Chapter 12

Ordinary Differential Equation of Second Order

Contents

A second-order differential equation has the general form $F(x,y,y',y'')=0$. The equation solved for y'' is of the form $y''=f(x,y,y')$.
A second-order differential equation is said to be linear if it can be written as

$$y''+P(x)y'+Q(x)y=R(x) \tag{12.1}$$

Here
1. If $R(x)=0$, the equation (12.1) is called homogeneous.
2. If $R(x)\neq 0$, the equation (12.1) is called non-homogeneous.
3. If $P(x)$ and $Q(x)$ are constants and $R(x)=0$, the equation (12.1) is called a homogeneous having constant coefficients.
4. Any differential equation of the second order which can not be written in the form of equation (12.1) is said to be non-linear.

■ **Example 12.1** $y''+4y=\sin x$ and $y''+5\sin xy=x^5$ are particular examples of non- homogeneous. ■

■ **Example 12.2** $x^2y''+xy'+(x^2-1)y=0$ and $y''+6xy'+3xy=0$ are particular examples of homogeneous. ■

■ **Example 12.3** $y''y+y'=0$ and $y''y+4xy'+6xy=7x$ are not linear. ■

Definition 12.0.1 A solution of $F(x,y,y',y'')=0$ on an interval I is a function $\varphi(x)$ that satisfies the

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differential equation at each point of I; that is, for all x in I

$$F(x,\varphi(x),\varphi'(x),\varphi''(x)) = 0$$

■ **Example 12.4** Consider $x^2y'' - 5xy' + 10y = 0, x > 0$. Here $\varphi(x) = x^3\cos(\ln x)$ is a solution of the second ordinary differential equation since

$$\begin{aligned}
\varphi(x) &= x^3\cos(\ln x)\\
\varphi'(x) &= 3x^2\cos(\ln x) - x^2\sin(\ln x)\\
\varphi''(x) &= 5x\cos(\ln x) - 5x\sin(\ln x)
\end{aligned}$$

Now, if we substitute φ'', φ' and φ in the equation

$$\begin{aligned}
x^2\varphi''(x) - 5x\varphi'(x) + & \ 10\varphi(x) = x^2(5x\cos(\ln x) - 5x\sin(\ln x)) - 5x(3x^2\cos(\ln x)\\
& -x^2\sin(\ln x)) + 10(x^3\cos(\ln x))\\
&= 5x^3\cos(\ln x) - 5x^3\sin(\ln x) - 15x^3\cos(\ln x)\\
& +5x^3\sin(\ln x) + 10x^3\cos(\ln x)\\
&= 0
\end{aligned}$$

■

12.1 Homogeneous Equation with Constant Coefficients

12.1.1 General Solution, Basis, Initial Solution

Basis

Theorem 12.1.1 If $y_1(x)$ and $y_2(x)$ are both solutions of the linear homogeneous equation $y'' + P(x)y' + Q(x)y = 0$, then the function $y(x) = C_1y_1(x) + C_2y_2(x)$ is also a solution where C_1 and C_2 are only a constants.

Definition 12.1.1 The set $\{y_1(x), y_2(x)\}$ is called a basis or a fundamental system of solution for $y'' + P(x)y' + Q(x)y = 0$, if $y_1(x)$ and $y_2(x)$ are linear independent.

Definition 12.1.2 The wronskian of n functions $y_1, y_2, y_3, \cdots, y_n$ of (n-1) times differentiable on an interval I, is defined as

$$\begin{aligned}
W(x) &= W(y_1, y_2, y_3, \cdots, y_n)\\
&= \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n \\ y_1'(x) & y_2'(x) & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}
\end{aligned}$$

Theorem 12.1.2 If $W(y_1, y_2, y_3, \cdots, y_n) \neq 0$ for some $x_0 \in I$, then $\{y_1, y_2, y_3, \cdots, y_n\}$ is linearly independent.

■ **Example 12.5** Consider $\{1, \sin^2 t, \cos^2 t\}$

$$\begin{aligned}
W(x) &= W(1, \sin^2 t, \cos^2 t)\\
&= \begin{vmatrix} 1 & \sin^2 t & \cos^2 t \\ 0 & \sin(2t) & -\sin(2t) \\ 0 & 2\cos(2t) & -2\cos(2t) \end{vmatrix}\\
&= -2\sin(2t)\cos(2t) + 2\sin(2t)\cos(2t)\\
&= 0
\end{aligned}$$

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Therefore, $\{1, \sin^2 t, \cos^2 t\}$ is linearly dependent but $\{t, t^2, \sin^2 t\}$ is linearly independent because

$$\begin{aligned}
W(x) &= W(t, t^2, \sin^2 t) \\
&= \begin{vmatrix} t & t^2 & \sin^2 t \\ 0 & 2t & \sin(2t) \\ 0 & 2 & 2\cos(2t) \end{vmatrix} \\
&= t(4t\cos(2t) - 2\sin(2t)) - 1(2t^2\cos(2t) - 2\sin^2 t) \\
&= 4t^2\cos(2t) - 2t\sin(2t)) - 2t^2\cos(2t) + 2\sin^2 t \\
&= 2t^2\cos(2t) - 2t\sin(2t)) + 2\sin^2 t \neq 0 \text{ for all t}
\end{aligned}$$

■

Theorem 12.1.3 If $y_1(x)$ and $y_2(x)$ are any two solutions of $y'' + P(x)y' + Q(x)y = 0$ and $W(x) = W(y_1, y_2)$, then $W' + P(x)W = 0$.

Remark 12.1.4
1. The function $y_1(x)$ and $y_2(x)$ constitute a basis of solution for $y'' + P(x)y' + Q(x)y = 0$ in an interval I of the x-axis if and only if they are linearly independent and if neither function is a constant multiple of the other on I.
2. Wronskian Test is a simple test whether two solutions; that is, $y_1(x)$ and $y_2(x)$, of the equation $y'' + P(x)y' + Q(x)y = 0$ are linearly dependent or independent on the interval

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

$y_1(x)$ and $y_2(x)$ are linearly independent on I if and only if $W(x) \neq 0$ on I.

■ **Example 12.6** Consider the solution $y_1(x) = \cos x$ and $y_2(x) = \sin x$ of the equation $y'' + y = 0$ for all x

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \cos^2 x + \sin^2 x = 1 \neq 0$$

Therefore, $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are linearly independent. ■

General Solution

Definition 12.1.3 The general solution of a differential equation of second order is a function $y = \Phi(x, C_1, C_2)$ which is dependent on the arbitrary constants C_1 and C_2 such that it satisfies the equation for any values of the constants C_1 and C_2.

Theorem 12.1.5 If $y_1(x)$ and $y_2(x)$ are linearly independent solution of $y'' + P(x)y' + Q(x)y = 0$, then the general solution given by $y(x) = C_1y_1(x) + C_2y_2(x)$ where C_1 and C_2 are arbitrary constants.

■ **Example 12.7** Consider the solution $y_1(x) = \cos x$ and $y_2(x) = \sin x$ of the equation $y'' + y' = 0$ for all x, then $y = C_1\cos x + C_2\sin x$ is the general solution for any arbitrary constants C_1 and C_2 for the equation $y'' + y' = 0$; since $y_1(x) = \cos x$ and $y_2(x) = \sin x$ are linearly independent. ■

Initial Condition

Definition 12.1.4 A function obtained from the general solution for specific values of the constants C_1 and C_2 for specified initial conditions $y_{x=x_0} = y_0$; $y'_{x=x_0} = y'_0$ is called a particular solution.

An initial value problem for the second order differential equation consists of finding a solution y of the differential equation that also satisfies initial condition of the form $y(x_0) = y_0$, $y'(x_0) = y_1$, where y_0 and y_1 are given constants.

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Theorem 12.1.6 If in the equation $y'' = f(x,y,y')$. The function $f(x,y,y')$ and its partial derivatives with respect to the arguments y, y' are continuous in some region containing the values $x = x_0, y = y_0, y' = y'_0$, then there is one and only one solution , $y = y(x)$, of the equation that satisfies the conditions: $y_{x=x_0} = y_0$ and $y'_{x=x_0} = y'_0$. These conditions are called initial conditions.

The geometrical meaning of the initial condition is that only one curve passes through a given point (x_0, y_0) of the plane with given slope of the tangent line y'_0.

