GRAFTING: DECOUPLED SCALE FACTORS AND MODULUS IN RNS-CKKS

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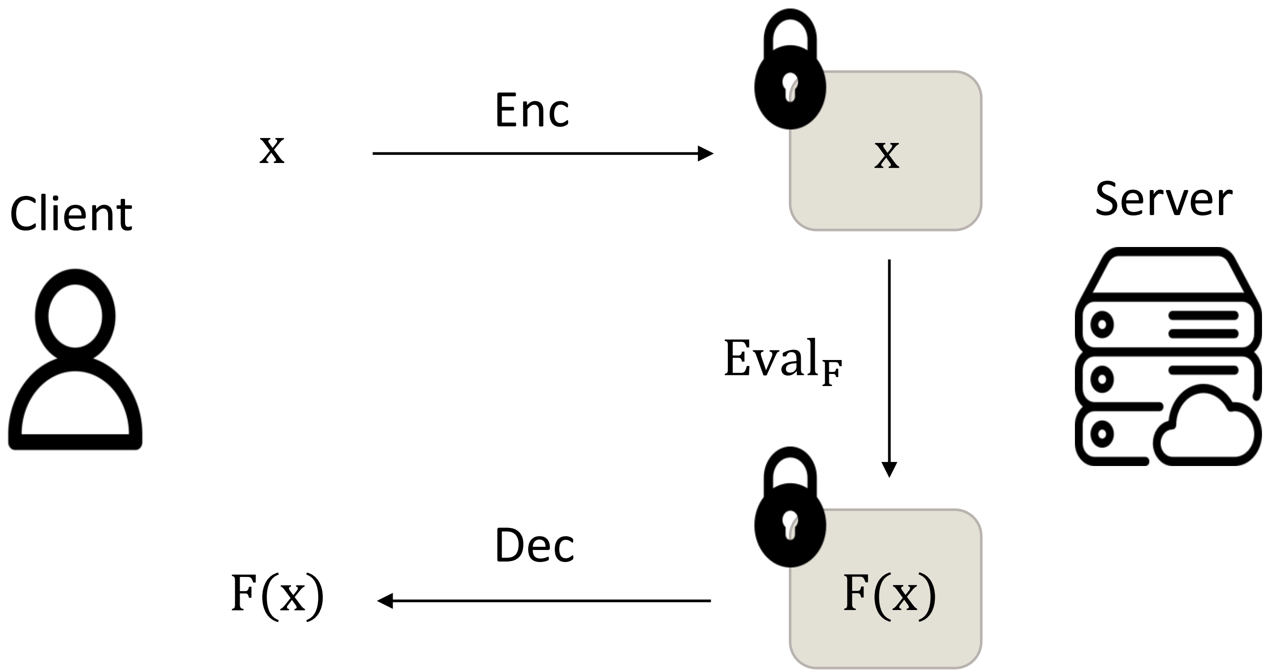
Seonghak Kim[2], Johannes Mono[2,4], Taeyeong Noh[2]

FULLY HOMOMORPHIC ENCRYPTION (FHE)



- FHE enables computations on encrypted data without decryption.
- Provides efficient privacy-preserving computation.
- CKKS supports approximate computations on real/complex numbers.

RNS-CKKS

- CKKS encodes $\vec{z} \in \mathbb{C}^{N/2}$ with scale factor Δ as:
  - Plaintext: $\Delta m = \lfloor \Delta \cdot DFT^{-1}(\vec{z}) \rceil \in R_Q = \mathbb{Z}_Q[X]/(X^N + 1)$.

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where s: secret, e: error, $\Delta m \ll Q$.

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- For modulus $Q = \prod_{i=0}^{\ell} q_i$, the CRT: $\mathbb{Z}_Q \cong \prod_{i=0}^{\ell} \mathbb{Z}_{q_i}$ allows

$$R_Q \cong R_{q_0} \times R_{q_1} \times \cdots \times R_{q_\ell}$$

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$\Rightarrow$ Filling Q with machine's word-size primes is the most efficient!

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- Rescale:

$$(c_0, c_1) = \left(\left\lfloor \frac{d_0}{q_\ell} \right\rceil, \left\lfloor \frac{d_1}{q_\ell} \right\rceil\right) \in R_{Q/q_\ell}^2, \quad c_0 + c_1 s \approx \frac{\Delta^2}{q_\ell} m_1 m_2 + \frac{\Delta}{q_\ell}(m_1 e_2 + m_2 e_1) \ \ \left(mod\ \frac{Q}{q_\ell}\right).$$

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∴ Hence, each modulus should match the scale factor:

$q_1 \approx \cdots \approx q_\ell \approx \Delta$ for multiplication levels ℓ.

