Lecture 5a: Performance Analysis

CS 70: Data Structures and Program Development

Tuesday, February 18, 2020

Learning Goals for Today

1. I can describe strengths and weaknesses of timing code
2. I can describe strengths and weaknesses of counting operations
3. I can describe strengths and weaknesses of asymptotic analysis
4. I can contrast $o(g)$, $O(g)$, $\Theta(g)$, $\Omega(g)$, $\omega(g)$.
5. I can do asymptotic analyses for iterative (looping) functions.

Why Algorithms Matter

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$$\text{Sorting 100 items by brute force} = 100! \text{ permutations} \approx 10^{158} \text{ steps}$$

But how can we *compare* different algorithms?

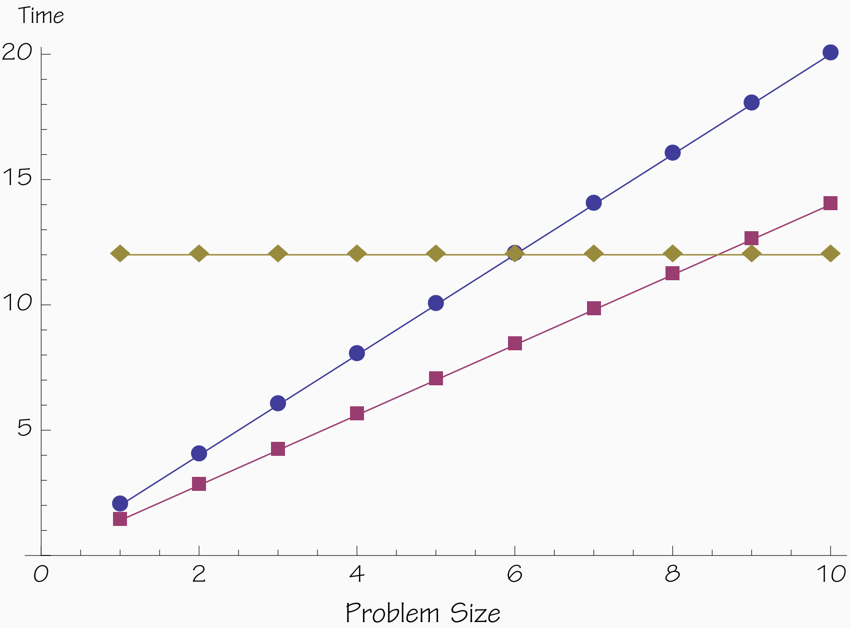
Benchmarking (Measuring Runtime)

Imagine this case study

- At Xcomp '15, Professor Bauer of UCNY announces that the new sorting algorithm WildSort takes only 0.16 seconds to sort a list of 100,000 names.
- Later, at the same conference, Professor Taylor of SUNY SD reports the new algorithm SneakerSort takes only 0.03 seconds to sort a list of 100,000 names.

Your conclusion(s)? Which is better?

Which Program is “Fastest” ?



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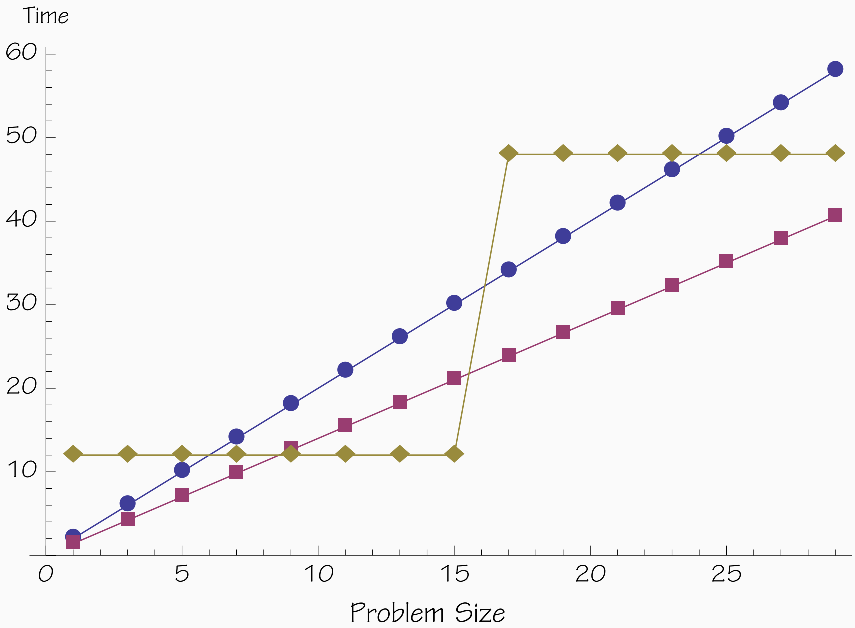

