CONTINUOUS STATE VARIABLES: NEOCLASSICAL GROWTH MODEL

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MOTIVATION

- We analyzed finite Markov decision processes (MDP).
  1. X is a finite set: state space.
  2. A is a finite set: action space.
- In many interesting applications, state space and action space are not finite.
- Example: a household chooses how much to consume and how much to save: real numbers.
- Instead of studying general results I will focus on a specific example – neoclassical growth model – and illustrate some computational issues.

NEOCLASSICAL GROWTH MODEL

- The planner maximizes

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \quad \text{subject to} \quad k_{t+1} = z_t f(k_t) + (1-\delta)\, k_t - c_t \text{ for all } t \geq 0$$

- We restrict $k_t, c_t \geq 0$ for all $t \geq 0$. k_0 is given.
- $\beta \in (0,1)$ is the discount factor, $\delta \in (0,1]$ is the depreciation rate.
- u is a utility function and f is a production function.
- $z_t > 0$ is a productivity shock that follows a Markov chain with values in Z. For simplicity assume Z is finite and P is a transition matrix.

NEOCLASSICAL GROWTH MODEL

- We assume
  - $u : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strictly increasing, strictly concave and twice continuously differentiable with

$$\lim_{c \to 0} u'(c) = \infty, \ \lim_{c \to \infty} u'(c) = 0.$$

  - $f : \mathbb{R}_+ \to \mathbb{R}$ is continuous, strictly increasing, strictly concave and twice continuously differentiable with

$$\lim_{k \to 0} f'(k) = \infty, \ \lim_{k \to \infty} f'(k) = 0, \ f(0) = 0.$$

- These assumptions will allow us to ignore corner solutions.

NEOCLASSICAL GROWTH MODEL

- The state variables are

$$x_t := (k_t, z_t) \in \mathsf{X} := \mathbb{R}_+ \times \mathsf{Z}.$$

- The state space is not finite.
- Let action a be the choice of capital next period:

$$a_t := k_{t+1} \in \mathsf{A} := \mathbb{R}_+.$$

- The action space is not finite.
- It is also possible to associate a_t with consumption c_t.

NEOCLASSICAL GROWTH MODEL

- Bellman equation is

$$V(k,z) = \max_{k' \in \Gamma(k,z)} \left\{ u\left(zf(k) - k'\right) + \beta \sum_{z' \in Z} P\left(z,z'\right) V\left(k',z'\right) \right\},$$

where

$$\Gamma(k,z) = \left[0, zf(k) + (1-\delta)\, k\right].$$

NEOCLASSICAL GROWTH MODEL

- Problem: things we learned about finite state MDPs do not apply here!
- Easier part: theorems that do not require finiteness of state space and action space (see Stokey and Lucas, 1989).
- Harder part: how to solve it numerically in an efficient way?

DISCRETIZATION

DISCRETIZATION

- One possible idea is to discretize the state space and action space and then solve the problem numerically using the methods we learned for finite state MDPs.
- We know we will find an exact solution to the discretized problem, but will it be a good approximation to the original problem?
- We first need to choose a grid for k and z.
- We focus on k – assume that z is finite state Markov chain (if it is not, we can discretize it as well using Tauchen or Rouwenhurst method).

DISCRETIZATION

- First step: choose $k_{\min}$ and $k_{\max}$ and the number of points N_k.
- This often requires some experimentation.
  - Too many points – it will make the problem costly to solve.
  - Too few points – it will make the solution to the discretized problem far from the true solution of the original problem.
  - The interval $[k_{\min}, k_{\max}]$ too wide – it will make the problem costly to solve.
  - The interval $[k_{\min}, k_{\max}]$ too narrow – it will impose a constraint that was not present in the original problem.

DISCRETIZATION

- Do not choose $k_{\min} = 0$: recall that $f(0) = 0$, so if $\lim_{c \to 0} u(c) = -\infty$ then you will run into problems.
- To choose $k_{\max}$, note that because $\lim_{k \to \infty} f'(k) = 0$ for each z there exists a $k_{\max,z}$ such that $zf(k) + (1 - \delta) k < k$ for all $k > k_{\max,z}$.

RESIDUALS

- How to verify that the solution to the discretized problem is a good approximation to the solution of the original problem?
- Recall the RHS of the Bellman equation:

$$\max_{k' \in \Gamma(k,z)} \left\{ u\left(zf(k) + (1-\delta)k - k'\right) + \beta \sum_{z' \in Z} P\left(z, z'\right) V\left(k', z'\right) \right\}.$$

- The first order condition with respect to k' is

$$-u'\left(zf(k)\,(1-\delta)\,k - k'\right) + \beta \sum_{z' \in Z} P\left(z, z'\right) V'\left(k', z'\right) = 0.$$

where $V'\left(k', z'\right)$ is the derivative of V with respect to k'.

RESIDUALS

- Evaluate the LHS of the first order condition at the solution to the discretized problem at various k (including these that did not belong to the grid) and z.
- If the solution is a good approximation, the LHS should be close to zero.
- What does "close" mean?