STRUCTURE ON CKKS MODULUS CHAIN

- Modulus chain in RNS-CKKS is constructed as follows:

$$Q_0 \mid Q_1 \mid \cdots \mid Q_L,$$

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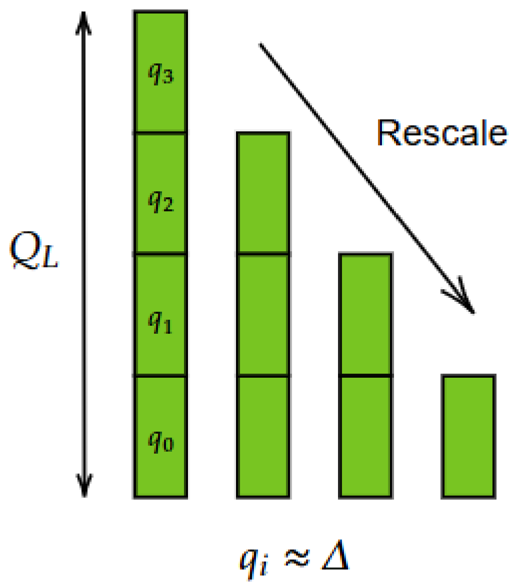

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  - The auxiliary modulus P allows key-switching at any level ℓ.
  - The main property enabling this is:

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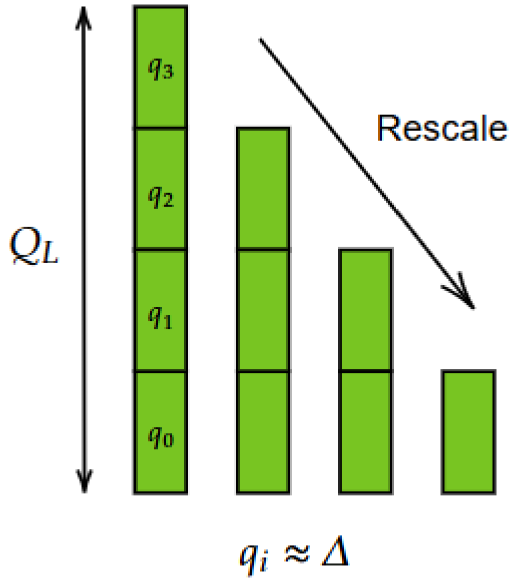

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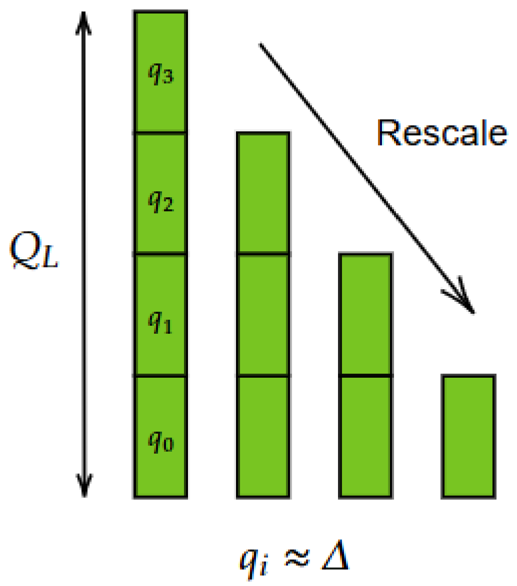

Why don't we set $Q_\ell \mid Q_L$ — not necessarily $Q_\ell \mid Q_{\ell+1}$,

while each Q_ℓ is filled up with machine's word-size primes?

GRAFTING:
A NOVEL MODULUS MANAGEMENT SYSTEM

RATIONAL RESCALE WITH SPROUT

- We set the top-modulus as $Q_{top} = q_0 q_1 \cdots q_{L-1} \cdot r_{top}$.
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  - r_{top} is called a *sprout,* reusable modulus factor of Q_{top}.

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⇒ We call it *Rational Rescale*, a generalized Rescale in RNS-CKKS.

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    $\Rightarrow$ Universal sprout, operating within 2 machine words.

