On certain Lagrangian subvarieties in minimal resolutions of Kleinian singularities

overview + motivation

1) Reminder on Kleinian singularities

2) Anti-Poisson involutions & their fixed loci

3) Preimage of fixed loci

Overview:

minimal resolution Kleinian singularity.

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\pi"] & X \arrow[loop right, "\theta"] \\
\cup & \cup \\
\pi^{-1}(X^\theta) & X^\theta
\end{tikzcd}$$

anti-Poisson involution

Lagrangian subvariety

Goal: Describe X^θ and $\pi^{-1}(X^\theta)$ as subschemes.

Motivation: classification of irreducible $HC(\mathfrak{g}.K)$-mod.

- G $\overset{\text{simply-connected}}{\vee}$ simple alg group / $\mathbb{C}$; $\mathfrak{g}$. $\overset{\text{nilp cone}}{\mathcal{N}}$

  $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ Lie alg involution $\rightsquigarrow$ $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$

  $\hookrightarrow K$. connected alg subgrp

  $\theta := -\sigma$ anti-Poisson involution

- $\mathbb{O}$ nilpotent orbit. $\mathbb{O}' \subset \bar{\mathbb{O}}$, $\operatorname{codim}_{\mathbb{C}}(\mathbb{O}', \bar{\mathbb{O}}) = 2$

  $e' \in \mathbb{O}'$, assume e' is normal in $\bar{\mathbb{O}}$

  $\wr$

  S' Slodowy slice $\rightsquigarrow$ $S' \cap \bar{\mathbb{O}} \simeq X$. a Kleinian singularity

  $\shortparallel$

  $e' + \mathfrak{z}_{\mathfrak{g}}(f')$. where $\{e', h', f'\}$ $\mathfrak{sl}_2$-triple.

$\theta \curvearrowright S' \cap \bar{\mathbb{O}}$ $\rightsquigarrow$ $(S' \cap \bar{\mathbb{O}})^{\theta} \simeq X^{\theta}$. fixed locus

$\shortparallel$

$S' \cap \bar{\mathbb{O}} \cap \mathfrak{p}$ related to $AV(M)$. associated variety of a $HC(\mathfrak{g}.K)$-mod M annihilated by the unipotent ideal $J(\mathbb{O})$ s.t. $\operatorname{codim}_{\mathbb{C}}(\partial\bar{\mathbb{O}}, \bar{\mathbb{O}}) = 2$

$\operatorname{codim}_{\mathbb{C}}(\partial\bar{\mathbb{O}}, \bar{\mathbb{O}}) \geq 4$. classification done by [Losev & Yu 23]

$\operatorname{codim}_{\mathbb{C}}(\partial\bar{\mathbb{O}}, \bar{\mathbb{O}}) = 2$. classification unknown

Example: $\mathfrak{g} = \mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$

$\sigma : X \longmapsto -X^T$. $\mathfrak{k} = \mathfrak{g}^{\sigma} = SO(2)$

$\theta = -\sigma : X \longmapsto X^T$, $\mathfrak{p} = \mathfrak{g}^{\theta}$ = symmetric matrices

$\mathbb{O}$ = conj. class of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\underset{\underset{S' \cap \bar{\mathbb{O}}}{\shortparallel}}{\bar{\mathbb{O}}} = \mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| \underbrace{a^2 + bc = 0}_{} \right\} = X$

$e' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $S' = \mathfrak{g}$.

Kleinian singularity of type A_1

$X^{\theta} = (S' \cap \bar{\mathbb{O}})^{\theta} = S' \cap \bar{\mathbb{O}} \cap \mathfrak{p} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle| a^2 + bc = 0.\ b = c \right\}$

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- $HC(\mathfrak{g}, K)$-mod : f.g. $\mathcal{U}(\mathfrak{g})$-mod. M. $K \curvearrowright M$ locally finitely & integrates to $K \curvearrowright M$.

