# HETEROGENOUS AGENT MODELS

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# MOTIVATION

- We introduce a class of models with heterogeneous agents.
- We focus on models with household heterogeneity and incomplete markets: Aiyagari-Bewley-Hugget-Imrohoroglu models.
  - Infinitely lived households face idiosyncratic shocks.
  - Markets are incomplete – households cannot trade all assets they would like to trade.
- We will find general equilibrium of these models.

# AGENTS

- In all these models there is a continuum of agents/households indexed by $i \in [0,1]$.
- Agents are infinitely lived, have a standard period utility function $u(c)$, where $c$ is consumption, discount future at $\beta \in (0,1)$.
- Labor endowment (productivity) is stochastic and follows a stationary Markov process (same process for all $i$), with a finite set of states Z and transition matrix $P$.
- Agents consume and trade a single asset $a$ that pays (net) return $r$. Wage per unit of labor is $w$.
- Agents face a borrowing constraint, $a \geq -\phi$.

# AGENTS

- State variables: beginning of period assets $a$ and labor endowment $z$.
- Recursive formulation of a problem of an agent with assets $a$ and labor endowment $z$ is

$$V(a,z) = \max_{c,a'} \left\{ u(c) + \beta \sum_{z' \in Z} P(z,z') V(a',z') \right\}$$

$$\text{subject to } c + a' = zw + (1+r)\,a,$$

$$a' \geq -\phi.$$

- The solution consists of the value function $V(a,z)$ and the policy functions $a'(a,z), c(a,z)$.
- Note: everything (e.g. $V$) depends on $(r,w)$, we supress it in notation.

# DISTRIBUTION

- Suppose that there a finite grid of asset levels $(a_1, \ldots, a_N)$ and define the unconditional distribution $\lambda_t(a, z) := \mathbb{P}(a_t = a, z_t = z)$.
- Given the policy function $a'(a, z)$ and $P$ we have

$$\lambda_{t+1}(a', z') = \sum_{a,z} \lambda_t(a, z) \cdot P(z, z') \cdot \mathcal{I}(a' = a'(a, z)),$$

  where $\mathcal{I}$ is the indicator function.
- What if assets are not restricted to a grid? Similar logic, but a more messy formula (+ measure theory).
- We will need to discretize the state space and action space anyway.

## AGGREGATION

- Given a distribution $\lambda_t$ and policy functions $a'(a,z)$ and $c(a,z)$ we can calculate aggregate variables.
- We have asset demand and consumption

$$
\begin{aligned}
A'_t &= \int a'(a,z) \cdot d\lambda_t(a,z), \\
C_t &= \int c(a,z) \cdot d\lambda_t(a,z).
\end{aligned}
$$

- I switched notation again, for a finite grid we have $A'_t = \sum_{a,z} a'(a,z) \cdot \lambda_t(a,z)$ and $C_t = \sum_{a,z} c(a,z) \cdot \lambda_t(a,z)$.

# THE REST OF THE ECONOMY

- How are $r$ and $w$ determined?
- What/who supplies goods?
- What/who supplies assets?
- We will start with the Hugget model:
  - Endowment economy: $w$ exogenous, $wz$ is the amount of goods received by an agent.
  - Nothing/nobody else in the economy. Supply of assets is zero.
- Market clearing conditions:
  - Goods market: $C_t = w \int z_i di$.
  - Asset market: $A'_t = 0$.

## HUGGET MODEL

- Assets are loans from/to agents.
- Asset market clears if the total amount of loans (negative $a$) is equal to the total amount of savings (positive $a$).
- We will be looking for a stationary equilibrium: the distribution of agents and prices (here only $r$) are constant over time.
- This was already anticipated by how we wrote the Bellman equation – no time subscripts anywhere.

# RCE IN HUGGET MODEL

**Definition (Stationary recursive competitive equilibrium (RCE))**

A stationary recursive competitive equilibrium is a rate of return $r$, a value function $V(a,z)$, policy functions $a'(a,z)$ and $c(a,z)$, and a distribution $\lambda(a,z)$ such that

1. Given $r$, the value function $V(a,z)$ satisfies the Bellman equation, and associated policy functions $a'(a,z)$, $c(a,z)$, solve the agent's maximization problem;
2. The probability distribution $\lambda(a,z)$ is the stationary distribution of the Markov process $(a_t, z_t)$ induced by $P$ and $a'(a,z)$;
3. Markets clears:

$$\int a'(a,z) \cdot d\lambda(a,z) = 0, \quad \int c(a,z) \cdot d\lambda(a,z) = w \int z d\lambda(a,z).$$

## COMMENTS

- We now need to find $r$ such that aggregate asset demand resulting from optimal decisions of agents is zero.
- Alternatively, we can find $r$ such that the goods market clears.
- Recall: the Walras law says that if there are $N$ markets and $N-1$ clear, the $N$-th market also clears.

## COMMENTS

- The usual procedure is:
  1. Guess $r$.
  2. Solve the Bellman equation for $V(a,z)$ and $a'(a,z)$.
  3. Find the stationary distribution $\lambda(a,z)$.
  4. Calculate aggregate asset demand $A'$.
  5. If $A' > 0$, decrease $r$, if $A' < 0$, increase $r$.
- This combines several things:
  - solving the Bellman equation
  - finding the stationary distribution
  - (new) finding the equilibrium price $r$.

## COMPUTATION

- When we solve the model on a computer we discretize the state space and have a finite grid of points for assets: $(a_1, \ldots, a_N)$ and for productivity $(z_1, \ldots, z_M)$.
- We usually want to allow maximizers of the RHS of the Bellman equation to not necessarily belong to the grid.
- We cannot use the formula

$$\lambda\left(a', z'\right) = \sum_{a,z} \lambda(a,z) \cdot P\left(z, z'\right) \cdot \mathcal{I}\left(a' = a'(a,z)\right),$$

  because $a'(a,z)$ might not belong to the grid.
- How to find the stationary distribution $\lambda(a,z)$ in this case?

