Leveraging Discrete CKKS to Bootstrap in High Precision

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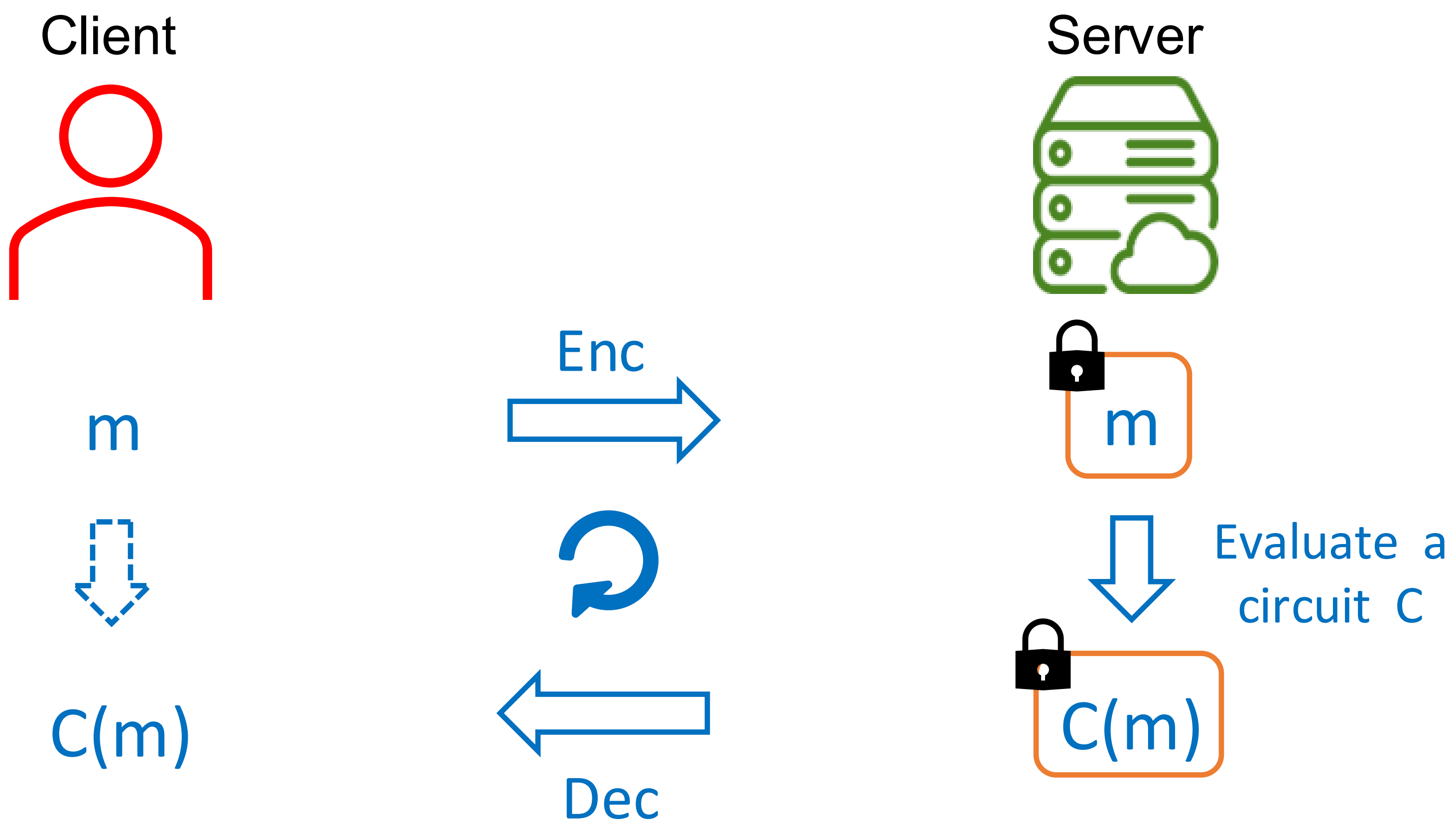
CRYPTO LAB

주식회사 크립토랩

Fully Homomorphic Encryption

- **FHE** enables computations on encrypted data without decryption.
  - Allows *Evaluation on Data* while keeping *Privacy*
  - One of the most popular Privacy Enhancing Technologies (PETs)



CKKS FHE Scheme

- **CKKS** FHE Scheme
  - Support arithmetic computation on $(\mathbb{C}, +, \times)$
    - $\mathrm{m} \in \mathbb{C}^N$ is encoded via scale factor Δ: $\mathrm{Ecd}_\Delta(\mathrm{m}) = \lceil \Delta \cdot \mathrm{iDFT}(\mathrm{m}) \rfloor \in \mathbb{Z}[x]/(x^N + 1)$

  ➔ Bounded-depth arbitrary computation

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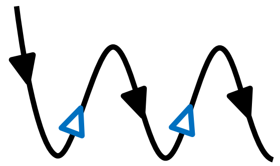
  ➔ Bounded-depth arbitrary computation

  - Arbitrary computation thanks to **Bootstrapping**
    - **Mult** consumes modulus Q for ciphertext
      - $\mathrm{Ecd}_\Delta(\mathrm{m}_1) \cdot \mathrm{Ecd}_\Delta(\mathrm{m}_2) \approx \Delta \cdot \mathrm{Ecd}_\Delta(\mathrm{m}_1 \odot \mathrm{m}_2)$: need $\times\ \Delta^{-1}$: modulus Q becomes Q/Δ
    - Bootstrapping refreshes Q and allows further computation
      - Usually, $\leq$ 20 bits of precision



Why High-Precision?

- For *IND-CPA^D Security, Circuit Privacy or Threshold Decryption*, we add *exponential noise* after evaluations (a.k.a. noise flooding):

   1) (High-Precision) CKKS computation,
   2) Decrypt,
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Then, only a few bits of the *correct message* remain:

Incorrect digit

	Decrypted: 3.1415926535897932**7**
+	Exponential Noise: 0.00150964880763503
	Result: 3.14**310230239742830**

Incorrect digits

Why High-Precision?

- For *IND-CPA^D Security, Circuit Privacy or Threshold Decryption*, we add *exponential noise* after evaluations (a.k.a. noise flooding):

  $\approx$ 64 bits for $\lambda = 128$[1]

  1) (High-Precision) CKKS computation, ➔ Need precision $\geq (64 + \alpha)$ bits
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	Decrypted: 3.14159265358979327
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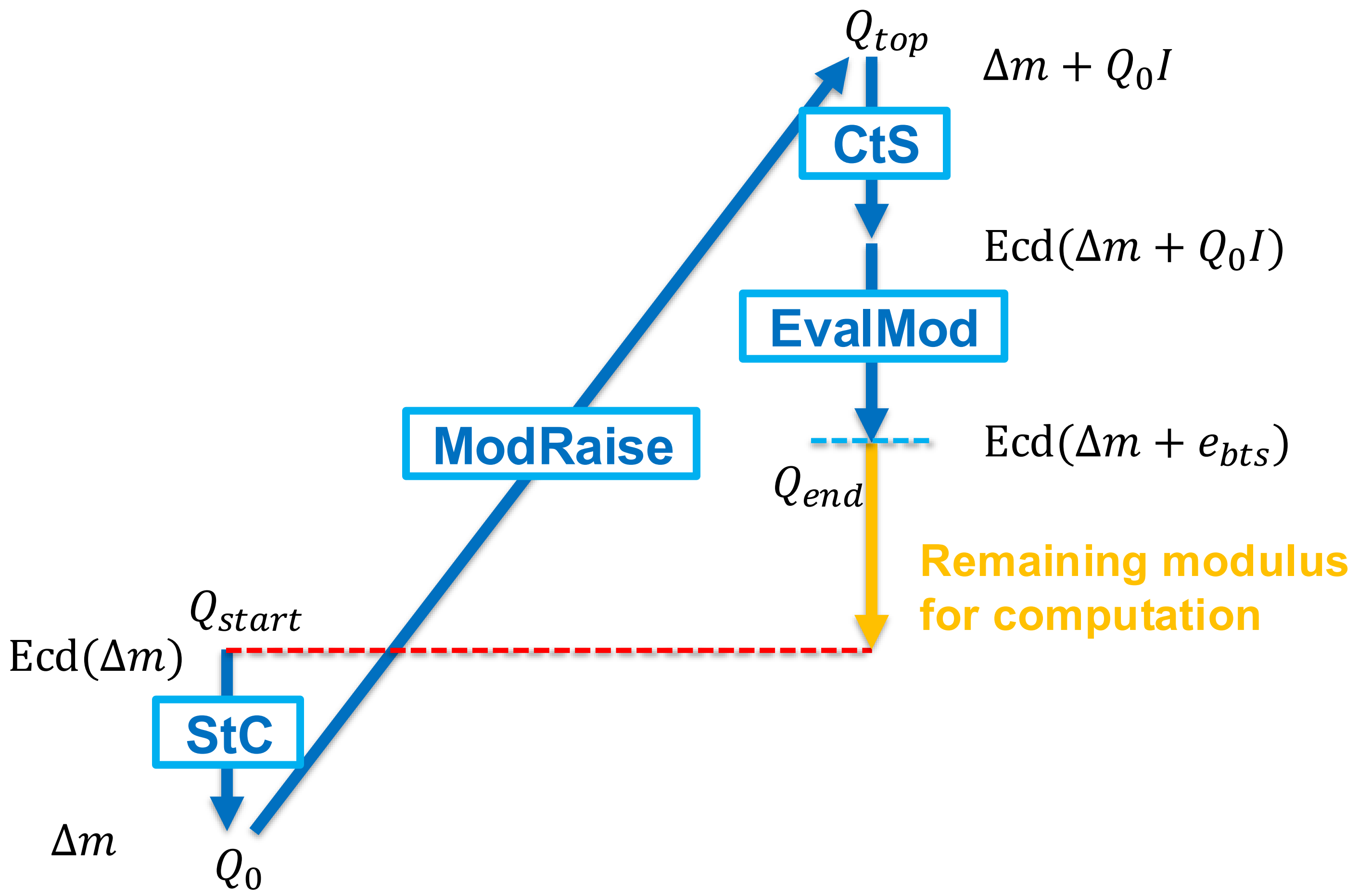
1. B. Li, D. Micciancio, M. Schultz-Wu, J. Sorrell, "Securing Approximate Homomorphic Encryption Using Differential Privacy," CRYPTO 2022.

