On certain Lagrangian subvarieties in minimal resolutions of Kleinian singularities

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Motivations

Kleinian singularities are the only normal Gorenstein singularities in dimension 2. Anti-Poisson involutions and their fixed loci appear naturally when we want to classify irreducible Harish-Chandra modules over Kleinian singularities.

Overview:

$$\begin{array}{cccc} \text{minimal resolution} & & \text{Kleinian singularity} & \\ \widetilde{X} & \xrightarrow{\pi} & X \circlearrowleft \theta & \text{anti-Poisson involution} \\ \cup & & \cup & \\ \pi^{-1}(X^{\theta}) & & X^{\theta} & \text{Lagrangian subvariety} \end{array}$$

Goal: Describe X^{θ} and $\pi^{-1}(X^{\theta})$ as schemes.

1 Kleinian singularities

2 Anti-Poisson involutions and their fixed loci

3 Preimage of fixed loci under minimal resolutions

Kleinian singularities

Kleinian singularities are quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup. The algebra of invariant functions $\mathbb{C}[u,v]^\Gamma$ is finitely generated. *Kleinian singularities* are the quotient varieties $X := \mathbb{C}^2/\Gamma = \operatorname{Spec} \mathbb{C}[u,v]^\Gamma$.

Example

When $\Gamma = \{\pm I_2\}$, we have $\mathbb{C}[u,v]^\Gamma =$ even degree polynomials $=$ $\mathbb{C}[x = u^2, y = v^2, z = uv] = \mathbb{C}[x,y,z]/(xy - z^2)$.

Fact (Klein, 1884)

$\mathbb{C}^2/\Gamma \hookrightarrow \mathbb{C}^3$ (one relation), and $\mathbb{C}^2/\Gamma$ is a singular surface with the only singularity at 0.

Kleinian singularities

Classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$:

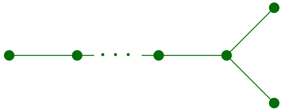
- The cyclic group of order $n+1$. A_n
  $xy - z^{n+1} = 0$
- The binary dihedral group of order $4(n-2)$, $n \geq 4$. D_n
  $x^{n-1} + xy^2 + z^2 = 0$
- The binary tetrahedral group of order 24. E_6
  $x^4 + y^3 + z^2 = 0$
- The binary octahedral group of order 48. E_7
  $x^3y + y^3 + z^2 = 0$
- The binary icosahedral group of order 120. E_8
  $x^5 + y^3 + z^2 = 0$

McKay correspondence: Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ are in bijection with *ADE* Dynkin diagrams.

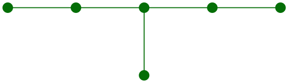
Kleinian singularities

A_n

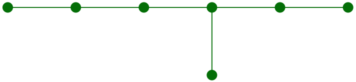
D_n



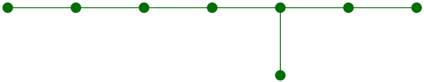
E_6



E_7



E_8



Kleinian singularities

How to attach a Dynkin diagram to a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$?

1. McKay Correspondence [McKay, 1979].
2. Minimal Resolution [Du Val, 1934]. By resolution we mean a smooth variety $\widetilde{X}$ equipped with a projective birational morphism $\pi\colon \widetilde{X} \to X := \mathbb{C}^2/\Gamma$. The minimality condition means that any other resolution factors through $\widetilde{X}$. The exceptional locus

$$\pi^{-1}(0) = C_1 \cup \cdots \cup C_n, \;\; C_i \simeq \mathbb{P}^1$$

is a connected union of $\mathbb{P}^1$'s. We can construct the dual graph of $\pi^{-1}(0)$ by replacing each C_i by a vertex i and joining vertices i and j by an edge if C_i intersects with C_j.

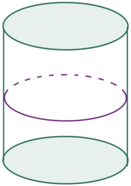
Fact (Du Val, 1934)

The dual graph of $\pi^{-1}(0)$ is the corresponding type Dynkin diagram.

