MATHEMATICS

TEACHER'S GUIDE
GRADE 10

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INTRODUCTION

The study of mathematics at this cycle, grades 9 -10, prepares our students for the future, both practically and philosophically. Studying mathematics provides them not only with specific skills in mathematics, but also with tools and attitudes for constructing the future of our society. As well as learning to think efficiently and effectively, our students come to understand how mathematics underlies daily life and, on a higher level, the dynamics of national and international activity. The students automatically begin to apply high-level reasoning and values to daily life and also to their understanding of the social, economic, political and cultural realities of the country. In turn, this will help them to actively and effectively participate in the ongoing process of developing the nation.

At this cycle, our students gain a solid knowledge of the fundamental mathematical theories, theorems, rules and procedures. They also develop reliable skills for using this knowledge to solve problems independently. To this end, the objectives of mathematics learning at this cycle are to enable students to

- gain a solid knowledge of mathematics.
- appreciate the power, elegance and structure of mathematics.
- use mathematics in daily life.
- understand the essential contributions of mathematics to the fields of engineering, science, economics and so on.

Recent research gives strong arguments for changing the way in which mathematics has been taught. The rote-learning paradigm has been replaced by the student-centered model. A student-centered classroom stimulates student inquiry, and the teacher serves as a mentor who guides students as they construct their own knowledge base and skills. A primary goal when you teach a concept is for the students to discover the concept for themselves, particularly as they recognize threads and patterns in the data and theories that they encounter under your guidance.

One of our teaching goals is particularly fostered by the student-oriented approach. We want our students to develop personal qualities that will help them in real life.

For example, student-oriented teachers encourage students' self confidence and their confidence in their knowledge, skills and general abilities. We motivate our students to express their ideas and observations with courage and confidence. Because we want them to feel comfortable addressing individuals and groups and to present themselves and their ideas well, we give them safe opportunities to stand before the class and present their work. Similarly, we help them learn to learn to answer questions posed directly to them by other members of the class.

Teamwork is also emphasized in a student-centered classroom. For example, the teacher creates favorable conditions for students to come together in groups and exchange ideas about what they have learned and about material they have read. In this process, the students are given many opportunities to openly discuss the knowledge they have acquired and to talk about issues raised in the course of the discussion.

This teacher's guide will help you teach well. For example, it is very helpful for budgeting your teaching time as you plan how to approach a topic. The guide suggests tested teaching-time periods for each subject you will teach. Also, the guide contains answers to the review questions at the end of each topic.

Each section of your teacher’s guide includes student-assessment guidelines. Use them to evaluate your students’ work. Based on your conclusions, you will give special attention to students who are working either above or below the standard level of achievement. Check each student’s performance against the learning competencies presented by the guide. Be sure to consider both the standard competencies and the minimum competencies. Note that the *minimum requirement level* is not the *standard level of achievement*. To achieve the standard level, your students must fulfill all of their grade-level’s competencies successfully.

When you identify students who are working either below the standard level or below the minimum level, give them extra help. For example, give them supplementary presentations and reviews of the material in the section, give them extra time to study, and develop extra activities to offer them. You can also encourage high-level students in this way. You can develop high-level activities and extra exercises for them and can offer high-level individual and group discussions. Be sure to show the high-level students that you appreciate their good performance, and encourage them to work hard. Also, be sure to discourage any tendencies toward complacency that you might observe.

Some helpful reference materials are listed at the end of this teacher’s guide. For example, the internet is a rich resource for teachers, and searching for new web sites is well worth your time as you investigate your subject matter. Use one of the many search engines that exist – for example, Yahoo and Google.

Do not forget that, although this guide provides many ideas and guidelines, you are encouraged to be innovative and creative in the ways you put them into practice in your classroom. Use your own knowledge and insights in the same way as you encourage your students to use theirs.

Contents

Active Learning and Continuous Assessment required!

Dear mathematics teacher! For generations the technique of teaching mathematics at any level was dominated by what is commonly called the **direct instruction.** That is, students are given the exact tools and formulas they need to solve a certain mathematical problem, sometimes without a clear explanation as to why, and they are told to do certain steps in a certain order and in turn are expected to do them as such at all times. This leaves little room for solving varying types of problems. It can also lead to misconceptions and students may not gain the full understanding of the concepts that are being taught.

You just sit back for a while and try to think the most common activities that you, as a mathematics teacher, are doing in the class.

> *Either you explain (lecture) the new topic to them, and expect your students to remember and use the contents of this new topic or you demonstrate with examples how a particular kind of problem is solved and students routinely imitate these steps and procedures to find answers to a great number of similar mathematical problems.*

But this method of teaching revealed little or nothing of the meaning behind the mathematical process the students were imitating.

We may think that teaching is telling students something, and learning occurs if students remember it. But research reveals that teaching is not “pouring” information into students’ brain and expecting them to process it and apply it correctly later.

Most educationalists agree that learning is an active meaning-making process and students will learn best by trying to make sense of something on their own with the teacher as a guide to help them along the way. This is the central idea of the concept Active Learning.

> **Active learning**, *as the name suggests, is a process whereby learners are actively engaged (involved) in the learning process, rather than "passively" absorbing lectures. Students are rather encouraged to think, solve problems, do activities carefully selected by the teacher, answer questions, formulate questions of their own, discuss, explain, debate, or brainstorm, explore and discover, work cooperatively in groups to solve problems and workout projects.*

The design of the course materials (student textbooks and teachers guides) for mathematics envisages active learning to be dominantly used. With this strategy, we feel that you should be in a position to help students understand the concepts through relevant, meaningful and concrete activities. The activities should be carried out by students to explore the world of mathematics, to learn, to discover and to develop interest in the subject. Though it is your role to exploit the opportunity of using active learning at an optimal level, for the sake of helping you get an insight, we recommend that you do the following as frequently as possible during your teaching:

- Engage your students in more relevant and meaningful activities than just listening.
- Include learning materials having examples that relate to students life, so that they can make sense of the information.
- Let students be involved in dialog, debate, writing, and problem solving, as well as higher-order thinking, e.g., analysis, synthesis, evaluation.
- Encourage students' critical thinking and inquiry by asking them thoughtful, open-ended questions, and encourage them to ask questions to each other.
- Have the habit of asking learners to apply the information in a practical situation. This facilitates personal interpretation and relevance.
- Guide them to arrive at an understanding of a new mathematical concept, formula, theorem, rule or any generalization, by themselves. You may realize this by giving them an activity in which students sequentially uncover layers of mathematical information one step at a time and discover new mathematics.
- Select assignments and projects that should allow learners to choose meaningful activities to help them apply and personalize the information. These need to help students undertake initiatives, discover mathematical results and even design new experiments to verify results.
- Let them frequently work in peers or groups. Working with other learners gives learners real-life experience of working in a group, and allows them to use their metacognitive skills. Learners will also be able to use the strengths of other learners, and to learn from others. When assigning learners for group work membership, it is advisable if it is based on the expertise level and learning style of individual group members, so that individual team members can benefit from one another's strengths.

In general, if mathematics is to develop creative and imaginative mathematical minds, you must overhaul your traditional methods of presentation to the more active and participatory strategies and provide learning opportunities that allow your students to be actively involved in the learning process. While students are engaged with activities, group discussions, projects, presentations and many others they need to be continuously assessed.

Continuous Assessment

You know that continuous assessment is an integral part of the teaching learning process. Continuous assessment is the periodic and systematic method of assessing and evaluating a person's attributes and performance. Information collected from continuous behavioral change of students will help teachers to better understand their strengths and weaknesses in addition to providing a comprehensive picture of each student over a period of time. Continuous assessment will afford student to readily see his/her development pattern through the data. It will also help to strengthen the parent teacher relationship and collaboration. It is an ongoing process more than giving a test or exam frequently and recording the marks.

Continuous assessment enables you to assess a wide range of learning competencies and behaviors using a variety of instruments some of which are:

- Tests/ quizzes (written, oral or practical)
- Class room discussions, exercises, assignments or group works.
- Projects
- Observations
- Interview
- group discussions
- questionnaires

Different competencies may require different assessment techniques and instruments. For example, oral questions and interviews may serve to assess listening and speaking abilities. They also help to assess whether or not students are paying attention, and whether they can correctly express ideas. You can use oral questions and interviews to ask students to restate a definition, note or theorem, etc. Questionnaires, observations and discussions can help to assess the interest, participation and attitudes of a student. Written tests/exams can also help to assess student's ability to read, to do and correctly write answers for questions.

When to Assess

Continuous assessment and instruction are integrated in three different time frames namely, Pre-instruction, During-instruction and Post-instruction. To highlight each briefly

1. Pre-instruction assessment

This is to assess what students luck to start a lesson. Hence you should start a lesson by using opportunities to fill any observed gap. If students do well in the pre-instruction assessment, then you can begin instructing the lesson. Otherwise, you may need to revise important concepts.

The following are some suggestions to perform or make use of pre-instruction assessment.

i. assess whether or not students have the prerequisite knowledge and skill to be successful, through different approaches.
ii. make your teaching strategies motivating.
iii. plan how you form groups and how to give marks.
iv. create interest on students to learn the lesson.

2. Assessment During Instruction:

This is an assessment during the course of instruction rather than before it is started or after it is completed. The following are some of the strategies you may use to assess during instruction.

i. observe and monitor students' learning.

ii. check that students are understanding the lesson. You may use varying approaches such as oral questions, asking students to do their work on the board, stimulate discussion, etc.

iii. identify which students need extra help and which students should be left alone.

iv. ask a balanced type of exercise problems according to the students ability, help weaker students and give additional exercise for fast students.

v. monitor how class works and group discussions are conducted

3. Post Instruction Assessment:

This is an assessment after instruction is completed. It is conducted usually for the purpose of documenting the marks and checking whether competencies are achieved. Based on the results students scored, you can decide whether or not there is anything the class didn't understand because of which you may revise some of the lessons or there is something you need to adjust on the approach of teaching. This also help you analyze whether or not the results really reflect what students know and what they can do, and decide how to treat the next lesson.

Forming and managing groups

You can form groups through various approaches: mixed ability, similar ability, gender or other social factors such as socioeconomic factors. When you form groups, however, care need to be taken in that you should monitor their effort. For example, if students are grouped by mixed ability the following problems may happen.

1. Mixed ability grouping may hold back high-ability students. Here, you should give enrichment activities for high ability students.
2. High ability students and low ability students might form a teacher – student relationship and exclude the medium ability students from group discussion. In this case you should group medium ability students together.

When you assign group work, the work might be divided among the group members, who work individually. Then the members get together to integrate, summarize and present their finding as a group project. Your role is to facilitate investigation and maintain cooperative effort.

Highlights about assessing students

You may use different instruments to assess different competencies. For example, consider each of the following competencies and the corresponding assessment instruments.

Competency 1. Define exponential and logarithmic functions.

Instrument: Oral question.

Question: What are exponential and logarithmic functions?

Competency 2 - Students will evaluate logarithm of a number.

Instrument: class work/homework/quiz/test

Question: Find the value of x where a. $x = \log_2 32$ b. $\log_2 x = 4$

Competency 3 – Apply logarithmic functions in their daily life problems.

Instrument: Assignment/project.

Question: Chaltu deposited 10,000 Birr in a bank at her surroundings. Go to the bank and ask the rate of interest the bank pays and determine the time that takes Chaltu to have an amount to be 15,000 Birr?

How often to assess

Here are some suggestions which may help you how often to assess.

- Class activities / class works: Every day (when convenient).
- Homework/Group work: as required.
- Quizzes: at the end of every one (or two) sub topics.
- Tests: at the end of every unit.
- Exams: once or twice in every semester.

How to Mark

The following are some suggestions which may help you get well prepared before you start marking:

- use computers to reduce the burden for record keeping.
- although low marks may diminish the students motivation to learn, don't give inflated marks for inflated marks can also cause reluctance.

The following are some suggestions on how to mark a semester's achievement.

1. One final semester exam 30%.
2. Tests 25%
3. Quizzes 10%
4. Homework 10%
5. Class activities, class work, presentation demonstration skills 15%
6. Project work, in groups or individually 10%.

Moreover

In a group work allow students to evaluate themselves as follows using format of the following type.

	A	***B***	***C***	***D***	
The ability to communicate					
The ability to express written works					
Motivation					
Responsibility					
Leadership quality					
Concern for others					
Participation					
Over all					

You can shift the leadership position or regroup the students according to the result of the self evaluation. You can also consider your observation.

Reporting students' progress and marks to parents

Parents should be informed about their children's progress and performance in the class room. This can be done through different methods.

1. The report card: two to four times per year.
2. Written progress report: Per week/two weeks/per month/two months.
3. Parent – teacher conferences (as scheduled by the school).

The report should be about the student performance say, on tests, quizzes, projects, oral reports, etc that need to be reported. You can also include motivation or cooperation behavior. When presenting to parents your report can help them appraise fast learner, pay additional concern and care for low achieving student, and keep track of their child's education. In addition, this provides an opportunity for giving parents helpful information about how they can be partners with you in helping the student learn more effectively.

The following are some suggested strategies that may help you to communicate with parents concerning marks, assessment and student learning.

1. Review the student's performance before you meet with parents.
2. Discuss with parents the students good and poor performances.
3. Do not give false hopes. If a student has low ability, it should be clearly informed to his/her parents.
4. Give more opportunities for parents to contribute to the conversation.
5. Do not talk about other students. Don't compare the student with another student.
6. Focus on solutions

NB. ***All you need to do is thus plan what type of assessment and how many of each you are going to use beforehand (preferably during the beginning of the year/semester).***

UNIT 1 POLYNOMIAL FUNCTIONS

INTRODUCTION

This unit requires a firm understanding of function. The unit gives much emphasis to the definition of polynomial function, theorems on polynomial, zero(s) of a polynomial function and graph of polynomial function. Each topic is presented by giving succinct explanation and illustrative examples worked out in details.

The activities and exercises given in each sub unit are designed to encourage students to think critically about the lesson presented and to explore the key concept in more details:

Unit Outcomes

After completing this unit, students will be able to:

- *define polynomial functions.*
- *perform the four fundamental operations on polynomials.*
- *apply the theorems on polynomials to solve related problems.*
- *sketch and analyze the graphs of polynomial functions.*
- *determine the number of rational and irrational zeros of a polynomial.*

Suggested Teaching Aids in Unit 1

You can present different charts and a model graph for polynomial function that demonstrate the end behavior of polynomial function. You can also encourage students to prepare different representative graphs of polynomial function by themselves. Apart from the use of the student textbook, you need to elaborate more real life problems from your surroundings so that students can best appreciate and see how useful polynomial function is.

1.1 INTRODUCTION TO POLYNOMIAL FUNCTIONS

Periods allotted: 5 periods

Competencies

At the end of this sub-unit, students will be able to:

- *define the polynomial function of one variable.*
- *identify the degree, leading coefficient and constant terms of a given polynomial function.*
- *give different forms of polynomial functions.*
- *perform the four fundamental operations on polynomials.*

Vocabulary: Function, Domain, Constant function, Linear function, Quadratic function, Polynomial function, Degree, Leading coefficient, Constant term.

Introduction

This sub-unit begins with discussing the concept of function and special types of function such as constant function, linear function and quadratic function which are special cases of the general class of polynomial function.

It also provides students with definition of polynomial function and operations on polynomials: addition, subtraction, multiplication and division of polynomial functions.

Teaching Notes

Students are expected to have some background on the concept of function, domain and range of a function.

For the purpose of revision, you could ask students questions like:

1. Determine whether the following relations are functions or not. Let them give reasons.

   a. $R = \{(x, y): x \geq y + 1\}$ b. $R = \{(x, y): x^2 + y^2 = 1\}$

   c. $R = \{(x, y): y = x^2+1\}$ d. $R = \{(x, y): y = x(x-2)(x+2)\}$

2. Find the domain and range of the functions:

   a. $R = \{(x, y): x = 2y - \frac{2}{3}\}$ b. $R = \{(x, y): y = x^2 - 2x + 5\}$

You may ask students to describe constant function, linear function and quadratic function which they studied in grade 9. After deliberation by students, you may start the lesson by discussing the formal definition of constant, linear and quadratic function given in the student textbook and illustrate with examples. You can give exercise 1.1 as class work. This exercise might be used as a guide into the definition of polynomial function to gauge students' abilities and good understanding of the polynomial function. The possible answers to Exercise 1.1 are as follows.

Answers to Exercise 1.1

1.
a.	Quadratic	b.	None	c.	None	d.	Linear
e.	Constant	f.	Constant	g.	None	h.	Quadratic
i.	Linear	j.	None	k.	None	l.	None

2. Constant function, if $a = 0$, $b = 0$ and $c \in \mathbb{R}$
   Linear function, if $a = 0$, $b \neq 0$ and $c \in \mathbb{R}$
   Quadratic function, if $a \neq 0$ $b, c \in \mathbb{R}$

Before giving the formal definition of polynomial function, you should encourage students to define the terms monomial, binomial, trinomial and polynomial by providing examples. Let them determine the following expressions as monomial, binomial, trinomial or none of these.

a. $2\sqrt{3}\,x^2 + 5x^2$ b. $ax^2 + 6x + c,\ a \neq 0$

c. $\frac{2}{3}x(1 - x)\,(2 - x) + 8x$ d. $10 - x^4 + 5x + \frac{2x^2 - 6}{3}$

After giving the definition of polynomial function, encourage students, through question and answer, to revise the terms such as degree, coefficients, leading coefficients and constant term.

After understanding these, let the students consider example 1 in the student textbook to find these terms. Give some exercises or the Activity 1.1 in the student textbook and group the students so that they can practice to solve. You are supposed to round in the class to facilitate their work and help students who could not solve it properly.

The purpose of Activity 1.1 is to help students to revise terms related to polynomial function and check their level of understanding of polynomial function and its relevant terms. The possible answer to Activity 1.1 is as follows.

Answers to Activity 1.1

1. a. 4 b. $\frac{-3}{2}$ c. -1 d. $\frac{3}{4}$

2. a. Let the surface area of the match box be A, then
   $A = ph$, where p is perimeter of the base and h is its height.

$$\begin{aligned} A(x) &= 2[3x + 3(x + 1) + x(x + 1)\ ] \\ &= 2[3x + 3x + 3 + x^2 + x] \\ &= 2[x^2 + 7x + 3] \\ \Rightarrow A(x) &= (2x^2 + 14x + 6)\ \text{cm}^2 \end{aligned}$$

   b. Degree of $A = 2$
   Constant term of $A = 6$.

After ensuring the ability of the students in conducting Activity 1.1, encourage students to reflect upon and engage in discussion about polynomial function similar to example 2 and 3 on the board, you may also give exercises for fast learners to determine whether the following functions are polynomial or not.

a. $f(x) = 2x - \log_2 x$ b. $g(x) = \frac{3}{5}(2 - x)\left(\frac{2}{3} - 5x\right)$

c. $h(x) = \sqrt[3]{8x^3} + \sqrt{2x^2} + 8$ d. $f(x) = x + \frac{x^2+1}{x^2+1} - 6$

With active participation of students, you need to assist them to tell about polynomial function over integer, rational number and real numbers. You may then give questions 1 and 2 of Exercise 1.2 as class discussion so that each student will do individually. Select the rest of questions in Exercise 1.2 as homework for all students.

Assessment

At the end of this lesson, apart from Exercise 1.2, you can give class activities, assignments and quiz or test, to assess their level of understanding. For fast learners or interested students, you can also give the following additional exercise problems.

1. For each of the following polynomial function, let students find the degree, the leading coefficient, the constant term, a_{n-1}, a_{n-3} and a_2.
   a. $f(x) = \frac{2}{3}\left(2 - 6x^5 + \frac{9}{4}x\right) + 8x - x^7$
   b. $f(x) = (1 - 2x)(x^2 - 1)(x + 2)$
   c. $f(x) = (1 - x^3)\left(2x + \frac{1}{3}\right)(x - 1) - 12x$
   d. $f(x) = (8x - x^4 + x^3) - (2x + x^3 - x^4 + 8)$
2. List the coefficients of x^n, x^{n-1}, x^{n-2}, … x^2, x in that order for the following polynomials of degree n.
   a. $16x^8 + 3x^7 - x^4 + x^3 - 5x$
   b. $\frac{3}{2}x^4 - \frac{1}{4}x^6 + 8x^2 - \frac{4}{5}$
3. A rectangular container measured 1 m by 2 m by 1 m is covered with a layer of lead shielding of uniform thickness. Find the volume v of the lead shielding as a function of the thickness x (in meter) of the shielding.
   (**Answer:** $v(x) = (1 + 2x)(2 + 2x)(1 + 2x) - 2$)

Answers to Exercise 1.2

1. a, b, d, f, g, h, j, n, o, q, r, s and t are polynomial functions. But, c, e, i, k, m, l, and p are not polynomial functions

2.

No	degree	Leading Coefficient	Constant term
a	4	3	−9
b	25	1	1
d	2	$\frac{1}{3}$	1
f	1	−95	108
g	6	312	0
h	3	−1	$\sqrt{2}$
j	3	$-\frac{5}{8}$	$\frac{3}{4}$
n	1	$\frac{1}{12}$	0
o	no degree	0	0
q	97	$\sqrt{54}$	π
r	0	——	$\frac{4}{7}-2\pi$
s	2		2
t	2	−1	$-\frac{1}{2}$
		1	

3. a. a, b, f, g, o and s

   b. a, b, d, f, g, j, n, o, s and t.

   c. a, b, d, f. g, h, j, n, o, q, r, s and t

4. a, b, e, g, h, and i are polynomial expressions.

5. a. Since the volume of a box is length × width × height

   We have that $V(x) = (20 - 2x)(20 - 2x)x$

   $= 4x^3 - 80x^2 + 400x$

   $\Rightarrow V(x) = 4x^3 - 80x^2 + 400x$

   b. $V(x) = (20–2x)(20–2x)x$

   Domain: $x \geq 0$ or $20 - 2x \geq 0$

   $= 0 \leq x$ or $10 \geq x$

   $= \{x \in \mathbb{R} \mid 0 \leq x \leq 10\}$

After completion of Exercise 1.2, you should allow time for students to look at the fundamental operations in algebra and let students express their thoughts or ideas on commutative, associative and distributive laws.

Addition of polynomials is done by adding the like terms. Polynomial can be added horizontally or vertically as shown in the next example.

Let $f(x) = x^3 - 7x + 12$ and $g(x) = 4x^2 + 3x - 8$

When adding horizontally, use commutative and associative property:

$$f(x) + g(x) = (x^3 - 7x + 12) + (4x^2 + 3x - 8)$$
$$= x^3 + 4x^2 + (-7+3)x + (12 - 8)$$
$$= x^3 + 4x^2 - 4x + 4$$

When adding vertically, first line up like terms.

$$\begin{array}{r} x^3 \qquad\quad - 7x + 12 \\ \underline{+\, 4x^2 + 3x - 8} \\ x^3 + 4x^2 - 4x + 4 \end{array}$$

Therefore, $f(x) + g(x) = x^3 + 4x^2 - 4x + 4$.

Start the new lesson by giving questions 1-5 of Activity 1.2 as class work and round to identify those who need further assistance and those who are fast enough to solve each question, and put on record. For those who are fast enough, you can give questions 6 of Activity 1.2 additionally. Finally, for all students, you can solve each of the questions of Activity 1.2 on the board by giving each student a chance to participate. You may then give students the chance to answer question 6 of Activity 1.2 by providing examples on the board. With participation, you need to assist them to tell the reason for each step while solving Example 4.

Answers to Activity 1.2

1. Like terms are terms having the same variables to the same powers but possibly different coefficients.eg. $8a^2$, $-4a^2$ . . .
2. They are not like terms because they have no variables to the same powers.
3. (b) and (c) are true statements but (a) and (d) are false statements.
4. a. $(4x + a) + (2a - x) = 4x + a + 2a - x$
   $= (4x - x) + (a + 2a) = 3(x + a)$.

   b. $5x^2y + 2xy^2 - (x^2y - xy^2) = 5x^2y + 2xy^2 - x^2y + xy^2$
   $= (5x^2y - x^2y) + (2xy^2 + xy^2)$
   $= 4x^2y + 3xy^2$.

   c. $8a - (b + 9a) = 8a - b - 9a = -a - b = -(a + b)$.

   d. $2x - 4(x - y) + (y - x) = 2x - 4x + 4y + y - x$
   $= (2x - 5x) + (4y + y) = 5y - 3x$.
5. a. (False $\therefore f(x) + g(x) = (x^3 - 2x^2 + 1) + (x^2 - x - 1)$
   $= x^3 - 2x^2 + 1 + x^2 - x - 1 = x^3 - x^2 - x$.

   b. (True $\therefore f(x) - g(x) = (x^3 - 2x^2 + 1) - (x^2 - x - 1)$
   $= x^3 - 2x^2 + 1 - x^2 + x + 1 = x^3 - 3x^2 + x + 2$.

   c. (False $\therefore g(x) - f(x) = (x^2 - x - 1) - (x^3 - 2x^2 + 1)$
   $= x^2 - x - 1 - x^3 + 2x^2 - 1 = 3x^2 - x^3 - x - 2$.

   d. True
6. c and d are necessary true.

You need to discuss and solve Activity 1.3 with active participation of students. The purpose of this activity is to help students to revise addition of polynomials. After the completion of Activity 1.3, you may start subtraction of polynomials by presenting examples such as Example 5.

Answers to Activity 1.3

1. In part (a) of Example 4 on the student textbook, the degree of $f + g$ is 2 is lower than the common degree 3 and in part (b), the degree of $f + g$ is 5. Which is equal to the degree of f.
2. Yes.
3. In part (a) of Example 4 in student textbook, the degree of $f + g$ is 2 which is lower than the common degree 3 because their leading coefficients are equal but opposite in sign.
4. The domain of $(f + g)(x)$ is the set of all real numbers.

Before students do Activity 1.4, you can give hint to students to use Example 6 as a guide for doing Activity 1.4.

Revise the distributive property of multiplication over addition by using several examples such as

$$\begin{aligned}(x + 5)(2x + 3) &= x(2x + 3) + 5(2x + 3)\\ &= 2x^2 + 3x + 10x + 15\\ &= 2x^2 + 13x + 15 \text{ … combine like terms}\end{aligned}$$

Product of two polynomials can also be found by arranging the multiplication vertically like multiplication of whole numbers as follows:

$$\begin{array}{r} x + 5\\ \underline{2x + 3}\\ 3x + 15\\ \underline{2x^2 + 10x\quad\;\;}\\ 2x^2 + 13x + 15\end{array}$$

At this stage, you can also give application problems involving polynomial function similar to Example 8. For example, you can give the application problem given below. An open box is to be made from a rectangular piece of cardboard that measure 80 cm by 50 cm, by cutting out a square of the same size from each corner and bending up the sides. Find the volume v of the box as a function of x where x is the length of the side of the square cutting out.

Answers to Activity 1.4

1. The degree of $f.g$ is $m + n$
2. Not defined
3. Yes

It is possible to encourage students, through question and answer, to revise division of polynomials and illustrate with examples from division of positive integers which they studied in lower grades. For instance, let them consider what happens when 13 is divided by 5. Undoubtedly, they will get the answer 2 and have a remainder of 3. They could write this as $\frac{13}{5} = 2 + \frac{3}{5}$

Clearly, another way of thinking about this division is

$$13 = 2 \times 5 + 3$$

So, let students generalize that division of polynomials is something like division of positive integers.

After discussing these, let the students observe what value is obtained when 24 is divided by 8. Clearly, the value is 3, that is $24 = 3 \times 8$. So, you can say that the division is exact.

To divide polynomials using long division, first divide the first term of the dividend by the first term of the divisor. This is the first term of the quotient. Multiplying the new term of the quotient by the divisor, subtract this product from the dividend. Repeat these steps using the difference as the new dividend until the first term of the divisor is of a greater degree than the new dividend. The last new dividend whose degree is less than the divisor is the remainder. Let students practice division of polynomials by doing the following division.

a. Divide $x^3 + 2x - 4x^2 + 8$ by $x^2 - 2$.

b. Divide $\frac{1}{2}x^6 + 4x^4 + 8x - 2$ by $x^3 + x - 1$.

Following this, students are supposed to do Activity 1.5. After doing this activity, consider, example 9 and 10 to discuss division of polynomial by giving special emphasis to students' participation.

Answers to Activity 1.5

1. $x - 2$ is the divisor, $x^2 - x + 2$ is dividend, $x + 1$ is the quotient and 4 is the remainder.
2. The result when $x^3 + 1$ is divided by $x + 1$ is equal to $x^2 - x + 1$. The remainder is 0.
3. When the remainder is zero, the division is said to be exact (or if the divisor is the factor of the dividend, then the division is exact).
4. The degree of the dividend must be greater than or equal to the degree of the divisor.
5. The degree of the quotient is $n - m$.

You may assign the students a task to solve Group Work 1.1 given in the student textbook to work in groups. Of course, you may be interested in the correct answer. However, you also want to see how they understand the concept, the procedures they use and how they communicate the solution to the problems. Let some of the group present and demonstrate their solution on the board to encourage their participation in the class discussion. Ask other students to comment on the issue raised. The possible answers for Group Work 1.1 are given as follows.

Answers to Group work 1.1

1. $f(x) = x^3 + 2x^2 + x + 2$ and $g(x) = -x^3 - 2x^2 - 4x + 2$.
   The leading coefficients of the terms with degree ≥ 2, f and g are equal but opposite in sign.
2. $a = 0, b \neq 0$
3. $f(x) = x^4 - 2x^2 + 2x - 14$.

You can also add examples to find $f+g$, $f-g$, fg and $\dfrac{f}{g}$ where

$$f(x) = \frac{2}{3}x - x^3 + \frac{4}{7}x^2 + 1 \text{ and } g(x) = \frac{8x - x^4 + 6}{2}$$

In which the solutions are:

$$f + g = \frac{14}{3}x - \frac{1}{2}x^4 - x^3 + \frac{4}{7}x^2 + 4$$

$$f - g = \frac{1}{2}x^4 - x^3 + \frac{4}{7}x^2 - \frac{10}{3}x - 2$$

$$fg = \frac{1}{2}x^7 - \frac{2}{7}x^6 - \frac{1}{3}x^5 - \frac{9}{2}x^4 - \frac{5}{7}x^3 + \frac{92}{21}x^2 + 6x + 3$$

$$\frac{f}{g} = \frac{28x - 42x^3 + 24x^2 + 42}{168x - 21x^4 + 126}$$

So, the students can develop more critical thinking on the operations of polynomials.

After deliberating on this lesson, since all students may not go in parallel, it is necessary to develop additional exercises of different capacity apart from the ones given in the student textbook that need to be solved by themselves.

You can also encourage students by providing different exercises, either as a group work or as an assignment and share each with another.

Exercise 1.3 might be useful as a stimulus in order to encourage students to reflect upon and engage in doing class work or home work.

This exercise might also be used as a means of checking student's participation, understanding and how well they achieved, what was expected by the end of the lesson checking their work. It is also important to maintain some sort of a record keeping system.

Assessment

Continuous assessment addresses various strategies that teachers can use in order to ensure that all students in their class can fully participate in meaningful discussion. Apart from the details mentioned above, you can also give class activities, group discussion, assignments, exercise problem and quiz or test for assessing students learning. This is because assessment helps you to obtain useful feedback on what, how much, and how well your students are learning.

Finally, you can ask students the following additional exercise problems to check if they have gained the insight and for consolidating the entire sub-unit.

Let students find $f + g,\ f - g,\ f\ .\ g$ and $\frac{f}{g}$ where $f(x) = x - \frac{2}{3}x^5 + \sqrt{2}x^2 + \frac{x+\sqrt{2}}{4}$ and $g(x) = x^5 + \frac{8x - x^4 + 6}{4}$

For gifted students, you can give extra question to divide like:

a. $x^5 \div x - 1$ b. $x^7 \div x + 1$

Answers to Exercise 1.3

1.
   a. $x^2 - 2x - 8$
   b. $x^3 + x^2 - 8x - 12$
   c. $2x - x^2 + 8$
   d. Not a polynomial
   e. Not a polynomial
   f. $x^4 - 2x^3 - 11x^2 + 12x + 36$
   g. Not a polynomial
   h. $4x^2 + 12x + 9$
   i. $x^4 - 4x^3 + 9x^2 - 8x + 5$
   j. $-2x^4 + 2x^2 + 2x - 7$
2. In general, the set of polynomial functions is closed under addition, subtraction and multiplication. Therefore, all are polynomial functions except (d) and (g).
3. All the expressions represent polynomial functions except (d) and (g).
4.
   a. $f + g: f(x) + g(x) = (3x - \frac{2}{3}) + (2x + 5) = 5x + \frac{13}{3}$

      $f - g: f(x) - g(x) = (3x - \frac{2}{3}) - (2x + 5) = x - \frac{17}{3}$

Function	f	g	$f + g$	$f - g$
Degree	1	1	1	1

   b. $f + g: f(x) + g(x) = (-7x^2 + x - 8) + (2x^2 - x + 1)$
      $= -7x^2 + x - 8 + 2x^2 - x + 1 = -5x^2 - 7$

      $f - g: f(x) - g(x) = (-7x^2 + x - 8) - (2x^2 - x + 1)$
      $= -7x^2 + x - 8 - 2x^2 + x - 1 = -9x^2 + 2x - 9$

Function	f	g	$f + g$	$f - g$
Degree	2	2	2	2

c. $f + g$: $f(x) + g(x) = (1 - x^3 + 6x^2 - 8x) + (x^3 + 10)$
$= 1 - x^3 + 6x^2 - 8x + x^3 + 10 = 6x^2 - 8x + 11$
$f - g$: $f(x) - g(x) = (1 - x^3 + 6x^2 - 8x) - (x^3 + 10)$
$= 1 - x^3 + 6x^2 - 8x - x^3 - 10 = 6x^2 - 2x^3 - 8x - 9$

Function	f	g	$f + g$	$f - g$
Degree	3	3	2	3

5. a. $f \,.\, g = f(x) \,.\, g(x) = (2x + 1)(3x - 5)$
$= 2x(3x - 5) + 1\,(3x - 5)$
$= 6x^2 - 10x + 3x - 5 = 6x^2 - 7x - 5$

Function	f	g	$f\,.\,g$
Degree	1	1	2

b. $f \,.\, g$: $f(x) \,.\, g(x) = (x^2 - 3x + 5)\,(5x + 3)$
$= x^2(5x + 3) - 3x(5x + 3) + 5(5x + 3)$
$= 5x^3 + 3x^2 - 15x^2 - 9x + 25x + 15$
$= 5x^3 - 12x^2 + 16x + 15$

Function	f	g	$f\,.\,g$
Degree	2	1	3

c. $f \,.\, g$: $f(x) \,.\, g(x) = (2x^3 - x - 7)\,(x^2 + 2x)$
$= 2x^3(x^2 + 2x) - x(x^2 + 2x) - 7(x^2 + 2x)$
$= 2x^5 + 4x^4 - x^3 - 2x^2 - 7x^2 - 14x$
$= 2x^5 + 4x^4 - x^3 - 9x^2 - 14x$

Function	f	g	$f\,.\,g$
Degree	3	2	5

d. $f \cdot g$: $f(x) \,.\, g(x) = 0(x^3 - 8x^2 + 9) = 0$
$f \,.\, g$ is a polynomial function of no degree.

6.

a.
$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x - 1 & x^3 - 1 \\ & \underline{x^3 - x^2} \\ & \quad x^2 - 1 \\ & \quad \underline{x^2 - x} \\ & \qquad x - 1 \\ & \qquad \underline{x - 1} \\ & \qquad\quad 0 \end{array}$$

b.
$$\begin{array}{r|l} & x + 1 \\ \hline x^2 - x + 1 & x^3 + 1 \\ & \underline{x^3 - x^2 + x} \\ & \quad x^2 - x + 1 \\ & \quad \underline{x^2 - x + 1} \\ & \qquad 0 \end{array}$$

c.
$$\begin{array}{r|l} & x^2 - 1 \\ \hline x^2 + 1 & x^4 - 1 \\ & \underline{x^4 + x^2} \\ & \quad -x^2 - 1 \\ & \quad \underline{-x^2 - 1} \\ & \qquad 0 \end{array}$$

d.

$$
\begin{array}{r|l}
 & x^4 - x^3 + x^2 - x + 1 \\
\hline
x + 1 & x^5 + 1 \\
 & \underline{x^5 + x^4} \\
 & \quad -x^4 + 1 \\
 & \quad \underline{-x^4 - x^3} \\
 & \qquad x^3 + 1 \\
 & \qquad \underline{x^3 + x^2} \\
 & \qquad\quad -x^2 + 1 \\
 & \qquad\quad \underline{-x^2 - x} \\
 & \qquad\qquad x + 1 \\
 & \qquad\qquad \underline{x + 1} \\
 & \qquad\qquad 0
\end{array}
$$

e.

$$
\begin{array}{r|l}
 & -x^3 + 2x^2 - x + 2 \\
\hline
x^3 - x - 2 & -x^6 + 2x^5 + 2x^3 + 6 \\
 & \underline{-x^6 + x^4 + 2x^3} \\
 & \quad 2x^5 - x^4 + 6 \\
 & \quad \underline{2x^5 - 2x^3 - 4x^2} \\
 & \quad -x^4 + 2x^3 + 4x^2 + 6 \\
 & \quad \underline{-x^4 + x^2 + 2x} \\
 & \qquad 2x^3 + 3x^2 - 2x + 6 \\
 & \qquad \underline{2x^3 - 2x - 4} \\
 & \qquad\qquad 3x^2 + 10
\end{array}
$$

7.

	Quotient	Remainder
a	$8x + 2$	7
b	$x^2 + x + 1$	0
c	$2y - 1$	y
d	$3x^3 - 7x^2 + 21x - 67$	200
e	$3x^2 - 3x + 3$	0

1.2 THEOREMS ON POLYNOMIALS

Periods allotted: 6 periods

Competencies

At the end of this sub-unit, students will be able to:

- *state and prove the Polynomial Division Theorem.*
- *apply the Polynomial Division Theorem.*
- *state and prove the Remainder Theorem.*
- *apply the Remainder Theorem.*
- *state and prove the Factor Theorem.*
- *apply the Factor Theorem.*

Vocabulary: Dividend, divisor, quotient, remainder, factor, remainder theorem, factor theorem.

Introduction

In elementary arithmetic, students learned to divide a positive integer by another to obtain a quotient and remainder.

This sub-unit provides a similar procedure called long division, for dividing one polynomial by another. Two important theorems, that is, the remainder theorem and factor theorem, which pertain to long division of polynomials, are well treated.

Teaching Notes

You can introduce the lesson by revising division algorithm of positive integers and discuss the terms such as divisor, dividend, quotient and the remainder. Let students divide 89 by 12. Clearly, they will get the answer 7 and remainder 5. They could write this as

$$89 \div 12 = 7 + \frac{5}{12}$$

Here, 89 is the dividend, 12 is the divisor, 7 is the quotient and 5 is the remainder. Let students think of another way of writing this division as

$$89 = 7 \times 12 + 5$$

And this enables students to generalize that

Dividend = (Divisor) × (Quotient) + Remainder.

When you divide one whole number by another, you continue the procedure until you obtain a remainder zero or number less than the divisor. Likewise, when students divide a polynomial by another, they should continue the long division procedure until the remainder is either the zero polynomial or a polynomial of lower degree than the divisor.

After pointing out these, let the students do Activity 1.6 in student textbook individually. This activity allows students to consider what points may be most relevant when one polynomial is divided by another.

Answers to Activity 1.6

1.

	$q(x)$	$r(x)$
a	$x + 4$	5
b	$x^2 - 3x + 6$	-4
c	$x^3 + 1$	0

2. The degree of $f(x)$ is greater than the degree of $d(x)$. In general, if the degree of $f(x)$ is m and degree of $d(x)$ is n, then $m > n$. But in general, it can be true that $m = n$.

3. Note that a fractional expression is said to be improper if the degree of numerator is greater than or equal to the degree of denominator. Likewise, the fractional expression $\frac{f(x)}{d(x)}$ is improper if the degree of $f(x)$ is greater than or equal to the degree of $d(x)$.

4. The fractional expression $\frac{r\ (x)}{d\ (x)}$ is proper because the degree of r (x) is less than the degree of d (x).

You may encourage students to state polynomial division theorem. With close participation of students, you need to prove the theorem.

After discussing the proof of polynomial division theorem, it is possible to encourage students to engage them with doing exercises similar to example 1 in the student textbook, in group and you can pick one from the group to solve and demonstrate on the board.

You can hint them to use example 1 as a guide for doing questions 1 and 2 of exercise 1.4. After conducting this, you may round to identify those who need support. And, for those who are fast enough, you can give them question 3 of exercise 1.4. For all students, you need to give the correct answer of each of the questions of exercise 1.4 on the board by giving each student a chance to participate.

Assessment

After completing this section, you can use anyone of the following techniques for assessing student learning. Class activities, group discussion, assignment and quiz or test. To help students refocus, you can give the following questions.

In each of the following, divide and find the quotient and the remainder.

a. $(4a^3 - 1) \div (3a - 1)$ b. $(11x - 2 + 12x^3) \div (3x + 2)$

c. $(3x - x^2 + 2x^3 - 1) \div (x + 2)$ d. $(3 + 4^3 - y) \div (y - 3)$

For fast learners, you can give the following questions to perform divisions assuming that n is a positive integer.

a. $(ax^2 + bx + c) \div (x - r)$ b. $\dfrac{x^{3n} + 3x^{2n} + 18}{x^n + 9}$

c. $\dfrac{ax^3 + bx^2 + cx + D}{ax + b}$ d. $(3x^5 + 2x^4 + 5x^3 - 7x - 3) \div (x + 0.8)$

Answers to Exercise 1.4

1.

	$q\ (x)$	$r\ (x)$
a	$x - 2$	9
b	$x + 1$	$4 - 5x$
c	$\frac{1}{2}x^2 + 4x - 6$	0

2. a. $f(x) = (x + 2)(x^2 - 7x + 13) - 18$

b. $f(x) = (x - \frac{1}{2})(x^2 + \frac{5}{2}x - \frac{3}{4}) - \frac{115}{8}$

3. a.

$$
\begin{array}{r|l}
 & x^{2n} + 2x^{n} + 6 \\
\hline
x^{n} + 3 & x^{3n} + 5x^{2n} + 12x^{n} + 18 \\
 & \underline{x^{3n} + 3x^{2n}} \\
 & \qquad 2x^{2n} + 12x^{n} + 18 \\
 & \qquad \underline{2x^{2n} + 6x^{n}} \\
 & \qquad\qquad 6x^{n} + 18 \\
 & \qquad\qquad \underline{6x^{n} + 18} \\
 & \qquad\qquad\quad 0
\end{array}
$$

b.

$$
\begin{array}{r|l}
 & x^{2n} + x^{n} + 5 \\
\hline
x^{n} - 2 & x^{3n} - x^{2n} + 3x^{n} - 10 \\
 & \underline{x^{3n} - 2x^{2n}} \\
 & \qquad x^{2n} + 3x^{n} - 10 \\
 & \qquad \underline{x^{2n} - 2x^{n}} \\
 & \qquad\qquad 5x^{n} - 10 \\
 & \qquad\qquad \underline{5x^{n} - 10} \\
 & \qquad\qquad\quad 0
\end{array}
$$

You can start the discussion on remainder theorem by introducing division of polynomial function $f(x)$ by $(x - r)$ to obtain the result of the form

$f(x) = (x - r)\, q(x) + R$

where $q(x)$ is the quotient and R is the remainder. For instance, if $f(x) = 2x^3 - 3x^7 + 8x + 2$ is divided by $(x - 2)$, you obtain the result

$2x^3 - 3x^2 + 8x + 2 = (x-2)\left(2x^2 + x + 10\right) + 27$.

At this stage, students may do Activity 1.7 as class work or you may group students so that they can solve. By rounding, you need to facilitate their work. The purpose of this activity is to help the students to revise the remainder theorem.

Let students consider $f(x) = (x - r)\, q(x) + R$.

Encourage students to let $x = r$ so that the expression becomes

$$f(r) = (r - r)\, q(r) + R = R$$

This leads them to the remainder theorem as stated in the student textbook.

Answers to Activity 1.7

1. a. $f(-2) = 20; f(2) = 0$ b. 20 c. Yes
   d. $f(2) = 0$ e. Yes

2. a.

$$
\begin{array}{r|l}
 & 2x + 1 \\
\hline
x + 1 & 2x^2 + 3x + 1 \\
 & \underline{2x^2 + 2x} \\
 & \quad x + 1 \\
 & \quad \underline{x + 1} \\
\end{array}
$$

Remainder $\rightarrow 0$

$f(-1) = 2(-1)^2 + 3(-1) + 1$

$= 2 - 3 + 1 = 0$

So, the remainder is equal to $f(-1)$.

b.

$$
\begin{array}{r|l}
 & x^5 - x^4 + x^3 - x^2 + x - 1 \\
\hline
x + 1 & x^6 + 1 \\
 & \underline{x^6 + x^5} \\
 & -x^5 + 1 \\
 & \underline{-x^5 - x^4} \\
 & x^4 + 1 \\
 & \underline{x^4 + x^3} \\
 & -x^3 + 1 \\
 & \underline{-x^3 - x^2} \\
 & x^2 + 1 \\
 & \underline{x^2 + x} \\
 & -x + 1 \\
 & -\underline{x - 1} \\
\end{array}
\qquad
\begin{array}{r|l}
 & x^3 + x^2 + x + 1 \\
\hline
x - 1 & x^6 + 1 \\
 & \underline{x^6 - x^5} \\
 & x^5 + 1 \\
 & \underline{x^5 - x^4} \\
 & x^4 + 1 \\
 & \underline{x^4 - x^3} \\
 & x^3 + 1 \\
 & \underline{x^3 - x^2} \\
 & x^2 + 1 \\
 & \underline{x^2 - x} \\
 & x + 1 \\
 & \underline{x - 1} \\
\end{array}
$$

Remainder $\rightarrow 2$ Remainder $\rightarrow 2$

$f(-1) = (-1)^4 + 1 = 1 + 1 = 2$ $f(1) = (1)^4 + 1 = 1 + 1 = 2$

So, the remainder is equal to $f(-1)$. So, the remainder is equal to $f(1)$.

c.

$$
\begin{array}{r|l}
 & -x^3 + x^2 + 2x + 4 \\
\hline
x - 2 & -x^4 + 3x^3 + 2 \\
 & -\underline{x^4 + 2x^3} \\
 & x^3 + 2 \\
 & \underline{x^3 - 2x^2} \\
 & 2x^2 + 2 \\
 & \underline{2x^2 - 4x} \\
 & 4x + 2 \\
 & \underline{4x - 8} \\
 & 10 \leftarrow \text{remainder} \\
\end{array}
$$

$f(2) = -(2)^4 + 3(2)^3 + 2$

$= -16 + 24 + 2 = 10$

So, the remainder is equal to $f(2)$.

d.

$$\begin{array}{r|l} & x^2 - x \\ \hline x+1 & x^3 - x + 1 \\ & \underline{x^3 + x^2} \\ & \quad -x^2 - x + 1 \\ & \quad \underline{-x^2 - x \qquad} \\ & \qquad\qquad 1 \leftarrow \text{remainder} \end{array}$$

$f(-1) = (-1)^3 - (-1) + 1$

$= -1 + 1 + 1 = 1$

So, the remainder is equal to $f(-1)$.

$$\begin{array}{r|l} & x^2 + x \\ \hline x-1 & x^3 - x + 1 \\ & \underline{x^3 - x^2 \qquad} \\ & \quad x^2 - x + 1 \\ & \quad \underline{x^2 - x \qquad} \\ & \qquad\qquad 1 \leftarrow \text{remainder} \end{array}$$

$f(1) = (1)^3 - 1 + 1$

$= 1 - 1 + 1 = 1$

So, the remainder is equal to $f(1)$.

