GSP. Jan 27

On certain Lagrangian subvarieties in minimal resolution of Kleinian singularities

1) Kleinian singularities

2) Anti-Poisson involution & their fixed loci

3) Preimage of fixed loci under minimal resolution

Overview:

$$\begin{tikzcd}
\text{minimal resolution} & & \text{Kleinian singularity} & \\
\tilde{X} \arrow[rr, "\pi"] & & X \arrow[loop above, "\theta"] & \theta \ \text{anti-Poisson involution} \\
\cup & & \cup & \\
\pi^{-1}(X^\theta) & & X^\theta & \text{(singular) Lagrangian subvariety.}
\end{tikzcd}$$

Goal: Describe X^θ and $\pi^{-1}(X^\theta)$ as schemes. $\left\{ \begin{array}{l} \text{irreducible component} \\ \text{intersection pattern} \\ \text{reduced or not (multiplicity)} \end{array} \right.$

Motivation: Classification of irreducible $HC(\mathfrak{g}, K)$-mod

- G simple algebraic group; $\mathfrak{g}$, $\mathcal{N}$.

  $\sigma: \mathfrak{g} \longrightarrow \mathfrak{g}$, Lie algebra involution $\rightsquigarrow$ $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. K, corresponding alg group

A $HC(\mathfrak{g}, K)$-module is a f.g. $U(\mathfrak{g})$-mod, M, s.t $\mathfrak{k} \curvearrowright M$ locally finitely & integrates to $K \curvearrowright M$.

associated variety $AV(M) = Supp(gr M)$

M irreducible $\rightsquigarrow$ $AV(M)$ is finite union of K-orbits in $\mathcal{N} \cap \mathfrak{p}$

- Take $\mathbb{O} \subset \mathcal{N}$, nilpotent orbit, J, unipotent ideal associated to $\mathbb{O}$
- $\operatorname{codim}_{\mathbb{C}}(\partial\bar{\mathbb{O}}, \bar{\mathbb{O}}) \geq 4$

[Losev & Yu, 23] classified irred M s.t. $J \subset Ann(M)$ & $AV(M) = \overline{\mathbb{O}_K}$ $\quad$ $\mathbb{O}_K$ is a K-orbit in $\mathbb{O} \cap \mathfrak{p}$.

- $\operatorname{codim}_{\mathbb{C}}(\partial\bar{\mathbb{O}}, \bar{\mathbb{O}}) = 2$, classification unknown!

  Take $\mathbb{O}' \subset \bar{\mathbb{O}}$ s.t. $\operatorname{codim}_{\mathbb{C}}(\mathbb{O}', \bar{\mathbb{O}}) = 2$

Take $e' \in \mathbb{O}'$, e', a normal point in $\bar{\mathbb{O}}$ $\rightsquigarrow$ S', Slodowy slice $\rightsquigarrow$ $S' \cap \bar{\mathbb{O}} \simeq X$, a Klenian singularity

$\Theta \curvearrowright S' \cap \bar{\mathbb{O}} \rightsquigarrow \underbrace{S' \cap \bar{\mathbb{O}} \cap \mathfrak{p}}_{\text{local picture of } AV(M) \text{ near } e'} = (S' \cap \bar{\mathbb{O}})^{\Theta} \simeq X^{\Theta}$, fixed loci

$\Theta := -\sigma$, $\mathfrak{p} = \mathfrak{g}^{\Theta}$

anti-Poisson involution.

§1. Reminder on Kleinian singularities

Let $\Gamma \subset SL_2(\mathbb{C})$ be a finite subgroup.

Kleinian singularity $X := \mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[u,v]^\Gamma$

Example: $\Gamma = \{\pm I_2\}$ (A_1)

$\mathbb{C}[u,v]^\Gamma =$ even degree polynomials $= \mathbb{C}[x=u^2, y=v^2, z=uv]$
$= \mathbb{C}[x,y,z]/(xy-z^2)$

Theorem (Klein, 1884): $X = \mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$ (single relation)
with with an isolated singularity at 0.

