
$v = \overset{2}{p}_2 + 4\overset{1}{p}_1$
$u = (p_{-1} \times p_{-1})$
Step 3 (Replacing Symbols):

$$p_{-1} \circ ((\overset{2}{p}_2 + 4\overset{1}{p}_1) \circ (p_{-1} \times p_{-1})) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times p_{-1})$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{2}{p}_2$
$g = 4\overset{1}{p}_1$
$h = (p_{-1} \times p_{-1})$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (\overset{2}{p}_2 \circ (p_{-1} \times p_{-1}) + 4\overset{1}{p}_1 \circ (p_{-1} \times p_{-1})) \cdot (\overset{1}{p}_2 - 4\overset{2}{p}_1) \circ (p_{-1} \times p_{-1})$$

Step 0 (Identify Formula):
Formula 12
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{1}{p}_2$

$g = -4\overset{2}{p}_1$
$h = (p_{-1} \times -p_{-1})$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (\overset{2}{p}_2 \circ (p_{-1} \times p_{-1}) + 4\overset{1}{p}_1 \circ (p_{-1} \times p_{-1})) \cdot (\overset{1}{p}_2 \circ (p_{-1} \times p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_2$
$i = 2$
$a^2 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (p_{-1}) + 4\overset{1}{p}_1 \circ (p_{-1} \times p_{-1})) \cdot (\overset{1}{p}_2 \circ (p_{-1} \times p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 4p_1$
$i = 1$
$a^1 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (p_{-1}) + 4p_1 \circ (p_{-1})) \cdot (\overset{1}{p}_2 \circ (p_{-1} \times p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = p_2$
$i = 1$
$a^1 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (p_{-1}) + 4p_1 \circ (p_{-1})) \cdot (p_2 \circ (p_{-1}) - 4\overset{2}{p}_1 \circ (p_{-1} \times p_{-1}))$$

Step 0 (Identify Formula):
Formula 10
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 4p_1$

$i = 2$
$a^1 = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_2 \circ (p_{-1}) + 4p_1 \circ (p_{-1})) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (p_{-1}))$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_2$
$p_l = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_1 \circ (p_{-1})) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (p_{-1}))$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
$p_l = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_2 \circ (p_{-1}) - 4p_1 \circ (p_{-1}))$$

Step 0 (Identify Formula):
Formula 16
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_2$
$p_l = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_{-2} - 4p_1 \circ (p_{-1}))$$

Step 0 (Identify Formula):
Formula 21
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_k = p_1$
$f = p_{-1}$
Step 3 (Replacing Symbols):

$$p_{-1} \circ (p_{-2} + 4p_{-1}) \cdot (p_{-2} - 4p_{-1}) = p_{-1} \circ (-4p_1 + 1p_0) \cdot (4p_1 + 1p_0)$$

Functions of Several Variables

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Problems 2

1b). Use the sequential definition to show that the given limit does not exist:

$\lim\limits_{(0,2)} \mathbf{f}$ if

$$\mathbf{f} = \left[\mathbf{p}_{-1} \circ \left(\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2 - 4\,\overset{2}{\mathbf{p}}_1 + \mathbf{4}\right)\right] \cdot \left(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 - 2\,\overset{1}{\mathbf{p}}_1\right) : \mathbb{R}^2 - \{(0,2)\} \to \mathbb{R}$$

Choose $x_1 = (\mathbf{p}_{-1} \times 2\mathbf{p}_0 + \mathbf{p}_{-1})$

$$y^{x_1} = \left(\left[\mathbf{p}_{-1} \circ \left(\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2 - 4\,\overset{2}{\mathbf{p}}_1 + \mathbf{4}\right)\right] \cdot \left(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 - 2\,\overset{1}{\mathbf{p}}_1\right)\right) \circ (\mathbf{p}_{-1} \times 2\mathbf{p}_0 + \mathbf{p}_{-1})$$

We run Program 1 (How to use Formulas) multiple times, omitting many of the details because we know how to do them:

$$= \mathbf{p}_{-1} \circ (\mathbf{p}_{-2} + \mathbf{p}_2 \circ (2\mathbf{p}_0 + \mathbf{p}_{-1}) - 4(2\mathbf{p}_0 + \mathbf{p}_{-1}) + \mathbf{4}) \cdot \left[\mathbf{p}_{-1} \cdot (2\mathbf{p}_0 + \mathbf{p}_{-1}) - 2\mathbf{p}_{-1}\right]$$

$$= \mathbf{p}_{-1} \circ (\mathbf{p}_{-2} + 4\mathbf{p}_0 + 4\mathbf{p}_{-1} + \mathbf{p}_{-2} - 8\mathbf{p}_0 - 4\mathbf{p}_{-1} + \mathbf{4}) \cdot \left(2\mathbf{p}_{-1} + \mathbf{p}_{-2} - 2\mathbf{p}_{-1}\right)$$

$$= \mathbf{p}_{-1} \circ (2\mathbf{p}_{-2}) \cdot (\mathbf{p}_{-2}) = \frac{\mathbf{p}_{-2}}{2\mathbf{p}_{-2}} \Rightarrow lim\ \ y^{x_1} = \frac{1}{2}$$

Choose $x_2 = (2\mathbf{p}_{-1} \times 2\mathbf{p}_0 + \mathbf{p}_{-1})$

$$y^{x_2} = \left(\left[\mathbf{p}_{-1} \circ \left(\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2 - 4\,\overset{2}{\mathbf{p}}_1 + \mathbf{4}\right)\right] \cdot \left(\overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 - 2\,\overset{1}{\mathbf{p}}_1\right)\right) \circ (2\mathbf{p}_{-1} \times 2\mathbf{p}_0 + \mathbf{p}_{-1})$$

Again, we run Program 1:

$$= \mathbf{p}_{-1} \circ (4\mathbf{p}_{-2} + 4\mathbf{p}_0 + 4\mathbf{p}_{-1} + \mathbf{p}_{-2} - 8\mathbf{p}_0 - 4\mathbf{p}_{-1} + \mathbf{4}) \cdot \left(2\mathbf{p}_{-1} \cdot (2\mathbf{p}_0 + \mathbf{p}_{-1}) - 4\mathbf{p}_{-1}\right)$$

$$= \mathbf{p}_{-1} \circ (5\mathbf{p}_{-2}) \cdot (2\mathbf{p}_{-2}) = \frac{2\mathbf{p}_{-2}}{5\mathbf{p}_{-2}} \Rightarrow lim\ \ y^{x_2} = \frac{2}{5}$$

$lim\, y^{x_1} \neq lim\, y^{x_2}. \Rightarrow \underset{(0,2)}{lim}\, \mathbf{f}\ \ DNE.$

2a) Use the $\delta - \varepsilon$ - definition to verify the limits
$\lim\limits_{(1,2)} \mathbf{f} = 1$ if

$$\mathbf{f} = 7\,\overset{1}{\mathbf{p}}_1 - 3\,\overset{2}{\mathbf{p}}_1 : \mathbb{R}^2 \to \mathbb{R}$$

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta_\varepsilon>0}{\exists} \left(\underset{(x,y)}{\forall}\ \boldsymbol{\rho}\left((x,y),(1,2)\right) < \delta_\varepsilon\right) \Rightarrow\ \left|\mathbf{f}\,(x,y) - 1\right| < \varepsilon$$

(SIDE WORK)$|x-1| < \rho\left((x,y),(1,2)\right)$
(SIDE WORK)$|y-2| < \rho\left((x,y),(1,2)\right)$
(SIDE WORK) $|(7x-3y)-1| = |7(x-1)+3(2-y)| < 7|x-1|+3|2-y| = 7|x-1|+3|y-2| < 7\delta_\varepsilon + 3\delta_\varepsilon = 10\delta_\varepsilon$

$$|\mathbf{f}(x,y)-1| < 10\delta_\varepsilon \Rightarrow |\mathbf{f}(x,y)-1| < 10\rho\left((x,y),(1,2,)\right) < \varepsilon$$

$$\text{Let } \delta_\varepsilon = \frac{1}{10}\varepsilon. \text{ Then } \underset{\varepsilon>0}{\forall}\ \underset{\delta_\varepsilon>0}{\exists}\left(\underset{(x,y)}{\forall}\ \boldsymbol{\rho}\ \left((x,y),(1,2)\right) < \frac{1}{10}\varepsilon\right) \Rightarrow\ |\mathbf{f}\ (x,y)-1| < \varepsilon$$

3. Verify if the given functions are continuous $\mathbf{f} = \left[\mathbf{p}_{-1} \circ \left(\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2\right)\right] \cdot \left(\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1\right) \cup \mathbf{0}_{\big|\{(0,0)\}} : \mathbb{R}^2 \to \mathbb{R}$,$

First, we will verify that the function is continuous at (0,0). To find $\underset{(0,0)}{lim} f$:

Choose $x_1 = (\mathbf{p}_{-1} \times \mathbf{p}_{-1})$

$$y^{x_1} = f \circ x_1 = \left(\left[\mathbf{p}_{-1} \circ (\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2)\right] \cdot \left(\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1\right)\right) \circ (\mathbf{p}_{-1} \times \mathbf{p}_{-1})$$

We run Program 1, "How to use Formulas",multiple times to get:

$$\left[\mathbf{p}_{-1} \circ (\mathbf{p}_{-2} + \mathbf{p}_{-2})\right] \cdot (\mathbf{p}_{-3} \cdot \mathbf{p}_{-1}) = \left[\mathbf{p}_{-1} \circ (2\mathbf{p}_{-2})\right] \cdot (\mathbf{p}_{-4}) = \frac{\mathbf{p}_{-4}}{2\mathbf{p}_{-2}} = \frac{\mathbf{p}_{-2}}{2}, lim\ y^{x_1} = 0$$

Choose $x_2 = (\mathbf{p}_{-1} \times 2\,\mathbf{p}_{-1})$

$$y^{x_2} = f \circ x_2 = \left(\left[\mathbf{p}_{-1} \circ (\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2)\right] \cdot \left(\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1\right)\right) \circ (\mathbf{p}_{-1} \times 2\,\mathbf{p}_{-1})$$

Again we run Program 1 "How to use Formulas" multiple times to get:

$$\left[\mathbf{p}_{-1} \circ (\mathbf{p}_{-2} + 4\,\mathbf{p}_{-2})\right] \cdot (\mathbf{p}_{-3} \cdot 2\,\mathbf{p}_{-1}) = \left[\mathbf{p}_{-1} \circ (5\,\mathbf{p}_{-2})\right] \cdot (2\,\mathbf{p}_{-4}) = \frac{2\,\mathbf{p}_{-4}}{5\,\mathbf{p}_{-2}} = \frac{2\,\mathbf{p}_{-2}}{5}, lim\ y^{x_2} = 0$$

We have now established that $\underset{0}{lim} f$ could be 0. We use the $\delta - \varepsilon$ definition to verify that 0 is indeed the limit.

$$\underset{\varepsilon>0}{\forall} \ \underset{\delta_\varepsilon>0}{\exists} \left(\underset{(x,y)}{\forall} \boldsymbol{\rho}\ ((x,y),(0,0)) < \delta_\varepsilon\right) \Rightarrow |\mathbf{f}\,(x,y) - 0| < \varepsilon$$

(SIDE WORK)$|x - 0| < \rho\,((x,y),(0,0))$
(SIDE WORK)$|y - 0| < \rho\,((x,y),(0,0))$
(SIDE WORK)$|f(x,y) - 0| < \varepsilon \equiv |\dfrac{x^3 \cdot y}{x^2 + y^2}| < \varepsilon$
(SIDE WORK)$|\dfrac{x^3 \cdot y}{x^2 + y^2}| = |x \cdot y||\dfrac{x^2}{x^2 + y^2}| < |x \cdot y| = |x| \cdot |y| < (\sqrt{x^2 + y^2}) \cdot (\sqrt{x^2 + y^2}) = x^2 + y^2 =$
$\mathbf{p}_2 \circ \rho((x,y),(0,0)) < \varepsilon$

$$\text{Let } \delta_\varepsilon = \sqrt{\varepsilon} : \underset{\varepsilon>0}{\forall} \ \underset{\delta_\varepsilon>0}{\exists} \left(\underset{(x,y)}{\forall} \boldsymbol{\rho}\ ((x,y),(0,0)) < \sqrt{\varepsilon}\right) \Rightarrow |\mathbf{f}\,(x,y) - 0| < \varepsilon$$

This proves that $\underset{(0,0)}{lim} \mathbf{f} = 0$, and since $\underset{(0,0)}{lim} \mathbf{f} = 0$ and $\mathbf{f} = 0$, the function is continuous at the point (0,0)

In order to verify that the function is continuous at all points within its domain, we choose an ABF point, (a_1, a_2).

To find $\underset{(a_1,a_2)}{lim} \mathbf{f}$: Choose $x_1 = (a_1\mathbf{p}_0 + \mathbf{p}_{-1} \times a_2\mathbf{p}_0 + \mathbf{p}_{-1})$

$$y^{x_1} = f \circ x_1 = \left(\left[\mathbf{p}_{_{-1}} \circ (\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2) \right] \cdot \left(\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1 \right) \right) \circ (a_{_1}\mathbf{p}_{_0} + \mathbf{p}_{_{-1}} \times a_{_2}\mathbf{p}_{_0} + \mathbf{p}_{_{-1}})$$

We run Program 1 multiple times to get:

$$\begin{gathered}\left[\mathbf{p}_{_{-1}} \circ \left((a_{_1})^2\mathbf{p}_{_0} + 2\,a_{_1}\mathbf{p}_{_{-1}} + \mathbf{p}_{_{-2}} + (a_{_2})^2\mathbf{p}_{_0} + 2\,a_{_2}\mathbf{p}_{_{-1}} + \mathbf{p}_{_{-2}}\right)\right] \cdot \\ \left[\left((a_{_1})^3\mathbf{p}_{_0} + 3\,(a_{_1})^2\mathbf{p}_{_{-1}} + 3(a_{_1})\mathbf{p}_{_{-2}} + \mathbf{p}_{_{-3}}\right) \cdot (a_{_2}\mathbf{p}_{_0} + \mathbf{p}_{_{-1}})\right]\end{gathered}$$

$$= \frac{\left(\left((a_{_1})^3\mathbf{p}_{_0} + 3\,(a_{_1})^2\mathbf{p}_{_{-1}} + 3(a_{_1})\mathbf{p}_{_{-2}} + \mathbf{p}_{_{-3}}\right) \cdot (a_{_2}\mathbf{p}_{_0} + \mathbf{p}_{_{-1}})\right)}{\left((a_{_1})^2\mathbf{p}_{_0} + 2\,a_{_1}\mathbf{p}_{_{-1}} + \mathbf{p}_{_{-2}} + (a_{_2})^2\mathbf{p}_{_0} + 2\,a_{_2}\mathbf{p}_{_{-1}} + \mathbf{p}_{_{-2}}\right)}$$

After some simplifying, we get that

$$lim\, y^{x_1} = \frac{(a_{_1})^3\mathbf{p}_0 \cdot a_{_2}\mathbf{p}_0}{(a_{_1})^2\mathbf{p}_0 + (a_{_2})^2\mathbf{p}_0} = \frac{(a_{_1})^3 \cdot a_{_2}}{(a_{_1})^2 + (a_{_2})^2}$$

Choose $x_{_2} = (a_{_1}\mathbf{p}_{_0} + \mathbf{p}_{_{-1}} \times a_{_2}\mathbf{p}_{_0} + 2\,\mathbf{p}_{_{-1}})$

$$y^{x_2} = f \circ x_2 = \left(\left[\mathbf{p}_{_{-1}} \circ (\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2) \right] \cdot \left(\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1 \right) \right) \circ (a_{_1}\mathbf{p}_{_0} + \mathbf{p}_{_{-1}} \times a_{_2}\mathbf{p}_{_0} + 2\,\mathbf{p}_{_{-1}})$$

We run Program 1 multiple times to get:

$$\left[\mathbf{p}_{_{-1}} \circ \left((a_{_1})^2\mathbf{p}_{_0} + 2\,a_{_1}\mathbf{p}_{_{-1}} + \mathbf{p}_{_{-2}} + (a_{_2})^2\mathbf{p}_{_0} + 4\,a_{_2}\mathbf{p}_{_{-1}} + 4\mathbf{p}_{_{-2}}\right)\right] \cdot \left(\left((a_{_1})^3\mathbf{p}_{_0} + 3\,(a_{_1})^2\mathbf{p}_{_{-1}} + 3(a_{_1})\mathbf{p}_{_{-2}} + \mathbf{p}_{_{-3}}\right.\right.$$

$$= \frac{\left(\left((a_{_1})^3\mathbf{p}_{_0} + 3\,(a_{_1})^2\mathbf{p}_{_{-1}} + 3(a_{_1})\mathbf{p}_{_{-2}} + \mathbf{p}_{_{-3}}\right) \cdot (a_{_2}\mathbf{p}_{_0} + 2\,\mathbf{p}_{_{-1}})\right)}{\left((a_{_1})^2\mathbf{p}_{_0} + 2\,a_{_1}\mathbf{p}_{_{-1}} + \mathbf{p}_{_{-2}} + (a_{_2})^2\mathbf{p}_{_0} + 4\,a_{_2}\mathbf{p}_{_{-1}} + 4\,\mathbf{p}_{_{-2}}\right)}$$

After some simplifying, we get that

$$lim\, y^{x_2} = \frac{(a_{_1})^3\mathbf{p}_0 \cdot a_{_2}\mathbf{p}_0}{(a_{_1})^2\mathbf{p}_0 + (a_{_2})^2\mathbf{p}_0} = \frac{(a_{_1})^3 \cdot a_{_2}}{(a_{_1})^2 + (a_{_2})^2}$$

We have now established that $\frac{(a_{_1})^3 \cdot a_{_2}}{(a_{_1})^2 + (a_{_2})^2}$ is a possible limit of the function $\mathbf{f}at(a_{_1}, a_{_2})$. We now use the $\delta - \varepsilon$ definition to verify this:

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta_\varepsilon>0}{\exists} \left(\underset{(a_1,a_2)}{\forall}\ \boldsymbol{\rho}\ ((x,y),(a_{_1},a_{_2})) < \delta_{_\varepsilon} \right) \Rightarrow\ |\mathbf{f}\,(x,y) - \left(\frac{(a_{_1})^3 \cdot a_{_2}}{(a_{_1})^2 + (a_{_2})^2} \right)| < \varepsilon$$

If you continue with the tedious algebraic manipulation, applying many "tricks" of absolute value functions, you can eventually show with the definition that the function is continuous at every point (a_1, a_2) in its domain.