Next, select voluntary students and encourage them to do some more questions such as example 2 and 3 on the board. Using these examples with student participation, you need to assist them to tell the relationship between polynomial division theorem and the remainder theorem. Ask and assist students to solve questions 1 and 2 of Exercise 1.5 on the board by applying remainder theorem. After you find out what students know and what they are able to do, give questions 3 and 4 of Exercise 1.5 as homework and check their answer to evaluate their learning.

Assessment

By assessing continuously in different ways, you can be confident about what students know and what they can do. For consolidating, you can give the following questions.

1. Find the remainder k when the polynomial $f(x)$ is divided by $x - c$ for the given number c.

   a. $f(x) = 4x^4 + 2x^3 - 6x^2 - 5x + 1;\ c - \frac{1}{2}$

   b. $f(x) = 3x^3 - x^2 + x + 2;\ x = -\frac{2}{3}$

   c. $f(x) = 4x^4 - 3x^3 + 5x^2 + 7x - 6;\ x = 0.7$

2. When the polynomial $p(x) = ax^4 - bx^3 + cx - 8$ is divided by $x - 1$ and $x + 1$, the remainders are 2 and 3 respectively. If $p\left(\frac{1}{2}\right) = -1$, then find the values a, b and c.

3. When $f(x) = \frac{2}{3}x^4 - ax^3 + \frac{4}{3}x + 1$ is divided by $x - 1$, the remainder is 5. Find the value of a.

Answers to Exercise 1.5

1. a. $f(x) = (x - 2)(x^2 + x + 9) + 29$ with remainder 29
   $f(2) = (2)^3 - 2^2 + 7(2) + 11 = 8 - 4 + 14 + 11 = 29$
   Therefore, $f(2) = 29$ = Remainder.

   b. $f(x) = (x + 1)(x^3 - x^4 + x^2 - x + 2) + (-1)$ with remainder -1.
   $f(-1) = 1 - (-1)^5 + 2(-1)^3 + (-1) = 1 + 1 - 2 - 1 = -1$
   Therefore, $f(-1) = -1$ = Remainder.

   c. $f(x) = (x + \frac{2}{3})(x^3 + \frac{4}{3}x^2 + \frac{37}{9}x - \frac{74}{27}) + \frac{229}{81}$
   $f(-\frac{2}{3}) = (-\frac{2}{3})^4 + 2(-\frac{2}{3})^3 + 5(-\frac{2}{3})^2 = \frac{229}{81}$
   Therefore, $f(\frac{-2}{3}) = \frac{229}{81}$ = Remainder.

2. If k is the remainder, then

   a. $k = f(1) = 0$ b. $k = f(-\frac{1}{2}) = 0$ c. $k = f(-1) = 0$

3. $a = -11$
4. $a = 1$ and $b = -4$

You may start the lesson by revising factorization of polynomials. For instance, ask students to factorize polynomials such as $x^2 + 2x - 3$ and $2x^2 + 7x^2 + 7x + 2$. Clearly, $x - 5$ is a factor of $x^2 + 2x - 3$ because $x^2 + 2x - 3 = (x-1)(x+3)$; and $x+1, x+2$ and $2x+1$ are factors of $2x^3 + 7x^2 + 7x + 2$ because $2x^3 + 7x^2 + 7x + 2 = (x+1)(x+2)(2x+1)$.

Assist students to find the remainder when $x^2 + 2x - 3$ is divided by $x - 1$ and when $2x^3 + 7x^2 + 7x + 2$ is divided by each factor. From the result you obtained above, guide students to do Activity 1.8 in the class and check their answer to see their performance or to see how well they have achieved.

Answers to Activity 1.8

1. a. $f(2) = 0$, Hence, $f(x) = (x - 2)\, q(x)$ b. The remainder is 0

   c. $x - 2$ is a factor of $f(x)$ d. $f(-1) = 0$ and $f(1) = 6$

   e. $f(x) = (x - 2)(x^2 - 3x - 4)$
   $f(x) = (x + 1)(x^2 - 6x + 8)$ and
   $f(x) = (x - 4)(x^2 - x - 2)$

2. a. $f(-1) = 0$, $f(1) = 0$ and $f(3) = 0$

   b. The remainder is 0 when $f(x)$ is divided by $x + 1$, $x - 1$ and $x - 3$.

   c. Since the remainder is 0 when $f(x)$ divided by $x + 1$, $x - 1$ and $x - 3$. Hence $x + 1$, $x - 1$ and $x - 3$ are factors of $f(x)$ and $f(x) = (x + 1)(x - 1)(x - 3)$.

At this stage, give some more examples of polynomial function $f(x)$ that can be written as

$$f(x) = (x - r)\, q(x) \text{ where } q(x) \text{ is the quotient.}$$

Ask students to find the value $f(r)$. Clearly, the value of $f(x)$ at $x = r$ is

$f(r) = (r - r)\, q(r) = 0$. So, the remainder is 0 and $x - r$ is a factor of $f(x)$.

After stating factor theorem, you are advised to group students so that they can practice to solve Group work 1.2. With active participation of students, pick one student from each group to present the solution on the board and keep a record about their participation. You need to give the correct answer of this group work.

Answers to Group work 1.2

1. a. $f(-1) = 4(-1)^4 - 5(-1)^2 + 1 = 0$

$\Rightarrow f(-1) = 0$

$\Rightarrow x + 1$ is a factor of $f(x)$

b. $f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^4 - 5\left(\frac{1}{2}\right)^2 + 1 = \frac{4}{16} - \frac{5}{4} + 1 = \frac{-16}{16} + 1 = 0$

$\Rightarrow f\left(\frac{1}{2}\right) = 0$

$\Rightarrow 2x - 1$ is a factor of $f(x)$.

c. $f(x) = (4x^2 - 1)(x^2 - 1) = (2x - 1)(2x + 1)(x - 1)(x - 1)$

2. Proof ($\Rightarrow$) suppose that $f(c) = 0$, $c =$ constant. Now, since $x - c = d(x)$ is a polynomial function and $f(x)$ is also another polynomial function, we have that

$f(x) = (x - c)\, q(x) + r(x)$, where

$r(x) = 0$ or degree $r(x) <$ degree $(x - c) = 1$ (by the polynomial division theorem)

But $f(c) = 0 \Rightarrow r(c) = 0$.

Now, since degree $r(x) < 1$, then $r(x) = k =$ constant but $r(c) = 0 = k \Rightarrow k = 0$

$\Rightarrow r(x) = 0$, for all real number $x \in \mathbb{R}$

$\Rightarrow f(x) = (x - c)\, q(x)$

$\Rightarrow x - c$ is a factor of $f(x)$.

($\Leftarrow$) suppose that $x - c$ is a factor of $f(x)$, then $f(x) = (x - c)\, q(x)$, where $q(x)$ is the quotient

$\Rightarrow f(c) = (c - c)\, q(c) = 0.\ q(c) = 0 \Rightarrow f(c) = 0$

Therefore, the hypothesis of the factor theorem holds true.

Before they do, you can suggest to them to use Examples 4 and 5 in the student textbook as a guide for doing questions 1 and 2 of Exercise 1.6 in the class. While students are doing these questions, you may round to identify those who need further assistance and those who are fast performers to solve each question and keep record.

For all other students, you can solve these questions on the board by giving each student the chance to participate. You may then give the rest of the questions of Exercise 1.6 as homework.

After deliberating on this lesson, it is advisable to give additional exercises apart from the ones given in the student textbook that need to be solved by students themselves.

You can use the following questions as additional exercise problems, particularly, for students who are fast learners.

1. Use the factor theorem to show that
   a. $x - 1$ is a factor of $P(x) = x^{2n+1} - 1$; $n \in \mathbb{N}$
   b. $x - 1$ is a factor of $f(x) = 3x^4 - 2x^3 + 5x - 6$
   c. $x - 1$ is a factor of $f(x) = 6x^4 - \frac{13}{4}x^3 - \frac{5}{2}x^5 - \frac{1}{4}$
2. Find the linear factors of:
   a. $p(x) = 4x^4 - 4x^3 - 9x^2 + x + 2$
   b. $f(x) = x^3 - 8x^2 + 17x - 4$
3. Find the polynomial function of degree 4 such that $f(-1) = 2$, and x, $x - 1$, $x + 1$ and $x + 2$ are factors of the polynomial.

If $x + 1$ and $x + 2$ are factors of the polynomial $f(x) = ax^3 - 2x^2 + bx + 1$. Find the values of a and b; and the remaining factor.

Assessment

After completing this lesson, you can use any one of the following for assessing students learning: class activities, group discussion, assignments and quiz or test. Assessment is used to refocus the learning process and to help students make their learning more efficient and more effective.

Answers to Exercise 1.6

1. a. Since $f(-1) = 0$, $g(x) = x + 1$ is a factor of $f(x)$
   b. Since $f(1) = 1 \neq 0$, $g(x) = x - 1$is not a factor of $f(x)$
   c. Since $f\left(\frac{3}{2}\right) = 14 \neq 0$, $g(x) = x - \frac{3}{2}$ is not a factor of $f(x)$.
   d. Since $f(-2) = -24 \neq 0$, $g(x) = x + 2$ is not a factor of $f(x)$.
2. a. $k = 11$ b. $k = -\frac{73}{81}$ c. $k = 0$
3. Let $f(x) = x^4 - 2ax^3 + ax^2 - x + k$

   Now, $x - 2$ is a factor of $f(x)$ means that $f(2) = 0$.

   $\Rightarrow f(2) = 0$

   $\Rightarrow (2)^4 - 2a(2)^3 + a(2)^2 - 2 + k = 0$

   $\Rightarrow 16 - 16a + 4a - 2 + k = 0$

$\Rightarrow -12a + k = -14$ ……………. (1)

and $f(-1) = 3$

$\Rightarrow (-1)^4 - 2a(-1)^3 + (-1)^2 - (-1) + k = 3$

$\Rightarrow 1 + 2a + a + 1 + k = 3$

$\Rightarrow 3a + k = 1$ ……………(2)

From equation(1) and equation(2) solving simultaneously, we get:

$$\begin{cases} -12a + k = -14 \\ \quad 3a + k = \;\;1 \end{cases}$$
$$\overline{-15a \quad = -15}$$
$$a = 1$$

Substituting $a = 1$ in equation (2), we get:

$$3(1) + k = 1 \Rightarrow k = -2$$

Therefore, $a = 1$ and $k = -2$

4. Let $f(x) = ax(x - 1)(x + 2)$

$f(2) = 24$ . . . (given).

$\Rightarrow a(2)(2 - 1)(2 + 2) = 24$

$\Rightarrow 8a = 24 \Rightarrow a = 3$

Therefore, $f(x) = 3x(x - 1)(x + 2) = 3x^3 + 3x^2 - 6x$.

5. Let $f(x) = x^n - a^n$

$x - a$ is a factor of $f(x)$ means that $f(a) = 0$

$f(a) = (a)^n - a^n = a^n - a^n = 0$.

Therefore, $x - a$ is a factor of $x^n - a^n$.

6. Let $f(x) = 2x^3 - x^2 - 2x + 1$

$f(1) = 2(1)^3 - (1)^2 - 2(1) + 1 = 2 - 1 - 2 + 1 = 0$. So $x - 1$ is a factor of $f(x)$

$f(0) = 2(0)^3 - (0)^2 - 2(0) + 1 = 1 \neq 0$. So, x is not a factor of $f(x)$.

$f(-1) = 2(-1)^3 - (-1)^2 - 2(-1) + 1 = -2 - 1 + 2 + 1 = 0$.

So, $x + 1$ is a factor of $f(x)$.

7. a. Let $f(x) = x^3 + 3x^2 - 3x + c$

$f(3) = 0$

$(3)^3 + 3(3)^2 - 3(3) + c = 0$

$27 + 27 - 9 + c = 0$

$45 + c = 0 \Rightarrow c = -45$

b. Let $f(x) = x^3 - 2x^2 + x + c$

$f(-2) = 0$

$(-2)^3 - 2(-2)^2 + (-2) + c = 0$

$-8 - 8 - 2 + c = 0$

$-18 + c = 0 \Rightarrow c = 18$.

8. $A(x) = x^2 + 13x + 36 = (x + 4)(x + 9)$, where $x > -4$ hence $(x + 9)$ is the length and $(x + 4)$ is the width.

Therefore, the length is 5 feet longer than the width.

1.3 ZEROS OF A POLYNOMIAL FUNCTION

Periods allotted: 4 periods

Competencies

At the end of this sub-unit, students will be able to:

- *determine the zero(s) of a given polynomial function.*
- *state the Location Theorem.*
- *apply the location theorem to approximate the zero(s) of a given polynomial function.*
- *apply the rational root test to determine the zero(s) of a given polynomial function.*

Vocabulary: Zero(s) of polynomial, multiplicity, location theorem, rational root test.

Introduction

This sub-unit gives much emphasis on finding zero(s) of polynomial function. Things like location theorem and rational root test are treated in this sub-unit. Basically, when you are finding zero(s) of a function, you are looking for input value that causes your functional value to be equal to zero.

The following steps are recommended when you are using rational root test.

Step 1. List all the factors of the constant term.

Step2. List all the factors of the leading coefficient.

Step 3. List all the possible rational zeros.

Teaching Notes

You may start the lesson by asking students to solve linear equations and quadratic equations they studied in grade 9.

In introducing the notion zero of function, students are expected to know the difference between the number 0 and zero(s) of a function.

They should note that zero(s) of a function f is any real number c such that $f(c) = 0$.
For example, –3, –2 and 1 are zeros of $f(x) = x^3 + 4x^2 + x - 6$, because

$$f(-3) = (-3)^3 + 4(-3)^2 + (-3) - 6 = -27 + 36 - 3 - 6 = 0$$

$$f(-2) = (-2)^3 + 4(-2)^2 + (-2) - 6 = -8 + 16 - 2 - 6 = 0$$

$$f(1) = (1)^3 + 4(1)^2 + (1) - 6 = 1 + 4 + 1 - 6 = 0$$

You can give some more examples so that students should be able to generalize that a polynomial function of degree n has at most n zeros.

Encourage students to solve question 1 of Activity 1.9 and assist each student to do questions 2, 3 and 4 of Activity 1.9 orally and discuss the correct answer with the active participation of the students. The purpose of this activity is to help students revise zero(s) of linear function and quadratic function.

Answers to Activity 1.9

1. a $x = -\frac{1}{3}$ b. $x = \frac{3}{2}$ c. $x = -\frac{5}{2}$ and $x = \frac{5}{2}$

   d. $x = -4$ and $x = 3$. e. $x = 0$ and $x = 1$ f. $x = -1$ and $x = 1$
2. Every quadratic function has at most two zeros.
3. Factorization method, completing the square method and the quadratic formula.
4. A polynomial function of degree three has at most three zeros.

   A polynomial function of degree four has at most four zeros.

Give students an example of finding the zeros and their multiplicities such as

a. $(1-x)-\frac{2}{3}(x+1)=0$ b. $(x-1)(x+2)^3(x+1)^2=0$

In the above two equations, in part (a), $\frac{1}{5}$ is the value of x that makes the equation true.

In other words, the number $\frac{1}{5}$ is a root of a polynomial function

$f(x)=(1-x)-\frac{2}{3}(x+1)$. And, in part (b) you can say that -2 is a zero of multiplicity 3 of the equation and –1 is a zero of multiplicity 2 of the equation.

By giving special emphasis to the participation of students, give them some questions similar to Examples 3 and 4 in the student textbook to demonstrate on the board. You may then give them questions 1 and 2 of Exercise 1.7 as class work to solve on their own. Check and assess whether each student is performing in accordance with the steps enlisted in the student textbook. Before they do the questions, you can hint and give examples that would serve as a guide for doing questions 3, 4 and 5 of Exercise 1.7. At this stage, you may round to identify those who need further assistance and those who are fast enough to solve, keep record and decide what to do next. For those who are fast enough, you may give questions 6 and 7 of Exercise 1.7 as additional problems and, for all students you can solve each question of Exercise 1.7 on the board by giving each student the chance to participate.

Assessment

By giving additional class activities, class discussion, assignment and quiz or test, you can be confident about what students know and what they can do. After identifying fast learner students, you might give them the following additional exercise problems.

1. If –2 is a double zero of $p(x)=x^4-7x^2+4x+20$, write $p(x)$ as a product of linear factors.
2. In each of the following polynomial function, find the zeros and indicate the multiplicity of each if over 1. What is the degree of each polynomial?

   a. $p(x)=(x+8)(x-(1+\pi))^2$

   b. $f(x)=3\left(x+\frac{2}{3}\right)^3(x+\pi)^2\left(x-\frac{1}{2}\right)$

3. In each of the following, find a polynomial $p(x)$ of lowest degree, with leading coefficient 1, that has the indicated zeros.

   3 (multiplicity 2) and –2

   $\frac{1}{3}$ (multiplicity 3), and 1 (multiplicity 2)

   –7 (multiplicity 2), $-3+\sqrt{2}, -3-\sqrt{2}$

   $-\frac{1}{2}$ (multiplicity 3), $1+\sqrt{2}, 1-\sqrt{2}$

Answers to Exercise 1.7

1. a. $x = \frac{5}{3}$ b. $x = \frac{-11}{6}$

   c. $x = -1, x = 2$ and $x = \frac{2}{3}$ d. { }

   e. $x = 1$ f. $t = -3$, $t = 1$ and $t = 2$

   g. $y = -1$, $y = 0$ and $y = 1$ h. $-x = -\frac{\sqrt{6}}{2}$ and $x = \frac{\sqrt{6}}{2}$

2. a. 0; multiplicity 12 and $\frac{2}{3}$; multiplicity 1

   b. $\sqrt{2}$; multiplicity 2
   –1; multiplicity 1

   c. 0; multiplicity 6
   π; multiplicity 5
   (π + 1); multiplicity 3

   d. $\sqrt{3}$; multiplicity 5
   –5; multiplicity 9
   $\frac{1}{3}$; multiplicity 1

   e. 1; multiplicity 3.

3. a. Let $f(x) = ax(x-5)(x-8)$. Then,
   $f(10) = 17$
   $\Rightarrow 10a(10-5)(10-8) = 17$
   $\Rightarrow 100a = 17 \Rightarrow a = \frac{17}{100}$.

   Therefore, $f(x) = \frac{17}{100}x(x-5)(x-8) = \frac{17}{100}x^3 - \frac{17}{20}x^2 - \frac{34}{25}x^2 + \frac{34}{5}x$

   $$= \frac{17}{100}x^3 - \frac{221}{100}x^2 + \frac{34}{5}x.$$

4. a. $f(x) = x^3 + x^2 - 5x + 3 = (x-1)^2(x+3)$.
   Therefore, 1 is the zero of multiplicity 2.

   b. $f(x) = x^4 - 3x^2 + 3x^2 + x = x(x+1)^3$
   –1 is the zero of multiplicity 3.

   c. $f(x) = 4x^3 - 4x^2 + x = x(2x-1)^2$
   Therefore, $\frac{1}{2}$ is the zero of multiplicity 2.

5. Let $f(x) = (3x + 4)\, q(x)$

$$f\left(-\frac{4}{3}\right) = \left(3\left(-\frac{4}{3}\right)+4\right) q(x) = (-4 + 4)\, q(x) = 0$$

Therefore, $-\frac{4}{3}$ is a zero of $f(x)$.

6. a. Let $f(x) = ax(x - 3)(x - 4)$

$f(1) = 5$

$a(1 - 3)(1 - 4) = 5$

$6a = 5 \Rightarrow a = \frac{5}{6}$.

Therefore, $f(x) = \frac{5}{6}x(x - 3)(x - 4)$

b. Let $f(x) = a(x + 1)\left(x-\left(1+\sqrt{2}\right)\right)\left(x-\left(1-\sqrt{2}\right)\right)$

$f(0) = 3$

$a(0 + 1)\left(0-\left(1+\sqrt{2}\right)\right)\left(0-\left(1-\sqrt{2}\right)\right) = 3$

$a\left(-1-\sqrt{2}\right)\left(\sqrt{2}-1\right) = 3$

$-a = 3 \Rightarrow a = -3$

Therefore, $f(x) = -3(x + 1)\left(x-\left(1+\sqrt{2}\right)\right)\left(x-\left(1-\sqrt{2}\right)\right)$.

7. For $a \in \mathbb{R}$ and $a > 0$, a polynomial function

$f(x) = -a(x + 2)\left(x-\frac{1}{2}\right)(x-3)$.

Since there are many values for a, many different polynomial functions.

8. Let $p(x) = ax(x - 1)(x + 1)$

$p(2) = 6$

$2a(2 - 1)(2 + 1) = 6$

$2a(1)(3) = 6$

$6a = 6 \Rightarrow a = 1$

Therefore, $p(x) = x(x - 1)(x + 1)$

a. $p(-x) = -x(-x - 1)(-x + 1)$

$= -x(-1)(x + 1)(-1)(x - 1)$

$= (-1)(-1)(-x)(x + 1)(x - 1)$

$= -x(x + 1)(x - 1) = -p(x)$

b. In the intervals $(-\infty, -1)$ and $(0, 1)$ $P(x)$ is less than zero.

9. $p = \frac{11}{2}$ and $q = \frac{1}{2}$

10. The missile is 72 m above the ground at $\frac{\sqrt{13}-2}{2}$ and at 2 seconds after launch.

After ensuring the ability of students in solving Exercise 1.7, select voluntary students and encourage them to find real zero(s) of the polynomial function $f(x) = x^4 - 3x^2 + 2$ using a table values on the blackboard.

After doing this example, give students Activity 1.10 to do in group. You should monitor and assist each group to do this activity using any method of their own choice. Finally, at least one group should demonstrate how it dealt with each question in this activity on the board. You need to give the correct answer at the end of the discussion with active participation of the students.

Answers to Activity 1.10

1. a. Has no real roots.
   b. $-1, -\sqrt{2}$ and $\sqrt{2}$ are roots of $f(x)$. So, its roots are rational and irrational.
   c. $-\frac{3}{2}$, –1 and 1 are roots of $f(x)$. So, its roots are rational.
   d. $-\sqrt{3}, -\sqrt{2}, \sqrt{2}$ and $\sqrt{3}$ are roots of $f(x)$. So, its roots are all irrational.
2. a.

x	–4	–3	–2	–1	0	1	2	3	4
$y = f(x)$	–182	–77	–24	–5	–2	3	28	91	210

   b.

x	–4	–3	–2	–1	0	1	2	3	4
$y = f(x)$	602	210	38	–10	–6	2	–10	–42	–70

You may then give Exercise 1.8 as homework so that each student will do it on his/her own. The exercise is prepared for the students to apply location theorem in finding the zeros of a polynomial function.

Before they do that, you can encourage and assist them to use examples 5 and 6 in the student textbook as a guide for doing Exercise 1.8. Check their work to determine their level of understanding.

Assessment

In order to ensure that all students can fully participate in their learning, you can give class activities, conduct class discussion, assignment and quiz or test. You might also give additional exercise problems.

Answers to Exercise 1.8

1. As shown in the table in the student textbook.
   a. f has zeros between –1 and 0; 0 and 2; and 2 and 5.
   b. f has zeros between –5 and –4; –4 and –3; –2 and –1; 0 and 1; 1 and 2.
2. a. $f(-3) = -20 < 0$ and $f(-2) = 5 > 0$.
   Therefore, f has a zero between –3 and –2.
   b. $f(1) = -8 < 0$ and $f\left(\frac{3}{2}\right) = \frac{117}{8} > 0$.
   Therefore, f has a zero between 1 and $\frac{3}{2}$.

c. $f(-1) = -1 < 0$ and $f(1) = 1 > 0$. Therefore, f has a zero between -1 and 1.
d. $f(1) = -2 < 0$ and $f(2) = 15 > 0$.
Therefore, f has a zero between 1 and 2.

3. a. f has a real zero between 0 and 1; 3 and 4 and 4 and 5.
b. f has real zeros between 0 and 1.
c. f has a real zero between -2 and -1.
d. f has real zeros between 0 and 1; and between 1 and 2.

4. a. 2, $\frac{1-\sqrt{3}}{2}$ and $\frac{1+\sqrt{3}}{2}$ are real zeros of f.
b. $-1, -\sqrt{2}, \sqrt{2}$, 1 and 2 are real zeros of f.
c. $-2, -\sqrt{2}, \sqrt{2}$ and 1 are real zeros of f.
d. $-\frac{1}{2}, -\sqrt{5}$, 0 and $\sqrt{5}$ are real zeros of f.

5. Approximately at 0.5 seconds and at 2.2 seconds after launch, the missile is 50m above the ground.
6. No

Rational Root Test

You may start the lesson now by defining an alternative method for finding the zeros of polynomial function having rational zeros which is called rational root test. After stating rational root test theorem, encourage students to answer questions of Activity 1.11 orally in the class. Consider their level of participation and keep record.

Answers to Activity 1.11

1. Checking if the coefficients are integers, if not, making them integers by multiplying them by their least common multiple.
2. The leading coefficient must be 1 so that the possible rational zeros are factors of the constant term.
3. Multiply each term by the least common multiple (LCM) of the coefficients to clear fractions. The resulting polynomial has the same zero as the original.
4. The number 0 is at least one rational zero of a polynomial whose constant term is 0.

After doing this, with active participation of students, you need to discuss all the steps enlisted in Example 7 in the student textbook. Finally, you can hint them to use this example as a guide for solving questions in Exercise 1.9 as an assignment and give feedback concerning the answers to these questions. This exercise might be used as a means of evaluation of how well they achieved, and what was expected by the end of the sub-unit.

After deliberating on this section, since all students may not have equal level of understanding, it is quite necessary to develop additional exercises of different capacity apart from the ones given in the student textbook that need to be solved on their own.

You can also encourage students, either as a group work or assignment, to come up with different exercises and share each with one another.

More challenging exercise problems and application problems could be given to students who are succeeding and moving quickly. Some of these problems are as follows.

1. Find all rational and irrational roots for each of the following polynomial equation.

   a. $2x^3 - 5x + 2 = 0$ b. $2x^5 - 3x^4 - 2x + 3 = 0$

   c. $2x^5 + x^4 - 6x^3 - 3x^2 - 8x - 4 = 0$ c. $x^4 - 2x^3 - 5x^2 + 8x + 4 = 0$

2. In each of the following, find the rational roots of the polynomial function

   a. $p(x) = x^4 - 5x^3 + \frac{15}{2}x^2 - 2x - 2$ b. $p(x) = x^4 + \frac{7}{6}x^3 - \frac{7}{3}x^2 - \frac{5}{2}x$

   b. $p(x) = 3x^5 - 5x^4 - 8x^3 + 16x^2 + 21x + 5$ d. $p(x) = x^4 - \frac{21}{10}x^2 + \frac{2}{5}x$

3. A rectangular box has dimensions 2 by 2 by feet. If each dimension is increased by the same amount, how much would this amount be to create a new box with volume three times the old?

A rectangular water tank has dimensions 2 by 3 by 4 meter. If each dimension is decreased by the same amount, how much should this amount be to create a new water tank with volume one-fourth of the old?

Assessment

By assessing continuously in different ways, you can be confident about what students know and what they can do. A number of different assessments will give you a better picture of the knowledge and skills acquired by the students. From among the assessments, you can use the following: class activities, group discussion, assignment, exercise problems and quiz or a test.

Answers to Exercise 1.9

1. a. 3; multiplicity 2
   –6; multiplicity 1
   Its degree is 3.

   b. –3; multiplicity 1
   –2; multiplicity 3
   1; multiplicity 2
   Its degree is 6

   c. 2; multiplicity 4
   –3; multiplicity 3
   1; multiplicity 1
   Its degree is 8

   d. 1; multiplicity 3
   2; multiplicity 1
   Its degree is 4

   e. 2; multiplicity 2
   Its degree is 4

2. a. –2, 1 and 3 are rational zeros of *p*.

   b. it has no rational zeros.

c. $\frac{-1}{3}$ and 2 are rational zeros of p.

d. -1 and $\frac{3}{2}$ are rational zeros of p.

e. $\frac{3}{2}, \frac{1}{3} -\frac{1}{2}$ are the rational zeros of p.

3. a. 1, –1 and 5 and the factorized form is $f(x) = (x - 1)(1 + 1)(x - 5)$

b. $-1, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $g(x) = (x+1)\left(x+\frac{\sqrt{3}}{3}\right)\left(x-\frac{\sqrt{3}}{3}\right)$ and

$p(t) = (t+1)(t-2)(t^2+1)$

c. –1 and 2 are the only two rational zeros of p.

4. a. $-2, -\frac{1}{3}$ and $\frac{1}{2}$ are rational zeros of p.

b. $\frac{-3}{2}, -2, \frac{3}{2}$ and 2 are rational zeros of p.

c. 0 is the only rational zero of h.

d. $0, -1, -\frac{5}{3}$ and $\frac{3}{2}$ are the only rational zero of p.

5. a. $\frac{1}{2}$ is the only rational root of the equation.

b. $-2, -\frac{1}{2}, \frac{1}{2}, 1$ are the rational root of the equation.

c. -1 and $1, \frac{3}{2}$ are the rational roots of the equation.

1.4 GRAPHS OF POLYNOMIAL FUNCTION

Periods allotted: 5 periods

Competencies

At the end of this sub-unit, students will be able to:

- *sketch the graph of a given polynomial function.*
- *describe the properties of the graph of a given polynomial function.*

Vocabulary: x-intercept, y-intercept, Turning points, Local extremum, Continuous curve, Smooth curve, Peak, Valley.

Introduction

This sub-unit presents graphs of polynomial function. Basically, the graph of a polynomial function is a smooth and continuous curve. The sub-unit gives much emphasis on how to use the sign of the leading coefficient of polynomial function to determine the end behavior of its graph. Finding the zeros, x-intercept as well as the y-intercept of the graph are treated in the sub-unit. Another important concept, the possible number of turning points, is also presented.

Teaching Notes

To begin with the discussion of the lesson, you may ask students to state the properties of the graph of constant, linear and quadratic function which was studied in grade 9. After deliberation by students, you may encourage students, through question and answer, to revise terms such as straight line, slope, y-intercept, x-intercept, parabola and vertex using the graphs of $f(x) = 2x - 1$ and $g(x) = x^2 + x - 2$. After working out these, ask students to do questions 1 and 2 of Activity 1.12 in student textbook as class exercise. By rounding, you need to see their work. The purpose of this class exercise is to help students to revise the graphs of linear function and quadratic functions, and check how they perform what was expected. Identify students who are incapable of doing these questions, give the correct answer and conduct the solution of question 3 of Activity 1.12 on the board together with the active participation of all students.

Answers to Activity 1.12

1. a.

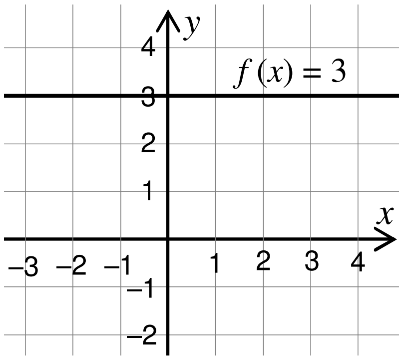


Figure 1.2

b.

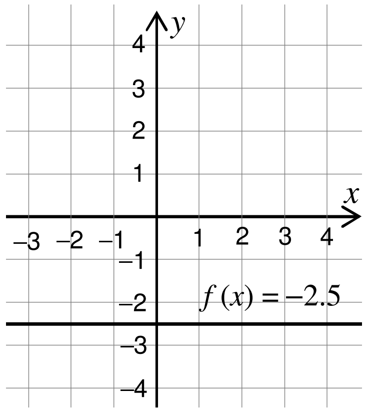


Figure 1.3

c.

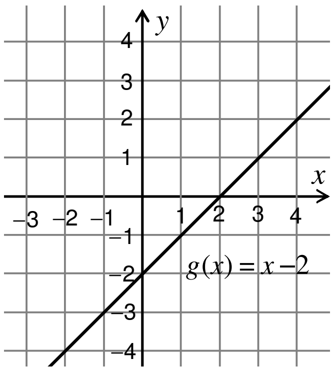


Figure 1.4

d.

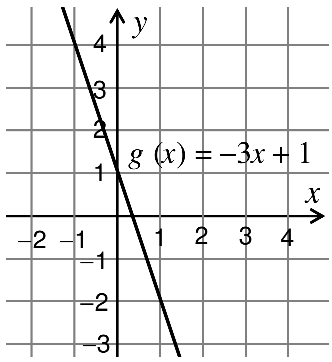


Figure 1.5

2. a.

x	-2	-1	0	1	2	3	4
$f(x) = x^2 - 4x + 5$	17	10	5	2	1	2	5

b.

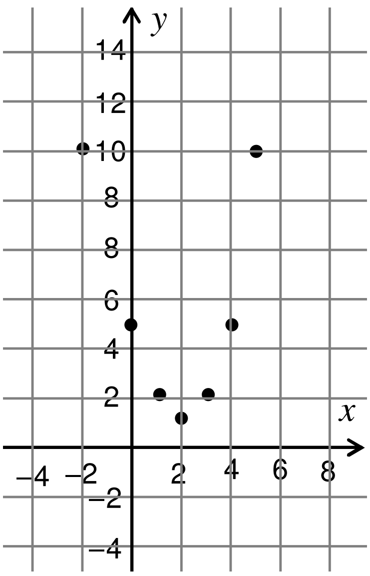


Figure 1.6

c.

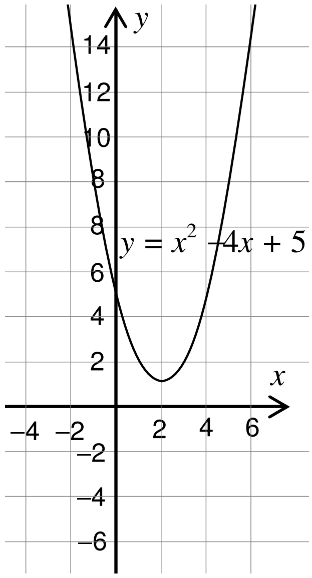


Figure 1.7

The graph is called parabolic.

- Domain $f = \mathbb{R}$
- Range of f is $\{y: \ y \geq 1\}$

3.

x	–3	–2	–1	0	1	2	3
$f(x)$	6	1	–2	–3	–2	1	6
$g(x)$	–2	1	2	1	–2	–7	–14
$h(x)$	–27	–8	–1	0	1	8	27
$p(x)$	–80	–15	0	1	0	–15	–80

a.

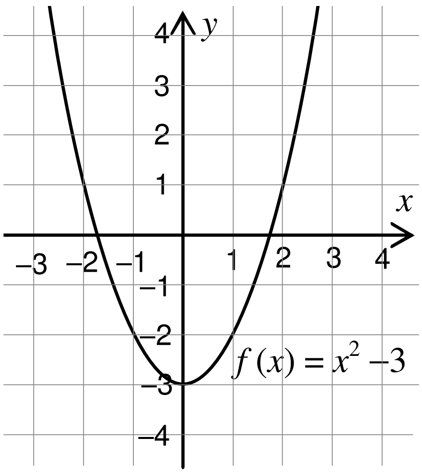


Figure 1.8

b

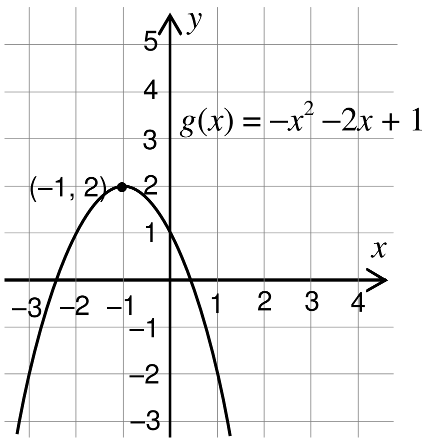


Figure 1.9

c.

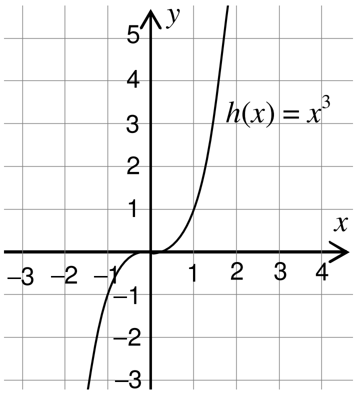


Figure 1.10

d.

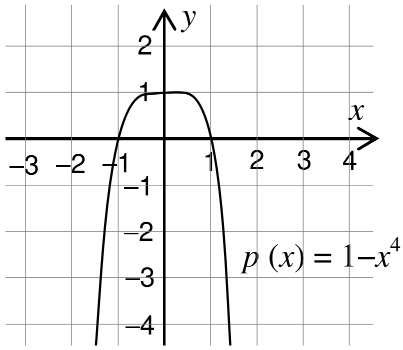


Figure 1.11

After ensuring the ability of the students in conducting Activity 1.12, you may discuss each step of Examples 1, 2, 3 and 4 in the student textbook. You may then ask questions on activity 1.13 by giving chance to each student to participate. Hint students to use the properties, what they have seen in the previous examples as a guide for doing Group work 1.3. Give Group work 1.3 as an assignment to present on separate paper. You should encourage and assist each group to do the Group work. Finally, let each group demonstrate how it worked out the group work on the board. You need to check and keep record whether or not each group is performing as expected to decide what to do next.

Answers to Activity 1.13

1. a. The set of all real numbers.
   b. In Figure 1.11 (a), the graph of g rises in the right and falls in the left. In Figure 1.11 (b), the graph of f rises both in the left and in the right.
   c. The term x^3 in $g\ (x)$ is positive when $x = 2^{10}$ and negative when $x = –2^{10}$.
   The term x^4 in $f\ (x)$ is positive when $x = 2^{10}$ or $x = –2^{10}$
2. a. No, because the range of every polynomial function is a subset of the set of real numbers. Example, the range of $f\ (x)$ in Q1 is not $\mathbb{R}$.
   b. Yes, the graph of every polynomial function crosses y-axis at exactly one point. Because for $x = 0$, there is a corresponding single value for y. We can easily find the value of y by letting $x = 0$.

Answers to Group work 1.3

1. a. The graph crosses x-axis at $x = –2$, $x = –1$, $x = 1$ and $x = 2$.
   b. The value of $g\ (x)$ is 0 at each of these points.
   c. The truth set of $g\ (x) = 0$ is the values of x at which the graph crosses x-axis. Or T.S = $\{–2, –1, 1, 2\}$
2. a. The coordinates of the point where the graph crosses the x-axis are $(–2, 0)$, $(–1, 0)$, $(1, 0)$ and $(2, 0)$. The coordinate of the point where the graph crosses y-axis is $(0, 4)$.

b. Yes, $(x + 2)(x + 1)(x – 1)(x – 2) = (x + 2)(x – 2)(x + 1)(x – 1)$
$= (x^2 – 4)(x^2 – 1) = x^4 – 5x^2 + 4$

Therefore, $h(x) = f(x)$.

3. a. The graph of $p(x) = 2x + 1$ intersects x-axis one time. Similarly,
   b. The graph of $p(x) = x^2 + 4$ never intersects x-axis because $p(x) \neq 0$.
   c. The graph of $p(x) = x^2 – 8$ intersects x-axis two times at points $(–2\sqrt{2}, 0)$ and $(2\sqrt{2}, 0)$.
   d. two times.
4. Every fourth-degree polynomial function not necessarily intersects x-axis 4 times.

After discussing Examples 7 and 8 in the student textbook, group students and give Group work 1.4 as an assignment. You can pick one from each group to present the correct answer to this activity.

Answers to Group work 1.4

1. i. Increases as x takes values far to the right and to the left.
   ii. Decreases as x takes values far to the right and to the left.
   iii. Increases as x takes values far to the left and decreases far to the right.
   iv. Increased as x takes values far to the right and decreases far to the left.
2. a. 1 for even degree and 0 for odd degree
   b. at most n c. at most n
   d. Odd degree 1, even degree 0.

Finally, let students generalize the end behavior of the graph of a polynomial function:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0, a_n \neq 0$$

When i. $a_n > 0$ and n is even or odd.
ii. $a_n < 0$ and n is even or odd

You can add more exercises for consolidating the behavior of the graph of polynomial function. At this stage, you may help students to realize that for any polynomial function:

$$f(x) = a_n x^n + a_n + x^{n-1} + ... + a_1 x + a_0, a_n \neq 0$$

i. For n is even
   a. If $a_n > 0$, then the graph of f rises to the left and to the right.
   b. If $a_n < 0$, then the graph of f falls to the left and to the right.

ii. For n is odd
   a. If $a_n > 0$, then the graph of f falls to the left and rises to the right.
   b. If $a_n < 0$, then the graph of f falls to the right and rises to the left.

iii. For a polynomial function of degree n,
   a. The maximum number of x-intercept (or zeros) is n.
   b. The number of turning point is $n – 1$

You can hint students by asking them to determine the end behavior of f, the maximum number of x-intercept and the number of turning point of the graph of the polynomial function:

$f(x) = 3x^4 - x^5 - 9x^2 - 2x + 6$

In which the answer is:

The graph of f falls to the right and rises to the left. Its maximum number of x-intercept is 5 and it has 4 turning points so that the students develop more critical thinking on the graph of polynomial function.

After deliberating on the lesson, since the level of understanding of each student is different, it is necessary to develop additional exercises of different capacity apart from the ones given in the textbook that need to be solved by students themselves. You can also encourage students either as group work or as an assignment to come up with different exercises and share each with one another.

For the purpose of checking the participation of students and their understanding, you may give Exercise 1.10 as homework, check their work and put a record.

Assessment

Apart from the details mentioned above, you may use a different approach to look at student performance and other assessment activities in order to determine the learning of the students. You can use different assessment techniques such as class activities, group discussion, assignment, exercise problems and quiz or test.

Answers to Exercise 1.10

1. a.

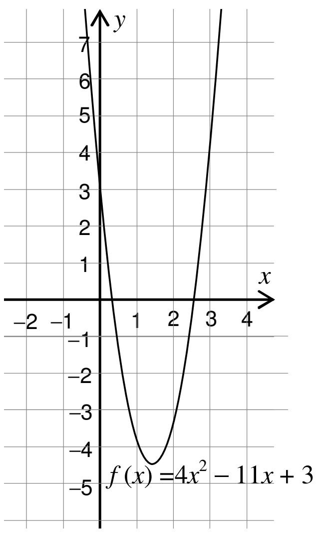


Figure 1.11

d.

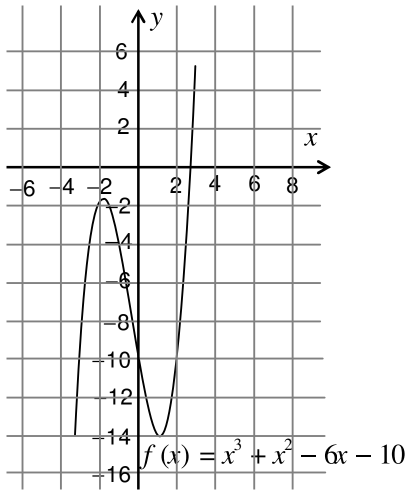


Figure 1.12

b.

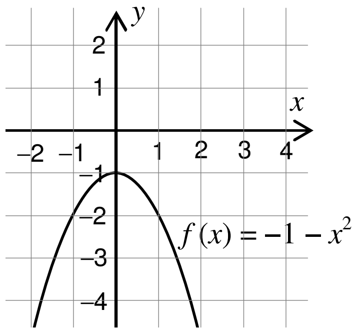


Figure 1.13

e.

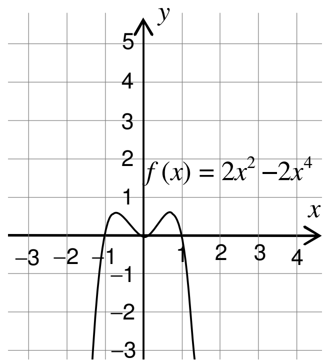


Figure 1.14

c

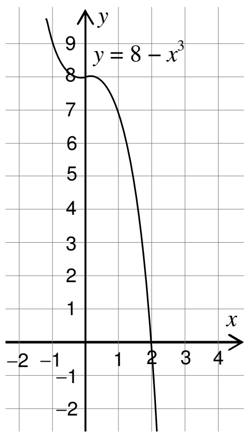


Figure 1.15

f.

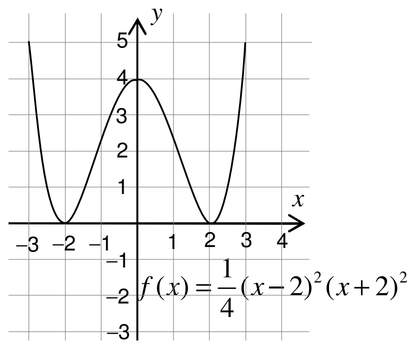


Figure 1.16

2.

N<u>o</u>	Values far to the right	Values far to the left	II	III	IV
a	Increases	Increases	1	Even	Positive
b	Increases	Increases	2	Even	Positive
c	Decreases	Increases	1	Odd	Negative
d	Increases	Decreases	3	Odd	Positive
e	Decreases	Increases	3	Odd	Negative
f	Increase	Decreases	2	Odd	Positive
g	Increases	Decreases	3	Odd	positive

3. i. In (a) f increases without bound as x takes values far to the right and left.

In (b) f decreases without bound as x takes values both far to the right and to the left.