$\pi : \tilde{X} \longrightarrow X$, minimal resolution

exceptional locus $\pi^{-1}(0)_{red} = \boxed{C_1} \cup \cdots \cup C_n$. $C_i \simeq \mathbb{P}^1$.
irreducible components

Dual graph of $\pi^{-1}(0)_{red}$: $C_i \rightsquigarrow \overset{\bullet}{i}$

$C_i \cap C_j \neq \phi \rightsquigarrow \underset{i}{\bullet}\!\!-\!\!-\!\!-\!\!\underset{j}{\bullet}$

Fact (Du Val, 1934): Dual graphs of $\pi^{-1}(0)_{red}$ are ADE Dynkin diagrams.
(bijection)

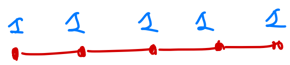
Rem: $\pi^{-1}(0)$ is not reduced in general!

$\mathfrak{g}$: simple Lie algebra of types ADE, simple root system $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$

δ – unique maximal root. $\delta = \sum_{i=1}^{n} m_i \alpha_i$. in the adjoint rep $\mathfrak{g} \curvearrowright \mathfrak{g}$.

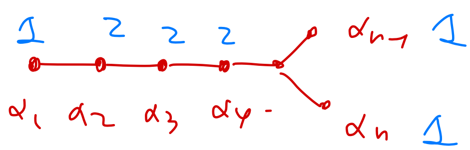
Then we have $\pi^{-1}(0) = \sum_{i=1}^{n} m_i C_i$ as a divisor. [Artin 66].

Type A_n: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. $\delta = \varepsilon_1 - \varepsilon_{n+1} = \alpha_1 + \cdots + \alpha_n \rightsquigarrow \pi^{-1}(0)$ reduced



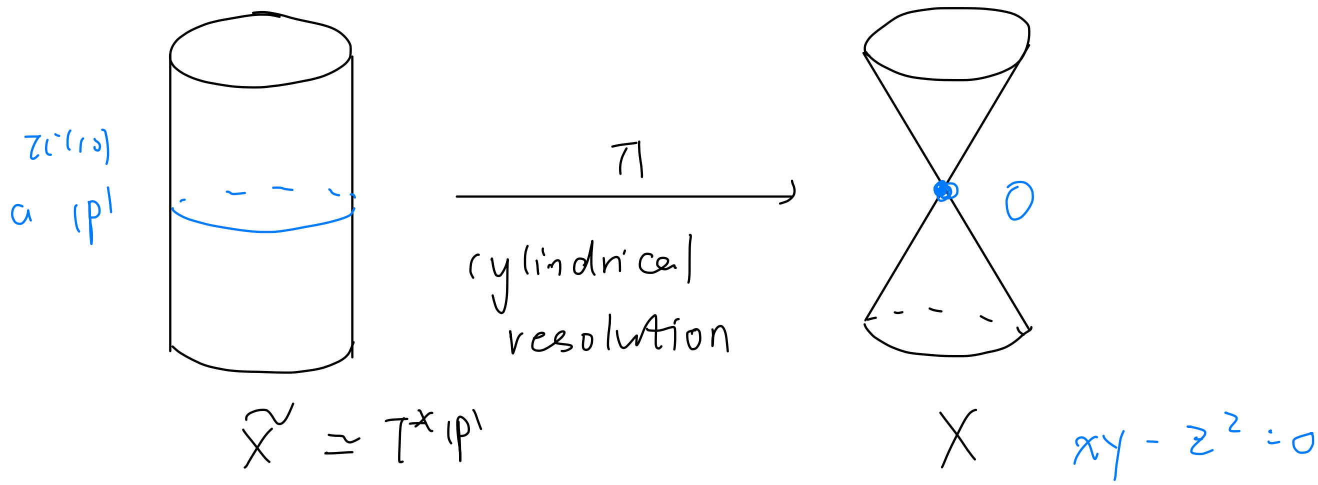
$(m_1 \cdots m_n) = (1 \cdots 1)$

Type D_n $(m_1 \cdots m_n) = (1, 2, \cdots 2, 1, 1) \rightsquigarrow \pi^{-1}(0)$ not reduced

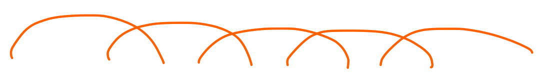


Examples:

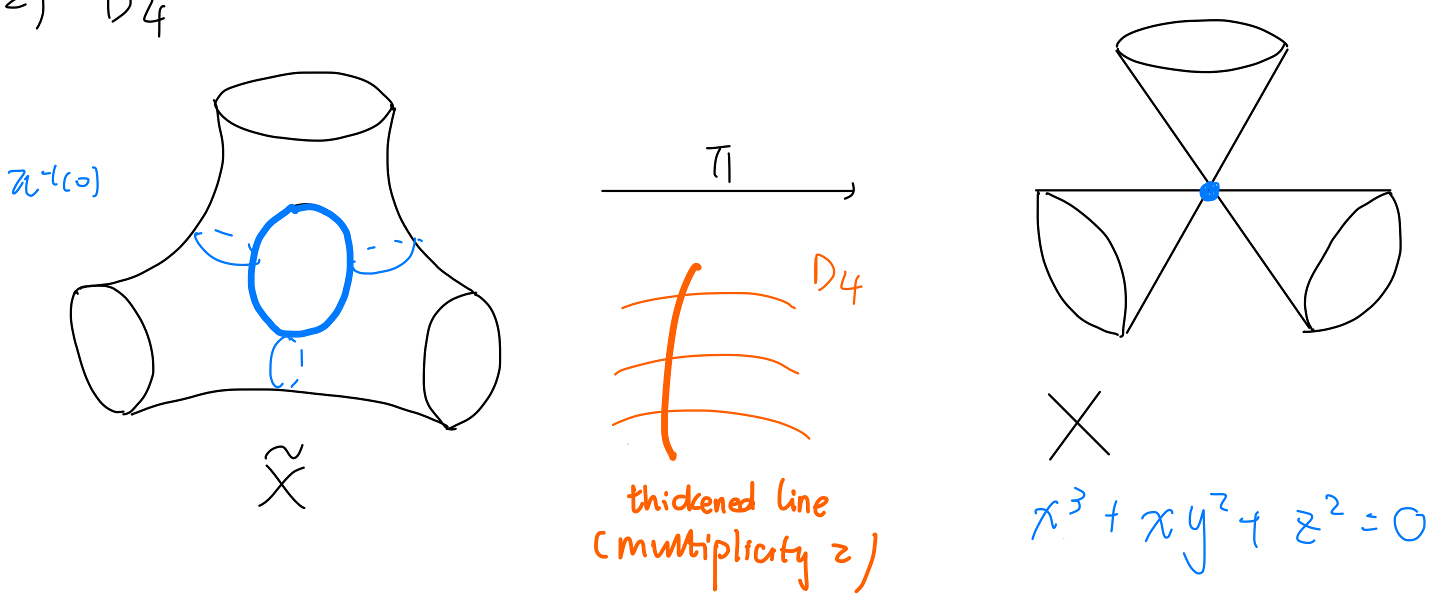
1) A_1

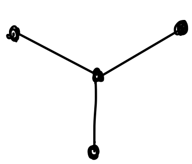


$\pi^{-1}(0)_{red} = \mathbb{P}^1$ dual graph $\bullet$ A_1 Dynkin diagrams

More generally, type A: $\pi^{-1}(0)$  (reduced)

2) D_4



dual graph  D_4 Dynkin diagram

§2. Anti-Poisson involution

$\mathbb{C}[u,v]$, Poisson bracket $\{f_1, f_2\} = \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}$

(deg -2)

$\Gamma \subset SL_2(\mathbb{C}) \rightsquigarrow \Gamma$ preserves $\{\cdot,\cdot\}$.

$\rightarrow$ $\mathbb{C}[u,v]^\Gamma \subset \mathbb{C}[u,v]$, graded Poisson subalgebra (deg -2)

<u>Example</u>: Type A_n: $\mathbb{C}[X] = \mathbb{C}[x,y,z]/(xy - z^{n+1})$

$\{x, y\} = (n+1)^2 z^n$

$\{x, z\} = (n+1)x$

$\{y, z\} = -(n+1)y$

<u>Def</u>: An <u>anti-Poisson involution</u> of a Kleinian singularity $X \simeq \mathbb{C}^2/\Gamma$ is a graded algebra involution $\theta: \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$ such that

$$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\} \quad \forall f_1, f_2 \in \mathbb{C}[X].$$

<u>Example</u>: Type A_n: $\theta: \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$

$x \longmapsto y$, $y \longmapsto x$, $z \longmapsto z$

is an anti-Poisson involution.

<u>Def (fixed locus)</u>:

$X^\theta := \operatorname{Spec} \mathbb{C}[X]/I$, where $I = (\theta(f) - f,\ f \in \mathbb{C}[X])$

<u>Example (continued)</u>

$X^\theta = \operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^{n+1}, x-y) \simeq \operatorname{Spec} \mathbb{C}[x,z]/(x^2 - z^{n+1})$ reduced.