Standard CKKS Bootstrapping

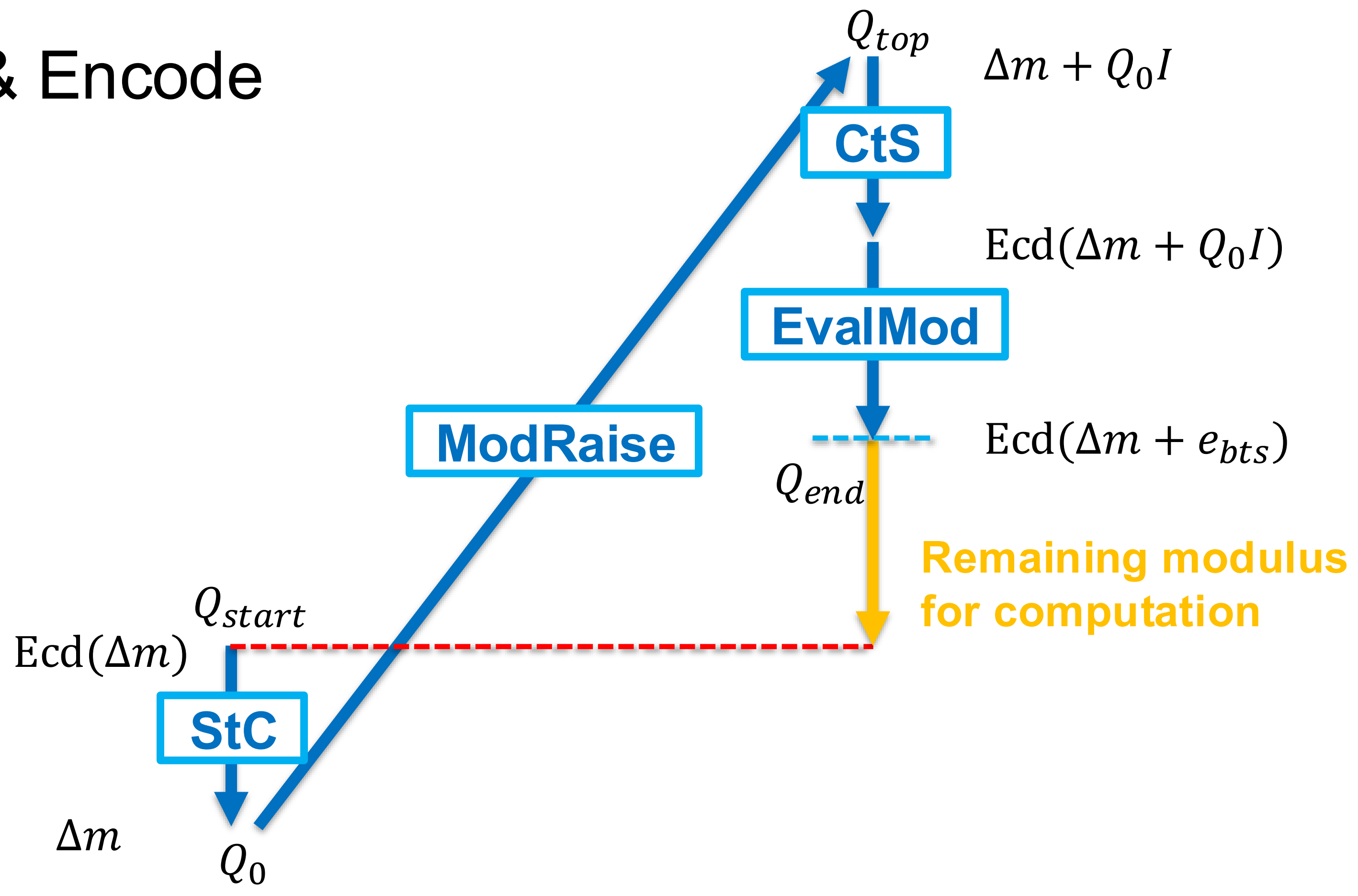
- **CKKS Bootstrapping:** Increase modulus for further computation



Bootstrapping Procedure

Standard CKKS Bootstrapping

- **CKKS Bootstrapping:** Increase modulus for further computation
  - **StC & CtS:** homomorphic Decode & Encode
  - **EvalMod:** homomorphic "mod Q_0"
    - Polynomial approx. of $\mathrm{x} \mapsto (x \; mod \; Q_0)$



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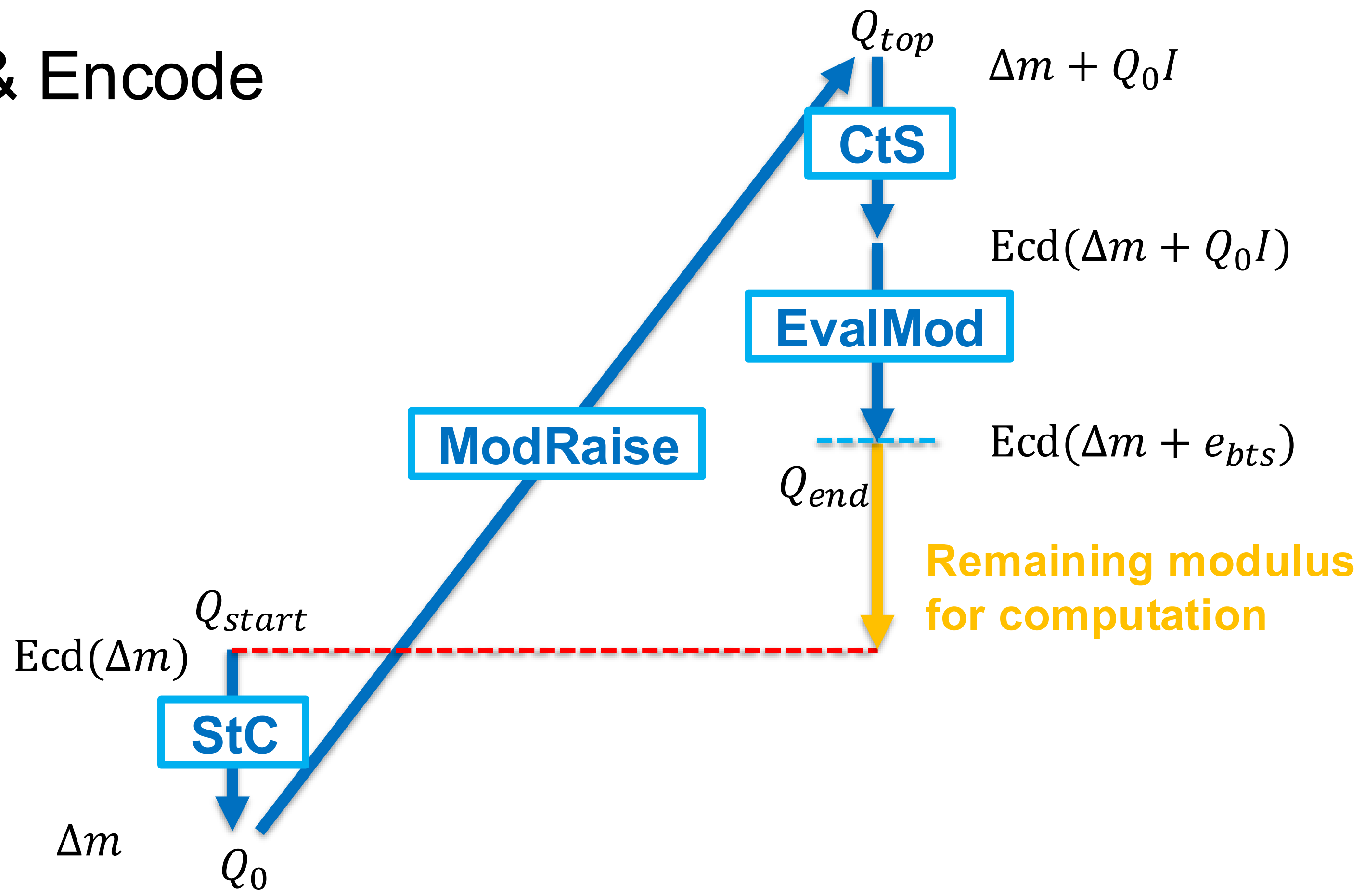

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  - For t-bit precision,
    - Polynomial degree $\sim \Theta(\log t)$
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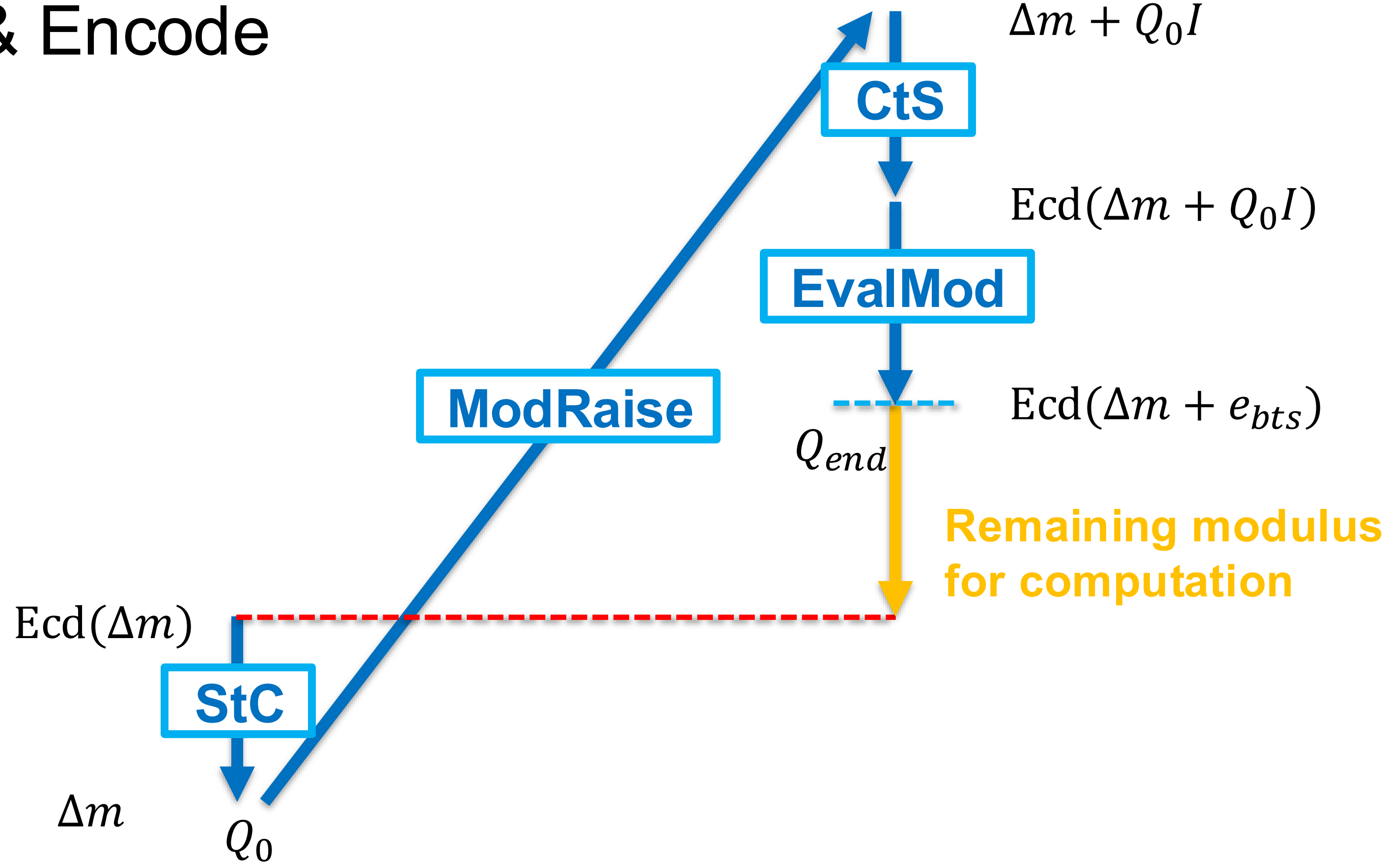

