Numerical Analysis Homework 2

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Program Usage:

$./solver -i matrix.txt -m {GESP or JCB}

1 Apply Gaussian and Jaboci's method (50 points)

1.1 Use Gaussian Elimination with Scaled Partial Pivoting

1.1.1 Question 6.2.1.c

$$2x_1 - 3x_2 + 2x_3 = 5 \tag{1}$$
$$-4x_1 + 2x_2 - 6x_3 = 14 \tag{2}$$
$$2x_1 + 2x_2 + 4x_3 = 8 \tag{3}$$

$$\left[\begin{array}{c|c} A & b \end{array}\right]^{(1)} = \left[\begin{array}{ccc|c} 2 & -3 & 2 & 5 \\ -4 & 2 & -6 & 14 \\ 2 & 2 & 4 & 8 \end{array}\right]$$

Find Max values for each lines:

$$S_i = max_{i=1}^{n} \|a_{ij}\| \tag{4}$$

$$S_1 = max\{\|2\|, \|-3\|, \|2\|\} = 3 \tag{5}$$
$$S_2 = max\{\|-4\|, \|2\|, \|-6\|\} = 6 \tag{6}$$
$$S_3 = max\{\|2\|, \|2\|, \|4\|\} = 3 \tag{7}$$

Now, find scaled partial pivots of first column to check if need interchanges

$$max_{i=1}^{n}\{\frac{\|a_{ij}\|}{S_i}\} \tag{8}$$
$$\frac{\|a_{11}\|}{S_1} = 0.666667 \tag{9}$$
$$\frac{\|a_{21}\|}{S_2} = 0.666667 \tag{10}$$
$$\frac{\|a_{31}\|}{S_3} = 0.500000 \tag{11}$$

Maximum pivot index=1. There is no row interchange Now calculate m_{21} and m_{31} to reduce rows

$$m_{21} = \frac{a_{21}}{a_{11}} = -2 \tag{12}$$
$$m_{31} = \frac{a_{31}}{a_{11}} = 1 \tag{13}$$

Apply this operations on matrix:

$$E_2 - m_{21} * E_1 \rightarrow E_2 \tag{14}$$
$$E_3 - m_{31} * E_1 \rightarrow E_3 \tag{15}$$

$$\left[\begin{array}{c|c} A & b \end{array}\right]^{(2)} = \left[\begin{array}{ccc|c} 2 & -3 & 2 & 5 \\ 0 & -4 & -2 & 24 \\ 0 & 5 & 2 & 3 \end{array}\right]$$

Now, find scaled partial pivots of second column to check if need interchanges

$$\frac{\|a_{22}\|}{S_2} = 0.666667 \tag{16}$$
$$\frac{\|a_{32}\|}{S_3} = 1.250000 \tag{17}$$

Maximum pivot index=3 and pivot=1.25. Change row 3 with 2.

$$\left[\begin{array}{c|c} A & b \end{array}\right]^{(2)} = \left[\begin{array}{ccc|c} 2 & -3 & 2 & 5 \\ 0 & 5 & 2 & 3 \\ 0 & -4 & -2 & 24 \end{array}\right]$$

Now calculate m_{32} to reduce row

$$m_{32} = \frac{a_{32}}{a_{22}} = -2 \tag{18}$$

Apply this operations on matrix:

$$E_3 - m_{32} * E_2 \rightarrow E_3 \tag{19}$$

$$\left[\ A \mid b\ \right]^{(3)} = \left[\begin{array}{ccc|c} 2 & -3 & 2 & 5 \\ 0 & 5 & 2 & 3 \\ 0 & 0 & -0.4 & 26.4 \end{array}\right]$$

Solve matrix with backward substitution.
Results:

$$\begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} 109 & 27 & -66 \end{bmatrix}^t$$

1.1.2 Question 7.3.1.a

$$3x_1 - 1x_2 + 1x_3 = 1 \tag{20}$$
$$3x_1 + 6x_2 + 2x_3 = 0 \tag{21}$$
$$3x_1 + 3x_2 + 7x_3 = 4 \tag{22}$$

$$\left[\ A \mid b\ \right]^{(1)} = \left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 3 & 6 & 2 & 0 \\ 3 & 3 & 7 & 4 \end{array}\right]$$

Find Max values for each lines:

$$S_i = max_{i=1}^{n} \|a_{ij}\| \tag{23}$$

$$S_1 = max\{\|3\|, \|-1\|, \|1\|\} = 3 \tag{24}$$
$$S_2 = max\{\|3\|, \|6\|, \|2\|\} = 6 \tag{25}$$
$$S_3 = max\{\|3\|, \|3\|, \|7\|\} = 7 \tag{26}$$