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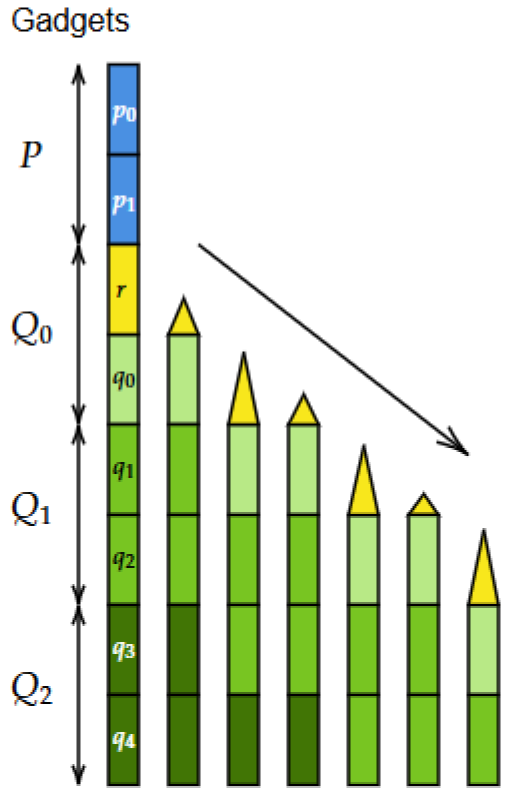

Modulus descending scenario during HE computations

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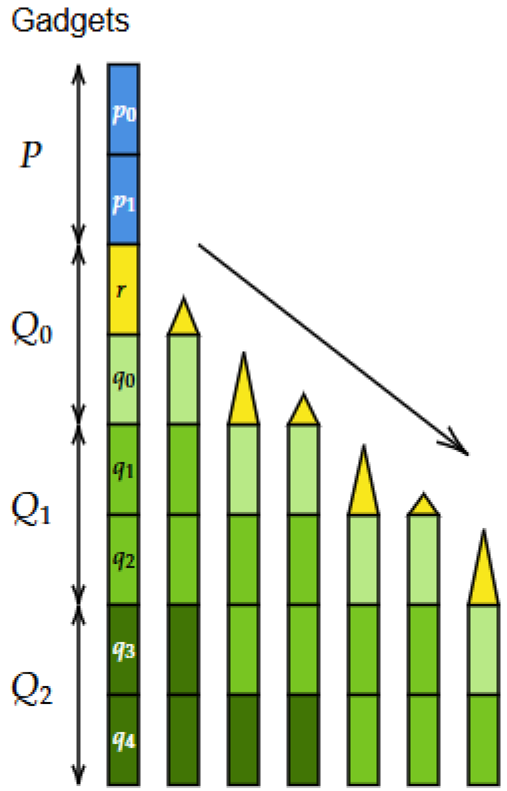

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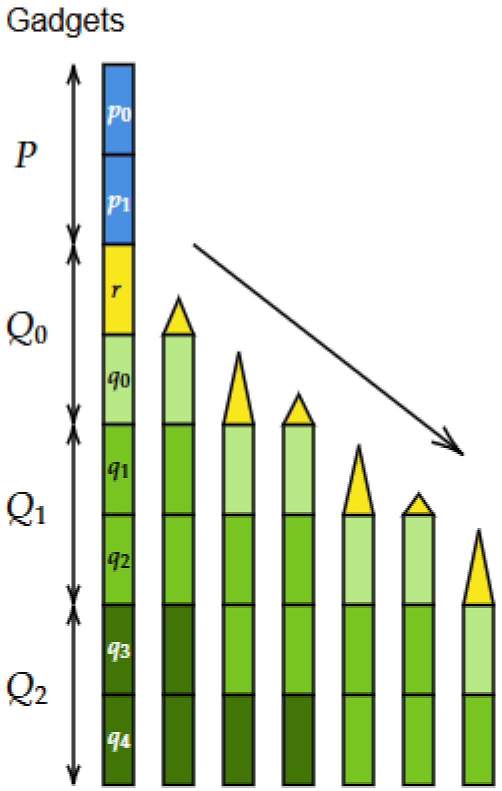