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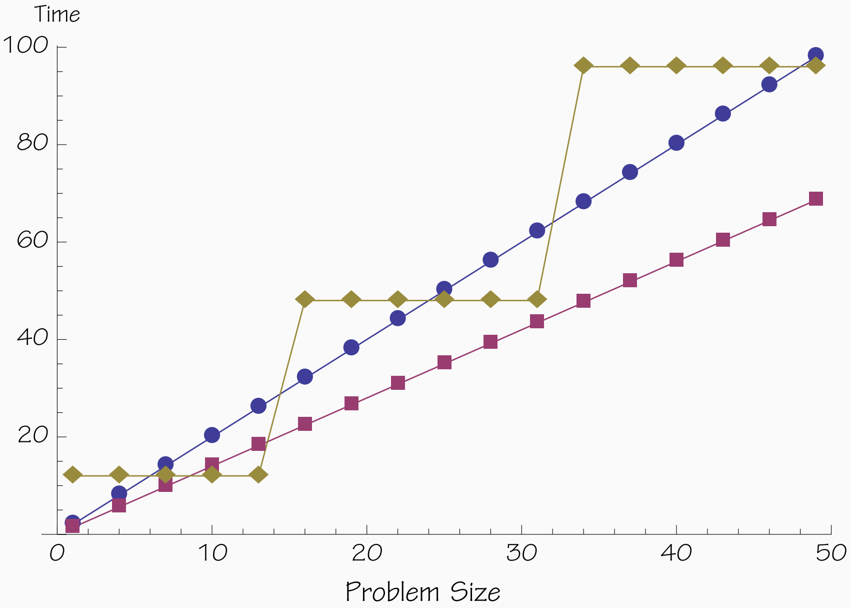

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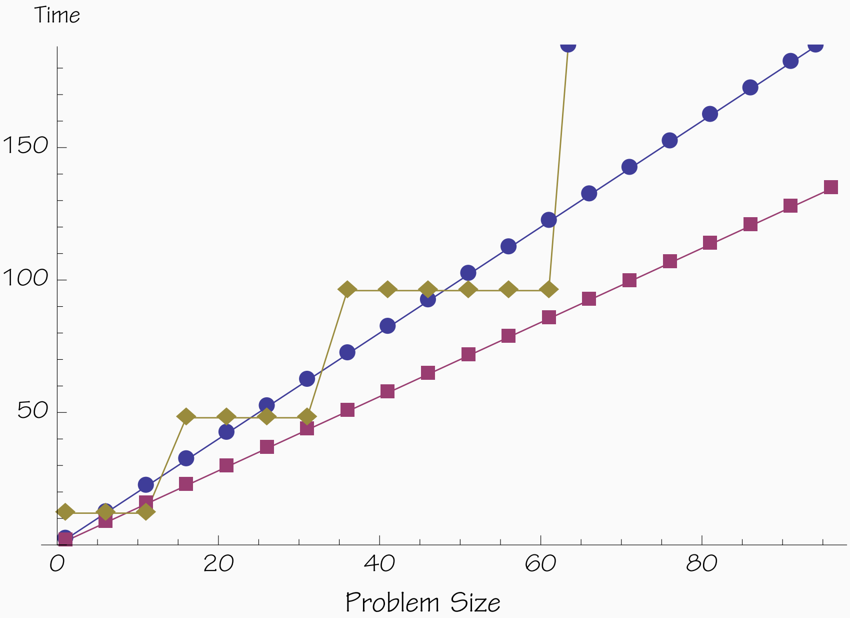