Now, find scaled partial pivots of first column to check if need interchanges

$$max_{i=1}^{n}\{\frac{\|a_{ij}\|}{S_i}\} \tag{27}$$
$$\frac{\|a_{11}\|}{S_1} = 1.000000 \tag{28}$$
$$\frac{\|a_{21}\|}{S_2} = 0.500000 \tag{29}$$
$$\frac{\|a_{31}\|}{S_3} = 0.428571 \tag{30}$$

Maximum pivot index=1 and pivot=1.0. There is no row interchange Now calculate m_{21} and m_{31} to reduce rows

$$m_{21} = \frac{a_{21}}{a_{11}} = 1 \tag{31}$$
$$m_{31} = \frac{a_{31}}{a_{11}} = 1 \tag{32}$$

Apply this operations on matrix:

$$E_2 - m_{21} * E_1 \rightarrow E_2 \tag{33}$$
$$E_3 - m_{31} * E_1 \rightarrow E_3 \tag{34}$$

$$\left[\ A \mid b\ \right]^{(2)} = \left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 7 & 1 & -1 \\ 0 & 4 & 6 & 3 \end{array}\right]$$

Now, find scaled partial pivots of second column to check if need interchanges

$$\frac{\|a_{22}\|}{S_2} = 1.166667 \tag{35}$$
$$\frac{\|a_{32}\|}{S_3} = 0.571429 \tag{36}$$

Maximum pivot index=2 and pivot=1.166667. No Row Change
Now calculate m_{32} to reduce row

$$m_{32} = \frac{a_{32}}{a_{22}} = 0.571429 \tag{37}$$

Apply this operations on matrix:

$$E_3 - m_{32} * E_2 \rightarrow E_3 \tag{38}$$

$$\left[\ A \mid b\ \right]^{(3)} = \left[\begin{array}{ccc|c} 3 & -1 & 1 & 1 \\ 0 & 7 & 1 & -1 \\ 0 & 0 & 5.429 & 3.571 \end{array}\right]$$

Solve matrix with backward substitution.
Results:

$$\begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} 0.0351 & -0.2368 & 0.6579 \end{bmatrix}^t$$

1.1.3 Question 7.3.1.b

$$10x_1 - x_2 + 0 = 9 \tag{39}$$
$$-x_1 + 10x_2 - 2x_3 = 7 \tag{40}$$
$$0 - 2x_2 + 10x_3 = 6 \tag{41}$$

$$\left[\begin{array}{c|c} A & b \end{array}\right]^{(1)} = \left[\begin{array}{ccc|c} 10 & -1 & 0 & 9 \\ -1 & 10 & -2 & 7 \\ 0 & -2 & 10 & 6 \end{array}\right]$$

Find Max values for each lines:

$$S_i = max_{i=1}^{n} \|a_{ij}\| \tag{42}$$

$$S_1 = max\{\|10\|, \|-1\|, \|0\|\} = 10 \tag{43}$$
$$S_2 = max\{\|-1\|, \|10\|, \|-2\|\} = 10 \tag{44}$$
$$S_3 = max\{\|0\|, \|-2\|, \|10\|\} = 10 \tag{45}$$

Now, find scaled partial pivots of first column to check if need interchanges

$$max_{i=1}^{n}\{\frac{\|a_{ij}\|}{S_i}\} \tag{46}$$
$$\frac{\|a_{11}\|}{S_1} = 1.000000 \tag{47}$$
$$\frac{\|a_{21}\|}{S_2} = 0.100000 \tag{48}$$
$$\frac{\|a_{31}\|}{S_3} = 0.000000 \tag{49}$$

Maximum pivot index=1 and pivot=1.0. There is no row interchange.
Now calculate m_{21} and m_{31} to reduce rows

$$m_{21} = \frac{a_{21}}{a_{11}} = -0.1 \tag{50}$$
$$m_{31} = \frac{a_{31}}{a_{11}} = 0 \tag{51}$$

Apply this operations on matrix:

$$E_2 - m_{21} * E_1 \rightarrow E_2 \tag{52}$$
$$E_3 - m_{31} * E_1 \rightarrow E_3 \tag{53}$$

$$\left[\begin{array}{c|c} A & b \end{array}\right]^{(2)} = \left[\begin{array}{ccc|c} 10 & -1 & 0 & 9 \\ 0 & 9.9 & -2 & 7.9 \\ 0 & -2 & 10 & 6 \end{array}\right]$$