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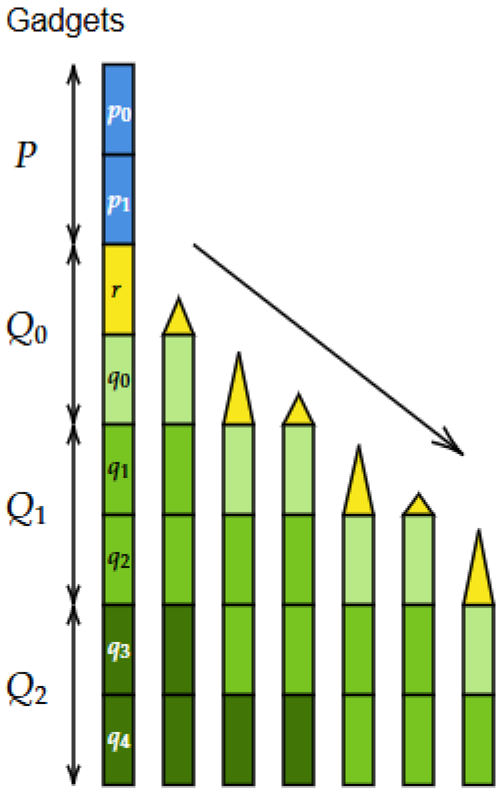
  - Some factors of r_{top} are resurrected during Rational Rescale.



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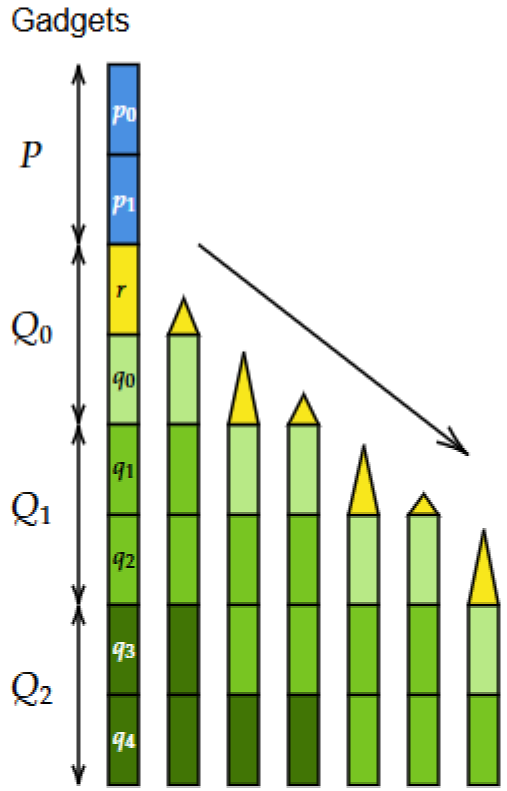
$\Rightarrow$ We call this process Modulus Resurrection.



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APPLICATION OF GRAFTING & EXPERIMENTAL RESULTS

WHEN APPLIED TO STANDARD CKKS

Parameter	N	log PQ	# factors	Mult (ms)	Boot (s)	Key size (MB)
HEaaN [Cry22]	2^{15}	777	22	102.20	14.5	115.34
		780	13	57.28 **(1.8x)**	7.6 **(1.9x)**	44.04 **(62% ↓)**

- CKKS parameters can benefit from fewer RNS factors by Grafting.
- Operations gain up to 2.0x speed-up, with key size reduced by up to 62%.

[Cry22] CrypoLab, HEaaN library, 2022.
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  - Parameters with small scale factors → accelerates well!
  - Parameters with large scale factors → not by much

[Cry22] CrypoLab, HEaaN library, 2022.
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WHEN APPLIED TO BIT-CKKS [BCKS24]

Parameter	N	log PQ	# factors	Mult (ms)	GateBoot (s)	Key size (MB)
Bit-CKKS [BCKS24]	2^{14}	424	14	16.1	5.18	47.71
		426	8	16.2 **(1.8x)**	2.74 **(1.9x)**	16.52 **(65% ↓)**
	2^{16}	1598	46	884.8	102.10	144.70
		1522	25	428.1 **(2.1x)**	56.41 **(1.8x)**	109.05 **(25% ↓)**

- Low-precision parameters or Bit/Discrete-CKKS can also benefit from fewer RNS factors!
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[BCKS24] Youngjin Bae et. al., Bootstrapping bits with CKKS, Eurocrypt 2024.

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- **Δ-Changeability**: Decoupled Δ and Q allows changing Δ freely during computation!

[CKK20] Jung Hee Cheon et. al., Efficient homomorphic comparison methods with optimal complexity, Asiacrypt 2020.

