CONTINUOUS STATE VARIABLES: NEOCLASSICAL GROWTH MODEL

QUANTITATIVE ECONOMICS 2024

Piotr Żoch

January 10, 2025

MOTIVATION

- We analyzed finite Markov decision processes (MDP).
 - 1. X is a finite set: state space.
 - 2. A is a finite set: action space.
- In many interesting applications, state space and action space are not finite.
- Example: a household chooses how much to consume and how much to save: real numbers.
- Instead of studying general results I will focus on a specific example neoclassical growth model – and illustrate some computational issues.

The planner maximizes

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t), \quad \text{subject to} \quad k_{t+1} = z_t f(k_t) + (1-\delta) k_t - c_t \text{ for all } t \ge 0$$

- We restrict $k_t, c_t \ge 0$ for all $t \ge 0$. k_0 is given.
- $\beta \in (0,1)$ is the discount factor, $\delta \in (0,1]$ is the depreciation rate.
- *u* is a utility function and *f* is a production function.
- z_t > 0 is a productivity shock that follows a Markov chain with values in Z. For simplicity
 assume Z is finite and P is a transition matrix.

- We assume
 - $-u:\mathbb{R}_+\to\mathbb{R}$ is continuous, strictly increasing, strictly concave and twice continuously differentiable with

$$\lim_{c\to 0} u'(c) = \infty, \lim_{c\to \infty} u'(c) = 0.$$

 $-f: \mathbb{R}_+ \to \mathbb{R}$ is continuous, strictly increasing, strictly concave and twice continuously differentiable with

$$\lim_{k\to 0} f'(k) = \infty, \lim_{k\to \infty} f'(k) = 0, f(0) = 0.$$

These assumptions will allow us to ignore corner solutions.

The state variables are

$$x_t := (k_t, z_t) \in X := \mathbb{R}_+ \times Z.$$

- The state space is not finite.
- Let action *a* be the choice of capital next period:

$$a_t \coloneqq k_{t+1} \in \mathsf{A} \coloneqq \mathbb{R}_+.$$

- The action space is not finite.
- It is also possible to associate a_t with consumption c_t .

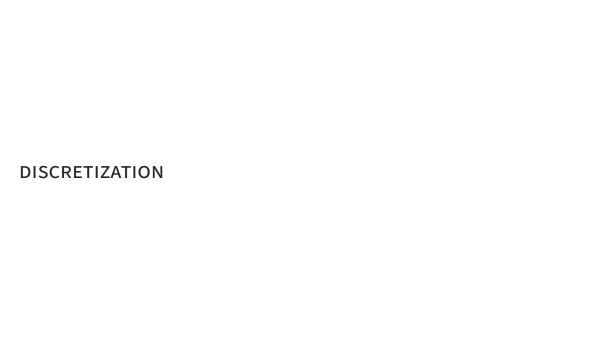
• Bellman equation is

$$V(k,z) = \max_{k' \in \Gamma(k,z)} \left\{ u(zf(k) - k') + \beta \sum_{z' \in \mathbb{Z}} P(z,z')V(k',z') \right\},\,$$

where

$$\Gamma(k,z) = [0,zf(k) + (1-\delta)k].$$

- Problem: things we learned about finite state MDPs do not apply here!
- Easier part: theorems that do not require finiteness of state space and action space (see Stokey and Lucas, 1989).
- Harder part: how to solve it numerically in an efficient way?



DISCRETIZATION

- One possible idea is to discretize the state space and action space and then solve the problem numerically using the methods we learned for finite state MDPs.
- We know we will find an exact solution to the discretized problem, but will it be a good approximation to the original problem?
- We first need to choose a grid for k and z.
- We focus on k assume that z is finite state Markov chain (if it is not, we can discretize it as well using Tauchen or Rouwenhurst method).

DISCRETIZATION

- First step: choose k_{\min} and k_{\max} and the number of points N_k .
- This often requires some experimentation.
 - Too many points it will make the problem costly to solve.
 - Too few points it will make the solution to the discretized problem far from the true solution of the original problem.
 - The interval $[k_{min}, k_{max}]$ too wide it will make the problem costly to solve.
 - The interval $[k_{min}, k_{max}]$ too narrow it will impose a constraint that was not present in the original problem.

DISCRETIZATION

- Do not choose $k_{\min} = 0$: recall that f(0) = 0, so if $\lim_{c \to 0} u(c) = -\infty$ then you will run into problems.
- To choose k_{\max} , note that because $\lim_{k\to\infty} f'(k) = 0$ for each z there exists a $k_{\max,z}$ such that $zf(k) + (1-\delta)k < k$ for all $k > k_{\max,z}$.

- How to verify that the solution to the discretized problem is a good approximation to the solution of the original problem?
- Recall the RHS of the Bellman equation:

$$\max_{k'\in\Gamma(k,z)}\left\{u(zf(k)+(1-\delta)k-k')+\beta\sum_{z'\in\mathbb{Z}}P(z,z')V(k',z')\right\}.$$

• The first order condition with respect to k' is

$$-u'\left(zf(k)\left(1-\delta\right)k-k'\right)+\beta\sum_{z'\in\mathcal{T}}P(z,z')V'(k',z')=0.$$

where V'(k',z') is the derivative of V with respect to k'.

- Evaluate the LHS of the first order condition at the solution to the discretized problem at various *k* (including these that did not belong to the grid) and *z*.
- If the solution is a good approximation, the LHS should be close to zero.
- What does "close" mean?

- The expression has derivatives of the value function
- We want to have a measure that is easier to interpret.
- · Easy to show (envelope condition) that

$$V'(k,z) = \left(zf'(k) + 1 - \delta\right)u'(c)$$

where *c* is the optimal consumption.

