HAETAE

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2024 KpqC Winter Camp

Table of Contents

HAETAE

- Digital signature scheme
- Secure against quantum attacks!
  - based on **lattice hard problems** MLWE and MSIS
  - follows **Fiat-Shamir with aborts** framework, secure in QROM
- Simple but short!
  - simpler than Falcon[1] & shorter than Dilithium[1]
  - optimal rejection rate with simple rejection condition
- Design rationale
  - **Bimodal** rejection sampling
  - **Hyperball** distribution
- Candidate in *KpqC 2nd round & NIST PQC Additional Signatures*[2]

[1] NIST 2022 PQC signature standards
[2] NIST's on-ramp PQC signature competition, from 2023.

HAETAE

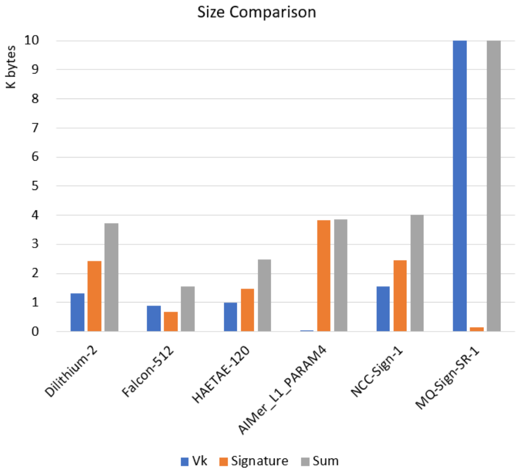


Figure: KpqC round 2, signature schemes

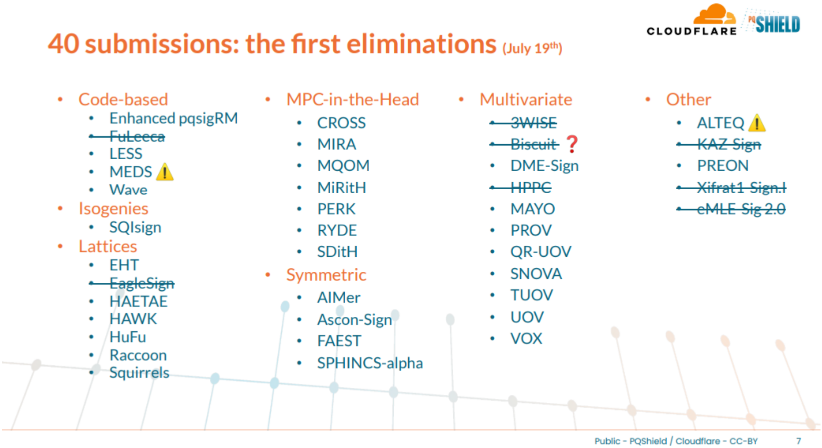
HAETAE

40 submissions

- Code-based
  - Enhanced pqsigRM
  - FuLeeca
  - LESS
  - MEDS
  - Wave
- Isogenies
  - SQISign
- Lattices
  - EHT
  - EagleSign
  - HAETAE
  - HAWK
  - HuFu
  - Raccoon
  - Squirrels
- MPC-in-the-Head
  - CROSS
  - MIRA
  - MQOM
  - MiRitH
  - PERK
  - RYDE
  - SDitH
- Symmetric
  - AIMer
  - Ascon-Sign
  - FAEST
  - SPHINCS-alpha
- Multivariate
  - 3WISE
  - Biscuit
  - DME-Sign
  - HPPC
  - MAYO
  - PROV
  - QR-UOV
  - SNOVA
  - TUOV
  - UOV
  - VOX
- Other
  - ALTEQ
  - KAZ-Sign
  - PREON
  - Xifrat1-Sign.I
  - eMLE-Sig 2.0

Slide from `https://datatracker.ietf.org/meeting/117/materials/slides-117-tls-new-post-quantum-signature-algorithms-on-the-horizon-00`