RESIDUALS

- The expression has derivatives of the value function
- We want to have a measure that is easier to interpret.
- Easy to show (envelope condition) that

$$V'(k,z) = \left(zf'(k) + 1 - \delta\right) u'(c)$$

  where c is the optimal consumption.
- Calculate

$$\mathcal{H}(k,z) := c - \left(u'\right)^{-1} \left(\beta \sum_{z' \in Z} P(z,z') z' f'\left(k'\right) u'\left(c'\right) \right).$$

RESIDUALS

- Euler equation residuals are

$$\mathcal{R}(k,z) := \frac{\mathcal{H}(k,z)}{c(k,z)}.$$

- Interpretation: agents make $\mathcal{R}(k,z)\%$ mistakes when choosing c.
- Example: if $\mathcal{R}(k,z)$ is 0.01, then agents spend 1 dollar per 100 dollars "incorrectly".
- Bounded rationality interpretation.

RESIDUALS

- Euler equation residuals should be quite small.
- They should be even smaller for points (k, z) that are visited frequently.
- You can immediately see the problem with a grid that is too coarse: grid points are unlikely to correspond to the exact maximizer k'.

MAXIMIZATION

BELLMAN EQUATION

- Recall the Bellman equation

$$V(k,z) = \max_{k' \in \Gamma(k,z)} \left\{ u\left(zf(k) + (1-\delta)\,k - k'\right) + \beta \sum_{z' \in Z} P\left(z,z'\right) V\left(k',z'\right) \right\},$$

- Solve the maximization problem directly using techniques we learned when we talked about optimization.
- We do not need to assume that k' belongs to the grid.
- This gives us hope to finding the exact maximizer k'.
- Problem: we do not know the exact value of V at points that do not belong to the grid.

APPROXIMATION

- We will use an approximation of V at points that do not belong to the grid, $\hat{V}$.
- Continuous piecewise linear interpolation is a safe choice.
- Then for each k and z we have to solve for k' using

$$\max_{k' \in \Gamma(k,z)} u\left(zf(k) + (1-\delta)\,k - k'\right) + \beta \sum_{z' \in Z} P\left(z, z'\right)\hat{V}\left(k', z'\right).$$

MODIFICATION

- In the above we have

$$\sum_{z' \in Z} P(z, z') \hat{V}(k', z')$$

- We can approximate V and then calculate the sum (the expected value)
- It is usually cheaper to approximate the expected value directly.
- Let

$$W(k', z) := \sum_{z' \in Z} P(z, z') V\prime(k', z')$$

- Obtain $\hat{W}(k', z)$ and then solve

$$\max_{k' \in \Gamma(k,z)} u(zf(k) + (1 - \delta) k - k') + \beta \hat{W}(k', z).$$

SUMMARY

- Start with some grid, as in the previous section.
- Instead of finding k' that belongs to the grid (by comparing continuation values directly), solve the maximization problem directly.
- This requires using some optimizer for each k and z.
- This requires knowing $\hat{V}$ at points that do not belong to the grid.
- Important: $\hat{V}$ must preserve the concavity of V.

FIRST ORDER CONDITION

FIRST ORDER CONDITION

- The maximization problem

$$\max_{k' \in \Gamma(k,z)} u\left(zf(k) + (1-\delta)\,k - k'\right) + \beta \sum_{z' \in Z} P\left(z, z'\right) V\left(k', z'\right),$$

  has a first order condition

$$-u'\left(zf(k) + (1-\delta)\,k - k'\right) + \beta \sum_{z' \in Z} P\left(z, z'\right) V'\left(k', z'\right) = 0.$$

- Instead of solving the maximization problem directly, we can solve the first order condition (a nonlinear equation) for k'.
- To find it, we can use for example the bisection method or Newton's method.

FIRST ORDER CONDITION

- Same comments as with the maximization problem apply,
- We need to approximate V' at points that do not belong to the grid.
- We can do it by approximating V and then calculating the derivative or by approximating the derivative directly.
- We can also approximate W or W' instead.

ENDOGENOUS GRID METHOD

FIRST ORDER CONDITION

- Recall the idea of the previous approach: fix a grid for k and z and then solve the nonlinear equation for k'.
- The nonlinear equation is the most costly part.
- In some cases we can avoid it by using the endogenous grid method.

ENDOGENOUS GRID METHOD

- The first order condition was

$$u'\left(zf(k) + (1-\delta)\,k - k'\right) = \beta W'\left(k', z\right)$$

  and we wanted to find k' as a function of k and z.
- The endogenous grid method (by Chris Carroll) inverts the logic.
- Suppose you want to go to some k' given z. What k do you need to start with?

ENDOGENOUS GRID METHOD

- Start with a grid for k' and z.
- Invert the first order condition as

$$zf(k) + (1-\delta)\,k - k' = (u')^{-1}\left(\beta W'\left(k', z\right)\right)$$

  and solve for k.
- Although you might need to use a nonlinear solver, you do not need to evaluate W' repeatedly.
- Now we know k as a function of k' and z.
- After we did it for all k' and z, we have an endogenous grid for k.

ENDOGENOUS GRID METHOD

- We now know $k(k', z)$ for each k' and z in an exogenous grid.
- We now use interpolation to get $k'(k, z)$ for each k and z in the same exogenous grid.
- After that we have V for each (k,z) in the exogenous grid.
- Get W and repeat the process.