For now, we may simply say that the function is continuous across its entire domain by virtue of the continuity of the Six-Functions from which it is made of. We say that the sum, product, and composition of any of these six functions is thus continuous. This can be easily proven elsewhere, but here, we shall just take it as truth and end at that.

Functions of Several Variables

Problems 2

Yitong Chen, Preeti Mohan, Caron Song, Divya Velury, Julie Zhu

1c) Use the sequential definition to show that the given limit does not exist.

$\lim\limits_{(0,0)}\mathbf{f}$ if $\mathbf{f} = [\mathbf{p}_{-1} \circ (\overset{1}{\mathbf{p}}_6 + \overset{2}{\mathbf{p}}_2)] \cdot (\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1) : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}$,

$(x,y) \mapsto \mathbf{f}(x,y) = \frac{x^3 y}{x^6+y^2}$

Using formula (4) download $\mathbf{x} = \mathbf{p}_{-1} \times \mathbf{p}_{-1}$

$\lim\limits_{(0,0)}\mathbf{f} = g^{\mathbf{x}} \equiv \lim \mathbf{f} \circ \mathbf{x} = g^{\mathbf{x}} \equiv \lim \mathbf{f}(\mathbf{p}_{-1}, \mathbf{p}_{-1}) = \lim \frac{\mathbf{p}_{-3}\mathbf{p}_{-1}}{\mathbf{p}_{-6}+\mathbf{p}_{-2}}$
$= \lim \frac{\mathbf{p}_{-2}\mathbf{p}_{-2}}{\mathbf{p}_{-2}(\mathbf{p}_{-3}+1)} = \lim \frac{\mathbf{p}_{-2}}{\mathbf{p}_{-3}+1} = 0$ which is the partial limit in this case.

Now download $\mathbf{x} = \mathbf{p}_{-1} \times \mathbf{p}_{-3}$

$\lim\limits_{(0,0)}\mathbf{f} = g^{\mathbf{x}} \equiv \lim \mathbf{f} \circ \mathbf{x} = g^{\mathbf{x}} \equiv \lim \mathbf{f}(\mathbf{p}_{-1}, \mathbf{p}_{-3}) = \lim \frac{\mathbf{p}_{-3}\mathbf{p}_{-3}}{\mathbf{p}_{-6}+\mathbf{p}_{-6}}$
$= \lim \frac{\mathbf{p}_{-6}}{2\mathbf{p}_{-6}} = \frac{1}{2}$ which is the partial limit in this case.

Since these 2 partial limits are different, the limit $\lim\limits_{(0,0)}\mathbf{f}$ does not exist.

Functions of Several Variables

Problems 2

Yitong Chen, Preeti Mohan, Caron Song, Divya Velury, Julie Zhu

2. d) Run How to Check if $\lim\limits_{(a}\mathbf{f} = g$

To find $\mathbf{e}$ use the inequalities: $\sin x < 1$
$|x^i - a^i| < \rho(x,a)$
$i = 1, 2, ..., n$
$x = (x^1, ..., x^n)$
$a = (a^1, ..., a^i)$
Using formula 6 we get $\underset{\epsilon>0}{\forall}\ \underset{\delta_\epsilon}{\exists}\ \underset{(x,y)|(0,0)}{\forall} \rho((x,y),(0,0)) < \delta_\epsilon \Rightarrow |\mathbf{f}(x,y)|$

Side Work
$|\mathbf{f}(x,y) - 0| < (x^2+y^2)\sin\frac{1}{x^2+y^2} \le (x^2+y^2) = \mathbf{p}_2(\rho((x,y),(0,0)) < \epsilon$

Let $\delta_\epsilon = \sqrt{\epsilon}$, then $\underset{\epsilon>0}{\forall}\ \underset{\delta_\epsilon=\sqrt{\epsilon}}{\exists}\ \underset{(x,y)|(0,0)}{\forall} \rho((x,y),(0,0)) < \sqrt{\epsilon} \Rightarrow |\mathbf{f}(x,y) - 0| < \epsilon$

Functions of Several Variables

Sooraj Boominathan, Vincent Po, Ben Zhao, Nathan Yu, Steve Zhang

Problems 3

3. Let $\mathbf{f} = \overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 \cdot \left[\mathbf{p}_{-1} \circ \left(\overset{1}{\mathbf{p}}_2 + \overset{2}{\mathbf{p}}_2\right)\right] \cdot \left(\overset{1}{\mathbf{p}}_2 - \overset{2}{\mathbf{p}}_2\right) \cup \mathbf{0}_{\big|\{(0,0)\}} : \mathbb{R}^2 \to \mathbb{R}$,

$$(x,y) \mapsto \mathbf{f}(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ \quad 0 & \text{if} \quad (x,y) = (0,0) \end{cases}.$$

a) Form the quotient functions ${}^{i}\mathbf{Q}^{\mathbf{f}}_{0}$ if $i = 1, 2$, $0 = (0,0) \in \mathbf{R}^2$.

b) Find $\mathbf{f}''_{12}$, $\mathbf{f}''_{21}$ for $\mathbf{f}$.

c) Show that $\mathbf{f}''_{12}(0,0) \neq \mathbf{f}''_{21}(0,0)$ for $\mathbf{f}$. Does this result contradict the equality of mixed partial derivatives $\mathbf{f}''_{12}$ and $\mathbf{f}''_{21}$? Explain. (Check continuity of $\mathbf{f}''_{12}$).

Proposition 5': $\mathbf{f}''_{12}(a) \neq \mathbf{f}''_{21} \equiv \mathbf{f}''_{12}, \mathbf{f}_{21}\, are\, \mathbf{not}\, continuous\, at\, (a)$

PROGRAM

Step 1: Define $\mathbf{f}(x,y)_{|\mathbb{R}^2-\{0\}}$

Step 2: Find $\dfrac{d}{dx}(\mathbf{f}(x,y)) = \mathbf{f}'_1(x,y)$; Find $\dfrac{d}{dy}(\mathbf{f}(x,y)) = \mathbf{f}'_2(x,y)$

Step 3: Form ${}^{1}\mathbf{Q}^{\mathbf{f}}_{(0,0)}$ for $\mathbf{f}$; Form ${}^{2}\mathbf{Q}^{\mathbf{f}}_{(0,0)}$ for $\mathbf{f}$

Step 4: Find $\lim\limits_{0} {}^{1}\mathbf{Q}^{\mathbf{f}}_{(0,0)}$ for $\mathbf{f}$; Find $\lim\limits_{0} {}^{2}\mathbf{Q}^{\mathbf{f}}_{(0,0)}$ for $\mathbf{f}$

Step 5: Find $\dfrac{d}{dy}(\mathbf{f}'_1(x,y)) = \mathbf{f}''_{12}(x,y)$; Find $\dfrac{d}{dx}(\mathbf{f}'_2(x,y)) = \mathbf{f}''_{21}(x,y)$

Step 6: Form ${}^{2}\mathbf{Q}^{\mathbf{f}'_1}_{(0,0)}$ for $\mathbf{f}'_1$; Form ${}^{1}\mathbf{Q}^{\mathbf{f}'_2}_{(0,0)}$ for $\mathbf{f}'_2$

Step 7: Find $\lim\limits_{0} {}^{2}\mathbf{Q}^{\mathbf{f}'_1}_{(0,0)}$ for $\mathbf{f}$; $\lim\limits_{0} {}^{1}\mathbf{Q}^{\mathbf{f}'_2}_{(0,0)}$ for $\mathbf{f}$

Step 8: Check if two limits are equal (i) if equal, end program (ii) if not equal, go to Step 9

Step 9: Run sequential limit program

1) Download function $\mathbf{f}''_{12|\mathbb{R}^2-\{0\}}$
2) Download point (0,0)
3) Choose sequence $x_1 : \mathbb{N} \to \mathbb{R}^2$, $lim\, x_1 = (0,0), x_1 \neq 0$
4) Compose $\mathbf{f}''_{12}$ with x_1 to form y^x
5) Find $lim\, y^{x_1}$
6) Repeat (3) - (5) with another sequence x_2
7) Compare $lim\, y^{x_1}$ and $lim\, y^{x_2}$
   (i)if equal, run $\delta - \varepsilon$ program.
   (ii) if not equal, $\mathbf{f}''_{12}$ is not continuous at (0,0)

Step 1: Define $\mathbf{f}(x,y)$

$$\mathbf{f}(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$$

Step 2: Find $\dfrac{d}{dx}(\mathbf{f}(x,y)) = \mathbf{f}'_1(x,y)$; Find $\dfrac{d}{dy}(\mathbf{f}(x,y)) = \mathbf{f}'_2(x,y)$

$$\frac{d}{dx}(\mathbf{f}(x,y)) = \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}; \frac{d}{dy}(\mathbf{f}(x,y)) = \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$$

Step 3: Form ${}^1\mathbf{Q}^{\mathbf{f}}_{(0,0)}$ for $\mathbf{f}$; Form ${}^2\mathbf{Q}^{\mathbf{f}}_{(0,0)}$ for $\mathbf{f}$

$${}^1\mathbf{Q}^{\mathbf{f}}_{(0,0)} = \mathbf{0}; {}^2\mathbf{Q}^{\mathbf{f}\,(0,0)} = \mathbf{0}$$

Step 4: Find $\lim_0 {}^1\mathbf{Q}^{\mathbf{f}}_{(0,0)}$ for $\mathbf{f}$; Find $\lim_0 {}^2\mathbf{Q}^{\mathbf{f}}_{(0,0)}$ for $\mathbf{f}$

$$\lim_0 {}^1\mathbf{Q}^{\mathbf{f}}_{(0,0)} = \mathbf{0}; \lim_0 {}^2\mathbf{Q}^{\mathbf{f}}_{(0,0)} = \mathbf{0}$$

Step 5: Find $\dfrac{d}{dy}(\mathbf{f}'_1(x,y))$; Find $\dfrac{d}{dx}(\mathbf{f}'_2(x,y))$

$$\frac{d}{dy}(\mathbf{f}'_{_1}(x,y)) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}; \frac{d}{dx}(\mathbf{f}'_{_2}(x,y)) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}$$

Step 6: Form ${}^2\mathbf{Q}^{\mathbf{f}'_1}_{(0,0)}$ for $\mathbf{f}'_{_1}$; Form ${}^1\mathbf{Q}^{\mathbf{f}'_2}_{(0,0)}$ for $\mathbf{f}'_{_2}$

$${}^2\mathbf{Q}^{\mathbf{f}'_1}_{(0,0)} = -\mathbf{1}; {}^1\mathbf{Q}^{\mathbf{f}'_2}_{(0,0)} = \mathbf{1}$$

Step 7: Find $\underset{0}{lim}\, {}^2\mathbf{Q}^{\mathbf{f}'_1}_{(0,0)}$ for $\mathbf{f}$; $\underset{0}{lim}\, {}^1\mathbf{Q}^{\mathbf{f}'_2}_{(0,0)}$ for $\mathbf{f}$

$$\underset{0}{lim}\, {}^2\mathbf{Q}^{\mathbf{f}'_1}_{(0,0)} = \mathbf{f}''_{_{12}} = -1; \underset{0}{lim}\, {}^1\mathbf{Q}^{\mathbf{f}'_2}_{(0,0)} = \mathbf{f}''_{_{21}} = 1$$

Step 8: Check if two limits are equal (i) if equal, end program (ii) if not equal, go to Step 9

The limits are not equal. Proceed to Step 9 in the program.

Step 9: Run sequential limit program

1) Download function

$$\mathbf{f}''_{_{12}} = [\mathbf{p}_{_{-3}} \circ (\overset{1}{\mathbf{p}}_{_2}\overset{1}{\mathbf{p}}_{_2} + \overset{2}{\mathbf{p}}_{_2})] \cdot [\overset{1}{\mathbf{p}}_{_6} + 9\overset{1}{\mathbf{p}}_{_4}\overset{2}{\mathbf{p}}_{_2} - 9\overset{1}{\mathbf{p}}_{_2}\overset{2}{\mathbf{p}}_{_4} - \overset{2}{\mathbf{p}}_{_6}]$$

2) Download point (0,0)

Point (0,0) is downloaded.

3) Choose sequence $x_1 : \mathbb{N} \to \mathbb{R}^2(lim\, x = (0,0), x \neq 0$

$$x_{_1} = \mathbf{p}_{_{-1}} \times \mathbf{p}_{_{-1}}$$

4) Compose $f'_{_1}$ with $x_{_1}$ to form y^{x_1}

$$y^{x_1} = \mathbf{f}'_{_1} \circ x_{_1} = [\mathbf{p}_{_{-3}} \circ (2\,\mathbf{p}_{_{-2}})] \cdot [\mathbf{p}_{_{-6}} + 9\,\mathbf{p}_{_{-6}} - 9\,\mathbf{p}_{_{-6}} - \mathbf{p}_{_{-6}}] = 8\,\mathbf{p}_{_6} \cdot \mathbf{0} = \mathbf{0}$$

5) Find $lim\, y^{x_1}$

$$lim\, y^{x_1} = 0$$

6) Repeat (3) - (5) with another sequence x_2

$$x_2 = 2\mathbf{p}_{-1} \times \mathbf{p}_{-1} \to y^{x_2} = \mathbf{f} \circ x_2 = [\mathbf{p}_{-3} \circ (5\,\mathbf{p}_{-2})] \cdot [64\,\mathbf{p}_{-6} + 9\,(16\,\mathbf{p}_{-4})(\mathbf{p}_{-2}) - 9\,(4\,\mathbf{p}_{-2}(\mathbf{p}_{-4} - \mathbf{p}_{-6})] = \frac{1}{125}\mathbf{p}_6 \cdot [171\,\mathbf{p}_{-6}] = \frac{171}{125} \to lim\, y^{x_2} = \frac{171}{125}$$

7) Compare $lim\ y^{x_1}$ and $lim\ y^{x_2}$

$$lim\, y^{x_1} \neq lim\, y^{x_2}$$

Therefore, $\mathbf{f}''_{12}$ is not continuous at (0,0)

Since $\mathbf{f}''_{12}$ is not continuous at (0,0), this proves that $\mathbf{f}''_{12}$ and $\mathbf{f}''_{21}$ do not necessarily have the same value when evaluated at (0,0). Thus, the equality of mixed partial derivatives is not contradicted.