Remark 12.1.7 The zero or trivial solution $y = 0$ is always a solution to $y'' + P(x)y' + Q(x)y = 0$, it is the only solution that satisfies $y(x_0) = 0$ and $y'(x_0) = 0$.

■ **Example 12.8** Solve the initial value problem $y'' + y' - 6y = 0$, where $y(0) = 1$, $y'(0) = 0$ and the general solution of the differential equation is $y = C_1e^{2x} + C_2e^{-3x}$.
Solution: Differentiating the general solution we get $y'(x) = 2C_1e^{2x} - 3C_2e^{-3x}$. To satsfy the initial condition we required that

$$\begin{aligned} y'(0) &= 2C_1 - 3C_2 = 0 \Rightarrow 2C_1 - 3C_2 = 0 \\ y(0) &= C_1 + C_2 = 0 \Rightarrow C_1 + C_2 = 0 \end{aligned}$$

From the above equations, $C_1 = \frac{3}{5}$ and $C_2 = \frac{2}{5}$. Thus the required solution of the initial value problem is

$$y = \frac{3}{5}e^{2x} + \frac{2}{5}e^{-3x}$$

■

12.1.2 Reduction Order

Some differential equation of second order equation of free y; that is, $F(x,y',y'') = 0$ can be solved by reducing it to first order differential equation by letting $y' = u \Rightarrow y'' = u'$ and substituting in to the equation $F(x,y',y'') = 0$, where u as a function of x.

■ **Example 12.9** Solve $xy'' + 2y' = x^2 - 1$.
Solution: The differential equation is free of y. Let $y' = u \Rightarrow y'' = u'$, then

$$\begin{aligned} xy'' + 2y' &= x^2 - 1 \\ \Rightarrow xu' + 2u &= x^2 - 1 \\ u' + \frac{2}{x}u &= x - \frac{1}{x} \end{aligned}$$

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Let $p(x)=\frac{2}{x}$ and $r(x)=x-\frac{1}{x}$, so by first order differential equation

$$\begin{aligned}
u &= e^{-\int p(x)\,dx}[\int r(x)e^{\int p(x)\,dx}\,dx]\\
&= e^{-\int \frac{2}{x}\,dx}[\int (x-\frac{1}{x})e^{\int \frac{2}{x}\,dx}\,dx]\\
&= e^{\ln\frac{1}{x^2}}[\int (x-\frac{1}{x})e^{\ln x^2}\,dx]\\
&= \frac{1}{x^2}[\int (x-\frac{1}{x})x^2\,dx]\\
&= \frac{1}{x^2}\int (x^3-x)\,dx\\
&= \frac{1}{x^2}[\frac{1}{4}x^4-\frac{1}{2}x^2+c_1]\\
&= \frac{1}{4}x^2-\frac{1}{2}+\frac{c_1}{x^2}\\
y' &= \frac{1}{4}x^2-\frac{1}{2}+\frac{c_1}{x^2} \text{ since } y'=u\\
y &= \frac{1}{12}x^3-\frac{1}{2}x-\frac{c_1}{x}+c_2 \text{ is the required solution}
\end{aligned}$$

■

Similarly, some differential equation of second order equation of free x; that is, $F(y,y',y'')=0$ can be solved by reducing it to first order differential equation by letting $y'=u$; that is, $\frac{dy}{dx}=u$ and treat u as a function of y; that is

$$\begin{aligned}
u &= u(y)=u(y(x))\\
\Rightarrow y'' &= \frac{du}{dx}=\frac{d}{dx}u(y(x))=\frac{du}{dy}\frac{dy}{dx}=u\frac{du}{dy}\\
&= uu'
\end{aligned}$$

substituting $y'=u$ and $y''=u\frac{du}{dy}$ in to the equation $F(y,y',y'')=0$, we have $F(y,u',u\frac{du}{dy})=0$.

■ **Example 12.10** Solve $2yy''-(y')^2=1$.
Solution: The differential equation $2yy''-(y')^2=1$ is free of x. So, put $y'=u$ and $y''=u\frac{du}{dy}$ in to the equation, then

$$\begin{aligned}
& 2yy''-(y')^2=1\\
\Rightarrow\quad & 2yu\frac{du}{dy}-u^2=1\\
& 2yu\frac{du}{dy}=u^2+1\\
& \frac{2u}{u^2+1}du=\frac{1}{y}dy \text{ separable equation}\\
& \text{Integrating both side}\\
& \ln(u^2+1)=\ln|y|+c\\
& u^2+1=c_1y,\ \text{where } c_1=\pm e^c\\
& u=\sqrt{c_1y-1}\\
& \frac{dy}{dx}=\sqrt{c_1y-1}\\
& \frac{dy}{\sqrt{c_1y-1}}=dx\\
& 2\sqrt{c_1y-1}=x+c_2\\
& y=c_3x^2+c_4x+c_0
\end{aligned}$$

■

12.1.3 Given One Solution to Find Another

Consider the homogeneous differential equation

$$y'' + P(x)y' + Q(x)y = 0 \tag{12.2}$$

suppose a non trivial solution $y_1 = y_1(x)$ is given. Now we have to find $y_2 = y_2(x)$ which is linear independent with $y_1(x)$. So, put $u(x) = \frac{y_2(x)}{y_1(x)}$, since $y_1(x)$ and $y_2(x)$ are linearly independent and $u(x)$ is not constant.

$$\begin{aligned}
\Rightarrow y_2 &= uy_1 \\
\Rightarrow y_2' &= u'y_1 + uy_1' \\
\Rightarrow y_2'' &= u''y_1 + u'y_1 + u'y_1' + uy_1'' \\
&= u''y_1 + 2u'y_1' + +uy_1''
\end{aligned}$$

Substituting y_2, y_2' and y_2'' in equation (12.2), we have

$$\begin{aligned}
& y'' + P(x)y' + Q(x)y = 0 \\
& \Leftrightarrow (u''y_1 + 2u'y_1' + uy_1'') + P(x)(u'y_1 + uy_1') + Q(x)(uy_1) = 0 \\
& \Leftrightarrow u''y_1 + 2u'y_1' + P(x)u'y_1 + u(y_1'' + P(x)y_1' + Q(x)y_1) = 0 \\
& \Leftrightarrow u''y_1 + 2u'y_1' + P(x)u'y_1 = 0, \text{ since } y_1'' + P(x)y_1' + Q(x)y_1 = 0
\end{aligned}$$

Applying reduction of order by letting $z = u'$, $z' = u''$, then

$$\begin{aligned}
& z'y_1 + z(2y_1 + P(x)y_1) = 0 \\
& \Leftrightarrow \frac{z'}{z} = \frac{-2y_1'}{y_1} - P(x) \\
& \Leftrightarrow \frac{1}{z}\,dz = (\frac{-2y_1'}{y_1} - P(x))\,dx \text{ separable equation} \\
& \Leftrightarrow \ln|z| + \ln|y_1|^2 = -\int P(x)\,dx \\
& zy_1^2 = e^{-\int P(x)\,dx} \text{ or} \\
& z = \frac{e^{-\int P(x)\,dx}}{y_1^2} \\
& \frac{du}{dx} = \frac{e^{-\int P(x)\,dx}}{y_1^2} \\
& u = \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx
\end{aligned}$$

Therefore, $y_2(x) = uy_1(x) = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$ is another solution of equation (12.2) and which is linearly independent with y_1.

■ **Example 12.11** Consider $x^2y'' - xy' + y = 0$ on $I = (0,\infty)$ and $y_1(x) = x$ is a solution. So, based on the given one solution we can get the other. First we can rewrite the equation as

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$$

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So, let $P(x) = \frac{-1}{x}$, then

$$\begin{aligned} u &= \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx \\ &= \int \frac{e^{-\int(\frac{-1}{x})\,dx}}{x^2}\,dx \\ &= \int \frac{x}{x^2}\,dx \\ &= \ln|x| \\ &= \ln x,\ \text{since } I=(0,\infty) \\ y_2(x) &= y_1(x)u \\ &= x\ln x \end{aligned}$$

Therefore, $y = C_1x + C_2x\ln x$ is the general solution of the given differential equation. ■

12.1.4 Real Root, Complex Roots, Double Root of the Characteristic Equation

Definition 12.1.5 $y'' + Py' + Qy = 0$, where P and Q are constants are called homogeneous equations with constant coefficients.