In (c) f increases without bound as x takes values far to the left and decreases without bound as x takes values far to the right.

In (d) f increases without bound as x takes values far to the right and decreases without bound as x takes values far to the left.

In (e) f decreases without bound as x takes values far to the left and to the right. In (f) f increases without bound as x takes values far to the right and the left.

ii. In (a) 2 in (b) No, in (c) 1, in (d) 1, in (e) 3, in (f) 2

iii. For (a) 3, for (b) –1, for (c) 8, for (d) –10, for (e) 0, for (f) 4.

iv. In (a) 1, in (b) 1, in (c) 2, in (d) 2, in (e) 3, in (f) 3

4. a. Not b. Not c. Not
   d. Not e. Yes f. Yes

5. a. Positive coefficient, odd degree, 2 turning points
   b. Negative leading coefficient, odd degree and 2 turning points.
   c. Positive leading coefficient, odd degree, 4 turning points.
   d. Negative leading coefficient, even degree, 3 turning points.
   e. Positive leading coefficient, even degree and 3 turning points.
   f. Negative leading coefficient, even degree and 5 turning points.
   g. Negative leading coefficient, odd degree and 4 turning points.
   h. Negative leading coefficient, degree 1 (odd) and 0 turning points.
   i. Negative leading coefficient, degree 2 and 1 turning points.

6. a. True ; because a polynomial function f of degree n has n –1 turning points.
   b. True; for example, the polynomial function $f(x) = x^2$ touches x-axis at one point that is (0, 0)

At the end of this unit, you may give review exercise as an assessment so that students could review/ revise what they have learned in this unit.

Answers to Review Exercises on Unit 1

1. a. quotient = $x^2 + 6x - 12$, remainder = 7
   b. quotient = $3x^2 - 5x + 1$, remainder = 3
   c. quotient = $3x^3 + x^2 + x - 7$, remainder = 22
   d. quotient = $2x^2 + 5x - 1$, remainder = 0
   e. quotient = $x^3 + 3x^2 - x + 2$, remainder = $3x - 1$
   f. quotient = $3x - 2$, remainder = 0

2. $p(x) = (ax + b)q(x) + k$ (k = constant)

$$\Rightarrow p\left(\frac{-b}{a}\right) = k = \text{remainder.}$$

3. Since n is odd positive integer, we have that $(-1)^n + 1 = -1 + 1 = 0$

$\Rightarrow x = -1$ is a root of $x^n + 1$

$\Rightarrow x + 1$ is a factor of $x^n + 1$.

4. Consider the function f defined by $f(x) = x^2 - 2$.

Here, the rational roots of f are from $\pm 1, \pm 2$

But $f(1) = 1-2 = -1 \neq 0$

$f(-1) = 1-2 = -1 \neq 0$
$f(2) = 4-2 = 2 \neq 0$
$f(-2) = 4-2 = 2 \neq 0$
$\Rightarrow f$ has no a rational root.
But $\sqrt{2} = x$ is a root of f.
$\Rightarrow \sqrt{2}$ is not a rational number.
$\Rightarrow \sqrt{2}$ is an irrational number.

5. a. -3 is the only rational zero.
   b. 0, 1 and -3.

6. a. $f(x) = 2x^3 - 3x^2 - kx - 17$ divided by $x - 3$ has a remainder of -2 hence $f(3) = -2$.

$\Rightarrow f(3) = 2(3)^3 - 3(3)^2 - k(3) - 17 = -2$

$\Rightarrow 2(27) - 27 - 3k - 17 = -2$

$\Rightarrow 27 - 3k - 17 = -2$

$\Rightarrow 10 - 3k = -2$

$\Rightarrow k = 4$

   b. $f(x) = x^3 - 6x^2 + 2kx - 3$ and $(x - 1)$ is a factor of f means that

$f(x) = 0 \Rightarrow (1)^3 - 6(1)^2 + 2k(1) - 3 = 0$

$\Rightarrow 1 - 6 + 2k - 3 = 0$

$\Rightarrow 2k = 8$

$\Rightarrow k = 4$

   c. $5x - 2$ is a factor of $x^3 - 3x^2 + kx + 15$ means that.

$$f\left(\frac{2}{5}\right) = 0 \Rightarrow \left(\frac{2}{5}\right)^3 - 3\left(\frac{2}{5}\right)^2 + k\left(\frac{2}{5}\right) + 15$$

$$\Rightarrow \frac{8}{125} - \frac{12}{25} + \frac{2k}{5} + 15 = 0$$

$$\Rightarrow \frac{1823}{125} = \frac{-2}{5}k$$

$$\Rightarrow k = \frac{-1823}{50}$$

7. a.

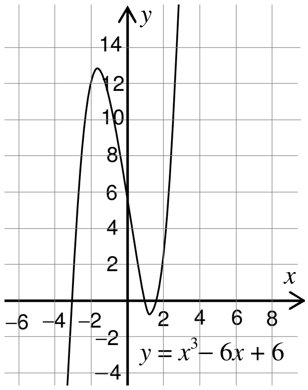


Figure 1.18

b.

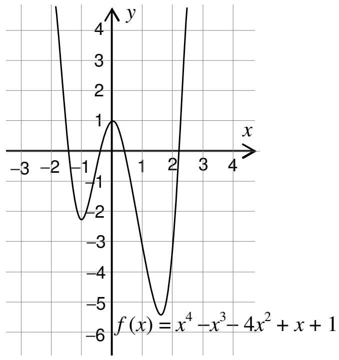


Figure 1.17

c.

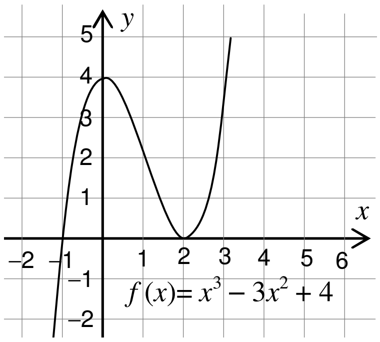


Figure 1.19

d.

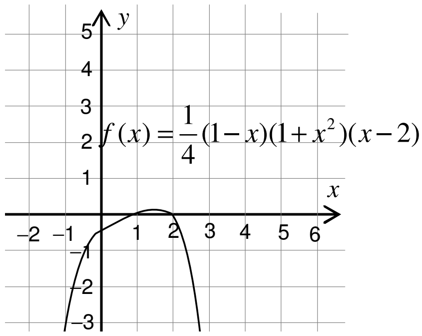


Figure 1.20

8. Sketch the graph of $f(x) = x^4$

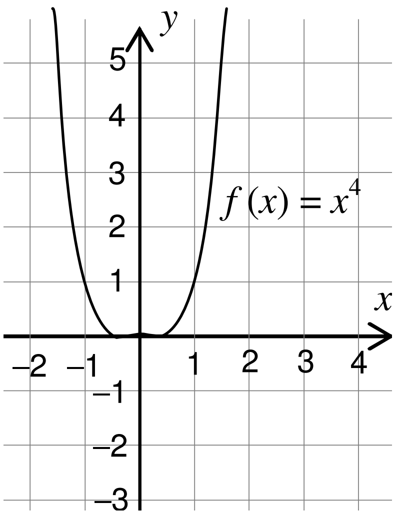


Figure 1.21

a. $g(x) = f(x) + 3 = x^4 + 3$ the graph of g is different from the graph f because the graph of g extends up by 3 units and g is even function

b. $g(x) = f(-x) = x^4$, the graph of g is the same to the graph of f
g is even function

c. $g(x) = -f(x) = -x^4$, the graph of g is the mirror image of the graph of f along x-axis and g is an even function.

d. $g(x) = f(x+3) = (x+3)^4$ the graph of g is different from the graph of f in such a way that it shifts horizontally by 3 units to the left. It is neither even nor odd.

9. $f(x) = A(x-1)^2 + B(x+2)^2$ is divided by $x+1$ and $x-2$ the remainder are 3 and -15 respectively from this $f(-1) = 3$ and $f(2) = -15$

$f(-1) = A(-1-1)^2 + B(-1+2)^2 = 3$

$A(-2)^2 + B(1)^2 = 3$

$4A + B = 3$ …………….. 1

$f(2) = A(2-1)^2 + B(2+2)^2 = -15$

$\Rightarrow A(2-1)^2 + B(4)^2 = -15$

$\Rightarrow A + 16B = -15$…………..2

From equation (1) and equation (2), we get

$$\left\{\begin{array}{l|l} 4A+B=3 & \times 1 \\ A+16B=-15 & \times -4 \end{array}\right. \Rightarrow \left\{\begin{array}{l} 4A+B=3 \\ \underline{-4A-64B=60} \end{array}\right.$$

$$-63B = 63$$

$$B = -\frac{63}{63} = -1$$

$B = -1$ and $4A - 1 = 3$

$4A = 4 \Rightarrow A = 1$

Therefore, $A = 1$ and $B = -1$, hence the equation is $f(x) = (x-1)^2 - (x+2)^2$

$\Rightarrow f(x) = -6x - 3$

10. $f(x) = x^2 + (c-2)x - c^2 - 3c + 5$ is divided by $x + c$ the remainder is -1

$f(-c) = (-c)^2 + (c-2)(-c) - c^2 - 3c + 5 = -1$

$\Rightarrow c^2 + (-c^2 + 2c) - c^2 - 3c + 5 = -1$

$\Rightarrow c^2 - c^2 + 2c - c^2 - 3c + 5 = -1$

$\Rightarrow -c^2 - c + 6 = 0$

$\Rightarrow c^2 + c - 6 = 0$

$\Rightarrow (c+3)(c-2) = 0$

$\Rightarrow c = -3$ or $c = 2$

11. $x - 2$ is a common factor of the expressions.

$x^2 (m+n)x - n$ and $2x^2 + (m-1)x + (m+2n)$ this implies

$(2)^2 (m+n)2 - n = 0$ ……………………… 1 and

$2(2)^2 + (m-1)\,2 + (m+2n) = 0$ ………..… 2

From 1, $4(2m + 2n) - n = 0$

$\Rightarrow 8m + 8n - n = 0$

$\Rightarrow 8m + 7n = 0$……………………………………..3

$8 + (2m - 2) + (m + 2n) = 0$

$\Rightarrow 8 + 2m - 2 + m + 2n = 0$

$\Rightarrow 6 + 3m + 2n = 0$

$\Rightarrow 3m + 2n = -6$ …………………………..….4

From 3, $m = -\frac{7n}{8}$. Substitute in equation 4

$$3\left(-\frac{7n}{8}\right) + 2n = -6$$

$$-\frac{21}{8}n + 2n = -6$$

$$\frac{-21n + 16n}{8} = -6$$

$$\frac{-5}{8}n = -6 \Rightarrow n = \frac{48}{5}$$

From equation (3), we have $8m + 7 \times \frac{48}{5} = 0$

$$8m = \frac{-336}{5} \Rightarrow m = -\frac{42}{5}$$

12 a. $x^3 - 4x^2 - 7x + 10 = (x - 1)(x^2 - 3x - 10)$

$= (x - 1)(x - 5)(x + 2)$

b. $2x^5 + 6x^4 + 7x^3 + 21x^2 + 5x + 15 = 2x^4(x + 3) + 7x^2(x + 3) + 5(x + 3)$

$= (2x^4 + 7x^2 + 5)(x + 3) = 0$

$= (x^2 + 1)(2x^2 + 5)(x + 3) = 0$

13. $R = y^4 + 2y^3 - 4y^2 - 5y + 14$ where r is response in microseconds y is age group in years.

R = 8 is given

$y^4 + 2y^3 - 4y^2 - 5y + 14 = 8$

$y^4 + 2y^3 - 4y^2 - 5y + 6 = 0$ where $y > 0$

$(y - 1)(y^3 + 3y^2 - y - 6) = 0$

$y = 1$

14. $p(t) = t^3 - 14t^2 + 20t + 120$.

Where t is the number of years after take over

$p(t) \leq 0$ is the club making a loss hence $t^3 - 14t^2 + 20t + 120 \leq 0$.

UNIT 2 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

INTRODUCTION

This unit begins by revising the properties of exponents discussed in grade 9. Then it discusses the concept of logarithms and their properties. Exponential and logarithmic functions and their graphs are the central topics of this unit. The unit also gives emphasis to the techniques of solving exponential and logarithmic equations. Finally students are encouraged to solve problems on practical applications of exponential and logarithmic functions from different fields such as population growth, compound interest, etc.

Unit Outcomes

After completing this unit, students will be able to:

- *understand that the laws of exponents are valid for real exponents.*
- *know specific facts about logarithms.*
- *know basic concepts about exponential and logarithmic functions.*
- *solve mathematical problems involving exponents and logarithms.*

Suggested Teaching Aids in Unit 2

In addition to the students' textbook and the teacher's guide, you are advised to prepare and bring to the class the following materials whenever the topic requires:

- Calculator to compute exponents and logarithms of numbers.
- Pre made graphs of common exponential and logarithmic functions like graphs of $f(x) = 2^x$, $f(x) = \left(\frac{1}{2}\right)^x$, $f(x) = b^x$ for $b > 1$ and for $0 < b < 1$, $f(x) = \log_2 x$, $g(x) = \log_b x$, for $b > 1$ and for $0 < b < 1$ drawn on big drawing papers. You can fix the pre made graph of an exponential or logarithmic function to the wall whenever you teach the topic. By so doing, you can save your time and display a very accurate and better looking graph to the students.
- Enlarged table of logarithms. Prepare a relatively larger logarithmic table and fix it to the wall whenever you teach how to use logarithmic tables.

2.1 EXPONENTS AND LOGARITHMS

Periods allotted: 6 Periods

Competencies

At the end of this subunit, students will be able to;

- *explain what is meant by exponential expression.*
- *state and apply the properties of exponents (where the exponents are real numbers).*
- *express what is meant by logarithmic expression by using the concept of exponential expression.*
- *solve simple logarithmic equation by using the properties of logarithm.*
- *recognize the advantage of using logarithm to the base 10 in calculation.*
- *identify the ″characteristic″ and ″mantissa″ of a given common logarithm.*
- *use the table for finding logarithm of a given positive number and antilogarithm of a number.*
- *compute using logarithm.*

Vocabulary: Exponent, Power, Exponential Expression, Logarithm, Logarithmic expression, Logarithm of a number, Common logarithm, Mantissa, Characteristic, Antilogarithm

Introduction

This subunit mainly focuses on exponents and logarithms of numbers and their laws. Students are encouraged to apply the laws of exponents and logarithms to simplify exponential and logarithmic expressions. After students have practiced finding logarithms of numbers, they will be introduced to common logarithms and the advantages of using common logarithms. Towards the end of the subunit, the techniques of using the table for finding logarithm of a given positive number and antilogarithm of a number are discussed.

Teaching Notes

Exponents

Introduce to the students the idea of multiplying a number by itself. Let them multiply a given number by itself repeatedly. For example, $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$. Tell them that this product is easily represented as $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$. Repeat this with other numbers.

Example: $3 \times 3 \times 3 \times 3 = 81$. Show them that this product is also represented as $3 \times 3 \times 3 \times 3 = 3^4$.Then explain that they can use the idea of *exponents* to easily represent a product involving the same factor. Generalize your specific considerations as follows:

If n is a positive integer, then, a^n is the product of n factors of a.

i.e.

$$a^n = \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ factor}}$$

Give attention on how to name a, n, and a^n.

Then give students time to do Activity 2.1. This activity is aimed at identifying the base and exponent of a given power. So, make sure that students identify the differences between powers of the form 2^6 and -2^6 and $-3x^2$. Let students sharing the same desk do question number 2 together. In number 2 of Activity 2.1 the bases are all negative but some exponents are odd and some others are even. Encourage the students to reach a conclusion that *a negative base raised to an odd exponent is negative and that a negative base raised to an even exponent is positive.*

Have an ongoing assessment to check if students can easily identify values of powers with negative base raised to odd or even exponents. For instance, they should immediately identify that $(-2)^{999}$ is a negative number whereas $(-2)^{1000}$ is a positive number. You can orally ask the students more questions of this type to check their level of understanding.

Answers to Activity 2.1

1.

	Base	**Exponent**	**Value**
a.	4	3	64
b.	-2	8	256
c.	$\frac{2}{7}$	4	$\frac{16}{2401}$
d.	-1	23	1
e.	5t	4	$625t^4$

2. a. -1 b. 1 c. -1 d. 1 e. -1
 f. 1 g. -2 h. 4 i. -8 j. 16
 k. -32 l. 64

3. A negative base raised to an odd exponent gives a negative value.

Laws of Exponents

Students are expected to remember the laws of exponents which they had learnt in the previous grades. To refresh their memory, they may start doing Group work 2.1.

Make the students discuss the laws that they have applied while simplifying each exponential expression. Give chance to the groups to state to the whole class the laws they applied. Then write the laws on the board and let them practice these exponential laws with more examples and exercises.

Answers to Group work 2.1

a. 2^8 b. 4^{11} c. 2^4 d. 2^4 e. 6^3

f. $15^{(-2)}$ g. 3^{10} h. $\frac{8}{27}$ i. a^{c+d}

If students seem not to remember the laws that they had learnt in grade 9, you may use the following inquiry approach to present the laws of exponents. That is, give them several specific examples as an activity which step by step leads the students to generalize each exponential law. Use questioning and answering technique until students are able to generalize or discover the law. For instance,

- ✓ $10^3 \times 10^2 = 1000 \times 100 = 100000 = 10^5 = 10^{3+2}$
- ✓ $2 \times 2^2 \times 2^3 = 2 \times 4 \times 8 = 64 = 2^6 = 2^{(1+2+3)}$
- ✓ $c^3 \times c^4 \times c^5 = (c \times c \times c) \times (c \times c \times c \times c) \times (c \times c \times c \times c \times c) = c^{12} = c^{3+4+5}$

From such an activity, they must be encouraged to generalize the law $a^m \times a^n = a^{m+n}$.

Enrich this law by considering more examples before going to discover the next law. Use a similar approach to discover other exponential laws. To help them reach at another law, you guide them to do Activity 2.2.

Activity 2.2 is aimed at helping students to discover the law $a^0 = 1$ $(a \neq 0)$. So give them specific tasks of the form $\frac{a^m}{a^m}$ $(a \neq 0)$. Students should observe that $\frac{a^m}{a^m} = 1$(by cancellation law). But $\frac{a^m}{a^m}$ is also equal to $a^{m-m} = a^0 = 1$. At this point you can raise the question "Is $\frac{0^0}{0^0} = 1$?" Explain this in relation to division by 0.

So, conclude $a^0 = 1 (a \neq 0)$. Let them practice this new law using examples given in the student text.

Answers to Activity 2.2

1. a. Yes because $\frac{2^3}{2^3} = 1 = 2^{3-3} = 2^0$

b. Yes because $\frac{10^5}{10^5} = 1 = 10^{5-5} = 10^0$

c. Yes because $\frac{(-8)^3}{(-8)^3} = 1 = (-8\ )^{3-3} = (-8\ )^0$

2. Any non-zero number raised to zero is one.

After doing this activity, you can form groups of students and let them do Group work 2.2. In group work 2.2, students are expected to derive a relationship between a^{-n} and $\frac{1}{a^n}$ by considering specific examples. For instance $\frac{3^3}{3^5} = \frac{3\times3\times3}{3\times3\times3\times3\times3} = \frac{1}{3^2}$.

But using the law, $\frac{a^m}{a^n} = a^{m-n}$, $\frac{3^3}{3^5} = 3^{3-5} = 3^{-2}$. Therefore, $3^{-2} = \frac{1}{3^2}$. So guide their activity properly by giving additional examples (if necessary) until the students reach at the desired generalization.

Answers to Group work 2.2

a. $\frac{3^5}{3^7} = 3^{-2}$ b. i. $\frac{1}{2^3} = 2^{-3}$ ii. $\frac{1}{3^2} = 3^{-2}$

c. $a^{-n} = \frac{1}{a^n}\ (a \neq 0, n > 0)$

Once the students have understood the laws of exponents, you can now let them apply the laws of exponents in simplifying exponential expressions. Give them time to practice by doing Exercise 2.1. You can take out volunteers to the board and give them questions from Exercise 2.1.Let others comment on the works of their peers. Ask them if they can agree or disagree on the solution given by their peers. This will help you to get a feedback about the students' level of understanding the laws so that you can take an immediate remedial measure.

Answers to Exercise 2.1

1. a. $t^2 \times t = t^{2+1} = t^3$ b. $t^3 \times t \times t^5 = t^{3+1+5} = t^9$

c. $r \times r^4 \times r^5 \times r = \text{r}^{1+4+5+1} = r^{11}$ d. $a^3 \times a \times a^{-5} = a^{3+1+(-5)} = a^{-1}$

e. $\frac{7^6}{7^4} = 7^{(6-4)} = 7^2 = 49$ f. $\frac{(-3y)^2}{(-3y)^5} = (-3y)^{(2-5)} = (-3y)^{-3}$

g. $\frac{(2x)^7}{(2x)^8} = (2x)^{(7-8)} = (2x)^{-1}$ h. $b^{2x} \div b = b^{(2x-1)}$

i. $(5^5)^{2n} = 5^{(5\times 2n)} = 5^{10n}$ j. $(b^y)^x = b^{(y\times x)} = b^{yx}$

k. $(7^3)^{-2} = 7^{(3\times -2)} = 7^{-6}$ l. $(a^{3x})^2 = a^{(3x\times 2)} = a^{6x}$

2\. a. $81 = 3^4$

b. $\dfrac{16^{2x+3}}{16^{2x-3}} = \dfrac{2^{4\times(2x+3)}}{2^{4\times(2x-3)}} = \dfrac{2^{(8x+12)}}{2^{(8x-12)}} = 2^{24}$

c. $\dfrac{49^x}{7^y} = 7^{(2x-y)}$

d. $64^a \times 4^a = 2^{6a} \times 2^{2a} = 2^{8a}$

3\. a. $(xyz)^2 = x^2 \times y^2 \times z^2$

c. $\left(\dfrac{9}{3}\right)^2 = \dfrac{9^2}{3^2} = \dfrac{3^4}{3^2} = 3^{4-2} = 3^2 = 9$

b. $(2ab^2)^5 = 2^5 \times a^5 \times (b^2)^5 = 2^5a^5b^{10}$

d. $\left(\dfrac{-2}{2n}\right)^6 = \dfrac{(-2)^6}{2^6 \times n^6} = \dfrac{1}{n^6}$

4\. a. $\left(\dfrac{3}{2}\right)^0 = 1$

b. $\left(\dfrac{8}{3}\right)^{-2} = \left(\dfrac{3}{8}\right)^2 = \dfrac{9}{64}$

c. $\left(\dfrac{1}{4^{-3}}\right)^{-1} = \left(\dfrac{4^{-3}}{1}\right)^1 = \dfrac{4^{-3}}{1} = \dfrac{1}{4^3} = \dfrac{1}{64}$

d. $(-2)^{-5} = \dfrac{1}{(-2)^5} = \dfrac{1}{-32} = -\dfrac{1}{32}$

e. $(3x^2)^{-3} = \dfrac{1}{(3x^2)^3} = \dfrac{1}{27x^6}$

Rational Exponents

Revise powers with integral exponents and make sure whether there is a common understanding among the students as to what powers such as 3^5, 2^{-3}, 7^0 mean. Then extend the laws of exponents to the rational numbers. Let the students do Activity 2.3 in groups. This activity helps them to discover a relationship between $a^{\frac{1}{n}}$ and $\sqrt[n]{a}$.

Ask them a question like *what do powers such as* $6^{\frac{1}{2}}$ *and* $6^{\frac{2}{3}}$ *mean?* Give them time to think and reflect. Give them hints like: What is $\sqrt{6}\times\sqrt{6}$? What is $6^{\frac{1}{2}}\times 6^{\frac{1}{2}}$? Which exponential rule helps you to simplify $6^{\frac{1}{2}}\times 6^{\frac{1}{2}}$? How do you then compare $\sqrt{6}$ and $6^{\frac{1}{2}}$? Repeat this with another rational exponent like $6^{\frac{1}{3}}$ until students see a relationship between $6^{\frac{1}{3}}$ and $\sqrt[3]{6}$. Consider additional examples of your own until students see a relationship between $a^{\frac{1}{n}}$ and $\sqrt[n]{a}$. Give students a chance to see whether they can generalize a rule. Finally, state $a^{\frac{1}{n}} = \sqrt[n]{a}$ (for $a > 0$ and an integer $n > 1$) and consider more examples.

Study different cases for the base and exponent. *What will happen when $a < 0$ and n is odd? What will happen when $a < 0$ and n is even*? Let them discuss and practice each case with examples.

Answers to Activity 2.3

1. a. i. 6 ii. 6 b. $6^{\frac{1}{2}} = \sqrt{6}$

2. a. i. 6 ii. 6 b. $6^{\frac{1}{3}} = \sqrt[3]{6}$

3. a. i. 2 ii. 2 b. $2^{\frac{1}{4}} = \sqrt[4]{2}$

4. $a^{\frac{1}{n}} = \sqrt[n]{a}$, n is an integer > 1

Irrational Exponents

Use oral question and answer method to revise what powers like 3^5, 2^{-3}, 7^0, $6^{\frac{1}{2}}$, $7^{\frac{2}{3}}$ mean, and make sure if students have understood them. Then extend the laws of exponents to the irrational numbers.

Ask a question like: *What do powers such as* $2^{\sqrt{2}}$, $2^{\sqrt{3}}$, 2^{π} and $2^{\sqrt{5}}$ *mean? Which of these numbers is the smallest? Which one is the largest*? Let the students talk to each other for a little while and see that the arrangement of these numbers from smallest to largest or vice versa will be difficult for them because they do not know the exact values of the numbers. Invite the students to use a calculator to approximate numbers $2^{\sqrt{3}}$, 2^{π} and $2^{\sqrt{5}}$. For example, $2^{\sqrt{5}} \approx 4.711$ which is a number between 4.5 and 5.

Next let the students consider the table of values for 2^x. (Give them a pre made table of values for 2^x or write it on the board). From the table of values for 2^x, they can see that if $x_1 < x_2$, then $2^{x_1} < 2^{x_2}$.

Since $2.2 < \sqrt{5} < 2.3$, $2^{2.2} < 2^{\sqrt{5}} < 2^{2.3}$. This suggests that the expression a^x is defined for irrational exponents. Then, give them the definition of irrational exponents and summarize once again the laws of exponents for real exponents. Practice the exponential laws with several examples and exercises. Finally, divide the students into groups. Students sharing the same desk may work together. Let them do Group work 2.3. Those who seem to be slow in their learning speed can do only the first question. Go round the groups and give them guidance and necessary explanations. Brief them that doing question number1 first helps them to easily answer question number 2.

It is quite natural that you may find students who are fast or slow in their learning speed. Since these students deviate from most of the students in the class, they need a very special treatment of the topic. Those who comprehend faster may need additional tasks which are relatively higher in their level of difficulty whereas those who are slow in their learning speed may need an individualized help of yours. Slow learners may prefer simpler tasks and more drills. Question number 1 (a, b, c, d, e) of Group work 2.3 may be appropriate to such students before going to question number 2.

Answers to Group work 2.3

1. a. Yes b. Yes c. No d. Yes e. No
2. a. Yes, No b. Yes, No

Then give them ample time to do Exercise 2.2 in groups. You can make students sharing the same desk work together.

This exercise requires all of the exponential rules discussed so far for real exponents. Observe the students while they are doing the exercise. Check if most of the students are able to solve the exercise correctly. Determine whether most of them have difficulties in doing Exercise2.2 or not. Have the relevant data about their performance so that you can decide to proceed to the next topic or to revise the lesson once again.

If you find few fast learners who seem idle because they have finished the exercise earlier, then you can choose any of the following problems and give them as additional tasks and support them whenever they require your help.

1. Prove that for any real numbers a and b, if $a > b$, then $2^a > 2^b$.

   (Hint: $a > b \Rightarrow a - b > 0$, $2^{(a-b)} > 1$)

2. Find x, if $a^x = b^x$, for $a > b > 1$
3. Prove the following laws of exponents($a > 0$, $b > 0$, m and n are integers)

   a. $a^m \times a^n = a^{m \times n}$ (Hint: $a^m = a \times a \times a \times ... \times a$ *(m factors)* and $a^n = a \times a \times a \times ... \times a$ *(n factors)*

   b. $\dfrac{a^m}{a^n} = a^{m-n}$ c. $(a^m)^n = a^{mn}$ d. $(a \times b)^n = a^n \times b^n$

   e. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$ f. $a^{-n} = \dfrac{1}{a^n}$ (Hint: $a^{-n} \times a^n = a^{n-n} = a^0 = 1$)

   g. $\dfrac{1}{a^{-n}} = a^n$ (Hint*: use the law* a^{-n} $\dfrac{1}{a^n}$)

   h. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$ (Hint*: use the law* $a^{-n} = \dfrac{1}{a^n}$) i. $\dfrac{a^{-m}}{b^{-n}} = \dfrac{b^n}{a^m}$

Answers to Exercise 2.2

a. a^6	b. $\dfrac{72}{17}$	c. $\dfrac{1}{49}$	d. $\dfrac{a^6}{4b^4}$	e. $3a$
f. $\dfrac{a^6}{b^3}$	g. $\dfrac{b^{10}}{a^6}$	h. n^3	i. $\dfrac{1}{n^4}$	j. $\dfrac{m^4}{n^3}$
k. $\dfrac{b}{\sqrt{a}}$	l. $3^{2\sqrt{2}}$	m. $2^{6\sqrt{3}}$	n. $\dfrac{1}{2}$	o. $2^{\sqrt{2}}$
p. $\dfrac{2a}{\sqrt[3]{b}}$				

Logarithms

Consider the exponential equation $3^2 = 9$ together with the students.

In $3^2 = 9$, ask them to identify the base and the exponent. Explain to them that this equation can be written in logarithm form (with identical meaning) as $\log_3 9 = 2$. Put stress on how they should read $\log_3 9 = 2$. Similarly, if $2^4 = 16$, then we can immediately say that $4 = \log_2 16$ and vice versa. If $10^3 = 1000$, we also say that $3 = \log_{10} 1000$.

Let them practice converting exponential statements to logarithmic statements and vice versa by doing Activity 2.4. Check if students are able to convert one into the other.

Finally, generalize your consideration as follows:

For a fixed positive number $b \neq 1$, and for any $a > 0$, $b^c = a$, if and only if $c = \log_b a$.

Tell them what had forced the Scottish mathematician, John Napier to develop Logarithms.

Answer to Activity 2.4

Exponential statement	Logarithmic statement
$2^3 = 8$	$\log_2 8 = 3$
$2^5 = 32$	$\log_2 32 = 5$
$2^6 = 64$	$\log_2 64 = 6$
$10^2 = 100$	$\log_{10} 100 = 2$
$2^x = y$	$\log_2 y = x$

Then let them do the examples that follow the activity. Give them Exercise 2.3 as a class work. Go round the students and observe how they are doing it. Let few students come out and do the exercises on the board. Involve the remaining students to comment on the work of their friends. This will help you to easily identify students with learning difficulties and help them on the spot.

Answer to Exercise 2.3

1. a. $\log_{100} 10000 = 2$ b. $\log_2 \frac{1}{32} = -5$

   c. $\log_{125} 5 = \frac{1}{3}$ d. $\log_8 \frac{1}{4} = \frac{-2}{3}$

2. a. $10^4 = 10000$ b. $7^1 = \sqrt{49}$

   c. $10^{-1} = 0.1$ d. $2^{-2} = \frac{1}{4}$

3. a. 3   b. 2   c. 2   d. $\frac{1}{2}$

Laws of Logarithms

Use an inquiry approach just like they did in discovering the laws of exponents. That is, give them several examples until students are able to generalize or discover the logarithmic rule. You can begin with Group work 2.4. For instance, you may encourage students to

a. Compare the results of $\log_2 8 + \log_2 2$ and $\log_2(8\times 2)$.

b. Compare the results of $\log_{10} 100 + \log_{10} 1000$ and $\log_{10}(100\times 1000)$.

c. Compare the result of $\log_3 9 + \log_3 \frac{1}{27}$ and $\log_3(9\times\frac{1}{27})$. Then, ask them if

they can suggest a possible simplification for $\log_b x + \log_b y$ from the above considerations. Generalize their activity and give them more examples on the rule discovered. Do all of group work 2.4 to discover all other logarithmic laws.

Students may also be given the logarithmic law first and are required to check its truthfulness by several specific instances. Though this method is not active, one can use it as an alternative approach to help particular learners with special needs.

Answers to Group work 2.4

1. a. $\log_2 8+\log_2 2=3+1=4$; $\log_2 8\times 2=\log_2 16=4$

   $\therefore \log_2 8+\log_2 2=\log_2 8\times 2$

   b. $\log_{10} 100+\log_{10} 1000=\log_{10}(100\times 1000)=5$

   c. $\log_3 9+\log_3 \frac{1}{27}=\log_3\left(9\times\frac{1}{27}\right)=-1$

   $\therefore \log_b x+\log_b y=\log_b xy$

2. a. $\log_2 8-\log_2 2=\log_2 \frac{8}{2}=2$

   b. $\log_{10} 100-\log_{10} 1000=\log_{10} \frac{100}{1000}=-1$

   c. $\log_3 9-\log_3 \frac{1}{27}=\log_3\left(9\div\frac{1}{27}\right)=5$

   $\therefore \log_b x-\log_b y=\log_b \frac{x}{y}$

3. a. $3\log_2 2=\log_2 2^3$ b. $2\log_{10} 100=\log_{10} 100^2=4$

   c. $\frac{1}{2}\log_2 16=\log_2 \sqrt{16}=2$ $\therefore k\log_b x=\log_b x^k$

4. a. $\log_3 3=1$ b. $\log_8 8=1$

   c. $\log_{100} 100=1$ d. $\log_{\frac{1}{3}} \frac{1}{3}=1$

   $\therefore \log_b b=1\,(b>0)$

5. a. $\log_3 1 = 0$ b. $\log_4 1 = 0$

   c. $\log_{\frac{1}{3}} 1 = 0$ d. $\log_{1000} 1 = 0$

   $\therefore \log_b 1 = 0 \ (b > 0)$

Let them practice the laws of logarithms by doing the examples given in the text and Exercise 2.4. This exercise helps you to assess whether they are able to simplify logarithmic expressions by applying the laws. Depending on the time you have, you can give them exercise 2.4 (1 and 2) as class work and (3 and 4) as a home work. Evaluate their work and determine whether they are able to find logarithm of a number.

Answers to Exercise 2.4

1. a. 2 b. 1 c. 5 d. 3

   e. $\frac{1}{2}$ f. $\frac{1}{2}$ g. $\frac{1}{5}$ h. –3

2. a. $\log_2 64\times1024 = 16$ b. $\log_2 \frac{32}{256} = -3$ c. 27

   d. $0.3010 + (-3) = -2.699$ e. 4 f. –1

   g. $\frac{12}{49}$

3. a. –4 b. –4 c. 3 d. 3 e. 10

4. a. 0.7925 b –2.3223 c. 5.6571

Logarithms in Base 10 (Common Logarithms)

Materials: pre-made logarithmic table (larger in size) or the logarithmic table attached at the end of the text book.

Introduce to them that many positive numbers can be easily written in the form 10^n.

For example, 10^4=10000. This is equivalent to saying $\log_{10}10000 = 4$. That is, $\log_{10}10000$ is the exponent needed on the base 10 to give 10000.

Students should note that the logarithm of any positive number to base 10 is called a **common logarithm**. They should also note that a common logarithm is written without indicating its base. For example, $\log_{10} x$ is simply denoted by log *x*. You may give them Activity 2.5 at this stage so that they can practice writing common logarithms without indicating the base.

Answers to Activity 2.5

a. $\frac{1}{2}$ b. – 4 c. 0 d. $1 - n$

Then, tell your students that one simple reason why we prefer logarithms with base10 is that we use a base 10 decimal system and many computations involving scientific notations are easily done using logarithms with base10. Any logarithm to base other than 10 can be expressed to a common logarithm. This helps the students to use a table of common logarithm while performing mathematical computations.

Give due attention to the method of using the common logarithmic table to find the specific common logarithmic values for numbers between 1 and 10. You can fix a relatively larger logarithmic table to the wall as a teaching aid while you are teaching how to read the logarithm of a number from the common logarithmic table.

Help them identify the characteristic and mantissa of a given common logarithm using several examples. After explaining what is meant by antilogarithm of a number, guide your students how the table is used in finding antilogarithm of a given number.

Encourage the students to compute expressions like $\frac{354\times 605}{8450}$, $380^{\frac{1}{3}}$, $\sqrt{35}$ etc. using common logarithms. Let them do Group work 2.5 so that they can appreciate the importance of using common logarithms over the other bases during mathematical computations.

Answers to Group work 2.5

1. Base 10 is preferable because
   i. We use the base 10 number system
   ii. Any logarithm to base other than 10 can be expressed to a logarithm with base 10 so that we can use a ready-made logarithm table.
2. Let $x = \sqrt{3}$

   $\log x = \log\sqrt{3} = \log 3^{\frac{1}{2}} = \frac{1}{2}\log 3 = \frac{1}{2}(0.4771) = 0.2385$

   $\Rightarrow x \approx 1.73$
3. Let $x = 10^{\sqrt{3}}$

   $\log x = \log 10^{\sqrt{3}} = \sqrt{3}\log 10 = \sqrt{3}$

   $\Rightarrow x = \text{anti}\log\sqrt{3} = 10^{\sqrt{3}}$ because $\log 10^{\sqrt{3}} = \sqrt{3}$.

Then let them do Exercise 2.5. This exercise is aimed at assessing students' ability of computing logarithms, identifying the mantissa and characteristic of a given logarithm and their ability of using logarithmic table. Therefore, follow their work very attentively. Check if they are able to find common logarithms of numbers and determine the mantissa and characteristic of a given logarithm. Using different assessment techniques ask them to find the antilogarithm of a given number. Check if they are able to use logarithmic table and compute values of mathematical expressions using logarithm.

If you find students who seem slow in their learning speed, then you can make them do most of the questions given in Exercise 2.5 with your little additional assistance. However, for those who are faster in their learning speed you may choose additional tasks from the ones given below and give them.

Prove the following logarithmic laws: (b, x and y are positive numbers, $b \neq 1$)

a. $\log_b xy = \log_b x + \log_b y$ [Hint: Let $\log_b x = u$ and $\log_b y = v$.
This implies $b^u = x$ and $b^v = y$. Multiply these last two equations to get $xy = b^{u+v}$]

b. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

c. $\log_b x^k = k\log_b x$

$\log_b 1 = 0$ [Hint: Any non zero number raised to 0 is 1]

d. $\log_b b = 1$ [Hint: A number raised to 1 is the number itself]

$\log_a c = \dfrac{\log_b c}{\log_b a}$ [Hint: let $\log_a c = x$. This implies $a^x = c$. Take the base b logarithm of both sides and solve for x]

e. $b^{\log_b c} = c$ [Hint: Take the base b logarithm of both sides and solve for $\log_b c$]

Assessment

Think of always the minimum learning competencies that are expected of the students at the end of the section. In addition to the ongoing assessments you made before and during each lesson, you can use different formal and informal assessment techniques to get feedback about their level of understanding of the subunit.

Ask them to apply the laws of exponents in simplifying exponential expressions. Let them change a given exponential expression to logarithmic expression. Give them several exercise problems to simplify logarithmic expressions, to find logarithm of a number, to determine the mantissa and characteristic of a given logarithm, to find the antilogarithm of a given number and to compute with logarithms.

Oral questions, group works, class activities, quizzes, home works and assignments will help you as continuous assessment techniques to collect relevant data about the performance of the students so that you can assist individual students. Whenever possible, ask them 2 or 3 questions in written form at the end of every class. Have the habit of keeping their records frequently so that you can use it as part of your summative continuous assessment. This will motivate students to attentively listen to the daily lesson and read about the topic in advance.

Check their responses and always give immediate feedback or corrections to their work. Give individual assistance to those who are lagging behind.

You can set questions of the following type to assess the students' level of understanding of the topic "exponents and logarithms:"

1. Explain briefly what is meant by an exponential expression.
2. Which of the following is an exponential expression?

   a. x^5 b. 5^x c. $5x$ d. $\frac{x}{5}$

3. In the power $(-2x)^4$
   - The base is ___________
   - The exponent is ___________
   - The value of the power is ___________
4. Which of the following is (are) true?

   a. $888^{100} \times 888^{50} = 888^{150}$ b. $\frac{888^{100}}{888^{50}} = 888^{100} - 888^{50}$

   c. $888^{100} \times 888^{50} = (888 \times 888)^{150}$ d. $\frac{888^{100}}{888^{50}} = 888^{(100-50)}$

5. The simplified form of $\left(\frac{a^{-2}a^3}{a^4a^2}\right)^{-2}$, $a \neq 0$ is ____________
6. Change the equation $a^k = d$ to logarithmic form.
7. Write an equivalent exponential statement for $\log_7 3 = \frac{1}{3}$
8. Which of the following is false?

   a. $\log_8 99 + \log_8 80 = \log_8(99 \times 80)$ b. $\log_8 \frac{99}{80} = \log_8 99 - \log_8 80$

   c. $\log_{80} 99 = \frac{\log 99}{\log 80}$ d. $8^{\log_{80} 99} = 99$

   e. $\log_8(99-80) = \log_8 99 - \log_8 80$

Answers to Exercise 2.5

1. a. $\frac{5}{4}$ b. $\frac{3}{2}$ c. $-\frac{1}{4}$ d. $m - n$
2. a. The characteristic is –4 and the mantissa is log 4.02
   b. The characteristic is 2 and the mantissa is log 2.03
   c. The characteristic is 0 and the mantissa is log 5.5
   d. The characteristic is 3 and the mantissa is log 2.190
   e. The characteristic is –1 and the mantissa is log 2.5
   f. The characteristic is 0 and the mantissa is log 8
   g. The characteristic is 1 and the mantissa is log 2.3
   h. The characteristic is 1 and the mantissa is log 3.5902

3. a. log 3.12 = 0.4942 b. log 1.99 = 0.2989
   c. log 7.2 = 0.8573 d. log 5.436≈log5.44 = 0.7356
   e. log 0.12 = 0.0792+(–1) f. log 9.99 = 0.9996
   g. log 0.00007 = 0.8451+(–5) h. log300 = 2.4771
4. a. antilog 0.8998 = 7.94 b. antilog 0.8 = 6.31
   c. antilog 1.3010 = 20 d. antilog0.9953 ≈ 9.89
   e. antilog 5.721 = 526000 f. antilog 1.9999 ≈ 100
   g. antilog –6 = 0.000001 h. antilog–0.2 = 0.6310
5. a. 234 b. 1.71 c. 2.68
   d. 0.000524 e. 399 f. 0.577

2.2 THE EXPONENTIAL FUNCTIONS AND THEIR GRAPHS

Periods allotted: 5 Periods

Competencies

At the end of this subunit, students will be able to:

- *define an exponential function.*
- *draw the graph of a given exponential function.*
- *describe the graphical relationships of exponential functions having bases reciprocal to each other.*
- *describe the properties of an exponential function by using its graph.*

Materials Required:

Calculator, Graphs drawn on a big graph paper

Vocabulary: Exponential function, Domain, Range, Asymptote, Intercepts

Introduction

In this subunit, major emphasis is given to the concept of exponential functions, their graphs and the properties of the graphs of exponential functions. So, assist students to define exponential functions and give examples of such functions. Then discuss the respective properties of the function by sketching the graph of each.

Teaching Notes

Let the students begin the lesson by doing Activity 2.6. Give them appropriate explanation and guidance on how to do the activity. In this activity, a single Amoeba cell divides itself into two every hour. That means, the number of cells created after the first, second, third, fourth, fifth and t^{th} hour will respectively be 2, 4 , 8, 16, 32 and 2^t. So the table looks like the following:

Time in hr(t)	0	1	2	3	4	5	t
No of cells(y)	$1 = 2^0$	$2 = 2^1$	$4 = 2^2$	$8 = 2^3$	$16 = 2^4$	$32 = 2^5$	2^t

The formula which helps to estimate the number of cells after t hours is given by $y = 2^t$

Now tell them that $y = 2^t$ is known as an exponential function.

Next, let the students compare the functions $f(x) = 2^x$ and $g(x) = x^2$. Ask them if they can see any difference between the two. They should be able to say that f and g are not the same functions. They must further explain that whether a variable appears as an exponent with a constant base or as a base with a constant exponent should make a big difference. Then tell them that the function g is a quadratic function, whereas the function f is a new type of function called an *exponential function.* Then generalize it like the following:

Functions of the form $f(x) = b^x$, $b > 0$, $b \neq 1$ are called *exponential functions.*

Answer to Activity 2.6

For a and b

Time in hr(t)	0	1	2	3	4	5	t
No of cells(y)	$1 = 2^0$	$2 = 2^1$	$4 = 2^2$	$8 = 2^3$	$16 = 2^4$	$32 = 2^5$	2^t

c. $y = 2^t$

Graphs of exponential functions

Let the students give more examples of exponential functions.

Then consider the graphs of some exponential functions so that students can explore some of the properties of exponential functions. You may start with $f(x) = 2^x$.

Demonstrate to the students how they should first construct a table of values for some integral values in the domain. You also demonstrate how they should join the points by a smooth curve. You can bring a previously drawn graph of $f(x) = 2^x$ and use it as a teaching aid for your demonstration. Let the students closely study the graph of $f(x)= 2^x$ and discuss the properties of the graph [Activity2.7] with the whole class. Use question and answer technique to do Activity 2.7. Make sure that most of the students are participating in doing the activity.

Answer to Activity 2.7

1. The set of real numbers
2. For no value of x (it cannot be negative)
3. No
4. The set of positive real numbers
5. At (0, 1)
6. For all $x > 0$
7. $0 < 2^x < 1$
8. Yes
9. It increases without bound
10. It approaches to 0
11. No
12. The line $y = 0$

Then consider additional examples like the graph of $g(x) = \left(\frac{3}{2}\right)^x$ and discuss its properties in the same way as you did for $f(x) = 2^x$. Finally generalize the properties of all the graphs of $y = b^x$ for any $b > 1$ together with the students.

Use the same treatment for the functions of the form $y = b^x$ for $0 < b < 1$.

Finally, students should note that the graphs of $y = b^x$ for any $b > 1$ have similar shapes and similarly, graphs of $y = b^x$ for any $0 < b < 1$ have also similar shapes, but changed direction.