Austin Fehr, Mehdi Drissi, Rahul Rajala, Cody Poteet, Kenneth Teel

Program Limit of Function

1. Download a function **f** and a point a.
2. Create an arbitrary but fixed sequence whose limit is a but is not constant.
3. Create the composition of **f** with 2.
4. Check the limit of 3 by running the limit of a sequence program.

Program for 3 - 3
1. Download the given function **f**.
2. Form the quotient function with **f** = **f**, i = 1, and a = (0,0).
3. Form the quotient function with **f** = **f**, i = 2, and a = (0,0).
4. Form the quotient function with **f** = **f**, i = 1, and a is an arbitrary but fixed point (a^1, a^2).
5. Form the quotient function with **f** = **f**, i = 2, and a is an arbitrary but fixed point (a^1, a^2).
6. Find limit of (4) by running the program limit of function where f and a are as has been previously stated.

7. Find limit of (3) by running the program limit of function where f and a are as has been previously stated.
8. Form the quotient function with **f** = (6), i = 1, and a = (0,0).
9. Form the quotient function with **f** = (7), i = 2, and a = (0,0).
10. Form the quotient function with **f** = (6), i = 1, and a is an arbitrary but fixed point (a^1, a^2).
11. Form the quotient function with **f** = (7), i = 2, and a is an arbitrary but fixed point (a^1, a^2).
12. Form the union of (8) and (10).
13. Form the union of (9) and (11).
14. Find limit at (0,0) of 12 using the program limit of a function.
15. Find value of (12) at (0,0).
16. Check if (14) = (15).
16.a. If (16) is true continue the program.
16.b. If (16) is false exit the program and print as output f''_{21} is not continuous at (0,0) so the theorem $f''_{12}(a) = f''_2 1(a)$ will not necessarily hold.
17. Find limit at (0,0) of 13 using the program limit of a function.
18. Find value of (13) at (0,0).
19. Check if (17) = (18).
19.a. If (19) is true exit the program and print as output that both second derivatives are continuous at (0,0) so the theorem $f''_{12}(a) = f''_2 1(a)$ will hold.
19.b. If (19) is false exit the program and print as output f''_{12} is not continuous at (0,0) so the theorem $f''_{12}(a) = f''_2 1(a)$ will not necessarily hold.

PROOF FOR EXTREME VALUES ON A COMPACT SET

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The purpose of this proof is to illustrate the existance of a minimum and a maximum on a closed interval

Completness Axiom: $\forall$ S $\subset \mathbb{R}$ and if $\forall\ x \in S$ are bounded from above then $\exists$ M such that M= sup(S) and if $\forall\ x \in S$ are bounded from below then $\exists N$ such that N= inf(S)

Defintion 1: (Supremum) Let E $\subset \mathbb{R} \quad and \quad |E| > 0$. If E is bounded from above then $\exists M \in \mathbb{R} \quad such \quad that \quad \forall \quad x \in E, x < M$ and M is the smallest of all such upper bounds or M is the supremum of E, so we write M = sup(E)

Defintion 2: (Infimum) Let E $\subset \mathbb{R} \quad and \quad |E| > 0$. If E is bounded from below then $\exists M \in \mathbb{R} \quad such \quad that \quad \forall \quad x \in E, M < x$ and M is the largest of all such lower bounds or M is the infimum of E, so we write M = inf(E)

Defintion 3: (Closed Set) Let E be a set of real numbers. the set E is said to be closed provided that $\forall$ $c > 0$ and x $\in$ E the intersection (x-c, x+c) $\cap$ E contains infinitly many points of E, $\forall$ x $\in$ E.

Defintion 4: (Compact Set) Let E be a set of real numbers. the set E is said to be compact provided E is both closed and bounded.

Defintion 5: (Subsequences) let the sequence

$$s_1, s_2, s_3, ...$$

be any sequence. Then by a subsequence we mean any sequence

$$s_{n_1}, s_{n_2}, s_{n_3}, ...$$

where

$$n_1 < n_2 < n_3 < ...$$

$n \in \mathbb{N}$

Theorem 1: *Every Sequence contains a monotonic subsequence.*
Proof: We construct first a nonincreasing subsequence if possible. We call the mth element x_m of the sequence x_n a turn-back point if $x_m \leq x_n$ $\forall$ $n > m$ If there is an infinite number of turn-back points $x_{m_1}, x_{m_2}, x_{m_3}, ...$ the we have found our non-increasing subsequence since

$$x_{m_1} \leq x_{m_2} \leq x_{m_3} \leq ...$$

This would not be possibile if there are only finatly many turn back points. Let us suppose that x_M is the last turn-back point such that any element x_n for n > M is not a turn-back point. Thus $x_m > x_n$ for some $m > n$. From this we can choose $x_{m_1} > x_{M+1}$ with $m_1 > M+1$, then $x_{m_2} > x_{m_1}$ with $m_2 > m_1$, and then $x_{m_3} > x_{m_2}$ with $m_3 > m_2$, and so on to obtain an increasing subsequence

$$x_{M+1} < x_{m_1} \leq x_{m_2} < \ldots$$

as required

Theorem 2: (Bolzano-Weierstrass) *Every bounded sequence contains a convergent sequence.*
Proof: By Theorem 1 every sequence contains a monotonic subsequence. Here that sequence would be both monotonic an bounded, and hence convergent.

Now for the Actual Proof:

Theorem 3:(Existance of Extreme Values on a Compact Set)
Let f be continous on closed set A$\in \mathbb{R}$.
f : E $\rightarrow \mathbb{R}$
Then f possesses both a absloute maximum and absloute minimum. Proof: Because the set is compact it is also closed and bounded. Because the set is bounded then there must exist a least upper bound and a greatest lower bound. By the completness axiom $\exists$ a sup(*f(x)*) x$\in$A which we will say is equalivalent to abf $M \in \mathbb{R}$ We will show that $\exists x_0$ such that $f(x_0)$ = M. because M is the supremum then

$$\exists x_1 \text{ such that } f(x_1) = \text{M-1}$$
$$\exists x_2 \text{ such that } f(x_2) = \text{M-}1/2$$
$$\exists x_3 \text{ such that } f(x_3) = \text{M-}1/3$$
$$.$$
$$..$$
$$\ldots$$
$$\exists x_n \text{ such that } f(x_n) = \text{M-}1/n$$

Then applying Theorem 2 we are able to fing a convergent subsequence x_{n_k} such that as $\lim_{k\to\infty} x_{n_k} = x_0 \in A$ Thus $\lim_{k\to\infty} f(x_{n_k}) = M = f(x_0)$ Thus the function reaches a maximum on the set, the same proof applies to the infinum or absloute minimum

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Problem 1 Let $\mathbf{f} = 3\overset{1}{p}_3 \cdot \overset{3}{p}_2 + 4\overset{1}{p}_2 \cdot \overset{2}{p}_2 + 6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1 : \mathbb{R}^{\nvDash} \to \mathbb{R}$
Find all three partial derivatives along with f''_{13} and f''_{31}

Unstated usage of the rule that $\overset{i}{f}{}'_i = \overset{i}{f}{}'$ occurred when needed.

Step 0 (Identify Formula):
Derivative 4 (Sum)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 3\overset{1}{p}_3 \cdot \overset{3}{p}_2 + 4\overset{1}{p}_2 \cdot \overset{2}{p}_2$
$g = 6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1$
Step 3 (Replacing Symbols):

$$f'_1 = (3\overset{1}{p}_3 \cdot \overset{3}{p}_2 + 4\overset{1}{p}_2 \cdot \overset{2}{p}_2)'_1 + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)'_1$$

Step 0 (Identify Formula):
Derivative 4 (Sum)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 3\overset{1}{p}_3 \cdot \overset{3}{p}_2$
$g = 4\overset{1}{p}_2 \cdot \overset{2}{p}_2$
Step 3 (Replacing Symbols):

$$f'_1 = (3\overset{1}{p}_3 \cdot \overset{3}{p}_2)'_1 + (4\overset{1}{p}_2 \cdot \overset{2}{p}_2)'_1 + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)'_1$$

Step 0 (Identify Formula):
Formula 39

Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{1}{p}_3$
$k = 3\overset{3}{p}_2$
Step 3 (Replacing Symbols):

$$f_1' = (3\overset{3}{p}_2)(\overset{1}{p}_3)_1' + (4\overset{1}{p}_2 \cdot \overset{2}{p}_2)_1' + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_1'$$

Step 0 (Identify Formula):
Formula 39
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{1}{p}_2$
$k = 4\overset{2}{p}_2$
Step 3 (Replacing Symbols):

$$f_1' = (3\overset{3}{p}_2)(\overset{1}{p}_3)_1' + (4\overset{2}{p}_2)(\overset{1}{p}_2)_1' + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_1'$$

Step 0 (Identify Formula):
Formula 39
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 6\overset{2}{p}_1 \cdot \overset{3}{p}_1$
$k = \overset{1}{p}_1$
Step 3 (Replacing Symbols):

$$f_1' = (3\overset{3}{p}_2)(\overset{1}{p}_3)_1' + (4\overset{2}{p}_2)(\overset{1}{p}_2)_1' + (6\overset{2}{p}_1 \cdot \overset{3}{p}_1)(\overset{1}{p}_1)_1'$$

Step 0 (Identify Formula):
Derivative 1 (Power)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_3$
Step 3 (Replacing Symbols):

$$f_1' = (3\overset{3}{p}_2)(3\overset{1}{p}_2) + (4\overset{2}{p}_2)(\overset{1}{p}_2)_1' + (6\overset{2}{p}_1 \cdot \overset{3}{p}_1)(\overset{1}{p}_1)_1' =$$
$$9\overset{1}{p}_2 \cdot \overset{3}{p}_2 + (4\overset{2}{p}_2)(\overset{1}{p}_2)_1' + (6\overset{2}{p}_1 \cdot \overset{3}{p}_1)(\overset{1}{p}_1)_1'$$

Step 0 (Identify Formula):
Derivative 1 (Power)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_2$
Step 3 (Replacing Symbols):

$$f_1' = 9\overset{1}{p}_2 \cdot \overset{3}{p}_2 + (4\overset{2}{p}_2)(2\overset{1}{p}_1) + (6\overset{2}{p}_1 \cdot \overset{3}{p}_1)(\overset{1}{p}_1)_1' = 9\overset{1}{p}_2 \cdot \overset{3}{p}_2 + 8\overset{1}{p}_1 \cdot \overset{2}{p}_2 + (6\overset{2}{p}_1 \cdot \overset{3}{p}_1)(\overset{1}{p}_1)_1'$$

Step 0 (Identify Formula):
Derivative 1 (Power)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_1$
Step 3 (Replacing Symbols):

$$f_1' = 9\overset{1}{p}_2 \cdot \overset{3}{p}_2 + 8\overset{1}{p}_1 \cdot \overset{2}{p}_2 + (6\overset{2}{p}_1 \cdot \overset{3}{p}_1)(1) = 9\overset{1}{p}_2 \cdot \overset{3}{p}_2 + 8\overset{1}{p}_1 \cdot \overset{2}{p}_2 + 6\overset{2}{p}_1 \cdot \overset{3}{p}_1$$

That completes the first partial derivative.

Step 0 (Identify Formula):
Derivative 4 (Sum)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 3\overset{1}{p}_3 \cdot \overset{3}{p}_2 + 4\overset{1}{p}_2 \cdot \overset{2}{p}_2$
$g = 6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1$
Step 3 (Replacing Symbols):

$$f_2' = (3\overset{1}{p}_3 \cdot \overset{3}{p}_2 + 4\overset{1}{p}_2 \cdot \overset{2}{p}_2)_2' + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_2'$$

Step 0 (Identify Formula):
Derivative 4 (Sum)
Step 1 (Identify Side):
Left Side

Step 2 (Identify Symbols):
$f = 3\overset{1}{p}_3 \cdot \overset{3}{p}_2$
$g = 4\overset{1}{p}_2 \cdot \overset{2}{p}_2$
Step 3 (Replacing Symbols):

$$f_2' = (3\overset{1}{p}_3 \cdot \overset{3}{p}_2)_2' + (4\overset{1}{p}_2 \cdot \overset{2}{p}_2)_2' + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_2'$$

Step 0 (Identify Formula):
Formula 39
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{2}{p}_2$
$k = 4\overset{1}{p}_2$
Step 3 (Replacing Symbols):

$$f_2' = (3\overset{1}{p}_3 \cdot \overset{3}{p}_2)_2' + (4\overset{1}{p}_2)(2\overset{2}{p}_1)_2' + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_2' = (3\overset{1}{p}_3 \cdot \overset{3}{p}_2)_2' + 8\overset{1}{p}_2 \cdot \overset{2}{p}_1 + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_2'$$

Step 0 (Identify Formula):
Derivative 1 (Power)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_2$
Step 3 (Replacing Symbols):

$$f_2' = (3\overset{1}{p}_3 \cdot \overset{3}{p}_2)_2' + (4\overset{1}{p}_2)(2\overset{2}{p}_1) + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_2' = (3\overset{1}{p}_3 \cdot \overset{3}{p}_2)_2' + 8\overset{1}{p}_2 \cdot \overset{2}{p}_1 + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_2'$$

Step 0 (Identify Formula):
Formula 40
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 3\overset{1}{p}_3 \cdot \overset{3}{p}_2$
Step 3 (Replacing Symbols):

$$f_2' = 8\overset{1}{p}_2 \cdot \overset{2}{p}_1 + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_2'$$

Step 0 (Identify Formula):
Formula 39
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{2}{p}_1$
$k = 6\overset{1}{p}_1 \cdot \overset{3}{p}_1$
Step 3 (Replacing Symbols):

$$f_2' = 8\overset{1}{p}_2 \cdot \overset{2}{p}_1 + (6\overset{1}{p}_1 \cdot \overset{3}{p}_1)(\overset{2}{p}_1)_2'$$

Step 0 (Identify Formula):
Derivative 1 (Power)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_1$
Step 3 (Replacing Symbols):

$$f_2' = 8\overset{1}{p}_2 \cdot \overset{2}{p}_1 + 6\overset{1}{p}_1 \cdot \overset{3}{p}_)$$

That completes the second partial derivative.
Step 0 (Identify Formula):
Derivative 4 (Sum)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 3\overset{1}{p}_3 \cdot \overset{3}{p}_2 + 4\overset{1}{p}_2 \cdot \overset{2}{p}_2$
$g = 6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1$
Step 3 (Replacing Symbols):

$$f_3' = (3\overset{1}{p}_3 \cdot \overset{3}{p}_2 + 4\overset{1}{p}_2 \cdot \overset{2}{p}_2)_3' + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_3'$$

Step 0 (Identify Formula):
Derivative 4 (Sum)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):

$f = 3\overset{1}{p}_3 \cdot \overset{3}{p}_2$
$g = 4\overset{1}{p}_2 \cdot \overset{2}{p}_2$
Step 3 (Replacing Symbols):

$$f_3' = (3\overset{1}{p}_3 \cdot \overset{3}{p}_2)_3' + (4\overset{1}{p}_2 \cdot \overset{2}{p}_2)_3' + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_3'$$

Step 0 (Identify Formula):
Formula 39
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{3}{p}_2$
$k = 3\overset{1}{p}_3$
Step 3 (Replacing Symbols):

$$f_3' = (3\overset{1}{p}_3)(\overset{3}{p}_2)_3' + (4\overset{1}{p}_2 \cdot \overset{2}{p}_2)_3' + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_3'$$

Step 0 (Identify Formula):
Formula 40
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 4\overset{1}{p}_2 \cdot \overset{2}{p}_2$
Step 3 (Replacing Symbols):

$$f_3' = (3\overset{1}{p}_3)(\overset{3}{p}_2)_3' + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \overset{3}{p}_1)_3'$$

Step 0 (Identify Formula):
Formula 39
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{3}{p}_1$
$k = 6\overset{1}{p}_1 \cdot \overset{2}{p}_1$
Step 3 (Replacing Symbols):

$$f_3' = (3\overset{1}{p}_3)(\overset{3}{p}_2)_3' + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1)(\overset{3}{p}_1)_3'$$

Step 0 (Identify Formula):
Derivative 1 (Power)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_2$
Step 3 (Replacing Symbols):

$$f'_3 = 6\overset{1}{p}_3 \cdot \overset{3}{p}_1 + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1)(\overset{3}{p}_1)'_3$$

Step 0 (Identify Formula):
Derivative 1 (Power)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_1$
Step 3 (Replacing Symbols):

$$f'_3 = 6\overset{1}{p}_3 \cdot \overset{3}{p}_1 + 6\overset{1}{p}_1 \cdot \overset{2}{p}_1$$

That completes the 3rd partial derivative.
Step 0 (Identify Formula):
Derivative 4 (Sum)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 6\overset{1}{p}_3 \cdot \overset{3}{p}_1$
$g = 6\overset{1}{p}_1 \cdot \overset{2}{p}_1$
Step 3 (Replacing Symbols):

$$f''_{31} = (6\overset{1}{p}_3 \cdot \overset{3}{p}_1)'_1 + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1)'_1$$

Step 0 (Identify Formula):
Formula 39
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{1}{p}_3$

$k = 6\overset{3}{p}_1$
Step 3 (Replacing Symbols):

$$f''_{31} = (\overset{1}{p}_3)'_1(6\overset{3}{p}_1) + (6\overset{1}{p}_1 \cdot \overset{2}{p}_1)'_1$$

Step 0 (Identify Formula):
Formula 39
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{1}{p}_1$
$k = 6\overset{2}{p}_1$
Step 3 (Replacing Symbols):

$$f''_{31} = (\overset{1}{p}_3)'_1(6\overset{3}{p}_1) + (\overset{1}{p}_1)'_1(6\overset{2}{p}_1)$$

Step 0 (Identify Formula):
Derivative 1 (Power)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_3$
Step 3 (Replacing Symbols):

$$f''_{31} = 18\overset{1}{p}_2 \cdot \overset{3}{p}_1 + (\overset{1}{p}_1)'_1(6\overset{2}{p}_1)$$

Step 0 (Identify Formula):
Derivative 1 (Power)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_1$
Step 3 (Replacing Symbols):

$$f''_{31} = 18\overset{1}{p}_2 \cdot \overset{3}{p}_1 + 6\overset{2}{p}_1$$

Now for the last partial derivative.