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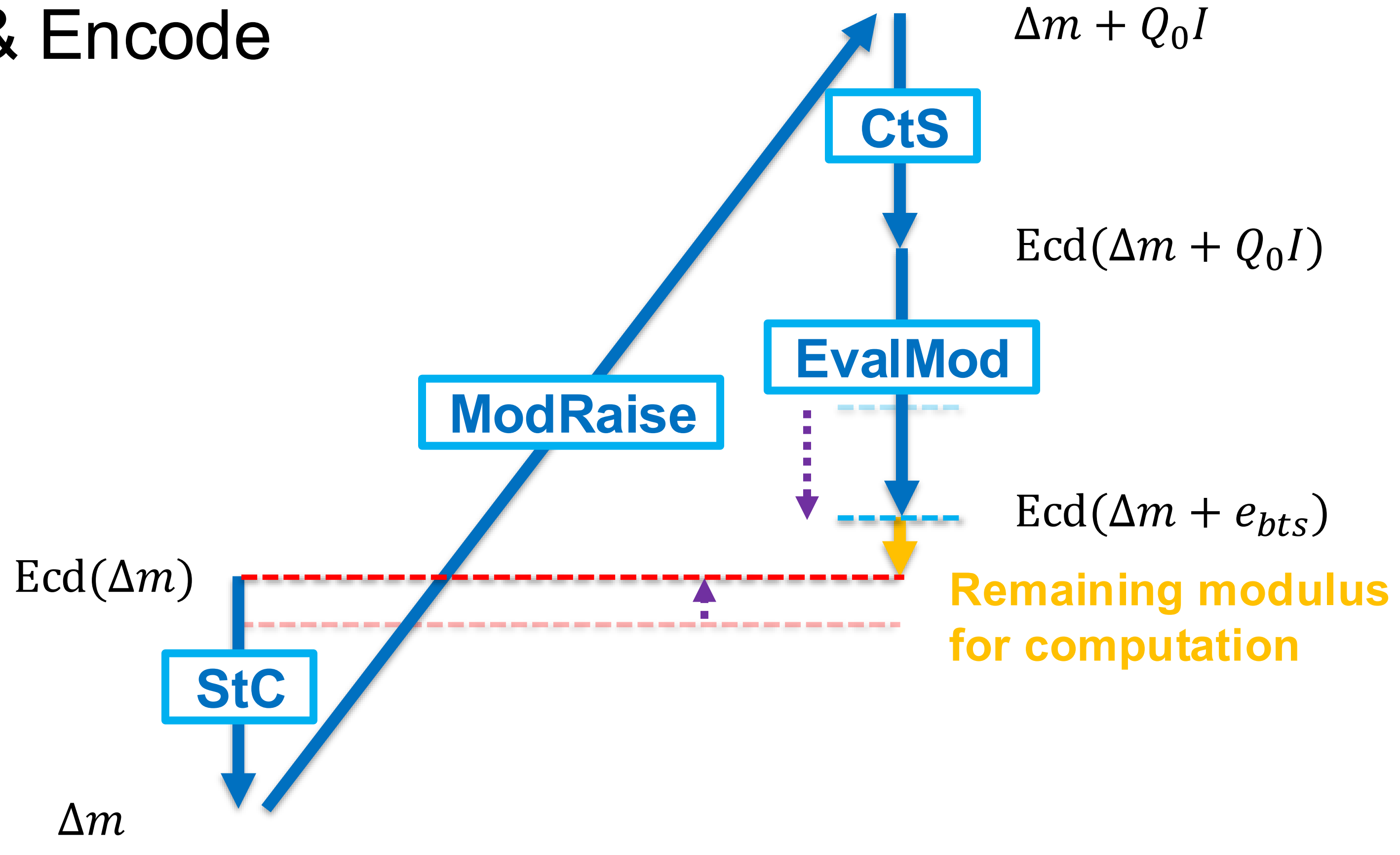

Bootstrapping Procedure w/ higher precision

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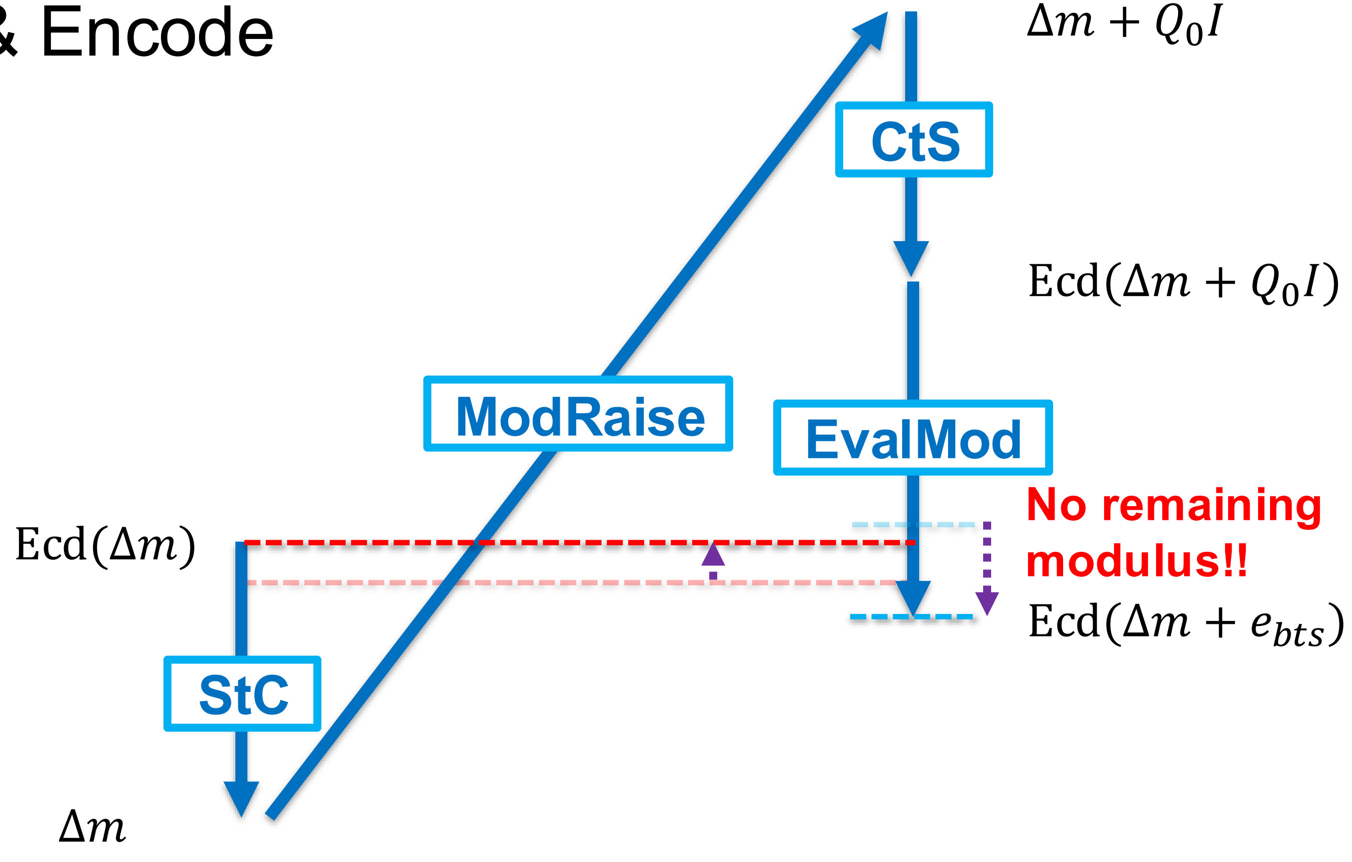
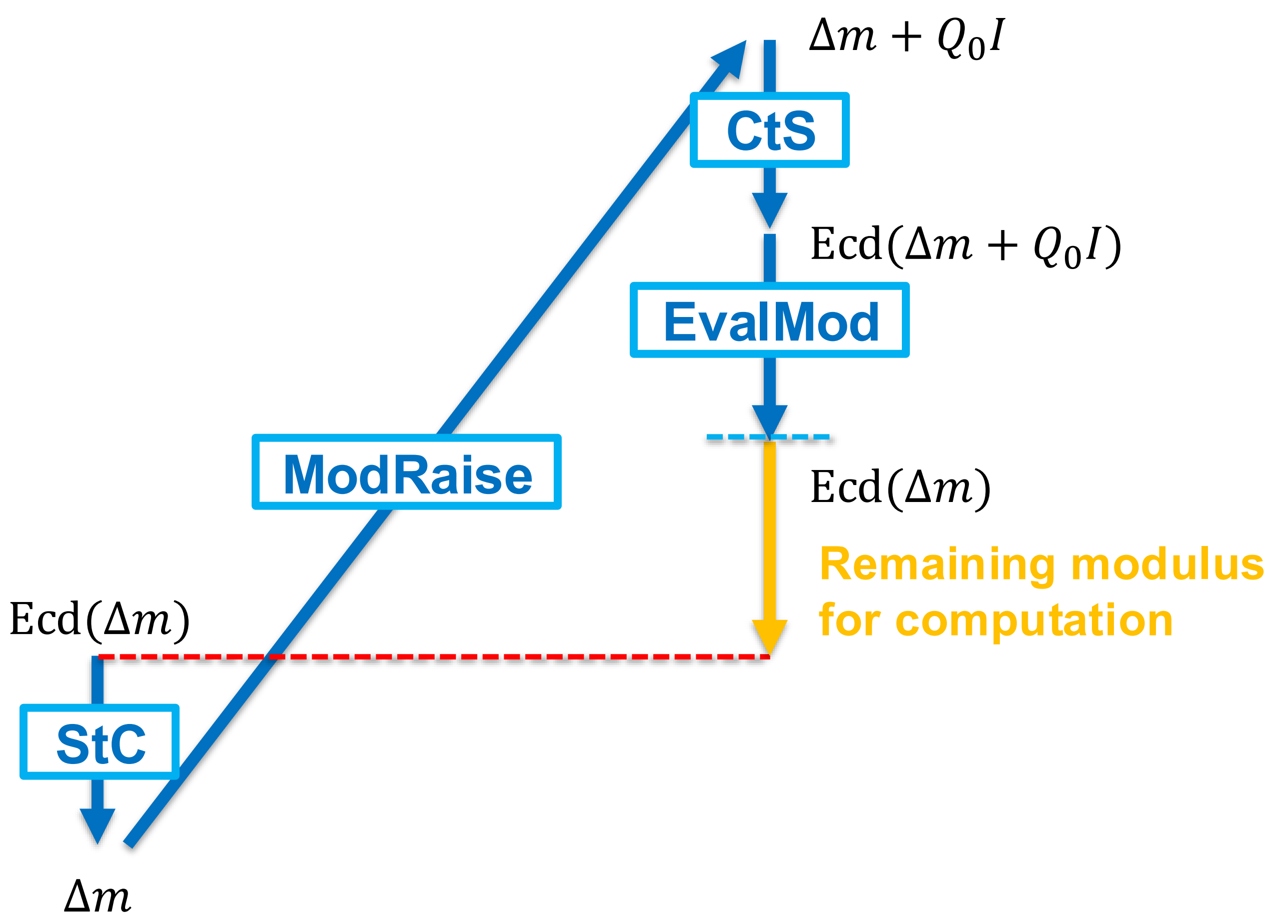


Bootstrapping Procedure w/ higher precision

CKKS Bootstrapping via Integer Cleaning

- **EvalRound**[2,3] instead of EvalMod
  - EvalMod: $x \mapsto (x \bmod Q_0)$

    $\Delta m + Q_0 I \mapsto \Delta m$



Bootstrapping Procedure

2. S. Kim, M. Park, J. Kim, T. Kim, and C. Min, “EvalRound Algorithm in CKKS Bootstrapping,” Asiacrypt 2022.
3. H. Sung, S. Seo, T. Kim, and C. Min, “EvalRound+ Bootstrapping and its Rigorous Analysis for CKKS Scheme, ” ePrint Archive 2024.