- union of two $\mathbb{A}^1$ when n odd
- a cusp when n even.

Prop 1 (Classification of θ)

There are finitely many anti-Poisson involutions on $\mathbb{C}^2/\Gamma$ up to conjugation by Poisson automorphisms.

Prop 2 (Description of X^θ)

(1) X^θ_{red} is either a (singular) Lagrangian subvariety of X, or only contains the singular point.

(2) X^θ is reduced and connected.
Each irreducible component is either an $\mathbb{A}^1$ or a cusp.

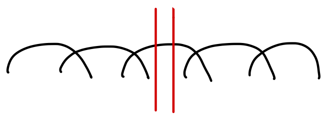
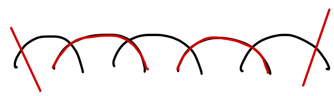
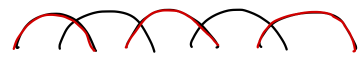
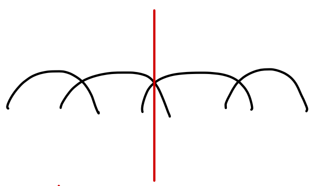
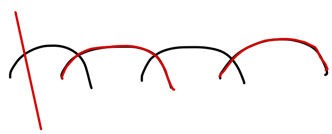
Proof (1) follows from

Lemma (Exercise) (M, ω) symplectic manifold with anti-symplectic involution τ ($\tau^*\omega = -\omega$)
↳ pf: look at tangent space
Then M^τ is either empty or a Lagrangian submanifold.

applied to $X^{reg} = X \setminus \{0\}$

(2) follows from classification in Prop 1 + case by case calculation.

Example Anti-Poisson involutions for Type An Kleinian singularity

	Case I	Case II	Case III
An. n odd $xy - z^{n+1} = 0$ X^{θ}	$x \leftrightarrow y$ 	$z \mapsto -z$ 	$x \mapsto -x$ $y \mapsto -y$ $z \mapsto -z$ 
$\pi^{-1}(X^{\theta})$			
An. n even $xy - z^{n+1} = 0$ X^{θ}	$x \leftrightarrow y$ 	$x \mapsto -x$ $z \mapsto -z$ 	Fixed loci in purple !
$\pi^{-1}(X^{\theta})$	 the only triple intersection		
Corresponding Cartan involution	outer involution $\sigma: X \mapsto -X^T$	inner involution $\sigma = Ad(I_{k,l})$	not seen.

Dn., E6, two cases of API ; E7, E8 one case .

§3 Preimage of fixed loci under minimal resolution

$\pi: \tilde{X} \longrightarrow X$, $0 \in X^\theta \rightsquigarrow \pi^{-1}(0) \subset \pi^{-1}(X^\theta)$

§3.1 lift.

$\theta : X \longrightarrow X$

Def: A <u>lift</u> of $\theta: X \to X$ is an anti-symplectic involution $\tilde{\theta}: \tilde{X} \longrightarrow \tilde{X}$ s.t. $\pi \circ \tilde{\theta} = \theta \circ \pi$

Thm 3: There exists a unique lift for any anti-Poisson involution θ on X.

Now let's see why the lift helps to determine the preimage. $\pi^{-1}(X^\theta)$

Claim: $\pi^{-1}(X^\theta) = \underbrace{\pi^{-1}(0)}_{\text{union of } \mathbb{P}^1\text{'s}} \cup \boxed{\tilde{X}^{\tilde{\theta}}}$ $\tilde{X}^{\tilde{\theta}}_{red}$ is smooth Lagrangian (by Lemma)

no intersection! X becomes $||$ in $\pi^{-1}(X^\theta)$;

no cusp! $\prec$ becomes $|$ in $\pi^{-1}(X^\theta)$

Let $L \subset X^\theta$ be an irreducible component

define $\tilde{L} := \overline{\pi^{-1}(L \setminus 0)}$, then $\tilde{L}$ is an irreducible component in $\pi^{-1}(X^\theta)$ &

$\tilde{L} \subset \tilde{X}^{\tilde{\theta}} \Rightarrow \tilde{L}$ smooth.