Step 0 (Identify Formula):
Derivative 4 (Sum)

Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 9\overset{1}{p}_2 \cdot \overset{3}{p}_2 + 8\overset{1}{p}_1 \cdot \overset{2}{p}_2$
$g = 6\overset{2}{p}_1 \cdot \overset{3}{p}_1$
Step 3 (Replacing Symbols):

$$f''_{13} = (9\overset{1}{p}_2 \cdot \overset{3}{p}_2 + 8\overset{1}{p}_1 \cdot \overset{2}{p}_2)'_3 + (6\overset{2}{p}_1 \cdot \overset{3}{p}_1)'_3$$

Step 0 (Identify Formula):
Derivative 4 (Sum)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 9\overset{1}{p}_2 \cdot \overset{3}{p}_2$
$g = 8\overset{1}{p}_1 \cdot \overset{2}{p}_2$
Step 3 (Replacing Symbols):

$$f''_{13} = (9\overset{1}{p}_2 \cdot \overset{3}{p}_2)'_3 + (8\overset{1}{p}_1 \cdot \overset{2}{p}_2)'_3 + (6\overset{2}{p}_1 \cdot \overset{3}{p}_1)'_3$$

Step 0 (Identify Formula):
Formula 39
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = \overset{3}{p}_2$
$k = 9\overset{1}{p}_2$
Step 3 (Replacing Symbols):

$$f''_{13} = (9\overset{1}{p}_2)(\overset{3}{p}_2)'_3 + (8\overset{1}{p}_1 \cdot \overset{2}{p}_2)'_3 + (6\overset{2}{p}_1 \cdot \overset{3}{p}_1)'_3$$

Step 0 (Identify Formula):
Derivative 1 (Power)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_2$
Step 3 (Replacing Symbols):

$$f''_{13} = 18\overset{1}{p}_2 \cdot \overset{3}{p}_1 + (8\overset{1}{p}_1 \cdot \overset{2}{p}_2)'_3 + (6\overset{2}{p}_1 \cdot \overset{3}{p}_1)'_3$$

Step 0 (Identify Formula):
Formula 40
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$k = 8\overset{1}{p}_1 \cdot \overset{2}{p}_2$
Step 3 (Replacing Symbols):

$$f''_{13} = 18\overset{1}{p}_2 \cdot \overset{3}{p}_1 + (6\overset{2}{p}_1 \cdot \overset{3}{p}_1)'_3$$

Step 0 (Identify Formula):
Formula 39
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$f = 6\overset{2}{p}_1$
$k = \overset{3}{p}_1$
Step 3 (Replacing Symbols):

$$f''_{13} = 18\overset{1}{p}_2 \cdot \overset{3}{p}_1 + (6\overset{2}{p}_1)(\overset{3}{p}_1)'_3$$

Step 0 (Identify Formula):
Derivative 1 (Power)
Step 1 (Identify Side):
Left Side
Step 2 (Identify Symbols):
$p_a = p_1$
Step 3 (Replacing Symbols):

$$f''_{13} = 18\overset{1}{p}_2 \cdot \overset{3}{p}_1 + 6\overset{2}{p}_1$$

And that completes the 5 partial derivatives desired.

Functions of Several Variables

Yitong Chen, Preeti Mohan, Caron Song, Divya Velury, Julie Zhu

Problems 4

7. Let $\mathbf{f} = \phi \circ \left(\overset{1}{\mathbf{p}}_{1} \cdot \overset{2}{\mathbf{p}}_{2}\right) : \mathbb{R}^2 \to \mathbb{R}$, where ϕ and ψ are twice differentiable single variable functions. Show that

$$\mathbf{F} = \overset{1}{\mathbf{p}}_{2} \cdot \mathbf{f}''_{11} - \overset{2}{\mathbf{p}}_{2} \cdot \mathbf{f}''_{22} + \overset{1}{\mathbf{p}}_{1} \cdot \mathbf{f}'_{1} - \overset{2}{\mathbf{p}}_{1} \cdot \mathbf{f}'_{2} = 0$$

<u>PROGRAM</u>

Step 1: Find values for all derivatives of $\mathbf{f}$ on both sides of the equality.

1) Download $\mathbf{f}$

2) Determine which variable to take the derivative with respect to. For a mixed partial derivative, start with the leftmost number subscript and move to the right with each repetition of this step. For this i^{th} variable derivative, components of $\mathbf{f}$ that are not of this variable are to be treated as constants and their derivatives are the same as the components themselves.

3) Break up $\mathbf{f}$ by using left hand side of Derivatives of Algebraic Functions formula (4) and/or formula (5)

$$(\mathbf{f} + \mathbf{g})' = \mathbf{f}' + \mathbf{g}'$$
$$(\mathbf{f} \cdot \mathbf{g})' = \mathbf{f}' \cdot \mathbf{g}'$$

4) Find the derivatives of each composition in the function using the Derivative of Composition Function if necessary, formula (6)

$$(\mathbf{f} \circ \mathbf{g})' = (\mathbf{f}' \circ \mathbf{g}) \cdot \mathbf{g}'$$

5) Repeat 1-4) for mixed partial derivatives. Use the derivative found in 4) as $\mathbf{f}$

Step 2: Compare left and right side of equations to prove or disprove equality.

<u>Step 1:</u> Find values for all derivatives of $\mathbf{f}$ on both sides of the equality.

1) Break up the **f** by using left hand side of Derivatives of Algebraic Functions

Saturday Presentation Medhi Drissi, Austin Fehr, Jerri Ni, Cody Poteet, Rahul Rajala, Kenneth Teel
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3 Directional Derivative

The goal is to find find the tangent vector in some direction on a surface.

First download $\mathbb{E}_3$, $\mathbb{R}^3$, $\mathbb{F}^3$. Next download a surface:

$S \subset \mathbb{E}_3 = \{X \equiv (x^1, x^2, x^3); f(x) = 0 : \mathbb{R}^3 \to \mathbb{R}\}$

A vector valued function can be formed from S being of the identification of a point with vector. The vector would be the position vector to the corresponding point and will be $\bar{S}$ which is $\mathbb{F}^3 \to \mathbb{R}$.

Next download any arbitrary but fixed point on the surface a. Form the position vector to that point $\bar{a}$.

$\bar{u} \in \mathbb{F}^3 \wedge |u| = 1$

Now, the vector $\bar{a} + \bar{u}h \wedge h \in \mathbb{R}$ is in the desired direction of movement. The multiplying by a scalar is allowed because:

$\bar{i} \parallel \bar{g} \equiv \bar{i} = \rho\bar{g} \wedge \rho \in \mathbb{R}$

So we can form a function analogous to the quotient function.

${}^{u}Q^{a}_{f} = p_{-1} \cdot \big(f \circ \left(a^1 p_0 + u^1 p_1\right) \times \left(a^2 p_0 + u^2 p_1\right) \times \left(a^3 p_0 + u^3 p_1\right) - f(a)\big) :$
$\mathbb{R} - \{0\} \to \mathbb{R}$

The main difference from the previous quotient function is that instead of one linear function you have all linear functions (assuming u^i does not equal 0) because adding two vectors involves adding each of their components.

Forming the limit of the quotient function at 0 gives the directional derivative in the direction of u notated $f'_{\bar{u}}$. This can be calculated to be:

$$\nabla f \cdot u$$

4 Tangent Plane

Next we'll form the plane tangent to a point on a surface with the point called a. Form the position vector to a called **a**. All prior downloads are still valid. A tangent plane can be defined in two ways.

$$\nabla f(a) \cdot (\mathbf{x} - \mathbf{a})$$

The first uses the fact that a plane can be thought of as a set of vectors from one point (a) whose normal vectors are parallel. The second involves having two tangent vectors. From the directional derivative we can find a tangent vector to a surface. Using the trivial directions of $\bar{u}^1 = i$ and $\bar{u}^2 = j$ we can form the directional derivatives f'_1 and f'_2.

$$\mathbf{x} = \lambda f'_1(a) + \kappa f'_2(a)$$

This second derivation is less easy to derive a tangent plane, but works for non inner product spaces.

5 Problem 3

The problem asks for a tangent plane and normal line to a surface at the point. First the tangent plane will be formed based on how the previous section. The surface that will be downloaded:

$$S = \{X \equiv (x, y, z); x^2 + 2y^2 + 3z^2 - 21 = 0\}$$

We next download the point $A \equiv (4, -1, 1)$ and then form its position vector $\mathbf{A}$.

Form ∇S which mathemathica can tell us is:

$\nabla S = (2x)i + (4y)j + (6z)k$

Now we have both ∇S and $\mathbf{A}$. So the tangent plane is:

$\nabla S \cdot (\mathbf{x} - \mathbf{A}) = 0 \equiv x^2 + 2y^2 + 3z^2 - 4x + 2y - 3z = 0$

6 Program for Solving 11

The problem asks for the maximum of the error for the surface area of a rectangular cube given the range the error of each side is in. We first download k which is the amount of possible error.

$|e| \leq k \wedge k \in R \equiv |x| \leq k \wedge |y| \leq k \wedge$

Functions of Several Variables

Sooraj Boominathan, Vincent Po, Steve Zhang, Varun Ennamuri, Nathan Yu, Ben Zhao

The Tangent Vector, Gradients, Highest and Lowest Values of Surfaces

We begin by defining some sets. The first set is:

$$\mathbf{E}_3 \supset S = \{X \equiv x = (x^1, x^2, x^3); \mathbf{f}(x) = 0 \wedge \mathbf{f} : \mathbf{R}^3 \to \mathbf{R}\}$$

Where S is a surface in euclidian 3-space. We can also write this as:

$$\mathbf{E}_3 \supset S = \{X \equiv x = (x^1, x^2, x^3); \mathbf{x} = \mathbf{g}(t, s) \wedge s, t \in \mathbb{R} \wedge \mathbf{g} = \mathbf{g}^1 \times \mathbf{g}^2 \times \mathbf{g}^3 \wedge \mathbf{g} : \mathbf{R}^2 \to \mathbf{R}^3 \wedge \mathbf{g}^i : \mathbf{R}^2 \to \mathbf{R}\}$$

If the functions $\mathbf{f}$ and $\mathbf{g}$ are linear, then the surface S is a plane.

The second set is:

Let $\mathbf{G_f}$ be the graph of the function $\mathbf{f} = \overset{1}{\mathbf{sin}} \cdot [\mathbf{exp} \circ (-\overset{2}{\mathbf{p}}_1)] : \mathbb{R}^2 \to \mathbb{R}$ in the Euclidean space $\mathbb{E}_3$. Find an equation of the tangent plane to $\mathbf{G_f}$ at the point $A \equiv (x, y, z)$, where $x = \frac{\pi}{2}$ and $y = 0$.

$$\mathbf{E}_3 \supset \Gamma = \{X \equiv x = (x^1, x^2, x^3); \mathbf{x} = \boldsymbol{\gamma}(t) \wedge t \in \mathbb{R} \wedge \gamma = \gamma^1 \times \gamma^2 \times \gamma^3 \wedge \gamma : \mathbf{R} \to \mathbf{R}^3 \wedge \gamma^i : \mathbf{R} \to \mathbf{R}\}$$

$$\equiv \{X \equiv x = (x^1, x^2, x^3); \mathbf{x} = \bar{\boldsymbol{\gamma}}(t) \wedge \bar{\gamma} = \bar{\gamma}^1 \hat{i} + \bar{\gamma}^2 \hat{j} + \bar{\gamma}^3 \hat{k} : \mathbf{R} \to \mathbf{F}_3\}$$

Where Γ is a curve. If γ is linear, then this curve represents a straight line.

We can now begin our discussion on the tangent vector, gradients, and maximum/minimum values of a surface.

The Tangent Vector

Step 1: Download a curve $\Gamma \subset \mathbf{E}_3$ based on our definition above.

Step 2: Choose ABF point A on the curve, such that:

$$A \in \Gamma \wedge \bar{a} \equiv \bar{\gamma}(t_0) \wedge t_0 \in \mathbf{R}$$

Step 3: Choose another point B on the curve:

$$B \in \Gamma \wedge \bar{b} \equiv \bar{\gamma}(t_0 + h) \wedge t_0, h \in \mathbf{R}$$

Step 4: Form the equivalence class [AB]:

$$[AB] = \bar{b} - \bar{a} = \bar{\gamma}(t_0 + h) - \bar{\gamma}(t_0)$$

If we multiply this by any constant, we will get a vector that is "parallel" to this vector [AB]. We thus choose to multiply [AB] by $\frac{1}{h}$, the reason for which will become evident shortly:

$$\frac{1}{h}\,[AB] = \frac{\bar{\gamma}(t_0 + h) - \bar{\gamma}(t_0)}{h}$$

We recognize that this is in the same format as the quotient fuction, and thus:

$$\mathbf{Q}^{\bar{\gamma}}_{t_0}(h) = \frac{\bar{\gamma}(t_0 + h) - \bar{\gamma}(t_0)}{h}$$

But since we have never defined the quotient function for vectors, we will make γ into a function again, and then at the end convert it into a vector. The quotient function thus becomes:

$$\mathbf{Q}_{t_0}^{\gamma^i}(h) = \frac{\gamma^i(t_0 + h) - \gamma^i(t_0)}{h}$$

If we add a limit, this becomes the derivative, and then we go into vector space:

$$\underset{0}{lim}\ \mathbf{Q}_{t_0}^{\gamma^i}(h) = \underset{0}{lim}\ \mathbf{Q}_{t_0}^{\gamma^1}\,\hat{i} + \underset{0}{lim}\ \mathbf{Q}_{t_0}^{\gamma^2}\,\hat{i} = \bar{\gamma}(t_0)$$

Thus, our interpretation of the tangent is that:

$$''\bar{\gamma}'(t_0) \text{ is parallel to the tangent line'' at the point } \gamma(t_0)$$

The equation of this Tangent Line at A is:

$$\mathbf{E}_3 \supset l = \{X \equiv x = (x^1, x^2, x^3); \bar{\mathbf{x}} = \boldsymbol{\gamma}(t_0) + \alpha(\boldsymbol{\gamma}')(t_0) \wedge \alpha \in \mathbf{R}\}$$

<u>The Gradient</u>

We propose that:

$$\Gamma \subset S \equiv \mathbf{f} \circ \boldsymbol{\gamma} = \mathbf{0}$$

In english, take the curve to be a subset of the surface. If an ABF point is chosen from Γ:

$$\Gamma \subset S \equiv x \in \Gamma \equiv x \in S$$

$$\mathbf{f} \circ \boldsymbol{\gamma} = \mathbf{0} : \mathbf{R} \to \mathbf{R}$$

$$(\mathbf{f} \circ \boldsymbol{\gamma})' = \mathbf{0} : \mathbf{R} \to \mathbf{R}$$

$$\text{Where } \mathbf{f} : \mathbf{R}^3 \to \mathbf{R} \wedge \boldsymbol{\gamma} : \mathbf{R} \to \mathbf{R}^3$$

$$(\mathbf{f} \circ \boldsymbol{\gamma})' = \sum_{i=1}^{3} \left[\mathbf{f}'_i \circ \boldsymbol{\gamma}\right] \cdot (\boldsymbol{\gamma}^i)' : \mathbf{R} \to \mathbf{R} = 0$$

If we go into $\mathbf{F}_3$:

$$\boldsymbol{\gamma}' = \gamma^{1'} + \gamma^{2'} + \gamma^{3'} \equiv (\boldsymbol{\gamma}^1)'\hat{i} + (\boldsymbol{\gamma}^2)'\hat{j} + (\boldsymbol{\gamma}^3)'\hat{k}$$

Thus:

$$(\nabla \mathbf{f}(t)) \bullet \bar{\gamma}'(t) = \mathbf{0} : \mathbf{R} \to \mathbf{R}$$

This gives us an interpretation of what the gradient is: it is a vector that is perpendicular to the surface, which equivalently means that it is perpendicular to a curve on that surface.