CKKS Bootstrapping via Integer Cleaning

- **EvalRound**[2,3] instead of EvalMod
  - EvalRound: $x \mapsto x - (x \bmod Q_0)$

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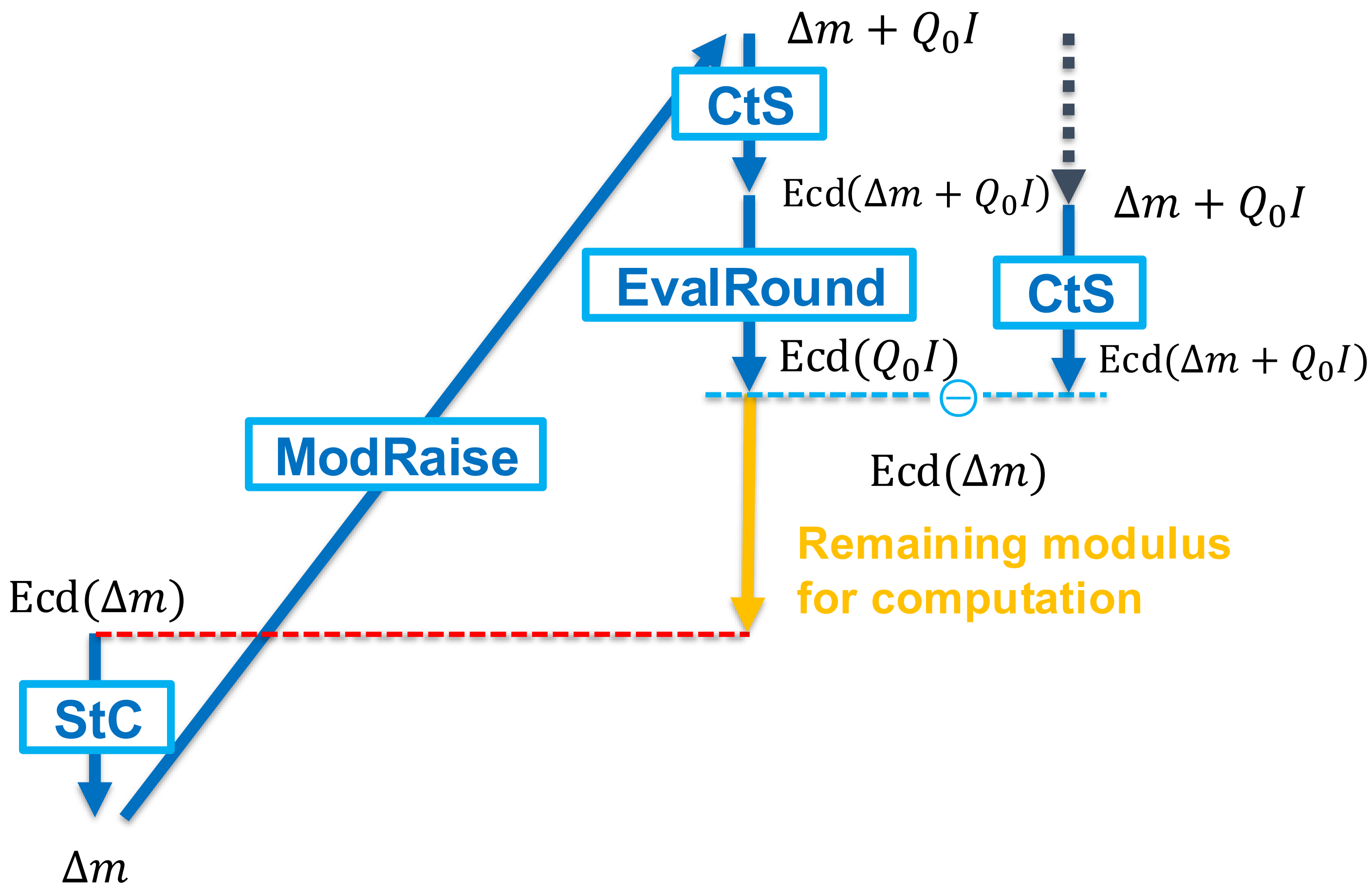

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CKKS Bootstrapping via Integer Cleaning

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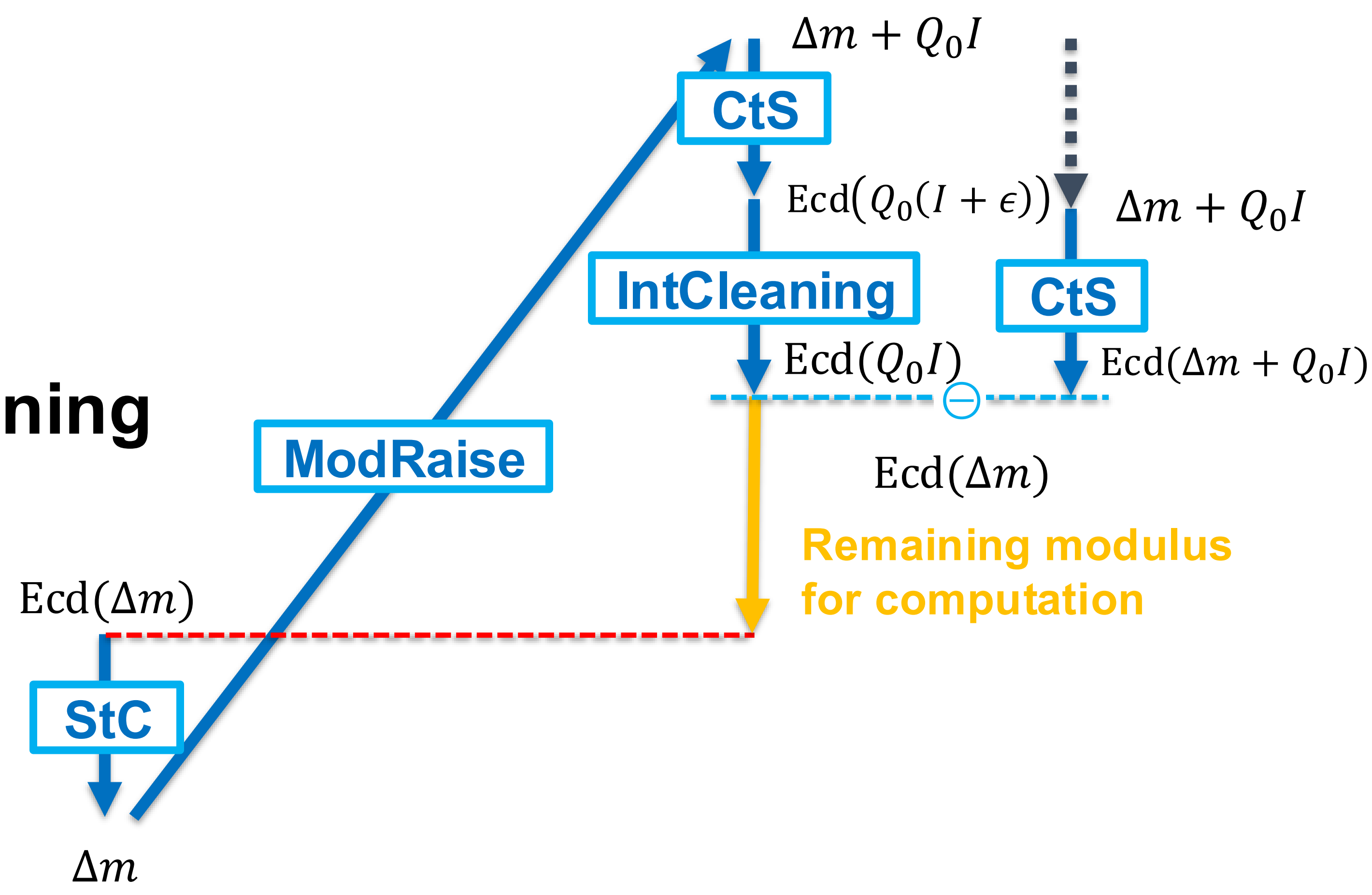
  - EvalRound: $x \mapsto x - (x \bmod Q_0)$

    $$\Delta m + Q_0 I \mapsto Q_0 I$$

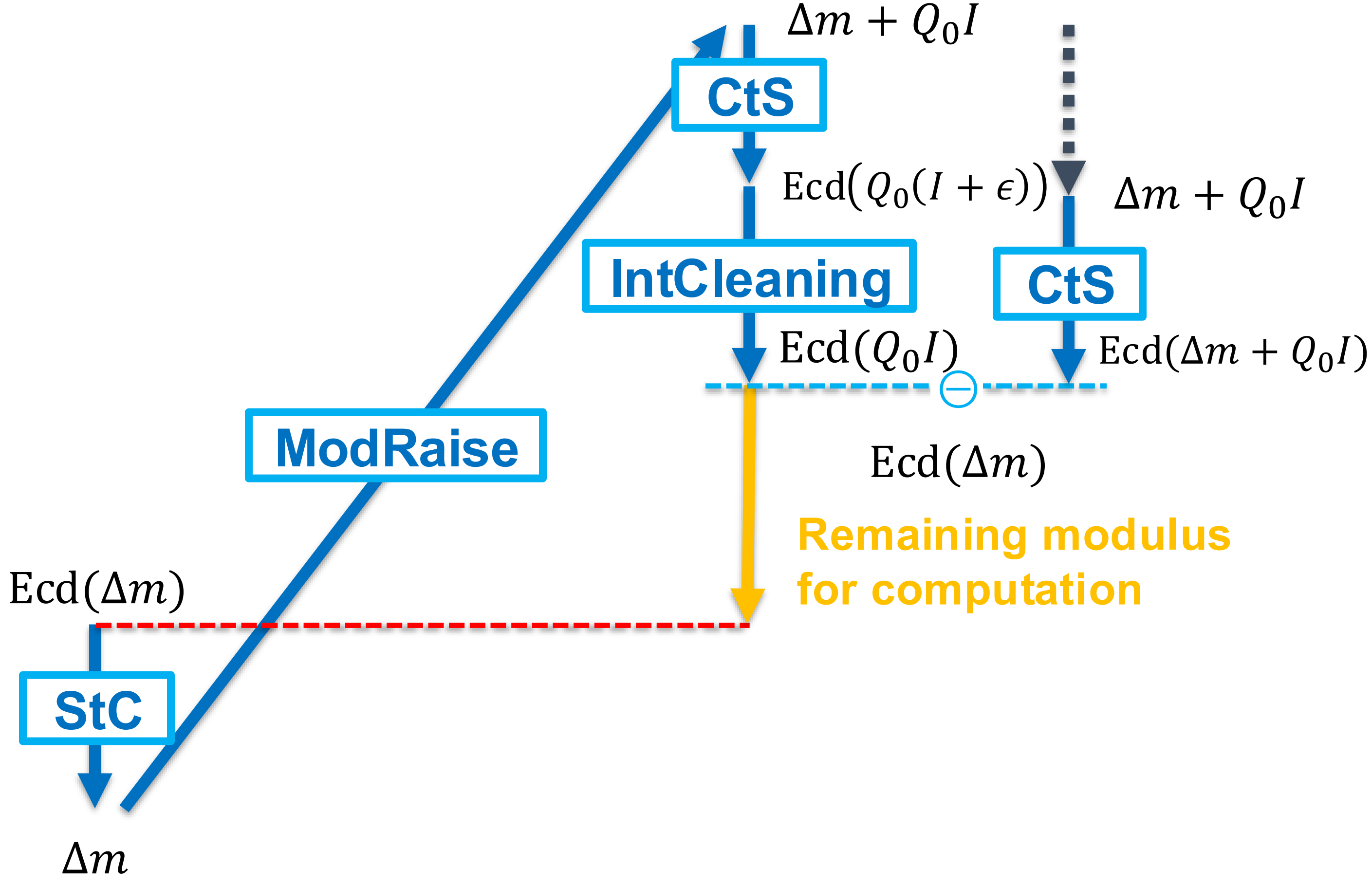
- One can regard this as **Integer Cleaning**
  - $Q_0(I+\epsilon) \mapsto Q_0 \cdot I$
    - I: integer, $\epsilon = \Delta m / Q_0$
    - Regard Q_0 as a scale factor
  - It cleans *erroneous integer* $I + \epsilon$.



