Kleinian singularities and their anti-Poisson involutions

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Motivations

Kleinian singularities are the only normal Gorenstein singularities in dimension 2.

Anti-Poisson involutions appear naturally when we want to classify irreducible Hairsh-Chandra modules over Kleinian singularities.

Kleinian singularities

Kleinian singularities are quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup acting on $\mathbb{C}[u, v]$. The algebra of invariant functions $\mathbb{C}[u, v]^\Gamma$ is finitely generated. Denote the corresponding variety by $X := \mathbb{C}^2/\Gamma$.

Example

When $\Gamma = \{\pm I_2\}$, we have $\mathbb{C}[u, v]^\Gamma =$ even degree polynomials $=$ $\mathbb{C}[x = u^2, y = v^2, z = uv] = \mathbb{C}[x, y, z]/(xy - z^2)$.

Fact [Klein, 1884]: $\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$ (one relation), and $\mathbb{C}^2/\Gamma$ is a singular surface with only singularity at 0.

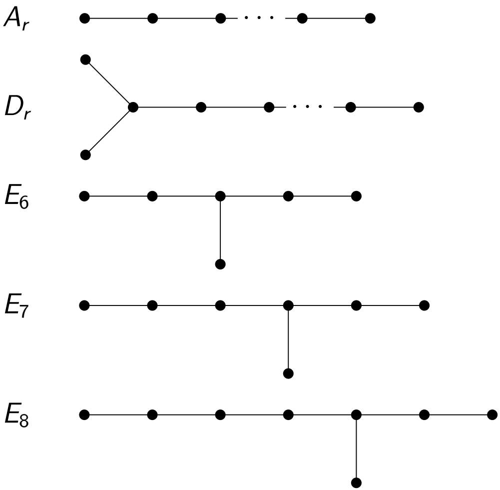
Kleinian singularities

Q: How to classify Γ's?

(A_r) The cyclic group of order $r+1$.
$$xy - z^{r+1} = 0$$
(D_r) The binary dihedral group of order $4(r-2)$, $r \geq 4$.
$$x^{r-1} + xy^2 + z^2 = 0$$
(E_6) The binary tetrahedral group of order 24.
$$x^4 + y^3 + z^2 = 0$$
(E_7) The binary octahedral group of order 48.
$$x^3y + y^3 + z^2 = 0$$
(E_8) The binary icosahedral group of order 120.
$$x^5 + y^3 + z^2 = 0$$

Kleinian singularities

Observation: They are in bijection with *ADE* type Dynkin diagrams.



Kleinian singularities

There are two ways to construct this bijection.

1. McKay Correspondence [McKay, 1979].
2. Minimal Resolution [Du Val, 1934]. By resolution we mean a smooth variety $\widetilde{X}$ equipped with a projective birational morphism $\pi\colon \widetilde{X} \to X := \mathbb{C}^2/\Gamma$. The minimality conditional means other resolution factors through $\widetilde{X}$. The exceptional locus

$$\pi^{-1}(0) = C_1 \cup \cdots \cup C_r, \;\; C_i \simeq \mathbb{P}^1$$

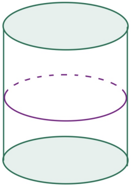
is a connected union of $\mathbb{P}^1$'s.

Fact: Construct the dual graph of $\pi^{-1}(0)$ by replacing each C_i by a vertex and drawing a line to connect C_i with C_j if they intersect. Then the dual graph is exactly the corresponding type Dynkin diagram.

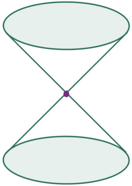
Kleinian singularities

For example,

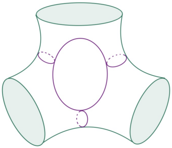
$T^*\mathbb{P}^1$



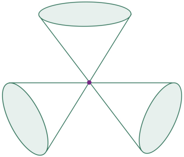
$\xrightarrow{\pi}$



A_1 singularity
$xy - z^2 = 0$



$\xrightarrow{\pi}$



D_4 singularity
$x^3 + xy^2 + z^2 = 0$

Anti-Poisson involutions

Consider the standard symplectic structure on $\mathbb{C}^2$ given by $\omega(u, v) = 1$. The ring of functions $\mathbb{C}[u, v]$ becomes a Poisson algebra.

The finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C}) = \mathrm{Sp}_2(\mathbb{C})$ preserves this Poisson bracket, so $\mathbb{C}[u, v]^\Gamma$ is a graded Poisson subalgebra of $\mathbb{C}[u, v]$ with $\deg\{\cdot, \cdot\} = -2$.

Example

Recall type A_r: $\mathbb{C}[x, y, z]/(xy - z^{r+1})$. The degree of the generators are $\deg x = \deg y = r + 1$, $\deg z = 2$. The Poisson brackets are given by

$$\begin{aligned} \{x, y\} &= (r+1)^2 z^r, \\ \{x, z\} &= (r+1)x, \\ \{y, z\} &= -(r+1)y. \end{aligned}$$

Anti-Poisson involutions

Set $X := \mathbb{C}^2/\Gamma$ to denote a Kleinian singularity.