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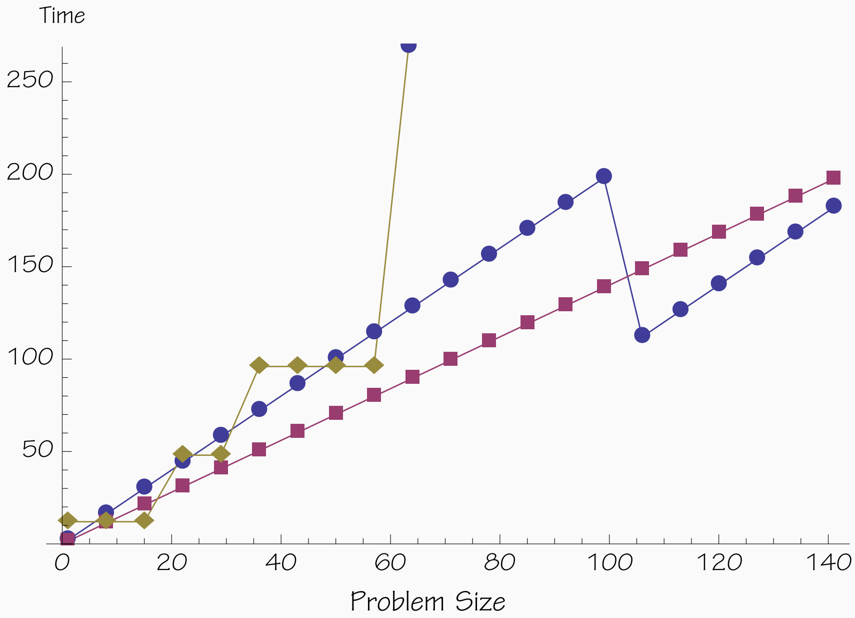

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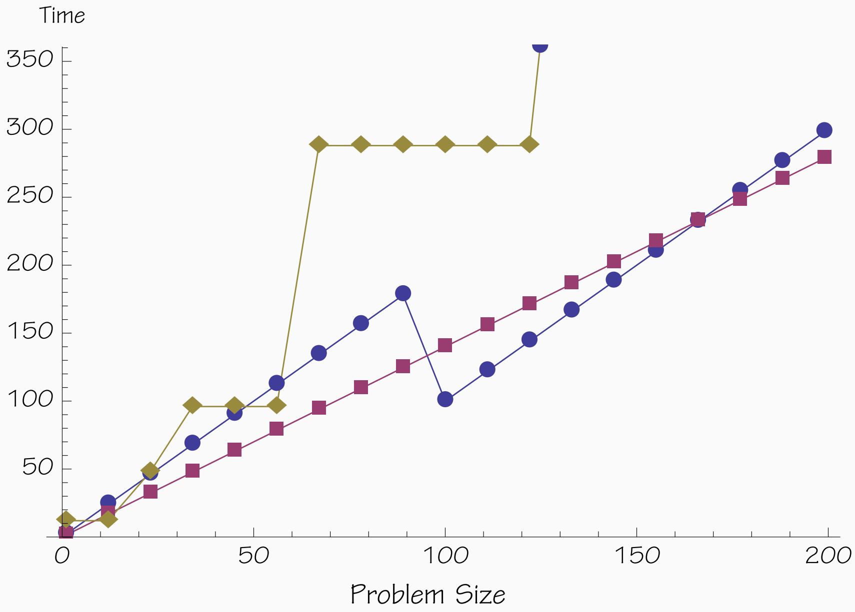