Bootstrapping Procedure w/ EvalRound

High-precision CKKS Bootstrapping

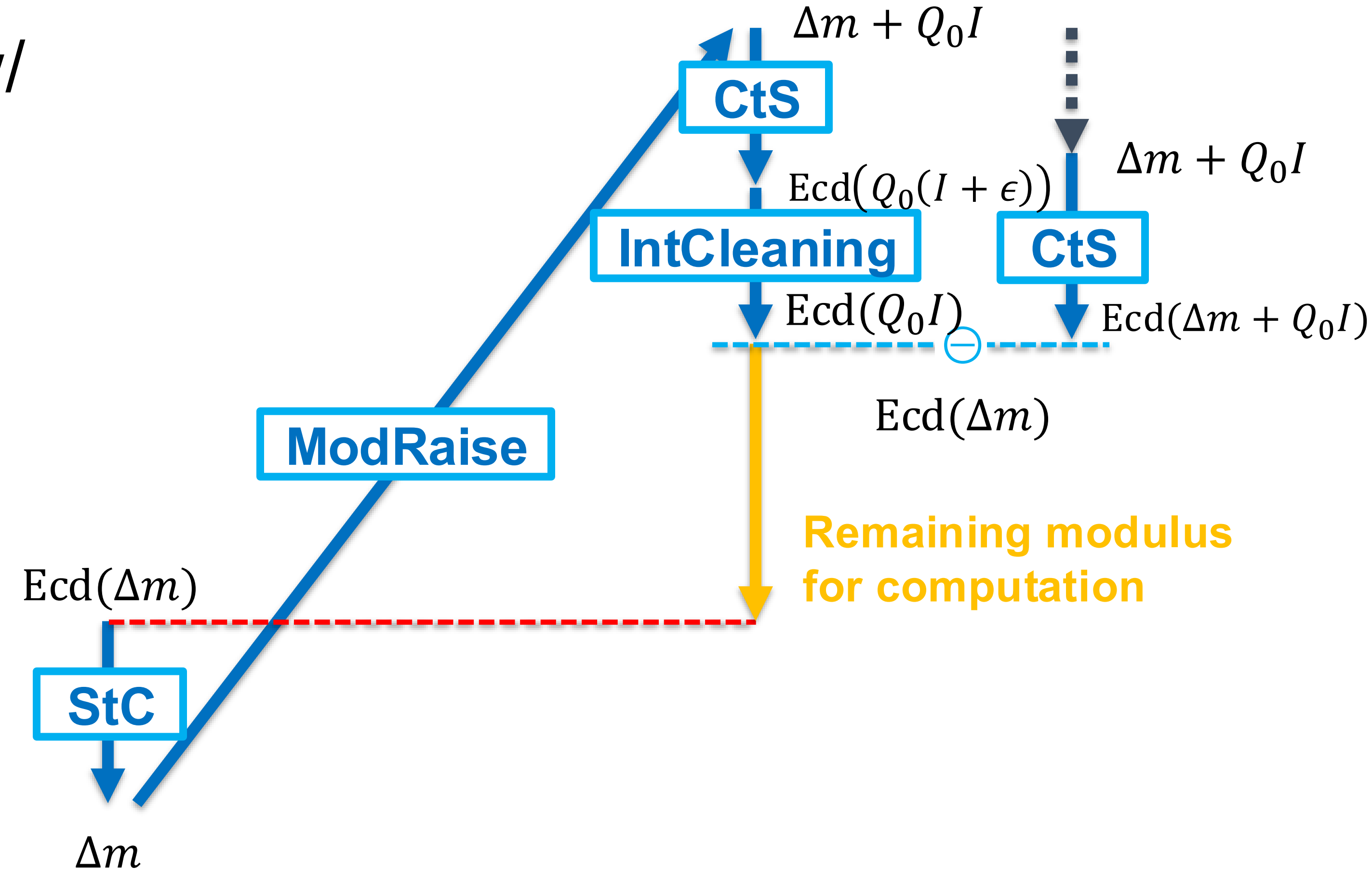
- We introduce new integer cleaning method a.k.a. **Iterative Integer Cleaning**



Bootstrapping Procedure w/ EvalRound

High-precision CKKS Bootstrapping

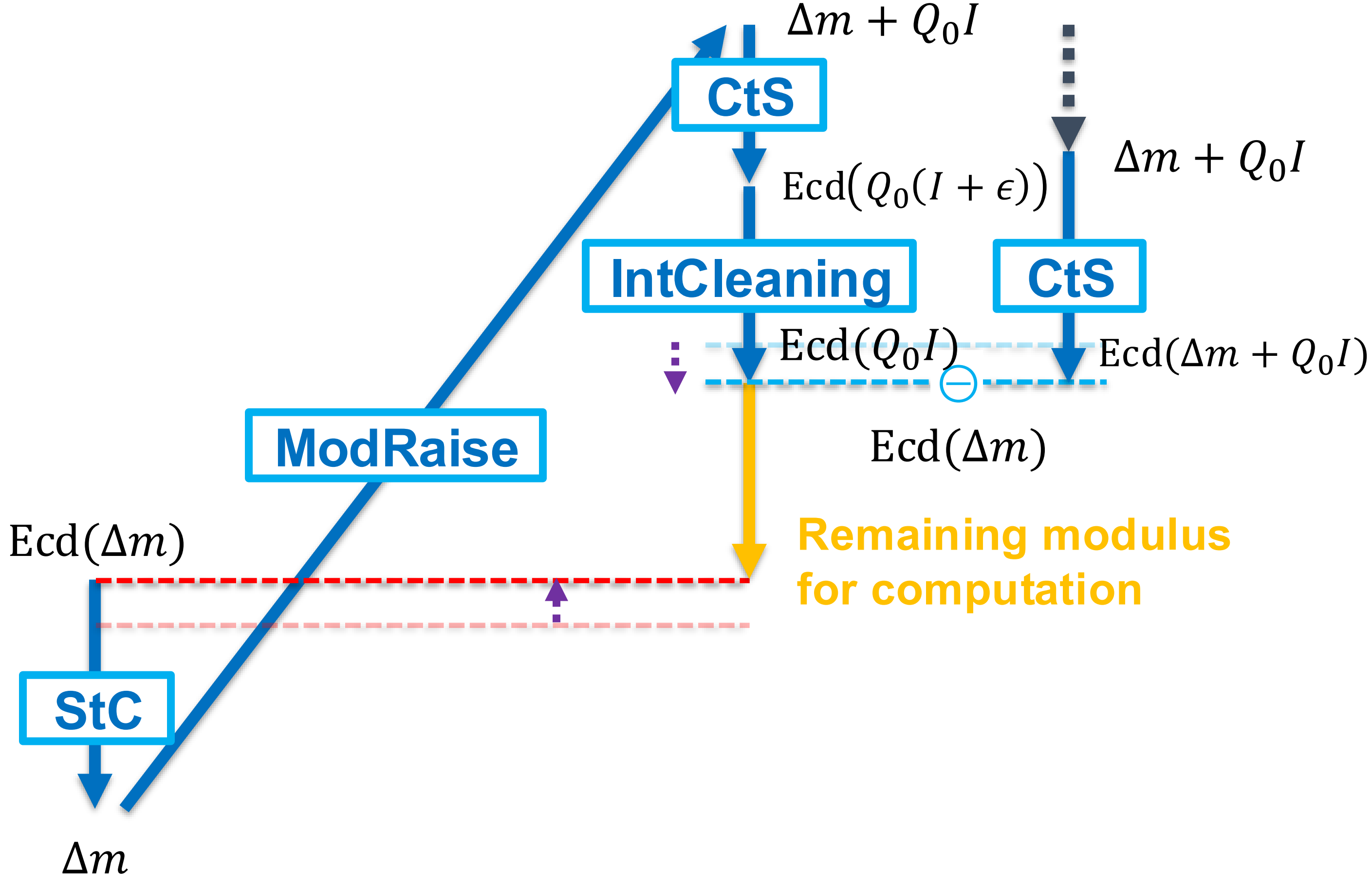
- We introduce new integer cleaning method a.k.a. **Iterative Integer Cleaning** w/
  - Low modulus consumption



Bootstrapping Procedure w/ EvalRound

High-precision CKKS Bootstrapping

- We introduce new integer cleaning method a.k.a. **Vectorized Cleaning** w/
  - Low modulus consumption
  - Low even for high precisions



Bootstrapping Procedure w/ EvalRound

Vectorized Cleaning

- For a digit-β representation $I = \sum_{\ell=0}^{\kappa} I_\ell \beta^\ell$,

Vectorized Cleaning

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  ① Digit Extraction

  - $I + \epsilon \mapsto \{I_\ell + \epsilon_\ell\}_{\ell=0..\kappa}$

$\boldsymbol{I+\epsilon}$ **: Noisy integer**

$\downarrow$ $\boxed{\text{Digit Extraction}}$ $\downarrow$

$\boldsymbol{I_\kappa+\epsilon_\kappa}$ $\cdots$ $\boldsymbol{I_0+\epsilon_0}$ **: Noisy digits**