Now, find scaled partial pivots of second column to check if need interchanges

$$\frac{\|a_{22}\|}{S_2} = 0.990000 \tag{54}$$
$$\frac{\|a_{32}\|}{S_3} = 0.200000 \tag{55}$$

Maximum pivot index=2 and pivot=0.990000. No Row Change
Now calculate m_{32} to reduce row

$$m_{32} = \frac{a_{32}}{a_{22}} = 0.202020 \tag{56}$$

Apply this operations on matrix:

$$E_3 - m_{32} * E_2 \rightarrow E_3 \tag{57}$$

$$\left[\begin{array}{c|c} A & b \end{array}\right]^{(3)} = \left[\begin{array}{ccc|c} 10 & -1 & 0 & 9 \\ 0 & 9.9 & -2 & 7.9 \\ 0 & 0 & 9.596 & 7.596 \end{array}\right]$$

Solve matrix with backward substitution.
Results:

$$\begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} 0.9958 & 0.9579 & 0.7916 \end{bmatrix}^t$$

1.2 Use Jacobi's Method

$$x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j!=i}^{n}\left(a_{ij}x_j^k\right)\right), i = 1, 2, 3, ...n \tag{58}$$

1.2.1 Question 6.2.1.c

Iterations:

Iter	X_1	X_2	X_3
0	0.0000	0.0000	0.0000
1	2.5000	7.0000	2.0000
2	11.0000	18.0000	-2.7500
3	32.2500	20.7500	-12.5000
4	46.1250	34.0000	-24.5000
5	78.0000	25.7500	-38.0625
6	79.1875	48.8125	-49.8750
7	125.5938	15.7500	-62.0000
8	88.1250	72.1875	-68.6719
9	179.4531	-22.7656	-78.1562
10	46.5078	131.4375	-76.3438
11	276.0000	-129.0156	-86.9727
12	-104.0508	298.0820	-71.4922
13	521.1152	-415.5781	-95.0156
14	-525.8516	764.1836	-50.7686
15	1199.5439	-1197.0088	-117.1660
16	-1675.8472	2054.5898	0.7324

Couldn't find specific solution for this problem with Jacobi.
Have to change stopping criterion which is 0.001.

1.2.2 Question 7.3.1.a

Iterations:

Iter	X_1	X_2	X_3
0	0.0000	0.0000	0.0000
1	0.3333	0.0000	0.5714
2	0.1429	-0.3571	0.4286
3	0.0714	-0.2143	0.6633
4	0.0408	-0.2568	0.6327
5	0.0368	-0.2313	0.6640
6	0.0349	-0.2398	0.6548
7	0.0352	-0.2357	0.6592
8	0.0350	-0.2373	0.6574
9	0.0351	-0.2366	0.6581
10	0.0351	-0.2369	0.6578

$X_1^{10} = 0.0351, X_2^{10} = -0.2369, X_3^{10} = 0.6578$

Found exact solution with stopping criteria=0.001

1.2.3 Question 7.3.1.b

Iterations:

Iter	X_1	X_2	X_3
0	0.0000	0.0000	0.0000
1	0.9000	0.7000	0.6000
2	0.9700	0.9100	0.7400
3	0.9910	0.9450	0.7820
4	0.9945	0.9555	0.7890
5	0.9956	0.9572	0.7911

$X_1^5 = 0.9956, X_5^{10} = 0.9572, X_5^{10} = 0.7911$

Found exact solution with stopping criteria=0.001

2 Image Registration (50 Points)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

B(x,y)=[1,2],[2,1],[3,1]
F(x',y')=[2,2],[-1,4],[-4,4]

$$A * B = F : \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix}$$

$$A * B^1 = F^1 : \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

Multiply matrix A and B^1

$$a_{11} + 2a_{12} + a_{13} = 2 \tag{59}$$
$$a_{21} + 2a_{22} + a_{23} = 2 \tag{60}$$

$$A * B^2 = F^2 : \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

Multiply matrix A and B^2

$$2a_{11} + a_{12} + a_{13} = -1 \tag{61}$$
$$2a_{21} + a_{22} + a_{23} = 4 \tag{62}$$

$$A * B^3 = F^3 : \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ 1 \end{bmatrix}$$

Multiply matrix A and B^3

$$3a_{11} + a_{12} + a_{13} = -4 \tag{63}$$
$$3a_{21} + a_{22} + a_{23} = 4 \tag{64}$$