Let D stands for the operator $\frac{d}{dx}$; that is, $D = \frac{d}{dx}$. Then $Dy = \frac{dy}{dx} = y'$ and $(D-c)y = (\frac{d}{dx} - c)y$, where c is a constant. Similarly, for higher derivatives, $D^2y = \frac{d^2y}{dx^2}$, $D^3y = \frac{d^3y}{dx^3}$ and so on. For example, $D\sin\theta = \cos\theta$, $D^2\sin\theta = -\sin\theta$ etc.
For a homogeneous linear ordinary differential equation $y'' + Py' + Qy = 0$ with constant coefficients we can introduce the second order differential operator

$$f(D) = D^2 + PD + QI$$

where I is the identity operator defined by $Iy = y$. Then we can write that ordinary equation as

$$f(D)y = (D^2 + PD + QI)y = 0$$

If $f(D) = 0$ or $D^2 + PD + QI = 0$ is called the auxiliary equation for $y'' + Py' + Qy = 0$. Since $De^{\lambda x} = \lambda e^{\lambda x}$ and $D^2e^{\lambda x} = \lambda^2 e^{\lambda x}$, then we obtain

$$\begin{aligned} f(D)e^{\lambda x} &= (D^2 + PD + QI)e^{\lambda x} \\ &= (\lambda^2 + P\lambda + Q)e^{\lambda x} \\ &= f(\lambda)e^{\lambda x} \\ &= 0 \end{aligned}$$

This confirms that $e^{\lambda x}$ is a solution of the differential equation $y'' + Py' + Qy = 0$ if and only if λ is a solution of the auxiliary equation $f(\lambda)$.

■ **Example 12.12** The differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

has symbolic form $(D^2 + 5D + 6I)y = 0$ and $D^2 + 5D + 6I = 0$ is its auxiliary equation. ■

Theorem 12.1.8 If $f(D)$ is a polynomial in D and $f(\lambda) = 0$, where $\lambda \in R$, then $f(D)e^{\lambda x} = 0$.

Corollary 12.1.9 If $f(D)$ is a polynomial in D with constant coefficients, then $e^{ax}f(D)y = f(D-a)e^{ax}y$.

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■ **Example 12.13** Factor $f(D) = D^2 - 3D - 40I$ and solve $f(D)y = 0$.
Solution: $D^2 - 3D - 40I = (D - 8I)(D + 5I)$ because $I^2 = I$. Now $(D - 8I)y = y' - 8y = 0$ has the solution $y_1 = e^{8x}$. Similarly, the solution of $(D + 5I)y = 0$ is $y_2 = e^{-5x}$. This is a basis of $f(D)y = 0$ on any interval. From the factorization we obtain the ordinary differential equation, as expected

$$\begin{aligned}(D-8I)(D+5I)y &= (D-8I)(y'+5y)\\ &= D(y'+5y) - 8(y'+5y)\\ &= y'' + 5y' - 8y - 40y\\ &= y'' - 3y' - 40y\\ &= 0\end{aligned}$$

Verify that this agree with the result of the previous method. This is not unexpected because we factored $f(D)$ in the same way as the characteristic polynomial $f(\lambda) = \lambda^2 - 3\lambda - 40$. ■

Remark 12.1.10 We shall seek a solution of $y'' + Py' + Qy = 0$ in the form of $y = e^{\lambda x}$. Substituting $y = e^{\lambda x}$ in to $y'' + Py' + Qy = 0$, we find that number λ must satisfy the equation

$$\lambda^2 + P\lambda + Q = 0$$

Since $y'' = \lambda^2 e^{\lambda x}$, $y' = \lambda e^{\lambda x}$ and $y = e^{\lambda x}$. So, here

1. The equation $\lambda^2 + P\lambda + Q = 0$ is called a characteristic equation (auxiliary equation) like $D^2 + PD + QI = 0$
2. λ is a root of the equation $\lambda^2 + P\lambda + Q = 0$.

We have three cases for the solution of the second order homogeneous linear equations with constant coefficients.

- Case I: If $(\frac{P}{2})^2 - Q > 0$, the characteristic equation $\lambda^2 + P\lambda + Q = 0$ has two distinct roots λ_1 and λ_2; that is,

$$\lambda_1 = \frac{-P}{2} + \sqrt{(\frac{P}{2})^2 - Q} \text{ and } \lambda_2 = \frac{-P}{2} - \sqrt{(\frac{P}{2})^2 - Q}$$

  In this case we have two linearly independent solution $y = e^{\lambda_1 x}$ and $y = e^{\lambda_2 x}$. Therefore, the general solution will be $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.
- Case II: If $(\frac{P}{2})^2 - Q = 0$, the characteristic equation $\lambda^2 + P\lambda + Q = 0$ has double roots; that is, $\lambda_1 = \lambda = \lambda_2$. So, $y_1(x) = e^{\lambda x}$ is one of solution and by reduction method $y_2(x) = xe^{\lambda x}$ is another solution. Therefore, the general solution will be

$$y = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$$

  Consider $y_1(x) = e^{\lambda x}$ and $y_2(x) = xe^{\lambda x}$, here y_1 and y_2 are linearly independent with each other.
- Case III: If $(\frac{P}{2})^2 - Q < 0$, the characteristic equation $\lambda^2 + P\lambda + Q = 0$ has complex roots; that is,

$$\lambda_1 = \frac{-P}{2} \pm i\sqrt{Q - (\frac{P}{2})^2}$$

  For convenience, write $\alpha = \frac{-P}{2}$ and $\beta = \sqrt{\frac{4Q-P^2}{2}}$. So, the roots of the characteristic equation are $\lambda = \alpha \pm i\beta$; that is, $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. This yields two solutions

$$y_1(x) = e^{(\alpha+i\beta)x} \text{ and } y_2(x) = e^{(\alpha-i\beta)x}$$

  They are linearly independent because the wronskian is

$$W(x) = \begin{vmatrix} e^{(\alpha+i\beta)x} & e^{(\alpha-i\beta)x} \\ (\alpha+i\beta)e^{(\alpha+i\beta)x} & (\alpha-i\beta)e^{(\alpha-i\beta)x} \end{vmatrix} = -2i\beta e^{2\alpha x} \neq 0$$

  Therefore, the general solution will be $y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$.

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Remark 12.1.11 $e^{i\theta} = \cos\theta + i\sin\theta$ is Euler's Equation
By using Euler's Equation, we can rewrite the general solution

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}$$

as $y = e^{\alpha x}(C_3\cos\beta x + C_4\sin\beta x)$, where $C_3 = C_1 + C_2$ and $C_4 = i(C_1 - C_2)$

■ **Example 12.14** Find the general solution $y'' + y' - 6y = 0$.
Solution: Let $y(x) = e^{\lambda x}$. Then $y'(x) = \lambda e^{\lambda x}$ and $y''(x) = \lambda^2 e^{\lambda x}$. If we substitute those values in to $y'' + y' - 6y = 0$

$$\begin{aligned} y'' + y' - 6y = 0 \quad &\Rightarrow \quad \lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 6e^{\lambda x} = 0 \\ &\Rightarrow \quad (\lambda^2 + \lambda - 6)e^{\lambda x} = 0 \\ &\Rightarrow \quad (\lambda^2 + \lambda - 6) = 0 \text{ since } e^{\lambda x} > 0 \text{ for all } x \end{aligned}$$