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## YOUNG (2010)

- Let

$$q(a,z,a_n) = \mathbb{I}\left(a'(a,z) \in [a_{n-1}, a_n]\right)\frac{a'(a,z) - a_{n-1}}{a_n - a_{n-1}} + \mathbb{I}\left(a'(a,z) \in [a_n, a_{n+1}]\right)\frac{a_{n+1} - a'(a,z)}{a_{n+1} - a_n}$$

  be the distribution of agents with assets $a_n$ and $a_{n+1}$.

- Then

$$\lambda\left(a', z'\right) = \sum_{a,z} \lambda(a,z) \cdot P\left(z, z'\right) \cdot q\left(a, z, a'\right).$$

# YOUNG (2010)

- This the same as saying that agent with assets $a$ and productivity $z$ will choose $a_n$ with probability $q(a, z, a_n, )$.
- Given there is a continuum of agents, this is the same as saying that the fraction $q(a, z, a_n, )$ of agents with assets $a$ and productivity $z$ will choose $a_n$.
- This approach (due to Young (2010)) is useful because it makes aggregates unbiased.

# AGGREGATION AND EQUILIBRIUM

- Once we have the distribution $\lambda(a, z)$ we can calculate aggregates.
- For example:

$$A' = \sum_{a,z} a'(a, z) \cdot \lambda(a, z).$$

- We can repeat the same procedure for various $r$ to find the equilibrium $r$, such that $A' = 0$.
- Better: write a function that takes $r$ as an input and returns $A'$ as an output. Then use a root finding algorithm to find the equilibrium $r$.

# DISCUSSION

- Suppose we want to use a bracketing method to find the equilibrium $r$.
- What are the appropriate bounds? Probably $r < -1$ does not make sense (nobody would save). The upper bound is more tricky.
- We can show that $r < \beta^{-1} - 1$ in equilibrium for the Hugget model (just see what happens if $r > \beta^{-1} - 1$).

# DISCUSSION

- Here you need to solve the Bellman equation for each $r$: possibly many times.
- This can be costly - try to optimize the code.
  - Use HPI or OPI.
  - Use EGM.
  - Get the transition matrix for $(a, z)$ and use it to find the stationary distribution (do not simulate anything!)
  - Check out Simon Mongey's slides – general notation but similar models.

# AIYAGARI MODEL

- In Aiyagari model there is a representative firm that hires labor and rents capital from households.
- The firm has a constant returns to scale production function $F(K, L)$, where $K$ is capital and $L$ is labor.
- The firm is competitive, so it takes $r$ and $w$ as given.
- The firm's problem is

$$\max_{K,L} F(K, L) - (r + \delta) K - wL.$$

# AIYAGARI MODEL

- Assets accumulated by households are capital and loans to other agents.
- Labor market clearing: $L = \int z \cdot d\lambda(a, z)$.
- Asset market clearing: $K = \int a \cdot d\lambda(a, z)$.
- Goods market clearing: $F(K, L) = C + \delta K$.

# AIYAGARI MODEL

- Notice that here we have

$$r = F_K(K, L) - \delta, \quad w = F_L(K, L).$$

- Because of the constant returns to scale

$$r = F_K\left(\frac{K}{L}, 1\right) - \delta, \quad w = F_L\left(\frac{K}{L}, 1\right).$$

- We can solve for $K/L$ as a function of $r$. This also allows us to solve for $w$ as a function of $r$.
- We know $L$ (it is exogenous) so we have $K(r)$

# AIYAGARI MODEL

- This suggests a following strategy:
  - Guess $r$. Repeat all the steps from the Hugget model to find $A'$.
  - Calculate $K$.
  - Check if $A' = K$. If yes, we found the equilibrium $r$. If not, adjust (how?) $r$ and repeat.
- We can simply find the root of $A'(r) - K(r) = 0$.

# ADDING GOVERNMENT

- In many applications we are interested in the effects of some government policies.
- Example: how does an increase in taxation affect the wealth distribution?
- Example: how does an increase in government debt crowd out capital accumulation?
- We will now consider a simple extension of the Aiyagari model with government.

## ADDING GOVERNMENT

- The intertemporal budget constraint of the government is

$$B_{t+1} = (1 + r)\, B_t + T_t - G_t.$$

  where $B_t$ is the government debt, $T_t$ is tax revenue net of transfers and $G_t$ is government purchases of goods.
- The government collects taxes on labor and capital income. Linear tax system with rates $\tau^w, \tau^r$. Tax revenue net of transfers is

$$T = \int \tau^w w z_i di + \int \tau^r r a_i - d.$$

## ADDING GOVERNMENT

- In a stationary equilibrium the government budget constraint becomes

$$rB = T - G.$$

- Asset market clearing: $K + B = \int a \cdot d\lambda(a, z)$.
- Goods market clearing: $F(K, L) = C + \delta K + G$.
- Key difference: assets available in the economy are $K + B$, not just $K$.

## ADDING GOVERNMENT

- We also need to modify the household problem.
- Recursive formulation of a problem of an agent with assets $a$ and labor endowment $z$ is

$$V(a,z) = \max_{c,a'} \left\{ u(c) + \beta \sum_{z' \in Z} P(z,z') V(a',z') \right\}$$

$$\text{subject to } c + a' = (1 - \tau^{w})\, zw + \left(1 + (1 - \tau^{r})\, r\right) a + d,$$

$$a' \geq -\phi.$$

- Note: dependence on government policies, we supress it in notation.