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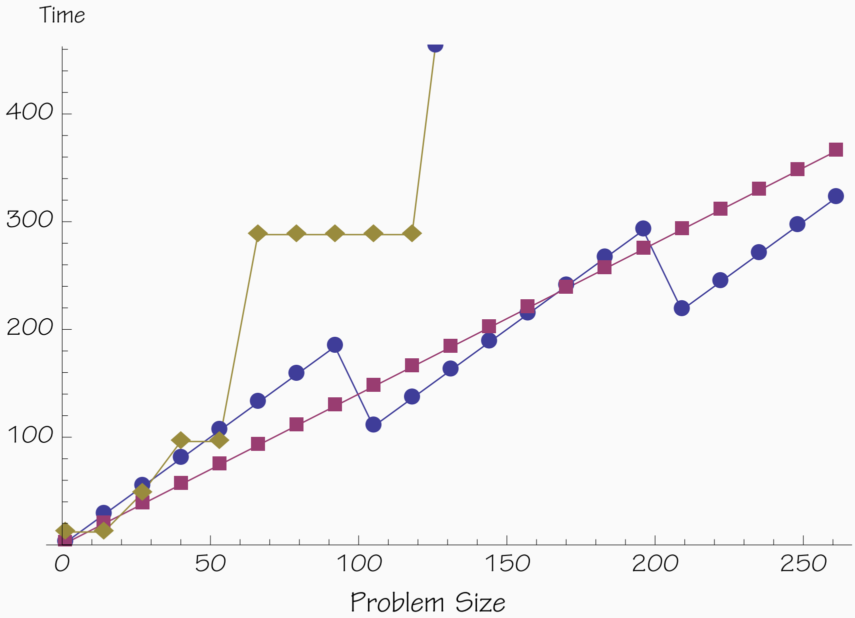

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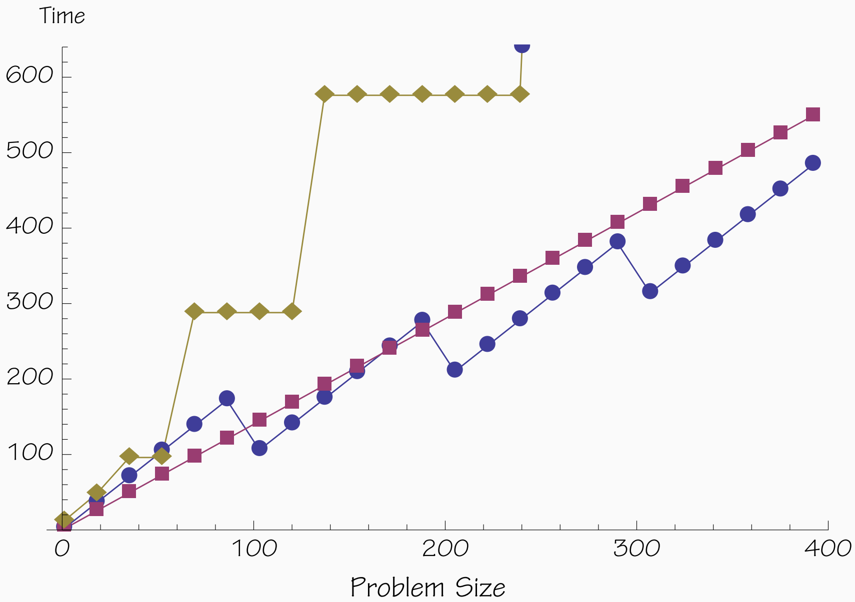