Calculate

$$\mathcal{H}(k,z) := c - \left(u'\right)^{-1} \left(\beta \sum_{z' \in \mathbb{Z}} P(z,z') z' f'(k') u'(c')\right).$$

Euler equation residuals are

$$\Re(k,z) \coloneqq \frac{\Re(k,z)}{c(k,z)}.$$

- Interpretation: agents make $\Re(k,z)\%$ mistakes when choosing c.
- Example: if $\Re(k,z)$ is 0.01, then agents spend 1 dollar per 100 dollars "incorrectly".
- Bounded rationality interpretation.

- Euler equation residuals should be quite small.
- They should be even smaller for points (k, z) that are visited frequently.
- You can immediately see the problem with a grid that is too coarse: grid points are unlikely to correspond to the exact maximizer k'.



BELLMAN EQUATION

• Recall the Bellman equation

$$V(k,z) = \max_{k' \in \Gamma(k,z)} \left\{ u(zf(k) + (1-\delta)k - k') + \beta \sum_{z' \in \mathbb{Z}} P(z,z')V(k',z') \right\},\,$$

- Solve the maximization problem directly using techniques we learned when we talked about optimization.
- We do not need to assume that k' belongs to the grid.
- This gives us hope to finding the exact maximizer k'.
- Problem: we do not know the exact value of V at points that do not belong to the grid.

APPROXIMATION

- We will use an approximation of V at points that do not belong to the grid, \hat{V} .
- Continuous piecewise linear interpolation is a safe choice.
- Then for each k and z we have to solve for k' using

$$\max_{k' \in \Gamma(k,z)} u \Big(z f(k) + \big(1-\delta\big) \, k - k' \Big) + \beta \sum_{z' \in \mathbb{Z}} P \Big(z,z' \Big) \hat{V} \Big(k',z' \Big).$$

MODIFICATION

In the above we have

$$\sum_{z'\in Z} P(z,z') \hat{V}(k',z')$$

- We can approximate V and then calculate the sum (the expected value)
- It is usually cheaper to approximate the expected value directly.
- Let

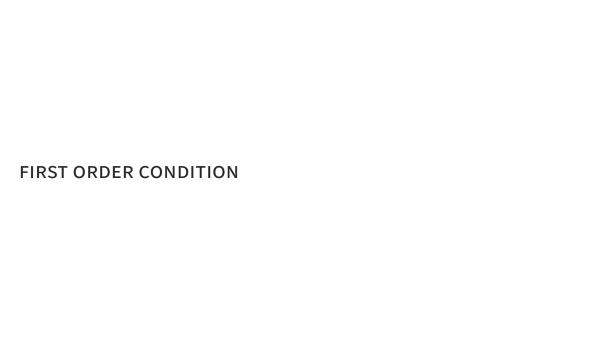
$$W(k',z) := \sum_{z' \in \mathcal{I}} P(z,z') V \iota(k',z')$$

• Obtain $\hat{W}(k',z)$ and then solve

$$\max_{k' \in \Gamma(k,z)} u(zf(k) + (1-\delta)k - k') + \beta \hat{W}(k',z).$$

SUMMARY

- Start with some grid, as in the previous section.
- Instead of finding k' that belongs to the grid (by comparing continuation values directly), solve the maximization problem directly.
- This requires using some optimizer for each k and z.
- This requires knowing \hat{V} at points that do not belong to the grid.
- Important: \hat{V} must preserve the concavity of V.



FIRST ORDER CONDITION

The maximization problem

$$\max_{k' \in \Gamma(k,z)} u \Big(z f(k) + \big(1-\delta\big) \, k - k' \Big) + \beta \sum_{z' \in \mathbb{Z}} P \Big(z,z' \Big) V \Big(k',z' \Big),$$

has a first order condition

$$-u'\left(zf(k)+(1-\delta)\,k-k'\right)+\beta\sum_{z'\in\mathbb{Z}}P(z,z')V'(k',z')=0.$$

- Instead of solving the maximization problem directly, we can solve the first order condition (a nonlinear equation) for k'.
- To find it, we can use for example the bisection method or Newton's method.

FIRST ORDER CONDITION

- Same comments as with the maximization problem apply,
- We need to approximate V' at points that do not belong to the grid.
- We can do it by approximating V and then calculating the derivative or by approximating the derivative directly.
- We can also approximate W or W' instead.



FIRST ORDER CONDITION

- Recall the idea of the previous approach: fix a grid for k and z and then solve the nonlinear equation for k'.
- The nonlinear equation is the most costly part.
- In some cases we can avoid it by using the endogenous grid method.

ENDOGENOUS GRID METHOD

The first order condition was

$$u'(zf(k) + (1-\delta)k - k') = \beta W'(k',z)$$

and we wanted to find k' as a function of k and z.

- The endogenous grid method (by Chris Carroll) inverts the logic.
- Suppose you want to go to some k' given z. What k do you need to start with?

ENDOGENOUS GRID METHOD

- Start with a grid for k' and z.
- Invert the first order condition as

$$zf(k) + (1 - \delta)k - k' = (u')^{-1}(\beta W'(k', z))$$

and solve for k.

- Although you might need to use a nonlinear solver, you do not need to evaluate W' repeatedly.
- Now we know k as a function of k' and z.
- After we did it for all k' and z, we have an endogenous grid for k.

ENDOGENOUS GRID METHOD

- We now know k(k',z) for each k' and z in an exogenous grid.
- We now use interpolation to get k'(k, z) for each k and z in the same exogenous grid.
- After that we have V for each (k,z) in the exogenous grid.
- Get W and repeat the process.