Remember that the properties of exponential functions require due attention. The domain and the range, the y-intercept of the functions, whether the function is increasing or decreasing, the asymptote of the function, all need their own time of consideration. Let the students be involved in doing Exercise 2.6. You can also apply different assessment techniques (oral questions, group discussions, assignments, tests, etc) to make sure whether your students have understood all these points or not. Before proceeding to the next topic, check if they can draw and list the properties of exponential functions by examining their graphs.

Answer to Exercise 2.6

1. $f(x) = 6^x$; $g(x) = \left(\frac{1}{6}\right)^x$; $h(x) = 2e^x$

2. a. 1.4 b. 1.8 c. 2.8 d. 0.3

3. a.

x	-4	-3	-2	-1	0	1	2	3	4
$y = 3^x$	$\frac{1}{81}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81
$y = \left(\frac{1}{3}\right)^x$	81	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$	$\frac{1}{81}$

Graphs of $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$ using the same coordinate axes.

a.

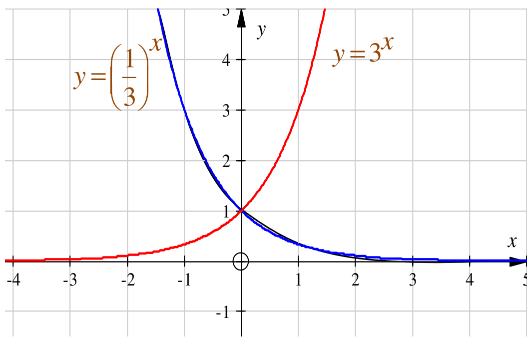


Figure 2.1

b.

x	-3	-2	-1	0	1	2	3
$y = 10^x$	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$	1	10	100	1000
$y=\left(\frac{1}{10}\right)^x$	1000	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$

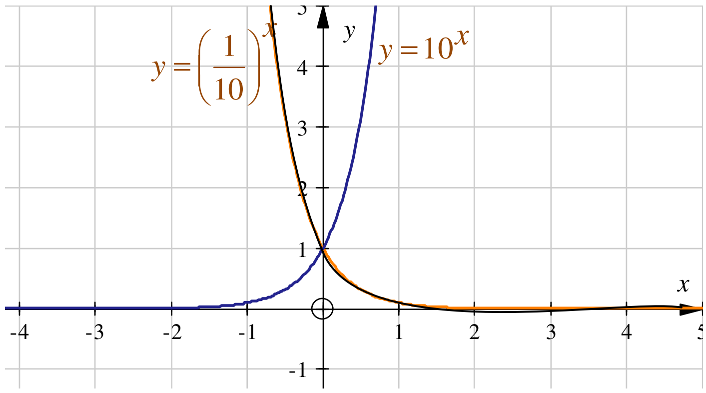


Figure 2.2

c.

x	-4	-3	-2	-1	0	1	2	3	4
$y = 4^x$	$\frac{1}{256}$	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64	256
$y=\left(\frac{1}{4}\right)^x$	256	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$

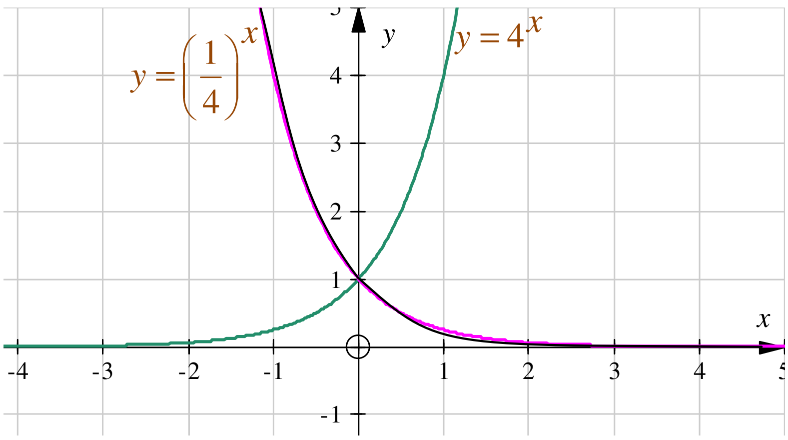


Figure 2.3

4. a. The domain is the set of all real numbers.

   The range is the set of positive real numbers.

b. The y-intercept is 1.

c. $h(x) = 3^x$, $k(x) = 10^x$ and $f(x) = 4^x$ are increasing whereas $g(x) = \left(\frac{1}{3}\right)^x$,

$f(x) = \left(\frac{1}{10}\right)^x$ and $g(x) = \left(\frac{1}{4}\right)^x$ are decreasing functions.

d. The x-axis is the asymptote for the graphs.

The Exponential Function with Base *e*

Introduce the irrational number e and its historical background. Then sketch its graph and discuss the properties (domain, range, whether it is increasing or decreasing) of the natural exponential function $y = e^x$. Use oral question to check their understanding. Then you can group the students and give them Exercise 2.7 as a class work (Neighboring students may work together). Go round the class and check their discussion and work. Give them feedbacks and hints whenever necessary. Make sure that they are correctly constructing table of values and drawing graphs. Check if they are correctly stating the domain and the range. Give them immediate feedback and support.

Answers to Exercise 2.7

1. a.

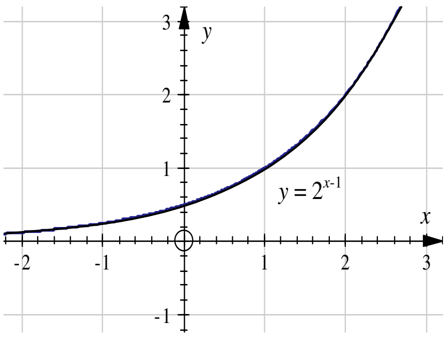


Figure 2.4

b.

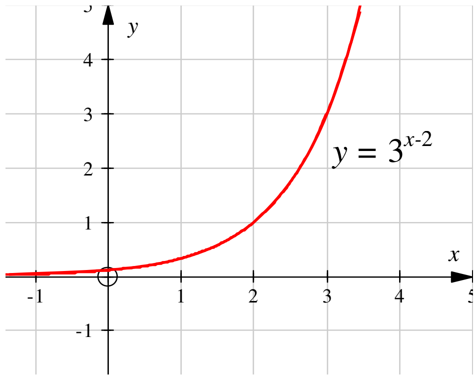


Figure 2.5

c.

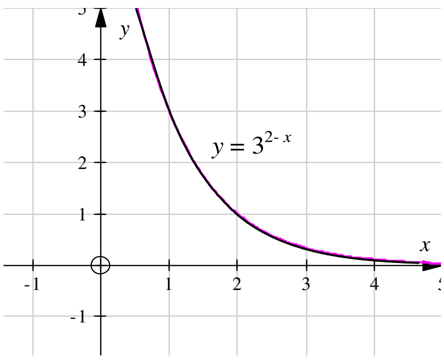


Figure 2.6

2. a. $e^3 = 20.0855369$ b. $e^{\sqrt{3}} = 5.6522337$

   c. $e^{-7.3011} = 0.0006748$ d. $e^{\sqrt{5}} = 9.3564693$

3. a. b.

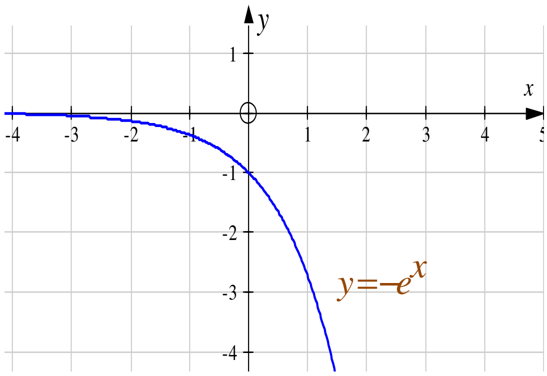


Figure 2.7

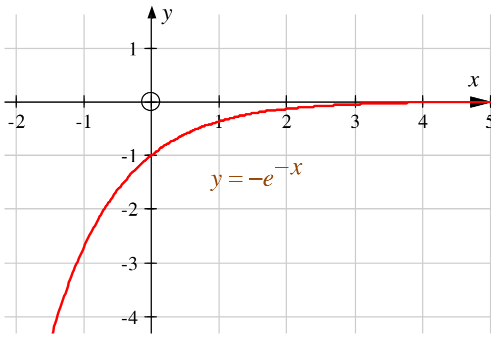


Figure 2.8

c.

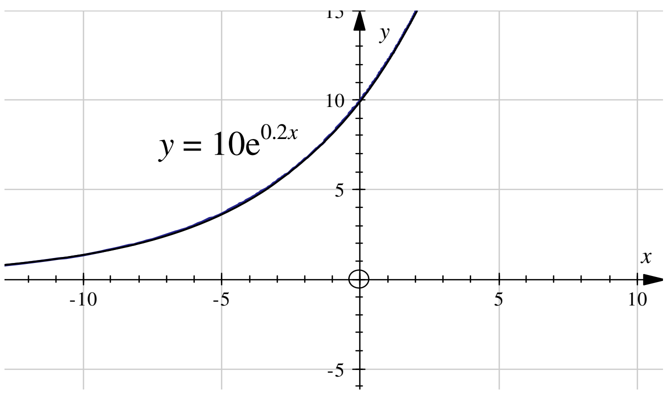


Figure 2.9

4. a. The domain of $y = -e^x$ is the set of real numbers and the range is the set of negative real numbers.

   b. The domain of $y = -e^{-x}$ is the set of real numbers and the range is the set of negative real numbers.

   c. The domain of $y = 10e^{0.2x}$ is the set of real numbers and the range is the set of positive real numbers

Assessment

Remember the minimum learning competencies expected of the students. As part of the assessment technique, let your students define exponential and logarithmic functions. Ask them to draw exponential functions of the form $y = b^x$, where $b > 1$ and $y = b^x$ where $0 < b < 1$ on the same coordinate system. Ask them to list the properties of such exponential functions by examining their graphs. Always check their answers and give immediate feedback. Questions of the following type may also help you for such purpose:

- ✓ Construct tables of values for $y = 6^x$ and $y = \left(\frac{1}{6}\right)^x$
- ✓ Draw $y = 6^x$ *and* $y = \left(\frac{1}{6}\right)^x$ on the same coordinate system.
- ✓ Find the domain and the range of each function.
- ✓ Which function is increasing and which is decreasing?
- ✓ Determine the y - intercept of each.
- ✓ What is the asymptote of each function?

2.3 LOGARITHMIC FUNCTIONS AND THEIR GRAPHS

Periods allotted: 6 Periods

Competencies

At the end of this subunit, students will be able to:

- *define logarithmic function.*
- *draw the graph of a given logarithmic function.*
- *describe the properties of a logarithmic function by using its graph.*
- *describe the graphical relationship of logarithmic functions having bases reciprocal to each other.*
- *describe how the domains and ranges of exponential and logarithmic functions are related.*
- *describe the relationships of the graphs of exponential and logarithmic functions.*

Vocabulary: Logarithmic function, Inverse of logarithmic function, Natural logarithm

Introduction

Meaning of logarithmic functions, their graphs and properties of the graphs of logarithmic functions are among the ones which need due attention in this subunit. After introducing the concept of logarithmic functions, students will be assisted to give their own examples and draw graphs of logarithmic functions. Finally, the properties (the domain, the range, the intercepts, and the behavior) of the graphs of logarithmic functions are presented.

Teaching Notes

Revise the relationship between exponential equation and its corresponding logarithmic equation. Let the students practice changing an exponential expression to logarithmic and vice versa. Then state the formal definition of a logarithmic function as $y = \log_b x$, where $x > 0$, $b > 0$ and $b \neq 1$.

Let your students give more examples of logarithmic functions. Activity 2.8 is a real life problem taken from chemistry. It is to be solved by applying the concept of logarithmic functions. Let students do this activity in groups so that they can attach meaning to what they are going to learn. After their attempt you are expected to show them the proper way of solving the activity. Tell them that the concept of a logarithmic function is applied to solve the activity.

Answers to Activity 2.8

a. Since pH of beer $y = 4.3 < 7$ beer is an acid and
pH of wine $y = 3.4 < 7$, so wine is also an acid

b. $[H^+] = 1.585 \times 10^{-8}$

Next encourage the students to draw the graph of $y = \log_2 x$. First, let them find values of $y = \log_2 x$ for some integral values of x and construct a table of values. Show them how to plot the corresponding points. After the students have drawn the graph, show the whole class the graph of $y = \log_2 x$ which you have already prepared in a relatively large drawing paper as a teaching aid.

Let them do exactly the same for the function g(x) $= \log_{\frac{3}{2}} x$. Draw the graph together with the students.

Next let them do Activity 2.9 in groups. In this activity students should concentrate on the properties of the graphs of the two functions $y = \log_2 x$ and g(x) $= \log_{\frac{3}{2}} x$. Go around the class and check if most of the students are correctly stating the properties of the two graphs. Finally summarize the properties of the two graphs to the whole class.

Answers to Activity 2.9

1. Domain of f = Domain of g = $(0, \infty)$
2. $\log_2 x < 0$ when $0 < x < 1$ and $\log_2 x > 0$ when $x > 1$
3. $\log_{\left(\frac{3}{2}\right)} x < 0$ when $0 < x < 1$ and $\log_{\left(\frac{3}{2}\right)} x > 0$ when $x > 1$
4. Range of f = Range of g = The set of real numbers or $(-\infty, \infty)$
5. $x = 1$
6. The values of $\log_2 x$ and $\log_{\left(\frac{3}{2}\right)} x$ increase when x increases.
7. No
8. The y- axis

Study the properties of more functions like $f(x) = \log_3 x$, $g(x) = \log_{10} x$ so that students can generalize the properties of the functions of the form $f(x) = \log_b x$, where $b > 1$.

In order to study the properties of the graph of the functions $y = \log_b x$, where $0 < b < 1$, first consider $y = \log_{\left(\frac{1}{2}\right)} x$. Graph $y = \log_{\left(\frac{1}{2}\right)} x$ and discuss its properties with your students. Then sketch the graphs of $f(x) = \log_{\left(\frac{1}{10}\right)} x$, $g(x) = \log_{\left(\frac{1}{3}\right)} x$, $h(x) = \log_{\left(\frac{2}{3}\right)} x$.

Finally, students should note that the graphs of $y = \log_b x$ for any $b > 1$ have similar shapes and similarly graphs of $y = \log_b x$ for any $0 < b < 1$ have also similar shapes.

Now it is time for the students to do Exercise 2.8. Pair of students may work together. This exercise helps you assess your students ability of graphing and describing properties of functions of the form $y = \log_b x$ for any $b > 1$ and $y = \log_b x$ for any $0 < b < 1$. Therefore, evaluate their work to make sure that your students have correctly identified the properties of logarithmic functions of $\log_b x$ for $b > 1$ and $0 < b < 1$ from their graphs.

Answers to Exercise 2.8

1. a. b.

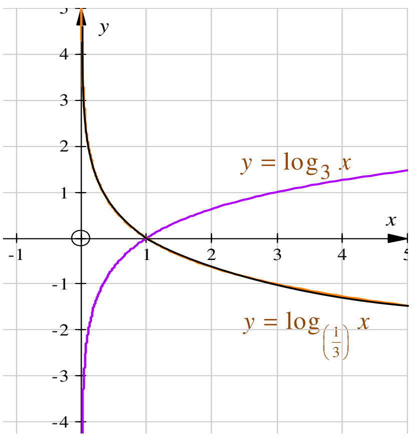


Figure 2.10

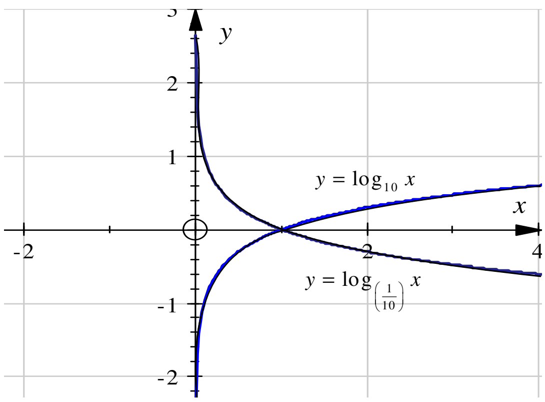


Figure 2.11

2. a The domain is the set of all positive real numbers
   The range is the set of real numbers
   b. The x-intercept is 1

c. $h(x) = \log_3 x$ and $k(x) = \log_{10} x$ are increasing whereas $g(x) = \log_{\left(\frac{1}{3}\right)} x$ and $f(x) = \log_{\left(\frac{1}{10}\right)} x$ are decreasing functions.

d. The y-axis is the asymptote for the graphs.

The Relationship between the Function $y = b^x$ and $y = \log_b x$ ($b > 0$, $b \neq 1$)

You may start the lesson by considering the functions $y = 2^x$ and $y = \log_2 x$ [Activity 2.10]. Let the students study the table of values of the two functions and see how the values of x and y are interchanged. (Give them the table of values of the two functions on the board).

Give them chance to tell the domains and ranges of each. Let them find a relationship between their domains and ranges. Encourage them to draw the graphs of $y = 2^x$ and $y = \log_2 x$ on the same coordinate system. Let them observe the relationship between the two graphs. Do the same consideration for the functions $y = 10^x$ and $y = \log_{10} x$ and $y = \left(\frac{1}{2}\right)^x$ *and* $y = \log_{\left(\frac{1}{2}\right)} x$. Finally, summarize the relationship between $y = b^x$ and $y = \log_b x$ ($b > 0$, $b \neq 1$) as a final remark. Use oral and written assessment techniques to check if the students are able to explain the relationship of the graphs of $y = b^x$ and $y = \log_b x$.

Answers to Activity 2.10

1. If $y = 2^x$, then $x = 2^y$ (Values of x and y are interchanged)
2.

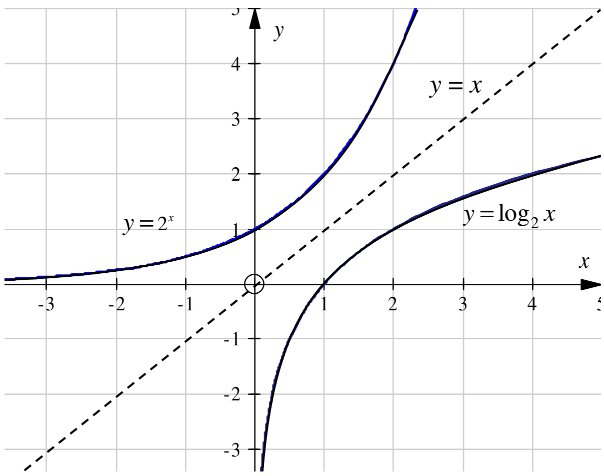


Figure 2.12

3. Domain of $y = 2^x$ = Range of $y = \log_2 x$

4. Graphed in 2.
5. One is the mirror image of the other along the line $y = x$.
6. Serves as a line of reflection

The Natural Logarithm

Introduce $y = \log_e x$ starting from the natural exponential function $y = e^x$. Explain to your students that interchanging x and y in $y = e^x$, gives us $x = e^y$ which is the same as $y = \log_e x$.

$y = \log_e x$ is called the natural logarithm of x and is usually represented by $\ln x$. Then, discuss the graph of $y = \ln x$ and its properties together with your students. Ask the students to draw $y = e^x$ and $y = \ln x$ on the same coordinate system and tell the properties of each.

Then give them Exercise 2.9 as a class work. In this exercise students are required to draw function of the form $y = b^x$ and $y = \log_b x$ $(b > 0, b \neq 1)$ and tell the domain and range of each. So carefully check their work so that you can have information about their level of understanding. Give them the necessary corrections.

Assessment

Use a combination of various assessment techniques to check if your students are able to define logarithmic functions. Assess if they can draw functions of the form $y = \log_b x$ where $b > 1$ and $0 < b < 1$ on the same coordinate system. Check if they are able to list properties of such logarithmic functions. Questions of the following type may help you to assess the above competencies:

1. Draw graphs of each of the following pair of functions on the same coordinate system:
   a. $y = \log_6 x$ and $y = \log_{\frac{1}{6}} x$ b. $y = 6^x$ and $y = \log_6 x$
2. What are the domain and the range of each of the functions given in 1 above?

Try to identify students who seem not to have understood the main points. In addition to the assistance you are giving in each period, think of arranging a makeup or tutorial class for such students. Keep records of their achievements regularly.

Depending on their level of understanding you may also give additional exercises of the following type:

For Slow Learners

Answer each the Following:

1. $4 > 2$. Is $\log_4 16 > \log_2 16$?
2. $9 > 3$ Is $\log_9 81 > \log_3 81$?

3. $\frac{1}{2} > \frac{1}{4}$. Is $\log_{\left(\frac{1}{2}\right)} 16 > \log_{\left(\frac{1}{4}\right)} 16$?

4. $10 < 100$. Is $\log_{10} 100 < \log_{100} 100$?

5. $\frac{1}{9} < \frac{1}{3}$. Is $\log_{\left(\frac{1}{9}\right)} 27 < \log_{\left(\frac{1}{3}\right)} 27$?

The following are optional depending on their performance of 1- 5

6. Let $a > b > 1$. Is $\log_a x > \log_b x$? (for $x > 0$)[*Hint: Refer to 1&2 above*]
7. Let $0 < a < b < 1$. Is $\log_a x < \log_b x$? (for $x > 0$) [*Hint: Refer to 3&5 above*]

For fast Learners

Which of the following statements are true and which are false? [Take $x > 0$]

1. If $a > b > 1$, then $\log_a x > \log_b x$
2. If $a > b > 1$, then $\log_a x < \log_b x$
3. If $0 < a < b < 1$, then $\log_a x < \log_b x$
4. If $0 < a < b < 1$, then $\log_a x > \log_b x$
5. If $a > 1$, then $\log_a x > 1$.
6. If $a > b > 1$, then $\log_a x = \log_b x$ for some x

Answers to Exercise 2.9

1. a b.

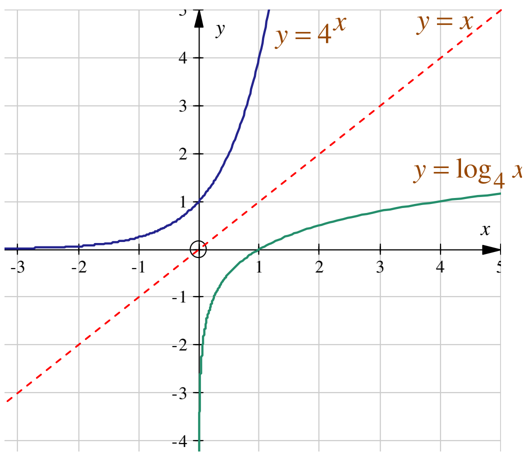


Figure 2.13

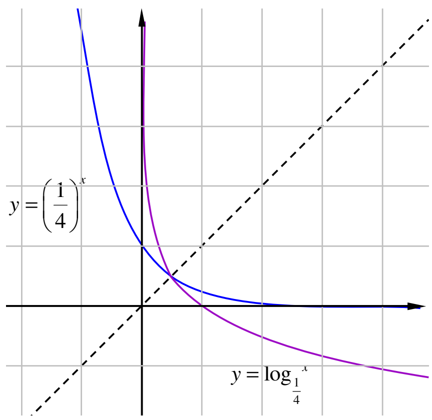


Figure 2.14

c. Domain of f = Range of g
Range of f = Domain of g

d. Domain of h = Range of k
Range of h = Domain of k

2. a. $\frac{1}{3}$ b. -2 c. $3x$ d. 3

3. a. a b. 2 c. $x + y$ d. $x - y$

2.4 EQUATIONS INVOLVING EXPONENTS AND LOGARITHMS

Periods allotted: 7 periods

Competencies

At the end of this sub-unit, students will be able to:

- *solve equations involving exponents.*
- *solve equations involving logarithms.*

Vocabulary: Exponential equation, Logarithmic equation

Introduction

Remember that the main purpose of this subunit is to enable the students to solve equations involving exponents and logarithms. So, throughout the subunit, due attention is given to the techniques of solving exponential and logarithmic functions.

Solving Exponential Equations

Teaching Notes

Start the lesson by helping your students identify exponential and logarithmic equations. Ask them questions like: What kinds of equations are called exponential? What kinds of equations are called logarithmic? Can you give examples?

If students are not able to give examples of their own, then suggest to them a few examples. In order to solve exponential and logarithmic equations, properties of exponents and logarithmic laws play a major role. Therefore, students should refresh their memory with these laws. Ask them to tell you the laws of exponents and logarithms. Write them all on the chalk board. Then begin with simple examples like $3^x = 81$. Explain and give reason for each step in the process of solving the equation. Let students solve more equations. Choose equations from Exercise 2.10 for fast and slow learners. For example, slow learners may do the first question whereas fast learners can do both number 1 and 2.

Let the students do some of the equations from Exercise 2.10 as a class work and the remaining as a home work. Check their work. Give them appropriate feedback. You can even give a value for their work.

Answers to Exercise 2.10

1. a. 4 b. -1 c. $\frac{25}{3}$ d. $\frac{3}{4}$

   e. -4 f. -2 or 2 g. -2 or 1 h. -2 i. $\frac{1}{6}$

2. a. 3.91 b. 1.1553 c. 0.3213 d. 14.20
   e. 0.2994 f. 1.4827 g. 0.9083 h. 2.0710

Solving Logarithmic Equations

Before solving logarithmic equations revise the laws of logarithms. Let few students come out and write them all on the chalk board. Then begin solving logarithmic equations with simple examples like $\log_3 x = 4$. Explain and give reason to each step during solving the equation. Let students solve more equations. You can solve the equations given as an example in the text book.

When solving logarithmic equations students should take into account the universe for which every term in the equation is valid. They should note that, $\log_b x$ is valid when $x > 0$, $b > 0$ and $b \neq 1$.

Similarly, $\log(x - 3)$ is valid when $x - 3 > 0$, i.e. when $x > 3$.

Let students do Exercise 2.11(a – f) in the class. Give exercise 2.11(g – m) as a home work and question number 2 can be given as an assignment. Check their work and give them the necessary correction every time.

Answers to Exercise 2.11

a. a. $U = \left(\frac{1}{2}, \infty\right)$; $x = 122$ b. $U = (0, \infty)$; $x = \frac{1}{8}$

c. $U = (-\infty, 0) \cup (2, \infty)$; $x = -1$ or 3

d. $U = (-\infty, -2) \cup (-1, \infty)$; $x = 0$ or -3

e. $U = (-\infty, -1) \cup (0, \infty)$; $x = \frac{1}{7}$ f. $U = (1, \infty)$; $x = \frac{11}{3}$

g. $U = (11, \infty)$; $x = 21$ h. $U = (1, \infty)$; $x = \frac{13}{8}$

i. $U = (0, \infty)$; $x = \frac{2}{3}$ j. $U = (0, \infty)$; $x = 2$

k. $U = (-1, \infty)$; $x = 0$, (NB $-4 \notin (-1, \infty)$) l. $U = \left(\frac{5}{3}, \infty\right)$; $x = 2$

m. $U = (-6, 1) \cup (1, \infty)$; $x = 3$

2. a. $\frac{1}{5}$ b. 5 (NB. $-5 \notin (0, \infty)$) c. 1 (NB. $-2 \notin (0, \infty)$)
   d. 4 e. -2 f. 7

Assessment

Use a combination of different assessment techniques (oral question, group work, group discussion, quiz, class work, home work, test, assignment, etc.) to make sure that your students are able to solve equations involving exponents and logarithms. Make sure that students can state the universe correctly and solve the given logarithmic equation.

To assess such competencies you may also give them exercise problems like the ones given below: Check their answers and give them immediate feedback.

Solve each of the following equations:

a. $2^{(x-1)}=8$ b. $9\left(3^{1-x}\right)=81$ c. $6^{\left(x^2+x\right)}=36$

d. $\log_3(2x-1)=4$ e. $\log_5(7x-2)=2$ f. $\log_4(x+3)-\log_4 x=1$

2.5 APPLICATIONS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Periods allotted: 6 periods

Competency

At the end of this sub-unit, students will be able to:

- *solve problems involving exponential and logarithmic functions from real life.*

Vocabulary: Population growth, Compound interest

Introduction

This is a sub-unit where students will be helped and encouraged to solve problems involving exponential and logarithmic functions from real life. So, problems from appropriate fields of study are selected and solved to show the students how exponential and logarithmic functions are applied in solving real life problems.

Teaching Notes

This is a good opportunity to show to your students the real life applications of exponential and logarithmic functions. Give orientations that exponential and logarithmic functions are used in describing and solving a wide variety of real-world problems. You can mention to them some of the areas of applications like estimating population growth, calculating compound interests and decay of a substance, etc. Pick simpler real life problems and solve them together with the students. Let the students identify the given and required ones in the problem. It is a good practice to use questioning and answering technique while solving problems as this technique enables the students to actively participate. First, do all of the examples given in the text book. Then, group the students and make them do Group work 2.6. Explain to them the essence of the activity. Ask them orally if this is an application of exponential or logarithmic function. What is given and what is required should clearly be identified first. After you have checked their attempt you are expected to solve the activity properly.

Answers to Group work 2.6

Temperature of an object at time t is given by $f(t) = cer^t + a$

Let us assume room temperature $a = 22°c$.

Therefore, $c = 95°c - 22°c = 73°c$. Hence, $f(t) = 73\,e^{rt} + 22$

$\Rightarrow f(5) = 65$ gives $73e^{5r} + 22 = 65$

$$73e^{5r} = 43 \Rightarrow e^{5r} = \frac{43}{73} \Rightarrow r \approx -0.1059$$

Therefore, the temperature function is $f(t) = 73e^{-0.1059t} + 22$.

The solution to the problem is a number t such that $f(t) = 40$.

Thus, $73e^{-0.1059t} + 22 = 40 \Rightarrow t \cong 13.22$. the tea will be drinkable approximately 13 minutes after it is brewed.

Give Exercise 2.12 as a group assignment for your students. Let them submit their work and arrange to them a special class to present their assignment. Correct their work and give it a certain value. Give them feedbacks and corrections for their work.

Assessment

Students must be assessed in terms of whether they are able to solve problems involving exponential and logarithmic functions from real life. Therefore, give them several real life exercise problems like the ones given in Exercise 2.12 and in the review exercises. Check their answers and give appropriate feedback.

Answers to Exercise 2.12

1 a. $f(t) = 3^t$.

b. The number of cells after one hour**:** $f(60) = 3^{60}$

c. It would take 10.48 minutes, rounded, for the population (number of cells) to reach 100,000.

2. a The population formula is $f(t) = 100{,}000\left(\frac{1}{2}\right)^t$

b. The number of cells after 10 minutes is almost 98.

c. It would take a little more than 13 minutes for the population of cells to reach 10.

3. Balance at the end of 10 years is rounded to Birr 3,300.39.

4. It will take about 29.61 minutes for the cell population to drop below a 1,000 count.

5. $y = 50 - 50e^{-0.3t} = y = 50 - 50e^{-0.3(9)} = 46.64$. The student is expected to learn about 47 words on the ninth hour.

6. $M = \frac{2}{3}\log\frac{E}{E_0} = \frac{2}{3}\log\frac{5.96\times10^{16}}{10^{4.40}} = \frac{2}{3}\log\left(5.96\times10^{11.6}\right) = \frac{2}{3}(0.775+11.6) = 8.25$

7. a. $L = 10\log\frac{I}{I_0} = 10\log\frac{3.2\times10^{-6}}{10^{-12}} = 10\log\left(3.2\times10^{6}\right) = 65\text{decibels.}$

b. $L = 10\log\frac{I}{I_0} = 10\log\frac{5.2\times10^{3}}{10^{-12}} = 10\log\left(5.2\times10^{15}\right) = 157\text{decibels.}$

After treating the particular topic or subunit, you can give part of the review exercises as group project work or assignment. The assignment will be submitted and, if possible, presented by the group members in a specially arranged tutorial class.

Answers to Review Exercises on Unit 2

1. a. 32 b. -32 c. $\frac{1}{32}$ d. $-\frac{1}{32}$
e. $\frac{4}{9}$ f. $\frac{9}{4}$ g. $\frac{9}{4}$ h. $\frac{4}{9}$

2. a. 2^7 b. 6 c. $(\sqrt{8})^3$ d. $\frac{1}{(ab)^3}$ e. $16n^{10}$
f. $\frac{x^2}{4y^2}$ g. $\frac{1}{d^2}$ h. $\frac{1}{x^6}$ i. e^{2x+3} j. 3^{2x-1}
k. 5 l. $2^{xz}3^{yz}$

3. a. $3^4 = 81$ b. $25^{\frac{1}{2}} = 5$ c. $2^{-2} = \frac{1}{4}$ d. $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$

4. a. 32 b. 2 c. 1 d. 4
e. 2 f. –2 g. $-\frac{1}{2}$
h. $\log_x 1000 = \frac{3}{2};\quad \frac{\log 1000}{\log x} = \frac{3}{2};\quad \frac{3}{\log x} = \frac{3}{2};\quad 3\log x = 6;\quad x = 100.$

5. a. $\log_{10} 50$ b. $\log_5 6$ c. $\log_3 (\frac{125}{49})$
d. $\log_a x^5 y^3$ e. $\log_a x^{\frac{8}{3}}b$ f. $\ln x^{\frac{5}{2}}$

6. a. 0.6243 b. -4.4×10^{-3} c. 0.9138
d. 0.6149 e. 0.5315+(–1) f. 0.9484
g. –5 h. 2.6990

7. a. 2.62 b. $\cong 2.24$ c. 3.49 d. 7.64
e. 941000 f. $\cong 57$ g. 0.0000000001
h. 0.50118

8. a. Domain = $(-\infty, \infty)$, Range = $(0, \infty)$

b. Asymptote : $y = 0$ (x-axis)

c. It is an increasing function.

d. The y intercept is at (0,1).

e. For all $x > 0$

f. It is less than 1 (between 0 and 1).

g. The functional value is never less than zero.

9. a. Domain = $(-\infty, \infty)$, Range = $(0, \infty)$

b. Asymptote : $y = 0$ (x-axis)

c. The function is a decreasing function.

d. The y-intercept is at (0, 1).

e. For all $x < 0$

f. It is less than 1 (between 0 and 1).

g. The functional value is never less than zero.

10. a.

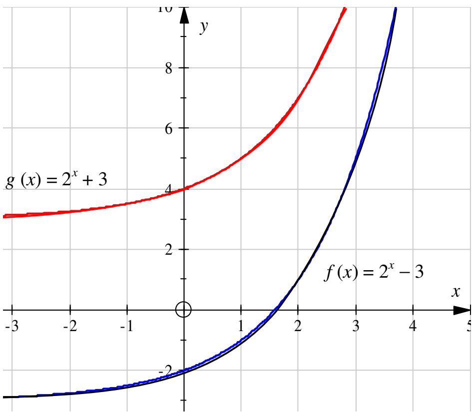


Figure 2.15

b.

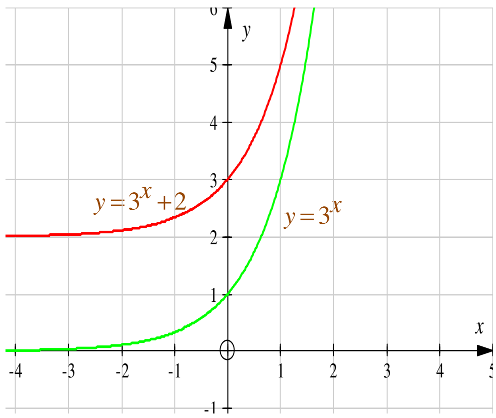


Figure 2.16

c. d.

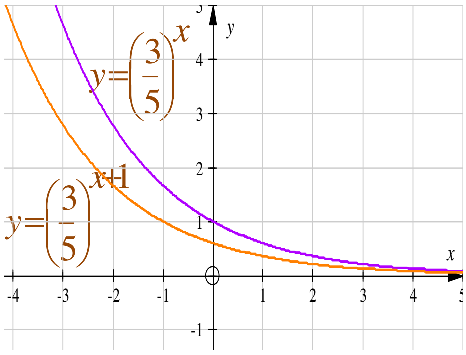


Figure 2.17

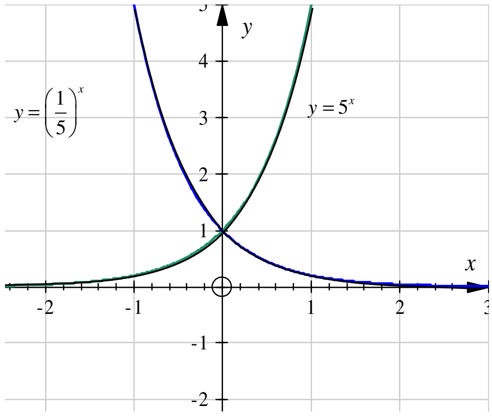


Figure 2.18

11. a. Domain = $(0, \ \infty)$ Range = $(-\infty, \infty)$ b. Asymptote: $x = 0$ (y- axis)
    c. It is an increasing function. d. The x-intercept is at $(1, 0)$.
    e. For all $x > 1$ f. When $0 < b < 1$

12. a. b.

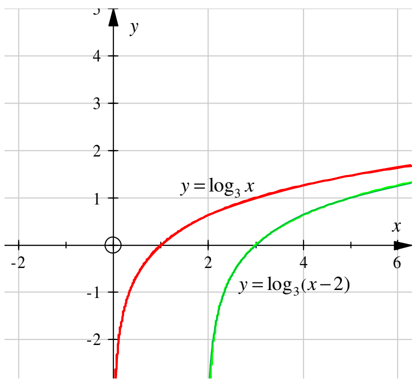


Figure 2.19

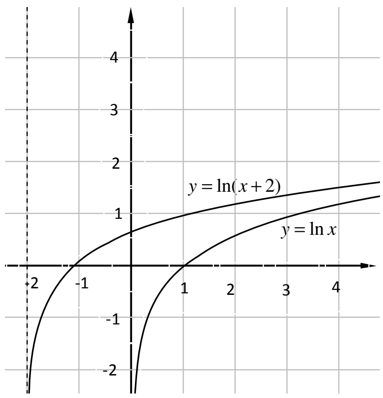


Figure 2.20

c. d.

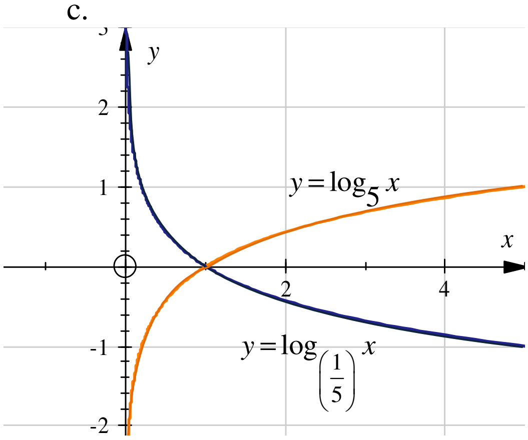


Figure 2.21

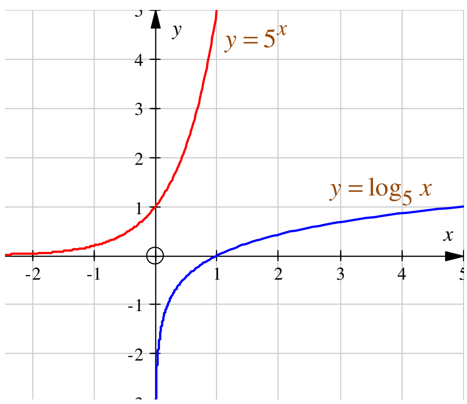


Figure 2.22

13. a. $U = (0, \infty)$ b. $U = (-3, \infty)$

c. $U = (-\infty, 3)$ d. $U = \left(\frac{12}{7},\ \infty\right)$

e. $U = (-3, 3)$ f. $U = (-\infty, 0) \cup (2, \infty)$

14. a. 3 b. -1 c. $\frac{3}{4}$ d. $\frac{25}{3}$

e. $\frac{1}{10}$ f. -2 or 1 g. -2 h. $\frac{1}{3}$

15. a. $x = 27$ b. $x = 64$

c. $x = e$ d. $x = 300$ [NB. $0 \notin U$]

e. $x = 2$ [NB. $-6 \notin U$] f. $x = 1$

g. $x = \frac{3+\sqrt{13}}{2}$ [NB. $\frac{3-\sqrt{13}}{2} \notin U$] h. $x = 2$

i. $x = 6$ j. $x = 32$

k. $x = \frac{1}{8}$

16. Amount after 5 years = 2433.31 Birr

17. a. 1050 b. 1102.5 c. 1157.625

d. 1628.895 e. $1000\,[1.05]^n$

18. Let town A and B have the same population after t years.

So $8.25 \times 10^7(1+5.2\%)^t = 1.11 \times 10^8(1+2.6\%)^t$

$$\frac{8.25\times10^7(1+5.2\%)^t}{1.11\times10^8(1+2.6\%)^t} = 1; \quad \frac{8.25\times10^7}{1.11\times10^8}\left(\frac{1.052}{1.026}\right)^t = 1;$$

$$\left(\frac{1.052}{1.026}\right)^t = \frac{1.11\times10^8}{8.25\times10^7}; \ (1.025)^t = \frac{11.1}{8.25}$$

$$t\log 1.025 = \log\frac{11.1}{8.25}; \quad t = \frac{\log\frac{11.1}{8.25}}{\log 1.025} \cong 12 \text{ years}$$

19. Suppose the depreciated value at the end of 10 years is V_t Birr. Then, using the formula $V_t = v_o\left(1 - \frac{r}{100}\right)^t$, we have $V_t = 30000\,(1 - 0.05)^{10}$

$\Rightarrow \log V_t = \log 30000\,(1 - 0.05)^{10}$

$= (0.4771 + 4 + 10(0.9777 - 1) = 4.\,2541$

$\therefore V_t = \text{antilog}\,(4.2541) = 18000$

Hence the required depreciated value is 18000 Birr.

UNIT 3 SOLVING INEQUALITIES

INTRODUCTION

This unit is designed to review inequalities. The unit gives more emphasis to solving inequalities involving absolute value, system of linear inequalities in two variables and quadratic inequalities. It presents methods of solving quadratic inequalities. In general, the concepts discussed in this unit enable students to solve the types of inequalities stated above and to perform some of their application.

Unit Outcomes

After completing this unit, students will be able to:

- *know and apply methods and procedures in solving problems on inequalities involving absolute value.*
- *know and apply methods in solving systems of linear inequalities.*
- *apply different techniques of solving quadratic inequalities.*

Suggested Teaching Aids in Unit 3

You can present different charts and graphs to demonstrate graphical solution to system of inequalities and quadratic inequalities.

You can also encourage students to prepare different representative graphs of system of linear inequalities and quadratic inequalities by themselves.

Apart from use of student textbook, you need to elaborate more real life problems from your surroundings so that students can best appreciate and see how useful systems of linear inequalities are in particular.

You can group students, give them hints on problems and let them assess such problems from their daily life to develop their mathematical form.

3.1 INEQUALITIES INVOLVING ABSOLUTE VALUE

Periods allotted: 4 periods

Competencies

At the end of this subunit, students will be able to:

- *describe sets using interval notation.*
- *solve inequalities involving absolute value of linear expression.*

Vocabulary: Open interval, Closed interval, Absolute value, Infinity, Solution set.

Introduction

This sub-unit mainly deals with inequalities involving absolute value. It starts by revising different ways of description of set. Activities and group work are presented for the purpose of revision. The concept discussed in this sub-unit enables students to solve inequalities involving absolute value.

Teaching Notes

Students are expected to have background on set description, intervals and equation involving absolute value. It is advisable to encourage students through questions and answers, to revise ways of set description such as complete listing method, partial listing method and set builder method, open interval and closed interval.

After pointing out these, let students write the solution set of the inequalities $2x - 1 < 3$ and $2 - x \leq 5$ in interval form. You may now ask students to do Activity 3.1 in the student textbook on their own independently so that they can practice to solve. By rounding, you need to see their work and give extra help to students who are lagging behind.

For fast learner students, you might give additional questions such as the following:

1. Find the value of x satisfying the following inequalities.

a. $\frac{1}{3}-2x<x+1$ b. $\frac{2}{5}(x-1)<x-(2x+1)$

c. $-3(2x-1)\le 2(5-2x)$ d. $-\frac{1}{2}(x-6)<\frac{1}{2}x+2$

e. $-3\left(\frac{1}{2}x-\frac{1}{4}\right)>\frac{x}{2}-\frac{1}{4}$ f. $-1<3-2x<9-x$

g. $\frac{1}{2}x-\frac{1}{3}\ge\frac{1}{6}<\frac{3}{2}x<2$ h. $\frac{3}{4}-\frac{1}{4}x\le x-1\le\frac{2}{7}x-\frac{1}{10}$

2. Solve each of the following inequalities and write the solution set in interval form.

a. $2x-3\le 5$ and $x-1>0$ b. $\frac{3}{4}x-9$ and $-\frac{1}{3}x\le\frac{-1}{5}$

c. $\frac{1}{2}x-\frac{1}{3}\ge-\frac{1}{6}$ or $\frac{2}{7}x-\frac{1}{7}<x$ d. $-1<3x+2\le 5$ or $\frac{3}{2}x-6>9$

e. $-3<\frac{x-1}{2}<5$ and $-1<\frac{1-x}{2}<2$ f. $-3<\frac{3x-1}{5}<\frac{1}{2}$ and $\frac{1}{3}<\frac{3-2x}{6}<\frac{9}{2}$

Answers to Activity 3.1

1. If it is possible to list all elements of a set within braces, { }, then it is called complete listing method.

   If a set consists of many elements that cannot be completely listed but have a certain pattern, then we list a few elements and we use three dots either before or after or both before and after listing a few elements. Such a method is called partial listing method. (Let students add some more examples about partial listing method and give them some exercises as homework). If a statement is used to describe the type of elements that belong to a set, then the set is said to be described in set-builder method. (Give some more exercises to students as homework). If a set A consists of elements satisfying a property say P, then A = {x: x had property P} is a set-builder method.