The definition of the gradient, in $\mathbf{F}_3$ is thus:

$$\nabla \mathbf{f} = grad(\mathbf{f}) = \mathbf{f}'_1\hat{i} + \mathbf{f}'_2\hat{j} + \mathbf{f}'_3\hat{k} : \mathbf{R}^3 \to \mathbf{F}_3$$

Maximum and Minimum Values of a Function

Our final topic concerns the maximum and minimum values of a surface, specifically in $\mathbf{E}_3$:

First, we have to assume that maximum or minimum value of the function exists within whatever constraints are laid down before hand. Then, download a function:

$$\mathbf{g} : \mathbf{R}^2 \to \mathbf{R},\ G_{\mathbf{g}} \subset \mathbf{E}_3.$$

This is an example of a surface. We can represent this as:

$$\mathbf{g}(a^1, a^2) = a^3; a^1, a^2, a^3 \in \mathbf{R}$$

We would like to represent this as a function $\mathbf{f} : \mathbf{R}^3 \to \mathbf{R}$, since this was part of one of the first sets defined, namely that of a surface S.

$$\mathbf{g}(a^1, a^2) = a^3 \Rightarrow \mathbf{g}(a^1, a^2) - a^3 = 0 = \mathbf{f}(a^1, a^2, a^3)$$

$$\mathbf{f} = \overset{12}{\mathbf{g}} - \overset{3}{\mathbf{p}}_1 \ \Rightarrow \ \nabla \mathbf{f} = \overset{12}{\mathbf{g}}{}'_1\,\hat{\mathbf{i}} + \overset{12}{\mathbf{g}}{}'_2\,\hat{\mathbf{j}} - \hat{\mathbf{k}} = \nabla \mathbf{g} - \hat{\mathbf{k}}$$

If we have a "highest" or "maximum" point on the surface, the vector normal to this point <u>must</u> be perpendicular to the "xy" plane (provided that we have already defined our orientation and assumed an orthonormal basis). This normal vector of the surface was mentioned earlier in the presentation of the gradient, namely:

$$''\nabla \mathbf{f} \;\perp S'', \mathbf{f} : \mathbf{R}^3 \to \mathbf{R}$$

If $''\nabla \mathbf{f} \perp$ S ":

$$\nabla \mathbf{f} = \alpha\, \hat{\mathbf{k}}, \alpha \in \mathbf{R}$$

Thus:

$$\nabla \mathbf{f} = \nabla \mathbf{g} - \hat{\mathbf{k}} \;\Rightarrow\; \nabla \mathbf{g} = \mathbf{0} \;\Rightarrow\; \mathbf{g}'_{_1}(a) = 0 \wedge \mathbf{g}'_{_2}(a) = 0$$

We now have two equations for two unknowns (a^1, a^2). a^3 follows directly from $\mathbf{g}(a^1, a^2) = a^3$, and we have found the maximum or minimum point(s) on the surface.

Functions of Several Variables

Sooraj Boominathan, Vincent Po, Ben Zhao, Nathan Yu, Steve Zhang, Varun Ennamuri

Gradients

We begin by defining some sets. The first set is:

$$\mathbf{E}_{_3} \supset S = \{X \equiv x = (x^1, x^2, x^3); \mathbf{f}(x) = 0 \wedge \mathbf{f} : \mathbf{R}^3 \to \mathbf{R}\}$$

Where S is a surface in educlidian 3-space. We can also write this as:

$$\mathbf{E}_{_3} \supset S = \{X \equiv x = (x^1, x^2, x^3); \mathbf{g} = \mathbf{g}^1 \times \mathbf{g}^2 \times \mathbf{g}^3 \wedge \mathbf{g} : \mathbf{R}^2 \to \mathbf{R}^3 \wedge \mathbf{g}^i : \mathbf{R}^2 \to \mathbf{R}\}$$

If the functions **f** and **g** are linear, then the surface S is a plane. The second set is:

$$\mathbf{E}_{_3} \supset \Gamma = \{X \equiv x = (x^1, x^2, x^3); \gamma = \gamma^1 \times \gamma^2 \times \gamma^3 \wedge \gamma : \mathbf{R} \to \mathbf{R}^3 \wedge \gamma^i : \mathbf{R} \to \mathbf{R}\}$$

Where Γ is a curve. If γ is linear, then this curve represents a straight line.

We can now begin our discussion on the gradient. We propose that:

$$\Gamma \subset S \equiv \mathbf{f} \circ \gamma = 0$$

In english, take the curve to be a subset of the surface. The composition of $\mathbf{f}$ and γ is 0 because we have already previously defined $\mathbf{f}(x) \equiv \mathbf{f} \circ x$ to equal 0 for any x. Thus:

$$\mathbf{f} \circ \gamma = 0 : \mathbf{R} \to \mathbf{R}$$

$$(\mathbf{f} \circ \gamma)' = 0 : \mathbf{R} \to \mathbf{R}$$

$$\text{Where } \mathbf{f} : \mathbf{R}^3 \to \mathbf{R} \wedge \gamma : \mathbf{R} \to \mathbf{R}^3$$

$$(\mathbf{f} \circ \gamma)' = \sum_{i=1}^{3} \left[\mathbf{f}'_i \circ \gamma\right] \cdot (\gamma^i)' : \mathbf{R} \to \mathbf{R} = 0$$

If we compose γ with t and then go into $\mathbf{F}_3$:

$$\gamma'(t) \equiv \bar{\gamma}(t) = \sum_{i=1}^{3} \left[(\bar{\gamma}^i)'\right] = (\gamma^1)'(t)\hat{i} + (\gamma^2)'(t)\hat{j} + (\gamma^3)'(t)\hat{k}$$

This is parallel to the tangent line to Γ at $\gamma(t)$.

$$(\mathbf{f}' \circ \gamma) \equiv (\bar{\mathbf{f}}' \circ \gamma) = \sum_{i=1}^{3} \left[\bar{\mathbf{f}}'_i \circ \gamma\right] = \mathbf{f}'_1(\gamma(t))\hat{i} + \mathbf{f}'_2(\gamma(t))\hat{j} + \mathbf{f}'_3(\gamma(t))\hat{k}$$

We define this to be the gradient of $\mathbf{f}$ at $\gamma(t)$, or $\nabla\mathbf{f}(\gamma(t))$

By assuming an orthonormal basis, we can then also say that $\nabla\mathbf{f}(\gamma(t))$ and $\gamma'(t)$ are perpendicular to each other, since their inner product is equal to zero, as state by our definition of the derivative of $\mathbf{f} \circ \gamma$

$$(\bar{\mathbf{f}}' \circ \gamma \bullet \bar{\gamma}'(t)) \equiv (\mathbf{f}'_1(\gamma(t))\hat{i} + \mathbf{f}'_2(\gamma(t))\hat{j} + \mathbf{f}'_3(\gamma(t))\hat{k}) \bullet ((\gamma^1)'(t)\hat{i} + (\gamma^2)'(t)\hat{j} + (\gamma^3)'(t)\hat{k}) = 0$$

This gives us an interpretation of what the gradient is: it is a vector that is perpendicular to the tangent line of a curve, which is a subset of a surface.

Removing the $\gamma(t)$, the gradient is:

$$\nabla \mathbf{f} = grad(\mathbf{f}) = \mathbf{f}'_1 \hat{i} + \mathbf{f}'_2 \hat{j} + \mathbf{f}'_3 \hat{k} : \mathbf{R}^n \rightarrow \mathbf{F}_n$$

Not sure on this next part?

It's worth noting that we could use this equation to find the equation for the plane that is tangent to the surface at any point, which we call U. This is represented by the vector equation:

$$\mathbf{E}_3 \supset p = \{X \equiv (x, y, z); (\bar{x} - \bar{U}) \bullet \nabla \mathbf{f} = 0\}$$

LAGRANGE MULTIPLIERS

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The purpose of this text is to illustrate the existance of a lagrange multiplier in a surface

Let us assume that $\exists$ a maximum and a minimum on a surface S_n .

To find the minimum and maximum of the set what we will do is we will select all the points given from $S_n \cap p_1 =$ X

This intersecion will give us a Set X which corresponds to the tangent curve.

Now what we will do is since we have a tangent curve we will take an abf Δ and add it from the "z" component of the plane thus creating a second plane p_{11} parallel to the first, we then find all points of X $= S_n \cap p_{11}$ and repeat

we then keep adding Δ until $|X| = 0$, the we thae the previous plane such that $|X|$ is not 0 and add $\frac{\Delta}{2}$ or $\frac{\Delta}{3}$...$\frac{\Delta}{n}$ until we find a plane such that $\exists$ points in set X

From this we ate able to contstruct a sequence of planes that converge to the maximum of the surface, By applying Bolzano-Wierstrass theorem we are able to find a convergent subsequence which converges to point A, the maximum.

so if we take the limit of the planes and the intersection with the surface S_n when $|X| = 1$, we will have found the maximum,

The same principle applies to the minimum.

Functions of Several Variables

Sooraj Boominathan, Vincent Po, Steve Zhang, Varun Ennamuri, Nathan Yu, Ben Zhao

Problems 7

1. Let $\mathbf{f} = \mathbf{ln} \circ (\overset{1}{\mathbf{p}}_3 \cdot \overset{2}{\mathbf{p}}_1 \cdot \overset{3}{\mathbf{p}}_2) : \mathbb{R}^3 \to \mathbb{R}$. Find the gradient of $\mathbf{f}$ at the point $a = (3, -1, 2)$. Find the directional derivative of $\mathbf{f}$ at a in the direction of the vector $\mathbf{w} = 2\,\mathbf{i} - 2\,\mathbf{j} - \mathbf{k}$. Find the maximum rate of change of $\mathbf{f}$ at a and the direction in which it occurs.

We download the previous presentation on the gradient and highest and lowest points of a surface, and apply what we presented there to these problems.

This is simply a direct application of the gradient. Using our definition of the gradient of $\mathbf{f}$:

$$\nabla \mathbf{f} = grad(\mathbf{f}) = \mathbf{f}'_1 \hat{\imath} + \mathbf{f}'_2 \hat{\jmath} + \mathbf{f}'_3 \hat{k} : \mathbb{R}^3 \to \mathbb{F}_3$$

$$\nabla \mathbf{f} = 3 \cdot \mathbf{p}_{-1} \circ \left(\overset{1}{\mathbf{p}}_1\right) \hat{\imath} + \mathbf{p}_{-1} \circ \left(\overset{2}{\mathbf{p}}_1\right) \hat{\jmath} + 2 \cdot \mathbf{p}_{-1} \circ \left(\overset{3}{\mathbf{p}}_1\right) \hat{k}$$

$$\nabla \mathbf{f}(a) = \hat{\imath} - \hat{\jmath} + \hat{k}$$

Directional Derivatives:
The derectional derivative is defined as being:

$$\frac{\nabla \mathbf{f} \bullet \mathbf{w}}{||\mathbf{w}||}$$

Thus, the directional derivative of $\mathbf{f}$ in the direction of $\mathbf{w}$ is:

$$\nabla \mathbf{f}(a) = \hat{\imath} - \hat{\jmath} + \hat{k} \to \frac{\nabla \mathbf{f} \bullet \mathbf{w}}{||\mathbf{w}||} = 1$$

The maximum rate of change at a is just the gradient at $\mathbf{f}$ It's direction is that of the gradient at a. Thus:

$$|\nabla \mathbf{f}| \text{ at a} = \sqrt{3}, \text{ and its direction is } \hat{\imath} - \hat{\jmath} + \hat{k}$$

6. Find the positive numbers a, b, and c whose sum is 120 such that the product $a^3 b^2 c$ is maximum.
Step 1: Download function $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}$

$$\mathbf{f} = \overset{1}{\mathbf{p}}_3 + \overset{2}{\mathbf{p}}_2 + \overset{3}{\mathbf{p}}_1 : \mathbb{R}^3 \to \mathbb{R}$$

$$\equiv \mathbf{f}\,(a, b, c) = a^3 b^2 c \ \wedge \ a, b, c \in \mathbb{R}$$

Step 2: Download function $\mathbf{g} : \mathbb{R}^3 \to \mathbb{R}$

$$\mathbf{g} = \overset{1}{\mathbf{p}}_3 + \overset{2}{\mathbf{p}}_2 + \overset{3}{\mathbf{p}}_1 \wedge \ \mathbf{g}\left(a, b, c\right) = 120 \wedge a, b, c \in \mathbb{R}$$

$$\equiv \mathbf{g}\left(a, b, c\right) = a + b + c \ \wedge \ \mathbf{g}\left(a, b, c\right) = 120 \ \wedge \ a, b, c \in \mathbb{R}$$

Step 3: Solve for c in $\mathbf{g}\left(a, b, c\right) : \mathbb{R}^3 \to \mathbb{R}$

$$\mathbf{g}\left(a, b, c\right) = a + b + c$$

$$\mathbf{g}\left(a, b, c\right) = 120$$

$$c = 120 - a - b$$

Step 4: Substitute c into function $\mathbf{f}\left(a, b, c\right) : \mathbb{R}^3 \to \mathbb{R}$

$$\mathbf{f}\left(a, b, c\right) = a^3 b^2 c \ \wedge \ a, b, c \in \mathbb{R}$$

This makes $\mathbf{f}$ a function from $\mathbb{R}^2 \to \mathbb{R}$, which we call $\mathbf{t}$

$$\equiv \mathbf{t}\left(a, b\right) = a^3 b^2 \left(120 - a - b\right) : \mathbb{R}^2 \to \mathbb{R} \ \wedge \ a, b \in \mathbb{R}$$

Step 5: Represent $\mathbf{t} : \mathbb{R}^2 \to \mathbb{R}$ as a function $\mathbf{h} : \mathbb{R}^3 \to \mathbb{R}$

$$\mathbf{t}\left(a, b\right) = d \wedge a, b, d \in \mathbb{R}$$

$$\equiv a^3 b^2 \left(120 - a - b\right) = d$$

$$\equiv a^3 b^2 \left(120 - a - b\right) - d = 0$$

$$\mathbf{h}\left(a, b, d\right) = a^3 b^2 \left(120 - a - b\right) - d \wedge \mathbf{h}\left(a, b, d\right) = 0$$

$$\equiv \mathbf{h}\left(a, b, d\right) = 120 a^3 b^2 - a^4 b^2 - a^3 b^3 - d \wedge \mathbf{h}\left(a, b, d\right) = 0$$

$$\equiv \mathbf{h} = 120 \overset{1}{\mathbf{p}}_3 \overset{2}{\mathbf{p}}_2 - \overset{1}{\mathbf{p}}_4 \overset{2}{\mathbf{p}}_2 - \overset{1}{\mathbf{p}}_3 \overset{2}{\mathbf{p}}_3 - \overset{3}{\mathbf{p}}_1 : \mathbb{R}^3 \to \mathbb{R} \wedge \mathbf{h}\left(a, b, d\right) = 0$$