$\downarrow$ PBW filtration $\quad \downarrow$ good filtration (K-stable)

always exists : $X_0 \subset M$ finite set of generators of $\mathcal{U}(\mathfrak{g})$-mod
$M_0 :=$ K-rep generated by X_0, M_0 f.dim'l b/c K acts locally finitely
$M_n := \mathcal{U}(\mathfrak{g})_n M_0 \rightsquigarrow \dim M_n < \infty$
& $M = \bigcup_{n\geq 0} M_n$ as M generated by M_0 as $\mathcal{U}(\mathfrak{g})$-mod

Associated variety $\underset{\text{set-theoretic}}{}$ $AV(M) := \operatorname{supp}(\underbrace{\operatorname{gr} M}_{\text{mod over } \operatorname{gr}\mathcal{U}(\mathfrak{g}) = S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]}) \subset \mathfrak{p}$

set-theoretic

M irreducible $\rightsquigarrow AV(M) \subset \mathcal{N} \cap \mathfrak{p}$

$\mathcal{O}$ nilpotent orbit $\rightsquigarrow J(\mathcal{O})$ unipotent ideal associated with $\mathcal{O}$

$\quad\quad J(\mathcal{O}) = \operatorname{Ker}[\mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{A}_0]$ $\quad \mathcal{A}_0 \to$ canonical quantization of $\mathbb{C}[\mathcal{O}]$

consider irreducible M annihilated by $J(\mathcal{O}) \rightsquigarrow AV(M) \subset \overline{\mathcal{O}} \cap \mathfrak{p}$

- $\operatorname{codim}(\partial\overline{\mathcal{O}}, \overline{\mathcal{O}}) \geq 4 \rightsquigarrow$ [Vogan 91] $AV(M)$ is irreducible $= \overline{\mathcal{O}_K}$
  ($\overline{\mathcal{O}}$ terminal) $\quad$ $\| $ closure of single K-orbit in $\mathcal{O} \cap \mathfrak{p}$

[Losev & Yu 23], classified irreducible $HC(\mathfrak{g}, K)$-mod annihilated by $J(\mathcal{O})$ s.t. $AV(M) = \overline{\mathcal{O}_K}$.

- $\operatorname{codim}(\partial\overline{\mathcal{O}}, \overline{\mathcal{O}}) = 2$. classification unknown,

$AV(M)$. may not be irreducible
$\cap$
$\overline{\mathcal{O}} \cap \mathfrak{p}$

S', slice to $e' \in \mathcal{O}'$, $\operatorname{codim}_{\overline{\mathcal{O}}} \mathcal{O}' = 2$

$S' \cap \overline{\mathcal{O}} \cap \mathfrak{p} = (S' \cap \overline{\mathcal{O}})^\theta = X^\theta$

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§1. Reminder on Kleinian singularities

$\Gamma \subset SL_2(\mathbb{C})$. finite subgroup

Kleinian singularity: $X := \mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[u,v]^\Gamma$

Example: $\Gamma = \{\pm I_2\}$

$\mathbb{C}[u,v]^\Gamma = \mathbb{C}[x=u^2, y=v^2, z=uv] = \mathbb{C}[x,y,z]/(xy-z^2)$

Fact (Klein): $X = \mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$ (single relation) with an isolated singularity at 0

minimal resolution: $\pi : \underset{\text{smooth}}{\tilde{X}} \longrightarrow X$, projective & birational.

exceptional locus: $\pi^{-1}(0)_{red} = \underset{\text{irreducible component}}{\boxed{C_1}} \cup \cdots \cup C_n$. $C_i \cong \mathbb{P}^1$

Dual graph of $\pi^{-1}(0)_{red}$: $C_i \rightsquigarrow \overset{}{\bullet}\ i$

$C_i \cap C_j \neq \phi \rightsquigarrow \underset{i}{\bullet}\!\!-\!\!-\!\!-\!\!-\!\!\underset{j}{\bullet}$

Fact (Du Val): Dual graphs of $\pi^{-1}(0)_{red} \xleftrightarrow{\text{1-to-1}}$ ADE Dynkin diagrams

Rem: $\pi^{-1}(0)$ is not reduced in general:

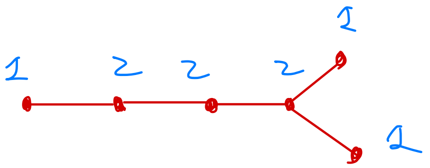
$\mathfrak{g}$ simple Lie alg of types ADE, simple root system $\{\alpha_1, \alpha_2, \cdots \alpha_n\}$