Definition

An *anti-Poisson involution* of X is graded algebra involution $\theta\colon \mathbb{C}[X] \to \mathbb{C}[X]$ such that

$$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\}, \ \forall\ f_1, f_2 \in \mathbb{C}[X].$$

Example

On type A_r: $\mathbb{C}[x, y, z]/(xy - z^{r+1})$ Kleinian singularity,

$$x \mapsto y,\ \ y \mapsto x,\ \ z \mapsto z$$

is an anti-Poisson involution.

Fact: There are finitely many anti-Poisson involutions on $\mathbb{C}^2/\Gamma$ up to conjugation by Poisson automorphisms.

The fixed loci

The fixed locus $X^\theta := \operatorname{Spec} \mathbb{C}[X]/I$ where $I = (\theta(f) - f,\ f \in \mathbb{C}[X])$

Proposition

The fixed locus X^θ is a reduced connected singular Lagrangian subvariety of X.

Example (continued)

On type A_r: $\mathbb{C}[x,y,z]/(xy - z^{r+1})$ Kleinian singularity,

$$x \mapsto y,\ y \mapsto x,\ z \mapsto z$$

is an anti-Poisson involution. The fixed locus is

$$\operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^{r+1}, x - y) \simeq \operatorname{Spec} \mathbb{C}[x,z]/(x^2 - z^{r+1}),$$

which is a union of two $\mathbb{A}^1$'s when r is odd; a cusp when r is even.

Preimage of fixed loci under minimal resolutions

Recall $\pi\colon \widetilde{X} \to X$ denote the minimal resolution. We would like to study the preimage of the fixed locus $\pi^{-1}(X^\theta)$.

Q1: How does the preimage $\pi^{-1}(X^\theta)$ look like in general?

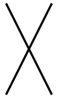
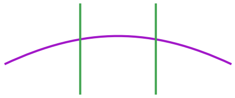
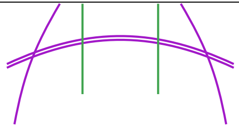
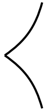
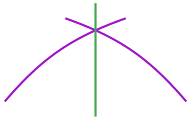
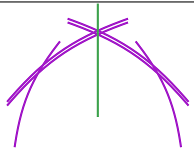
A1: It is a singular Lagragian subvariety of pure dimension 1 in $\widetilde{X}$. Each irreducible component is either a $\mathbb{P}^1$ and or an $\mathbb{A}^1$. The $\mathbb{P}^1$'s come from the exceptional locus $\pi^{-1}(0)$, while $\mathbb{A}^1$'s come from the fixed locus X^θ.

Q2: Is $\pi^{-1}(X^\theta)$ reduced?

A2: Not necessarily.

Preimage of fixed loci under minimal resolutions

Example: Type A_r Kleinian singularity with θ swapping $x \leftrightarrow y$. The fixed locus X^θ and its preimage looks like the following.

fixed locus	preimage	preimage
 A_r, r odd	 A_1	 A_3
 A_r, r even	 A_2	 A_4

Notation: straight green line is $\mathbb{A}^1$; curly purple line is $\mathbb{P}^1$; doubled line reflects non-reduced component.

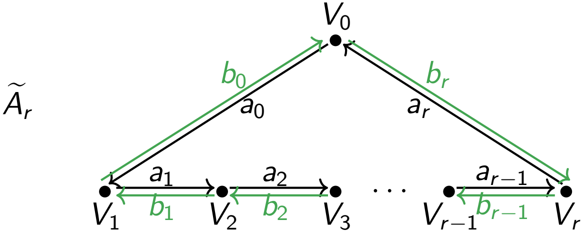
Nakajima quiver varieties

Q3: How do these $\mathbb{P}^1$'s and $\mathbb{A}^1$'s intersect with each other in general?

A3: To solve this problem, we need to find a good way to describe the minimal resolution. We will realize Kleinian singularities and their minimal resolutions as **Nakajima quiver varieties** in types A_r, D_r, E_6. (Types E_7 and E_8 is a bit tricky.)

Nakajima quiver varieties

Example: We realize type A_r: $xy - z^{r+1}$ singularity as quiver variety.



Each V_i is a 1-dimensional vector space. The group $G = \prod_{i=0}^{r} \mathrm{GL}(V_i)$ naturally acts on $M(V) := \bigoplus_{i,j=0}^{r} \mathrm{Hom}(V_i, V_j)$ with moment map $\mu = (\mu_i)_{0 \le i \le r} \colon M(v) \to \bigoplus_{0 \le i \le r} \mathfrak{gl}(V_i)$ where

$$\mu_i = a_{i-1} b_{i-1} - b_i a_i.$$

We have $\mu^{-1}(0)//G \simeq \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^{r+1})$ with

$$x := a_r a_{r-1} \cdots a_1 a_0, \;\; y := b_0 b_1 \cdots b_{r-1} b_r, \;\; z := a_0 b_0.$$

Thank you for your attention!