Step 6: Find gradient of function $\mathbf{h} : \mathbb{R}^3 \to \mathbb{R}$

$$\mathbf{E}_3 \supset \mathbf{G}_{\mathbf{h}} = \{X \equiv x = (x^1, x^2, x^3); \mathbf{h}(x) = 0 \wedge \mathbf{h} : \mathbf{R}^3 \to \mathbf{R}\}$$

We do this because, from our previous presentation of the maximum and minimum values of a function that the gradient of a function from $\mathbf{R}^3 \to \mathbf{R}$, we know that the gradient is perpendicular to the surface represented by the function in $\mathbf{E}_3$. At a maximum or a minimum, assuming an orthonormal basis, this vector is perpendicular to the "xy" plane. Thus, the "i" and "j" vectors are 0.

$$\nabla \mathbf{h} = \mathbf{h}'_1 \hat{i} + \mathbf{h}'_2 \hat{j} + \mathbf{h}'_3 \hat{k}$$

$$\left(360 \overset{1}{\mathbf{p}}_2 \overset{2}{\mathbf{p}}_2 - 4 \overset{1}{\mathbf{p}}_3 \overset{2}{\mathbf{p}}_2 - 3 \overset{1}{\mathbf{p}}_2 \overset{2}{\mathbf{p}}_3\right) \hat{i} + \left(240 \overset{1}{\mathbf{p}}_3 \overset{2}{\mathbf{p}}_1 - 2 \overset{1}{\mathbf{p}}_4 \overset{2}{\mathbf{p}}_1 - 3 \overset{1}{\mathbf{p}}_3 \overset{2}{\mathbf{p}}_2\right) \hat{j} - \hat{k}$$

Step 7: Set $\mathbf{h}_1'(a,b) = 0 \wedge \mathbf{h}_2'(a,b) = 0$

$$\equiv \mathbf{h}_1'\,(a,b) = 0 \wedge a, b \in \mathbb{R}$$

$$\equiv 360a^2b^2 - 4a^3b^2 - 3a^2b^3 = 0$$

$$\equiv \mathbf{h}_2'\,(a,b) = 0 \wedge a, b \in \mathbb{R}$$

$$\equiv 240a^3b - 2a^4b - 3a^3b^2 = 0$$

Step 8: Solve for a and b in the system of equations $a = 60, b = 40$

Step 9: Substitute values of a and b in expression for c

$$c = 120 - a - b$$
$$c = 120 - 60 - 40$$
$$c = 20$$

The positive numbers a, b, and c are 60, 40, and 20, respectively.

2. Let $\mathbf{G_f}$ be the graph of the function $\mathbf{f} = \overset{1}{\mathbf{sin}} \cdot [\mathbf{exp} \circ (-\overset{2}{\mathbf{p}}_1)] : \mathbb{R}^2 \to \mathbb{R}$ in the Euclidean space $\mathbb{E}_3$. Find an equation of the tangent plane to $\mathbf{G_f}$ at the point $A \equiv (x,y,z)$, where $x = \frac{\pi}{2}$ and $y = 0$.

Download $\mathbb{R}^3, \mathbb{E}^3, \mathbb{F}^3$. Download function $\mathbf{f}$:

$$\mathbf{f} = \overset{1}{\mathbf{sin}} \cdot [\mathbf{exp} \circ (-\overset{2}{\mathbf{p}}_1)] : \mathbb{R}^2 \to \mathbb{R}$$

Download a point A:

$$A \equiv (x,y,z)$$

Where x and y are given, and z is determined by $\mathbf{f}(x,y)$, which is equal to one. Our full point is thus:

$$A = (\frac{\pi}{2}, 0, 1)$$

In order to find the tangent plane, we must utilize the idea of the normal vector to the graph of a particular surface described by a function $\mathbb{R}^3 \to \mathbb{R}$. This was the set:

$$\mathbb{E}_3 \supset S = \{X \equiv x = (x^1, x^2, x^3); \mathbf{h}(x) = 0 \wedge \mathbf{h} : \mathbb{R}^3 \to \mathbb{R}\}$$

Download the "normal vector package" and run the program for that package. First we would like to describe our given $\mathbf{f}$ function as a function from $\mathbb{R}^3 \to \mathbb{R}$. We call this new function $\mathbf{h}$, and it is:

$$\mathbf{h} = \overset{1}{\mathbf{sin}} \cdot [\mathbf{exp} \circ (-\overset{2}{\mathbf{p}}_1)] - \overset{3}{\mathbf{p}}_1 : \mathbb{R}^3 \to \mathbb{R}$$

$$\nabla \mathbf{h} = \overset{1}{\mathbf{cos}} \cdot [\mathbf{exp} \circ (-\overset{2}{\mathbf{p}}_1)]\,\hat{i} - \overset{1}{\mathbf{sin}} \cdot [\mathbf{exp} \circ (-\overset{2}{\mathbf{p}}_1)]\,\hat{j} - \overset{3}{\mathbf{p}}_0\,\hat{k} : \mathbb{R}^3 \to \mathbb{F}^3$$

$$\nabla \mathbf{h}(A) = -\hat{j} - \hat{k}$$

This is normal vector to the surface described by the graph of the function $\mathbf{h}$ at point A. The equation of the tangent plane is thus:

$$\mathbb{E}_3 \supset p = \{X \equiv x = (x^1, x^2, x^3); \bar{\mathbf{x}} \bullet \nabla \mathbf{h}(A) = 0\}$$

4. The radius of the base of a cylinder is increasing at a rate of 0.5 cm/sec while its height is decreasing at a rate of 1.4 cm/sec. At what rate is the volume of the cylinder changing when the radius is 50 cm and the height is 80 cm?

Step 1: Download the function $\mathbf{V} : \mathbb{R}^2 \to \mathbb{R}$:

$$\mathbf{V} = \pi \cdot \overset{1}{\mathbf{p}}_2 \cdot \overset{2}{\mathbf{p}}_1 : \mathbb{R}^2 \to \mathbb{R}$$

$$\mathbb{E}_3 \supset \mathbf{G_V} = \{X \equiv x = (x^1, x^2, x^3); x^3 = \mathbf{V}(x^1, x^2) \wedge \mathbf{V} = \pi \cdot \overset{1}{\mathbf{p}}_2 \cdot \overset{2}{\mathbf{p}}_1 : \mathbb{R}^2 \to \mathbb{R}\}$$

Step 2: Download functions $\mathbf{r}$ and $\mathbf{h}$, both from $\mathbb{R} \to \mathbb{R}$
Step 3: From the cartesian product of these two functions $\mathbf{r}, \mathbf{h}$.
Step 4: Compose function $\mathbf{V}$ with $\mathbf{r} \times \mathbf{h}$:

$$\mathbf{V} \circ (\mathbf{r} \times \mathbf{h}) : \mathbb{R} \to \mathbb{R}$$

Step 5: Find the derivative of $\mathbf{V} \circ (\mathbf{r} \times \mathbf{h}) : \mathbb{R} \to \mathbb{R}$ using the formula for the derivative of a composition:

$$(\mathbf{V} \circ (\mathbf{r} \times \mathbf{h})) = [\mathbf{V}'_1 \circ (\mathbf{r} \times \mathbf{h})] \cdot (\mathbf{r}') + [\mathbf{V}'_2 \circ (\mathbf{r} \times \mathbf{h})] \cdot (\mathbf{h}') =$$
$$[2\pi \cdot \overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 \circ (\mathbf{r} \times \mathbf{h})] \cdot (\mathbf{r}') + [\pi \cdot \overset{1}{\mathbf{p}}_2 \circ (\mathbf{r} \times \mathbf{h})] \cdot (\mathbf{h}') : \mathbb{R}^2 \to \mathbb{R}$$

Step 6: Evaluate the function at ABF point $t \in \mathbb{R}$:

$$\mathbf{V}'(\mathbf{r} \times \mathbf{h}) = \Big[[2\pi \cdot \overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 \circ (\mathbf{r} \times \mathbf{h})] \cdot (\mathbf{r}') + [\pi \cdot \overset{1}{\mathbf{p}}_2 \circ (\mathbf{r} \times \mathbf{h})] \cdot (\mathbf{h}')\Big] \circ t$$

$$= [2\pi \cdot \overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 \circ (\mathbf{r}(t) \times \mathbf{h}(t))] \cdot (\mathbf{r}')(t) + [\pi \cdot \overset{1}{\mathbf{p}}_2 \circ (\mathbf{r}(t) \times \mathbf{h}(t))] \cdot (\mathbf{h}')(t)$$

$$= 2\pi \cdot \mathbf{r}(t) \cdot \mathbf{h}(t) \cdot \mathbf{r}'(t) + \pi \cdot \mathbf{p}_2 \circ (\mathbf{r}(t)) \cdot \mathbf{h}'(t)$$

Step 7: Substitute values provided in the problem:

$$\mathbf{V}'(\mathbf{r}(t) \times \mathbf{h}(t)) = 2\pi \cdot \mathbf{r}(t) \cdot \mathbf{h}(t) \cdot \mathbf{r}'(t) + \pi \cdot \mathbf{p}_2 \circ (\mathbf{r}(t)) \cdot \mathbf{h}'(t) = 500\pi \frac{cm^3}{s}$$

Problems 7, Problem 9 Medhi Drissi, Austin Fehr, Jerri Ni, Cody Poteet, Rahul Rajala, Kenneth Teel
Oklahoma School of Science and Mathematics

Let $\mathbf{f} = \overset{1}{\mathbf{p}}_3 + \overset{2}{\mathbf{p}}_3 + \overset{3}{\mathbf{p}}_3 : \mathbf{R}^3 \to \mathbf{R}$. Use Lagrange multipliers to find the minimum value of the function $\mathbf{f}$ on the planes $p_1 = \{X \equiv (x,\ y,\ z); x+y+z = 2\}$, and $p_2 = \{X \equiv (x,\ y,\ z); x+y-z = 6\}$.

7 For the Bound p_1

Download $\mathbf{f}$ and p_1.

Since $\mathbf{f}$ must be minimized while subject to the constraint of the plane p_1, at any minima, $\nabla\mathbf{f} = \lambda\nabla p_1$.

We will then create and solve a system of equations, including p_1 and the relations between the first, second, and third components derived from the above relation.

$\nabla\mathbf{f} = \mathbf{f}_1' i, \mathbf{f}_2' j, \mathbf{f}_3' k = \lambda\nabla p_1 = i + j + k$

$\mathbf{f}_1' = 3\overset{1}{\mathbf{p}}_2$
$\mathbf{f}_2' = 3\overset{2}{\mathbf{p}}_2$
$\mathbf{f}_3' = 3\overset{3}{\mathbf{p}}_2$

Since $\nabla\mathbf{f} = \lambda\nabla p_1$,

$$3\overset{1}{\mathbf{p}}_2 = \lambda$$
$$3\overset{2}{\mathbf{p}}_2 = \lambda$$
$$3\overset{3}{\mathbf{p}}_2 = \lambda$$

We also have

$$p_1 = x + y + z = 6$$

Solving the system of equations, we have four solutions:

(x, y, z, λ) = (-2, 2, 2, 12), (2, -2, 2, 12), (2, 2, -2, 12), (2/3, 2/3, 2/3, 4/3)

8 For the Bound $\boldsymbol{p}_2$

Download **f** and p_2.

∇ **f** is the same as above, but ∇p_2 will give i + j - k.
Thus, we have

$$\mathbf{f}'_1 i + \mathbf{f}'_2 j + \mathbf{f}'_3 = \lambda i + \lambda j - \lambda k$$

We can show this as a system of equations:

$$3\overset{1}{\mathbf{p}}_2 = \lambda$$
$$3\overset{2}{\mathbf{p}}_2 = \lambda$$
$$3\overset{3}{\mathbf{p}}_2 = -\lambda$$

We also have the relation p_2 = x + y - z - 6 = 0.

Solving this system of equations yields only imaginary answers, which are not part of **R**. Therefore, this function **f** does not have a minimum or maximum for the constraint p_2.

Functions of Several Variables

Sooraj Boominathan, Vincent Po, Steve Zhang, Varun Ennamuri, Nathan Yu, Ben Zhao

Problems 8

2. Evaluate the integral

$$\int_0^8 \left(\int_{\boxed{1}\ \mathbf{p}_{\frac{1}{3}} \times \mathbf{p}_1}^{2\,\mathbf{p}_0 \times \mathbf{p}_1} 4\,\mathbf{exp} \circ \overset{1}{\mathbf{p}}_4 \right)$$

Our calculators obviously cannot directly evaluate the integral of $\mathbf{exp} \circ \overset{1}{\mathbf{p}}_4$. We must therefore redefine the boundaries, integrating the inner function with respect to the 2nd variable instead of the first. Using a sketch to help (draw on board)

The new boundaries are thus:

$$\int_0^2 \left(\int_{\boxed{2}\ \mathbf{p}_1 \times \mathbf{0}}^{\mathbf{p}_1 \times \mathbf{p}_3} 4\,\mathbf{exp} \circ \overset{1}{\mathbf{p}}_4 \right)$$

We can now easily evaluate the inner integral and compose it with the limits of integration, and following the definition of the partial integrals:

$$(4\,\mathbf{exp} \circ \overset{1}{\mathbf{p}}_4 \cdot \overset{2}{\mathbf{p}}_1) \circ (\mathbf{p}_1 \times \mathbf{p}_3) - (4\,\mathbf{exp} \circ \overset{1}{\mathbf{p}}_4 \cdot \overset{2}{\mathbf{p}}_1) \circ (\mathbf{p}_1 \times \mathbf{p}_3) = (4\mathbf{exp} \circ \mathbf{p}_4) \cdot \mathbf{p}_3$$

We can now evaluate the outer integral and arrive at a final answer:

$$\int_0^2 (4\mathbf{exp} \circ \mathbf{p}_4) \cdot \mathbf{p}_3 = 8886109.$$

Extra Problem: Finding the Potential Function of a Conservative Vector Field

Our goal is to find the potential function of a conservative vector field, that is Theorem 2, where $\mathbf{f}$ is the potential function.

Step 1: Download the vector field $\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{F}_3$

$$\vec{\mathbf{F}} = \mathbf{P}\hat{\imath} + \mathbf{Q}\hat{\jmath} + \mathbf{R}\hat{k} \wedge \mathbf{P}, \mathbf{Q}, \mathbf{R} : \mathbb{R}^3 \to \mathbb{R}$$

Step 2: Check if the vector field is conservative using equality of mixed partial derivatives:

$$\mathbf{P} = \mathbf{f}'_1 \qquad \mathbf{Q} = \mathbf{f}'_2$$

$$\mathbf{P}'_2 = \mathbf{f}''_{12} \qquad \mathbf{Q}'_1 = \mathbf{f}'_{21}$$

$$\mathbf{P}'_2 = \mathbf{Q}'_1 \tag{1}$$

$$\mathbf{P} = \mathbf{f}'_1 \qquad \mathbf{R} = \mathbf{f}'_3$$

$$\mathbf{P}'_3 = \mathbf{f}''_{13} \qquad \mathbf{R}'_1 = \mathbf{f}'_{31}$$

$$\mathbf{P}'_3 = \mathbf{R}'_1 \tag{2}$$

$$\mathbf{Q} = \mathbf{f}'_2 \qquad \mathbf{R} = \mathbf{f}'_3$$

$$\mathbf{Q}'_3 = \mathbf{f}''_{23} \qquad \mathbf{R}'_2 = \mathbf{f}'_{32}$$

$$\mathbf{Q}'_3 = \mathbf{R}'_2 \tag{3}$$

If equations (1), (2), and (3) are all satisfied, then the vector field is conservative, as given by the Definitions 12 and 14.