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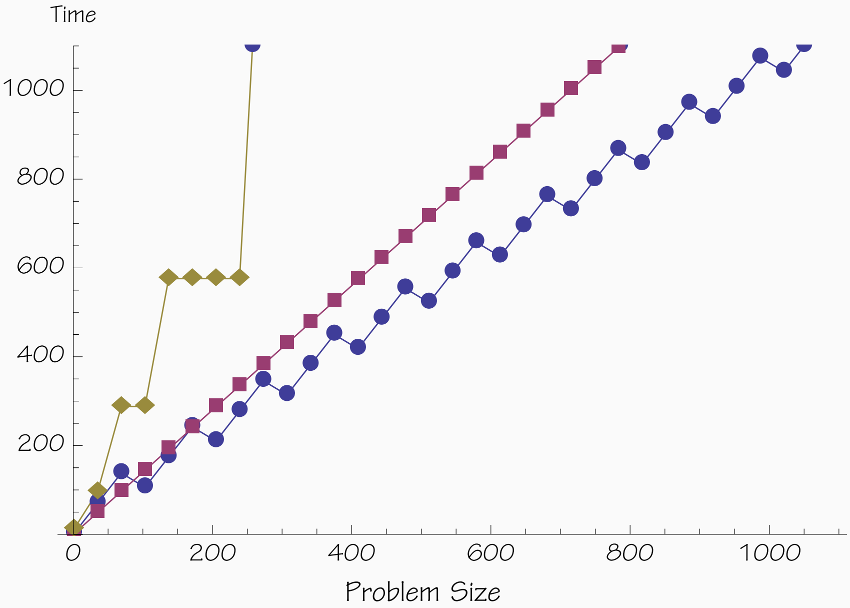
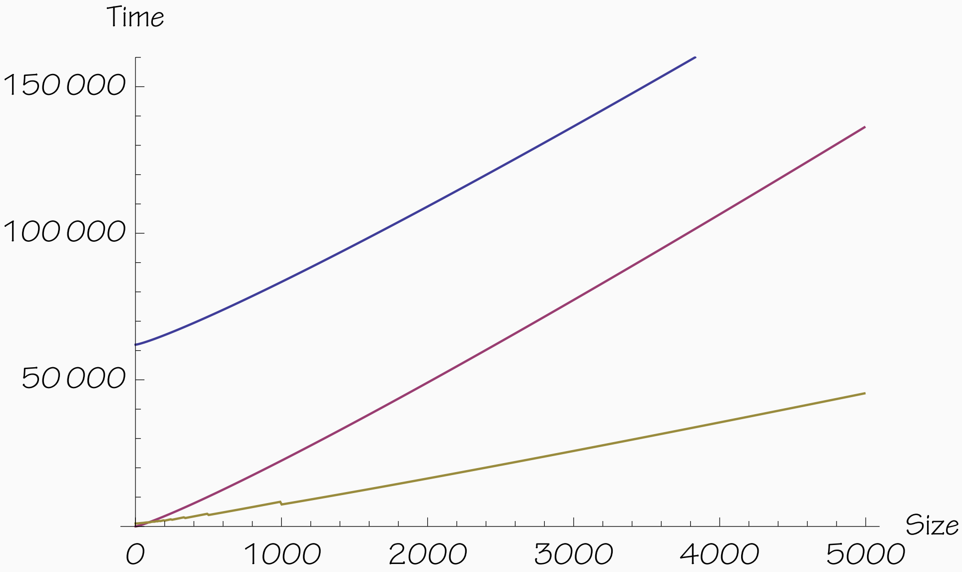