Combine equation 59, 61 and 63 to find a_{11}, a_{12} and a_{13} unknowns

$$\left[\begin{array}{c|c} H & h \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & -1 \\ 3 & 1 & 1 & -4 \end{array}\right]$$

Combine equation 60, 62 and 64 to find a_{21}, a_{22} and a_{23} unknowns

$$\left[\begin{array}{c|c} G & g \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 3 & 1 & 1 & 4 \end{array}\right]$$

Step 0. Let's apply gauss elimination to find unknowns for H.
Step 1. Find m_{21} and m_{31} to eliminate 1st column.

$$m_{21} = \frac{H_{21}}{H_{11}} = \frac{2}{1} = 2 \tag{65}$$
$$m_{31} = \frac{H_{31}}{H_{11}} = \frac{3}{1} = 3 \tag{66}$$
$$H_2 - m_{21} * H_1 \to H_2 \tag{67}$$
$$H_3 - m_{31} * H_1 \to H_3 \tag{68}$$

$$\left[\begin{array}{c|c} H & h \end{array}\right]^1 = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & -5 & -2 & -10 \end{array}\right]$$

Step 2: Pivot=2, eliminate 2nd column

$$m_{32} = \frac{H_{32}}{H_{22}} = \frac{-5}{-3} = 1.6667 \tag{69}$$
$$H_3 - m_{32} * H_2 \to H_3 \tag{70}$$
$$\tag{71}$$

$$\left[\begin{array}{c|c} H & h \end{array}\right]^2 = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & -5 \\ 0 & 0 & -0.333 & -1.665 \end{array}\right]$$

Step 3: Solve H^2 with backward substitution.

$$h_1 = a_{11} = -3 \tag{72}$$
$$h_2 = a_{12} = 0 \tag{73}$$
$$h_3 = a_{13} = 5 \tag{74}$$

Step 4. Let's apply gauss elimination to find unknowns for G.
Step 5. Find m_{21} and m_{31} to eliminate 1st column.

$$m_{21} = \frac{G_{21}}{G_{11}} = \frac{2}{1} = 2 \tag{75}$$
$$m_{31} = \frac{G_{31}}{G_{11}} = \frac{3}{1} = 3 \tag{76}$$
$$G_2 - m_{21} * G_1 \to G_2 \tag{77}$$
$$G_3 - m_{31} * G_1 \to G_3 \tag{78}$$

$$\left[\begin{array}{c|c} G & g \end{array}\right]^1 = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & 0 \\ 0 & -5 & -2 & -2 \end{array}\right]$$

Step 6: Pivot=2, eliminate 2nd column

$$m_{32} = \frac{G_{32}}{G_{22}} = \frac{-5}{-3} = 1.6667 \tag{79}$$
$$G_3 - m_{32} * G_2 \to G_3 \tag{80}$$
$$\tag{81}$$

$$\left[\begin{array}{c|c} G & g \end{array}\right]^2 = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & -0.333 & -2 \end{array}\right]$$

Step 7: Solve G^2 with backward substitution.

$$g_1 = a_{21} = 0 \tag{82}$$
$$g_2 = a_{22} = -2 \tag{83}$$
$$g_3 = a_{23} = 6 \tag{84}$$

A matrix is:

$$A = \begin{bmatrix} -3 & 0 & 5 \\ 0 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

Lastly, we need to find **inverse of matrix A(A^{-1}) to transform F to B**.

$$\left[\ A|I\ \right]^1 = \left[\begin{array}{ccc|ccc} -3 & 0 & 5 & 1 & 0 & 0 \\ 0 & -2 & 6 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A_1/-3 \rightarrow A_1 \tag{85}$$
$$A_2/-2 \rightarrow A_2 \tag{86}$$

$$\left[\ A|I\ \right]^2 = \left[\begin{array}{ccc|ccc} 1 & 0 & -1.667 & -0.333 & 0 & 0 \\ 0 & 1 & -3 & 0 & -0.5 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A_2 + 3A_3 \rightarrow A_2 \tag{87}$$
$$A_1 + 1.667A_3 \rightarrow A_1 \tag{88}$$

$$\left[\ A|I\ \right]^2 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.333 & 0 & 1.667 \\ 0 & 1 & 0 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Inverse of A(A^{-1})

$$A^{-1} = \begin{bmatrix} -0.333 & 0 & 1.667 \\ 0 & -0.5 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$