If any of the three equations are not satisfied, end the program, as there will not exist a potential function $\mathbf{f}$.

Step 3: Integrate $\mathbf{P}$ with respect to the 1st variable. [If not possible, integrate $\mathbf{Q}$ with respect to the 2nd variable. If not possible, integrate $\mathbf{R}$ with respect to the 3rd variable.]

$$\int_{\boxed{1}} \mathbf{P} = \overset{123}{\mathbf{g}} + \overset{23}{\mathbf{h}} \wedge \mathbf{g} : \mathbb{R}^3 \to \mathbb{R} \wedge \mathbf{h} : \mathbb{R}^2 \to \mathbb{R}$$

Step 4: Define $\int_{\boxed{1}} \mathbf{P}$ to be $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}$ and find $\mathbf{f}'_{_2}$

$$\mathbf{f} = \overset{123}{\mathbf{g}} + \overset{23}{\mathbf{h}}$$

$$\mathbf{f}'_{_2} = (\overset{123}{\mathbf{g}})'_{_2} + (\overset{23}{\mathbf{h}})'_{_2} = \mathbf{g}'_{_2} + \mathbf{h}'_{_2}$$

Step 5: Set $\mathbf{f}'_{_2} = \mathbf{Q}$ and solve for $\mathbf{h}'_{_2}$

$$\mathbf{Q} = \mathbf{g}'_{_2} + \mathbf{h}'_{_2}$$

$$\mathbf{h}'_{_2} = \mathbf{Q} - \mathbf{g}'_{_2}$$

Step 6: Integrate $\mathbf{h}'_{_2}$ with respect to the 2nd variable:

$$\int_{\boxed{2}} \mathbf{h}'_{_2} = \overset{23}{\mathbf{S}} + \overset{3}{\mathbf{t}} \wedge \mathbf{S} : \mathbb{R}^2 \to \mathbb{R} \wedge \mathbf{t} : \mathbb{R} \to \mathbb{R}$$

Step 7: Substitute $\overset{23}{\mathbf{h}}$ in $\mathbf{f}$:

$$\mathbf{f} = \overset{123}{\mathbf{g}} + \overset{23}{\mathbf{S}} + \overset{3}{\mathbf{t}} : \mathbb{R}^3 \to \mathbb{R}$$

Step 8: Find $\mathbf{f}'_{_3}$:

$$\mathbf{f}'_{_3} = (\overset{123}{\mathbf{g}})'_{_3} + (\overset{23}{\mathbf{S}})'_{_3} + (\overset{3}{\mathbf{t}})'_{_3} = \mathbf{g}'_{_3} + \mathbf{S}'_{_3} + \mathbf{t}'_{_3}$$

Step 9: Set $\mathbf{R} = \mathbf{f}'_{_3}$ and solve for $\mathbf{t}'_{_3}$:

$$\mathbf{R} = \mathbf{g}'_{_3} + \mathbf{S}'_{_3} + \mathbf{t}'_{_3}$$

$$\mathbf{t}'_{_3} = \mathbf{R} - \mathbf{g}'_{_3} - \mathbf{S}'_{_3}$$

Step 10: Find $\mathbf{t}'_{_3}$ with respect to the 3rd variable:

$$\int_{\boxed{3}} \mathbf{t}'_{_3} = \overset{3}{\mathbf{k}} + c\,\mathbf{p}_{_0} : \mathbb{R} \to \mathbb{R}$$

Step 11: Substitute $\overset{3}{\mathbf{t}}$ into $\mathbf{f}$:

$$\mathbf{f} = \overset{123}{\mathbf{g}} + \overset{23}{\mathbf{S}} + \overset{3}{\mathbf{k}} + c\,\mathbf{p}_{0} : \mathbb{R}^3 \to \mathbb{R} \tag{4}$$

This is the general form of the potential function given a conservative vector field.

Saturday Presentation Medhi Drissi, Austin Fehr, Jerri Ni, Cody Poteet, Rahul Rajala, Kenneth Teel
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9 Directional Derivative

The goal is to find find the tangent vector in some direction on a surface.

First download $\mathbb{E}_3$, $\mathbb{R}^3$, $\mathbb{F}^3$. Next download a surface:

$S \subset \mathbb{E}_3 = X \equiv (x^1, x^2, x^3); f(x) = 0 : \mathbb{R}^3 \to \mathbb{R}$

A vector valued function can be formed from S being of the identification of a point with vector. The vector would be the position vector to the corresponding point and will be $\bar{S}$ which is $\mathbb{F}^3 \to \mathbb{R}$.

Next download any arbitrary but fixed point on the surface a. Form the position vector to that point $\bar{a}$.

$\bar{u} \in \mathbb{F}^3 \wedge |u| = 1$

Now, the vector $\bar{a} + \bar{u}h \wedge h \in \mathbb{R}$ is in the desired direction of movement. The multiplying by a scalar is allowed because:

$\bar{i} \parallel \bar{g} \equiv \bar{i} = \rho\bar{g} \wedge \rho \in \mathbb{R}$

So we can form a function analogous to the quotient function.

${}^{u}Q_f^a = p_{-1} \cdot \big(f \circ \big(a^1 p_0 + hu^1 p_1\big) \times \big(a^2 p_0 + hu^2 p_1\big) \times \big(a^3 p_0 + hu^3 p_1\big) - f(a)\big) :$
$\mathbb{R} - \{0\} \to \mathbb{R}$

The main difference from the previous quotient function is that instead of one linear function you have all linear functions (assuming u^i does not equal 0) because adding two vectors involves adding each of their components.

Forming the limit of the quotient function at 0 gives the directional derivative in the direction of u notated $f'_{\bar{u}}$.

10 Problems 7, 11

The problem asks for the maximum of the error for the surface area of a rectangular cube given the range the error of each side is in.

$|e| \leq k \wedge k \in R \equiv \forall$

Tangent Vectors

Mehdi Drissi, Rahul Rajala, Cody Poteet, Kenneth Teel, Austin Fehr, Jerry Ni

The first question to start off with is what is a tangent vector. First as a quick formality we'll define:

Definition 1

$\bar{a} \in \mathbb{F}^n \parallel b \subset \mathbb{E}_n \equiv \underset{b \in \mathbb{E}_n, c \in \mathbb{E}_n}{\forall} \underset{\alpha \in \mathbb{R}}{\exists} [CB] = \alpha \bar{a}$

Now the goal of the tangent vector is to be analogous to the tangent line and more specifically to be parallel to the tangent line. We'll start by looking at the case of our function is a curve and later look at the case where our function is a surface.

Now any differentiable curve can be defined in the form $f : \mathbb{R}^m \to \mathbb{R}^n$. The only way for a function to be to $\mathbb{R}^n$ is if it is the Cartesian product of n functions. These n functions can be each single variable functions causing the function to be from $\mathbb{R} \to \mathbb{R}^n$. Now for the curve let us say we have any arbitrary but fixed point on the curve $x = (f^1(x), f^2(x), ..., f^n(x))$. We can form a position vector to that point with its components being $(f^1(x), f^2(x), ..., f^n(x))$ which we'll call $\bar{x} = \sum_{i=1}^{n} f^i(x) a_i$ where x consists of n vectors that form the basis for the position vector. We can then repeat the process for a second point we'll call y and we'll reuse the same basis. After this we'll form the vector $\bar{s} = \bar{y} - \bar{x}$ which the direction of s will be in the direction from x to y. Now based on the knowledge $\bar{a}, \bar{b} \in \mathbb{F}^n; \bar{a} \parallel \bar{b} \equiv \bar{a} = \alpha\bar{b} \wedge \alpha \in \mathbb{R}$ we can multiply s by any scalar which includes $\dfrac{1}{x-y}$ and conserve its direction. This now has it in a form like that of the Quotient function with f as the function and x as the point. Specifically defining h = y-x will allow it to really become the same form. By composing this with the limit function we have formed the derivative. Thinking about what we've done we'll notice that as h becomes near 0, similar to the single variable case, the points become near each other and the direction of the vector converges to the direction of the tangent line forming the tangent vector. Lastly by focusing on the vector function we can see this is a derivative of a sum and so the derivative can be calculated without much difficulty by focusing on the vector function.

Now that we've defined the tangent vector for a curve let's imagine what happens for a surface. Surfaces can be described in the form $f : \mathbb{R}^3 \to \mathbb{R} \wedge f = 0$. This can be done by simply taking a three variable function and moving all the variables to the other side. For a surface we have the issue that we no longer are limited to moving only in two directions either forward or backward but in a variety of directions. This means depending upon the direction the tangent vector can be different. This idea is the key to dealing with the issue. By choosing a direction this issue disappears. The direction can be represent with a vector and for the sake of keeping calculations simple a normal vector is preferred (normalizing a vector can be done simply by multiplying it by the reciprocal of its norm). So let us have a unit direction vector with the number of components corresponding to the number of dimensions in the space. We'll call this arbitrary but constant direction vector $\bar{r}$. Now let us again start at a point x on the surface and form position vectors as needed. We will then find a point y on the surface in the direction of r from x. Considering how direction is conserved by scalar multiplication (we'll have the scalar be h) we can move as far or close as desired and simply say we are moving $h\bar{r}$.

Proof of Uniform Continuity of Sine Function, MLP

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Definition of Uniformly Continuous

Let $\mathbf{f}$ be defined on a set $A \subset \mathbb{R}$. We say that $\mathbf{f}$ is uniformly continuous (on A) if $\forall \epsilon > 0 \; \exists \; \delta > 0$; if x and $y \in A |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$

Proof for Sine Function

Let $f(x) = sin(x)$
$g(x) = \mathbf{1}$
$h(x) = \mathbf{-1}$
$h(x) \leq f(x) \leq g(x)$
Then since f(x) is bounded:
$\forall x$ and $y \; \exists \; A; |x - y| < \delta \implies |f(x) - f(y)| < 2$

Thus the sine function is continuous.

Line Integrals

Programs to find the Line Integral with Respect to Arc

Yitong Chen, Preeti Mohan, Caron Song, Divya Velury, Julie Zhu

Program: How to form $\|\vec{\gamma'}\|$

1. Find a'
2. Find b'
3. Form $\mathbf{p}_2 \circ a'$
4. Form $\mathbf{p}_2 \circ b'$
5. (**3.**) + (**4.**)
6. Form $\|\vec{\gamma'}\| = \mathbf{p}_{\frac{1}{2}} \circ$ (**5.**)

Program: How to form $\mathbf{F}$

1. Form $\gamma = a \times b$
2. Form $\mathbf{f} \circ \gamma : \mathbb{R} \to \mathbb{R}$
3. Run How to form $\|\vec{\gamma'}\|$
4. Form $\mathbf{F}$ =(**2.**)$\cdot$(**3.**)

Program: How to Find Line Integral with Respect to Arc

1. Download $\mathbf{f}:\mathbb{R}^2\to\mathbb{R}$ and $C=\{(x,y);x=\mathbf{a}(t)\wedge y=\mathbf{b}(t)\wedge t\in[0,2]\}$

2. Run How to form $\mathbf{F}$

3. Find $\int \mathbf{F}$

4. Evaluate $\int\limits_{t_1}^{t_2} \mathbf{F}$

Functions of Several Variables

Yitong Chen, Preeti Mohan, Caron Song, Divya Velury, Julie Zhu

Problems 4

7. Let $\mathbf{f}=\phi\circ\left(\overset{1}{\mathbf{p}}_1\cdot\overset{2}{\mathbf{p}}_2\right):\mathbb{R}^2\to\mathbb{R}$, where ϕ and ψ are twice differentiable single variable functions. Show that

$$\mathbf{F}=\overset{1}{\mathbf{p}}_2\cdot\mathbf{f}''_{11}-\overset{2}{\mathbf{p}}_2\cdot\mathbf{f}''_{22}+\overset{1}{\mathbf{p}}_1\cdot\mathbf{f}'_1-\overset{2}{\mathbf{p}}_1\cdot\mathbf{f}'_2=0$$

<u>PROGRAM</u>

Step 1: Find values for all derivatives of $\mathbf{f}$ on both sides of the equality.

1) Download $\mathbf{f}$

2) Determine which variable to take the derivative with respect to. For a mixed

partial derivative, start with the leftmost number subscript and move to the

right with each repetition of this step. For this i^{th} variable derivative, components

of $\mathbf{f}$ that are not of this variable are to be treated as constants and their

derivatives are the same as the components themselves.

3) Break up $\mathbf{f}$ by using left hand side of Derivatives of Algebraic Functions

formula (4) and/or formula (5)

$$(\mathbf{f}+\mathbf{g})'=\mathbf{f}'+\mathbf{g}'$$
$$(\mathbf{f}\cdot\mathbf{g})'=\mathbf{f}'\cdot\mathbf{g}'$$

4) Find the derivatives of each composition in the function using the Derivative of

Composition Function if necessary, formula (6)

$$(\mathbf{f} \circ \mathbf{g})' = (\mathbf{f}' \circ \mathbf{g}) \cdot \mathbf{g}'$$

5) Repeat 1-4) for mixed partial derivatives. Use the derivative found in 4) as **f**

Step 2: Compare left and right side of equations to prove or disprove equality.

Step 1: Find values for all derivatives of **f** on both sides of the equality.

1) Break up the **f** by using left hand side of Derivatives of Algebraic Functions

Problem 2a from Problems 8

Husayn Ramji, Andrew Chang, Remington Brooks

Oklahoma School of Science and Mathematics Goal: Evaluate the Double Integral

Step 1: Download the problem

$$\int_0^3 \left(\int_{\boxed{1}\ \mathbf{p}_2 \times \mathbf{p}_1}^{9\,\mathbf{p}_0 \times \mathbf{p}_1} 4\overset{2}{\mathbf{p}}_1 \cdot \left(\cos \circ \overset{1}{\mathbf{p}}_2\right) \right)$$

Step 2: Download the original bounds.

$$\{(x, y)\,; x \in \left[\ y^2, 9\ \right] \wedge y \in [\ 0, 3\ ]\}$$

If there is no error, end program. If there is an error in evaluating using these bounds, continue to Step 3.

Step 3: Redefine the bounds.

$$\{(x, y)\,; x \in [\ 0, 9\ ] \wedge y \in [\ 0, \sqrt{x}\ ]\}$$

Step 4: The problem, with the new bounds, is now written as

$$\int_0^9 \left(\int_{\boxed{2}\ \mathbf{p}_1 \times 0}^{\mathbf{p}_1 \times \mathbf{p}_{1/2}} 4\overset{2}{\mathbf{p}}_1 \cdot \left(\cos \circ \overset{1}{\mathbf{p}}_2\right) \right)$$

Step 5: Evaluate using the new bounds

$$\int_0^9 \left(\int_{\mathbf{p}_1 \times 0}^{\mathbf{p}_1 \times \mathbf{p}_{1/2}} \boxed{2}\ 4\overset{2}{\mathbf{p}}_1 \cdot \left(\cos \circ \overset{1}{\mathbf{p}}_2 \right) \right) = 82.39057647$$

Husayn Ramji, Andrew Chang, Remington Brooks
Oklahoma School of Science and Mathematics Download $\mathbf{E}_3, \mathbf{F}_3$, and $\mathbf{R}^3$

5. Let $\mathbf{f} = \overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 - \overset{1}{\mathbf{p}}_4 - \overset{2}{\mathbf{p}}_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}$. Find the exteme values of $\mathbf{f}$.

Extreme values are also known as the ”Highest and Lowest Values of Surfaces.”