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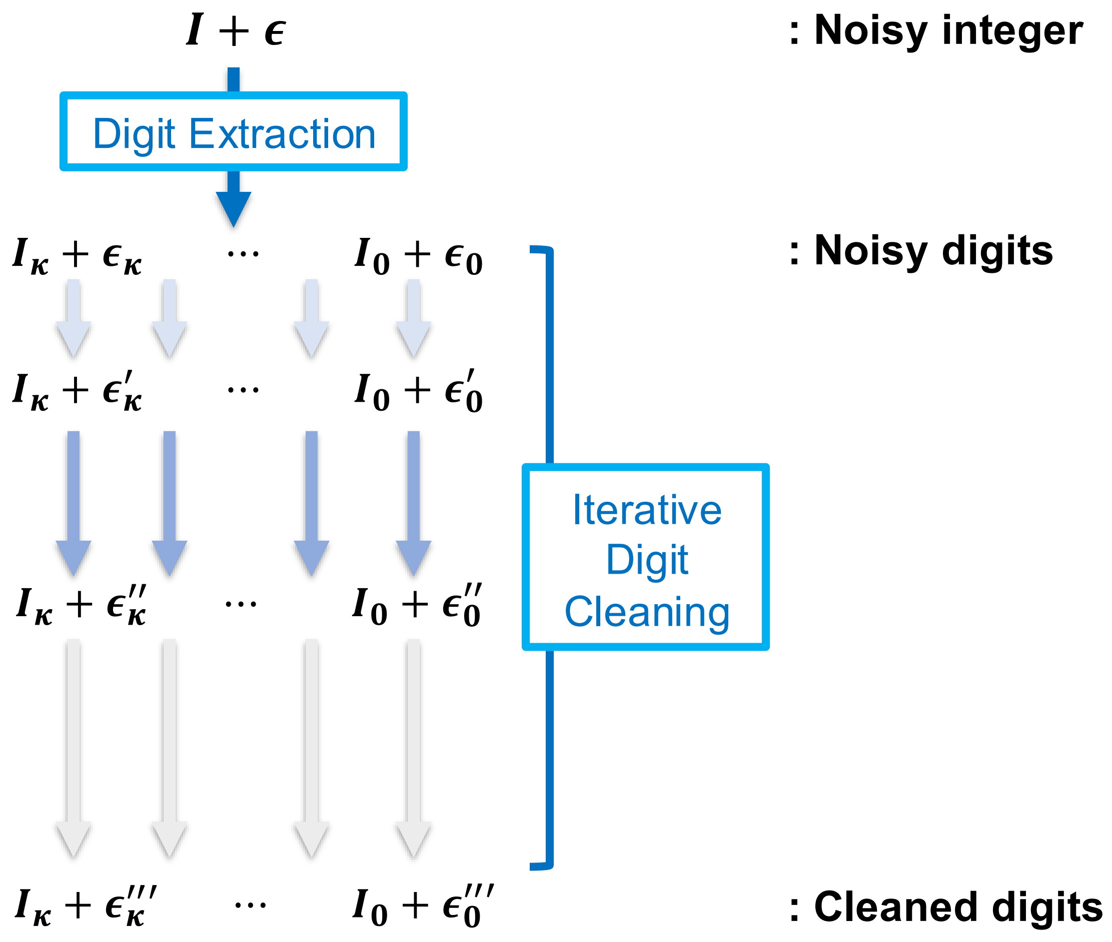
  ② Iterative Digit Cleaning

  - $I_\ell + \epsilon_\ell \;\mapsto\; I_\ell + \epsilon'_\ell \;\mapsto\; I_\ell + \epsilon''_\ell \;\;(\mapsto \cdots)$
    - $\epsilon''_\ell \ll \epsilon'_\ell \ll \epsilon_\ell$



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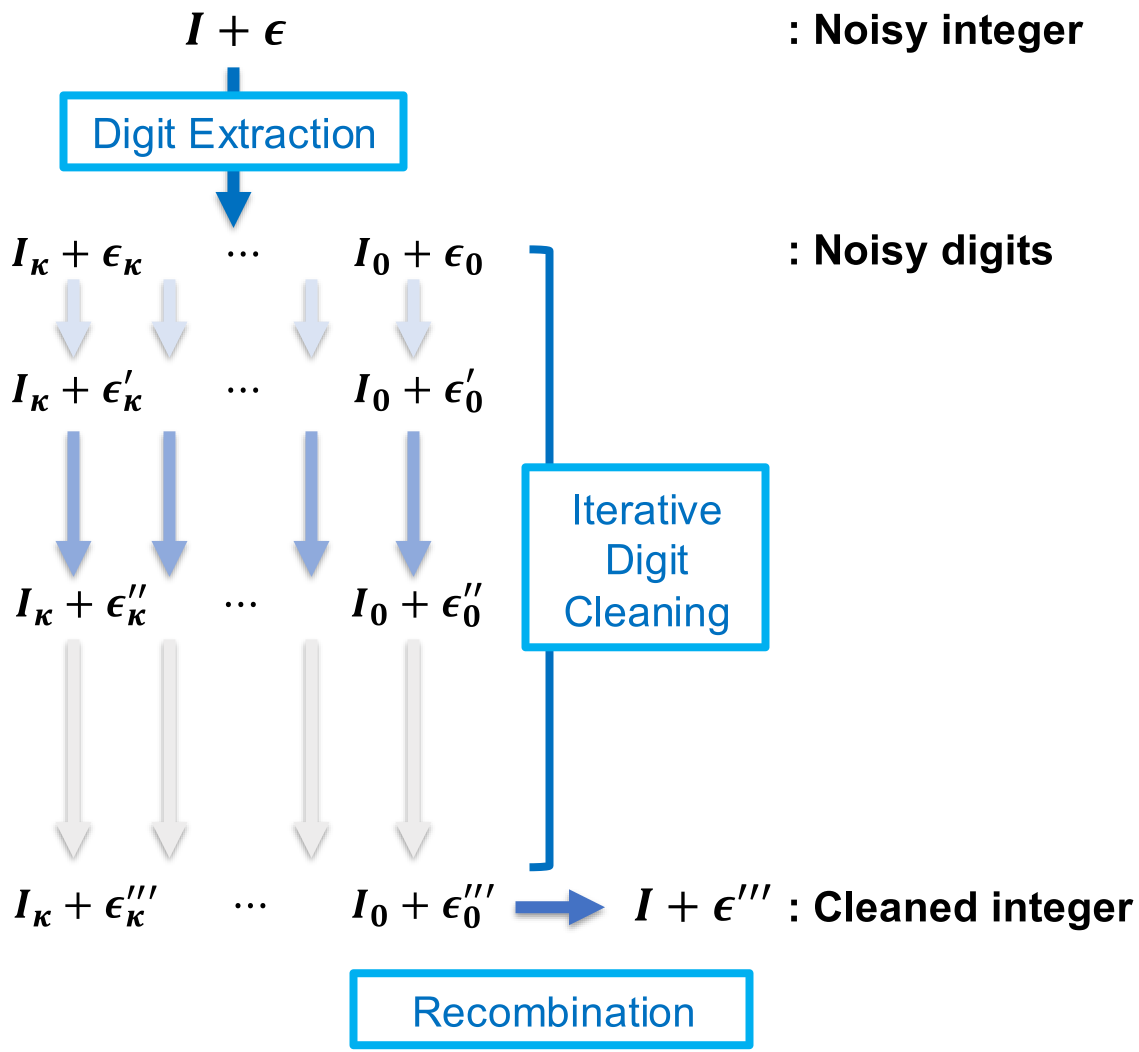
    - $\epsilon''_\ell \ll \epsilon'_\ell \ll \epsilon_\ell$

③ Recombination

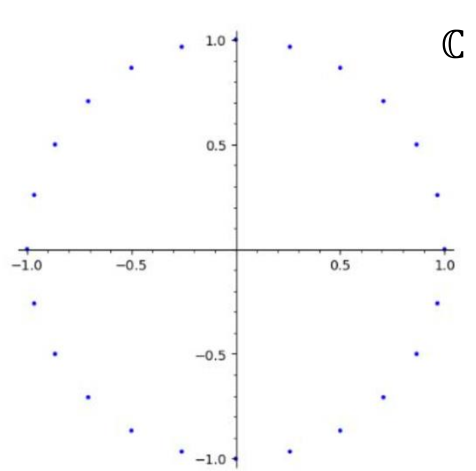
- $I + \epsilon''' = \sum_{\ell=0}^{\kappa}(I_\ell + \epsilon'''_\ell)\beta^\ell$
    - $\epsilon''' \ll \epsilon$

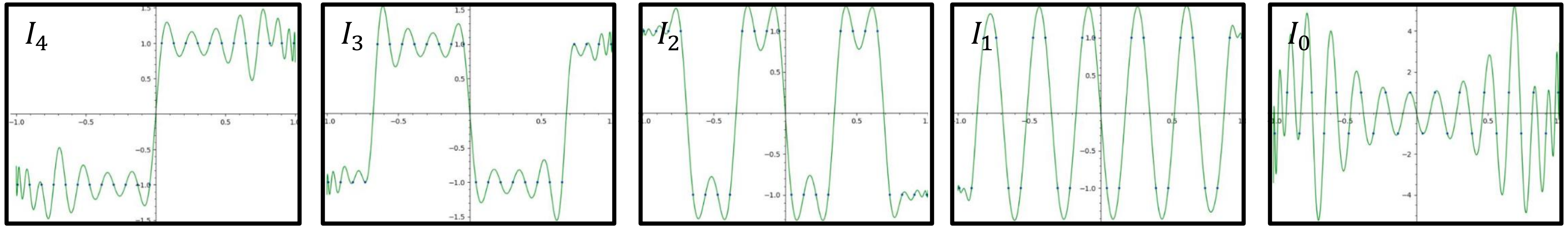


Vectorized Cleaning

Digit Extractions

- For $I \in [-K, K]$,
  - Integer → Roots of unity → Digits $\quad (j^2 = -1)$
    1. Approximate complex exponential $\mathrm{I} \mapsto e^{2\pi j \cdot I/(2K+1)}$
    2. Interpolate $e^{2\pi j \cdot I/(2K+1)} \mapsto I_\ell$ or $e^{2\pi j \cdot I_\ell/\beta}$ for all $0 \leq \ell \leq$





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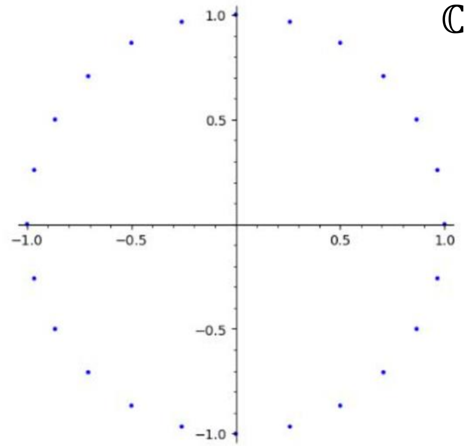

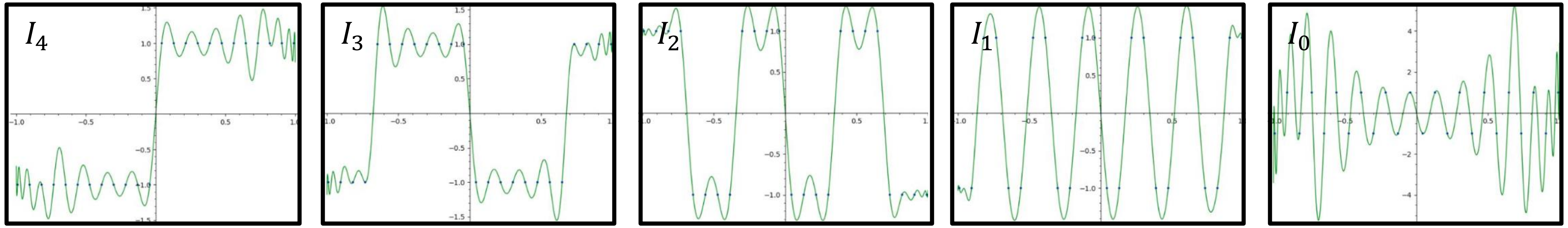

  - Or directly interpolate $I_\kappa I_{\kappa-1} \cdots I_1 I_{0(\beta)} \mapsto I_\ell$ or $e^{2\pi j \cdot I_\ell/\beta}$ for all $0 \le \ell \le \kappa$