So, the auxiliary equation is $\lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda_1 = 2$ and $\lambda_2 = -3$. Here $y_1(x) = e^{2x}$ and $y_2(x) = e^{-3x}$ are linearly independent with each other. The general solution of the given differential equation is

$$y(x) = C_1 e^{2x} + C_2 e^{-3x}$$

■

■ **Example 12.15** Solve the initial value problem $y'' + y' - 6y = 0$, where $y(0) = 1$ and $y'(0) = 0$.
Solution: The auxiliary equation is $\lambda^2 + \lambda - 6 = 0 \Rightarrow \lambda_1 = 2$ and $\lambda_2 = -3$. The general solution of the given differential equation is

$$y(x) = C_1 e^{2x} + C_2 e^{-3x}$$

Differentiating this solution, we get

$$y'(x) = 2C_1 e^{2x} - 3C_2 e^{-3x}$$

This satisfy the initial conditions we required that $y'(0) = 2C_1 - 3C_2 = 0$ and $y(0) = C_1 + C_2 = 1$. This implies $C_1 = \frac{3}{5}$ and $C_2 = \frac{2}{5}$. Thus the required solution of the initial value problem is

$$y(x) = \frac{3}{5}e^{2x} + \frac{2}{5}e^{-3x}$$

■

■ **Example 12.16** Solve the general solution for $y'' - 6y' + 9y = 0$.
Solution: Let $y(x) = e^{\lambda x}$. Then $y'(x) = \lambda e^{\lambda x}$ and $y''(x) = \lambda^2 e^{\lambda x}$. If we substitute those values in to $y'' - 6y' + 9y = 0$

$$\begin{aligned} y'' - 6y' + 9y = 0 \qquad &\Rightarrow \lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + 9e^{\lambda x} = 0 \\ &\Rightarrow (\lambda^2 - 6\lambda + 9)e^{\lambda x} = 0 \\ &\Rightarrow (\lambda - 3)^2 = 0, \text{ since } e^{\lambda x} > 0 \text{ for all } x \end{aligned}$$

So, by reduction order and Case II the general solution is

$$y(x) = C_1 e^{3x} + C_2 x e^{3x}$$

■

■ **Example 12.17** Solve the equation $y'' - 6y' + 13y = 0$.
Solution: Let $y(x) = e^{\lambda x}$. Then $y'(x) = \lambda e^{\lambda x}$ and $y''(x) = \lambda^2 e^{\lambda x}$. If we substitute those values in to $y'' - 6y' + 13y = 0$

$$\begin{aligned} y'' - 6y' + 13y = 0 \qquad &\Rightarrow \lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + 13e^{\lambda x} = 0 \\ &\Rightarrow (\lambda^2 - 6\lambda + 13)e^{\lambda x} = 0 \\ &\Rightarrow \lambda^2 - 6\lambda + 13 = 0 \text{ since } e^{\lambda x} > 0 \text{ for all } x \end{aligned}$$

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So, the auxiliary equation is $\lambda^2 - 6\lambda + 13 = 0 \Rightarrow \lambda = \frac{6\pm\sqrt{36-52}}{2} = 3 \pm 2i$; that is, $\alpha = 3$ and $\beta = 2$. The general solution of the given differential equation is

$$y(x) = e^{3x}(C_1 \cos 2x + C_2 \sin 2x)$$

■

■ **Example 12.18** Solve the initial value problem $y'' + y' = 0$, where $y(0) = 2$ and $y'(0) = 3$.
Solution: Let $y(x) = e^{\lambda x}$. Then $y'(x) = \lambda e^{\lambda x}$ and $y''(x) = \lambda^2 e^{\lambda x}$. If we substitute those values in to $y'' + y' = 0$

$$\begin{aligned} y'' + y' = 0 \qquad & \Rightarrow \lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \\ & \Rightarrow (\lambda^2 + 1)e^{\lambda x} = 0 \\ & \Rightarrow \lambda^2 + 1 = 0 \text{ since } e^{\lambda x} > 0 \text{ for all } x \end{aligned}$$

So, the auxiliary equation is $\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$; that is, $\alpha = 0$ and $\beta = 1$. The general solution of the given differential equation is

$$y(x) = C_1 \cos x + C_2 \sin x$$

If we differential the solution we have $y'(x) = -C_1 \sin x + C_2 \cos x$

$$\begin{aligned} y(0) &= C_1 \cos 0 + C_2 \sin 0 = 2 \Rightarrow C_1 = 2 \\ y'(0) &= -C_1 \sin 0 + C_2 \cos 0 = 3 \Rightarrow C_2 = 3 \end{aligned}$$

Therefore, the solution of the initial value problem is

$$y = 2\cos x + 3\sin x$$

■

■ **Example 12.19** Solve the initial value problem $y'' + y' - 2y = 0$, $y(0) = 4$, $y'(0) = -5$.
Solution: The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0$$

Its roots are $\lambda_1 = 1$ and $\lambda_2 = -2$. So that we can obtain the general solution

$$y = C_1 e^x + C_2 e^{-2x}$$

Now, we can to find the particular solution from the general solution and $y'(x) = C_1 e^x - 2C_2 e^{-2x}$. So.

$$\begin{aligned} y(0) &= C_1 + C_2 = 4 \\ y'(0) &= C_1 - 2C_2 = -5 \end{aligned}$$

Hence $C_1 = 1$ and $C_2 = 3$. This gives $y = e^x + 3e^{-2x}$ is the particular solution. ■

■ **Example 12.20** Solve the initial value problem $y'' + y' + \frac{1}{4}y = 0$, $y(0) = 3$, $y'(0) = -\frac{7}{2}$.
Solution: The characteristic equation is

$$\lambda^2 + \lambda + \frac{1}{4} = (\lambda + \frac{1}{2})^2 = 0$$

Its has the double root$\lambda = \frac{1}{2}$. This gives the general solution

$$y = (C_1 + C_2 x)e^{-\frac{1}{2}x}$$

We need is derivative $y'(x) = C_2 e^{-\frac{1}{2}x} - \frac{1}{2}(C_1 + C_2 x)e^{-\frac{1}{2}x}$. From this and the initial conditions we obtain

$$\begin{aligned} y(0) &= C_1 = 3 \\ y'(0) &= C_2 - \frac{1}{2}C_1 = -\frac{7}{2} \end{aligned}$$

Hence $C_1 = 3$ and $C_2 = -2$. This gives $y = (3 - 2x)e^{-\frac{1}{2}x}$ is the particular solution. ■

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12.2 Non-homogeneous Equation with constant Coefficients

The equation $y'' + Py' + Qy = R(x)$, where P and Q are constants and $R(x)$ depends solely on x (or is constant) is called a second-order linear differential equation with constant coefficients. If $R(x) = 0$, then the equation is called second-order homogeneous equation, otherwise, it is called second-order non homogeneous equation.

Theorem 12.2.1 The general solution of the non-homogeneous differential equation $y'' + Py' + Qy = R(x)$ can be

$$y(x) = y_p(x) + y_c(x)$$

where $y_p(x)$ is a particular solution of $y'' + Py' + Qy = R(x)$ and $y_c(x)$ is a complementary solution of $y'' + Py' + Qy = 0$

There are two methods for finding the particular solution
1. The method of undetermined coefficients
2. The method of variation of parameters

12.2.1 The method of undetermined coefficients

Let $y'' + Py' + Qy = R(x)$.
- Case I: If $R(x)$ is a polynomial function, then $y_p(x)$ is a polynomial of the same degree as $R(x)$.
- Case II:If $R(x)$ is of the form ae^{rx}, where a and r are constants, and
  1. If r is not a root of the auxiliary equation, then $y_p(x) = a_1 e^{rx}$
  2. If r is a root of multiple one, then $y_p(x) = a_1 x e^{rx}$
  3. If r is a double root(multiple two), then $y_p(x) = a_1 x^2 e^{rx}$
- Case III: If $R(x)$ is either $a\cos\gamma x$ or $a\sin\gamma x$, and
  1. If $i\gamma$ is not a root of the characteristic equation, then

  $$y_p(x) = a_1 \sin\gamma x + a_2 \cos\gamma x$$

  2. If $i\gamma$ is a root of the characteristic equation, then

  $$y_p(x) = a_1 x \sin\gamma x + a_2 x \cos\gamma x$$

■ **Example 12.21** Solve the equation

$$y'' + y' - 2y = x^2$$

Solution: The auxiliary equation is $\lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda_1 = 1$ and $\lambda_2 = -2$. Therefore $y_c = C_1 e^x + C_2 e^{-2x}$. For $R(x) = x^2$, we have $y_p(x) = c_0 + c_1 x + c_2 x^2$ with the same degree as $R(x)$. So,

$$\begin{aligned} y_p(x) &= c_0 + c_1 x + c_2 x^2 \\ \Rightarrow y_p'(x) &= c_1 + 2c_2 x \\ y''(x) &= 2c_2 \end{aligned}$$

Substituting the above equations in $y'' + y' - 2y = x^2$, we have $c_0 = \frac{-3}{4}$, $c_1 = \frac{-1}{2}$ and $c_2 = \frac{-1}{2}$. So, $y_p(x) = -\frac{3}{4} - \frac{1}{2}x - \frac{1}{2}x^2$. Therefore, the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1 e^x + C_2 e^{-2x} - \frac{3}{4} - \frac{1}{2}x - \frac{1}{2}x^2 \end{aligned}$$