HAETAE



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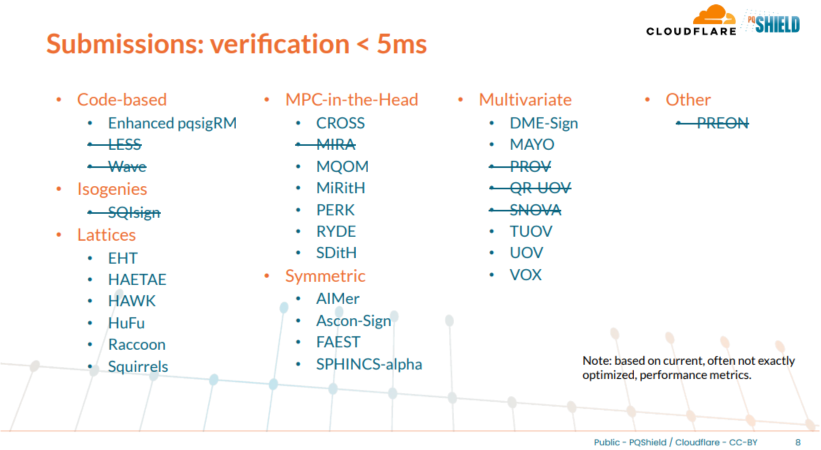
HAETAE



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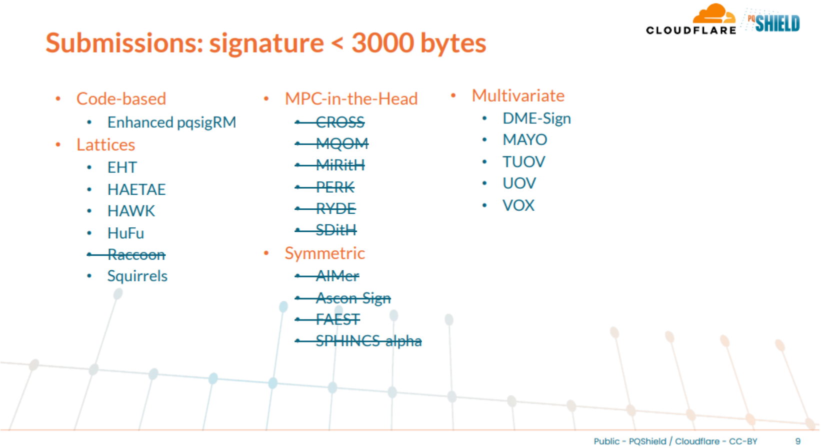
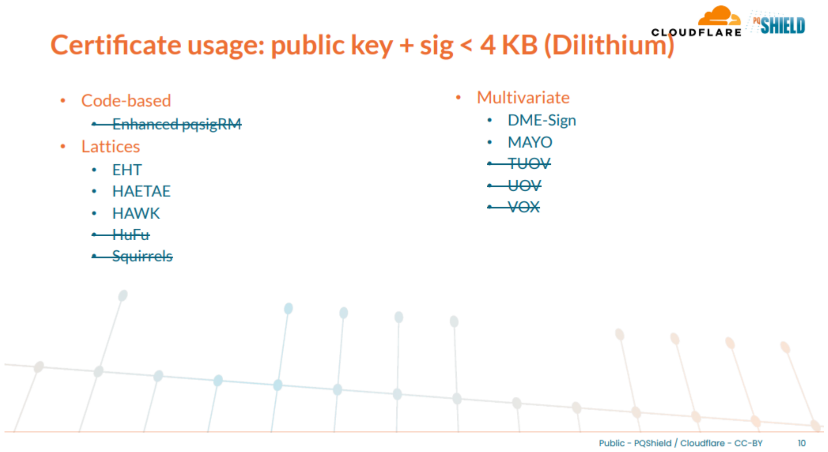
HAETAE



Slide from `https://datatracker.ietf.org/meeting/117/materials/slides-117-tls-new-post-quantum-signature-algorithms-on-the-horizon-00`

HAETAE



Slide from `https://datatracker.ietf.org/meeting/117/materials/slides-117-tls-new-post-quantum-signature-algorithms-on-the-horizon-00`

1. Brief Introduction to HAETAE

2. Preliminaries:
   - Digital signatures
   - Lattice hard problems
   - Lattice-based signatures
     - Bimodal rejection sampling

3. HAETAE:
   - Hyperball bimodal rejection sampling
   - Comparison to SotA lattice signatures

4. Changes after Round 1

Digital signatures

Conventional signatures:

Some images are from https://kr.freepik.com/search?format=search&last_filter=type&last_value=icon&query=magnifier&selection=1&type=icon

Digital signatures

Conventional signatures:

Alice

wants to sign
on a document

$\longrightarrow$

Bob

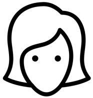
wants to verify
that Alice signed
on the document

verify by experts

Digital signatures:

$(\mathsf{sk}, \mathsf{vk}) \leftarrow \mathsf{KeyGen}$ and broadcast vk

Alice (knows sk)



signature $\sigma \leftarrow \mathsf{Sign}(\mathsf{sk}, m)$

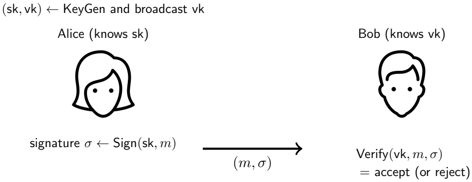
$\xrightarrow[(m, \sigma)]{}$

Bob (knows vk)

$\mathsf{Verify}(\mathsf{vk}, m, \sigma)$
$=$ accept (or reject)

Digital signatures

Digital signatures:



Anyone (who can access vk) can verify that (m, σ) is from Alice or not!

Correctness: $\mathsf{Verify}(\mathsf{vk}, m, \mathsf{Sign}(\mathsf{sk}, m)) = \mathsf{accept}$

Unforgeability: No one but Alice can make a new signature.

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Lattice hard problems

Lattice-based cryptography is ... are currently important candidates for post-quantum cryptography.

- *Wikipedia* -

Lattice-based cryptography bases its security on lattice hard problems, which have strong theoretical backgrounds:

- SVP and GapSVP$_\lambda$: NP-hard! [Ajt96, HR07]
- Worst-case to average-case **reductions** [Ajt96]
- Useful hard problems: NTRU, LWE, SIS, MLWE, MSIS, etc

Lattice hard problems

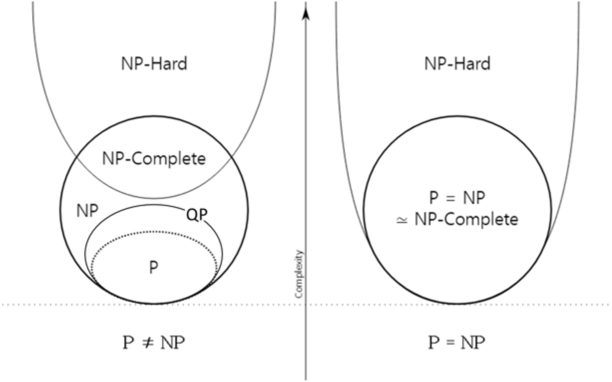

Figure: Category of hard problems when P≠NP and P=NP.

No proofs for Quantum Poly (QP), but is believed to be separated to NP-Hard problems.

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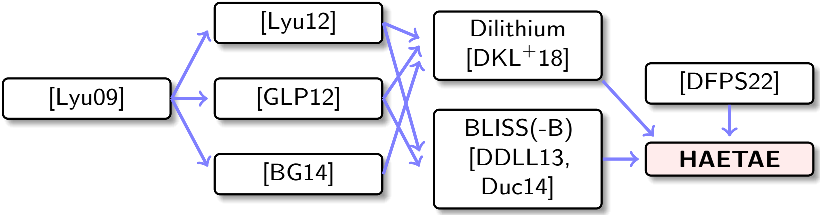
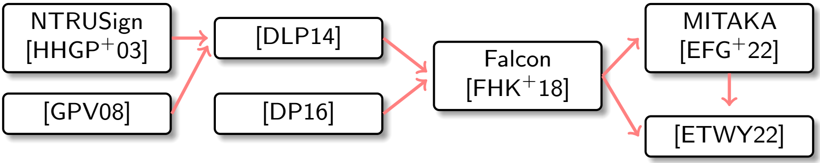
Lattice-based signatures

Fiat-Shamir with abort



Hash-and-Sign



Lattice-based signatures

Fiat-Shamir with abort:

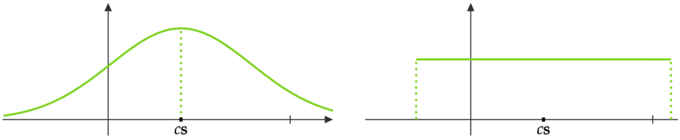
For secret $\mathbf{s}$, random $\mathbf{y}$, c, signature $\sigma = (c, \mathbf{z} = \mathbf{y} + c\mathbf{s})$

$$\left(\boxed{8}, \begin{bmatrix} 9 \\ -9 \\ 6 \\ 2 \\ 1 \\ 11 \\ 8 \\ -9 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 2 \\ 1 \\ 3 \\ 0 \\ -1 \end{bmatrix} + \boxed{8} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right)$$

Lattice-based signatures

Leakage from $(c, \mathbf{z} = \mathbf{y} + c\mathbf{s})$?

(High-level) With ∞ pairs of $(c, \mathbf{z} = \mathbf{y} + c\mathbf{s})$, we may collect $\mathbf{z}$ for same c:



($y \leftarrow \mathcal{N}(0, \sigma^2)$) ($y \leftarrow U[-a, a]$)

$\Rightarrow$ Recover $\mathbf{s}$ from $c\mathbf{s}$.

How to make it safe?

$$\underset{\text{not safe}}{(c, \mathbf{z} = \mathbf{y} + c\mathbf{z})} \xrightarrow[\text{Rejection Sampling}]{\text{several trials}} \underset{\text{safe}}{\sigma = (c, \mathbf{z} = \mathbf{y} + c\mathbf{z})}$$

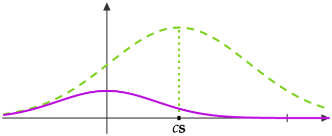
Rejection sampling

Rejection sampling

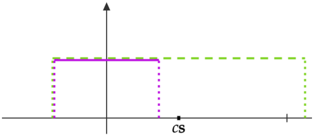
$$D_{\text{source}} = \{(c, \mathbf{z})\} \xrightarrow[\text{prob. } p(c,\mathbf{z})]{\text{reject with}} D_{\text{target}}$$

distribution of $(c, \mathbf{z})$, possibly leak $\mathbf{s}$

new distribution, independent of $\mathbf{s}$



$y \leftarrow \mathcal{N}(0, \sigma^2)$



$y \leftarrow U[-a, a]$

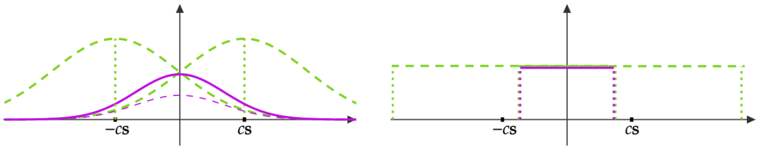
Bimodal rejection sampling

Run-time $\propto M$ ($\approx$ green area / purple area).

To decrease M, [DDLL13] uses

$$\mathbf{z} = \mathbf{y} + (-1)^b c\mathbf{s}$$

instead of $\mathbf{z} = \mathbf{y} + c\mathbf{s}$:



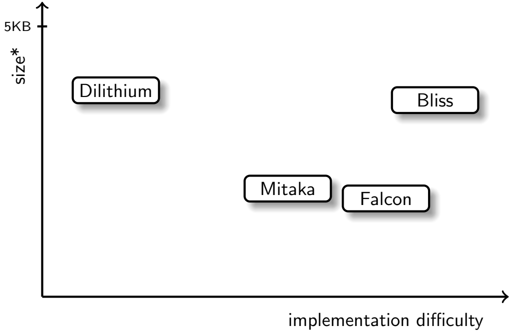
$y \leftarrow \mathcal{N}(0, \sigma^2)$ $y \leftarrow U[-a, a]$

Note, no change for the uniform case.

Bimodal rejection sampling

However, this makes "secure" implementation[3] much harder. It is basically due to "reject with probability a (transcendental) function of sk."