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    - ML training: FP16 ⇔ FP32
    - Iterative solvers: FP8 → FP16 → FP32
  - In **Encrypted World**, Grafting allows changing precision by changing **Δ**
    - E.g., Iterative solver with $\boldsymbol{\Delta}_0 \ll \boldsymbol{\Delta}_1 \ll \boldsymbol{\Delta}_2$
      - For each $x_{i+1} = f(x_i)$, we can use fine-tuned $\boldsymbol{\Delta}_i$
      - Mod consumption: $3\Delta_2 \cdot depth_f \ \rightarrow (\Delta_0 + \Delta_1 + \Delta_2) \cdot depth_f$

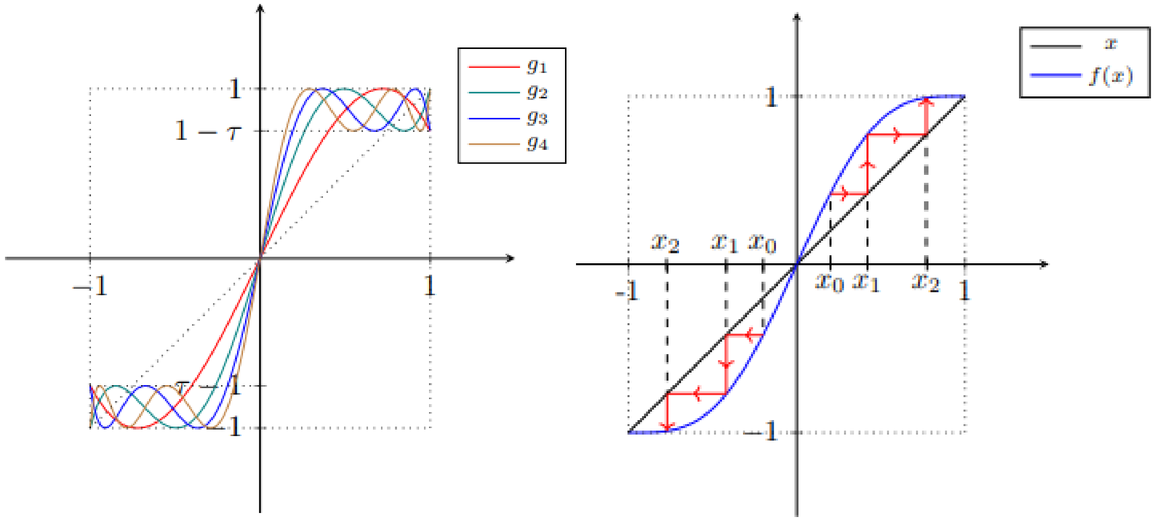
$(depth_f = \lfloor \log_2 \deg f + 1 \rfloor\,)$

[CKK20] Jung Hee Cheon et. al., Efficient homomorphic comparison methods with optimal complexity, Asiacrypt 2020.

APPLICATION TO HOMOMORPHIC COMPARISON [CKK20]

- Homomorphic Comparison [CKK20] use iterative method:
  - Evaluate $f^k \circ g^\ell : I_\epsilon \to I_{1-2^{-\alpha}}$,
    - for low-degree poly f (and g),
    - Iteratively narrowing the interval.

$(I_\epsilon := [-1, -\epsilon] \cup [\epsilon, 1])$



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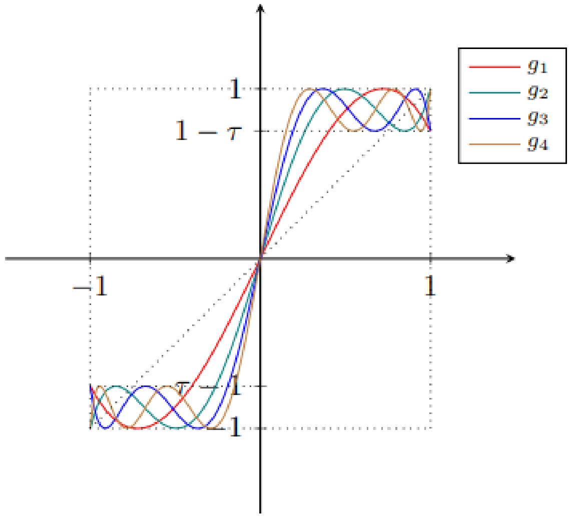