■

■ **Example 12.22** Find a particular solution of the equation $y'' - 6y' + 9y = e^{3x}$.
Solution: The corresponding characteristic equation is $\lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0$, which is a double root $\lambda = 3$. So, $y_p(x) = cx^2 e^{3x}$. By finding the differential values of y_p and substituting in the differential equation we will have $c = \frac{1}{2}$. Therefore,

$$y_p(x) = \frac{1}{2} x^2 e^{3x}$$

■

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Functions	Annihilator
$a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$	D^{m+1}
$e^{\alpha x}$	$D-\alpha$
$x^m e^{\alpha x}$	$(D-\alpha)^{m+1}$
$\sin(\beta x)$ or $\cos(\beta x)$	$D^2+\beta^2$
$x^m\sin(\beta x)$ or $x^m\cos(\beta x)$	$(D^2+\beta^2)^m$
$e^{\alpha x}\sin(\beta x)$ or $e^{\alpha x}\cos(\beta x)$	$D^2-2\alpha D+(\alpha^2+\beta^2)$
$x^m e^{\alpha x}\sin(\beta x)$ or $x^m e^{\alpha x}\cos(\beta x)$	$(D^2-2\alpha D+(\alpha^2+\beta^2))^{m+1}$

■ **Example 12.23** Find a particular solution of the equation $y''+2y'+2y=\sin x$.
Solution: The characteristic equation $\lambda^2+2\lambda+2=0 \Rightarrow \lambda=-1\pm i$. So, $\alpha=-1$ and $\beta=1$. Here $y_c=e^{-x}[C_1\cos x+C_2\sin x]$. The complex number i is not a solution of the characteristic equation. So, the term $\sin x$ on the right side of the differential equation contributes a term

$$\begin{aligned} y_p(x) &= c_1\sin x+c_2\cos x \\ y'_p(x) &= c_1\cos x-c_2\sin x \\ y''_p(x) &= -c_1\sin x-c_2\cos x \end{aligned}$$

By finding the differential values of y_p and substituting in the differential equation we will have $c_1=\frac{1}{5}$ and $c_2=-\frac{2}{5}$. Therefore,

$$y_p(x)=\frac{1}{5}\sin x-\frac{2}{5}\cos x$$

Thus, $y(x)=y_c(x)+y_p(x)=C_1e^{(-1-i)x}+C_2e^{(-1+i)x}+\frac{1}{5}\sin x-\frac{2}{5}\cos x$. ■

Annihilator

Sometimes, the function $R(x)$ in the second order differential equation $y''+Py'+Qy=R(x)$ is a combination of different function. In such a case, it is simple to use the annihilators.

Definition 12.2.1 Let $D=\frac{d}{dx}$(differential operator), if a differential operator, say A, applied to a function, say $R(X)$, results zero; that is, $AR(x)=0$, the operator A is said to be the annihilator of $R(x)$.

■ **Example 12.24** $f(x)=x$ annihilated by D^2; that is, $D^2x=0$, $g(x)=8x^3-5$ annihilated by D^4 since $D^4(8x^3-5)=0$. ■

The Annihilator Method

We can use the annihilator method if $R(x)$ and all of its derivatives are a finite set of linearly independent functions. That is, $R(x)$ must be one of Polynomial, Sine or cosine, Exponential (this includes hyperbolic sine and hyperbolic cosine like $x^m e^{\alpha}x$, $x^m\cos(\beta x)$, $x^m\sin(\beta x)$) or a linear combination of those.

Annihilator Table

It is difficult to find the annihilators for the functions like e^{x^2}, $\log x$, $\tan x$.

Steps to solve $f(D)y=R(x)$ using Annihilator

The differential equation $f(D)y=R(x)$ has constant coefficients and the function $R(x)$ consists of finite sums and products of constants, polynomials, exponential functions ($e^{\alpha x}$), sine and cosine functions.

- Step 1: Write the differential equation in factored operator form.
- Step 2: Solve the homogeneous case $f(D)=0$ and get y_c.
- Step 3: Apply the annihilator of $R(x)$ to both sides of the differential equation to obtain a new homogeneous differential equation.
- Step 4: Identify the basic form of the solution to the new differential equation and rewrite a particular candidate solution $y*_p$ as $y*_p=\sum L_i e^{\lambda_i x}$, where λ_i is unknown constants. .

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- Step 5: Delete all those terms $e^{\lambda_i x}$ from the the particular candidate solution $y*_p = \sum L_i e^{\lambda_i x}$ which is duplicated from the complementary solution y_c found in Step 2, the remaining sum will be y_p solution.
- Step 6: Determine the specific coefficients for the particular solution y_p.
- Step 7: With the particular solution y_p found in Step 6 and complementary solution y_c found in step 2 form the general solution $y = y_c + y_p$ of the differential equation.

Example 12.25 Solve $3y'' - y' - 2y = 2x^2 + 6x - 4$ with undetermined coefficient.
Solution:

- Step 1: The differential equation in factored operator form $3y'' - y' - 2y = 2x^2 + 6x - 4$ is
$$(3D^2 - D - 2)y = 2x^2 + 6x - 4$$
- Step 2: Now we can solve the homogeneous $f(D) = 0$ as
$$3D^2 - D - 2 = 0$$
So auxiliary equation for homogeneous is
$$\begin{aligned} 3\lambda^2 - \lambda - 2 &= 0 \\ (\lambda - 1)(3\lambda + 2) &= 0 \\ &\Rightarrow \lambda_1 = 1 \text{ and } \lambda_2 = -\frac{2}{3} \end{aligned}$$
Therefore, the complementary solution is
$$y_c = C_1 e^x + C_2 e^{-\frac{2}{3}x}$$
- Step 3: Now, we identify the annihilator of the right side $3y'' - y' - 2y = 2x^2 + 6x - 4$ of the non-homogeneous equation
$$D^3(2x^2 + 6x - 4) = 0$$
We apply the annihilator D^3 to both sides of the equation $(3D^2 - D - 2)y = 2x^2 + 6x - 4$ to obtain a new homogeneous equation
$$\begin{aligned} D^3(3D^2 - D - 2)y &= D^3(2x^2 + 6x - 4) \\ D^3(3D^2 - D - 2)y &= 0 \text{ since } D^3(2x^2 + 6x - 4) = 0 \\ D^3(D - 1)(3D + 2)y &= 0 \\ D^3(D - 1)(3D + 2) &= 0 \\ \Rightarrow \lambda^3(\lambda - 1)(3\lambda + 2) &= 0 \\ \Rightarrow \lambda_1 = 0, \lambda_2 = 0, \quad & \lambda_3 = 0, \lambda_4 = 1 \text{ and } \lambda_5 = -\frac{2}{3} \end{aligned}$$
- Step 4: The candidate particular solution is given by
$$\begin{aligned} \Rightarrow y*_p &= Ae^{\lambda_1 x} + Bxe^{\lambda_2 x} + Cx^2 e^{\lambda_3 x} + Dxe^{\lambda_4 x} + Ee^{\lambda_5 x} \\ & \quad \text{since at } \lambda = 0 \text{ is triple root by reduction order} \\ &= Ae^{0x} + Bxe^{0x} + Cx^2 e^{0x} + De^x + Ee^{-\frac{2}{3}x} \\ &= A + Bx + Cx^2 + De^x + Ee^{-\frac{2}{3}x} \end{aligned}$$
- Step 5: Delete the duplicate terms e^x and $e^{-\frac{2}{3}x}$ in $y*_p$, then $y_p = A + Bx + Cx^2$.
- Step 6: Find first and second derivatives of $y_p = A + Bx + Cx^2$,
$$\begin{aligned} y_p &= A + Bx + Cx^2 \\ \Rightarrow y'_p &= B + 2Cx \\ \Rightarrow y''_p &= 2C \end{aligned}$$
Substituting y_p, y'_p and y''_p in $3y'' - y' - 2y = 2x^2 + 6x - 4$, we have $3(2C) - (B + 2Cx) - 2(A + Bx + Cx^2) = 2x^2 + 6x - 4$, so $A = 0$, $B = -2$ and $C = -1$. Hence
$$y_p = -2x - x^2$$