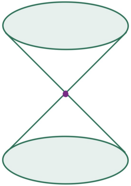
Kleinian singularities

For example,

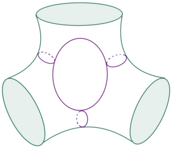
$T^*\mathbb{P}^1$



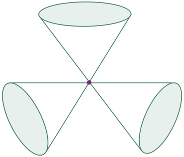
$\xrightarrow{\pi}$



A_1 singularity
$xy - z^2 = 0$



$\xrightarrow{\pi}$



D_4 singularity
$x^3 + xy^2 + z^2 = 0$

Anti-Poisson involutions

The algebra $\mathbb{C}[u, v]$ is a graded Poisson algebra with Poisson bracket

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u}.$$

The finite subgroup $\Gamma \subset SL_2(\mathbb{C})$ preserves this Poisson bracket, making $\mathbb{C}[u, v]^\Gamma$ into a graded Poisson subalgebra of $\mathbb{C}[u, v]$.

Example

On type $A_n : \mathbb{C}[x, y, z]/(xy - z^{n+1})$ Kleinian singularity singularity. The Poisson brackets are given by

$$\begin{aligned} \{x, y\} &= (n+1)^2 z^n, \\ \{x, z\} &= (n+1)x, \\ \{y, z\} &= -(n+1)y. \end{aligned}$$

Anti-Poisson involutions

Definition

An *anti-Poisson involution* of a Kleinian singularity $X := \mathbb{C}^2/\Gamma$ is a graded algebra involution $\theta\colon \mathbb{C}[X] \to \mathbb{C}[X]$ such that

$$\theta(\{f_1, f_2\}) = -\{\theta(f_1), \theta(f_2)\}, \ \forall \ f_1, f_2 \in \mathbb{C}[X].$$

Example

On type $A_n\colon \mathbb{C}[x, y, z]/(xy - z^{n+1})$ Kleinian singularity,

$$x \mapsto y, \ \ y \mapsto x, \ \ z \mapsto z$$

is an anti-Poisson involution.

Proposition

There are finitely many anti-Poisson involutions on $\mathbb{C}^2/\Gamma$ up to conjugation by Poisson automorphisms.

The fixed loci

The fixed locus $X^\theta := \operatorname{Spec} \mathbb{C}[X]/I$ where $I = (\theta(f) - f,\ f \in \mathbb{C}[X])$

Proposition (Description of X^θ)

1. *X^θ is a singular Lagrangian subvariety of X that contains* 0.
2. *X^θ is reduced and connected.*

Example (continued)

On type A_n: $\mathbb{C}[x,y,z]/(xy - z^{n+1})$ Kleinian singularity,

$$x \mapsto y,\ \ y \mapsto x,\ \ z \mapsto z$$

is an anti-Poisson involution. The fixed locus is

$$\operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^{n+1}, x - y) \simeq \operatorname{Spec} \mathbb{C}[x,z]/(x^2 - z^{n+1}),$$

which is a union of two $\mathbb{A}^1$'s when n is odd; a cusp when n is even.

Preimage of fixed loci under minimal resolutions

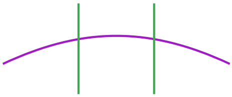
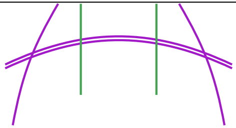
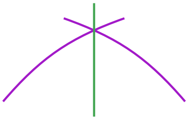
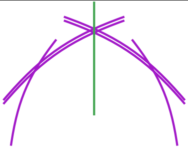
Recall $\pi\colon \widetilde{X} \to X$ denotes the minimal resolution. We would like to describe the preimage $\pi^{-1}(X^\theta)$.

Proposition

- *$\pi^{-1}(X^\theta)$ is a Lagrangian subvariety of $\widetilde{X}$ containing the exceptional locus $\pi^{-1}(0)$.*
- *Each irreducible component of $\pi^{-1}(X^\theta)$ is either a $\mathbb{P}^1$ or an $\mathbb{A}^1$.*
- *$\pi^{-1}(X^\theta)$ is connected.*
- *$\pi^{-1}(X^\theta)$ is usually non-reduced.*

Preimage of fixed loci under minimal resolutions

Example: Consider type A_n singularities with θ swapping $x \leftrightarrow y$. The fixed loci and their preimages look like the following.

X^θ	$\pi^{-1}(X^\theta)$	$\pi^{-1}(X^\theta)$
 A_n, n odd	 A_1	 A_3
 A_n, n even	 A_2	 A_4

Notation: straight green line is $\mathbb{A}^1$; curly purple line is $\mathbb{P}^1$; doubled line reflects non-reduced component.