For e.g., for ≈120 bits security[4][5],



[3] an implementation secure against physical attacks (side-channel attacks)
[4] core-SVP hardness
[5] size$= |\mathrm{sig}| + |\mathrm{vk}|$

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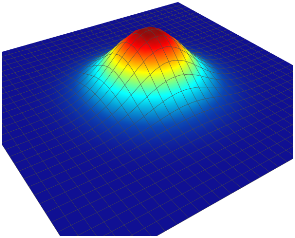
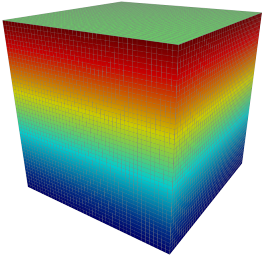
4. Changes after Round 1

Hyperball bimodal rejection sampling

Previously, the randomness $\mathbf{y}$ was chosen from either discrete Gaussian or uniform hypercube[6].





[6]The vectors $\mathbf{y}$ and $\mathbf{z}$ are high-dimensional vectors, so uniform in an interval is indeed a uniform hypercube.

Hyperball bimodal rejection sampling

We, instead, use **uniform hyperball** distribution for sampling $\mathbf{y}$ [DFPS22];

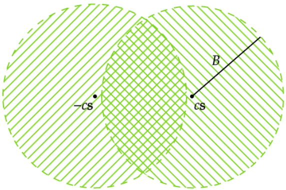
- to exploit optimal M,
- to reduce signature and verification key sizes,

based on the **bimodal approach** [DDLL13].

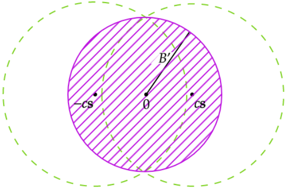
Hyperball bimodal rejection sampling

We reject $(c, \mathbf{z}) \sim D_{\mathsf{s}}$ (with p.d.f. p_{s}) to a target distribution D_{t} (with p.d.f. p_{t}), where

- p_{s}: uniform in hyperballs of radii B centered at $\pm c\mathbf{s}$
  - union of two large balls
- p_{t}: uniform in a smaller hyperball of radii B' centered at zero
  - a smaller ball in the middle



p_s



p_t

Hyperball bimodal rejection sampling

- $p_{\mathsf{s}}(\mathbf{x}) = \frac{1}{2\cdot\mathsf{vol}(\mathcal{B}(B))} \cdot \chi_{\|\mathbf{z}-c\mathbf{s}\|<B} + \frac{1}{2\cdot\mathsf{vol}(\mathcal{B}(B))} \cdot \chi_{\|\mathbf{z}+c\mathbf{s}\|<B}$,
- $p_{\mathsf{t}}(\mathbf{x}) = \frac{1}{\mathsf{vol}(\mathcal{B}(B))} \cdot \chi_{\|\mathbf{z}\|<B'}$.

$$\Rightarrow p(\mathbf{x}) = \frac{p_{\mathsf{t}}(\mathbf{x})}{M \cdot p_{\mathsf{s}}(\mathbf{x})} = \frac{\chi_{\|\mathbf{z}\|<B'}}{\chi_{\|\mathbf{z}-c\mathbf{s}\|<B} + \chi_{\|\mathbf{z}+c\mathbf{s}\|<B}}$$

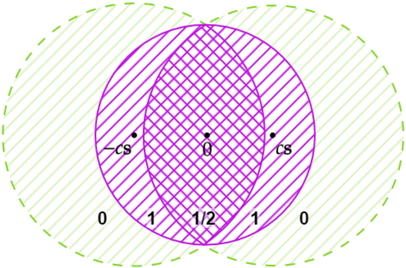
$$= \begin{array}{cl} 0 & \text{if } \mathbf{z} \notin \mathcal{B}(B'), \\ 1/2 & \text{if } \mathbf{z} \in \mathcal{B}(B') \cap \mathcal{B}(B, c\mathbf{s}) \cap \mathcal{B}(B, -c\mathbf{s}), \\ 1 & \text{if } \mathbf{z} \in \mathcal{B}(B') \setminus (\mathcal{B}(B, c\mathbf{s}) \cap \mathcal{B}(B, -c\mathbf{s})), \end{array}$$

for some $M > 0$.

Hyperball bimodal rejection sampling

That is, we return $\mathbf{x} = (c, \mathbf{z})$ with probability

- 0: if $\|\mathbf{z}\| \geq B'$,
- 1/2: else if $\|\mathbf{z} - c\mathbf{s}\| < B$ and $\|\mathbf{z} + c\mathbf{s}\| < B$,
- 1: otherwise.