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- Step 7: Therefore, the general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1e^x + C_2e^{-\frac{1}{2}x} - 2x - x^2 \end{aligned}$$

■

■ **Example 12.26** Solve $y'' - 4y' + 3y = 4e^x - e^{2x}$.
Solution:

- Step 1: The differential equation in factored operator form $y'' - 4y' + 3y = 4e^x - e^{2x}$ is

$$(D^2 - 4D + 3)y = 4e^x - e^{2x}$$

- Step 2: Now we can solve the homogeneous $f(D) = 0$ as

$$D^2 - 4D + 3 = 0$$

So auxiliary equation for homogeneous is

$$\begin{aligned} \lambda^2 - 4\lambda + 3 &= 0 \\ (\lambda - 1)(\lambda - 3) &= 0 \\ \Rightarrow & \quad \lambda_1 = 1 \text{ and } \lambda_2 = 3 \end{aligned}$$

Therefore, the complementary solution is

$$y_c = C_1e^x + C_2e^{3x}$$

- Step 3: Since $(D-1)e^x = 0$ and $(D-2)e^{2x} = 0$, the annihilator of $4e^x - e^{2x}$ is $(D-1)(D-2)$, then apply $(D-1)(D-2)$ to both sides of $y'' - 4y' + 3y = 4e^x - e^{2x}$, we have

$$\begin{aligned} y'' - 4y' + 3y &= 4e^x - e^{2x} \\ \Rightarrow (D^2 - 4D + 3)y &= 4e^x - e^{2x} \\ \Leftrightarrow (D-1)(D-2)(D^2 - 4D + 3)y &= (D-1)(D-2)(4e^x - e^{2x}) \\ \Leftrightarrow (D-1)(D-2)(D^2 - 4D + 3)y &= 0 \\ (D-1)(D-2)(D^2 - 4D + 3) &= 0 \\ (D-1)(D-2)(D-1)(D-3) &= 0 \\ (\lambda - 1)(\lambda - 2)(\lambda - 1)(\lambda - 3) &= 0 \\ \Rightarrow \lambda_1 = 1, \lambda_2 = 2, & \quad \lambda_3 = 1 \text{ and } \lambda_4 = 3 \end{aligned}$$

- Step 4: The candidate particular solution will be

$$\begin{aligned} \Rightarrow y*_p &= Ae^{\lambda_1 x} + Be^{\lambda_2 x} + Cxe^{\lambda_3 x} + Dxe^{\lambda_4 x} \\ & \quad \text{since at } \lambda = 1 \text{ is double root by reduction order} \\ &= Ae^x + Be^{2x} + Cxe^x + De^{3x} \end{aligned}$$

- Step 5: Delete the duplicate terms e^x and e^{3x} in $y*_p$, then
- Step 6: Find first and second derivatives of $y_p = Be^{2x} + Cxe^x$,

$$\begin{aligned} y_p &= Be^{2x} + Cxe^x \\ \Rightarrow y_p' &= 2B^{2x} + Ce^x + Cxe^x \\ \Rightarrow y_p'' &= 4B^{2x} + 2Ce^x + Cxe^x \end{aligned}$$

Substituting y_p, y_p' and y_p'' in $y'' - 4y' + 3y = 4e^x - e^{2x}$, we have

$$(4B^{2x} + 2Ce^x + Cxe^x) - 4(2B^{2x} + Ce^x + Cxe^x) + 3(Be^{2x} + Cxe^x) = 4e^x - e^{2x}$$

So $B = 1$ and $C = -2$. Hence

$$y_p = e^{2x} - 2xe^x$$

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- Step 7: Therefore, the general solution will be

$$\begin{aligned} y &= y_c + y_p \\ &= C_1e^x + C_2e^{3x} + e^{2x} - 2xe^x \end{aligned}$$

∎

∎ **Example 12.27** Solve $y'' + 8y = 5x + 2e^{-x}$ with undetermined coefficient.
Solution:

- Step 1: The differential equation in factored operator form $y'' + 8y = 5x + 2e^{-x}$ is

$$(D^2 + 8)y = 5x + 2e^{-x}$$

- Step 2: Now we can solve the homogeneous $f(D) = 0$ as

$$D^2 + 8 = 0$$

So auxiliary equation for homogeneous is $\lambda^2 + 8 = 0 \Rightarrow \lambda = \pm 2\sqrt{2}i$ it is a complex root. Therefore, the complementary solution is

$$y_c = C_1\cos(2\sqrt{2}x) + C_2\sin(2\sqrt{2}x)$$

- Step 3: Since $D^2 5x = 0$ and $(D+1)2e^{-x} = 0$, the annihilator of $5x + 2e^{-x}$ is $D^2(D+1)$, then apply $D^2(D+1)$ to both sides of $y'' + 8y = 5x + 2e^{-x}$, we have

$$\begin{aligned} y'' + 8y &= 5x + 2e^{-x} \\ \Rightarrow (D^2+8)y &= 5x + 2e^{-x} \\ \Leftrightarrow D^2(D+1)(D^2+8)y &= D^2(D+1)(5x + 2e^{-x}) \\ \Leftrightarrow D^2(D+1)(D^2+8)y &= 0 \\ D^2(D+1)(D^2+8) &= 0 \\ \lambda^2(\lambda+1)(\lambda^2+8) &= 0 \\ \Rightarrow \lambda_1 = 2\sqrt{2}i, \qquad & \lambda_2 = -2\sqrt{2}i, \lambda_3 = 0, \lambda_4 = 0 \text{ and } \lambda_5 = -1 \end{aligned}$$

- Step 4: The candidate particular solution will be

$$\begin{aligned} \Rightarrow y*_p &= Ae^{\lambda_1 x} + Be^{\lambda_2 x} + Ce^{\lambda_3 x} + Dxe^{\lambda_4 x} + Ee^{\lambda_5 x} \\ & \quad \text{since at } \lambda = 0 \text{ is double root by reduction order, we have} \\ &= Ae^{2\sqrt{2}ix} + Be^{-2\sqrt{2}ix} + Ce^{0x} + Dxe^{0x} + Ee^{-1x} \\ &= Ae^{2\sqrt{2}ix} + Be^{-2\sqrt{2}ix} + C + Dx + Ee^{-x} \\ &= F\cos(2\sqrt{2}x) + G\sin(2\sqrt{2}x) + C + Dx + Ee^{-x} \\ & \quad \text{since } F = A + B \text{ and } G = (A-B)i \end{aligned}$$

- Step 5: Delete the duplicate terms $\cos(2\sqrt{2}x)$ and $\sin(2\sqrt{2}x)$ in $y*_p$, then

$$y_p = C + Dx + Ee^{-1}$$

- Step 6: Find first and second derivatives of $y_p = C + Dx + Ee^{-x}$,

$$\begin{aligned} y_p &= C + Dx + Ee^{-x} \\ \Rightarrow y'_p &= D - Ee^{-x} \\ \Rightarrow y''_p &= Ee^{-x} \end{aligned}$$

Substituting y_p, y'_p and y''_p in $y'' + 8y = 5x + 2e^{-x}$, we have $Ee^{-1} + 8(C + Dx + Ee^{-x}) = 5x + 2e^{-x}$, so $C = 0$, $D = \frac{5}{8}$ and $E = \frac{2}{9}$. Hence, $y_p = \frac{5}{8}x + \frac{2}{9}e^{-1}$

- Step 7: Therefore, the general solution will be

$$\begin{aligned} y &= y_c + y_p \\ &= C_1\cos(2\sqrt{2}x) + C_2\sin(2\sqrt{2}x) + \frac{5}{8}x + \frac{2}{9}e^{-x} \end{aligned}$$

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12.2.2 The method of variation of parameters

Let $y_c = C_1y_1 + C_2y_2$ for $y'' + Py' + Qy = 0$, where y_1 and y_2 are any two linearly independent solution and C_1 and C_2 are any constant numbers. Then the method of variation of parameters says $y_p(x) = u_1y_1 + u_2y_2$, where u_1 and u_2 variables. So, by variation of parameters

$$u_1 = -\int \frac{y_2(x)R(x)}{W(y_1,y_2)}\,dx \text{ and } u_2 = \int \frac{y_1(x)R(x)}{W(y_1,y_2)}\,dx$$