2. The set:
   a. {–3, 1, a, b, 5} is described in complete listing method.
   b. { . . ., –2, –1, 0, 1, 2, . . . }, { . . ., –5, 0, 5, 10 } and { 1, 2, 3, . . .} are described in partial listing method.
   c. $\{x : x = 2n + 1 ; n\in\mathbb{Z}\}$ is described in set-builder method
3. a. {–3, –2, 1, 0, 2, 3} complete listing method
   b. {. . . , –4, –2} partial listing method or
   $\{x: x = 2n ; n$ is a negative integer$\}$ in set-builder method
   c. {7, 8, 9, . . . , 49 } partial listing method.
4. a. $\{n: n\in\mathbb{Z}\}$ b. $\{m: m = 3n; n\in\mathbb{W}\}$
   c. $\{x: -3\le x<5$ and $x\in\mathbb{R}\}$ d. $\{x: x\ge 2$ and $x\in\mathbb{R}\}$

5. a. $(-\infty, 0) \cup (0, \infty)$ b. $[-1, 2]$
   c. $(0.2, 0.8]$ d. $(-\infty, -1) \cup (-1, \infty)$
6. You may ask the students to try question number 6 in groups.
   a. $x < 4$ and the solution set is $(-\infty, 4)$
   b. $3 \le -x < 4$ implies $-4 < x \le -3$ or $(-4, -3]$

After conducting and doing Activity 3.1 with the active participation of the students, encourage students through question and answer to revise absolute value geometrically. After deliberation by students, you may give Activity 3.2 as class work. By rounding, give them a chance to work and give the correct answer by giving chance for each student to participate.

For interested students, you can give additional exercise questions such as:

If $x = -\frac{2}{3}$ and $y = \frac{1}{2}$, then evaluate each of the following.

a. $|2x + 6y|$ b. $\left|\frac{1}{3}x + \frac{2}{3}y\right| - 1$

c. $3\,|x - 1| + 4\,|2y + 1|$ d. $\left|\frac{9}{8}y - 17x\right|$

Answers to Activity 3.2

1. On a number line $|x|$ is the distance between the point with the coordinate x and the origin. You know that distance between two distinct points is measured by positive number. Therefore, it is always true that $|x| \ge 0$. If $x = 0$, then $|x| = 0$.
2. a. 3 b. 0 c. $\sqrt{5}$ d. 6
   e. $\sqrt{2} - 1$ (From this, you can conclude that $|a - b| = b - a$ if $a < b$.
   f. $\sqrt{5} - \sqrt{3}$ since $\sqrt{3} < \sqrt{5}$
3. a. 9 b. 15 c. 13
4. Let $x = 7$ and $y = 3$
   a. $|7-3| = 4 = |3-7|$ so, $|a-b| = |b-a|$ for $a = 7$ and $b = 3$

   To check from b) to e) take $x = -2$ and $y = 3$

   b. $|2(-2) - 3(3)| = |3(3) - 2(-2)|$
   $|-4 - 9| = |9 + 4|$
   $13 = 13$
   So, $|2x - 3y| = |3y - 2x|$

   c. $\sqrt{(-2)^2} = |-2|$
   $\sqrt{4} = 2$
   So, $\sqrt{x^2} = |x|$

d. $|-2|\ |3| = |-2\times 3|$

$2\times 3 = |-6| = 6$

So, $|x|\ |y| = |xy|$

e. $\left|\frac{-2}{3}\right| = \frac{|-2|}{|3|}$

$\frac{2}{3} = \frac{2}{3}$. So, $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

After practicing the rules with numbers, you can let them do the Examples 2, 3 and 4 so that they can solve equation involving absolute value.

Right after, with students participation, assist them to tell the reason for each of the steps when they solve questions such as Example 3 in the student textbook. You may then give Group work 3.1 as a group work so that each student can play an important role in the group discussion.

Answers to Group work 3.1

1. $a < 0$ and $b > 0$

a. $a - b < 0$ so, $|a-b| = -(a-b) = b-a$

b. since $a < 0$ and $b > 0$

$\Rightarrow ab < 0$

Thus $|ab-a| = \begin{cases} a-ab; \text{if } |ab| > |a| \\ ab-a; \text{if } |ab| < |a| \end{cases}$

c. $\frac{b}{a} < 0$. Hence $\left|\frac{b}{a}\right| = -\frac{b}{a}$

Allow them to try i. $|a-1|$ ii. $|a^2-ab|$

2. a. For $a \geq 0$, $|a| = a$. So $a = |a|$

For $a < 0$, $|a| = -a > 0$. Here $a < 0 < |a|$

$\therefore$ For any a, $a \leq |a|$

b. $a \leq |a|$ and $-a \leq |a|$

$-a \leq |a| \Rightarrow a \geq -|a|$ That is, $-|a| \leq a$. So, $-|a| \leq a \leq |a|$

3. a. $|x+y|^2 = (x+y)^2 = x^2 + 2xy + y^2 \leq x^2 + 2|xy| + y^2$

$x^2 + 2|x||y| + y^2 = |x|^2 + 2|x||y| + |y|^2$

Since $x^2 = |x|^2$ and $y^2 = |y|^2$

$\therefore |x+y|^2 = |x|^2 + 2|x||y| + |y|^2 = \left||x|+|y|\right|^2$

Taking square root of both sides, $|x+y| \leq |x| + |y|$

b. $|x| = |x - y + y| \leq |x - y| + |y|$ by (a)

$\Rightarrow |x - y| \geq |x| - |y|$

4. a. $x \leq \frac{15}{4}$ or $S = \left(-\infty, \frac{15}{4}\right]$

b. $x = 0$, $x = -3$ or S = {(–3, 0)}

c. $x < 4$ or $x > 12$ or S = $(-\infty, 4) \cup (12, \infty)$

d. This is for your excellent students. The solution needs cases on $2x - 1$.

Case 1: When $2x - 1 \geq 0$ i.e $x \geq \frac{1}{2}$

$$|2x - 1| = 2x - 1 < x + 3 \Rightarrow x < 4$$

$$\therefore S_1 = \left[\frac{1}{2}, 4\right)$$

Case 2: When $2x - 1 < 0$ i.e $x < \frac{1}{2}$

$$|2x - 1| = 1 - 2x < x + 3 \Rightarrow x > -\frac{2}{3}$$

$$\therefore S_2 = \left(-\frac{2}{3}, \frac{1}{2}\right). \text{Thus, } S = \left(-\frac{2}{3}, 4\right)$$

To consolidate what they have learned, you can give them Exercise 3.1 as an assignment. The purpose of this exercise is to check what students know, understand and what they can do.

The following questions can be given as additional exercise problems.

Solve each of the following absolute value inequalities.

a. $|x^2 - 2x + 1| < 0$ b. $-5\,|3x - 5| + 4 \leq 19$

c. $\left|\frac{2}{3}x - 8\right| < 0$ d. $\left|\frac{1}{3}x - \frac{3}{5}\right| < 1\frac{1}{4}$

e. $\left|\frac{2}{5} - \frac{5}{4}x\right| - -\frac{2}{3} < 1$ f. $|3 + 2x| > |-2x|$

Assessment

You need to assess students in order to be informed about their progress. Making assessment will enable you to find out what aspect of the lesson is difficult. Assessment will also give you a clear picture of the knowledge and skill of the students. So, at the end of this lesson, you can use class activities, group discussion, assignments, exercise problems and quiz or test in order to assess students.

Answers to Exercise 3.1

1. a. $(-\infty, -2) \cup (-2, \infty)$ b. $[1, 4]$ c. $(-1, \infty)$

d. $\left(-\infty, \frac{18}{5}\right]$ e. $\left[-\frac{3}{7}, \infty\right)$ f. $(-\infty, 1)$

2. a. $\{x : x \le -5\}$ b. $\left\{x : x > \frac{5}{2}\right\}$ c. $\{t : t > 4\}$

3. $y = 15 + x$ and $x + y \le 85$. Substituting $y - 15 + y \le 85 \Rightarrow y \le 50$;
$\therefore y \in (15, 50]$

4. a. 5 b. 0. c. 4 d. $\frac{36}{7}$

5. a. $\left\{-\frac{13}{3}, \frac{1}{3}\right\}$ b. $\left\{-\frac{6}{5}, \frac{12}{5}\right\}$ c. $\{\ \}$

d. $\left\{\frac{7}{2}\right\}$ e. $\{-1, 5\}$ f. $\left\{-\frac{5}{6}, \frac{1}{2}\right\}$

6. a. $\left\{x : -\frac{2}{5} \le x \le \frac{4}{5}\right\}$ or $\left[\frac{2}{5}, -\frac{4}{5}\right]$ b. $\{x : -2 < x < 2\}$ or $(-2, 2)$

c. $\left\{x : x \le -\frac{1}{3} \text{ or } x \ge \frac{2}{3}\right\}$
or $\left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{2}{3}, \infty\right)$

d. $\left\{x : x < \frac{1}{2} \text{ or } x > \frac{11}{2}\right\}$
or $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{11}{2}, \infty\right)$

e. $\left\{-\frac{5}{3}\right\}$ f. $(-\infty, \infty)$ or
because always $|x - 1| \ge 0$.

7. a. $\left\{x : \frac{-c-b}{a} < x < \frac{c-b}{a}\right\}$, if $a > 0$ and
$\left\{x : \frac{c-b}{a} < x < \frac{-c-b}{a}\right\}$, *if* $a < 0$

b. $\left\{x : \frac{-c-b}{a} \le x \le \frac{c-b}{a}\right\}$, if $a > 0$ and
$\left\{x : \frac{c-b}{a} \le x \le \frac{-c-b}{a}\right\}$, *if* $a < 0$

c. $\left\{x : x < \frac{-c-b}{a} \text{ or } x > \frac{c-b}{a}\right\}$, if $a > 0$ and
$\left\{x : x < \frac{c-b}{a} \text{ or } x > \frac{-c-b}{a}\right\}$, if $a > 0$.

d. $\left\{x : x \le \frac{-c-b}{a} \text{ or } x \ge \frac{c-b}{a}\right\}$, if $a > 0$ and
$\left\{x : x \le \frac{c-b}{a} \text{ or } x \ge \frac{-c-b}{a}\right\}$, if $a < 0$.

3.2 SYSTEMS OF LINEAR INEQUALITIES IN TWO VARIABLES

Periods allotted: 4 periods

Competency

At the end of this subunit, students will be able to:

- *solve system of linear inequalities in two variables by using graphical method*

Vocabulary: Linear equation, Linear inequalities, Region, Coordinate system.

Introduction

This sub-unit presents system of linear inequalities in two variables. Graphical method to determine a solution to such types of inequalities is elaborated with illustrative examples. Practical and application problems involving system of linear inequalities are presented in this sub-unit so that students will be able to solve some of the practical problems involving such inequalities.

Teaching Notes

You may start the lesson by encouraging students through question and answer, to revise concepts such as ordered pairs, linear equation, and system of linear equation and let them illustrate with examples.

After discussing these concepts, let students do Activity 3.3 on their own in the class. By rounding, check their work or performance in doing this activity. The purpose of this activity is to help the students to revise ordered pairs, linear equation and systems of linear equation they studied in grade 9.

For fast learner students, you can give the following questions to solve graphically.

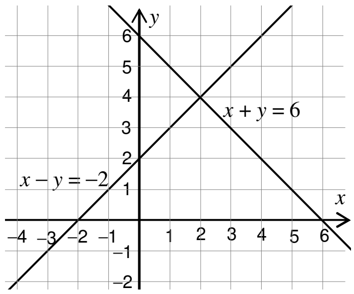
a. $\begin{cases} 1-2x > y \\ x > y \\ x+y < 1 \end{cases}$ b. $\begin{cases} x > 0 \\ y < 1 \\ x-2 \geq 1-y \end{cases}$ c. $\begin{cases} 2 < x-y \\ y < 4 \\ x > -1 \end{cases}$

d. $\begin{cases} 1-3 < 4+x < 0 \\ -4 < y-2 \leq 2 \end{cases}$ e. $\begin{cases} -\frac{1}{2} < y-1 < \frac{2}{3} \\ 0 < \frac{2}{3}x+1 < 3 \end{cases}$ f. $\begin{cases} x+y > 1 \\ 2x < y-1 \\ y-3x \leq 3 \end{cases}$

Answers to Activity 3.3

1. The solution set of two linear equations, if their graphs do not intersect, is empty set.

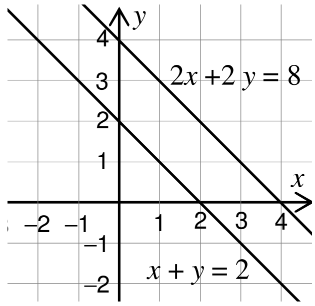
2. a.



S.S = {(2, 4)}

Figure 3.1

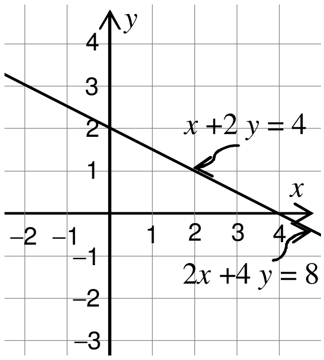
b.



S.S = { }.

Figure 3.2

c.



S.S={(x, y): $2x + 4y = 8$} or S.S={(x, y): $x + 2y = 4$}

Figure 3.3

3. The ordered pairs $(-1, -1)$, $(0, 0)$ and $(1, 1)$ are belonging to the relation R.

4.

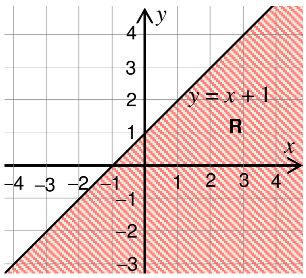


Figure 3.4

5. a.

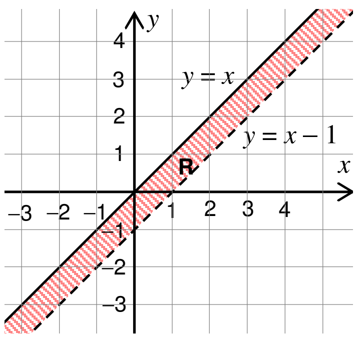


Figure 3.5

b.

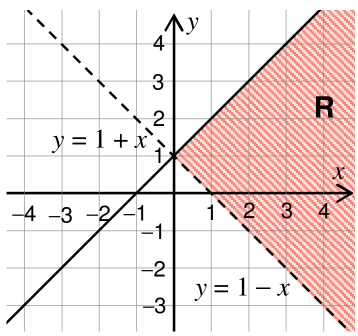


Figure 3.6

6. Try to make the students sketch the graph of the y terms.

   It would be good if you ask them to do a) and b) as assignment. Graphic solution is more appropriate.

   a. Solution set $S=\{(x, y): -1 \le x \le 3\}$ and $y \ge 0 = [-1, 3] \times [0, \infty)$

   b. $S=\{(x, y): y > x-3 \text{ for } x \ge 2\} = [2,\infty)\times[-1,\infty)$

After ensuring the ability of students in conducting Activity 3.3, encourage them by presenting and discussing Example 2 and 3 in the student textbook. You may then give Activity 3.4 as class discussion by giving each student a chance to participate.

Answers to Activity 3.4

1. The ordered pairs $\left(-\frac{1}{4},\frac{3}{4}\right),\left(\frac{1}{4},\frac{5}{4}\right),\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{3}{2}\right),\left(\frac{3}{2},\frac{1}{2}\right),(0, 1),(1,2),(1,0)$,$(2,0)$ and $(2,3)$ satisfy the inequality.
2. Domain is $x\in\left(-\frac{1}{2}, 2\right)$ and the range is $y\in(-2,3]$

With active participation of students, you need to present example 4 as an application problem and discuss each step. After discussing example 4, group students and give Group work 3.2 as group assignment to present in class.

The students should get their assignment feedback soon. Finally, at least a group should demonstrate how it worked out this group work on the board. You should also be able to see that the work being done is shared among each student.

Answers to Group work 3.2

1. Let the students practice searching for graphical solutions. The solution is the intersection of the three regions. The students have to sketch on a square lined paper.

a.

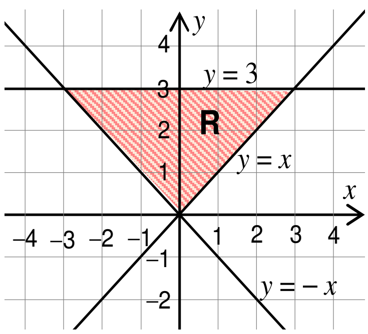


Figure 3.7

b.

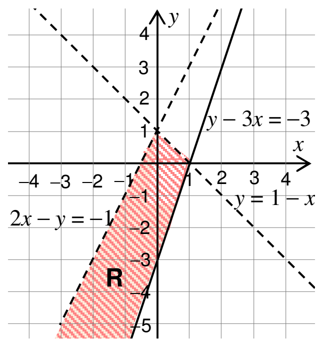


Figure 3.8

2. The answers for (2) can be read from the graphs in 1. From the sketching of the graphs and intersections of pairs of lines the domain and range are obtained.

   From R: when $y = 3$ from $y = x$ obtain $x = 3$

   $y = x$ and $y = -x$ intersect at (0, 0)

   ∴ Domain of R = [–3, 3], Range of R = [0, 3]

   Domain of $r = \{x: x < 1\}$, Range of $r = \{y: y < 1\}$.

For the purpose of checking the students' participation and level of understanding, you can give Exercise 3.2 as homework, check their work and put on record. Give question 7 of Exercise 3.2 as project work. Not all students learn at the same pace. Therefore, it is necessary to develop additional exercises of different level of difficulty apart from the one given in the textbook that needs to be solved by students who are interested to solve extra problems.

Assessment

Generally, it is not necessary to assess all students every day. What is important is to keep track of their learning so that students who are lagging behind are identified and given extra help. And students who are succeeding and moving along quickly are given more challenges to keep them being stimulated and learning.

You can give the following challenging problem to the students who are interested to solve.

Yohannes is in small scale business of constructing hen houses. A small house requires $8m^2$ of play wood and $6m^2$ of insulation. A large house requires $16m^2$ of play wood and $3m^2$ of insulation. Yohannes has available only $48m^2$ of play wood and $18m^2$ of insulation. If a small hen house sells for Birr 150 and a large hen house sells for Birr 200, then how many hen houses of each type should be built to maximize the revenue?

Hint: Let x represent the number of small hen house. Let y represent the number of large hen house. Let the revenue function R $(x, y) = 150x + 200y$

Answer to Exercise 3.2

1. a.

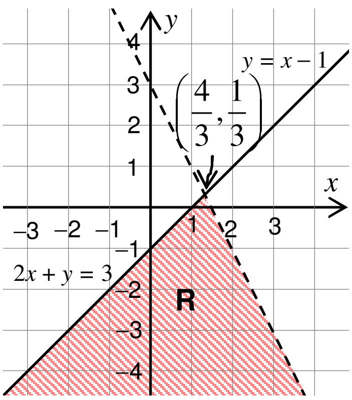


Figure 3.9

b.

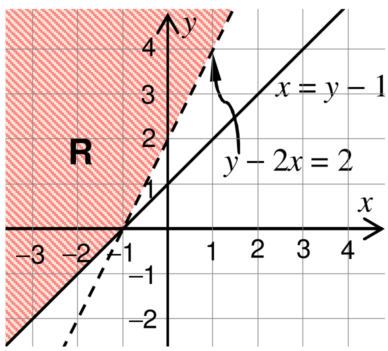


Figure 3.10

c.

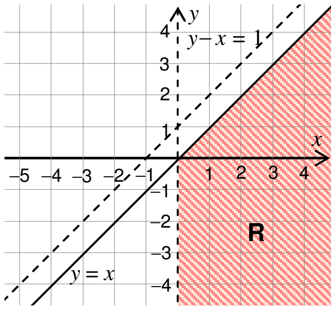


Figure 3.11

d.

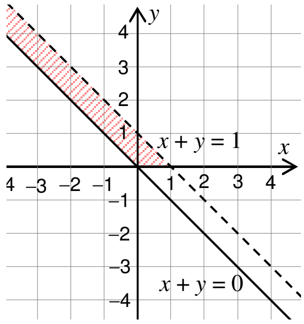


Figure 3.12

2. a.

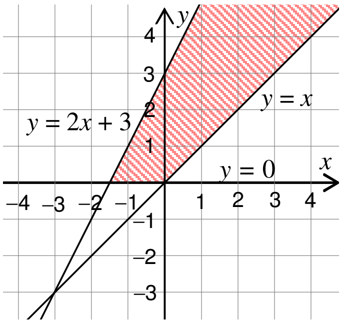


Figure 3.13

b.

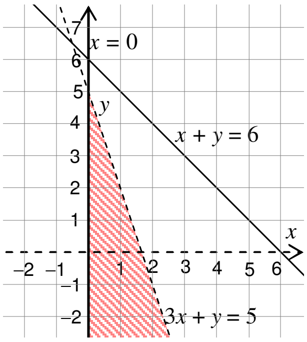


Figure 3.14

c.

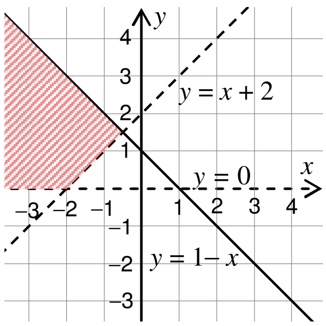


Figure 3.15

d.

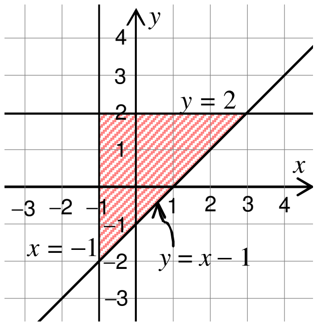


Figure 3.16

e.

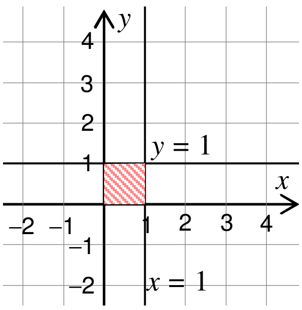


Figure 3.17

f.

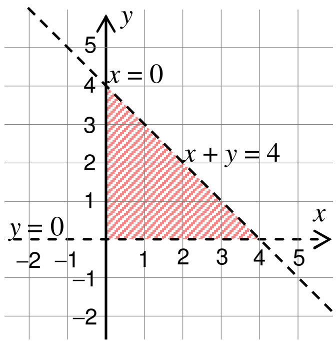


Figure 3.18

3. a. $\begin{cases} x \leq 0 \\ y \geq 0 \\ 2x - 3y + 6 \geq 0 \end{cases}$ b. $\begin{cases} y - x < 2 \\ y > x \end{cases}$

c. $\begin{cases} x \geq 0 \\ y \geq 0 \\ y + x > 2 \end{cases}$ d. $\begin{cases} x < 3 \\ y \geq 0 \\ y - x < 1 \end{cases}$

4. $\begin{cases} x \geq 0 \\ y \geq 0 \end{cases}$

5. $\begin{cases} 0 < x < 1 \\ 0 < y < 2 \end{cases}$

6. $\begin{cases} x + y < 10 \\ x + y > 5 \end{cases}$

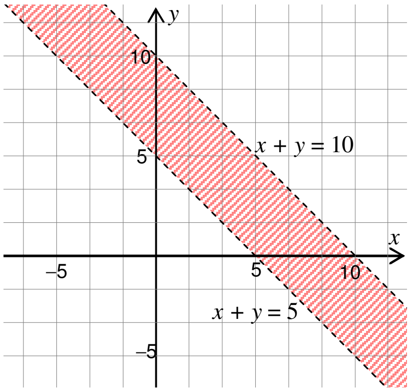


Figure 3.14

7. Let x denote the number of high grade pairs of shoes, and let y denote the number of low grade pairs of shoes produced in one day.

   Clearly $x \geq 0$ and $y \geq 0$.Suppose the factory produces at least twice as much low-grade pairs of shoes as high-grade pairs of shoes. Thus, $2y \geq x$.

   Suppose the maximum possible production is 500 pairs of shoes, so that $x + y \leq 500$. A dealer calls for delivery of at least 100 pairs of shoes of high-grade pairs of shoes per day, so that $x \geq 100$. The shaded area in **Figure 3.15** represents the set of all possible pairs (x, y) satisfying all these conditions.

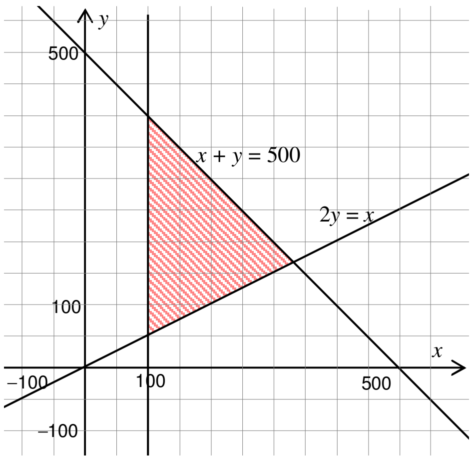


Figure 3.15

Suppose the operation made a profit of Birr 2.00 per pairs of shoes on high-grade and Birr 1.00 per pairs of shoes on low-grade. How many pairs of shoes of each type should be produced for maximum profit? The profit is described by $2x + y$. So we want to find maximum value that can be attained by $2x + y$ when the point (x, y) lies in the shaded region. It can be shown that the maximum value must be attained at a vertex of the given region.

We have to locate the vertices of the shaded region. The top vertex is at the intersection of the lines $x + y = 500$ and $x = 100$. Its coordinate is (100, 400).
Similarly, the right-hand vertex occurs at the intersection of the lines $x + y = 500$ and $2y = x$, so its coordinate is $\left(166\frac{2}{3},\ 333\frac{1}{3}\right)$. Finally, the bottom vertex is at (100, 50).
Evaluating $2x + y$ at each vertex, we obtain

$$2(100) + 400 = 600$$
$$2(100) + 50 = 250$$
$$2\left(\frac{1000}{3}\right)+\frac{500}{3}=833\frac{1}{3}$$

Since $833\frac{1}{3}$ is the largest of these numbers, the maximum profit is realized by producing $\frac{1000}{3}$ shoe of high-grade and $\frac{500}{3}$ shoe of low-grade.

3.3 QUADRATIC INEQUALITIES

Periods allotted: 11 periods

Competencies

At the end of this sub-unit, students will be able to:

- *solve quadratic inequalities by using product properties.*
- *solve quadratic inequalities using the sign chart method.*
- *solve quadratic inequalities using graphs.*

Vocabulary: Quadratic equation, Quadratic inequality

Introduction

This sub-unit gives special emphasis to quadratic inequalities. Different methods of solving quadratic inequalities are presented along with descriptive examples. The different methods discussed in this sub-unit are product properties; sign chart method and graphical method.

Teaching Notes

You can start the lesson by encouraging students to state methods of solving quadratic equation which they studied in grade 9 and illustrate with examples, like the following:

Solve each of the following quadratic inequalities.

a. $(1 - x)(2x + 1) > 0$ b. $(3x - 1)^2 > 9$

c. $2x^2 + 7x + 6 \leq 0$ c. $5x - 1 \geq 6x^2$

After discussing these, you may give some exercises or Activity 3.5 in the students textbook and group the students so that they can practice to solve. By rounding, you need to assist and facilitate their work. This activity might be useful as a stimulus in order to encourage students to engage in class discussion. Check whether they are capable of solving quadratic equations and quadratic inequalities.

Answers to Activity 3.5

1. a and c are quadratic equations. But b, d, e and f are not quadratic equations.
2. a, b, d and f are quadratic inequalities. But c, e and g are not quadratic inequalities.
3. If the product of two numbers is 0, then at least one of the factors must be 0.
4. The process of writing a quadratic expression as a product of two linear expressions is called factorization.
   a. $x(x + 6)$ b. $7x(5 - 4x)$
   c. $\left(\frac{1}{4} - 5x\right)\left(\frac{1}{4} + 5x\right)$ d. $(x + 1)(4x + 3)$
   e. Cannot be factorized f. $(x + 3)(x - 1)$
   g. $(3x + 1)(x - 4)$ h. $(x + 2)^2$
5. a. The expression $b^2 - 4ac$ is called the discriminate of the quadratic equation $ax^2 + bx + c = 0$; $a \neq 0$.
   b. For any quadratic equation $ax^2 + bx + c = 0$; $a \neq 0$
      - if $b^2 - 4ac = 0$, then the equation has exactly one real root.
      - if $b^2 - 4ac > 0$, then the equation has two distinct real roots.
      - if $b^2 - 4ac < 0$, then the equation has no real root.

After ensuring the ability of students in conducting Activity 3.5, encourage them to reflect upon and engage in discussion about solving quadratic inequalities using product properties similar to example 1 in the student textbook. With student active participation, you are required to assist them to tell about each step enlisted in examples in the student text book. You may then give Exercise 3.3 as homework and check their work to see the level of their understanding in solving quadratic inequalities using product properties.

Assessment

In order to ensure that all students in their class can fully participate in meaningful discussion, apart from Exercise 3.3 and examples given above, you need to give additional class activities, group discussion, assignment and quiz or test. You can give them the following questions as additional exercise problems.

1. Solve each of the following inequalities using product properties.
   a. $x^2 + 3x > 10$ b. $2x^2 + 5x \leq 3$
   c. $x^2 + 25 < 10x$ d. $x^2 + 6x + 9 \geq 0$

2. The monthly profit P (in Birr) that Mohammed makes on a sale of x pairs of shoes is determined by the formula $P(x) = x^2 + 5x - 50$. For what value of x is this profit positive?

Answers to Exercise 3.3

1. a. $\{x: x > 0 \text{ or } x < -5\}$ or $(-\infty, -5) \cup (0, \infty)$
   b. $\{1\}$ because $(1-1)^2 \leq 0 \Rightarrow 0 \leq 0$.
   c. $\{x: x > 4 \text{ or } x < -4\}$ or $\{-\infty, -4\} \cup (4, \infty)$
   d. $\{x: -7 < x < \frac{3}{5}\}$ or $\left(-7, \frac{3}{5}\right)$
   e. $\{x: -1 \leq x \leq \frac{3}{2}\}$ or $\left[-1, \frac{3}{2}\right]$
   f. $\{x: 3 \leq x \leq 5\}$ or $[3,5]$
2. a. $\{x: -4 < x < -1\}$ or $(-4, -1)$
   b. $\{x: x < -2 \text{ or } x > 2\}$ or $(-\infty, -2) \cup (2, \infty)$
   c. $\{x: x \leq -3 \text{ or } x \geq -2\}$ or $(-\infty, -3] \cup [-2, \infty)$
   d. $\{1\}$
   e. $\left\{x: x \leq -1 \text{ or } x \geq -\frac{1}{3}\right\}$ or $(-\infty, -1] \cup \left[-\frac{1}{3}, \infty\right)$
   f. $\left\{x: \frac{1}{2} < x < 3\right\}$ or $\left(\frac{1}{2}, 3\right)$
   g. $\left\{x: -\frac{1}{20} < x < \frac{1}{20}\right\}$ or $\left(-\frac{1}{20}, \frac{1}{20}\right)$
   h. $(-\infty, -2) \cup (-2, \infty)$ or $\mathbb{R} \setminus \{-2\}$
3. a. $\{x: -5 < x < 5\}$ or $(-5, 5)$
   b. $x < 5$ means all real number less than 5 but $x < -5$ is not part of the solution. Therefore, $\{x: x < 5\}$ is not the solution for $x^2 < 25$.
4. No, suppose $x = -8$ and $y = 3$, then $-8 < 3$. But $x^2 = (-8)^2 = 64$ and $y^2 = 3^2 = 9$ which implies that $x^2 > y^2$. Thus, $x < y$ does not follow that $x^2 < y^2$ (always).
5. For $\frac{1}{2} < t < 1$, the ball will be at a height of more than 8 meters.

After completion of Exercise 3.3, you should allow time for students to look at Example 2 to understand solving quadratic inequalities by using sign chart method. After a brief discussion of example 3 in the student textbook, encourage and group students to do Group work 3.3 as an assignment to present on a separate paper. Pick one student from any one group to demonstrate how to work out each question on the board, by giving each student the chance to participate.

Answers to Group work 3.3

1. a. $\left(0, \frac{2}{3}\right)$ b. $(-\infty, -3) \cup \left(\frac{1}{2}, \infty\right)$
   c. $[-1, 1]$ d. $[-5, 1]$
2. i. For $k \in (-\infty, -4) \cup (1, \infty)$ ii. For $k = -4$ or $k = 1$ iii. For $k \in (-4, 1)$
3. a. Between 0 and 5000.
   b. At 5000 units.

You can ask students to solve questions 1, 2 and 3 of Exercise 3.4 as class work and let students discuss the solution. You need to facilitate the discussion and give the correct answer on the board. Finally, give questions 4, 5 and 6 of Exercise 3.4 as home work. You are supposed to check their work and help students who could not solve them properly.

Assessment

After completing this lesson, you can use any of the following techniques for assessing students learning: class activities, group discussion, assignment and quiz or test. As additional exercise, you can ask students to solve the following inequalities.

a. $x^2 \geq 4(x+3)$ b. $x^2 \leq 3(2x-3)$

c. $(2x+4)(x-1) < (x+2)$ d. $0.23x^2 + 6.5x + 4.3 > 0$

Answers to Exercise 3.4

1. a. $\{x : x > 0 \text{ or } x < -5\}$ or $(-\infty, -5) \cup (0, \infty)$
   b. $\mathbb{R}$ because it is always true that $(x-3)^2 \geq 0$
   c. $\{x : x < -4 \text{ or } x > 4\}$ or $(-\infty, -4) \cup (4, \infty)$
   d. $\{x : x < -3 \text{ or } x > 5\}$ or $(-\infty, -3) \cup (5, \infty)$
   e. $\left\{x : -\frac{3}{2} < x < 1\right\}$ or $\left(-\frac{3}{2}, 1\right)$
   f. $\left\{x : x \leq -\frac{2}{3} \text{ or } x \geq \frac{1}{2}\right\}$ or $\left[-\infty, -\frac{2}{3}\right] \cup \left[\frac{1}{2}, \infty\right]$
   g. $\left\{x : x \leq -\frac{3}{2} \text{ or } x \geq -1\right\}$ or $\left(-\infty, -\frac{3}{2}\right] \cup [-1, \infty)$
   h. $\{x : -5 < x < 2\}$ or $(-5, 2)$
   i. $3x - x^2 < 4$ for all real numbers
2. a. $\{x : x > 3 \text{ or } x < -4\}$ or $(-\infty, -4) \cup (3, \infty)$
   b. $\{x : x > 3 \text{or } x < 3\}$ or $(-\infty, 3) \cup (3, \infty)$
   c. $\{x : -1 \leq x \leq 4\}$ or $[-1, 4]$
   d. $\{x : x < 2 \text{ or } x > 3\}$ or $(-\infty, 2) \cup (3, \infty)$
   e. $\varnothing$ because $(x+1)^2 \geq 0$.
   f. $x^2 - x + 3 \geq 0$ for all real numbers. Therefore, solution set $= \mathbb{R}$.

3. a. i. it has one real root for $k = -6$ or $k = -2$
   ii. it has two distinct real roots for $k < -6$ or $k > -2$
   iii. $(-6, -2)$
   b. i. it has one real root for $k = -1$ or $k = 11$
   ii. it has two distinct real roots for $k < -1$ or $k > 11$
   iii. it has no real root for $-1 < k < 11$
4. a. $\{k: k > 9\}$
   b. $\{k: k < -22\}$
5. The rocket is more than 3,200 km above the ground level in the time interval $10 < t < 20$.
6. Let the width and length of the reservoir be x unit. You know that $x > 0$, $10 - x > 0$ and $8 - x > 0$. Hence x must be $0 < x < 8$ or $(0, 8)$. Area of the remaining part $A_R = 80 - x^2$, and area of the reservoir:

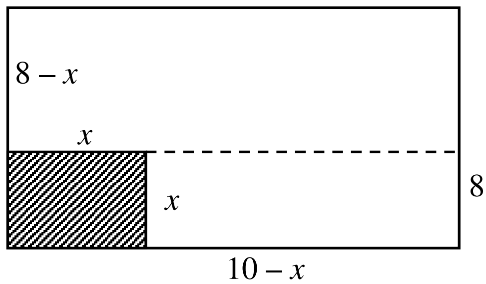


Figure 3.16

   $A_{res} = x^2$
   Required : $A_R < A_{res}$
   $80 - x^2 < x^2$
   $x^2 > 40$
   $(x+2\sqrt{10})(x-2\sqrt{10}) > 0$. Since $x > 0$ you should consider $x > 2\sqrt{10}$
   But x must be between 0 and 8 so that $8 - x$ will be positive.
   So, $x \in (2\sqrt{10}, 8)$

After giving the correct answers to Exercise 3.4, encourage students to do Activity 3.6 so that they will be able to see the nature of the graph of quadratic function. The purpose of this activity is to help students revise condition for the graph of quadratic function and enable them solve quadratic inequality graphically.

Answers to Activity 3.6

1. Recall that for quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$, the point at which the graph of f turns upward or downward is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right) \text{ or } \left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)\right)$.This point is called the vertex of the parabola or the turning point.
2. a.

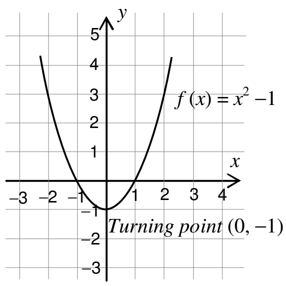


Figure 3.17

   b.

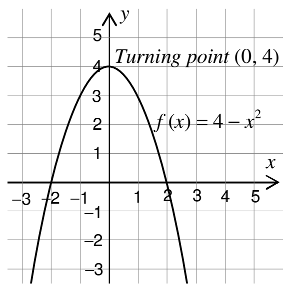


Figure 3.18

3. For any quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$
   if $a > 0$, it has minimum point.
   if $a < 0$, it has maximum point.
4. The value of x at which the quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$ attains its maximum or minimum point is $x = \frac{-b}{2a}$.

After discussing this activity, you need to discuss all the steps enlisted in examples 5, 6 and 7 in the student textbook with active participation of students. These examples are used as a guide for doing Exercise 3.5. At this stage you can give some more questions apart from the ones given in student textbook. After deliberating on the lesson, it is necessary to develop additional exercises because of the fact that all students are not learning in the same pace.

Assessment

Students understand the given lesson differently at different level. Therefore, in order to ensure that all students can fully participate in their learning, you may give various techniques of assessments. You can give class activities, assignments, group discussion, exercise problems, and quiz or test. These helps to check students how much they achieved.

As additional exercise problem, you may give the following to solve.

a. $23.8x^2 - 21.1x + 3.3 \geq 0$

b. $2.12x^2 - 9.87x + 6.79 < 0$

As an application problem, let students solve the following problem in group. A projectile is fired straight up ward from ground level at an initial velocity of 80 m/s. Its distance above the ground level t seconds later is given by the expression $80t - 2t^2$. For what time interval is the projectile more than 400 meters above the ground level?

Answers to Exercise 3.5

1. a. The solution set is $(-\infty, -5] \cup [-1, \infty)$

   b. As shown in **Figure 3.19**, The solution set for $x^2 + 6x + 5 < 0$ is $(-5, -1)$.

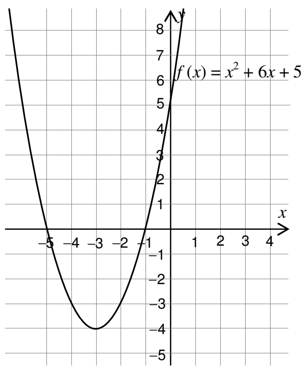


Figure 3.19

c.

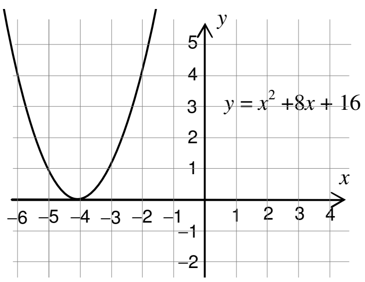


Figure 3.20

As shown in **Figure 3.20**, the graph of $f(x) = x^2 + 8x + 16$ is above x-axis and touches x-axis at a point $(-4, 0)$. Thus, the solution set of the quadratic inequality $x^2 + 8x + 16 < 0$ is empty set.

d.

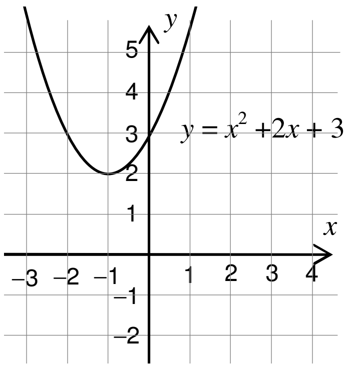


Figure 3.21

As shown in **Figure 3.21**, the graph of $f(x) = x^2 + 2x + 3$ does not cross x-axis and lies above x-axis. Thus, the solution set of this inequality consists of all real numbers, that is $(-\infty, \infty)$.

e.

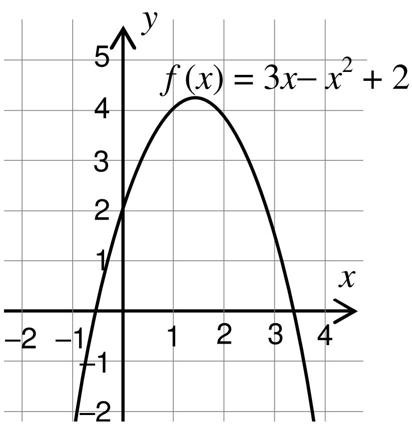


Figure 3.22

As shown in **Figure 3.22**, the graph of $f(x) = 3x - x^2 + 2$ lies below x –axis in the interval

$$\left(-\infty, \frac{3-\sqrt{17}}{2}\right) \cup \left(\frac{3+\sqrt{17}}{2}, \infty\right)$$

f.

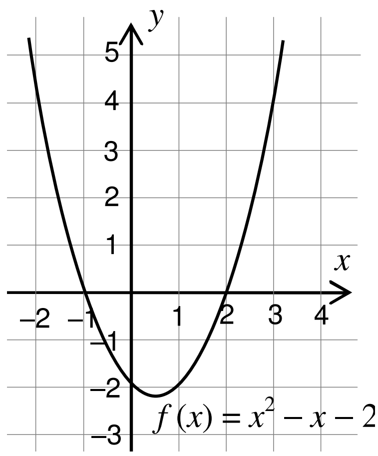


Figure 3.23

As shown in **Figure 3.23**, the graph of $f(x) = x^2 - x - 2$ lies below x-axis for $x \in (-1, 2)$.

Thus, the solution set of this inequality is $\{x: -1 \le x \le 2\}$ or $[-1, 2]$.

g. As shown in **Figure 3.24**, the graph of $f(x) = x(x-2)$ lies below x-axis for $x \in (0, 2)$. for $\in (-\infty, 0) \cup (2, \infty)$ Thus, the solution of this inequality is x: $x < -1$ or $x > 2\}$or $(-\infty, -1) \cup (2, \infty)$.

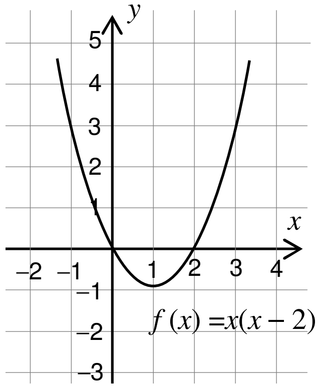


Figure 3.24

h. As shown in **Figure 3.25**, the graph of $f(x) = (x + 1)(x - 2)$ lies above x-axis for $x \in (-\infty, -1) \cup (2, \infty)$. Thus, the solution set of this inequality is $\{x : 0 < x < 1\}$ or $(0, 1)$.

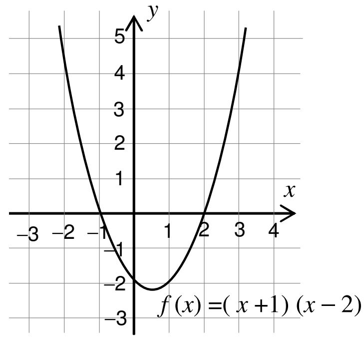


Figure 3.25

In similar method (or graphical method) the solution set of the inequality:

i. $3x^2 + 4x + 1 > 0$ is $\left\{x : x < -1 \text{ or } x > -\frac{1}{3}\right\}$ or $(-\infty, -1) \cup \left(-\frac{1}{3}, \infty\right)$

j. $x^2 + 3x + 3 < 0$ is empty set

k. $3x^2 + 22x + 35 \geq 0$ is $x : x \leq -5$ or $x \geq \frac{-7}{3}$ $\left\{x : x \leq -5 \text{ or } x \geq -\frac{7}{3}\right\}$ or $(-\infty, -5] \cup \left[-\frac{7}{3}, \infty\right)$

l. $6x^2 + 1 \geq 5x$ is $\left\{x : x \leq \frac{1}{2} \text{ or } x \geq \frac{1}{3}\right\}$ or $\left(-\infty, \frac{1}{2}\right] \cup \left[\frac{1}{3}, \infty\right)$

2. The solution set of $ax^2 + bx + c > 0$ consists of the set of all real numbers if $a > 0$ and $b^2 - 4ac < 0$. Thus, the solution set of $2x^2 + kx + 1 > 0$ consists of the set of real numbers if $k^2 - 4(2)(1) = k^2 - 8 < 0$. So, for $k \in \left(-\sqrt{8}, \sqrt{8}\right)$, the solution set of the inequality $2x^2 + kx + 1 > 0$ consists of the set of real numbers.

The Review Exercises on unit 3 might be useful as a stimulus in order to encourage students reflect upon and evaluate their level of understanding regarding solving quadratic inequalities.

Answers to Review Exercises on Unit 3

1. a. $(-1, 3)$ b. $\left(-\frac{9}{2}, 1\right)$

   c. $(-\infty, -\sqrt{2}) \cup (\sqrt{3}, \infty)$ d. $(-\infty, 0) \cup (1, \infty)$

   e. $(-\infty, -4] \cup [-1, \infty)$ f. $[1, 2]$

   g $(-4, 0)$ h. $\varnothing$ (because $2(x+1)^2 \geq 0$)

2. a. $(-\infty, 1) \cup (5, \infty)$ b. $[-3, 3]$

   c. $[-3, 7]$ d. $(-1, \frac{1}{2})$

   e. $\left(\frac{1}{3}, \frac{1}{2}\right)$ f. $\left[-\frac{5}{2}, 1\right]$

3. a. The solution set of the inequality $x^2 - x + 1 > 0$ consists of the set of real numbers, that is $(-\infty, \infty)$.

   b. $(-\infty, -2) \cup (3, \infty)$

   c. $(-\infty, 2-\sqrt{5}) \cup \left(2+\sqrt{5}, \infty\right)$

   d. The solution set of the inequalities $x^2 + 25 \geq 10x$ consists of the set of real numbers, that is, $(-\infty, \infty)$.

   e. $\left(-\infty, 6-\sqrt{10}\right) \cup \left(6+\sqrt{10}, \infty\right)$

   f. $\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{3}{2}, \infty\right)$

   g. $\left(-\infty, \frac{4}{3}\right) \cup (2, \infty)$

   h. $[2, 4]$

4. a. $\left(\frac{1-\sqrt{17}}{4}, \frac{1+\sqrt{17}}{4}\right)$ b. $\left(-\infty, \frac{3-\sqrt{39}}{2}\right) \cup \left(\frac{3+\sqrt{39}}{2}, \infty\right)$

   c. All real numbers d. $(2, 4)$

   e. $\left(-\frac{5}{2}, \frac{3}{5}\right)$ f. $\left(-\frac{3}{4}, \frac{1}{2}\right)$

5. a. $\{k: k < -5\}$

   b. It has only one real solution, if $k = 7$.