δ - unique maximal root $\delta = \sum_{i=1}^n m_i \alpha_i$

Then $\pi^{-1}(0) = \sum_{i=1}^n m_i C_i$ as a divisor [Artin]

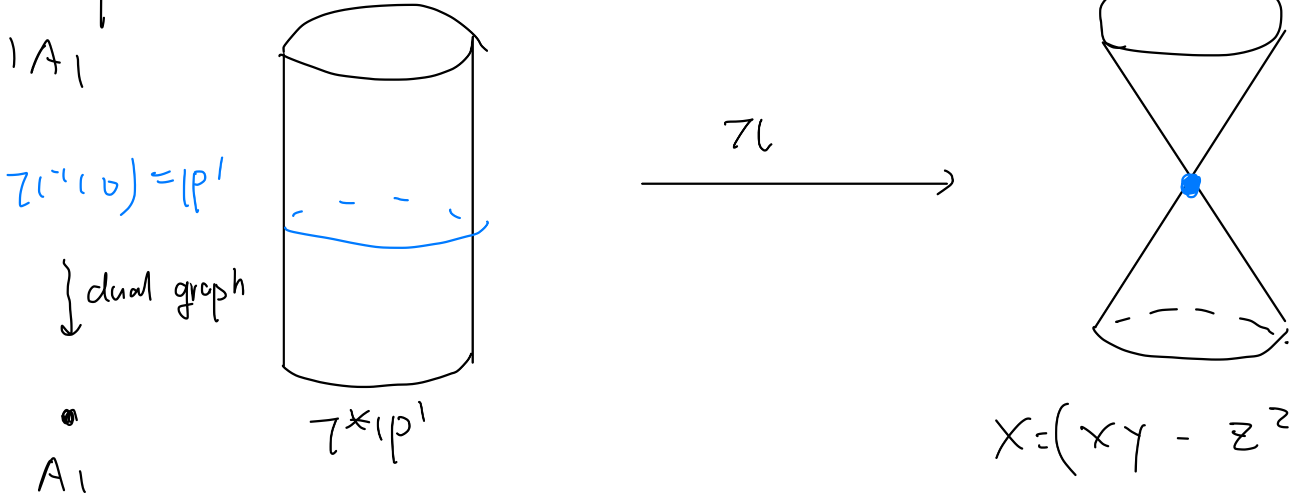
Type A_n. $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. $\delta = \varepsilon_1 - \varepsilon_{n+1} = \alpha_1 + \alpha_2 + \cdots + \alpha_n \rightsquigarrow \pi^{-1}(0)$ reduced

Type D_n $(\delta_1, \cdots, \delta_n) = (1, 2 \cdots 2, 1, 1) \rightsquigarrow \pi^{-1}(0)$ not reduced



Examples:

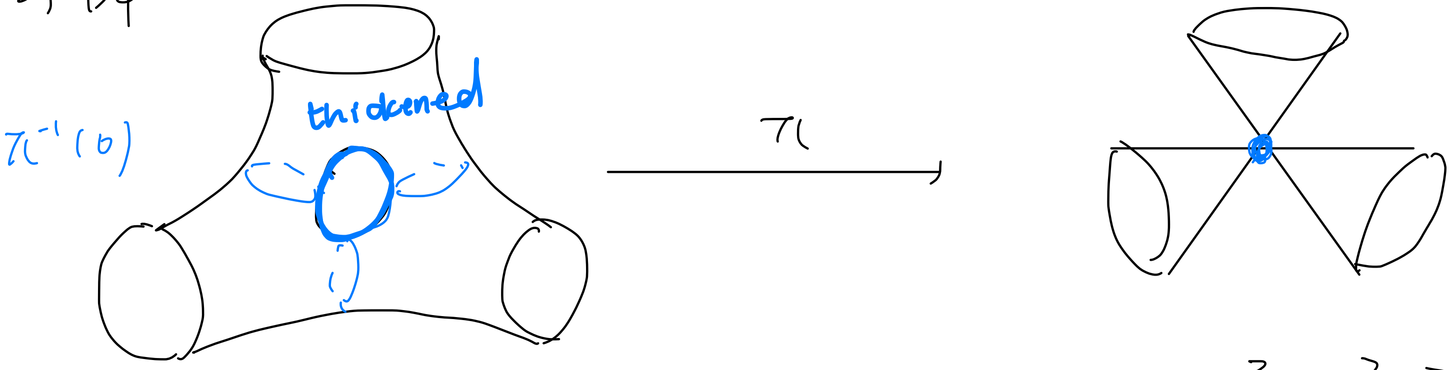
1) A_1



$X = (xy - z^2 = 0)$

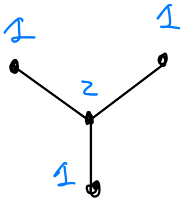
Similarly: Type A_n $\pi^{-1}(0)$:

2) D_4



$X = (x^3 + xy^2 + z^2 = 0)$

dual graph:



D_4 Dynkin diagram

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§2. Anti-Poisson involutions

$X = \mathbb{C}^2/\Gamma$, Kleinian singularity

$\mathbb{C}[u,v]$ graded Poisson alg

$\cup$

$\mathbb{C}[X] = \mathbb{C}[u,v]^\Gamma$ graded Poisson subalg.

Example: Type A_n: $\mathbb{C}[X] = \mathbb{C}[x,y,z]/(xy - z^{n+1})$

$\{x, y\} = (n+1)^2 z^n$

$\{x, z\} = (n+1) x$

$\{y, z\} = -(n+1) y$

Def: An anti-Poisson involution of $X \cong \mathbb{C}^2/\Gamma$ is a graded algebra involution $\theta : \mathbb{C}[X] \longrightarrow \mathbb{C}[X]$ s.t.

$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\}$, $\forall f_1, f_2 \in \mathbb{C}[X]$

Example: Type A_n: θ: $\mathbb{C}[X] \longrightarrow \mathbb{C}[X]$

$x \longmapsto y$

$y \longmapsto x$

$z \longmapsto z$

Def (scheme-theoretic fixed locus)

$X^\theta := \operatorname{Spec} \mathbb{C}[X]/I$ where $I = (\theta(f) - f \mid f \in \mathbb{C}[X])$

Example: (continued)

$$X^\theta = \mathrm{Spec}\,\mathbb{C}[x\ y\ z]/(xy - z^{n+1}, x-y) \simeq \mathrm{Spec}\,\mathbb{C}[x, z]/(x^2 - z^{n+1})$$

$\downarrow$

reduced

$$= \begin{cases} \text{union of two } \mathbb{A}^1, & n \text{ odd} \\ \text{cusp}, & n \text{ even} \end{cases}$$

Prop 2 (H.) Classification of θ

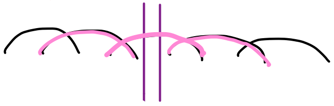
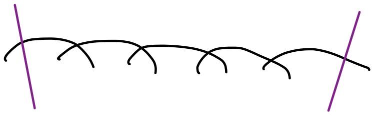
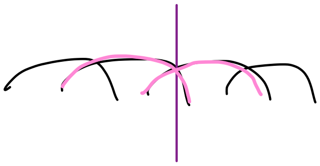
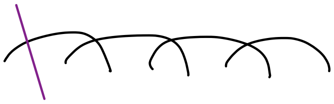
There are finitely many anti-Poisson involution on $X = \mathbb{C}^2/\Gamma$. up to conjugation by Poisson automorphism. $\longleftrightarrow$ $N_{GL_2(\mathbb{C})}(\Gamma)^- / N_{SL_2(\mathbb{C})}(\Gamma)$ up to conj

Prop 2 (H.) Description of X^θ

The scheme-theoretic fixed locus X^θ is reduced.

If $X^\theta \neq \{0\}$, each irreducible component of X^θ is either an $\mathbb{A}^1$ or a cusp.

