Solution to MuddMath Puzzle v9n1

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Problem: A permutation of a set is an ordering of all of the elements in the set for which each element appears exactly once. For example, 312 and 231 are two of the possible permutations of $\{1, 2, 3\}$. A local peak in a permutation is an element that exceeds the value of any of its neighbors. For example, the permutation 32516784 of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ contains three local peaks, namely 3,5 and 8. Determine, with justification, the average number of local peaks in all 40320 possible permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Solution: Let $X_1, X_2, \ldots, X_8$ be random variables given by

$$X_i = \begin{cases} 1 & \text{if position } i \text{ is a local peak.} \\ 0 & \text{otherwise,} \end{cases}$$

and let $X = X_1 + X_2 + \cdots + X_8$. The average number of peaks then is $\mathbb{E}[X]$. Since all integers in the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ are equally likely to be in any position in a permutation,

$$\mathbb{E}[X_1] = \mathbb{E}[X_8] = 1/2 \cdot 1 + 1/2 \cdot 0 = 1/2$$

and

$$\mathbb{E}[X_i] = 1/3 \cdot 1 + 2/3 \cdot 0 = 1/3$$

for all $i \in \{2, 3, 4, 5, 6, 7\}$. By linearity of expectation, the average number of local peaks is

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{8} X_i\right] = \sum_{i=1}^{8} \mathbb{E}[X_i] = 2 \cdot \frac{1}{2} + 6 \cdot \frac{1}{3} = 3.$$