   It has two real solutions, if $k \in \mathbb{R}$ and $k \neq 7$.

   For no real value of k will the equation have no real solution.

6. $\{x: 0 \leq x < 3\}$
7. $\{x: x \leq 8\} \cup \{x: x \geq 12\}$
8. Birr $1000 < x <$ Birr 6000

UNIT 4 COORDINATE GEOMETRY

INTRODUCTION

The students have already been introduced to arithmetic, algebra and geometry as different branches of mathematics in their previous mathematics lessons. Now, in this unit, they will be introduced to a new method of studying one branch of mathematics in relation to the other. In particular, arithmetic and algebra will be used to describe geometric relations and vice-versa. The students, in their study of algebra, have learnt how to draw graphs of some relations and functions. The graphs were geometrical representations of the algebraic ideas. This helps the students to have a better understanding of the algebra. In this unit, they will develop the ability to use algebra for a better understanding of geometry. The sub-units in the syllabus of this unit are reorganized as follows: Section 4.1 deals with distance between two points, Section 4.2 is devoted for introducing division of a line segment. Section 4.3 deals with equation of a line. Finally, Section 4.4 presents parallel and perpendicular lines. In the unit, explanations and justifications to concepts and proofs to theorems are given.

In general, the basic ideas presented in this unit enable students to grasp the fundamental methods in order to divide a given line segment into a given proportion and write different forms of equation of a line. Properties of lines like parallel lines and perpendicularity are useful to understand physical and practical applications in life.

Unit Outcomes

After completing this unit, students will be able to:

- *apply distance formula to find distance between any two given points in the coordinate plane.*
- *formulate and apply section formula to find a point that devises a given line segment in a given ratio.*
- *write different forms of equations of a line and understand related terms.*
- *describe parallel or perpendicular line in terms of their slopes.*

Suggested Teaching Aids in Unit 4

Students are familiar with points, lines and planes, models of line segments, parallel and perpendicular line segments around us. Edges of the textbook, tables, doors, windows and walls can be considered for describing line segments, parallel and perpendicular concepts. By extending a line segment in both sides indefinitely, one visualizes a line. Extending a sheet of paper in every direction, a plane is understood ideally.

As far as possible, students have to use the following teaching aids for this unit: A ruler, mathematical set, pieces of sticks, rope, charts representing parallel and perpendicular lines, and inclined lines, etc.

4.1 DISTANCE BETWEEN TWO POINTS

Periods Allotted: 2 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *derive the distance formula (to find distance between two points in the coordinate plane.*
- *apply the distance formula to solve related problems in the coordinate plane.*

Vocabulary: Ordinate, Abscissa, Distance formula

Introduction

This sub-unit is devoted to computing the distance between two given points. After working on Activity 4.1, a method to find distance is presented by the distance formula. Exercise 4.1 is given to apply the formula.

The students are already familiar with the notions of ordered pairs and coordinate plane. In this sub-unit, revision is needed to help students understand coordinates of a point and quadrants of a plane.

After students are familiarized with coordinate plane and coordinates of a point on a plane, you should help and ask them to do Activity 4.1. With active participation of the students, derive the distance formula.

Teaching Notes

It is important to continue discussing the idea of absolute value of a real number which the students had learned in grade 9. Revise the absolute of a real number x as:

$$|x| = \begin{cases} x, & if\ x \geq 0 \\ -x, & if\ x < 0 \end{cases}$$

For instance if $a < 0$ say $a = -7$, then $|a| = -a$.

So $|-7| = -(-7) = 7$.

For any $a, b \in \mathbb{R}$, $|a - b| = |b - a|$.

Let the students try the following types of questions.

1. If $a = 3$ and $b = 11$, what is $|a - b|$ equal to?
2. If $a = \sqrt{10}$ and $b = -\pi$, then $|a + b| =$ ___________
3. Suppose a and b are negative, then without absolute value $|a + b|$ is equal to ____.

In addition, give them an idea of how to locate the quadrant a point P (x, y) belongs. Any point of the form A$(x, 0)$ belongs to the x-axis and B$(0, y)$ lies on the y-axis of the coordinate plane.

Now, select some questions from Activity 4.1 and let them practice to refresh and strengthen the above ideas. In particular, Activity 4.1 is important in order to revise and apply distance between points on the number line and in the coordinate plane.

For fast learner students, you may give questions 7 and 8 while discussing the other questions.

Answers to Activity 4.1

1. a. Coordinate of P is –5, that of A is – 3, that of Q is 0 and that of B is 4.

   b. i. Distance between P and Q is $PQ = 0 - (-5) = 5$

   ii. $QB = 4 - 0 = 4$

   iii. $PB = 4-(-5) = 9$

2. a. $PQ = |x_2 - x_1|$ b. $QP = |x_1 - x_2|$

   c. They are the same since $|x_2 - x_1| = |x_1 - x_2|$ d. They are equal.

3. You can plot coordinates of a point (x, y) by measuring a known distance x units from y-axis and a known distance y units from the x-axis, taking sign into account.

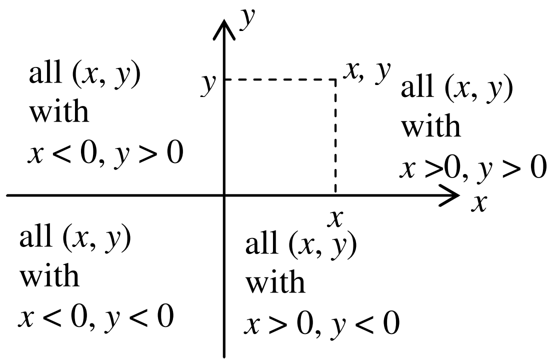


Figure 4.1

For any coordinates (x, y) on a plane, the sign of x and y are determined as shown in **Figure 4.1**.

4. (0, 0). The x-axis and the y-axis intersect at (0,0) dividing the coordinate plane into four regions called quadrants.

5.

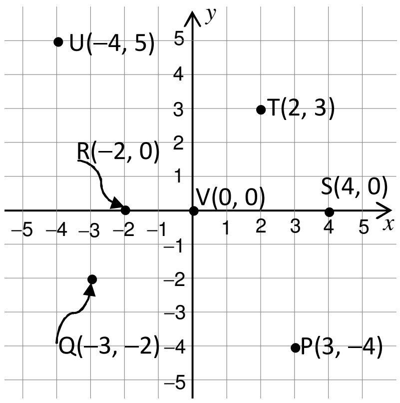


Figure 4.2

6. a. 0 b. 0

7. a. Every ordered pair (a, b) is associated to a unique point P. The converse is also true. As in 5 above, plot the points P (2, 3) and Q (2, 8)

   b. Vertical c. $PQ = 8 - 3 = 5$

8. a. Trivial (do as in 5) b. Horizontal c. $RT = 5 - (-2) = 7$

Horizontal lines are lines parallel to the x-axis. Two points $P(x_1, y_1)$ and $Q(x_2, y_1)$ for $x_1 \neq x_2$ determine a horizontal line. Similarly, vertical lines are parallel to the y-axis.

Next, ask the students to state the Pythagoras theorem to recall $a^2 + b^2 = c^2$, where c is the length of the hypotenuse of a right triangle. Give them the following question.

In ΔABC, if $m(\angle A) = 90°$, $AB = 3$ units and $BC = 7$ units, what is the length of $\overline{AC}$?

The students should be helped to widen their knowledge of distance between two points. After a brief discussion on a distance between P and Q on a line parallel to the x-axis and parallel to the y-axis, develop and guide the students to formulate the distance formula as presented in the textbook.

Let them be convinced that this formula works for any two points P and Q whether on a vertical or horizontal segment.

Select some questions form Exercise 4.1 and give them a class work. Check their performance and see how far they can use the formula to solve problems. Assign some questions for homework. Questions 5 and 6 could be attempted in group if situations allow.

Answers to Exercise 4.1

1. a. $AB = 10$ units b. $CD = \dfrac{\sqrt{34}}{2}$ units

   c. $EF = \sqrt{12 + 2\sqrt{3}}$ units d. By the distance formula,

   $$d = \sqrt{(-a-a)^2 + (b-(-b))^2}$$
   $$= 2\sqrt{a^2 + b^2} \text{ units}$$

e. 1 unit

f. $LM = \sqrt{6-4\sqrt{2}}$ units $= 2-\sqrt{2}$ units

g. $PQ = \sqrt{5}$ units

h. By the distance formula,

$$RT = d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{\left(\sqrt{2}b - \sqrt{2}a\right)^2 + (c-c)^2}$$
$$= \sqrt{2}\;|b - a| \text{ units}$$

2. a. The two points P (x_1, y_1) and Q (x_2, y_1) determine a horizontal line. Thus, the distance between *P* and *Q* is

$$PQ = d = \sqrt{(x_2 - x_1)^2 + (y_1 - y_1)^2} = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|.$$

b. The two points *P* (x_1 , y_1) and *Q* (x_1 , y_2) determine a vertical line. Thus, the distance between *P* and *Q* is

$$PQ = d = \sqrt{(x_1 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|.$$

3. $s = AB = \sqrt{(-1-3)^2 + (4+7)^2} = \sqrt{(-4)^2 + (11)^2} = \sqrt{16 + 121} = \sqrt{137}$

Area : $A = s^2 = \left(\sqrt{137}\right)^2 = 137$ sq. units

4. Diagonal : $d = PQ = \sqrt{(1-3)^2 + (-3-5)^2} = \sqrt{4+64} = \sqrt{68}$

Area of a square whose diagonal *d* is given by

$$A = \frac{d^2}{2} = \frac{1}{2}\left(\sqrt{68}\right)^2 = \frac{1}{2}(68) \text{ sq. units} = 34 \text{ sq. units.}$$

5. a. $AB = \sqrt{(2-5)^2 + (3+1)^2} = \sqrt{(-3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$ units

$BC = \sqrt{(1-2)^2 + (1-3)^2} = \sqrt{(-1)^2 + (-2)^2} = \sqrt{1 + 4} = \sqrt{5}$ units

$AC = \sqrt{(1-5)^2 + (1+1)^2} = \sqrt{(-4)^2 + (2)^2} = \sqrt{16 + 4} = \sqrt{20}$ units

Since $(AB)^2 = (BC)^2 + (AC)^2$, ΔABC is a right angled triangle.

b. $AB = \sqrt{(4+4)^2 + (-3-3)^2} = \sqrt{8^2 + (-6)^2} = \sqrt{64 + 36} = \sqrt{100} = 10$ units

$BC = \sqrt{(3\sqrt{3}-4)^2 + (4\sqrt{3}+3)^2} = \sqrt{27 - 24\sqrt{3} + 16 + 48 + 24\sqrt{3} + 9} = 10$ units

$AC = \sqrt{(3\sqrt{3}+4)^2 + (4\sqrt{3}-3)^2} = \sqrt{27 + 24\sqrt{3} + 16 + 48 - 24\sqrt{3} + 9}$

$= \sqrt{100} = 10$ units

Therefore, ΔABC is an equilateral triangle.

c. $AB = \sqrt{(6-2)^2 + (8-3)^2} = \sqrt{4^2 + 5^2} = \sqrt{16 + 25} = \sqrt{41}$ units

$BC = \sqrt{(7-6)^2 + (-1-8)^2} = \sqrt{1 + (-9)^2} = \sqrt{1 + 81} = \sqrt{82}$ units

$AC = \sqrt{(7-2)^2 + (-1-3)^2} = \sqrt{5^2 + (-4)^2} = \sqrt{25 + 16} = \sqrt{41}$ units

Therefore, ΔABC is an isosceles triangle since $AB = AC$.

6. Let the vertices of ΔABC be A (–4, 0), B (4, 0) and C (x, y).

   $AB = \sqrt{(4+4)^2} = \sqrt{64} = 8$ units

   $BC = \sqrt{(x-4)^2 + y^2}$; and $AC = \sqrt{(x+4)^2 + y^2}$

   Since ΔABC is an equilateral triangle, we have $AB = BC = AC$. Hence

   $\sqrt{(x-4)^2 + y^2} = \sqrt{(x+4)^2 + y^2}$

   $(x-4)^2 + y^2 = (x+4)^2 + y^2$

   $x^2 - 8x + 16 + y^2 = x^2 + 8x + 16 + y^2 \Rightarrow x = 0$

   You know that $BC = AB$.

   Hence $\sqrt{(x-4)^2 + y^2} = 8$

   $\Rightarrow \sqrt{(0-4)^2 + y^2} = 8$

   $16 + y^2 = 64$

   $y = \pm\sqrt{48} = \pm 4\sqrt{3}$

   Therefore, the coordinates of the third vertex are (0, $4\sqrt{3}$) or (0, $-4\sqrt{3}$).

7. $\sqrt{(0-b)^2 + (-2-4)^2} = 10 \Rightarrow b^2 = 64$. So, $b = \pm 8$.

   Therefore, the value of b is –8 or 8.

Assessment

Before you go to the next sub-topic, you are expected to assess the students' performance on the presented subject matter by either of the following methods:

- Asking oral questions.
- Giving class work (May be questions 1, 2 and 3 of exercise 4.1) and giving assignment. (May be question 4, 5a and 7). Give comments on their attempts.

4.2 DIVISION OF A LINE SEGMENT

Periods Allotted: 2 Periods

Competency

After completing this subunit, students will be able to:

- *determine the coordinates of points that divide a given line segment in a given ratio.*

Vocabulary: Ratio, Mid-point, Section formula.

Introduction

In this sub-unit, students should be able to understand and apply the notion of ratio and what is meant by a point R dividing a line segment internally. For a better understanding of these concepts, Activity 4.2 and Activity 4.3 are useful. To further apply section formula and find midpoint of a line segment, let students do Group work 4.1 and Exercise 4.2.

In this sub-unit, the students should be able to understand the notion of ratio and what is meant by a point *P* divides a line segment *AB* internally, or externally. For a better understanding of these concepts, let them do Activity 4.2 and Activity 4.3.

Teaching Notes

In order to write the division (section) formula, the students need to apply ratio on horizontal and vertical line segments. With active involvement of the students, derive the section formula as stated in the text. In some books, the ratio m:n is replaced by $r_1 : r_2$. As an immediate consequence with ratio 1:1, the mid-point of the line segment PQ is obtained.

$$M(\bar{x}, \bar{y}) = (x_0, y_0) = \left(\frac{1\cdot x_1 + 1\cdot x_2}{1+1}, \frac{1\cdot y_1 + 1\cdot y_2}{1+1}\right) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Activity 4.2 is designed to help students understand ratio and division of a line segment. Let the students think about the answers and discuss them with the active participation of all students.

For fast learners, you may give the following additional questions.

1. What is the mid-point of the line segment joining the points

   a. P (0, 0) and Q (6, 6)?

   b. P (–1, 0) and Q (–5, 7)?

   c. $P(1, \sqrt{2})$ and $Q(1, 5\sqrt{2})$?

2. Let $P(6,0)$, $Q(6, -10)$ and M be the mid-point of $\overline{PQ}$. Find the mid-point of the line segments

   a. $\overline{PM}$ b. $\overline{MQ}$

3. If P(0,1), Q(8, 5) and R(6, 4), find PR: RQ.

Answers to Activity 4.2

1. A ratio is a comparison of two quantities of the same kind by division.
2. The ratio of the lengths of two line segments is the quotient of the length of the first line segment and the length of the second line segment.
3. $AP = |-1-(-6)| = |-1+6| = 5$ units.

   $PB = |5-(-1)| = |6| = 6$ units.

   The ratio of the length of $\overline{AP}$ to the length of $\overline{PB}$ is $\frac{AP}{PB} = \frac{5}{6}$ or 5 : 6
4. The line segment *AB* is divided internally by *P* if *P* lies between *A* and *B* such that *AP* + *PB* = *AB*, as shown in **Figure 4.3**.

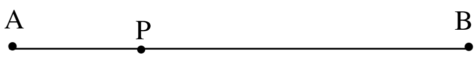


Figure 4.3

5. a.

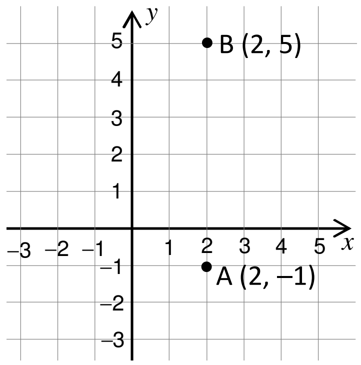


Figure 4.4

$$\text{mid-point } M(x_0, y_0) = \left(\frac{2+2}{2}, \frac{-1+5}{2}\right) = (2,\ 2)$$

b.

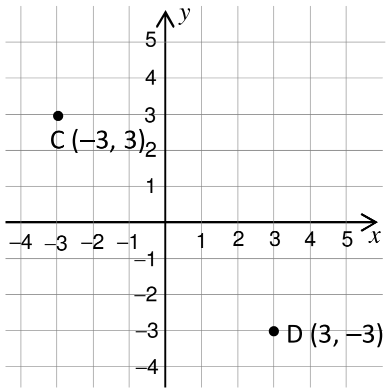


Figure 4.5

$$\text{mid-point } M(x_0, y_0) = \left(\frac{-3+3}{2}, \frac{(-3)+3}{2}\right) = (0,\ 0)$$

c.

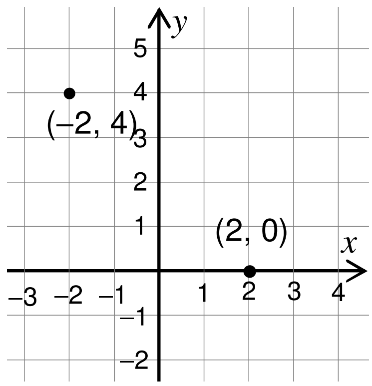


Figure 4.6

$$\text{mid-point } M(x_o, y_o) = \left(\frac{2+(-2)}{2}, \frac{0+4}{2}\right) = (0,\ 2)$$

After the students are familiarized with the notion of ratio and mid-point of a line segment internally or externally by a point, division of a line segment in a certain ratio has to be discussed with the help of Examples 2 and 3. Here you may produce your own examples. Finally, after the students are made to understand how to find the coordinates of a point that divides the line segment in a given ratio, they should be guided to derive the mid-point formula. The mid-point formula could be derived with the help of Example 3.

Answer to Activity 4.3

1. P (2, 1), Q (12, 1), R (7, 1)

   a. $PQ = |x_2 - x_1| = |12 - 2| = 10$ units

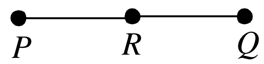
   b. i. $PR = |x_2 - x_1| = |7 - 2| = 5$ units

   ii. $RQ = |x_2 - x_1| = |7 - 12| = |-5| = 5$ units

   iii. Yes, $PQ = RQ$

   iv. Mid-point $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2+12}{2}, \frac{1+1}{2}\right) = (7, 1)$, R is the mid-point of $\overline{PQ}$.

   c.



, Where R (7, 1)

Figure 4.7

2. a. i. P (x_1, y_1) and Q (x_1, y_2), Mid-point is $\left(x_1, \frac{y_1 + y_2}{2}\right)$

   ii. R (x_1, y_1) and S (x_2, y_1), Mid-point is $\left(\frac{x_1 + x_2}{2}, y_1\right)$

   b. $\overline{RS}$ is a horizontal segment.

Group work 4.1 is designed to help the students practice and apply distance formula, section formula and mid-point formula. Make the group composition of slow and fast learners. Give them time to discuss the topic among themselves. Follow how they are participating in the discussion. If possible, let a group representative present the outcome obtained. Give credits and appreciations for active groups with a view to create competition.

Answer to Group Work 4.1

1. a. Length of the line segment is $\sqrt{(5-(-3))^2 + (7-1)^2} = 10$ units

   b. $M(x_0, y_0) = \left(\frac{5+(-3)}{2}, \frac{7+(1)}{2}\right) = (1, 4)$

2. Let the other end-point be (x_1, y_1). Then M $(x_0, y_0) = (1, -1) = \left(\frac{4+x}{2}, \frac{3+y_1}{2}\right)$

   $\Rightarrow 2(1, -1) = (4 + x_1, 3 + y_1)$ we get $x_1 = -2$ and $y_1 = -5$

   Therefore, $(x_1, y_1) = (-2, -5)$ is the other end-point.

3. If the points are R_1 and R_2 then, $R_1 = \left(\frac{2(4)+1(-6)}{2+1}, \frac{2(-3)+7}{2+1}\right) = \left(\frac{2}{3}, \frac{1}{3}\right)$

$$R_2 = \left(\frac{1(4)+2(-6)}{1+2}, \frac{1(-3)+2(7)}{2+1}\right) = \left(-\frac{8}{3}, \frac{11}{3}\right)$$

4.

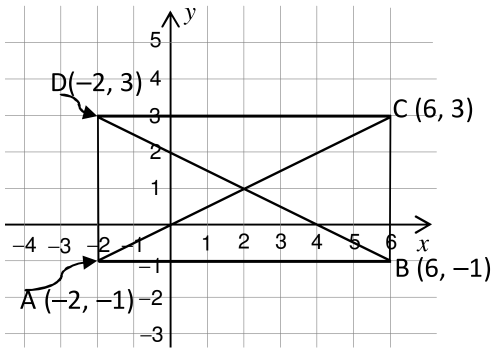


Figure 4.8

i. $A = 32$ unit^2 ii. $A = 16$ unit^2 iii. $32 : 16 = 2 : 1$

Here, the students can explain and apply section formula and mid-point formula. They can do similar problems like the examples given and the problems in Group work 4.1 independently.

For further exercise, give problems 1 and 3 of Exercise 4.2 as class exercise and the rest as homework to be marked.

Assessment

Collect and give corrections on the homework given from Exercise 4.2. Try to classify the students as slow and fast learners. This can help you follow the performances of the students. Use the following questions as additional exercises.

For slow learners: Given points P (–1, 3) and Q (5, y)

a. Find y so that the distance between P and Q is 10 units.

b. Using the value of y obtained in (a), find the mid-point of $\overline{PQ}$.

For fast learners:

a. Find the coordinates of the points R_1, R_2 and R_3 that divide $\overline{PQ}$ into four equal parts if P (2, –3) and Q (2, 3)

b. Let P (–1, 0) and Q (7, 8) be given points. Find the coordinates of the points that divide $\overline{PQ}$ into four equal parts.

c. Find points on the line $y = x - 2$ at a distance of $3\sqrt{2}$ units from point (1, –1).

Answers to Exercise 4.2

1. a. $\left(-\dfrac{1}{2}, 3\right)$ b. $\left(\dfrac{a}{2}, \dfrac{b}{2}\right)$ c. $\left(\dfrac{p+q}{2}, \dfrac{p+q}{2}\right)$

   d. $\left(-\dfrac{1}{2}, 0\right)$ e. $\left(\dfrac{3}{2}, \dfrac{3\sqrt{2}}{2}\right)$ f. $(2\sqrt{5}, 1)$

2. Note that $(a, b) = (c, d)$ if $a = c$ and $b = d$

$$M(x_o, y_o) = \left(\frac{x_1 + y_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$(-3, 2) = \left(\frac{x_2 + 1}{2}, \frac{y_2 - 3}{2}\right)$$

$$-3 = \frac{x_2 + 1}{2} \text{ and } 2 = \frac{y_2 - 3}{2} \Rightarrow x_2 = -7 \text{ and } y_2 = 7$$

Therefore, the coordinate of the other point is (–7, 7).

3. $$P(x_o, y_o) = \left(\frac{x_1 n + x_2 m}{n+m}, \frac{y_1 n + y_2 m}{n+m}\right)$$

$$= \left(\frac{1\times3+(-4\times2)}{3+2}, \frac{3\times3+(-3\times2)}{3+2}\right) = \left(-1, \frac{3}{5}\right)$$

4. Let the first point $P(x_o, y_o)$ divide the given line segment in the ratio 1: 2. Then,

$$P(x_o, y_o) = \left(\frac{x_1 n + x_2 m}{n+m}, \frac{y_1 n + y_2 m}{n+m}\right)$$

$$= \left(\frac{-1\times2+5\times1}{2+1}, \frac{5\times2+2\times1}{2+1}\right)$$

$$= \left(\frac{5-2}{3}, \frac{10+2}{3}\right) = (1, 4)$$

The second point $P'(x', y')$ divides the line segment in the ratio 2: 1.

$$P'(x', y') = \left(\frac{x_1 n + x_2 m}{n+m}, \frac{y_1 n + y_2 m}{n+m}\right)$$

$$= \left(\frac{-1+5\times2}{1+2}, \frac{5\times1+2\times2}{1+2}\right) = (3, 3)$$

Therefore, the points with coordinates (1, 4) and (3, 3) divide the line segment into three equal parts.

5. Let D, E and F be the mid-point of side $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively as shown in **Figure 4.9.**

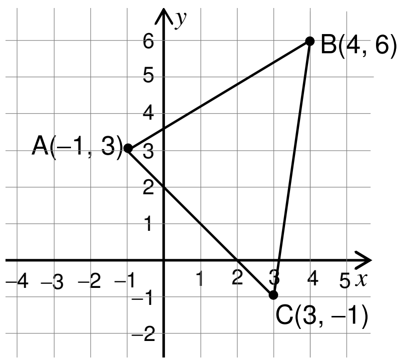


Figure 4.9

$$D(x_1, y_o) = \left(\frac{3}{2}, \frac{9}{2}\right)$$

$$E(x_1, y_1) = \left(\frac{7}{2}, \frac{5}{2}\right)$$

$$F(x_2, y_2) = (1, 1)$$

4.3 EQUATION OF A LINE

Periods Allotted: 8 Periods

Competencies

After completing this subunit, students will be able to:

- *define the gradient of a given line.*
- *determine the gradient of a given line (given two points on the line.*
- *express the slope of a line in terms of the angle formed by the line and the x-axis.*
- *determine the equation of a given line.*

Vocabulary: Gradient, Angle of inclination, Point-slope form, Slope intercept form, Two-point form.

Introduction

In this sub-unit, the students should be able to deepen their knowledge on the notion of gradient (slope), slope of a line in terms of angle of inclination and different forms of equation of a line. They should be able to use the concept of slope for writing equations of a given line. The different forms of equations of a line should be discussed by means of variety of examples. Focus on Definition 4.1 and computation of slope of a line through two points. The main target then is that the students should understand and write the point-slope form, slope-intercept form, two-point form and the general equation of a line supported by different examples and exercises.

Gradient (Slope) of a line

Teaching Notes

After an introduction using Activity 4.4 about gradient of a line, state the definition of gradient. If P (x_1, y_1) and Q (x_2, y_2) are two points of a line ℓ, then $m = \frac{y_2 - y_1}{x_2 - x_1}, x_1 \neq x_2$ is the gradient of ℓ. When $x_1 = x_2$, ℓ is vertical line and its gradient is not defined. We may say that a vertical line has no slope.

When $x_1 \neq x_2$ and $y_1 = y_2$, then $m = 0$. This can be seen from the slope formula.

Answers to Activity 4.4

a. i. 3 ii. 3 iii. 3 iv. 3

b. Yes. The value $\frac{y_2 - y_1}{x_2 - x_1}$ is called slope (gradient) of the line through the points.

After discussion with students on Activity 4.5, give examples like Examples 1 and 2. Let the students try Activity 4.5 to compute slope and check that same ratio is obtained by taking at least four points on a line. You can show them that the same holds true for any other point on the line of the given points.

Answers to Activity 4.5

1. There is no gradient for a line passing through (x_1, y_1) and (x_2, y_2) with $x_1 = x_2$. It is a vertical line.
2. The gradient of any horizontal line is zero.
3. Given equation of the line $f(x) = 3x - 1$. Taking any arbitrary points $P_1 = (1, 2), P_2 = (0, -1), P_3 = (-2, -7)$

   a. gradient using P_1 and P_2 is $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-1-2}{0-1} = 3$

   b. gradient using P_2 and P_3 is $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-7-(-1)}{-2-0} = \dfrac{-6}{-2} = 3$

   c. From a and b above, we observe that given any equation of a line passing through any arbitrary points, the gradient is independent of the points on the line.
4. a. $m_1 = \dfrac{y_2 - y_1}{x_2 - x_1} = a$ b. $m_2 = \dfrac{y_4 - y_3}{x_4 - x_3} = a$

   c. $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_4 - y_3}{x_4 - x_3}$ d. Yes; the slope of a line does not depend on the choice of coordinates of points on the line.

To familiarize students and do more practice, give them time to try some problems of Exercise 4.3 and check their performance. You may give problem 2 or 3 for fast learners. Give problems 4 and 5 as class work.

Answers to Exercise 4.3

1. a. 2 b. –1 c. $\sqrt{2}$ d. $-\dfrac{1}{2}$

   e. 0 f. no gradient g. $\dfrac{a-b}{b-1}$, $b \neq 1$
2. Slope of $\overline{AB}$: $m_1 = \dfrac{8}{5}$ Slope of $\overline{AC}$: $m_2 = -3$

   Slope of $\overline{BC}$: $m_3 = -\dfrac{4}{9}$
3. Slope of $\overline{PQ}$ is $\dfrac{3}{2}$ and slope of $\overline{QR}$ is $\dfrac{3}{2}$

   The two line segments $\overline{PQ}$ and $\overline{QR}$ have the same slope. So P, Q and R lie on the same line and they are called collinear points.
4. Given P_1 (–4, 6), P_2 (–1, 12) and P_3 (–7, 0)

   Slope of $\overline{P_1P_2}$ is 2, and Slope of $\overline{P_2P_3}$ is 2.

   Thus P_1, P_2 and P_3 lie on the same line.

Note: Three points are collinear if and only if their slopes calculated using any two of the points on the line is the same.

5. Slope of $\overline{AB}$ is $\frac{7}{6}$; and slope of $\overline{BC}$ is $\frac{7}{6}$. Since they have the same slope, the line passing through *A* and *B* also passes through *C*.

Slope of a line in terms of angle of inclination

Define angle of inclination θ of a line ℓ between ℓ and the positive *x*-axis in counter clockwise direction. Let the groups formed discuss Group work 4.2. With active participation of the students, formulate the relationship.

$m = \tan\theta$, which holds for a non-vertical line ℓ.

If ℓ is a vertical line, then θ = 90°. For any horizontal line θ = 0°. From the sign of tan θ, the graph of ℓ:

a. $m = \tan\theta > 0 \Rightarrow \ell$ rises from left to right.

b. $m = \tan\theta < 0 \Rightarrow \ell$ falls from left to right.

Answers to Group Work 4.2

1. $\text{OB} = \sqrt{(\text{AB})^2 + (\text{OA})^2} = \sqrt{(3)^2 + \left(3\sqrt{3}\right)^2} = \sqrt{36} = 6\,\text{units}$
2. The tangent of angle *BOA* is $\frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
3. $m\,(\angle BOA) = 30°$
4. 30°
5. $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
6. $O = (0, 0)$; $B = \left(3\sqrt{3},\ 3\right)$.

   The slope of the line ℓ is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-0}{3\sqrt{3}-0} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
7. The same

Once slope is defined, let them do Activity 4.6 so that they can relate slope and angle of inclination of a line.
Assist the students to try especially question 3.

Answers to Activity 4.6

1. The line is described by the equation $x = x_1$. It is a vertical line which is perpendicular to the *x*-axis. There is no tangent of the angle between this line and the *x*-axis.
2. It is a horizontal line and its slope is 0. So, the tangent of the angle between the line and *x*-axis is 0.
3. The angle of inclination of the line $y = x$ is 45° and $y = -x$ is 135°

Exercise 4.4 is a form of revision of what they did in the previous grades. Make them do the questions in class. Assist them and check what they did.

Answers to Exercise 4.4

1. a. Slope : $m =$ tan 30°. So $m = \frac{\sqrt{3}}{3}$

   b. Slope : $m = \tan 75^\circ = \frac{\sin 75^\circ}{\cos 75^\circ} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$

   c. Slope : $m = \tan 150^\circ = -\frac{\sqrt{3}}{3}$

   d. Since tan 90° is undefined, there is no slope.

   e. Slope : $m = \tan 0^\circ = 0$

2. a. 120° b. 150° c. 45°

   d. 30° e. 0°

3. Slope of $\overleftrightarrow{AB}$ is $\frac{2-0}{0+2} = 1$

   $\tan\theta = 1$, so $\theta = \tan^{-1}(1) = 45^\circ$ and $m(\angle BAC) = 45^\circ$

   Slope of $\overleftrightarrow{BC} = \frac{0-2}{2-0} = -1 \Rightarrow \tan\theta = -1$

   $\tan^{-1}(-1) = \theta = 135^\circ$, so $m(\angle BCA) = 45^\circ$

   Thus $\angle A + \angle B + \angle C = 180^\circ$

   $45^\circ + \angle B + 45^\circ = 180^\circ \Rightarrow \angle B = 90^\circ$

   Therefore, it is a right angled triangle.

Different Forms of Equations of Line

This sub-section helps to write different forms of equations of a line such as point-slope form, slope-intercept form, two point form and the general equation of a line which is linear in both x and y variables.

Ax + By + C = 0 where either A or B is non-zero.

This activity is a motivation for the next study to determine different forms of equations of a line in the rectangular coordinate plane. As a starting idea, let the students try the equations of Activity 4.7 in class. Follow their attempts and make your fast learner students work on questions 5 and 6.

Answers to Activity 4.7

1. Since $x = 2$ is the graph of a vertical line whose abscissa of a point is always unchanged, the points (2, 0), (2, –1), (2, 2) and $\left(2, \frac{1}{3}\right)$ lie on the graph of the line $x = 2$.

2. The points (–1, 0), $(0\ ,\ 1)$ and (–2 , –1) lie on the line $y - x = 1$.
3. Points (0, 4), (–1, 9) and $\left(\frac{2}{5}, 2\right)$ lie on $y = -5x + 4$
4. In equation of a line $y = ax + b$, the number b is called the y-intercept.
5. To find y-intercept, put $x = 0$. Thus $y = m\ (0) + b$ which gives
   $y = b$. So y – intercept is $y = b$
   To find x – intercept, put $y = 0$. Thus $0 = mx + b$
   So, x-intercept is $x = \frac{-b}{m}$, when $m \neq 0$.
6. a. To find the equation of the line through the points (–1, 3) and (4, 3), first we need to find the slope (gradient). Recall that $y = ax + b$ is an equation of a straight line.
   Substituting (–1, 3), $3 = -a + b$
   Substituting (4, 3), $3 = 4a + b$
   So $-a + b = 4a + b$
   $\therefore\ a = 0$.
   From $3 = -a + b$, $b = 3$.
   The equation of the line is $y = 3$
   b. Similarly,
   Substituting (–1, 1), $1 = -a + b$
   Substituting (1, –1), $-1 = -a + b$
   Adding $0 = 0 + 2b$. So $b = 0$ and $a = -1$
   $\Rightarrow\ y = -x$ is the equation of the line through the points (–1, 1) and (1, –1).

Suppose point P (x_1, y_1) and Q (x_2, y_2) are on a non-vertical line ℓ. Then, the slope (gradient) of ℓ is $m = \frac{y_2 - y_1}{x_2 - x_1}$. Using P, Q and m, for any point R (x, y) on ℓ some forms of equations of ℓ are the following:

i. Point-slope form
   Taking P (x_1, y_1) and m, $y = m\ (x - x_1) + y_1$
ii. Slope-intercept form
   with P $(x_1, y_1) = (0, b)$ on ℓ and m, $y = m\ (x - 0) + b \Rightarrow y = mx + b$.
iii. Two-point form
   Taking P (x_1, y_1) and Q (x_2, y_2) $\Rightarrow$ $y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$
   $y = m\ (x - x_1) + y_1$

In addition to the above, the general equation of a line is Ax + By + C = 0 where A $\neq$ 0 or B $\neq$ 0. This equation represents a vertical line if B = 0, a horizontal line if A = 0.
If B $\neq$ 0, $y = -\frac{Ax}{B} - \frac{C}{A}$ is an equation of a line in slope-intercept form.

Practice the above forms using the examples in the text. Consider the line ℓ containing P (3, 0) and Q(–1, 2). Ask the students to write the equation of ℓ in the above four forms. Give some of the questions of Exercise 4.5 as class work and some as assignment.

Assessment

You can evaluate the students' performance on this sub-topic, by giving either a quiz or a class work and giving them immediate feedback. You can select questions from Exercise 4.5 for this purpose. Check the assignment responses and discuss their problems if there are any, indicating major sources of error.

Answers to Exercise 4.5

1. a. $y = 9x + 14$ b. $y = -3x + 2$ or $y + 3x = 2$
   c. $y = 7$ (horizontal line) d. $y = -x + 2$ or $y + x = 2$
   e. $y = 0$ (or the x-axis) f. $x = 4$ (vertical line)
   g. $y + 3x = 4\pi$ h. $14x + 12y = -9$
2. a. $y = \frac{3}{2}x - 6$ b. $y = \frac{-\pi}{4}$
   c. $y = \frac{5}{3}x - \frac{2}{3}$ or $5x - 3y - 2 = 0$ d. $y = -\pi x$ or $\pi x + y = 0$
   e. $y = \sqrt{2}\,x - 2 - \sqrt{2}$ or $\sqrt{2}x - y - 2 - \sqrt{2} = 0$ f. $y = -x + \frac{11}{6}$
3. a. $y = 0.1x$ b. $y + \sqrt{2}x + 1 = 0$ c. $y = \pi x + 2$
   d. $4x - 3y - 5 = 0$ e. $y = \frac{-1}{4}x + 5$ f. $y = \frac{2}{3}x + \frac{3}{2}$
4. Evidently, the line passes through the points (a , 0) and (0 , b). Using these two points, you can find the equation of the line as follows:

   $\frac{y-0}{x-a} = \frac{b-0}{0-a} \Rightarrow \frac{y}{x-a} = \frac{-b}{a} \Rightarrow ay = -bx + ab$ or $ay + bx = ab$. Dividing both sides by ab.

   $\frac{x}{a} + \frac{y}{b} = 1$. This form of equation of a line is called two intercept form of equation of a line.
5. a. $\frac{3}{5}x - \frac{4}{5}y + 8 = 0$ b. $-y + 2 = 0$

   $y = \frac{3}{4}x + 10$ $\therefore\ y = 2$

   $\therefore$ Slope is $\frac{3}{4}$ and y-intercept is 10 $\therefore$ Slope = 0, y-intercept = 2

c. $2x - 3y + 5 = 0 \Rightarrow y = \frac{2}{3}x + \frac{5}{3}$ d. $x + \frac{1}{2}y - 2 = 0 \Rightarrow y = -2x + 4$

$\therefore$ Slope is $\frac{2}{3}$ and y-intercept is $\frac{5}{3}$ $\therefore$ Slope is -2 and y-intercept is 4

e. $y + 2 = 2(x - 3y + 1)$

$y = \frac{2}{7}x \Rightarrow$Slope is $\frac{2}{7}$ and y-intercept is 0.

6. Slope of the line passes through the points (5, –1) and (–3, 3) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-1)}{-3-5} = \frac{4}{-8} = \frac{-1}{2}$$

a. $y = m(x - x_1) + y_1$

$y = \frac{-1}{2}(x-5) + (-1)$ is the point-slope form of equation of the line.

b. From (a), we have slope $m = \frac{-1}{2}$

$\Rightarrow y = \frac{-1}{2}x + \frac{3}{2}$ is the slope intercept form of equation of the line.

c. $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow \frac{y-(-1)}{x-5} = \frac{3-(-1)}{-3-5}$

$\Rightarrow \frac{y+1}{x-5} = \frac{-1}{2} \Rightarrow y + 1 = \frac{-1}{2}(x-5)$ is the two-point form.

General form: $2y + x - 3 = 0$

7. a. Slope = $\frac{-2}{5}$, y-intercept = $\frac{3}{5}$

b. Slope = $\frac{5}{2}$, y-intercept = $\frac{1}{4}$

8. a. Equation of the line $\overline{AB}$ is $y = x + 2$

Equation of the line $\overline{BC}$ is $y = -x + 4$

Equation of the line $\overline{CA}$ is $y = 1$

b. Yes, it is.

c. (0, 4) and (4, 0)

4.4 PARALLEL AND PERPENDICULAR LINES

Periods Allotted: 3 Periods

Competencies

At the end of this sub-unit, students will be able to;

- *identify whether two lines are parallel or not.*
- *identify whether two lines are perpendicular or not.*
- *apply the properties of the slopes of parallel and perpendicular lines to solve related problems.*

Vocabulary: Non-vertical, Parallel, Perpendicular

Introduction

In this sub-unit, the students use the concept of slope for describing the properties of parallel and perpendicular lines. Before stating and proving the theorems on parallel and perpendicular lines, students should arrive at desired conclusions by doing Activity 4.8 and considering some questions for class discussion. Help your students to prove Theorems 4.1 and 4.2. After a discussion of Examples 1 and 2, you can assign questions of Exercise 4.6 from the student text book to be done in class.

Teaching Notes

Activity 4.8 will help to recall the basic definitions of parallel and perpendicular lines. Two lines in the same plane are parallel if they have no points in common. Similarly, ask the students to explain perpendicularity of lines. Discuss all questions in Activity 4.8 in class.

Answers to Activity 4.8

1. Two lines are parallel if they do not intersect (or do not have intersection point) and are on the same plane. Two line are perpendicular if they intersect at right angle and they are in the same plane.
2. a. The line ℓ_1, intersects x-axis at a point $(-1, 0)$ and y-axis at a point $(0, 1)$.

   Thus, its slope is 1.

   The line ℓ_2 intersects x- axis at a point $(1, 0)$ and y-axis at a point $(0, -1)$. Thus its slope is 1.

   b. Equation of the line ℓ_1 is $y = x + 1$, and ℓ_2 is $y = x - 1$

   c. Lines ℓ_1 and ℓ_2 have the same slope.
3. a. Slope of ℓ_1 is 1 and slope of ℓ_2 is -1.

   b. Equation of the line: ℓ_1 is $y = x + 1$ ℓ_2 is $y = -x + 1$

   c. The product of their slopes is -1. That is, $m_{1.}\ m_2 = 1 \times -1 = -1$

Assist the students to try the proofs of Theorems 4.1 and 4.2. Let them be aware of the reason why vertical lines are treated separately.

Let the students do Examples 1, 2 and 3 to practice the use of $m_1 = m_2$ and $m_1 . m_2 = -1$. Exercise 4.6 contains enough problems for class exercise and homework. Question 7 can be assigned for a group work and let representatives explain the solutions on the blackboard for the other groups. You can use the additional questions presented at the end of the answers to review exercises to motivate your active students.

Assessment

The Review Exercises on unit 4 consists of questions of the most important concepts discussed in the study. You can use and apply it in different ways such as class exercise, assignment or some comprehensive ones can be given as questions for an examination. Identify the questions and allocate them based on the strength of your students.

Answers to Exercise 4.6

1. a. Slope of $\overleftrightarrow{AB}$ is $\frac{-2-3}{2+1} = \frac{-5}{3}$ and slope of $\overleftrightarrow{PQ} = \frac{9-4}{-2-1} = \frac{-5}{3}$. Since $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ have the same slope, the line through A and B is parallel to the line through P and Q.

   b. Slope of $\overleftrightarrow{AB} = \frac{-5-5}{2+3} = -2$ and slope of $\overleftrightarrow{PQ} = \frac{5-4}{1+1} = \frac{1}{2}$. The product of the slope of the line through A and B and the slope of the line through P and Q is -1. Therefore, the line through A and B is perpendicular to the line through P and Q.

2. Let ℓ_1 be the line passing through the given points whose slope is:

   $m_1 = \frac{-2+3}{-3-2} = -\frac{1}{5}$. Thus, the slope m_2 of a line perpendicular to ℓ_1 is

   $m_2 = \frac{-1}{m_1}$, that is $m_2 = 5$

3. Slope of $\overleftrightarrow{AB} = \frac{1+2}{-3+5} = \frac{3}{2}$ and slope of $\overleftrightarrow{CD} = \frac{-3}{1-3} = \frac{3}{2}$

   Slope of $\overleftrightarrow{BC} = \frac{0-1}{3+3} = -\frac{1}{6}$ and slope of $\overleftrightarrow{AD} = \frac{-3+2}{1+5} = \frac{-1}{6}$

   Since slope of $\overleftrightarrow{AB}$ is equal to slope of $\overleftrightarrow{CD}$, $\overline{AB}$ and $\overline{CD}$ are parallel.

   Since slope of $\overleftrightarrow{BC}$ is equal to slope of $\overleftrightarrow{AD}$, $\overline{BC}$ and $\overline{AD}$ are parallel. Thus, the quadrilateral $ABCD$ is a parallelogram because its pairs of opposite sides are parallel.

4. The slope of the line ℓ: $2x - 3y = 6$ is $\frac{2}{3}$.

   a. The slope of the line through (2, –1) parallel to ℓ is $\frac{2}{3}$.

   So, $\frac{y+1}{x-2} = \frac{2}{3} \Rightarrow y = \frac{2}{3}x - \frac{7}{3}$ is the slope-intercept form of the line parallel to ℓ.

   b. Slope of the line perpendicular to ℓ is $\frac{-3}{2}$.

   So, $\frac{y+1}{x-2} = \frac{-3}{2} \Rightarrow y = \frac{-3}{2}x + 2$ is the slope-intercept form of the line perpendicular to ℓ.

5. a. Slope of the given line ℓ is $\frac{2}{5}$. Thus, equation of the line through (–1, 2) parallel to ℓ is $\frac{y-2}{x+1} = \frac{2}{5}$ which is $2x - 5y + 12 = 0$

   b. The given line ℓ has no slope. It is vertical line. Thus, equation of the line passing through (4, –6) parallel to ℓ is $\ell_1 : x = 4$.

6. a. Since the product of their slopes is – 1, they are perpendicular.

   b. They are neither parallel nor perpendicular.

   c. They are neither parallel nor perpendicular.

   d. Since their slopes are the same, they are parallel.