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4. Changes after Round 1

Comparison to SotA lattice signatures.

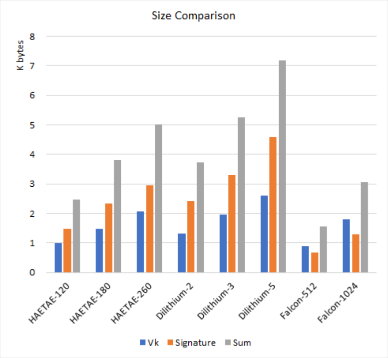
For 120-bit classical security. Sizes are in bytes.

Scheme	sig	vk	KeyGen	Sign	
				sampling	rejection
Dilithium-2	2420	1312	fast	Hypercube	$\|\cdot\|_\infty < B$
Bliss-1024[7]	1700	1792	fast	dGaussian at 0	reject with prob. f(sk, Sig)
HAETAE120	1468	1056	fast	dHyperball at 0	$\|\cdot\|_2 < B$
Mitaka-512[8]	713	896	slow	dGaussian at 0 & intGaussian at $H(m)$	none
Falcon-512	666	897	slow	dGaussian at $H(m)$	none

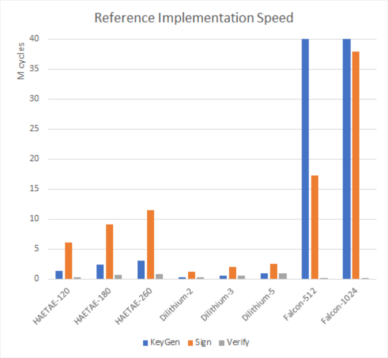
Table: Comparison between different lattice-based signature schemes.

[7] modified Bliss (to ≥ 120 bit-security) in Dilithium paper.
[8] Mitaka-512 has 102 bits of security

Numbers - Updated Reference Implementation

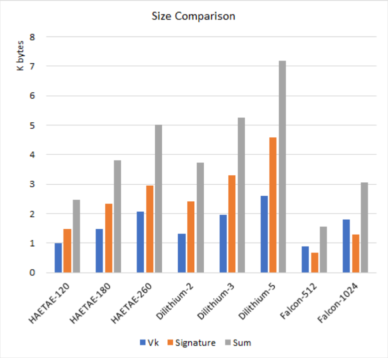


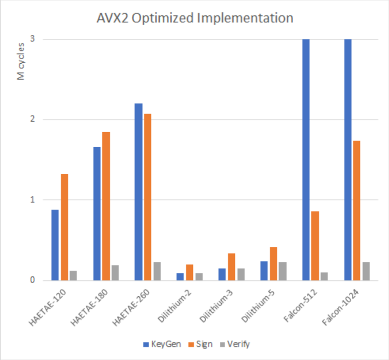
Size



Performance

Numbers - AVX2 optimized Implementation



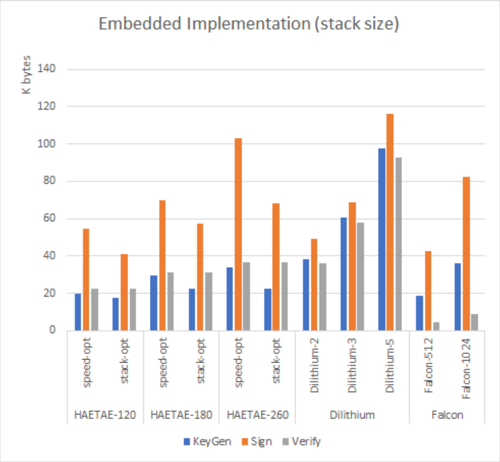


Size

Performance

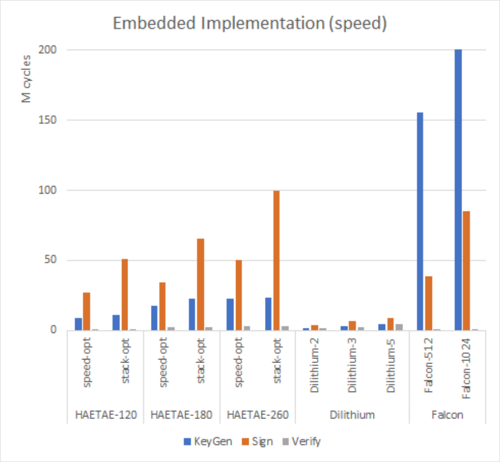
Numbers - Embedded Implementation on Cortex-M4

Stack-size of HAETAE and others on Cortex-M4.