Claim $\tilde{L} \simeq \mathbb{A}^1 \Rightarrow \tilde{L} \cap \pi^{-1}(0)$ is a single point in $\pi^{-1}(0)^{\tilde{\theta}}$

- The lift $\tilde{\theta}$ sends $\pi^{-1}(0)$ to $\pi^{-1}(0)$ b/c ($\theta(0)=0$). by permuting the components $C_i \simeq \mathbb{P}^1$. (swapping or preserve)

Lemma/Exercise. : An involution on $\mathbb{P}^1$ either acts trivially or has exactly two fixed points.

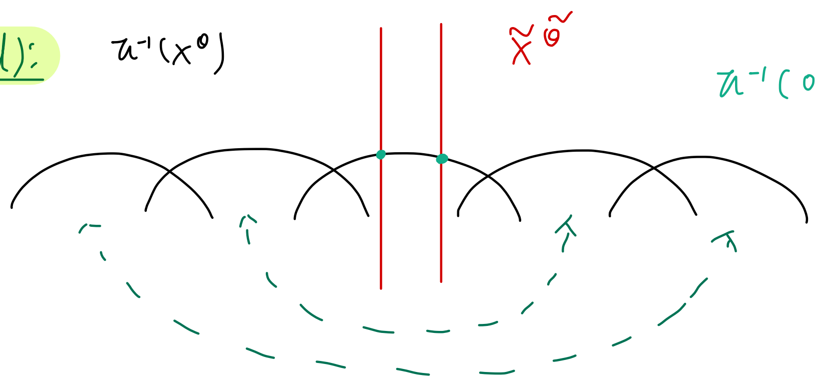
(comes from Möbius transformation)

Example (continued):

Type A: $\theta: x \longleftrightarrow y$

n odd

$\pi^{-1}(x^0)$

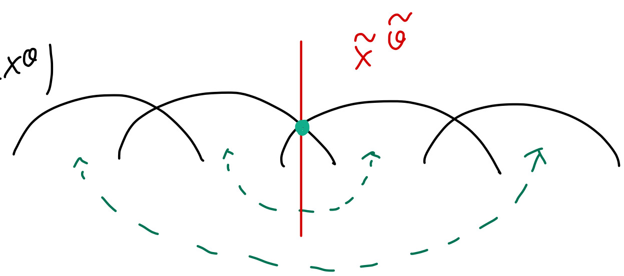


$\pi^{-1}(0)^{\tilde{\theta}}$ = two discrete points

action of $\tilde{\theta}$

n even

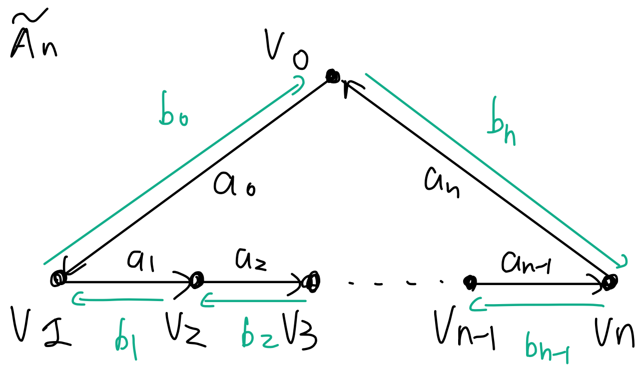
$\pi^{-1}(x^0)$



$\pi^{-1}(0)^{\tilde{\theta}}$ = a discrete point

§3.2 Nakajima quiver variety.

Example: Nakajima quiver variety type A.



Attach to each vertex a 1-dim'l vector space V_i

$\dim V_i = m_i$ ($=1$ in type A)
$m_0 = 1$

coefficient in $\delta = \sum_{i=1}^{n} m_i \alpha_i$

Consider the representation space

$$M(v) = T^*\left(\bigoplus_{i=0}^{n} \mathrm{Hom}(V_i, V_{i+1})\right) = \bigoplus_{i=0}^{n} \mathrm{Hom}(V_i, V_{i+1}) \oplus \bigoplus_{i=0}^{n} \mathrm{Hom}(V_{i+1}, V_i)$$

symplectic vector space

$G = \prod_{i=0}^{n} GL(V_i) \curvearrowright M(v, \omega)$

moment map $\mu = (\mu_i)_{i=0}^{n}$: $\mu_i = a_{i-1}b_{i-1} - b_i a_i$

We have $\mu^{-1}(0)/\!/G = \underbrace{\mathrm{Spec}\, \mathbb{C}[\mu^{-1}(0)]^G}_{\text{is also graded}} \simeq \underbrace{\mathrm{Spec}\, \mathbb{C}[x, y, z]/(xy - z^{n+1})}_{\text{Type } A_n \text{ Kleinian singularity.}}$

with $x := a_n a_{n-1} \cdots a_1 a_0$, $y := b_0 b_1 \cdots b_{n-1} b_n$, $z := a_0 b_0$ #