■ **Example 12.28** Use the variation method to obtain a particular solution of the ordinary differential equation $y'' - 2y' + y = \frac{e^x}{x}$.
Solution: The auxiliary equation is

$$\begin{aligned} \lambda^2 - 2\lambda + 1 = 0 \\ \Rightarrow y_c = C_1e^x + C_2xe^x \end{aligned}$$

and $R(x) = \frac{e^x}{x}$ and

$$W(x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x}$$

Then

$$\begin{aligned} u_1 &= -\int \frac{y_2(x)R(x)}{W(y_1,y_2)}\,dx \\ &= -\int \frac{xe^x \frac{e^x}{x}}{e^{2x}}\,dx \\ &= -\int dx \\ &= -x \end{aligned}$$

$$\begin{aligned} u_2 &= \int \frac{y_1(x)R(x)}{W(y_1,y_2)}\,dx \\ &= \int \frac{e^x \frac{e^x}{x}}{e^{2x}}\,dx \\ &= \int \frac{1}{x}\,dx \\ &= \ln|x| \end{aligned}$$

Therefore,

$$\begin{aligned} y_p(x) &= u_1y_1 + u_2y_2 \\ &= -xe^x + \ln|x|(xe^x) \\ &= (-1 + \ln|x|)xe^x \end{aligned}$$

■

■ **Example 12.29** Use the variation method to obtain a general solution of the ordinary differential equation $y'' - 5y' + 6y = 4e^{2x}$.
Solution: The auxiliary equation is $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda_1 = 2$ or $\lambda_2 = 3$. Therefore, $y_1(x) = e^{2x}$, $y_2 = e^{3x}$ and $R(x) = 4e^x$. So, the complementary solution is $y_c = C_1e^{2x} + C_2e^{3x}$. Next, compute the wronskian function of y_1 and y_2

$$W(x) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x}$$

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Then

$$\begin{aligned} u_1 &= -\int \frac{y_2(x)R(x)}{W(y_1,y_2)}\,dx \\ &= -\int \frac{e^{3x}4e^{2x}}{e^{5x}}\,dx \\ &= -\int 4\,dx \\ &= -4x \end{aligned}$$

$$\begin{aligned} u_2 &= \int \frac{y_1(x)R(x)}{W(y_1,y_2)}\,dx \\ &= \int \frac{e^{2x}4e^{2x}}{e^{5x}}\,dx \\ &= 4\int e^{-x}\,dx \\ &= -4e^{-x} \end{aligned}$$

Therefore,

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 \\ &= -4xe^{2x} - 4e^{-x}e^{3x} \\ &= -4xe^{2x} - 4e^{2x} \\ &= -4e^{2x}(x+1) \end{aligned}$$

The general solution is

$$\begin{aligned} y &= y_c + y_p \\ &= C_1e^{2x} + C_2e^{3x} - 4e^{2x}(x+1) \end{aligned}$$

■

12.3 Euler's Cauchy Equation

Definition 12.3.1 Any differential equation of the form

$$a_nx^nD^ny + a_{n-1}x^{n-1}D^{n-1}y + \cdots + a_1xDy + a_0y = 0$$

where $a_n, a_{n-1}, \cdots, a_1, a_0$ are constants is called Euler's differential equation.

The second order Euler Equation (Cauchy equation) has a form

$$x^2y'' + axy' + by = 0 \tag{12.3}$$

where a and b are constants and $x \neq 0$. It can be solved by purely algebraic manipulations. Indeed substitution $y = x^\lambda$ and its derivatives in to the differential equation (12.3),we get

$$\begin{aligned} &x^2\lambda(\lambda-2)x^{\lambda-2} + ax\lambda x^{\lambda-1} + bx^\lambda = 0 \\ &\Rightarrow \lambda(\lambda-1)x^\lambda + a\lambda x^\lambda + bx^\lambda = 0 \\ &\Rightarrow (\lambda(\lambda-1) + a\lambda + b)x^\lambda = 0 \\ &\Rightarrow (\lambda(\lambda-1) + a\lambda + b) = 0 \end{aligned}$$

Thus,

$$\lambda^2 + (a-1)\lambda + b = 0 \tag{12.4}$$

Therefore, $y = x^\lambda$ is a solution if $\lambda^2 + (a-1)\lambda + b = 0$
Let λ_1 and λ_2 be roots of equation (12.4)

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- Case I: If $\lambda_1 \neq \lambda_2$ and real. The general solution is

$$y = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}, x > 0$$

If λ is not rational number, then x^λ is denoted by $x^\lambda = e^{\lambda \ln x}$.

- Case II: If $\lambda_1 = \lambda_2$ and real, then $y_1(x) = x^{\lambda_1}$, $y_2(x) = x^{\lambda_1} \ln x$ by reduction of order. Hence the general solution is

$$\begin{aligned} y &= C_1 y_1 + C_2 y_2 \\ &= C_1 x^{\lambda_1} + C_2 x^{\lambda_1} \ln x \\ &= (C_1 + C_2 \ln x) x^{\lambda_1}, x > 0 \end{aligned}$$

- Case III: If λ_1 and λ_2 complex roots, suppose that the roots λ_1 and λ_2 are complex conjugates say $\lambda_1 = \alpha - \beta i$ and $\lambda_2 = \alpha + \beta i$ with $\beta \neq 0$. We must now explain what is meant by x^λ when λ is complex.
Remember that $x^\lambda = e^{\lambda \ln x}$. So,

$$\begin{aligned} x^{\alpha \pm \beta i} &= e^{(\alpha \pm \beta i) \ln x} \\ &= e^{\alpha \ln x} \cdot e^{(\beta \ln x) i} \\ &= x^\alpha (\cos(\beta \ln x) \pm i \sin(\pm \beta \ln x)) \end{aligned}$$

The general solution $y = C_1 y_1 + C_2 y_2 = C_1 x^\alpha \cos(\beta \ln x) + C_2 x^\alpha (\sin(\beta \ln x)$

■ **Example 12.30** Solve $2x^2 y'' + 3xy' - y = 0$ where $x > 0$.
Solution: Substitute $y = x^\lambda$ gives

$$\begin{aligned} &2x^2(\lambda(\lambda-1)x^{\lambda-2}) + 3x(\lambda x^{\lambda-1}) - x^\lambda = 0 \\ &\Rightarrow 2\lambda(\lambda-1)x^\lambda + 3\lambda x^\lambda) - x^\lambda = 0 \\ &\Rightarrow x^\lambda(2\lambda(\lambda-1) + 3\lambda - 1) = 0 \\ &\Rightarrow 2\lambda^2 - 2\lambda + 3\lambda - 1 = 0 \end{aligned}$$

$$\begin{aligned} &\Rightarrow 2\lambda^2 + \lambda - 1 = 0 \\ &\Rightarrow (2\lambda - 1)(\lambda + 1) = 0 \\ &\Rightarrow \lambda_1 = \frac{1}{2} \text{ or } \lambda_2 = -1 \end{aligned}$$

Therefore, $y(x) = C_1 x^{\frac{1}{2}} + C_2 x^{-1}$, where $x > 0$. ■

■ **Example 12.31** Solve $4x^2 y'' - 4xy' + 3y = 0$, where $x > 0$.
Solution: Substitute $y = x^\lambda$ gives

$$\begin{aligned} &4x^2(\lambda(\lambda-1)x^{\lambda-2}) - 4x(\lambda x^{\lambda-1})3x^\lambda = 0 \text{ of } x > 0 \\ &\Rightarrow 4\lambda(\lambda-1)x^\lambda - 4\lambda x^\lambda) + 3x^\lambda = 0 \\ &\Rightarrow (4\lambda^2 - 4\lambda)x^\lambda - 4\lambda x^\lambda + 3x^\lambda = 0 \\ &\Rightarrow x^\lambda((4\lambda^2 - 4\lambda) - 4\lambda + 3) = 0 \end{aligned}$$

$$\begin{aligned} &\Rightarrow 4\lambda^2 - 8\lambda + 3 = 0 \\ &\Rightarrow (2\lambda - 3)(2\lambda - 1) = 0 \\ &\Rightarrow \lambda_1 = \frac{3}{2} \text{ or } \lambda_2 = \frac{1}{2} \end{aligned}$$