Example Anti-Poisson involution for type A_n Kleinian singularities

		Type I	Type II	Type III
Lie alg involution		$\sigma: X \longmapsto -X^t$	$\sigma = \mathrm{Ad}(I_{k,\ell})$	Not seen unless in $sl_2(\mathbb{C})$. where $\sigma = id$.
A_n. n odd $xy - z^{n+1} = 0$	X^θ	$x \leftrightarrow y$ 	$z \longmapsto -z$ 	$x \longmapsto -x$ $y \longmapsto -y$ $z \longmapsto -z$ 
$\pi^{-1}(X^\theta)$		$(b_1\ b_2\ b_3\ b_4\ b_5) = (0.0.2.0.0)$ $\rightsquigarrow (a_1.a_2.a_3\ a_4\ a_5) = (1.2.3.2.1)$  non-reduced components		
A_n. n even $xy - z^{n+1} = 0$	X^θ	$x \leftrightarrow y$ 	$x \longmapsto -x$ $z \longmapsto -z$ 	N/A.
$\pi^{-1}(X^\theta)$				

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§3. Preimages

X Kleinian singularity, θ: anti-Poisson involution

$\pi: \tilde{X} \longrightarrow X \simeq \mathbb{C}^2/\Gamma$ minimal resolution

$0 \in X^\theta \rightsquigarrow \pi^{-1}(0) \subset \pi^{-1}(X^\theta)$. but there are more

§3.1 lift

Main Thm (H.) There exists a unique anti-symplectic involution $\tilde{\theta}: \tilde{X} \to \tilde{X}$ s.t. $\pi \circ \tilde{\theta} = \theta \circ \pi$. We call $\tilde{\theta}$ a lift of θ

Proof of Main Thm is through the realization of Kleinian singularities as Nakajima quiver varieties

Why the lift helps?

$\tilde{X}$ smooth $\rightsquigarrow$

$\tilde{X}^{\tilde{\theta}}$ smooth Lagrangian

Claim: $\pi^{-1}(X^\theta)_{red} = \pi^{-1}(0)_{red} \cup \boxed{\tilde{X}^{\tilde{\theta}}}$

no intersection

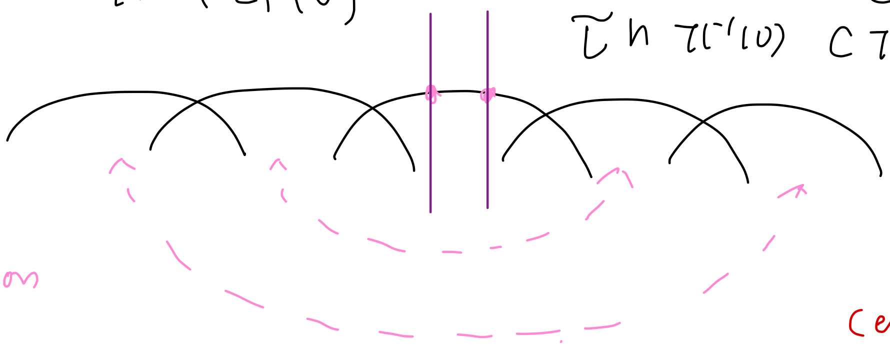
no cusp

$L \subset X^\theta$, irreducible component

Define $\tilde{L} := \overline{\pi^{-1}(L_i \setminus 0)} \rightsquigarrow \tilde{L} \simeq \mathbb{A}^1$ (Claim)

$\tilde{L} \cap \pi^{-1}(0) \subset \pi^{-1}(0)^{\tilde{\theta}}$

Example:



$\tilde{\theta}$ action

Type I.

in Type A_n, n odd

(even case skipped)

§3-2 multiplicities

X^0 . irreducible components $L_1, \cdots L_m$ ('A' or cusp)

$\pi^{-1}(X^0)$ irreducible components $\underbrace{\tilde{L}_1 \cdots \tilde{L}_m}_{\text{'A'}}$, $\underbrace{C_1 \cdots C_n}_{\text{'P'}}$

reduced could be non-reduced.

$$\pi^{-1}(X^0) = \sum_{j=1}^{m} \tilde{L}_j + \sum_{i=1}^{n} a_i C_i \text{ as a divisor}$$

determ a_i

Define $b_i := \# \{ \tilde{C}_j \mid \tilde{L}_j \cap C_i \neq \phi \}$.

Prop 3 (H-) multiplicities If $X^0 \subset X$ is a principal divisor.

→ almost always true (modulo A_n n odd Type III, n even Type II)

then
$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = C^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

inverse of Cartan matrix