(product of determinant maps)

Take $\chi : G \longrightarrow \mathbb{C}^*$, a generic character.

consider $\mu^{-1}(0)^s \subset \mu^{-1}(0)$ (semi) stable locus. $\rightsquigarrow G \curvearrowright \mu^{-1}(0)^s$ freely

Then $\tilde{X} \simeq \mu^{-1}(0)^s /\!/ G \longrightarrow \mu^{-1}(0)/\!/G \simeq X$

is the minimal resolution of Kleinian singularities [Kronheimer 89. Nakajima 94]

Example (continued)

$\Theta : M(V) \longrightarrow M(V)$

$\Theta(a_i) = b_{n+1-i}$ $\Theta(b_i) = a_{n+1-i}$. anti-symplectic involution

then
(1) Θ preserves $\mu^{-1}(0)$
(2) Θ preserves $\mu^{-1}(0)^s$
(3) Θ normalizes G

routine check

(4) $\Theta(x) = y$. $\Theta(y) = x$. $\Theta(z) = z$ $\rightsquigarrow$ Θ descends to θ on X.

Thm (Nakajima 96) $C_i = \{ (a, b) \in \mu^{-1}(0)^s /\!/ G \mid a_i = b_{i-1} = 0 \}$

so $\tilde{\theta}(C_i) = C_{n+1-i}$ (swapping the components C_i C_{n+1-i})

Corresponds to folding of Dynkin diagram A_n.

Rmk: types D. E can be done similarly; but main difficulty is that $\dim V_i = m_i$. not just 1-dim'l making it hard to find invariant functions x.y.z.
& no triple intersection in types D.E

§3.3 multiplicities

X^0 irreducible components $L_1, \dots, L_m$

reduced

$\pi^{-1}(X^0)$ irreducible components $\underbrace{\tilde{L}_1, \dots, \tilde{L}_m}_{|A|}$. $\underbrace{C_1, \dots, C_n}_{|P|}$

$$\pi^{-1}(X^0) = \sum_{j=1}^{m} L_j + \sum_{i=1}^{n} \boxed{a_i} C_i \quad \text{as a divisor.}$$

want to determine a_i

Define $b_i := \#\{\tilde{L}_j \mid \tilde{L}_j \cap C_i \neq \phi\}$

Prop 4 (multiplicity) If $X^0 \subset X$ is a principal divisor, then

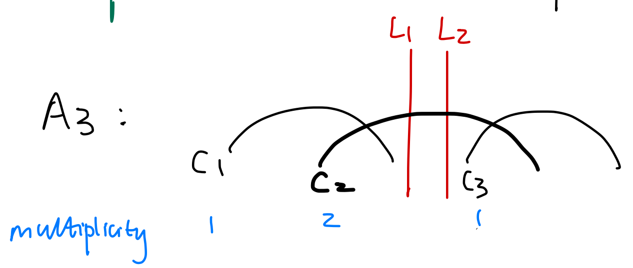
$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \boxed{\mathcal{C}^{-1}} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

inverse of Cartan matrix

Rmk: $X^0 \subset X$ is principal modulo two cases $\to$ have other method to determine multiplicity

Example (continued) Type A_n. $\theta: X \hookrightarrow Y$ (case I)

A_3:

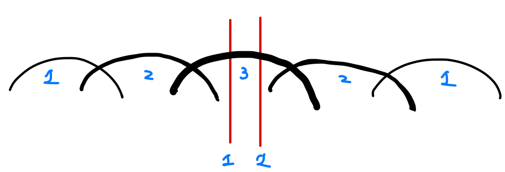


$b_1 = 0$, $b_2 = 2$. $b_3 = 0$

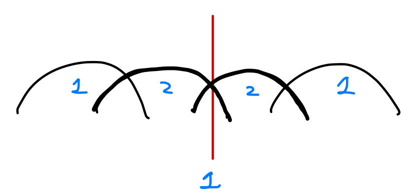
$$\mathcal{C}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$\Rightarrow a_1 = 1$. $a_2 = 2$. $a_3 = 1$

Similarly A_5,



; A_4



multiplicity

Not reduced unless for A_1.