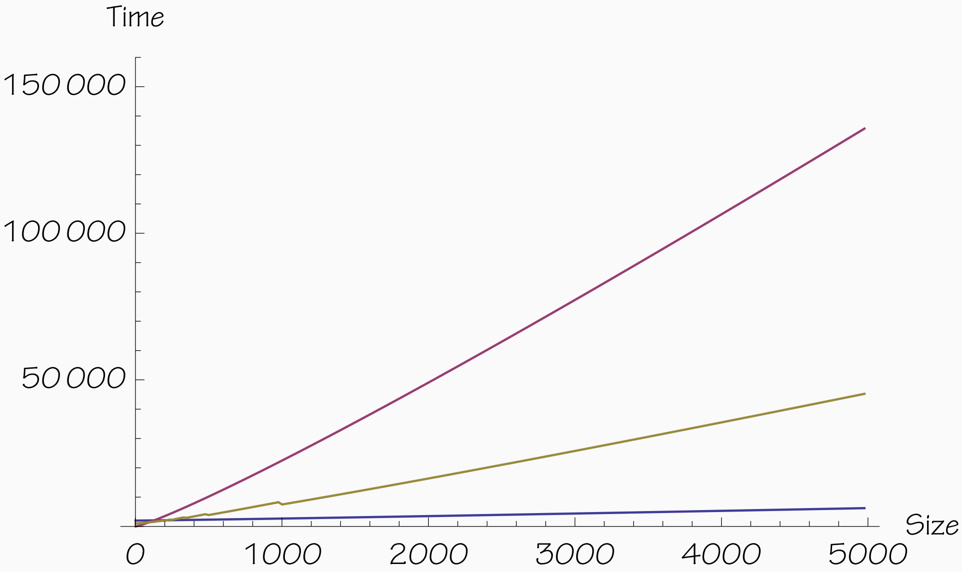
We can directly measure (benchmark)

- a specific algorithm
- written in a specific language
- as a specific program
- compiled using a specific version of a specific compiler
- with specific compiler flags settings (e.g., −g or not)
- running on a specific data set
- on a specific computer
- with a specific cpu (or cpus), memory, bus, hard drive, network card, ...
- under a specific version of a specific operating system
- while specific other programs run in the background.

Another example — oh, but blue is in Python



Switch blue to C++ for a 31x speedup:



Counting Steps

Donald Knuth, *Art of Computer Programing, Vol. 2*

[T]he determinant of almost all matrices ... can be computed with at most $(2n^3 - 3n^2 + 7n - 6)/6$ multiplications, $(2n^3 - 3n^2 + n)/6$ additions, and $(n^2 - n - 2)/2$ divisions.

Recall: 1D vs. 2D Arrays

Suppose we want to take in a sequence of characters like

10101110110111010001

and print these as 4 rows of 5 characters?

10101
11011
01110
10001

Outputs? comparisons? additions? multiplications?

```
int original[NUM_ROWS * NUM_COLS];

// ...load the digits into the array...

for (size_t row = 0; row < NUM_ROWS; ++row) {
    for (size_t col = 0; column < NUM_COLS; ++col) {
        cout << original[row * NUM_COLS + col];
    }
    cout << endl;
}
```

Concerns

Is incrementing the same as addition?

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Sum of operation times != total run-time

Michael Abrash, *The Zen of Assembly Language*

A few years back, I came across an article called "Optimizing for Speed". [The author] had clearly fine-tuned the code with care adding up cycles until he arrived at an implementation he calculated to be nearly 50% faster. There was, in fact, only one slight problem: it ran slower than the original version!

Asymptotic Analysis

Asymptotic Analysis (Big-O, Big-Θ, etc.)

Answers an abstract question about an algorithm:

- How do costs *scale* as input sizes become arbitrarily large?

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Not

- How many seconds do we need for an input of size n
- How many bytes of memory are required?
- How many additions are performed?
- How easy is it to implement the algorithm?
- Is this the best algorithm for my problem?

Costs

Answers an abstract question about an algorithm:

- How do costs *scale* as input sizes become arbitrarily large?

"Cost" might refer to

- Time spent (today's focus)
- Bytes of memory required
- Bits transmitted over the network
- Watts of electricity consumed
- etc.

Asymptotic Analysis

Answers an abstract question about an algorithm:

- How do costs *scale* as input sizes become arbitrarily *large*?

Suppose a function with input size n takes $63n$ steps when it runs.

What is the ratio?

$$\frac{\#\text{steps on some input twice as big}}{\#\text{steps for some input}}$$

Asymptotic Analysis

Answers an abstract question about an algorithm:

- How do costs *scale* as input sizes become arbitrarily *large*?