Therefore, $y(x) = C_1 x^{\frac{3}{2}} + C_2 x^{\frac{1}{2}}$. ■

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■ **Example 12.32** Solve $x^2y''+5xy'+4y=0$, where $x>0$.
Solution: Substitute $y=x^\lambda$ gives $x^\lambda(\lambda^2+4\lambda+4)=0$, hence, $\lambda_1=\lambda_2=-2$ and the general solution

$$\begin{aligned} y &= (C_1+C_2\ln x)x^{-2} \\ &= x^{-2}(C_1+C_2\ln x) \text{ where } x>0 \end{aligned}$$

■

■ **Example 12.33** Solve $x^2y''+xy'+y=0$, where $x>0$.
Solution: Substitute $y=x^\lambda$ gives

$$\begin{aligned} & x^2(\lambda^2-1)x^\lambda+x\lambda x^{\lambda-1}+x^\lambda=0 \\ & \Rightarrow (\lambda^2-\lambda)x^\lambda+\lambda x^\lambda+x^\lambda=0 \\ & \Rightarrow x^\lambda(\lambda^2-\lambda+\lambda+1)=0 \\ & \Rightarrow \lambda^2+1=0 \\ & \Rightarrow \lambda=\pm i \end{aligned}$$

From the above $\alpha=0$ and $\beta=1$. There, the general solution is

$$y(x)=C_1x^i+C_2x^{-i}=C_3\cos(\ln x)+C_4\sin(\ln x)$$

■

12.4 Linear ODE of Higher Order; System of ODE of Higher Order

A system of differential equations is a collection of equations in several unknown functions and their derivatives, each equation having at least one derivative. A system is called linear if the unknown functions and their derivatives enter each of the equations only to the first power. A linear system is of normal form when it is solved for all derivatives.

■ **Example 12.34** The system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= x-y+\frac{3}{2}t^2 \\ \frac{dy}{dt} &= -4x-2y+4t+1 \end{aligned}$$

is linear and is of normal form. We can eliminate from the linear system all unknowns (and their derivatives) except one, by adjoining to it the equations derived by differentiation. The resulting equation will contain one unknown function and its derivative of first and higher order. Finding the unknown function of this equation, we substitute its expression into the given equation and find the remaining unknown functions. ■

■ **Example 12.35** Solve the linear system

$$\frac{dx}{dt}=x-y+\frac{3}{2}t^2 \tag{12.5}$$

$$\frac{dy}{dt}=-4x-2y+4t+1 \tag{12.6}$$

Solution: To eliminate y and $\frac{dy}{dt}$, differentiate equation (12.5). This yields

$$\frac{d^2x}{dt^2}=\frac{dx}{dt}-\frac{dy}{dt}+3t \tag{12.7}$$

From equation (12.5) we find the expression of y in terms of t, x and $\frac{dx}{dt}$. Substitute in to equation (12.6), we get the expression $\frac{dy}{dt}$ in terms of the same quantities. Substitute this expression in to equation (12.7), we get a second-order linear equation

$$\frac{d^2x}{dt^2}+\frac{dx}{dt}-6x=3t^2-t-1 \Leftrightarrow x''+x'-6x=3t^2-t-1 \tag{12.8}$$

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First we have to find the homogeneous solution for $x'' + x' - 6x = 3t^2 - t - 1$. The characteristic equation is $\lambda^2 + \lambda - 6 = 0$.

$$\begin{aligned} \lambda^2 + \lambda - 6 &= 0 \\ (\lambda - 2)(\lambda + 3) &= 0 \\ \lambda_1 = 2 \text{ and } \lambda_2 &= -3 \end{aligned}$$

Therefore, $x(t) = C_1 e^{2x} + C_2 e^{-3x}$ is a complementary solution.
Next, we have to find a particular solution for $x'' + x' - 6x = 3t^2 - t - 1$. Since $R(x)$ is a polynomial function of degree 2, our particular solution will be $x_p = a_2 x^2 + a_1 x + a_0$. So,

$$\begin{aligned} x_p &= a_2 x^2 + a_1 x + a_0 \\ x_p' &= 2a_2 x + a_1 \\ x_p'' &= 2a_2 \end{aligned}$$

Substituting this, we have $a_2 = -\frac{1}{2}$. Hence, $x_p(t) = -\frac{1}{2}x^2$. Therefore, the general solution is

$$x = C_1 e^{2t} + C_2 e^{-3t} - \frac{1}{2}t^2$$

This expression is substituted in to equation (12.5) and we find the second unknown function

$$\begin{aligned} y &= -\frac{dx}{dt} + x + \frac{3}{2}t^2 \\ &= -C_1 e^{2t} + 4C_2 e^{-3t} + t^2 - t \end{aligned}$$

∎

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12.5 Exercise

1. Find the differential equation of the family of circles of radius r whose center lies on the x-axis.

   Ans. $y^2[(\frac{dy}{dx})^2+1]=r^2$

2. Find the differential equation of the family of parabolas with foci at the origin and axis along the x-axis.

   Ans. $y(\frac{dy}{dx})^2+2x\frac{dy}{dx}-y=0$

   Hint: the equation of the parabolas with foci at the origin and axis along the x-axis is given by $\sqrt{x^2+y^2}=\frac{x+c}{1^2}$.

3. Find the solution of the differential equation $y'=\frac{1}{2}(y^2-1)$, then satisfy the initial condition $y(0)=2$.

   Ans. $y=\frac{3+e^x}{3-e^x}$

4. Find a particular solution for the differential equation $y'=1+4y^2$, $y(0)=0$

   Ans. $\frac{1}{2}\tan^{-1}(\frac{y}{2})=x+c_0$

5. Find a particular solution for the differential equation$y'=y-x$, $y(0)=3$

   Ans. $y=x+1+2e^x$

6. Find the general solution for the differential equation $yy'+36x=0$.

   Ans. $y^2+36x^2=c_0$

7. Find the general solution for the differential equation $y'=\frac{4x^2+y^2}{xy}$.

   Ans. $y=\sqrt{2x^2(c+4\ln x)}$

8. Find a particular solution for the differential equation $2xyy'=3y^2+x^2$, $y(1)=2$

   Ans. $\frac{x^3}{y^2+x^2}=\frac{1}{5}$

9. Find the general solution $(2x+\frac{1}{y}-\frac{y}{x^2})dx+(2y+\frac{1}{x}-\frac{x}{y^2})dy=0$.

   Ans. $c_0=x^2=\frac{x}{y}+\frac{y}{x}+y^2$

10. Find the general solution for the differential equation $\frac{dy}{dx}+xy=x^3y^3$.

    Ans. $y=\frac{1}{\sqrt{x^2+1+c_0e^{x^2}}}$

11. Find the general solution for the differential equation $(x^2-y)dx+xdy=0$.

    Ans. $x-\frac{y}{x}=c_0$

12. Find the general solution for the differential equation $(x^2+y^2)dx+(x-2y)dy=0$.

    Ans. $\frac{x^3}{3}-xy-y^3=c_0$

13. Find the general solution for the differential equation $(y-3x^2)dx-(4y-x)dy=0$.

    Ans. $2y^2-xy+xy+x^3=c_0$

14. Find the general solution for the differential equation $(2x\tan y dx+(x^2-2\sin y)dy=0$.

    Ans. $x^2\sin y+2\sin^2 y=c$

15. Find the general solution for the differential equation $3y^2y'-ay^3-x-1=0$.

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Ans. $a^2y^3 = ce^{ax} - a(x+1) - 1$

16. Find the general solution for the differential equation $y' + 4x^2y = (4x^2 - x)e^{-\frac{x^2}{2}}$.

Ans. $y = e^{-\frac{x^2}{2}} + ce^{-\frac{4x^3}{3}}$

17. Find the initial value problem for the differential equation $y' + 2y\sin 2x = 2e^{\cos 2x}$, $y(0) = 0$.

Ans. $y = 2xe^{\cos 2x}$

18. Find the initial value problem for the differential equation $y' + \frac{y}{x^2} = 2xe^{\frac{1}{x}}$, $y(1) = 14$

Ans. $y = e^{\frac{1}{x}}(x^2 + \frac{14}{e} - 1)$

19. Solve the for the differential equation $y^{'} - (\frac{3}{x+1})y = (x+1)^3$.

Ans. $y = (x+1)^3(x + c_0)$

20. Solve the differential equation $xy' + y = \sqrt{x}$.

Ans. $y = \frac{2}{3}\sqrt{x} + \frac{c_0}{x}$

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