Numbers - Embedded Implementation on Cortex-M4

Speed of HAETAE and others on Cortex-M4.



Update Logs after Round 1

Nov, 2022 (v0.9): KpqC round 1

May, 2023 (v1.0)

- spec: missing parts inclusion, min-entropy analysis
- improved: rANS, secret key rejection
- implementation: fixed-point, constant-time

Nov, 2023 (v2.0)

- spec: implementation security
- improved: reduced precomputation table for rANS
- implementation: Bug-fix, AVX2 optimized, embedded (Cortex-M4)

Feb, 2024 (v2.1): KpqC round 2

- spec: HVZK for compressed HAETAE, more precise security bound, "refined" security estimation

Update Logs after Round 1

May, 2023 (v1.0)

- spec: missing parts inclusion, min-entropy analysis
- improved: hint compression, secret key rejection
- implementation: fixed-point, constant-time

- **Hint vector compression:** The hint vector h, a part of the signature, is compressed via $\mathsf{LowBits}^h$, $\mathsf{HighBits}^h$, and rANS encoding.

- **Secret key rejection:** Bounding $\|c\mathbf{s}\|_2 \leq \gamma\sqrt{\tau}$ via bounding $\mathcal{N}(\mathbf{s}) \leq \gamma^2 n$:

$$\mathcal{N}(\mathbf{s}) := \tau \cdot \sum_{i=1}^{m} \max_{0\leq j<2n}^{i\text{-th}} \|\mathbf{s}(\omega_j)\|_2^2 + r \cdot \max_{0\leq j<2n}^{(m+1)\text{-th}} \|\mathbf{s}(\omega_j)\|_2^2,$$

which can be efficiently checked.

- **Fixed-Point everywhere!**

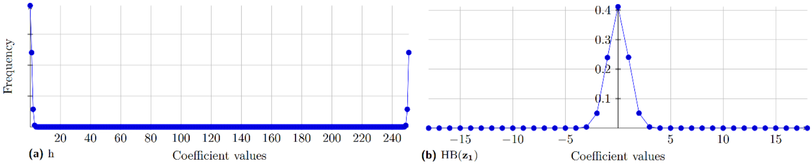
Update Logs after Round 1

Nov, 2023 (v2.0)

- spec: implementation security
- improved: reduced precomputation table for rANS
- implementation: Bug-fix, AVX2 optimized, embedded (Cortex-M4)

- **Implementation security:** Mainly on *Fix-Point Arithmetic* with no fix-point multiplication and *Protecting the Hyperball Sampler*.

- **Reduced precomputation table:** Cut off the extremely low-frequency symbols ($< 0.1\%$ in total):



(a) h (b) $\mathrm{HB}(\mathbf{z_1})$

Update Logs after Round 1

Nov, 2023 (v2.0)

- spec: implementation security
- improved: reduced precomputation table for rANS
- implementation: AVX2 optimized, embedded (Cortex-M4)

- **Bug-fix:** Implementation-specific bugs (reported via KpqC workshops/ bulletin/PQC forum/ourselves) are fixed.

- **AVX2 optimization:** Mainly on *Vectorized Hyperball Sampling* (as Keccak and NTT use existing optimized code) via parallel polynomial samplings. For HAETAE-120, 4.6x speed-up.

- **Embedded Cortex-M4 implementation:** Stack/speed optimizations, resulting in 40 to 54 KiB maximum stack size for HAETAE-120.

Update Logs after Round 1

Feb, 2024 (v2.1)

- spec: HVZK for compressed HAETAE, more precise security bound, "refined" security estimation

- **HVZK for compressed HAETAE:** Proof for HVZK is extended to cover the compressed HAETAE.

- **Precise security bound and "refined" security estimation:** Along with the original security bounds with BKZ block size based on GSA and Core-SVP analysis, we also give a security estimation for MLWE based on the leaky LWE estimator [DDGR20].