7. a. Slope of the line passing through the points (3, 1) and (–1, 3) is:

   $m_1 = \frac{3-1}{-1-3} = \frac{-2}{4} = -\frac{1}{2}$

   Thus, equation of the line through the point (2, 5) parallel to ℓ_1 is

   $\frac{y-5}{x-2} = -\frac{1}{2}$, that is $y = \frac{-1}{2}x + 6$

   b. Slope of ℓ is –1. Thus, equation of the line through the point (2, 5) parallel to ℓ is $y = -x + 7$ or $x + y = 7$.

   c. The slope of the line ℓ_1 joining the points (–1, 2) and (4, –2) is $m_1 = -\frac{4}{5}$

   Slope of the line perpendicular to ℓ_1 is $\frac{5}{4}$. Thus, equation of the line through (2, 5) perpendicular to ℓ_1 is $\frac{y-5}{x-2} = \frac{5}{4} \Rightarrow 4y - 5x - 10 = 0$.

d. Slope of ℓ is 1

Slope of the line perpendicular to ℓ is –1. Thus, equation of the line through (2, 5) perpendicular to ℓ is $\frac{y-5}{x-2} = -1 \Rightarrow y = -x + 7$

8. a. Let $\ell_1 : 4x + ky = 12$ and let $\ell_2 : x = 3y$

$ky = -4x + 12$ and $y = \frac{1}{3}x$

Slope of ℓ_1 is $\frac{-4}{k}$ and slope of ℓ_2 is $\frac{1}{3}$. Since ℓ_1 is parallel to ℓ_2.

$-\frac{4}{k} = \frac{1}{3}$. So $k = -12$. Note that $k \neq 0$.

Therefore, ℓ_1 is parallel to ℓ_2 for $k = -12$.

b. Let $\ell_2 : x - 3y = 5$ with slope $m_2 = \frac{1}{3}$.

Since ℓ_1 and ℓ_2 are perpendicular, $\left(\frac{-4}{k}\right)\left(\frac{1}{3}\right) = -1 \Rightarrow k = \frac{4}{3}$

Therefore, ℓ_1 is perpendicular to ℓ_2 for $k = \frac{4}{3}$.

9. a.

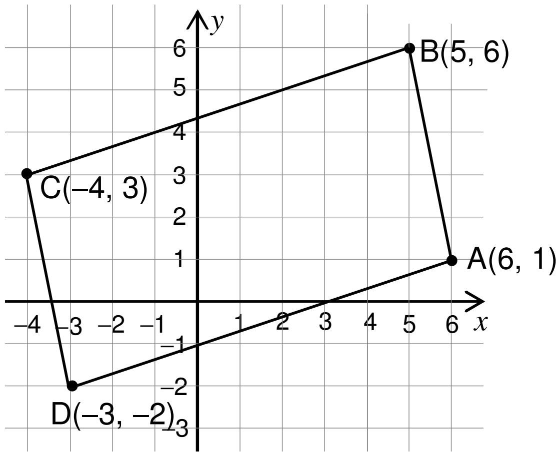


Figure 4.10

$AB = \sqrt{(5-6)^2 + (6-1)^2} = \sqrt{26}$; $BC = \sqrt{(5+4)^2 + (6-3)^2} = \sqrt{90}$

$CD = \sqrt{(-4+3)^2 + (3+2)^2} = \sqrt{26}$; $AD = \sqrt{(6+3)^2 + (1+2)^2} = \sqrt{90}$

Slope of $\overleftrightarrow{AB} = \frac{6-1}{5-6} = -5$ and slope of $\overleftrightarrow{CD} = \frac{3+2}{-4+3} = -5$

Thus, $\overline{AB}$ is parallel to $\overline{CD}$

Slope of $\overleftrightarrow{BC} = \frac{3-6}{-4-5} = \frac{1}{3}$ and slope of $\overleftrightarrow{AD} = \frac{-2-1}{-3-6} = \frac{1}{3}$

Thus, $\overline{BC}$ is parallel to $\overline{AD}$.
Therefore, since two opposite sides are parallel and equal, then the figure with the given vertices is a parallelogram.
b.

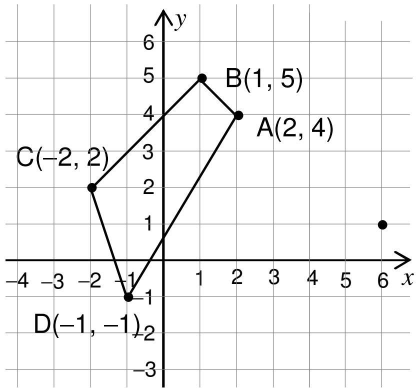


Figure 4.11

$AB = \sqrt{(1-2)^2+(5-4)^2} = \sqrt{2}$; $CD = \sqrt{(-1+2)^2+(1-2)^2} = \sqrt{2}$

Thus, $AB = CD$.

$BC = \sqrt{(1+2)^2+(5-2)^2} = \sqrt{9+9} = 3\sqrt{2}$; $AD = \sqrt{(2+1)^2+(4-1)^2} = \sqrt{9+9} = 3\sqrt{2}$.

Thus, $BC = AD$.

Slope of $\overleftrightarrow{AB} = -1$ and slope of $\overleftrightarrow{AD} = 1$

(Slope of $\overleftrightarrow{AB}$) $\times$ (Slope of $\overleftrightarrow{AD}$) $= -1$. Hence, $\overline{AB}$ is perpendicular to $\overline{AD}$.

Slope of $\overleftrightarrow{BC} = 1$ and hence $\overline{AB}$ is perpendicular to $\overline{BC}$.

Therefore, the figure with the given vertices is a rectangle.

10.

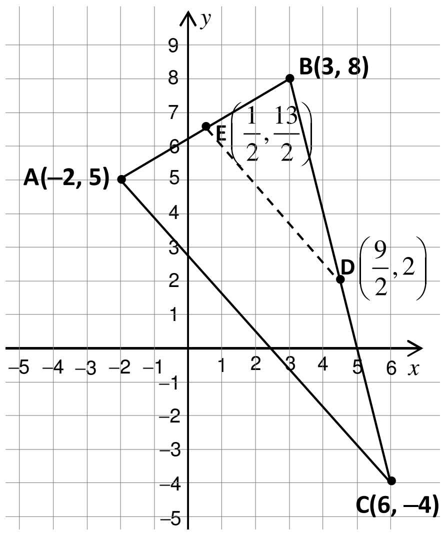


Figure 4.12

The mid-point of side $\overline{AB}$ is say $E\ (x_o, y_o) = \left(\frac{1}{2}, \frac{13}{2}\right)$.

The mid-point of side $\overline{BC}$ is say $D\ (x'_o, y'_o) = \left(\frac{9}{2}, 2\right)$.

Slope of $\overleftrightarrow{AC} = \frac{-4-5}{6+2} = -\frac{9}{8}$ and slope of $\overleftrightarrow{ED} = \frac{2-\frac{13}{2}}{\frac{9}{2}-\frac{1}{2}} = \frac{-9}{8}$.

Since $\overleftrightarrow{AC}$ and $\overleftrightarrow{ED}$ have the same slope, $\overline{ED}$ is parallel to $\overline{AC}$.

$$ED = \sqrt{\left(\frac{9}{2} - \frac{1}{2}\right)^2 + \left(2 - \frac{13}{2}\right)^2} = \sqrt{(4)^2 + \left(-\frac{9}{2}\right)^2} = \sqrt{16 + \frac{81}{4}} = \frac{1}{2}\sqrt{145}$$

$$AC = \sqrt{(6+2)^2 + (-4-5)^2} = \sqrt{8^2 + (-9)^2} = \sqrt{64+81} = \sqrt{145}.$$

Hence $ED = \frac{1}{2}\ AC$ and $\overleftrightarrow{ED}$ is parallel to $\overleftrightarrow{AC}$

Answers to Review Exercises on Unit 4

1. $AB = \sqrt{(-1-1)^2 + (1+1)^2} = \sqrt{(-2)^2 + (2)^2} = \sqrt{4+4} = \sqrt{8}$ units

   $BC = \sqrt{\left(\sqrt{3}+1\right)^2 + \left(\sqrt{3}-1\right)^2} = \sqrt{3 + 2\sqrt{3} + 1 + 3 - 2\sqrt{3} + 1} = \sqrt{8}$ units

   $AC = \sqrt{\left(\sqrt{3}-1\right)^2 + \left(\sqrt{3}+1\right)^2} = \sqrt{3 - 2\sqrt{3} + 1 + 3 + 2\sqrt{3} + 1} = \sqrt{8}$ units

   Therefore, ΔABC is equilateral.

2. The three points $\left(-\frac{1}{2}, 3\right)$, $(3, -1)$ and $\left(\frac{13}{2}, -5\right)$ divide the line segment into four equal parts.

3. $\frac{y+2}{x+4} = \frac{6+2}{3+4} = \frac{8}{7}$ Thus, $7y - 8x - 18 = 0$.

   The equation of the line passing through the given points is ℓ: $7y - 8x - 18 = 0$

4. a. $y + 3x = 27$ b. $2y - x - 12 = 0$

5. a. $AB = \sqrt{1^2 + 1^2} = \sqrt{2}$ $BC = \sqrt{(2-1)^2 + (0-1)^2} = \sqrt{2}$

   $AC = \sqrt{(2)^2 + (0)^2} = \sqrt{4} = 2$

   $AB^2 + BC^2 = \left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2 = 2 + 2 = 4 = AC^2$

   Therefore, ΔABC is right angled triangle.

   Alternative method: slope of $\overleftrightarrow{AB}$ is $m_1 = \frac{1-0}{1-0} = 1$

   $m_1 = m_2 = 1\ (-1) = -1$. $\overline{AB} \perp \overline{BC}$ and ΔABC is right angled at B.

b. $PQ = \sqrt{(-3-3)^2 + (4-1)^2} = \sqrt{(-6)^2 + 3^2} = \sqrt{36+9} = \sqrt{45}$

$QR = \sqrt{(-3-(-3))^2 + (1-4)^2} = \sqrt{0^2 + (-3)^2} = 3$

$PR = \sqrt{(-3-3)^2 + (1-1)^2} = \sqrt{(-6)^2 + 0^2} = 6$

$PR^2 + QR^2 = (6)^2 + (3)^2 = \left(\sqrt{45}\right)^2 = PQ^2$

Therefore, ΔPQR is a right angled triangle.

6. a. Slope is $\frac{2}{3}$ and y-intercept is $-\frac{4}{3}$.

b. Slope is $\frac{5}{2}$ and y-intercept is 1.

c. Slope is $\frac{-6}{5}$ and y-intercept is $\frac{4}{5}$.

d. Slope is $\frac{7}{3}$ and y-intercept is $\frac{1}{3}$.

7. a. $3y - 2x = 7$

b. $6y - 5x = 16$

8. Slope of the line through (–4, 5) and (3, t) is $m_1 = \frac{t-5}{3+4} = \frac{t-5}{7}$

Slope of the line through (1, 3) and (–4, 2) is $m_2 = \frac{2-3}{-4-1} = \frac{1}{5}$

Since the two lines are perpendicular, we have $m_1.m_2 = -1$

$\left(\frac{t-5}{7}\right)\left(\frac{1}{5}\right) = -1$. So, $t = -30$

9. Slope of the line through (4, –3) and (t, –2) is $m_1 = \frac{-2+5}{t-4} = \frac{1}{t-4}, t \neq 4$

Slope of the line through (–2, 4) and (4, –1) is $m_2 = \frac{-1-4}{4+2} = \frac{-5}{6}$.

Since the two lines are parallel, we have $m_1 = m_2$.

$\frac{1}{t-4} = \frac{-5}{6} \Rightarrow t = \frac{14}{5}$.

10. Write each general equation in slope-intercept form equation of a line.

Let ℓ_1: $Ax + By + C = 0, B \neq 0$

$By = -Ax - C \Rightarrow y = -\frac{A}{B}x - \frac{C}{B}$

Hence, its slope: $m_1 = -\frac{A}{B}, \ B \neq 0$.

Let ℓ_2 : $ax + by + c = 0$, $b \neq 0$. Then $y = \frac{-a}{b}x - \frac{c}{b}$

Hence, its slope: $m_2 = \frac{-a}{b}$, $b \neq 0$

Since ℓ_1 and ℓ_2 are perpendicular, we have $m_1 . m_2 = -1$

$$\left(\frac{-A}{B}\right)\left(\frac{-a}{b}\right) = -1$$

$$\frac{Aa}{Bb} = -1 \Rightarrow Aa + Bb = 0.$$

Once the chapter is completed, give a summary of the major concepts based on the competences listed. To motivate and see their follow ups, give them a test covering the unit. You may use some questions from the review exercises.

Additional problems for fast learners

1. Check whether or not the point

   i. P (4, 2) ii. Q (–6, 4)

   iii. R$\left(-\frac{3}{2}, \frac{7}{20}\right)$ iv. S$\left(\frac{1}{3}, -\frac{1}{5}\right)$ lies on the line $10y - 3x = 8$.

2. Let ℓ be the line through P (1, –1) and Q (4, 5), find the equation of the line perpendicular to $\overline{PQ}$ and through

   i. P ii. The mid-point of $\overline{PQ}$ iii. Q

   Are these lines parallel?

3. Find the equations of the line containing the diagonals of the parallelogram with vertices A (–4, 3), B (–3, –2), C (6, 1) and D (5, 6). Are these lines perpendicular?

4. Given lines ℓ_2: $(1 - a)x - y = -1$ and ℓ_2: $6x + ay = 9$, find the value of a so that

   i. $\ell_1 \parallel \ell_2$ ii. $\ell_1 \perp \ell_2$

UNIT 5 TRIGONOMETRIC FUNCTIONS

INTRODUCTION

The **trigonometric functions** are functions of angles. They are originally used to relate the angles of a triangle to the lengths of the sides of a triangle. **Trigonometry** simply means **triangle measure**. Trigonometric functions are important in the study of triangles and different phenomena in real life.

The most familiar trigonometric functions are the **sine, cosine**, and **tangent**. Hence, in this unit, major emphasis shall be given to the discussion of these functions, their trigonometrical values and graphs. The reciprocals of the basic trigonometric functions, simple trigonometric identities and real life applications of the trigonometric functions are also given due attention.

Unit Outcomes

After completing this unit, students will be able to:

- *know principles and methods in sketching graphs of basic trigonometric functions.*
- *understand important facts about reciprocals of basic trigonometric functions.*
- *identify trigonometric identities.*
- *solve real life problems involving trigonometric functions.*

Suggested Teaching Aids in Unit 5

In addition to the student's textbook and the teacher's guide, the teacher is advised to prepare and bring into the class the following teaching aids whenever the topic requires.

- ✓ Model of pair of rays to demonstrate counter-clockwise and clockwise rotations. You can use it when you teach positive and negative angles.
- ✓ Pre made graphs of the sine, cosine and tangent functions (larger size). You can fix the pre made graph of sine, cosine and tangent functions to the wall whenever you teach the particular topic. This not only saves your time but also helps you to display a very accurate and better looking graph to the students.
- ✓ An enlarged trigonometric table. Prepare a relatively larger sample trigonometric table and fix it to the wall whenever you teach how to use the trigonometric table.
- ✓ Compass to draw circles and arcs. This helps you to draw a unit circle.
- ✓ Protractors to measure angles whenever necessary.
- ✓ Straight edge or ruler. (You can use them whenever you need to draw a coordinate system or simply a line or line segment)
- ✓ Set square to check perpendicularity of line segments or to draw them.
- ✓ A thread to measure circumference of a circle. Students should prepare these before doing Group work 5.1.

5.1 BASIC TRIGONOMETRIC FUNCTIONS

Periods allotted: 15 Periods

Competencies

At the end of this sub-unit students will be able to;

- *define the sine, cosine and tangent functions of an angle in the standard position.*
- *determine the values of the functions for and angle in the standard position, given the terminal side of that angle.*
- *determine the values of the sine, cosine and tangent functions for quadrantal angles.*
- *locate negative and positive angles by identifying the direction of rotation.*
- *determine the values of trigonometric functions for some negative angles.*
- *determine the algebraic signs of the sine, cosine and tangent functions of angles in different quadrants.*
- *describe the relationship between trigonometrical values of complementary angels.*
- *describe the relationship between trigonometrical values of supplementary angles.*
- *determine the relationship between trigonometrical values of co-terminal angles.*
- *determine the trigonometrical values of large angles.*

- *construct of table of values for y = sin θ where $-2\pi \le \theta \le 2\pi$.*
- *draw the graph of y = sin θ*
- *determine the domain, range and period of the sine function.*
- *draw the graph of y = cos θ*
- *determine the domain, range and period of the cosine function.*
- *construct a table of values for y = tan θ where $-2\pi \le \theta \le 2\pi$.*
- *draw the graph the tangent function y = tan θ.*
- *determine the domain, range and period of the tangent function.*
- *discuss the behavior of the graph of tangent function.*

Vocabulary: Positive angle, Negative angle, Angle in standard position, Quadrantal angle, Radian, Degree, Unit circle, Special angle, Reference angle, Complementary angles, Supplementary angles, Co–terminal angles, Period

Introduction

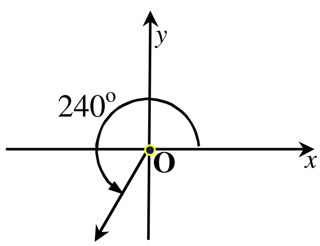
In this sub-unit, the three basic trigonometric functions, that is, the sine, cosine and tangent functions are thoroughly discussed. Values of these trigonometric functions for complementary, supplementary, co-terminal and large angles and the graphs of the three trigonometric functions are also the central points of the subunit.

The Sine, Cosine and Tangent Functions

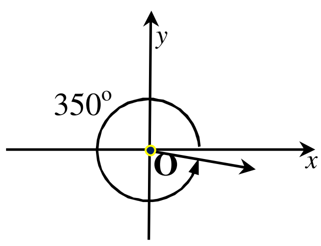
Teaching Notes

You can start the lesson by introducing the concept of an angle formed by a ray rotating in counter- clockwise or clockwise direction. Demonstrate this using a model of a ray. Then tell them that an angle is **positive** for anti-clockwise rotation and **negative** for clockwise rotation. Give examples of positive and negative angles and then introduce to them angles in standard position, first, second, third and fourth quadrant angles and quadrantal angles. Give many examples of each and let them do Exercise 5.1 as a class work. Pair or triple of students sitting on the same desk may work together. Go round the class and check their work. Invite few students to come to the board and show their work. Let others comment on their peer's work.

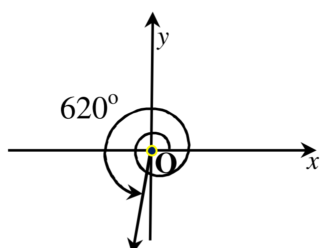
Answers to Exercise 5.1



Quadrant III
a



Quadrant IV
b



Quadrant III
c

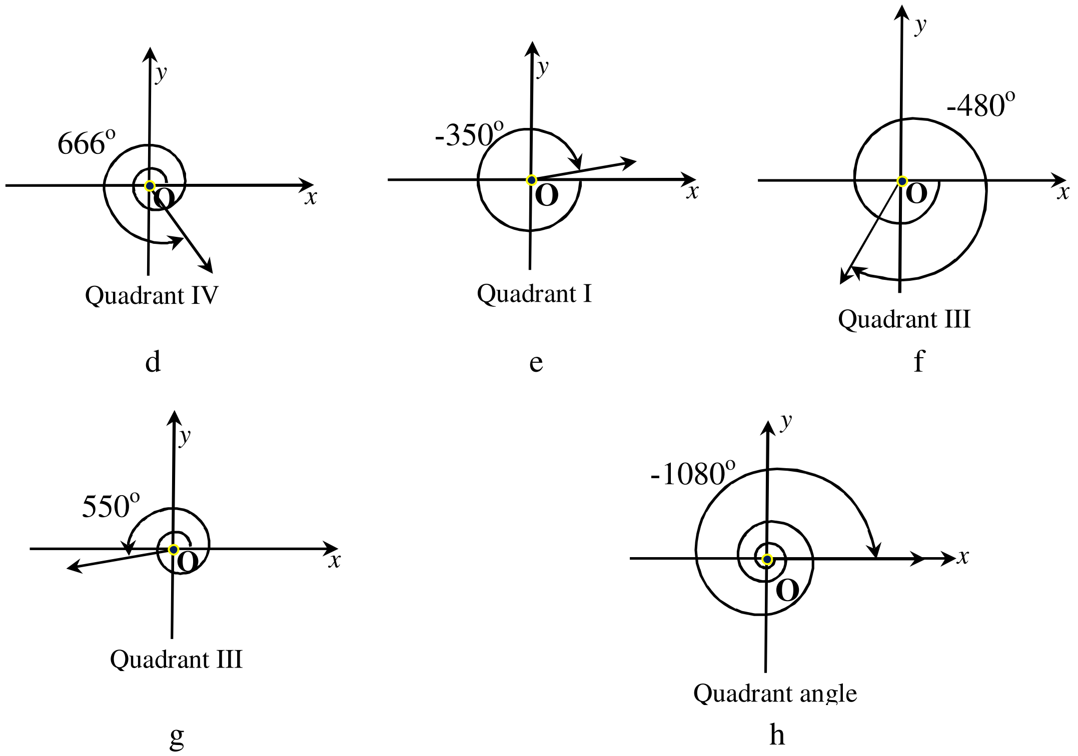


Figure 5.1

Then, introduce the concept **radian** as a unit of measuring angles. Explain the concept in relation to the size of an angle subtended at the center of a circle by an arc length ***s*** units and radius *r*. Let students do Group work 5.1 so that they can approximate 1 radian measure in degrees using a central angle subtended by an arc equal in length to the radius of the circle. Remember that students should be told to bring compasses, a thread, a ruler and protractors ahead of time.

After doing Group work 5.1, they are also expected to guess a formula which helps them convert degrees to radians and vice versa.

Answers to Group work 5.1

1–5. Measuring the circumference of a circle of radius 5 cm and dividing the circumference by 10 (length of diameter) gives a value which is approximately equal to $\pi (\approx 3.14)$.

6 and 7. The measure of the central angle AOB which is subtended by an arc equal in length to the radius is approximately equal to 57.3°.

8. $180° = \dfrac{180^o}{57.3^o} \approx 3.14 \text{ radians}$

$$360° = \frac{360^o}{57.3^o} \approx 6.28 \text{ radians}$$

Summarize their group work by writing the formula to convert degrees to radians and vice versa. Let them do the examples given in the text first and then you can give them Exercise 5.2 as a class work. Let this be an individual task. After giving them a reasonable time to do the exercises, ask volunteer students to tell the answer one by one. Every time let the other students agree or disagree by raising their hands.

Since there can be fast learning students in your class, you can use the following questions for those students.

1. Explain why tan 575° is positive but cos 575° is negative.
2. Study the graphs of sine and cosine functions and identify the quadrant in which both functions are increasing.
3. Complete the following table and graph $y = \sin\theta$ and $y = 2\sin\theta$ on the same coordinate system:

θ	0°	45°	90°	135°	180°	225°	225°	270°	315°	360°
$\sin\theta$										
$2\sin\theta$										

4. The period of the functions $y = \sin\theta$ and $y = \cos\theta$ is 360°. What is the period of
   i. $y = \sin 2\theta$?
   ii. $y = \cos 2\theta$?
5. The period of the function $y = \tan\theta$ is 180°. What is the period of $y = \tan 3\theta$?

Answers to Exercise 5.2

1. a. $\frac{1}{3}\pi$ rad b. $\frac{1}{4}\pi$ rad c. $-\frac{5}{6}\pi$ rad
   d. $\frac{1}{2}\pi$ rad e. $-\frac{3}{2}\pi$ rad f. $\frac{3}{4}\pi$ rad
2. a. 15^{o} c. 120^{o} e. -600^{o}
   b. -30^{o} d. 150^{o} f. 171.9^{o}

Definition of the Sine, Cosine and Tangent Functions

Draw a relatively larger right angle triangle ABC on the board. Then ask the students orally to identify the sides opposite to angle A and angle B, the sides adjacent to angle A and angle B and the hypotenuse of the triangle.

Then define to them sin θ, cos θ and tan θ by considering an angle θ in standard position and a point P (x, y) on its terminal side. Give them several examples to evaluate the three trigonometric functions. Then you can give them the first three problems of Exercise 5.3 as a class work and the last three of the same exercise as a home work depending on the time that you have. Give appropriate corrections to their work and have your own judgment if you can proceed to the next topic or not.

Answers to Exercise 5.3

a. $\sin\theta = -\frac{4}{5}$; $\cos\,\theta = \frac{3}{5}$; $\tan\theta = -\frac{4}{3}$

b. $\sin\theta = -\frac{8}{10}$; $\cos\,\theta = \frac{-6}{10}$; $\tan\theta = \frac{8}{6}$

c. $\sin\theta = -\frac{1}{\sqrt{2}}$; $\cos\,\theta = \frac{1}{\sqrt{2}}$; $\tan\theta = -1$

d. $\sin\theta = \frac{\sqrt{2}}{2}$; $\cos\,\theta = \frac{-\sqrt{2}}{2}$; $\tan\theta = -1$

e. $\sin\theta = \frac{-2\sqrt{5}}{10}$; $\cos\,\theta = \frac{4\sqrt{5}}{10}$; $\tan\theta = -\frac{1}{2}$

f. $\sin\theta = 0$; $\cos\,\theta = 1$; $\tan\theta = 0$

After introducing a unit circle, guide students to construct a unit circle. Using the unit circle let them determine the values of the sine, cosine and tangent functions for quadrantal angles (i.e. 0^o, 90^o, 180^o, 270^o, 360^o). Give them Exercise 5.4 as a class work. Let them do it in groups. Neighboring students may work together. Give them ample time to do and present their work. Involve the others to comment on each presentation. You are also expected to give corrections on the spot. Here, you can use software (if available) to help students visualize the concept and practice determining trigonometric values of angles.

Answers to Exercise 5.4

	θ	**sin θ**	**cos θ**	**tan θ**
a.	0^o	0	1	0
b.	360^o	0	1	0
c.	450^o	1	0	undefined
d.	540^o	0	−1	0
e.	630^o	−1	0	undefined

Next assist them to approximate the values of the trigonometric functions for special angles, i.e. 30^o, 45^o and 60^o. Give them time to do Group work 5.2. This group work is aimed at helping students find the trigonometric values of the special angle 45^o through their own effort. Their effort should certainly be well guided by you as a teacher.

Before going to the next lesson use different assessment techniques like oral and written questions, class and home works to make sure that the students are able to define the three trigonometric functions of an angle in standard position. Check whether they can find the values of the sine, cosine and tangent functions for quadrantal and special angles.

Answers to Group work 5.2

a. AB = $\sqrt{2}$ b. m($\angle A$)= 45° c. Yes

d. Side BC is opposite to angle A. Side AC is adjacent to angle A.

e. $\sin A = \frac{1}{\sqrt{2}}$; $\cos A = \frac{1}{\sqrt{2}}$; $\tan A = 1$

Trigonometric Values of Negative Angles

Revise negative and positive angles using clockwise and anti-clockwise rotations. Using a unit circle, let students find in groups trigonometric values of some negative quadrantal angles like –90°, –180°, –270° and –360°. Then give them time to do Activity 5.1. The main purpose of Activity 5.1 is to guide the student so that they can discover the relationship:

$$\sin(-\theta) = -\sin\theta,\ \cos(-\theta) = \cos\theta,\ \text{and } \tan(-\theta) = -\tan\theta, \text{ for any angle } \theta$$

So encourage the students to do every question in the activity until they are able to find the relationship mentioned above. Go round the class and check if they are completing the table correctly. Tell them that they can do question number 2 referring to the table in question number1.The third question is to be answered from their observation of the answers of the second question.

Make sure that your students are able to locate positive and negative angles by identifying the direction of rotation. Assess (using oral or written questions) if they can determine trigonometrical values of angles having equal measures but opposite in sign.

Answers to Activity 5.1

1. a

θ	**0°**	**30°**	**45°**	**60°**	**90°**	**180°**	**270°**	**360°**
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	–1	0
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	–1	0	1
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	—	0	—	0

b.

θ	-30^{o}	-45^{o}	-60^{o}	-90^{o}	-180^{o}	-270^{o}	-360^{o}
sin θ	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	0	1	0
cos θ	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1
tan θ	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	—	0	—	0

2. a. No b. Yes c. No d. No e. Yes
   f. No g. No h. Yes i. No

3. $\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$

Algebraic signs of the sine, cosine and tangent functions

Let the students discover the algebraic signs of the sine, cosine and tangent functions in all of the four quadrants. Before students are engaged in doing Activity 5.2, let them do the first example given in the textbook. Ask the whole class the following question orally:

If θ is a first quadrant angle, then what is the sign of the x co- ordinate? The sign of the y co- ordinate? What is the sign of $\sin\theta$, $\cos\theta$ and $\tan\theta$? Give them an appropriate explanation in the following way:

If θ is a first quadrant angle then,

the sign of $\sin\theta = \frac{opp}{hyp} = \frac{y}{r} = \frac{+ve}{r} = +ve$,

$\cos\theta = \frac{adj}{hyp} = \frac{x}{r} = \frac{+ve}{r} = +ve$ and

$\tan\theta = \frac{opp}{adj} = \frac{y}{x} = \frac{+ve}{+ve} = +ve$.

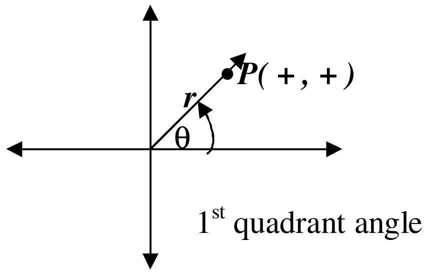
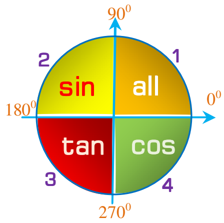


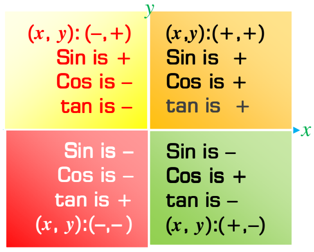
Figure 5.2

Therefore, in quadrant I all the three trigonometric functions are positive.

If θ is a second quadrant angle, then what is the sign of the x co- ordinate? The y co- ordinate? What is the sign of $\sin\theta$, $\cos\theta$ and $\tan\theta$?

Let the students do the same consideration for the remaining two quadrants. Activity 5.2 helps them for such purposes. Divide them in groups. Students sharing the same desk may work together. Let them do Activity 5.2 and reflect their responses. Summarize their work on the board diagrammatically like the following:





For easier memory, students may keep in mind the following statement:

All **S**tudents **T**ake **C**hemistry
Taking the first letter of each word we have
All: **A**ll are positive
Students: **S**ine is positive
Take: **T**angent is positive
Chemistry: **C**osine is positive

Answers to Activity 5.2

1. a. sin θ is –ve, cos θ is –ve and tan θ is + ve.

   b. sin θ is –ve, cos θ is + ve, tan θ is –ve.

2.

	θ has terminal side in quadrant			
	I	II	III	IV
sin θ	+	+	–	–
cos θ	+	–	–	+
tan θ	+	–	+	–

Finally, give them time to do Group work 5.3. To do the first question, they can refer to their work in Activity 5.2. To determine the sign of cos 267°, help them first to determine the quadrant where the 267° is located. Then, they can determine the sign of the given trigonometric function in that particular quadrant.

Give them additional exercise problems if necessary to check if they can determine the algebraic signs of the sine, cosine and tangent functions. Give them a group task so that they can summarize the signs of the values of the three basic trigonometric functions in each of the four quadrants.

Answers to Group work 5.3

1. a. III b. II c. IV d. III

a. a. Negative b. Positive c. Negative

b. sin θ is negative, cos θ is positive, tan θ is negative

Trigonometrical Values of Complementary Angles

Assess their prior knowledge of complementary angles in question and answer form. Let the students do Activity 5.3. Remember that the main purpose of this activity is to enable the students discover a relationship between trigonometrical values of complementary angles. Guide them until they can generalize the following relationship:

If α and β are any two complementary angles, then

$$\mathbf{\sin\alpha = \cos\beta} \qquad \mathbf{\cos\alpha = \sin\beta} \qquad \mathbf{\tan\alpha = \frac{1}{\tan\beta}}$$

Give them some more examples and let them practice these relationships by doing Exercise 5.5. You can do it in a whole group discussion you being the group leader. You can even ask them orally to check if they are able to explain the relationship between trigonometrical values of complementary angles.

Answers to Activity 5.3

1. a. $\sin 30^\circ = \frac{1}{2}$; $\cos 30^\circ = \frac{\sqrt{3}}{2}$; $\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

   $\cos 60^\circ = \frac{1}{2}$; $\sin 60^\circ \frac{\sqrt{3}}{2}$; $\tan 60^\circ = \sqrt{3}$

   b. i. $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$ ii. $\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$

   iii. $\tan 30^\circ = \frac{1}{\tan 60^\circ} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

2. a. $\sin\alpha = \frac{3}{5}$; $\cos\alpha = \frac{4}{5}$; $\tan\alpha = \frac{3}{4}$

   $\sin\beta = \frac{4}{5}$; $\cos\beta = \frac{3}{5}$; $\tan\beta = \frac{4}{3}$

   b. i. $\sin\alpha = \cos\beta = \frac{3}{5}$ ii. $\sin\beta = \cos\alpha = \frac{4}{5}$

   iii. $\tan\alpha = \frac{1}{\tan\beta} = \frac{3}{4}$

   c. If $\alpha + \beta = 90^\circ$, then $\sin\alpha = \cos\beta$, $\cos\alpha = \sin\beta$, and $\tan\alpha = \frac{1}{\tan\beta}$ or

   $\tan\beta = \frac{1}{\tan\alpha}$.

Give them time to do Exercise 5.5. Let each student does the exercise individually. Check if a particular student needs your assistance. Check if most of them are on the right track or not. This assessment should help you whether to continue to the next lesson or repeat the lesson once.

Answers to Exercise 5.5

a. 0.5150 b. $\frac{3}{5}$ c. $\frac{4}{5}$

d. k e. r f. $\frac{m}{n}$

Reference angle (θ_R)

First explain the concept reference angle using several examples. Demonstrate the corresponding reference angles for various given angles on the black board. Make sure if students are now able to identify the reference angle θ_R for a given angle θ before going to consider their trigonometric values. To this end they may do Exercise 5.6. After giving them some reasonable time you may invite students to come to the board and do the activity. Others should comment, agree or disagree.

Answers to Exercise 5.6

a. 30^o b. 10^o c. 60^o d. 40^o

e. 81^o f. 45^o g. 45^o h. 60^o

Next, let students find the sine, cosine and tangent values of a given angle θ and its reference angle. Encourage them to find trigonometric values of some more angles like 120^o, 150^o, 225^o, -300^o etc. by considering their reference angles. Let them compare the trigonometric values of angle θ and its reference angle. From such an activity they are expected to generalize the following by their own:

> *The values of the trigonometric function of a given angle θ and the values of the corresponding trigonometric functions of the reference angle θ_R are the same in absolute value but they may differ in sign.*

Let them practice this by doing the first question of Exercise 5.7

Supplementary Angles

Revise the concept of supplementary angles in question and answer form and demonstrate how to determine the reference angle for a given angle β if β is in quadrant II or III using the concept of supplementary angles. Next, consider the trigonometric values of an angle and its reference in different quadrants. Assess if your students can tell the relationship between trigonometrical values of supplementary angles using question 2 of Exercise 5.7. Give them the exercise as a class work or home work depending on the time that you have. Check their work and give appropriate corrections.

Answers to Exercise 5.7

1. a. $\sin 75^\circ$, $-\cos 75^\circ$, $-\tan 75^\circ$ b. $\sin 5^\circ$, $-\cos 5^\circ$, $-\tan 5^\circ$
   c $-\sin 40^\circ$, $-\cos 40^\circ$, $\tan 40^\circ$ d. $\sin 80^\circ$, $-\cos 80^\circ$, $-\tan 80^\circ$
   e $\sin 60^\circ$, $\cos 60^\circ$, $\tan 60^\circ$ f. $\sin 20^\circ$, $\cos 20^\circ$, $\tan 20^\circ$

2. a. $\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}$ and -1 b. -0.7986 c. -0.9004
   d. 0.9063 e. -0.3839 f. 0.9135

Co - terminal Angles

Explain the concept co-terminal angles and demonstrate it on the black board. Give them several examples on positive and negative co-terminal angles for a given angle θ. Using a color chalk (to represent an angle and its co-terminal) may help for such purpose. Let students do Activity 5.4 in pairs or those sharing the same desk may work together. Check if they can derive a formula which gives all angles which are co-terminal with 60°. Guide their work. Assist them. Give them hints and clues.

Answers to Activity 5.4

1.

$60^\circ + 1(360^\circ) = 420^\circ$	$60^\circ - 1(360^\circ) = -300^\circ$
$60^\circ + 2(360^\circ) = 780^\circ$	$60^\circ - 2(360^\circ) = -660^\circ$
$60^\circ + 3(360^\circ) = 1140^\circ$	$60^\circ - 3(360^\circ) = -1020^\circ$
$60^\circ + 4(360^\circ) = 1500^\circ$	$60^\circ - 4(360^\circ) = -1380^\circ$
$60^\circ + 5(360^\circ) = 1860^\circ$	$60^\circ - 5(360^\circ) = -1740^\circ$
$60^\circ + 6(360^\circ) = 2220^\circ$	$60^\circ - 6(360^\circ) = -2100^\circ$
.	.
.	.
.	.
$60^\circ + n(360^\circ)$	$60^\circ - n(360^\circ)$

2. $\theta \pm n(360^\circ)$ where $n = 1, 2, 3, \ldots$.

Let students practice how to find several positive and negative co-terminal angles for a given angle θ by doing Exercise 5.8. You can give it as an individual activity to assess whether or not every student is able to find positive and negative co-terminal angles of a given angle.

Answers to Exercise 5.8

a. $70^\circ \pm 360^\circ$ b. $110^\circ \pm 360^\circ$ c. $220^\circ \pm 360^\circ$

d. $270^\circ \pm 360^\circ$ e. $-90^\circ \pm 360^\circ$ f. $-37^\circ \pm 360^\circ$

g. $-60^\circ \pm 360^\circ$ h. $-70^\circ \pm 360^\circ$

Next, let the students do Activity 5.5 to find trigonometric values of co-terminal angles in small groups. Each group should find out the relationship between the trigonometric values of a given angle and its co-terminal angles. For example 60^0, -300^0 and 780^0 are all co-terminal angles. So, all have the same trigonometric values. Similarly, θ and β in this activity are co-terminal and they have the same trigonometric values.

Answers to Activity 5.5

a. Yes because both are angles in standard position and have a common terminal side.

b. θ is positive whereas β is negative.

c. $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$ $\tan\theta = \frac{y}{x}$

d. $\sin\beta = \frac{y}{r}$, $\cos\beta = \frac{x}{r}$ $\tan\beta = \frac{y}{x}$

e. Yes, Yes, Yes

f. Two co-terminal angles have the same trigonometric values.

Let students practice trigonometric value of co-terminal angles by doing the examples given in the text and Exercise 5.9. Create different ability groups and give them the exercise. This will help some learners to use the strengths of other learners, and to learn from others. Make sure that students are able to find trigonometrical values of co-terminal as well as large angles before going to the next topic or lesson.

Answers to Exercise 5.9

1. a. $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}$ b. $-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -1$

   c. $\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{3}$ d. $-\frac{\sqrt{2}}{2}$ $\frac{\sqrt{2}}{2}$ -1

2. a. $\sin 50^o$ b. $-\sin 20^o$ c. $-\cos 15^o$ d. $\cos 50^o$

   e. $-\tan 35^o$ f. $-\sin 80^o$ g. $\cos 55^o$ h. $\tan 55^o$

   i. $-\sin 80^o$ j. $\tan 45^o$ k. $\sin 30^o$ l. $\cos 40^o$

Graphs of the Sine, Cosine and Tangent Functions

Give them a general orientation that in this subunit they are going to draw the graphs of the three trigonometric functions and study their properties.

Before drawing the graph of the sine function encourage your students to construct a table of values for some common values of θ. Activities 5.6 is meant to guide them step by step to the graph of the sine function. To do this activity students may use the unit circle, or the concept of reference and/or co-terminal angles or even their calculator to complete the table of values given. Once they have completed the table, encourage them to mark the values of θ on the horizontal axis and the values of y on the vertical axis. Then they have to plot the points very carefully and join them by a smooth curve to sketch the graph of $\sin\theta$.

It is always advisable if you prepare the graph of $y = \sin\theta$ in advance and bring it to the class for demonstration. Fix it to the wall for few days so that students can be familiarized with the graph. Give ample time and serious consideration for the domain and the range, the period and the values for which $y = \sin\theta$ increases or decreases.

Answers to Activity 5.6

1.

θ in deg	-360	-330	-270	-240	-180	-120	-90
$y = \sin\theta$	0	0.5	1	0.87	0	-0.87	-1

θ in deg	0	90	120	180	240	270	330	360
$y = \sin\theta$	0	1	0.87	0	-0.87	-1	-0.5	0

2 and 3

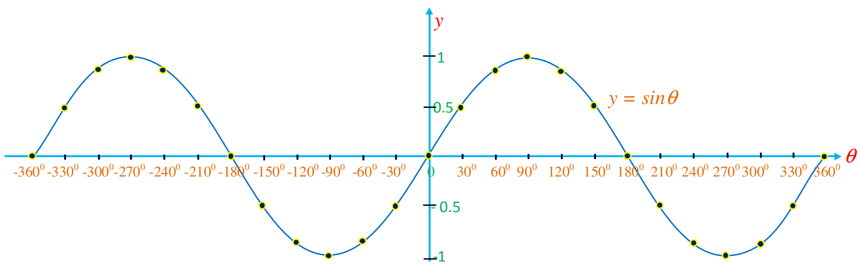


Figure 5.3

4. Domain = the set of real numbers

Range = $\{y \mid -1 \le y \le 1\}$

Now is time to discuss the graph of the cosine function and its properties. Approach this topic in exactly the same way you treated the graph of the sine function. Let them begin by doing Activity 5.7. Guide them; give them hints and assistance while they are completing the table of values. Tell them that they have to use the same approach they used in drawing the sine function. Go round the class and check their drawing. If the majority has finished drawing, then display your pre made graph of $y = \cos\theta$ and fix it to the wall so that they can compare it with that of their drawing. Now start investigating the behavior, the domain and range and the period of the graph together with the students. Finally summarize to them the basic points of the topic.

Answers to Activity 5.7

1.

θ in deg	–360	–300	–270	–240	–180	–120	-90	-60
$y = \cos\theta$	1	0.5	0	–0.5	–1	-0.5	0	0.5

2.

θ in deg	0	60	90	120	180	240	270	300	360
$y = \cos\theta$	1	0.5	0	-0.5	-1	-0.5	0	0.5	1

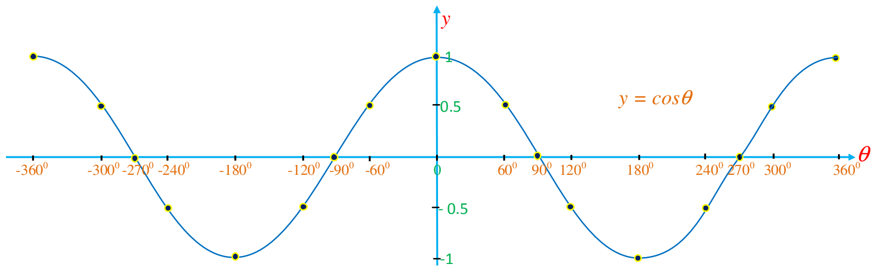


Figure 5.4

3. Domain = the set of real numbers

   Range = $\{y|\ -1 \le y \le 1\ \}$

4. 360^o or 2π radians is the period

The next activity is determining the graph of the tangent function and its properties. You may guide your students to do Activity 5.8 in almost the same fashion as the previous two activities. After they have attempted sketching the graph of y = tan θ show them the graph that you have prepared in advance as a teaching aid. Now get their attention and begin considering its properties like the domain and the range, the values for which y = tan θ is undefined, the period of the function, etc.

Answers to Activity 5.8

1. a.

θ in deg	–360	–315	–270	–225	–180	–135	–90	–45
y = tan θ	0	1	–	–1	0	1	–	–1

   b.

θ in deg	0	45	90	135	180	225	270	315	360
y = tan θ	0	1	–	–1	0	1	–	–1	0

2.

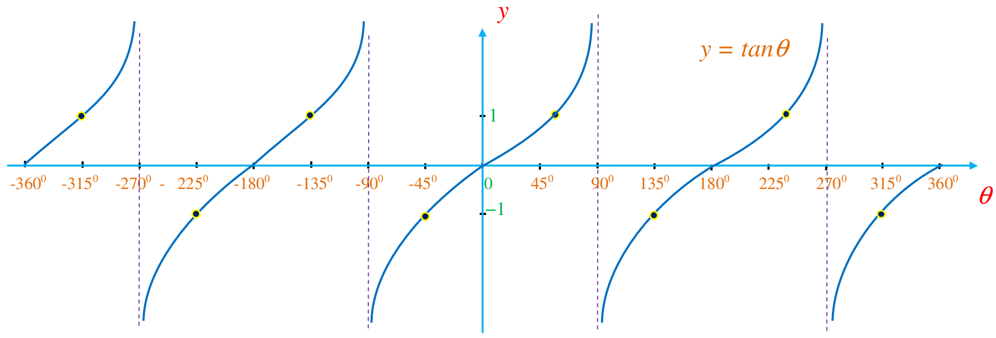


Figure 5.5

3. tan θ is undefined for $\theta = 90^o \pm n(180^0)$, n = 1, 2 , 3, … or

   tan θ is undefined for $\theta = n(\ 90^o\ )$ where n is an odd integer.

4. Domain = $\{\ \theta|\ \ \theta\ \ne n(90^o\ )$ where n is an odd integer$\}$.

   Range = The set of all real numbers

5. 180^o or π radians is the period

Group work 5.4 gives them a chance to practice what has been discussed so far about graphing of trigonometric functions. This task might be a little frustrating for slow learners. Therefore it is recommended that you create a mixed ability group so that slow learners get the necessary support and benefit from fast learners.

Answers to Group work 5.4

1. $\cos\theta = 0$, if $\theta = n\,(90°)$, n is an odd integer

2. $\sin\theta = -1$, if $\theta = -90° + n\,(360°)$, n is an integer or if $\theta = \left(\frac{4n+3}{2}\right)\pi; n \in \mathbb{Z}$

3.