Nakajima quiver varieties

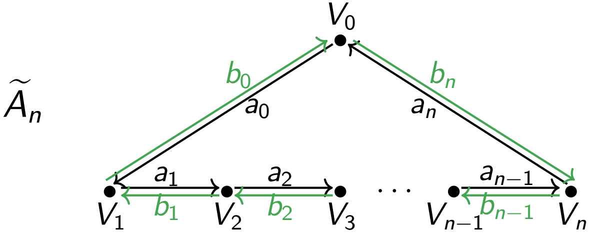
Question

How do the $\mathbb{P}^1$'s and $\mathbb{A}^1$'s in $\pi^{-1}(X^\theta)$ intersect with each other in other cases?

To solve this in general, we need to find a good way to describe the minimal resolution. We will realize Kleinian singularities and their minimal resolutions as **Nakajima quiver varieties** in types ADE.

Nakajima quiver varieties

Example: We realize type A_n: $xy - z^{n+1}$ singularity as quiver variety.



Each V_i is a 1-dimensional vector space. The group $G := \prod_{i=0}^{n} \mathrm{GL}(V_i)$ naturally acts on $M(V) := \bigoplus_{i=0}^{n} \mathrm{Hom}(V_i, V_{i+1}) \oplus \bigoplus_{i=0}^{n} \mathrm{Hom}(V_{i+1}, V_i)$ with moment map $\mu = (\mu_i)_{i=0}^{n} \colon M(V) \to \bigoplus_{i=0}^{n} \mathfrak{gl}(V_i)$ where

$$\mu_i = a_{i-1} b_{i-1} - b_i a_i.$$

We have $\mu^{-1}(0)/\!/G \simeq \operatorname{Spec} \mathbb{C}[x, y, z]/(xy - z^{n+1})$ with

$$x := a_n a_{n-1} \cdots a_1 a_0, \;\; y := b_0 b_1 \cdots b_{n-1} b_n, \;\; z := a_0 b_0.$$

Nakajima quiver varieties

Fixing a character $\chi\colon G \to \mathbb{C}^*$, we can consider the stable locus $\mu^{-1}(0)^s \subset \mu^{-1}(0)$ with respect to χ.

Theorem (Kronheimer 1989, Nakajima 1994)

For χ generic, the action of G on $\mu^{-1}(0)^s$ is free. The projective morphism

$$\mu^{-1}(0)^s//G \to \mu^{-1}(0)//G \simeq \mathbb{C}^2/\Gamma$$

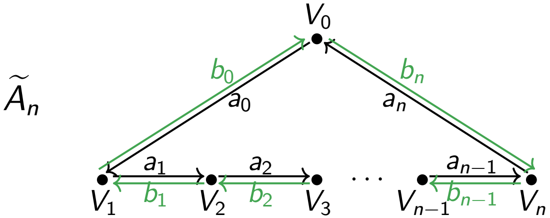
gives the minimal resolution of the Kleinian singularity.

Proposition (Nakajima 1998)

Fix the character $\chi(g) = \prod_{i=0}^n \det(g_i)$. Then $(a_i, b_i)_{i=0}^n \in \mu^{-1}(0)^s$ iff V_0 generates the entire $V = \oplus_{i=0}^n V_i$ under the action of $(a_i, b_i)_{i=0}^n$.

Nakajima quiver varieties

Example (continued): We describe $\mu^{-1}(0)^s$ for type A_n.



Set $U_i := \{(a_j, b_j)_{j=0}^n \in \mu^{-1}(0) | a_{i-1} \cdots a_1 a_0 \neq 0, b_{i+1} \cdots b_{n-1} b_n \neq 0\}$. Then we have

- U_i is G-invariant and χ-stable.
- $\{U_i\}_{i=0}^n$ forms an affine open cover of $\mu^{-1}(0)^s$.
- $U_i /\!/ G \simeq \operatorname{Spec} \mathbb{C}[u_i, v_i]$, where $u_i = \frac{b_i \cdots b_n}{a_{i-1} \cdots a_0}$, $v_i = \frac{a_i \cdots a_0}{b_{i+1} \cdots b_n}$.
- The morphism $U_i /\!/ G \to \mu^{-1}(0) /\!/ G \simeq \mathbb{C}^2 / \Gamma$ is given by

$$(u_i, v_i) \mapsto (x = u_i^{n-i} v_i^{n-i+1}, y = u_i^{i+1} v_i^i, z = u_i v_i)$$

Thank you for your attention!