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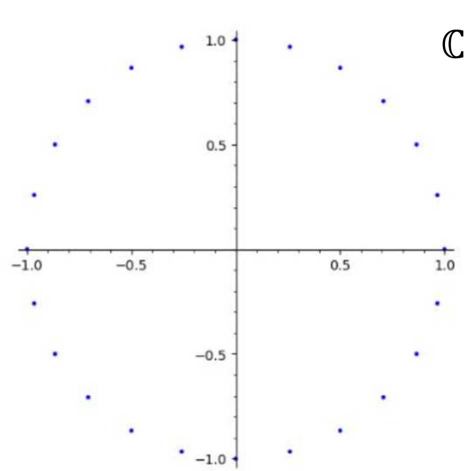
Better latency

$(j^2 = -1)$



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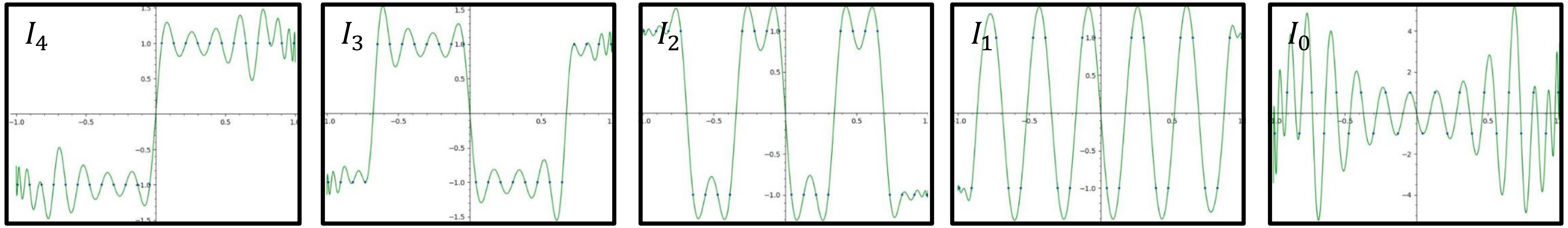

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Less modulus consumption

Vectorized Cleaning

Iterative Digit Cleaning

- For each digit, we iteratively apply:
  - Cleaning functions from BKSS24[4], CKKL24[5] maps $b + \epsilon \mapsto b + O(\epsilon^2)$,
    - $h_1(x) = 3x^2 - 2x^3$ for $b \in \{0, 1\}$ $(\beta = 2)$
    - $f_1(\mathrm{x}) = (\bar{x}^2 + 4x - 2x^2\bar{x})/3$ for $b \in \left\{e^{-2\pi j/3}, 1, e^{2\pi j/3}\right\}$ $(\beta = 3)$

4. Y. Bae, J. Kim, D. Stehlé, and E. Suvanto, “Bootstrapping Small Integers With CKKS,” Asiacrypt 2024.
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  - Start with $b + \epsilon$ w/ scale factor Δ

    → $b + O(\epsilon^2)$ w/ scale factor Δ^2

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    → $b + O(\epsilon^8)$ w/ scale factor Δ^8 …

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    → $b+O(\epsilon^8)$ w/ scale factor Δ^8 …

First cleaning consumes $4\log_2\Delta$ bits.
It doubles for later iterations... **quite a bit!**

We introduce **Thrifty approach** which consumes only $\log_2\Delta$ bits!

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Experiments

- We implemented for $\mathrm{N} = 2^{16}$ and failure probability $\leq 2^{-128}$.
- We used Grafting[6] implemented in C++.

Parameter set	# iter	Bit precision	Remaining modulus (bits)	Time (s)	Throughput (bits/s)
FGb*[7]	-	≈ 20	550	8.5	64.6
Ours	2		742	14.3	51.8

Low to High Precision CKKS Bootstrapping

6. J. H. Cheon, H. Choe, M. Kang, J. Kim, S. Kim, J. Mono, and T. Noh, "Grafting: Decoupled Scale Factors and Modulus in RNS-CKKS," ACM CCS 2025.
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$-\log_2|\mathrm{max\ error}|$ from 100 iterations

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2x FGb* (Meta-BTS[8])	-	≈ 40	530	17.0	31.1
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$\approx$ **15%** ↑

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4x FGb* (Meta-BTS[8])	-	≈ 80	490	34.0	14.4	$\approx$ 64% ↑
Ours	3		494	20.9	23.7	

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Thank You!

Blogpost

IACR ePrint: https://ia.cr/2025/1786

CKKS.org Blogpost: https://ckks.org/blog/2025/high_prec_bootstrap

Bonus – *Thrifty Approach*

- Notation:

$$ct(x,Q) \text{ implies } \langle \underbrace{ct}_{\in R_Q^2}, sk\rangle = x \mod Q, \qquad ct^{(\ell)}(x,Q) \text{ implies } \left\langle \underbrace{ct^{(\ell)}}_{\in R_Q^{\ell+1}}, \overbrace{sk \otimes \cdots \otimes sk}^{\ell\text{ - tensor}} \right\rangle = x \mod Q.$$

- Evaluate $h_1(x) = 3x^2 - 2x^3$: $ct(\Delta x, Q) \implies ct(\Delta^2 h_1(x), ?)$.
  - $x = b + \epsilon$ with $\epsilon = O(\Delta^{-1})$ → $h_1(x) + O(\Delta^{-2}) = b + O(\epsilon^2 + \Delta^{-2}) = b + O(\epsilon^2)$

Basic (a.k.a. Black-box)

1) $ct_1(\Delta^2 x, Q) = \Delta \cdot ct$
2) $ct_2(\Delta^2 x^2 + e, Q/\Delta^2) = \mathrm{Mult}(ct_1, ct_1)$
3) $ct_3(\Delta^2 x^3 + e', Q/\Delta^4) = \mathrm{Mult}(ct_1, ct_2)$
4) $ct_{res}(\Delta^2 h_1(x) + e'', Q/\Delta^4) = 3ct_2 - 2ct_3$

- All error e, e', e'' are $O(1)$.
- Δ should be an integer.

Bonus – *Thrifty Approach*

- Notation:

$$ct(x,Q) \text{ implies } \langle \overset{\in R_Q^2}{ct}, sk\rangle = x \mod Q, \qquad ct^{(\ell)}(x,Q) \text{ implies } \langle \overset{\in R_Q^{\ell+1}}{ct^{(\ell)}}, \overbrace{sk \otimes \cdots \otimes sk}^{\ell\text{ - tensor}}\rangle = x \mod Q.$$

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<u>Inverse Rescale Approach</u>

1) $ct_1(\Delta^2 x, \Delta Q) = \Delta \cdot ct$
2) $ct_2(\Delta^2 x^2 + e, Q/\Delta) = \text{Mult}(ct_1, ct_1)$
3) $ct_3(\Delta^2 x^3 + e', Q/\Delta^3) = \text{Mult}(ct_1, ct_2)$
4) $ct_{res}(\Delta^2 h_1(x) + e'', Q/\Delta^3) = 3ct_2 - 2ct_3$

$\langle ct, sk\rangle = \Delta x \mod Q$
$\to \langle \Delta \cdot ct, sk\rangle = \Delta^2 x \mod \Delta Q$

Bonus – *Thrifty Approach*

- Notation:

$$ct(x,Q) \text{ implies } \langle \underset{}{\overset{\in R_Q^2}{ct}}, sk\rangle = x \mod Q, \qquad ct^{(\ell)}(x,Q) \text{ implies } \langle \overset{\in R_Q^{\ell+1}}{ct^{(\ell)}}, \overbrace{sk \otimes \cdots \otimes sk}^{\ell\text{ - tensor}} \rangle = x \mod Q.$$

- Evaluate $h_1(x) = 3x^2 - 2x^3$: $ct(\Delta x, Q) \implies ct(\Delta^2 h_1(x), ?)$.
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Basic (a.k.a. Black-box)

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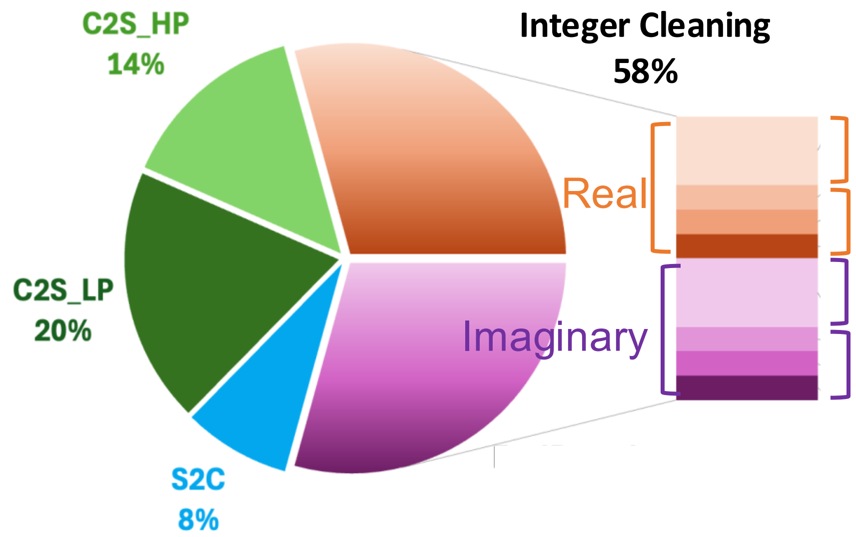
$\langle ct, sk\rangle = \Delta x \mod Q$
$\to \langle \Delta \cdot ct, sk\rangle = \Delta^2 x \mod \Delta Q$

Thrifty Approach

1) $ct_1^{(2)}(\Delta^2 x^2, Q) = ct \otimes ct$
2) $ct_2^{(2)}(\Delta^3 x^2, Q) = \Delta \cdot ct_1^{(2)}$
3) $ct_3^{(3)}(\Delta^3 x^3, Q) = ct \otimes ct \otimes ct$
4) $ct_4^{(3)}(\Delta^3 h_1(x), Q) = 3ct_2^{(2)} - 2ct_3^{(3)}$
5) $ct_{res}(\Delta^2 h_1(x) + e, Q/\Delta)$
$= \text{Rescale}(\text{ Relin}(\ ct_4^{(3)}\ )\ )$

Bonus – *Cheese et Bar*

- We implemented for $\mathrm{N} = 2^{16}$ and fail prob. $\leq 2^{-128}$.
- We used Grafting[6] implemented in C++.