First, download the function:

$$\mathbf{f} = \overset{1}{\mathbf{p}}_1 \cdot \overset{2}{\mathbf{p}}_1 - \overset{1}{\mathbf{p}}_4 - \overset{2}{\mathbf{p}}_4 : \mathbb{R}^2 \longrightarrow \mathbb{R}, G_f \subset \mathbf{E}_3$$

It can be represented as this as well:

$$(a^1, a^2) \longrightarrow \mathbf{f}(a^1, a^2) = a^3; a^1, a^2, a^3 \in \mathbf{R}$$

The graph of the function forms a surface, S, in Euclidian 3-space.

$$\mathbf{E}_3 \supset S = \{X \equiv x = (x^1, x^2, x^3); \mathbf{f}(x) = 0; \mathbf{f} = \mathbf{f}^1 \times \mathbf{f}^2 \times \mathbf{f}^3 \wedge \mathbf{f} : \mathbf{R}^2 \longrightarrow \mathbf{R}^3\}$$

$$\mathbf{f}(a^1, a^2) = a^3 \rightarrow \mathbf{f}(a^1, a^2) - a^3 = 0 = \mathbf{g}(a^1, a^2, a^3); \mathbf{g} : \mathbf{R}^3 \longrightarrow \mathbf{R}; \mathbf{g}(x) = 0$$

Form a gradient:

$$\mathbf{g} = \overset{12}{\mathbf{f}} - \overset{3}{\mathbf{p}}_3 \rightarrow \nabla \mathbf{g} = \mathbf{f}'_1 \hat{\mathbf{i}} + \mathbf{f}'_2 \hat{\mathbf{j}} - \hat{\mathbf{k}} = \nabla \mathbf{f} - \mathbf{k}$$

And finding points of maximum and minimum of the surface by assuming that:

$$\nabla \mathbf{g} \perp S$$

If it is true, then, the using the package, it finds a point where:

$$\nabla \mathbf{g} = a\hat{\mathbf{k}}, a \in \mathbf{R}$$

And finally

$$\nabla \mathbf{g} = \nabla \mathbf{f} - \hat{\mathbf{k}} = \nabla \mathbf{f} = 0 \rightarrow \mathbf{f}'_1 = 0 \wedge \mathbf{f}'_2 = 0$$

$\mathbf{f}_1' = 0$ at two points, $(-1/2, -1/2)$ and $(1/2, 1/2)$. These must be local minimum or maximums. In order to clarify which one it is, is must be plugged into the original function, as well as a point next to it to see if it is greater than or less than the original value.
When we plug in the points

It yields:

$$f(-1/2, -1/2) = 0.125$$
$$f(1/2, 1/2) = 0.125$$

And then some points near it to check if it is a minimum or maximum.

$$f(-51/100, -51/100) = 0.124$$
$$f(51/100, 51/100) = 0.124$$
$$f(-49/100, -49/100) = 0.124$$
$$f(49/100, 49/100) = 0.124$$

They are all less than 0.125. Thus, the points $(-1/2, -1/2)$ and $(1/2, 1/2)$ are both local maxima.

The steps to find these minimum and maximums are exemplified in the package "Highest and Lowest Values of Surfaces."

Vector Space of Free Vectors

Yitong Chen, Preeti Mohan, Caron Song, Divya Velury, Julie Zhu

Problems 6

8. Find a **vector equation** and a **number equation** for the line l passing through the point $D \equiv (1, 3, -2)$ and perpendicular to the plane $p = \{X \equiv (x, y, z); x - 2y + 3z - 3 = 0\}$.

PROGRAM

Step 1: Find normal $\mathbf{n}$ to plane p

1) Download $p : \mathbb{R}^3$
2) Find $\mathbf{n} = (n^1, n^2, n^3)$

Step 2: Find l
1) Download point D
2) Find the vector equation $\mathbb{F}_3$
3) Find the number equation

Step 1:Find normal $\mathbf{n}$ to plane p
1) Download p

$$\mathbb{E}^3 \supset p = \{X \equiv (x, y, z); x - 2y + 3z - 3 = 0\} \to \mathbf{n} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} \text{ "}\mathbf{n} \perp \mathbf{p}\text{"}$$

2) Find $\mathbf{n} = (n^1, n^2, n^3)$. $p = n^1x + n^2y + n^3z + c = 0 : \mathbb{R}^3$
With the given p: $n^1 = 1, n^2 = -2, n^3 = 3, c = -3$
So $\mathbf{n} \equiv (1, -2, 3)$
Step 2: Find l
1) Download point $D \equiv (1, 3, -2)$
2) Find vector equation: $\mathbf{x} = \mathbf{a} + t\mathbf{v}$
$A = D$ is on the line l we need to find. Let $\mathbf{d}$ represent the vector form of D just as $\mathbf{a}$ represent the vector form of A

$$\mathbf{a} = \mathbf{d} \equiv (1, 3, -2)$$

l is perpendicular the plane and therefore parallel to the normal $\mathbf{n}$

$$\mathbf{v} = \mathbf{n} \equiv (1, -2, 3)$$

Vector equation of l

$$\mathbb{E}^3 \supset l = X; \mathbf{x} = \mathbf{d} + t\mathbf{n}^t \ni \mathbb{R} =$$
$$X \equiv (x^1, x^2, x^3); x^1 = d^1 + tn^1, x^2 = d^2 + tn^2, x^3 = d^3 + tn^3, A \in \mathbb{R}^3$$

3) Find number equation by using the $\mathbb{R}^3$ forms of the vectors
Number equation of l

$$x;$$

Partial Derivatives - Applications

Problems 7

Yitong Chen, Preeti Mohan, Caron Song, Divya Velury, Julie Zhu

Linear Approximation

Linear approximation is for historical purposes. Nowadays we have calculators so it has little practical purpose. Nevertheless, we shall express how they were done back in the day.

Program:

Step 1: Download entire package of "The Tangent Vector, Gradients, Highest and Lowest Values of Surfaces"; if there is not enough memory, get the definition of a gradient and the property that it is"perpendicular" to the tangent vector at a point.

Step 2: Download function

Step 3: Find gradient of function.

Step 4: Choose point on graph of g "close" to the point to be approximated

Step 5: Find tangent plane to surface using the gradient and a point.

Step 6: Find the linear approximation function of **g** at the point.

Step 7: Use linear approximation function to approximate values.

Solution

Step 1:

Download the previous group's discussion. Notably, the explanation that the gradient is "$\perp$" to the tangent vector at the point.

Download the surface

$$\mathbb{E}_3 \supset S = \{X \equiv x = (x^1, x^2, x^3); f(x) = 0 \wedge \mathbf{f} : \mathbb{R}^3 \longrightarrow \mathbb{R}\}$$

We shall be dealing with a special surface $G_g \subset \mathbb{E}_3$, the graph of $\mathbf{g}$ where $\mathbf{g} : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

Step 2:

$$\mathbf{g}(x^1, x^2) = x^3 \qquad \mathbf{g}(x^1, x^2) - x^3 = 0$$

This is $\mathbf{f}$ in the definition of the surface

$$\mathbf{g}(x^1, x^2) - x^3 = 0 = \mathbf{f}(x^1, x^2, x^3) : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

Step 3:

$$\mathbf{f} = \overset{1,2}{\mathbf{g}} - \overset{3}{\mathbf{p}}_1$$

$$\nabla \mathbf{f} = \mathbf{g}'_1 \hat{\imath} + \mathbf{g}'_2 \hat{\jmath} - \hat{k} = \nabla \mathbf{g} - \hat{k}$$

"Tangent vector" $\gamma'(t)$ as defined in the package will be in our tangent plane. $\nabla \mathbf{f} \bullet \gamma'(t) = 0$ So $\nabla \mathbf{f}$ is " $\perp$ " to the tangent plane and is the plane's normal.

Step 4:

To find the tangent plane at a point $(a^1, a^2, \mathbf{g}(a^1, a^2))$ on the graph of $\mathbf{g}$,

Download $A \in G_{\mathbf{g}} \wedge \mathbf{a} \equiv a^1\hat{\imath} + a^2\hat{\jmath} + \mathbf{g}(a^1, a^2)\hat{k}$

Step 5: $p : \mathbf{n} \bullet (\mathbf{x} - \mathbf{a}) = 0$

$$\nabla \mathbf{f} \bullet \mathbf{x} = \nabla \mathbf{f} \bullet \mathbf{a}$$

$$\mathbf{g}_1' x^1 + \mathbf{g}_2' x^2 - x^3 = \mathbf{g}_1' a^1 + \mathbf{g}_2' a^2 - \mathbf{g}(a^1, a^2)$$

Step 6: $\mathbf{la}_{\mathbf{a}}^{\mathbf{g}} \equiv x^3 = \mathbf{g}_1'(x^1 - a^1) + \mathbf{g}_2'(x^2 - a^2) + \mathbf{g}(a^1, a^2)$

Step 7: This is a function that approximates the special surface $G_{\mathbf{g}}$ at points near A. To approximate $\mathbf{g}(x^1, x^2)$, substitute in x^1 and x^2 and find x^3.

Husayn Ramji, Andrew Chang, Remington Brooks

Oklahoma School of Science and Mathematics Goal: Evaluate the Double Integral

Step 1: Download the problem

$$\int_0^3 \left(\int_{\mathbf{p}_2 \times \mathbf{p}_1}^{9\,\mathbf{p}_0 \times \mathbf{p}_1} 4\overset{2}{\mathbf{p}_1} \cdot \left(\cos \circ \overset{1}{\mathbf{p}_2}\right) \boxed{1} \right)$$

Step 2: Download the original bounds.

$$\{(x, y)\,; x \in \left[\ y^2, 9\ \right] \wedge y \in [\ 0, 3\ ]\}$$

If there is no error, end program. If there is an error in evaluating using these bounds, continue to Step 3.

Step 3: Redefine the bounds.

$$\{(x, y)\,; x \in [\ 0, 9\ ] \wedge y \in [\ 0, \sqrt{x}\ ]\}$$

Step 4: The problem, with the new bounds, is now written as

$$\int_0^9 \left(\int_{\mathbf{p}_1 \times 0}^{\mathbf{p}_1 \times \mathbf{p}_{1/2}} 4\overset{2}{\mathbf{p}_1} \cdot \left(\cos \circ \overset{1}{\mathbf{p}_2}\right) \boxed{2} \right)$$

Step 5: Evaluate using the new bounds

$$\int_0^9 \left(\int_{\mathbf{p}_1 \times 0}^{\mathbf{p}_1 \times \mathbf{p}_{1/2}} 4\overset{2}{\mathbf{p}_1} \cdot \left(\cos \circ \overset{1}{\mathbf{p}_2}\right) \boxed{2} \right) = 82.39057647$$

Mehdi Drissi, Rahul Rajala, Cody Poteet, Kenneth Teel, Austin Fehr, Jerry Ni

Multivariate Final Presentation

Problem Set 2, Number 3, part a

Program - How to Check Continuity

0 Download function f. If first time called remember as original f.
1 Use lemmas to check for base case. If not a base case move on.
2 If $f = g \cdot h$ where g and h are functions then check continuity of g and h. If g and h are true than f is continuous. If $f \neq g \cdot h$ move on.
3 If $f = g + h$ where g and h are functions then check continuity of g and h. If both are true than f is continuous. If $f \neq g + h$ move on.
4 If $f = g \circ h$ where g and h are functions then check continuity of g and h. If both are true than f is continuous. If $f \neq g \circ h$ move on.
5 If $f = g \cdot \dfrac{1}{h}$ where g and h are functions at points were h is 0 use $\delta - \epsilon$ definition to check continuity at those points of original f. If $\delta - \epsilon$ definition fails output discontinuous. Then check continuity of g and h. If the three parts are true than f is continuous. If $f \neq g \cdot \dfrac{1}{h}$ output unknown continuity.

Step 0

Download $\mathbf{f} = \overset{1}{p}_1 \cdot \overset{2}{p}_1 \cdot \left[p_{-1} \circ \left(\overset{1}{p}_2 + \overset{2}{p}_2\right)\right] \cdot \left(\overset{1}{p}_2 - \overset{2}{p}_2\right) \cup \mathbf{0}_{\{(0,0)\}} : \mathbb{R}^2 \to \mathbb{R}$ and remember this f as original function.

Step 1

Move on.

Step 2

Call check continuity for $\overset{1}{p}_1$ and $\overset{2}{p}_1 \cdot \left[p_{-1} \circ \left(\overset{1}{p}_2 + \overset{2}{p}_2\right)\right] \cdot \left(\overset{1}{p}_2 - \overset{2}{p}_2\right)$.

Step 0

Download $f = \overset{1}{p}_1$.

Step 1

f is continuous by lemma 19.

Step 0

Download $f = \overset{2}{p_1} \cdot \left[p_{-1} \circ \left(\overset{1}{p_2} + \overset{2}{p_2}\right)\right] \cdot \left(\overset{1}{p_2} - \overset{2}{p_2}\right)$.

Step 1

Move on.

Step 2

Call check continuity for $\overset{2}{p_1}$ and $\left[p_{-1} \circ \left(\overset{1}{p_2} + \overset{2}{p_2}\right)\right] \cdot \left(\overset{1}{p_2} - \overset{2}{p_2}\right)$.

Step 0

Download $f = \overset{2}{p_1}$

Step 1

f is continuous by lemma 19.

Step 0

Download $f = \left[p_{-1} \circ \left(\overset{1}{p_2} + \overset{2}{p_2}\right)\right] \cdot \left(\overset{1}{p_2} - \overset{2}{p_2}\right)$

Step 1

Move on.

Step 2

Move on.

Step 3

Move on.

Step 4

Move on.

Step 5

The point where h is 0 is only (0,0). The value at that point is 0. The limit is also 0 based on:

$$(0,0)\mathbf{f} = 0 \text{ if } \mathbf{f} = \overset{1}{p_1} \cdot \overset{2}{p_1} \cdot \left[p_{-1} \circ \left(\overset{1}{p_2} + \overset{2}{p_2}\right)\right] \cdot \left(\overset{1}{p_2} - \overset{2}{p_2}\right) \cup \mathbf{0}_{\{(0,0)\}} : \mathbb{R}^2 \to \mathbb{R}$$

$$\left|xy\frac{x^2 - y^2}{x^2 + y^2}\right| < e(\sqrt{x^2 + y^2})$$

$$\left|xy\frac{x^2 - y^2}{x^2 + y^2}\right| \leq |x|\,|y|\,\frac{|x^2 - y^2|}{|x^2 + y^2|} \leq |x|\,|y|\,\frac{|x^2| + |-y^2|}{|x^2| + |y^2|} \leq |x||y| \leq \sqrt{x^2 + y^2}\sqrt{x^2 + y^2} \leq x^2 + y^2$$

Based on the final part $\underset{\epsilon}{\forall} \sqrt{\epsilon} = \delta$ will satisy the inequalities necessary for the limit to be verified.

So at (0,0) the original function is continuous.

Call check continuity for $\overset{1}{p_2} + \overset{2}{p_2}$ and $\overset{1}{p_2} - \overset{2}{p_2}$.

Step 0

Download $f = \overset{1}{p}_2 + \overset{2}{p}_2$.

Step 1

Move on.

Step 2

Move on.

Step 3

Call check continuity for $\overset{1}{p}_2$ and $\overset{2}{p}_2$.

Step 0

Download $f = \overset{1}{p}_2$.

Step 1

f is continuous by lemma 20.

Step 0

Download $f = \overset{2}{p}_2$.

Step 1

f is continuous by lemma 20.

Step 0

Download $f = \overset{1}{p}_2 - \overset{2}{p}_2$.

Step 1

Move on.

Step 2

Move on.

Step 3

Call check continuity for $\overset{1}{p}_2$ and $\overset{2}{p}_2$.

Step 0

Download $f = \overset{1}{p}_2$.

Step 1

f is continuous by lemma 20.

Step 0

Download $f = \overset{2}{p}_2$.

Step 1

f is continuous by lemma 20.

So since the original two pieces were both continuous the whole function is continuous.

Checking for a Conservative Vector Field

MPF Theorem

If and only if F is a conservative vector field with $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ than $P_2' = Q_1' \wedge P_3' = R_1'$.

The proof for if F is conservative vector field with $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ than $P_2' = Q_1' \wedge P_3' = R_1'$ along with the program to check that a vector field is conservative based on this proposition are on a previous pdf.

The proof for if $P_2' = Q_1' \wedge P_3' = R_1'$ than F is a conservative vector field with $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is:

$$P_2' = Q_1' \wedge P_3' = R_1'$$

$$\equiv \int_1 P = \int_2 Q \wedge \int_1 P = \int_3 R$$

$$\equiv \int_2 Q = \int_3 R$$

Let $f = \int_1 P$ than:

$$f = \int_1 P \wedge f = \int_2 Q \wedge f = \int_3 R$$

$$\equiv \nabla f = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

$$F = \nabla f$$

The last line is the definition of a conservative vector field and so the proof is complete. Proof of an implication of two propositions and its converse proves that two propositions are equivalent.