Suppose a function with input size n takes $5n^3$ steps when it runs.

What is the ratio?

$$\frac{\#\text{steps on some input three times as big}}{\#\text{steps for some input}}$$

Asymptotic Analysis

Answers an abstract question about an algorithm:

- How do costs *scale* as input sizes become arbitrarily *large*?

Suppose a function with input size n takes $n^3 + 17$ steps when it runs.

What is the ratio

$$\frac{\#\text{steps on some input three times as big}}{\#\text{steps for some input}}$$

as n gets very large?

Consequences

If we only care about *scalability* for arbitrarily large inputs:

Constant factors don't matter.

- $0.01n$ and n and $9999n$ scale linearly
- $0.01n^2$ and n^2 and $9999n^2$ scale quadratically
- $\ln n$ and $\log_2 n$ and $\log_{10} n$ scale logarithmically

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"Small" summands can be ignored

- $(n^2 + 100n + 1000000) \approx n^2$ for very large n
- $(n^n + n! + 2^n) \approx n^n$ for very large n

Comparing Growth Rates

	n	$n \log n$	n^2	n^3	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 18 min	10^{25} years
$n = 100$	< 1 sec	< 1 sec	1 sec	1s	10^{17} years	very long
$n = 1000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long
$n = 10,000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long
$n = 100,000$	< 1 sec	2 sec	3 hours	32 years	very long	very long
$n = 1,000,000$	1 sec	20 sec	12 days	31,710 years	very long	very long

(adapted from [2], Table 2.1, pg. 34)

Simplify

1. $O(3n^2 + 2n + 2 + \cos(\pi n))$
2. $O(\log_{10}(n^3))$
3. $O(5n^{1.5} + 2n \log n)$
4. $O(n^2 + 2^n)$

More Consequences

We can count “steps” rather than “instructions” or “clock cycles”

- Both give the same asymptotic result
- A step can be a lot of work, as long as it’s bounded by a constant.

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- Both give the same asymptotic result
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```
const int ARR_LENGTH = 80;
for (size_t i = 0; i < 50; ++i) {
  size_t arr[ARR_LENGTH]{i};
  for (size_t j = 0; j < ARR_LENGTH; ++j) {
    cout << arr[j] << " ";
  }
  cout << endl;
}
```

Be careful about identifying single steps!

```
string output;
for(size_t i = 0; i < n; ++i) {
    output += " " + to_string(i);
}
cout << output << endl;
```

Coarse-Grained Group: $O(1)$

- Takes 6 steps
- Takes 1 (big) step
- No more than 4000 steps
- Somewhere between 2 and $47,000$ steps, depending on the input

Coarse-Grained Group: $O(n)$

- Takes $100n + 3$ steps
- Takes $n/20 + 10,000,000$ steps
- Anywhere from 3 to 68 steps per item, for n items

Coarse-Grained Group: $O(n^2)$

- Takes $2n^2 + 100n + 3$ steps
- Takes $n^2/17$ steps.
- Somewhere between 1 and 40 steps per item, for n^2 items
- Anywhere between 1 and $7n$ steps per item, for n items

Making Life Simpler

If there's any one step that dominates (asymptotically), we can ignore everything else, e.g.,