Cheeses

Bonus – *Cheese et Bar*

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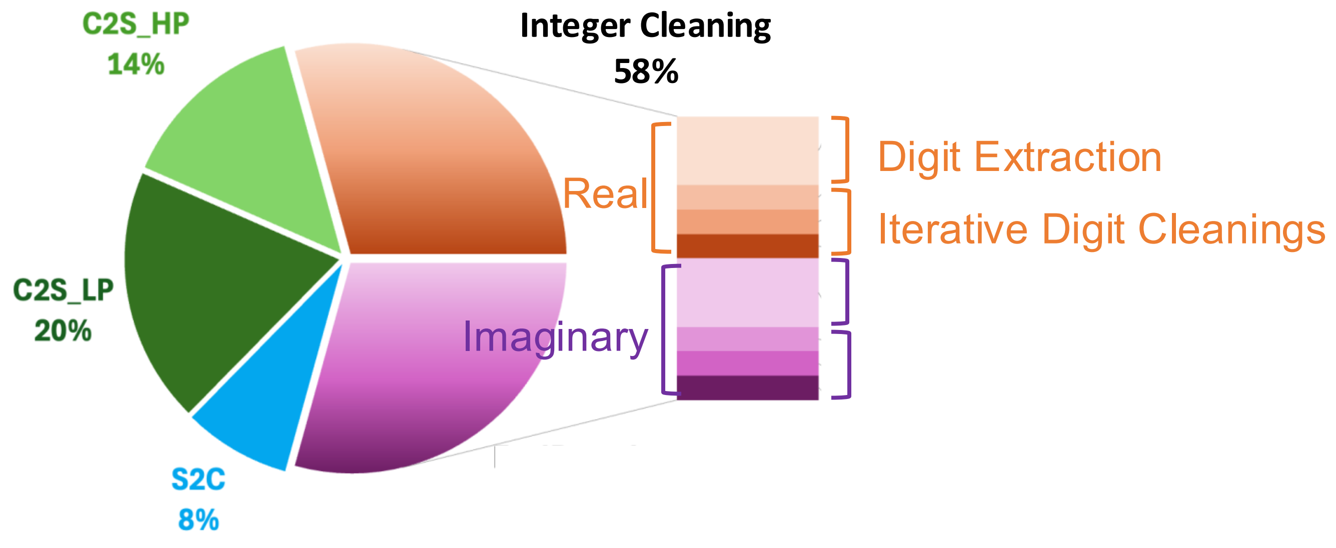
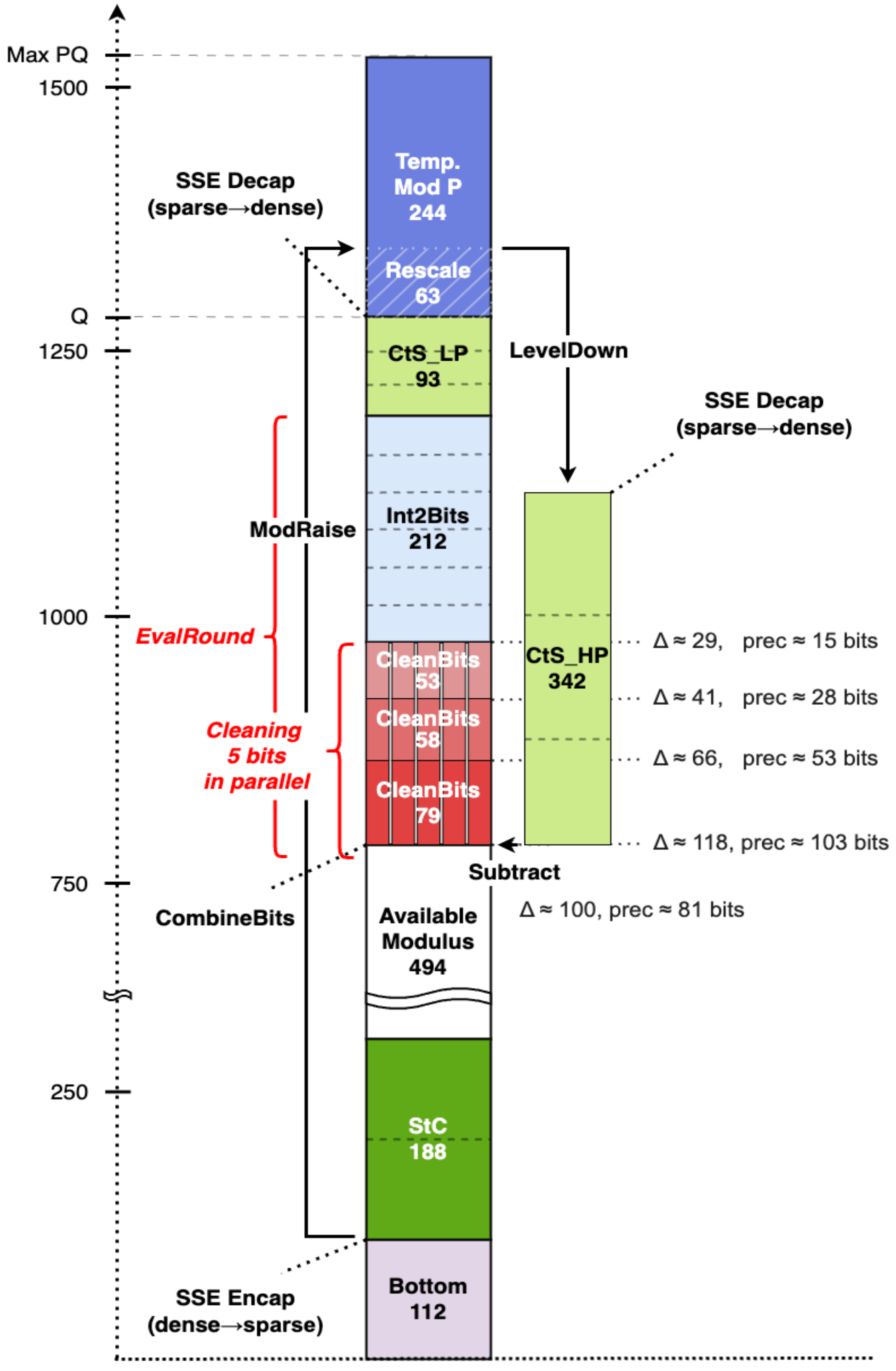


Cheeses



Bar

Bonus – *Cheese et Bar*

- We implemented for $\mathrm{N} = 2^{16}$ and fail prob. $\leq 2^{-128}$.
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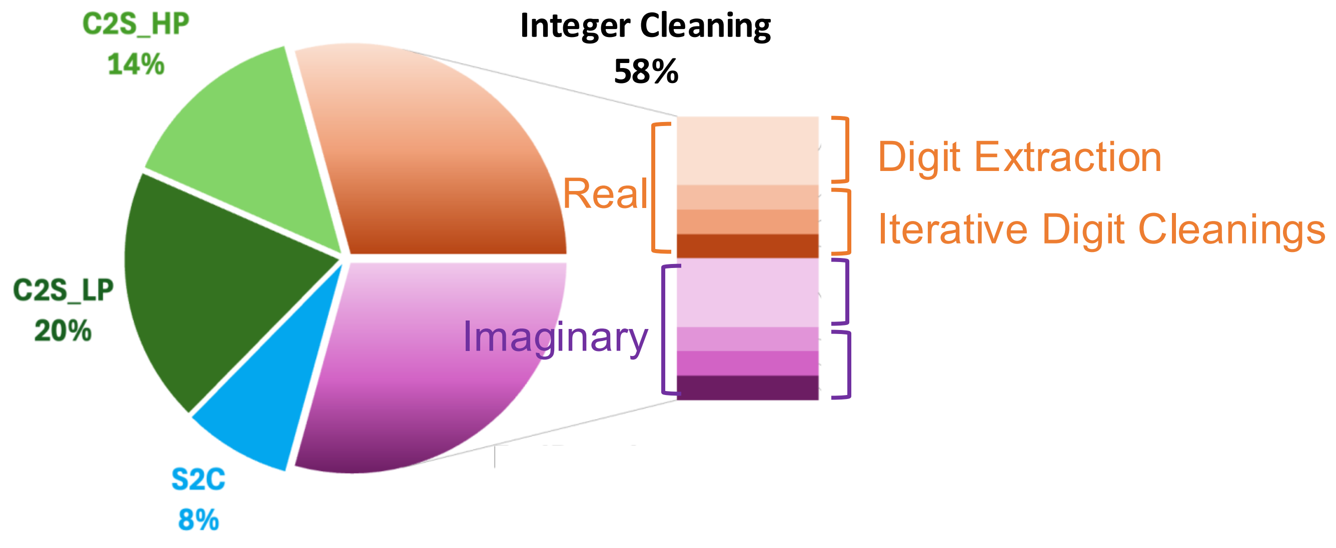
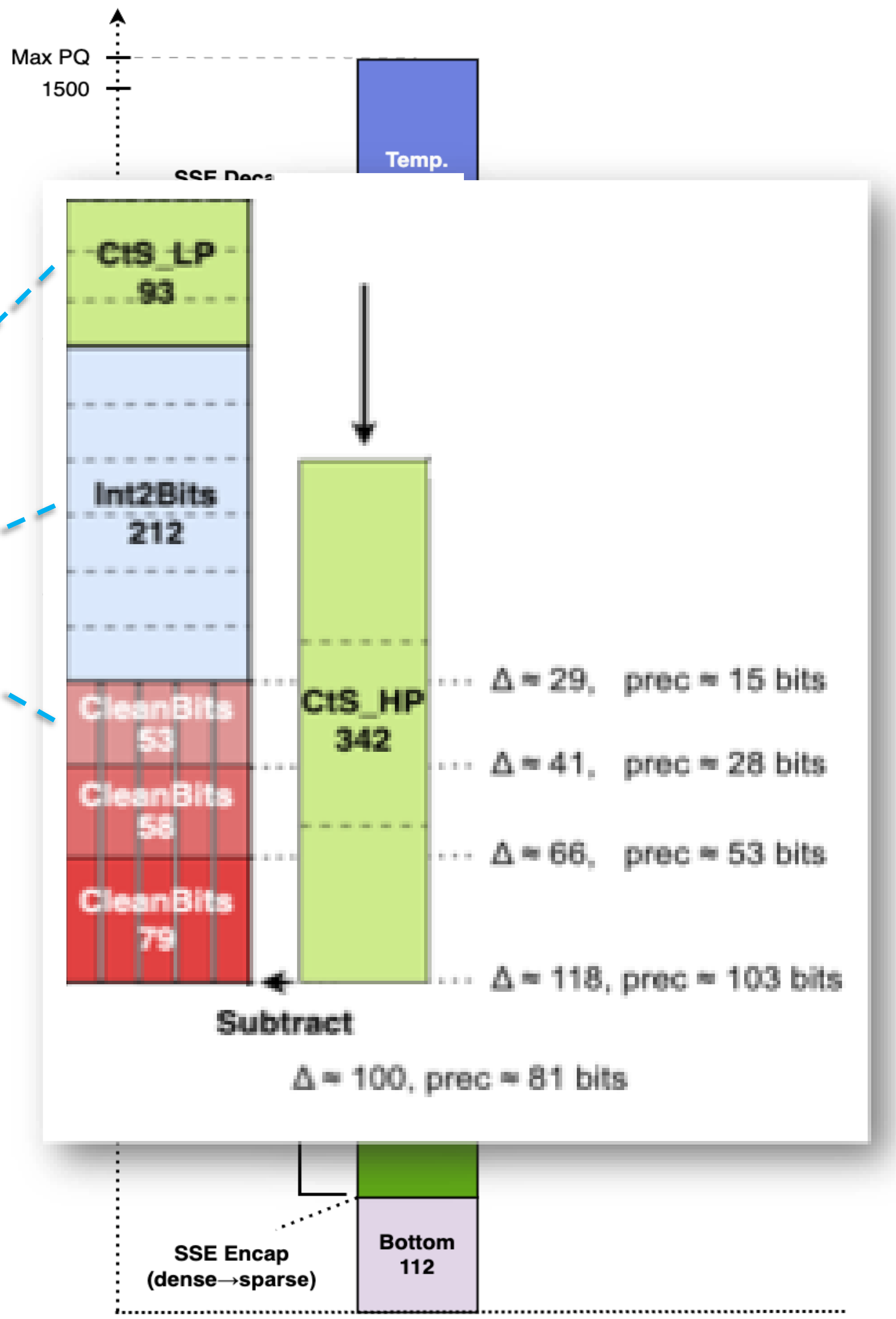
Scale factors of $\log_2 \Delta = 29{\sim}35$.

So, **Grafting** helps a lot !
Learn more in the next talk ☺



Cheeses



Bar