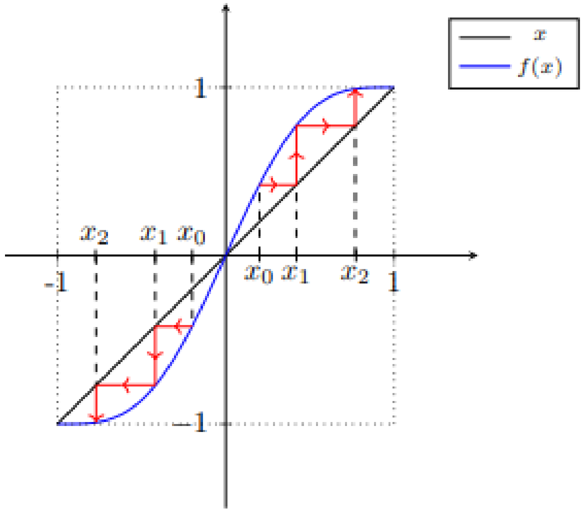

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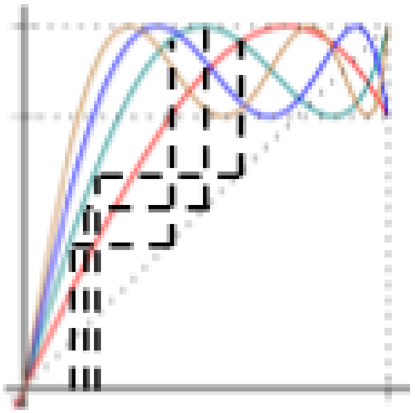

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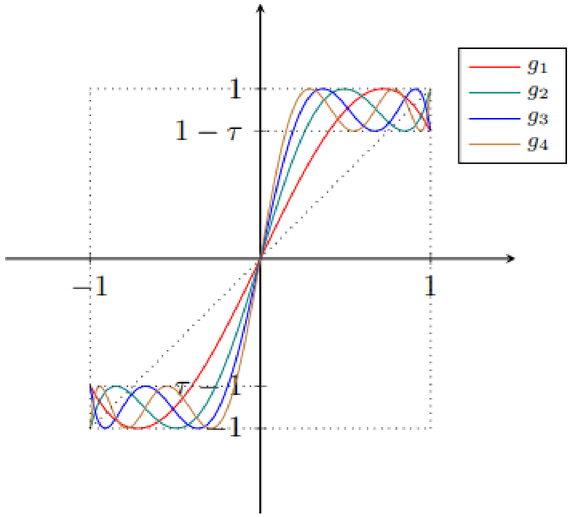

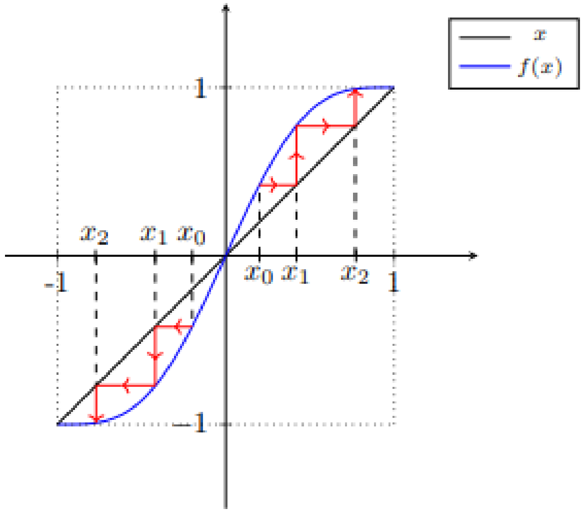

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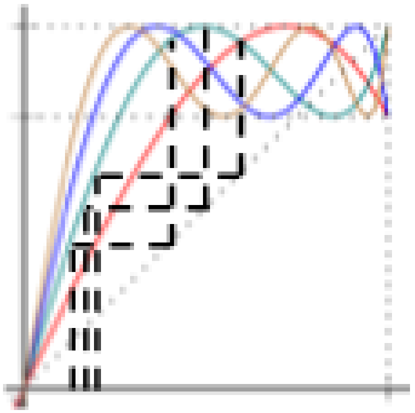

➔ We can save modulus by using smaller Δ

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Comparison Function	$f_3^{(2)} \circ g_3^{(4)}$		$f_3^{(2)} \circ g_3^{(8)}$	
Methods	Original	Changing Δ (28, 30, 42 bits)	Original	Changing Δ (31, 42 bits)
Consumed Modulus (bit)	(42 × 3) × 6 = 756	(28 × 3) × 4 + (30 × 3) + (42 × 3) = 552 **(27% ↓)**	(42 × 3) × 10 = 1260	(31 × 3) × 9 + (42 × 3) = 963 **(24% ↓)**
Precision (bit)	23.1	23.1	23.5	23.3

⌘ Bit-precision := $-\log_2|\text{max error}|$ from 100 iterations

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2. Flexible Scale factor
   - Scale adjustable independently of the modulus.
   - Modulus saving for iterative methods such as in [CKK20].

EPRINT: 2024/1014

THANK YOU!