```
for (int i = 0;  i < n;  ++i) {
    sum += 2;
}
```

Asymptotic notation, intuitively.

$f \in o(g)$ if f grows strictly less fast than g $\quad (<)$

$f \in O(g)$ if f grows no faster than g $\quad (<=)$

$f \in \Theta(g)$ if f grows at the same rate as g $\quad (=)$

$f \in \Omega(g)$ if f grows at least as fast as g $\quad (>=)$

$f \in \omega(g)$ if f grows strictly faster than g $\quad (>)$

What relationships hold between these classes?

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Asymptotic notation, more formally

$f \in o(g)$ if f stays below every multiple of g, eventually.

$f \in O(g)$ if f stays below some multiple of g, eventually.

$f \in \Theta(g)$ if f stays between two multiples of g, eventually.

$f \in \Omega(g)$ if f stays above some multiple of g, eventually,

$f \in \omega(g)$ if f stays above every multiple of g, eventually.

Asymptotic notation, even more formally

$f \in o(g)$ if $\forall c > 0.\ \exists N \geq 0.\ \forall n \geq N.\ f(n) \leq c \cdot g(n)$

$f \in O(g)$ if $\exists c > 0.\ \exists N \geq 0.\ \forall n \geq N.\ f(n) \leq c \cdot g(n)$

$f \in \Theta(g)$ if $\exists c_1, c_2 > 0.\ \exists N \geq 0.\ \forall n \geq N.\ c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$

$f \in \Omega(g)$ if $\exists c > 0.\ \exists N \geq 0.\ \forall n \geq N.\ f(n) \geq c \cdot g(n)$

$f \in \omega(g)$ if $\forall c > 0.\ \exists N \geq 0.\ \forall n \geq N.\ f(n) \geq c \cdot g(n)$

Exercises

1. Is $3n \in O(n)$?
2. Is $3n \in \Omega(n)$?
3. Is $3n \in \Theta(n)$?
4. Is $3n \in O(n^2)$?
5. Is $3n \in \Omega(n^2)$?
6. Is $3n \in \Theta(n^2)$?

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2. Is $3n \in \Omega(n)$?
3. Is $3n \in \Theta(n)$?
4. Is $3n \in O(n^2)$?
5. Is $3n \in \Omega(n^2)$?
6. Is $3n \in \Theta(n^2)$?
7. Is $3n \in O(2^n)$?

Which should we avoid saying?

1. This algorithm isn't scalable because it takes $O(n^3)$ time.
2. This algorithm isn't scalable because it takes $\Theta(n^3)$ time.
3. This algorithm isn't scalable because it takes $\Omega(n^3)$ time.
4. Wow! I'm surprised we can sort n `ints` in $O(n)$ time with Radix Sort.
5. Wow! I'm surprised we can sort n `ints` in $\Theta(n)$ time with Radix Sort.
6. Wow! I'm surprised we can sort n `ints` in $\Omega(n)$ time with Radix Sort.

Warning!

Many programmers say $O(g)$ when they mean $\Theta(g)$

- "It's too slow; it's $O(n^3)$"

and further assume that hidden factors are always small

- "If you double the input size of an $O(n^2)$ algorithm, it will take four times as long."

Calculating Asymptotically

$$O(f) + O(g) = O(f + g).$$

$$O(f) \cdot O(g) = O(f \cdot g)$$

$$O(max\{f, g\}) = O(f + g).$$

Calculating Asymptotically

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$O(f) \cdot O(g) = O(f \cdot g)$

$O(max\{f, g\}) = O(f + g)$.

(same for Θ and Ω)

A Very Common/Important Summation!

$$1 + 2 + 3 + \cdots + n \; = \frac{n(n+1)}{2} \in O(n^2)$$

Sample Calculations

Classify $T(n)$, the number of steps required for input n

```
for (int i = 0;  i < n;  ++i)
    ++sum;
```

Classify $T(n)$, the number of steps required for inputs n and m

```
for (int i = 0;  i < 2*n;  ++i)
    for (int j = 0;  j < m+1;  ++j)
        ++sum;
```

Classify $T(n)$, the steps required for input n

```
for (int i = 0;  i < n;  ++i)
    for (int j = 0;  j < i+1;  ++j)
        ++sum;
```

Classify $T(n, m)$, steps required for inputs n and m

```
for (int i = 0;  i < n;  ++i)
    ++sum;

for (int j = 0;  j < m;  ++j)
    ++sum;
```

Classify $T(n)$, the steps required for input n

```
for (int i = 0; i < n; ++i) {
    for (int j = 0; j < i; j += 2) {
        a[j+1] += 1;
        if (a[j+1] % 2 == 0) a[j] = 2*a[j];
    }
}
```