Thanks!

Check http://kpqc.cryptolab.co.kr/haetae!

Check https://github.com/mupq/pqm4 for the *embedded code*!

Any question?

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HAETAE description (high-level)

KeyGen(1^λ)

1: $\mathbf{A}_{\mathsf{gen}} \leftarrow \mathcal{R}_q^{k\times(\ell-1)}$ and $(\mathbf{s}_{\mathsf{gen}}, \mathbf{e}_{\mathsf{gen}}) \leftarrow S_\eta^{\ell-1} \times S_\eta^k$
2: $\mathbf{b} = \mathbf{A}_{\mathsf{gen}} \cdot \mathbf{s}_{\mathsf{gen}} + \mathbf{e}_{\mathsf{gen}} \in \mathcal{R}_q^k$
3: $\mathbf{A} = (-2\mathbf{b} + q\mathbf{j} \,|\, 2\mathbf{A}_{\mathsf{gen}} \,|\, 2\mathbf{Id}_k) \bmod 2q$ and write $\mathbf{A} = (\mathbf{A}_1 \,|\, 2\mathbf{Id}_k)$
4: $\mathbf{s} = (1, \mathbf{s}_{\mathsf{gen}}, \mathbf{e}_{\mathsf{gen}})$
5: if $\sigma_{\max}(\mathsf{rot}(\mathbf{s}_{\mathsf{gen}})) > \gamma$, then restart
6: Return $\mathsf{sk} = \mathbf{s}$, $\mathsf{vk} = \mathbf{A}$

Sign(sk, M)

1: $\mathbf{y} \leftarrow U(\mathcal{B}_{(1/N)\mathcal{R},(k+\ell)}(B))$
2: $c = H(\mathsf{HighBits}^{\mathsf{hint}}_{2q}(\mathbf{A}\lfloor \mathbf{y} \rceil, \alpha), \mathsf{LSB}(\lfloor y_0 \rceil), M) \in \mathcal{R}_2$
3: $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2) = \mathbf{y} + (-1)^b c \cdot \mathbf{s}$ for $b \leftarrow U(\{0,1\})$
4: $\mathbf{h} = \mathsf{HighBits}^{\mathsf{hint}}_{2q}(\mathbf{A}\lfloor \mathbf{z} \rceil - qc\mathbf{j}, \alpha) - \mathsf{HighBits}^{\mathsf{hint}}_{2q}(\mathbf{A}_1 \lfloor \mathbf{z}_1 \rceil - qc\mathbf{j}, \alpha) \bmod^+ \frac{2(q-1)}{\alpha}$
5: if $\|\mathbf{z}\|_2 \geq B'$, then restart
6: if $\|2\mathbf{z} - \mathbf{y}\|_2 < B$, then restart with probability $1/2$
7: Return $\sigma = (\mathsf{Encode}(\mathsf{HighBits}(\lfloor \mathbf{z}_1 \rceil, a)), \mathsf{LowBits}(\lfloor \mathbf{z}_1 \rceil, a), \mathsf{Encode}(\mathbf{h}), c)$

Verify$(\mathsf{vk}, M, \sigma = (x, \mathbf{v}, h, c))$

1: $\tilde{\mathbf{z}}_1 = \mathsf{Decode}(x) \cdot a + \mathbf{v}$ and $\tilde{\mathbf{h}} = \mathsf{Decode}(h)$
2: $\mathbf{w} = \tilde{\mathbf{h}} + \mathsf{HighBits}^{\mathsf{hint}}_{2q}(\mathbf{A}_1 \tilde{\mathbf{z}}_1 - qc\mathbf{j}, \alpha) \bmod^+ \frac{2(q-1)}{\alpha}$
3: $w' = \mathsf{LSB}(\tilde{z}_0 - c)$
4: $\tilde{\mathbf{z}}_2 = \left[\mathbf{w} \cdot \alpha + w'\mathbf{j} - (\mathbf{A}_1 \tilde{z}_1 - qc\mathbf{j})\right]/2 \bmod^{\pm} q$
5: $\tilde{\mathbf{z}} = (\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2)$
6: Return $(\ c = H(\mathbf{w}, w', M)\ ) \wedge (\ \|\tilde{\mathbf{z}}\| < B''\ )$