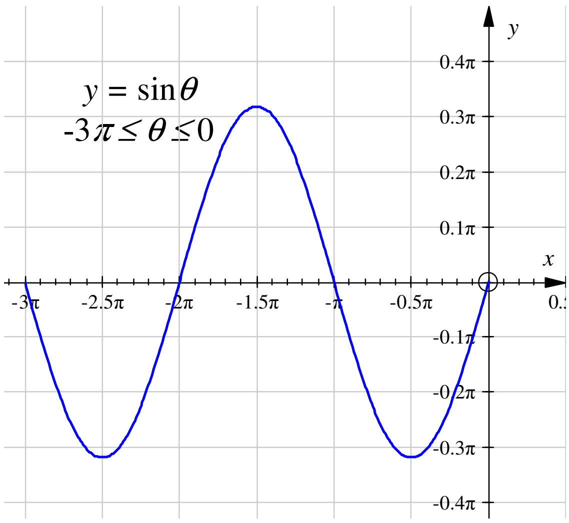


Figure 5.6 Graph of $y = \sin\theta;\ -3\pi \le \theta \le 0$

At this stage your students should be able to identify graphs of the sine and cosine functions just by looking at them. They should construct table of values for $y = \sin\theta$, $y = \cos\theta$ and $y = \tan\theta$ where $-360^{\circ} \le \pi \le 360^{\circ}$ and draw their graphs. They should also determine the domain, the range and the period of each trigonometric function. Apply multiple of techniques to have an ongoing assessment of these points. You can form mixed ability groups and give them Exercise 5.10. Check their responses and give them an immediate feedback. Take remedial action accordingly.

Answers to Exercises 5.10

1.	a.	0 to 1	b.	1 to 0	c.	0 to –1	d.	–1 to 0
2.	a.	1 to 0	b.	0 to –1	c.	–1 to 0	d.	0 to 1
3.	a.	0 to ∞	b.	– ∞ to 0	c.	0 to ∞	d.	Increases

Assessment

In addition to the ongoing assessment you did during each period, it is also a good practice to make a summative assessment towards the end of the subunit. In doing so, you always keep in mind the competencies of the subunit and try to measure them.

- Give them an angle θ in standard position and ask if they are able to
- describe the sin, cosine and tangent functions of θ.
- Give them the unit circle on the coordinate plane and ask them
- to find the trigonometrical values of different quadrantal angles.
- Give them angles having equal measure but opposite in signs and ask students to give their trigonometrical values.

Example: a. $\sin 135^o =$ ___ b. $\sin(-135^o) =$ ____

b. $\cos 135^o =$ ___ c. $\cos(-135^o) =$ ____

c. $\tan 135^o =$ ___ d. $\tan(-135^o) =$ ____

Check if the students have understood the signs of the values of the trigonometric functions in each quadrant.

Let them complete tables like the following:

θ has terminal side in quadrant				
	I	II	II	Iv
sin θ				
cos θ				
tan θ				

Set more questions of the following type to measure understanding:

1. a. If $\sin 20^o = a$, then $\cos 70^o =$ ____
   b. If $\cos \alpha = b$ and $\sin \beta = b$, then $\alpha + \beta =$ ___
2. a. If $\sin 40^o = x$, then $\sin 140^o =$ ___
   b. If $\cos 20^o = r$, then $\cos 200^o =$ _____
   c. If $\tan 50^o = t$, then $\tan 310^o =$____
   d. If $\sin(-970^o) = n$, then $\cos 70^o$_____

Concerning graphs of trigonometric functions , you always remember that students are expected to construct table of values, draw the graphs and determine the domain and range and the period of the functions $y = \sin \theta$, $y = \cos \theta$ and $y = \tan \theta$. So you may set assessment questions of the following type:

1. Construct a table of values and sketch the graph of $y = -\sin x$
2. a. Complete the following table:

θ in deg	0	30	45	60	90	120	135	150	180
2θ	0	60	90	120	180	240	270	300	360
Cos 2θ									

b. Sketch the graph of $y = \cos\theta$ and $y = \cos 2\theta$ using the same coordinate axes.

c. What is the period of $y = \cos 2\theta$?

3. Complete the table:

	As θ increases from			
	0° to 90°	90° to 180°	180° to 270°	270° to 360°
$\sin\theta$	increases			
$\cos\theta$		decreases		
$\tan\theta$				

5.2 THE RECIPROCALS OF THE BASIC TRIGONOMETRIC FUNCTIONS

Periods allotted: 7 periods

Competencies

At the end of this sub-unit, students will be able to;

- *define the cosecant function.*
- *determine the values of cosecant function for some angles.*
- *define the secant function.*
- *determine the values of secant functions for some angles.*
- *define the cotangent function.*
- *determine the value of cotangent function for some angles.*
- *explain the concept of co-functions.*

Vocabulary: Cosecant, Secant, Cotangent, Co-functions

Introduction

Definition of the cosecant, secant and cotangent functions, determining values of these functions for some angles and the concept of co-functions are the central points of this sub-unit.

Teaching Notes

Revise the definitions of the sine, cosine and tangent functions in question and answer form. To this end, consider an angle θ in standard position and a point P(x, y) on its terminal side. **See the adjacent figure.**

Referring to this figure, encourage students to define $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of x, y and r.

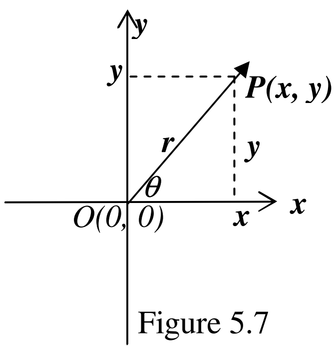


Figure 5.7

The Cosecant, Secant and Cotangent Functions

After the students revise the definitions of sine, cosine and tangent, you define the cosecant, secant and cotangent functions of the same angle θ in standard position in terms of *x, y* and *r* referring to the same figure given above. Then let them do Activity 5.9 to find values of $\csc\theta$, $\sec\theta$ and $\cot\theta$ for the given angle θ. Ask them to find the sine, cosine and tangent of the same angle θ. Let them compare the values of $\sin\theta$ and $\csc\theta$, $\cos\theta$ and $\sec\theta$ and $\tan\theta$ and $\cot\theta$. Encourage the students to discover the reciprocal relationship between the sine and cosecant, the cosine and secant, and the tangent and cotangent functions.

Answers to Activity 5.9

1. $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}$

2. $\sin\theta = \frac{1}{\csc\theta}$, if $\theta \neq n\pi, n \in \mathbb{Z}$

   $\cos\theta = \frac{1}{\sec\theta}$, *if* $\theta \neq \frac{n}{2}\pi$, where n is odd integer

   $\tan\theta = \frac{1}{\cot\theta}$, if $\theta \neq$ quadratntal angle.

3. They are reciprocals.

Give them more examples to practice these reciprocal relationships.

Finally, divide the class into small groups and let them do Group wok 5.5. Slow learners may be restricted to do only the first question or you can form a mixed ability group and make fast learners explain things to other peers. It is expected that group representatives briefly present their works.

Answers to Group work 5.5

1.

	θ has terminal side in quadrant			
	I	II	III	IV
$\csc\theta$	+	+	–	–
$\sec\theta$	+	–	–	+
$\cot\theta$	+	–	+	–

2.

θ in deg	–360	–300	–270	–240	–180	–120	–90	–60	0
θ in rad	-2π	$-\frac{5}{3}\pi$	$-\frac{3}{2}\pi$	$-\frac{4}{3}\pi$	$-\pi$	$-\frac{2}{3}\pi$	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	0
$y = \csc\theta$	—	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	—	$\frac{-2\sqrt{3}}{3}$	–1	$\frac{-2\sqrt{3}}{3}$	—
$y = \sec\theta$	1	2	—	–2	–1	–2	—	2	1

θ in deg	60	90	120	180	240	270	300	360
θ in rad	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	π	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	2π
$y = \csc\theta$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	—	$\frac{-2\sqrt{3}}{3}$	-1	$\frac{-2\sqrt{3}}{3}$	—
$y = \sec\theta$	2	—	-2	-1	-2	—	2	1

3.

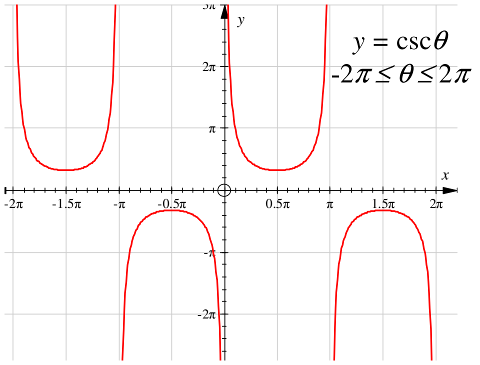


Figure 5.8 Graph of $y = \csc\theta$

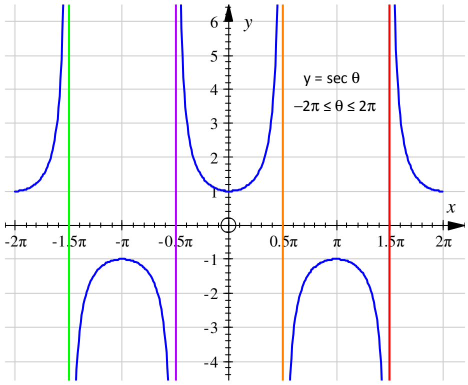


Figure 5.9 Graph of $y = \sec\theta$

4.

θ in deg	-360	-315	-270	-225	-180	-135	-90	-45	0
θ in rad	-2π	$-\frac{7}{4}\pi$	$-\frac{3}{2}\pi$	$-\frac{5}{4}\pi$	$-\pi$	$-\frac{3}{4}\pi$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0
$y = \cot\theta$	—	1	0	-1	—	1	0	-1	—

θ in deg	45	90	135	180	225	270	315	360
θ in rad	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{3}{2}\pi$	$\frac{7}{4}\pi$	2π
$y = \cot\theta$	1	0	-1	—	1	0	-1	—

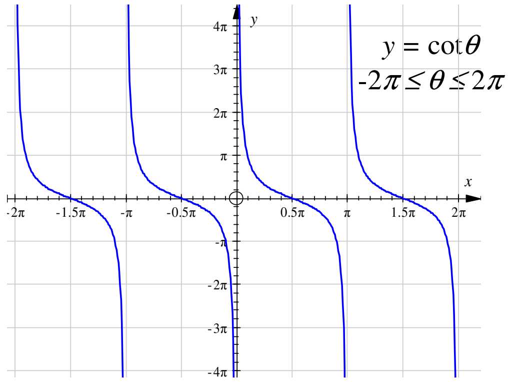


Figure 5.10 Graph of $y = \cot\theta$

5. sec θ is undefined if $\theta = n\,(90°)$, n is an odd integer.
   cot θ is undefined if $\theta = n\,(180°)$, n is an integer.

Give the students the first question of Exercise 5.11 as a class work and the remaining questions as a home work. Check their class work and give them corrections. Collect their home work and mark it or give it a certain value. Arrange time to do the corrections for the home work.

Using class works, home works, quizzes, tests or assignments, make sure that students are able to define the cosecant, secant and cotangent functions and determine the values of these functions for some angles.

Answers to Exercise 5.11

1.

	Point on the Terminal side	**csc θ**	**sec θ**	**cot θ**
a	(12, 5)	$\frac{13}{5}$	$\frac{13}{12}$	$\frac{12}{5}$
b	(–8, 15)	$\frac{17}{5}$	$\frac{17}{-8}$	$\frac{-8}{15}$
c	(–6, 8)	$\frac{10}{8}$	$-\frac{10}{6}$	$-\frac{6}{8}$
d	(5, 3)	$\frac{\sqrt{34}}{3}$	$\frac{\sqrt{34}}{5}$	$\frac{5}{3}$
e	(2, 0)	undefined	1	undefined
f	$\left(\frac{4}{5}, -\frac{3}{5}\right)$	$-\frac{5}{3}$	$\frac{5}{4}$	$-\frac{4}{3}$
g	$\left(\sqrt{2}, \sqrt{5}\right)$	$\sqrt{\frac{7}{5}}$	$\sqrt{\frac{7}{2}}$	$\sqrt{\frac{2}{5}}$
h	$\left(\sqrt{6}, \sqrt{3}\right)$	$\frac{3}{\sqrt{3}}$	$\frac{3}{\sqrt{6}}$	$\sqrt{2}$

2. a. $-\frac{1}{0.35}=-\frac{20}{7}$ b. $\frac{1}{2.6}=\frac{5}{13}$ c. $\frac{1}{30.5}=\frac{2}{61}$

d. 1 e. $\frac{3}{\sqrt{3}}=\sqrt{3}$ f. undefined

3.

	θ	csc θ	sec θ	cot θ
a.	30^o	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
b.	45^o	$\sqrt{2}$	$\sqrt{2}$	1
c.	60^o	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
d.	120^o	$\frac{2\sqrt{3}}{3}$	–2	$-\frac{\sqrt{3}}{3}$
e.	150^o	2	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
f.	210^o	–2	$-\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
g.	240^o	$-\frac{2\sqrt{3}}{3}$	– 2	$\frac{\sqrt{3}}{3}$
h.	300^o	$-\frac{2\sqrt{3}}{3}$	2	$\frac{-\sqrt{3}}{3}$
i.	-390^o	–2	$\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
j.	-405^o	$-\sqrt{2}$	$\sqrt{2}$	–1
k.	-420^o	$-\frac{2\sqrt{3}}{3}$	2	$-\frac{\sqrt{3}}{3}$
l.	780^o	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

4. $\sin\theta = \frac{8}{\sqrt{73}}$ $\cos\theta = \frac{3}{\sqrt{73}}$ $\tan\theta = \frac{8}{3}$

$\csc\theta = \frac{\sqrt{73}}{8}$ $\sec\theta = \frac{\sqrt{73}}{3}$

Co-functions

Before defining co-functions let students do Activity 5.10. Ask your students whether α and β are complementary or not. Let them give reasons to that. Ask them to find the values of the six trigonometric functions for angle α. Then let them find the values of the six trigonometric functions for angle β. Encourage them to compare the results of $\sin\alpha$ and cos β, $\cos\alpha$ and sin β, $\tan\alpha$ and cot β, etc. From their activity students are expected to generalize the relationship that for two complementary angles α and β ,

$\sin\alpha = \cos\beta$	$\cos\alpha = \sin\beta$	$\tan\alpha = \cot\beta$
$\csc\alpha = \sec\beta$	$\sec\alpha = \csc\beta$	$\cot\alpha = \tan\beta$

Then tell them that such pair of functions like the *sine* and *cosine, secant* and *cosecant* and *tangent* and *cotangent* are called co-functions.

Answers to Activity 5.10

$\sin\alpha = \frac{3}{5}$	$\sin\beta = \frac{4}{5}$	$\sin\alpha = \cos\beta$
$\cos\alpha = \frac{4}{5}$	$\cos\beta = \frac{3}{5}$	
$\tan\alpha = \frac{3}{4}$	$\tan\beta = \frac{4}{3}$	$\cos\alpha = \sin\beta$
$\csc\alpha = \frac{5}{3}$	$\csc\beta = \frac{5}{4}$	
$\sec\alpha = \frac{5}{4}$	$\sec\beta = \frac{5}{3}$	$\tan\alpha = \cot\beta$
$\cot\alpha = \frac{4}{3}$	$\cot\beta = \frac{3}{4}$	

Then let them do the example given in the textbook to find trigonometric values of co-functions. Then give them Exercise 5.12 so that they can practice finding trigonometric values of co-functions. Use a whole class discussion in form of question and answer to do the exercise. Have your own assessment of the students' performance.

For fast learning students you can give questions of the following type.

1. If $\sin 38° = \cos 2\theta$, then $\theta =$ _____________
2. If $\frac{\sin\theta}{\cos 55°} = 1$, then $\theta =$ _____________
3. Give the domain and the range of the following functions:
   i. $f(x) = \sec x$
   ii. $g(x) = \csc x$
4. Explain why the graphs of $y = \csc x$ and $y = \sec x$ never cross the x-axis.
5. State whether values of cosecant and cotangent functions are increasing or decreasing as values of θ increase from 0° to 90°.

Answers to Exercise 5.12

1. a. $\theta = 70^o$ b. $\theta = 10^o$ c. $\theta = 35^o$
   d. $\theta = 70^o$ e. $\theta = 15^o$ f. $\theta = 89^o$

2. a. 0.8387 b. 0.9744 c. 1
   d. x e. 90^o f. 35^o

5.3 SIMPLE TRIGONOMETRIC IDENTITIES

Periods allotted: 3 periods

Competencies

At the end of this sub-unit, students will be able to;

- *derive some of the trigonometric identities.*
- *identify the quotient identities.*
- *solve problems related to trigonometrical identities.*

Vocabulary: Pythagorean identities, Quotient identities

Introduction

In this sub-unit, the Pythagorean and quotient identities take the major consideration. Throughout the subunit students are encouraged to derive these trigonometric identities and apply them in solving related problems.

Pythagorean and Quotient Identities

Teaching Notes

Consider an angle θ in standard position and a point P(x, y) on its terminal side. Let the students do the following activities which step by step lead to discover the Pythagorean identities.

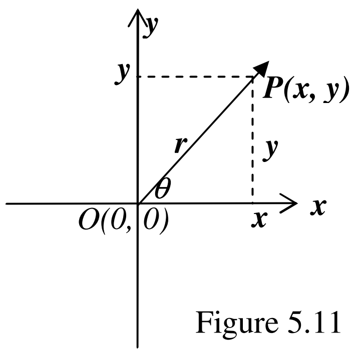


Figure 5.11

Answer the following in terms of x, y and r:

1. a. $\sin\theta =$ ------
   b. $\cos\theta =$ ------
   c. $\tan\theta =$ ------
2. Is $x^2 + y^2 = r^2$? (Why?)
   - What do you get if you divide both sides of $x^2 + y^2 = r^2$ by r^2 ?
   - What will the expression be if you substitute $\sin\theta$ for $\frac{y}{r}$ and $\cos\theta$ for $\frac{x}{r}$ in the equation $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$?
   - Use a similar approach to derive the other identities.

Practice the identities using more examples and give Exercise 5.13. Let each student do the activity individually. Go around the class and assist them. At the end ask individual volunteers to come out and show their work. The rest of the class has to comment on it. If you find fast learners who have finished their work earlier, then you can give them additional tasks like the ones given below:

1. If $\cos\alpha = 0.7$, find $\sec\alpha$.
2. If $\tan 84^\circ = 9.514$, find $\cot 6^0$.
3. If $\sin 45^\circ = \frac{\sqrt{2}}{2}$, find $\cos 45^0$.
4. If $\sin\beta = 0.5$, find $\tan\beta$.
5. If $\sec\theta = \frac{5}{3}$, find $\tan\theta$, $(0 \le \theta \le 90^0)$

Answers to Exercise 5.13

1. a. $\cos\theta = \frac{8}{17}$; $\tan\theta = \frac{15}{8}$; $\csc\theta = \frac{17}{15}$; $\sec\theta = \frac{17}{8}$; $\cot\theta = \frac{8}{15}$

   b. $\sin\theta = \frac{3}{5}$; $\tan\theta = -\frac{3}{4}$; $\csc\theta = \frac{5}{3}$; $\sec\theta = -\frac{5}{4}$; $\cot\theta = -\frac{4}{3}$

   c. $\sin\theta = -\frac{24}{25}$; $\cos\theta = -\frac{7}{25}$; $\tan\theta = \frac{24}{7}$; $\csc\theta = -\frac{25}{24}$; $\sec\theta = -\frac{25}{7}$

   d. $\sin\theta = -\frac{7}{25}$; $\tan\theta = -\frac{7}{24}$; $\csc\theta = -\frac{25}{7}$; $\sec\theta = \frac{25}{24}$; $\cot\theta = -\frac{24}{7}$

2. a. $\frac{a}{\sqrt{a^2+b^2}}$ b. $\frac{b}{\sqrt{a^2+b^2}}$ c. $\frac{b}{\sqrt{a^2+b^2}}$

   d. $\frac{a}{\sqrt{a^2+b^2}}$ e. $\frac{\sqrt{a^2+b^2}}{b}$ f. $\frac{a}{b}$

3. a. its complementary b. its complementary c. cotangent

Quotient identities

Quotient identities are expected to be derived from Activity 5.11. Let the students find the values of $\sin\theta$, $\cos\theta$, $\tan\theta$ and $\cot\theta$ in terms of x, y and r. Then let them divide the value of $\sin\theta$ by the value of $\cos\theta$. Tell them to compare this quotient with the value of $\tan\theta$. Let them do the same for $\frac{\cos\theta}{\sin\theta}$ and compare it with $\cot\theta$. Finally, encourage them to give you their conclusion.

Answers to Activity 5.11

a. $\sin\theta = \frac{y}{r}$; $\cos\theta = \frac{x}{r}$; $\tan\theta = \frac{y}{x}$; $\cot\theta = \frac{x}{y}$

b. $\frac{\sin\theta}{\cos\theta} = \frac{y}{x} = \tan\theta$ c. $\frac{\cos\theta}{\sin\theta} = \frac{x}{y} = \cot\theta$

After you give them the summary of the quotient identities let students practice the identities, by doing Group work 5.6 as class work or home work depending on the remaining time you have.

Answers to Group work 5.6

1. $\tan\alpha = \frac{-3}{4}$, $\cot\alpha = \frac{-4}{3}$
2. $\tan\alpha = \frac{8}{15}$, $\cot\alpha = \frac{15}{8}$
3. $\tan 330^\circ = -\frac{\sqrt{3}}{3}, \cot 330^\circ = -\sqrt{3}$
4. $\tan 150^\circ = \frac{-\sqrt{3}}{3}$, $\cot 150^\circ = -\sqrt{3}$
5. $\tan 60^\circ = \sqrt{3}$, $\cot 60^\circ = \frac{\sqrt{3}}{3}$
6. $\tan(90^\circ - \alpha) = \frac{y}{x}$, $\cot(90^\circ - \alpha) = \frac{x}{y}$

Using Tables of the trigonometric functions

Demonstrate to the students how they can use the trigonometric table and read values from it using a few examples. For this purpose, it is advisable to prepare a relatively larger trigonometric table for a few values of θ so that it will be visible to the whole class. Do all the examples given in the text together with the students. Then let them do some of the problems from Exercise 5.14 as a class work. Go round the class and check if they have mastered the skill of reading trigonometric values from the table. Give them the remaining problems of Exercise 5.14 as a home work. Check their home works and give them appropriate corrections.

For those that are fast learning students you can give questions of the following type.

1. Let $\csc\theta = \frac{\sqrt{5}}{2}$ and $\cot\theta < 0$, then find the values of the other five trigonometric functions of θ. (Hint: use Pythagorean identity)
2. Let θ be an angle such that $\cos(90° - \theta) = a$ and $\sin(90° - \theta) = b$. Then find
   i. $\tan\theta$ ii. $\cot\theta$
3. i. If $\sin\theta = \frac{1}{3}$, find $\frac{\cos\theta\tan\theta}{\csc\theta}$ (Hint: Apply quotient identity)

   ii. If $\tan\alpha = \frac{3}{4}$, find $\frac{\sin\alpha\sec\alpha}{\cot\alpha}$ (Hint: Apply quotient identity)
4. Show that $\frac{1+\tan^2\theta}{\csc^2\theta} = \tan^2\theta$ (Hint: use Pythagorean and quotient identities)
5. Using the concept of related angles and trigonometric table, find the value of csc 693°.

Answers to Exercise 5.14

1. a. 0.8572 b. 0.6018 c. 0.7265
   d. sec 162° = – sec 18° = –1.051
   e. sin 593° = – sin 53° = – 0.7986
   f. tan 593° = tan 53° = 1.327
   g. cos (–143°) = – cos 37° = –0.7986
2. a. 32° b. 46° c. 57°
   d. 90° e. 60° f. 29°

Assessment

Think of the competencies expected of the students. Check if they can derive the simple Pythagorean and quotient identities. Do they identify the Pythagorean and quotient identities? Are they able to solve problems related to trigonometrical identities?

Are they able to use the trigonometric table to read trigonometric values of angles?

Set appropriate instruments to measure the above competencies. You may also use questions like the following to assess their achievement:

1. Give them an angle θ in standard position [like the one given below] and ask them if they are able to restate the definition of csc θ, sec θ and tan θ in terms of a, b and r.
2. What is the relationship between

   csc θ and sin θ?___

   sec θ and cos θ? ____

   cot θ and tan θ? ____

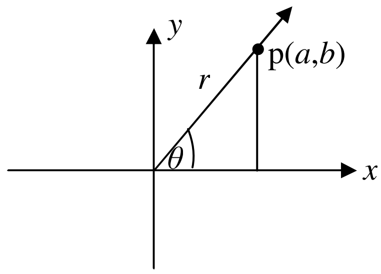


Figure 5.12

3. For which values of θ do you think is
   - csc θ not defined ?____
   - sec θ not defined ?____
   - cot θ not defined?_____
4. If csc 35° = m, then sec 55° =___

   If tan $\alpha = a$ and cot $\beta = a$, then $\alpha + \beta$ =___
5. Let θ be an obtuse angle with cos $\theta = \frac{-4}{5}$. Then find

   a. sin θ b. tan θ.
6. Using trigonometric table find

   a. sin 54° b. cos 111° c. tan 345°

5.4 REAL LIFE APPLICATION PROBLEMS

Periods allotted: 5 periods

Competency

At the end of the sub-unit, students will be able to:

- *solve real life problems using trigonomertical ratios.*

Vocabulary: Angle of elevation, Angle of depression

Introduction

This sub-unit is mainly about the real life applications of trigonometry. It deals with applications of trigonometrical ratios and trigonometrical identities in solving real life problems.

Teaching Notes

Give them an oral introduction that trigonometry was originally used to relate the angles of a triangle to the lengths of the sides of a triangle. So, trigonometric functions are important in the study of triangles and solving different problems in the real life. Therefore, give them orientation to the effect that in this section they are going to see some of the real life applications of trigonometry.

First explain to them the meaning of solving a triangle in relation to the sides, angles and area of a triangle. Then revise the basic properties on right angled triangles like the Pythagoras' theorem, the six trigonometric functions, and the Pythagorean and quotient identities. Next, let them to practice solving right angled triangles through different exercises. Activity 5.12 helps for such purpose.

Answers to Activity 5.12

1. $AC = \sqrt{13}$ cm, $m(\angle C) = 33.7^\circ, m(\angle A) \cong 56.3^\circ$, area $\Delta ABC = 3\text{cm}^2$
2. $AC \cong 21.9\text{cm}, BC \cong 8.9\text{cm}, m(\angle C)=66^\circ$, area $= 89\text{cm}^2$

Then give them a very clear explanation of an angle of depression and angle of elevation in relation to a line of sight. Give them examples from a different location of an object. Demonstrate it diagrammatically to the students. Select some practical problems on applying angle of elevation and angle of depression through solving right angled triangles. Do the examples given in the text together with the students before going to Exercise 5.15. Let the students carefully identify the given and required parts whenever solving a real life problem. If necessary let them represent the given word problem diagrammatically.

Give the first question of Exercise 5.15 as a home work and question 2 as a written assignment. Give the necessary feedback and corrections for the home work. You can take the assignment as part of your summative continuous assessment and give value for it.

Answers to Exercise 5.15

1. a. m ($\angle$A) = 40^o; a = 12.86; b = 15.32; Area(ΔABC) = 98.51 sq. units
   b. m ($\angle$B) = 36^o ; b = 8.72; c = 14.83; Area(ΔABC) = 52.32sq. units
   c. m ($\angle$B) = 54^o ; a = 5.81; c = 9.89; Area(ΔABC) = 23.24sq. units
   d. m ($\angle$A) = 35^o ; b = 14.28; c = 17.43; Area(ΔABC) = 71.41sq. units
   e. m ($\angle$B) = 52^o ; a = 12.31; b = 15.76; Area(ΔABC) = 97.00sq. units
   f. m ($\angle$B) = 73^o ; b = 45.79 ; c = 47.88; Area(ΔABC) = 320.53sq. units
2. a. Foot of the ladder is 2.44 m far from the building.
   b. Length of shadow = 28.87 meters
   c. The shadow will be 21.42 meters long.
   d. The height of the flagpole is 4.83meters.
   e. The boats are about 38.4 meters apart.
   f. The width of the canal is about 159.24 meters.

After treating the particular topic or sub-unit, you can give part of the review exercises as project work or assignment. The assignment will be submitted in written form and, if possible, presented by the group members in a specially arranged tutorial class. Mark the assignment and give value for it. Corrections and appropriate feedbacks should be given.

Assessment

Remember that students are expected to solve real life problems using trigonometical ratios. We hope that you have used multiples of formal and informal assessment techniques like group work, class work, home work, oral and written questions, assignments, quizzes, tests, etc. during each period. It is also necessary to conduct a summative assessment to measure the students' level of understanding the unit. Therefore, set real life problems like the ones given in exercise 5.15 and in the review exercise, and ask them. Make sure that students are clear with the concepts like "*angle of elevation*" and "*angle of depression*"

If most of the students seem not to have understood a given specific topic; re-teaching of that topic by using a different approach is recommended.

Answers to the Review Exercises on Unit 5

1. a. Quadrant III b. Quadrant IV c. Quadrant I
   d. Quadrant III e. Quadrant I f. Quadrantal angle
   g. Quadrant IV h. Quadrant III i. Quadrantal angle
   j. Quadrant III
2. a. 440^o and $-$280^o b. 500^o and $-$220^o c. 650^o and $-$70^o
   d. 15^o and $-$345^o e. 20^o and $-$340^o f. 315^o and $-$45^o
   g. 180^o and $-$180^o h. 202^o and $-$158^o i. 360^o and $-$360^o
   j. 150^o and $-$210^o

3. a. $\frac{2}{9}\pi$ rad b. $\frac{5}{12}\pi$ rad c. $\frac{4}{3}\pi$ rad
   d. $\frac{11}{6}\pi$ rad e. $\frac{-19}{36}\pi$ rad f. $-\pi$ rad
   g. $-\frac{11}{9}\pi$ rad h. $-\frac{7}{3}\pi$ rad i. -17π rad

4. a. 60^{o} b. -120^{o} c. 70^{o} d. 1290^{o}
   e. -80^{o} f. 900^{o} g. -45^{o} h. -7.5^{o}

5.

	sin θ	cos θ	tan θ
a.	1	0	undefined
b.	–1	0	undefined
c.	0	–1	0
d.	1	0	undefined
e.	–1	0	undefined
f.	1	0	undefined
g .	0	1	0
h.	–1	0	undefined

6.

	$\sin\theta$	$\cos\theta$	$\tan\theta$
a.	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
b.	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
c.	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
d.	–1	0	undefined
e.	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
f.	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
g.	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
h.	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$

7.

	a.	Negative	b.	Negative	c.	Negative	d.	Positive
	e.	Negative	f.	Positive	g.	Negative	h.	Negative
	i.	Negative	j.	Positive				
8.	a.	40^o	b.	80^o	c.	5^o	d.	54^o
	e	10^o	f.	24^o	g.	22^o	h.	0^o

9.

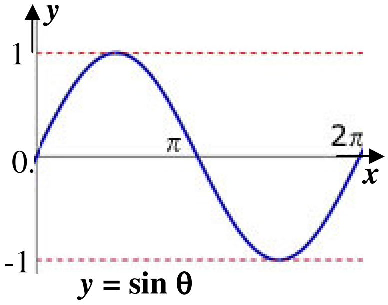


Figure 5.13

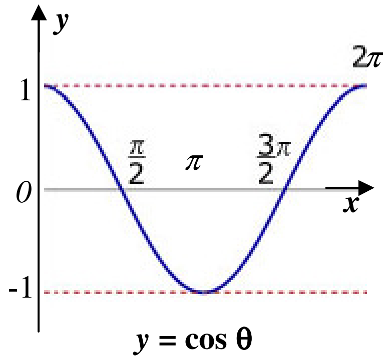


Figure 5.14

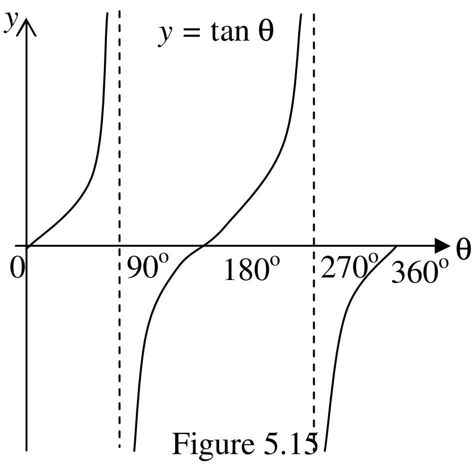


Figure 5.15

10.	a.	$-\frac{\sqrt{3}}{2}$	b.	$-\frac{1}{2}$	c.	$\sqrt{3}$
	d.	-1	e.	undefined	f.	$-\frac{\sqrt{3}}{3}$

11.

	sin θ	cos θ	tan θ	csc θ	sec θ	cot θ
a.	$\frac{12}{5}$	$\frac{5}{13}$	$\frac{12}{5}$	$\frac{13}{12}$	$\frac{13}{5}$	$\frac{5}{12}$
b.	$\frac{24}{25}$	$-\frac{7}{25}$	$-\frac{24}{7}$	$\frac{25}{24}$	$-\frac{25}{7}$	$-\frac{7}{24}$
c.	$-\frac{6}{\sqrt{61}}$	$\frac{5}{\sqrt{61}}$	$-\frac{6}{5}$	$-\frac{\sqrt{61}}{6}$	$\frac{\sqrt{61}}{5}$	$-\frac{5}{6}$
d.	$-\frac{17}{\sqrt{353}}$	$-\frac{8}{\sqrt{353}}$	$\frac{17}{8}$	$-\frac{\sqrt{353}}{17}$	$-\frac{\sqrt{353}}{8}$	$\frac{8}{17}$
e.	$\frac{8}{17}$	$\frac{15}{17}$	$\frac{8}{15}$	$\frac{17}{8}$	$\frac{17}{15}$	$\frac{15}{8}$
f.	$-\frac{8}{\sqrt{65}}$	$\frac{1}{\sqrt{65}}$	-8	$\frac{\sqrt{65}}{-8}$	$\sqrt{65}$	$-\frac{1}{8}$
g.	$-\frac{4}{5}$	$-\frac{3}{5}$	$\frac{4}{3}$	$-\frac{5}{4}$	$-\frac{5}{3}$	$\frac{3}{4}$
h.	1	0	Undefined	1	Undefined	0

12. a. Quadrant III b. Quadrant I c. Quadrant II
d. Quadrant IV e. Quadrant III f. Quadrant II

13. a. 60^o b 45^o c. 20^o
d. 10^o e. 55^o f. 20^o

14. a. $\frac{3}{5}$ b. $-\frac{3}{4}$ c. $\frac{5}{3}$ d. $-\frac{4}{3}$

15. $\frac{3}{\sqrt{13}}$

16. a. $m(\angle A) = 30^o$; $b = 18\sqrt{3}$; $c = 36$; Area(ΔABC) = $162\sqrt{3}$ sq. units
b. $m(\angle B) = 45^o$; $a = 8\sqrt{2}$; $b = 8\sqrt{2}$; Area(ΔABC) = 64sq. units
c. $m(\angle B) = 68^o$; $a = 4.04$; $c = 10.79$; Area(ΔABC) = 20.2sq. units
d. $m(\angle A) = 38^o$; $a = 28.94$; $b = 37.04$; Area(ΔABC) = 535.97sq. units

17 a. 11.49 meters
b. Height of building = 97.8 meters, height of pole = 27.1 meters
c. The angle of elevation of the sun is about 72^o

UNIT 6 PLANE GEOMETRY

INTRODUCTION

In this unit, the basic geometric notions learned in lower grades are strengthened. One of the objectives of this unit is to enable students to prove theorems on triangles, including Menelaus' theorem, and theorems on angles and arcs determined by lines intersecting outside, on and inside a circle. Students should recapitulate the definition of parallelogram, rectangle, rhombus, square and trapezium and should be able to prove theorems on parallelogram. Finally, they should be able to derive area and perimeter formulae for regular polygons.

Unit Outcomes

After completing this unit, students will be able to:

- *know more theorems special to triangles.*
- *know basic theorems specific to quadrilaterals.*
- *know theorems on circles and angles inside, on and outside a circle.*
- *solve geometrical problems on quadrilaterals, circles and regular polygons.*

Suggested Teaching Aids in Unit 6

Besides the usual teaching aids for teaching geometry like coloured chalks, ruler, compass and protractor, for this unit model of triangles, polygons and circles are important. You may display some models of regular polygons (could be diagrams) in classes.

Group the students so that they can produce either real models or diagrammatic models on triangles, quadrilaterals, polygons of more than four sides or on circles. Encourage or assign a group to work on inscribed or circumscribed polygons. Let them see the conditions to inscribe or circumscribe a rectangle. Each student is expected to participate in construction activities.

6.1 THEOREMS ON TRIANGLES

Periods allotted: 5 periods

Competency

At the end of this sub-unit, students will be able to:

- *apply the incidence theorems to solve related problems.*

Vocabulary: Median, Centroid, Altitude, Perpendicular bisector, Circumcentre, Orthocentre, Incentre, Collinear, Concurrent.

Introduction

The students are already familiar with triangle and its parts. It will be better if you revise the basic congruency and similarity concepts and theorems. The properties of right triangles are also useful in the study. Before you start discussions about theorems on triangles activities and group works should be done so that students can understand the theorems on triangles. The exercises are for students to use and apply the theorems and definitions.

Teaching Notes

As indicated in the textbook, introduce collinear points and concurrent lines. Describe how the students can bisect a line segment by construction. In order to help them understand the required concepts about the medians of a triangle, make them try Activity 6.1.With active participation of students, ask them to answer the questions by drawing triangles. They are required to use the Pythagoras Theorem. Observe that the medians cannot intersect outside the triangle for any triangle. You may show them the following case.

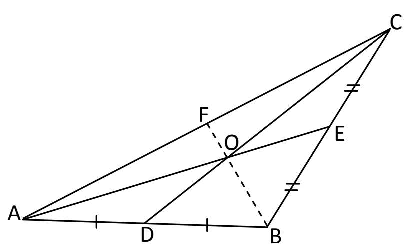
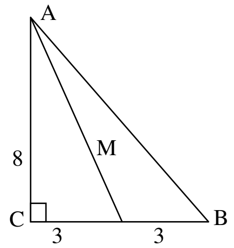


Figure 6.1

Taking the mid-point F of $\overline{AC}$, $\overline{BF}$ is a median containing O and O is inside ΔABC.

Answers to Activity 6.1

1. Median
2. Three
3. Consider **Figure 6.2**



By Pythagoras Theorem,

$$M^2 = (8\text{ cm})^2 + (3\text{ cm})^2$$
$$= 64\text{cm}^2 + 9\text{cm}^2$$
$$= 73\text{ cm}^2$$
$$\therefore M = \sqrt{73} \approx 8.5\text{cm}$$

Figure 6.2

4. Consider the following triangle

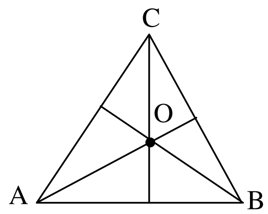


Figure 6.3

Figure 6.3 indicates that the medians passes through 'O' and hence they are concurrent. This is true for all triangles.

5. No

Now, state the definition of a median of a triangle and let them practice by working on Activity 6.2.

The main task of this activity is to make students practice using construction to construct medians of a triangle to see that the median are concurrent at a point $\frac{2}{3}$ of the distance from each vertex to the mid-point of the opposite side.

As an additional problem, let them measure the relationship for a right triangle. Where do the medians meet?

Mark their performances on the construction in Activity 6.2. Show a best construction to the class and encourage good working students.

Answers to Activity 6.2

1, 2.

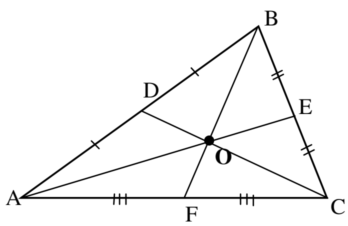


Figure 6.4

3. Yes

4. i. a. $AO = \frac{2}{3}AE$ b. $OE = \frac{1}{3}AE; AO : OE = 2:1$

ii. a. $CO = \frac{2}{3}CD$ b. $OD = \frac{1}{3}CD; CO : OD = 2:1$

iii. a. $BO = \frac{2}{3}BF$ b. $OF = \frac{1}{3}BF; BO : OF = 2:1$

5. The medians of a triangle are concurrent at a point $\frac{2}{3}$ of the distance from each vertex to the midpoint of the opposite side.

It will be important to revise ideas of congruent and similar triangles, which they studied in grade nine, before you state Theorem 6.1.

i. Two triangles ΔABC and ΔDEF are shown to be congruent using SAS congruency postulate, AAS and SSS Theorems. Here, corresponding angles and corresponding sides are congruent, and their measures are equal.

Note that if ABCD is a parallelogram $\Delta ABC \equiv \Delta CDA$

Thus, $\overline{AB} \equiv \overline{CD}$ and $\overline{BC} \equiv \overline{DA}$.

$\Rightarrow AB = CD$ and $BC = DA$

Similarly $\angle ABC \equiv \angle CDA$ and so m ($\angle CDA$)

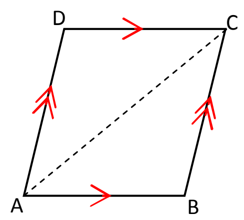


Figure 6.5

ii. Two triangles ΔABC and ΔDEF are shown to be similar using SAS, AA and SSS Theorems. In this case, the corresponding angles are congruent and the corresponding sides are proportional in length.

$$\Delta ABC \sim \Delta DEF \Rightarrow \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}.$$

After this, state Theorem 6.1 and discuss its proof. Ask the students to re-state the theorem and do Example 1.

At this position, students are to discuss about angle bisectors, altitude and perpendicular bisectors of a triangle. These terms are not new to the students. Let them state what an acute angle, obtuse angle and a right angle is. Let them discuss Activity 6.3 in class. You may ask them to answer the questions one by one since every student is expected to know questions 1– 3 immediately. Group the students and make them do the constructions given in 4, 5 and 6. Give different questions to different groups around.

This activity will be a good start to understand the Theorems to be discussed next.

Additional construction questions for fast learning students.

1. Give the steps to bisect an angle.
2. Draw a triangle and construct its angle bisectors. Are they concurrent? Can they meet outside the triangle?
3. Give a type of triangle whose angle bisectors and altitudes meet at a point.

Answers to Activity 6.3

1. A line or line segment that divides an angle into two equal parts is called an angle bisector.
2. A triangle may have three bases.
3. A triangle can have three altitudes.
4. a. Inside the triangle.
   b. Outside the triangle.
   c. On the triangle.
5. a. Inside the triangle
   b. Outside the triangle
   c. On the triangle.
6. From Figure 6.6 $\Delta AOE \equiv \Delta BOE$ (by SAS)
   $\therefore\ AO = BO$
   Similarly,
   $\Delta\, BOF \cong \Delta COF$
   $\therefore\ BO = CO$
   So, a. O is equidistant from A and B because $BO = AO$
   b. O is equidistant from B and C because $BO = CO$
   c. Yes, because the perpendicular bisector of sides of a triangle are concurrent.

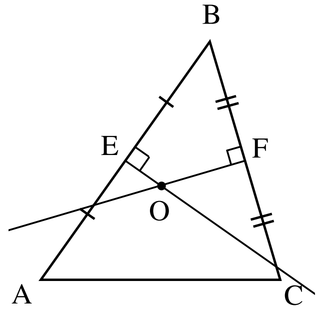


Figure 6.6

Give short explanations about perpendicular bisectors, altitudes and angle bisectors of a triangle. Discuss the theorems associated. Since Group work 6.1 incorporates most concepts stated in the theorems, encourage each group to do the questions. Follow each group and give comments.

State the Altitude and Menelaus' Theorems. With active participation of the students give their proofs. Ask them to prove Menelaus' Theorem by changing the position of F on the ray $\overrightarrow{AB}$. Let the students do Example 3 to apply Altitude Theorem resulting from similarity of triangles.

To farther practice the concepts in the topics discussed above, let the students do Exercise 6.1 select some questions to be done in class and give the rest as homework. Check their work on Altitude Triangle for an equilateral triangle. You can select some questions in particular to be done by excellent students.

Idea to Group work 6.1

Constructions and measurements from 1 to 6 are practical problems. Follow each group by using different types of triangles. It is better that student's use square paper.

Here you need to note that student's ability on construction and measurement problems hold true and students explain the centre of the inscribed circle is the incentre of the triangle and the centre of the circumscribed circle is the circumcentre of the triangle.

Additional problems

You can use the following problems to support the students on further exercise. Use Question A for slow learners and B for fast learners.

A. In ΔABC, let AB = 8 cm and O be the circumcentre of the triangle. If E is the foot of the perpendicular from O to $\overline{AB}$ such that OE = 3 cm, find the length of

a. $\overline{AO}$ b. $\overline{CO}$

B. a. In **Figure 6.7**, $\overline{AD} \equiv \overline{DC}$, $\overline{AE} = \overline{EB}$, F is the intersection of $\overline{CE}$ and $\overline{BD}$, $\overline{BG} // \overline{FC}$ and $\overline{CG} // \overline{BF}$. Prove that $BG = 2\ EF$.

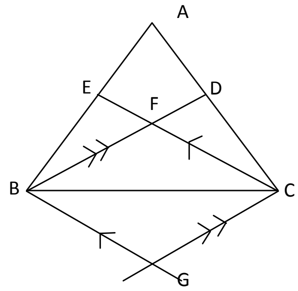


Figure 6.7

b. Consider **Figure 6.8** taken as one configuration for a proof of Menelaus' Theorem.

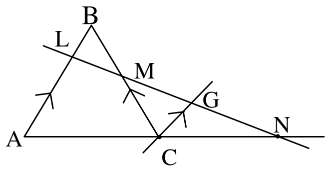


Figure 6.8

Show that $\frac{AL}{LB}.\frac{BM}{MC}.\frac{CN}{NA} = -1$

Hint for answers:

A. Refer to Theorem 6.2 on the student textbook on perpendicular bisectors.

B. a. Continue similar to the solution to Exercise 6.1, number 1.

b. $\Delta NCG \sim \Delta NAL$ and $\Delta MCG \sim \Delta MBL$

$$\frac{AL}{CG} = \frac{NA}{NC}\ and\ \frac{BM}{MC} = \frac{LB}{CG}$$

Multiplying the above equations

$$\frac{AL}{CG} = \frac{BM}{MC} = \frac{NA}{NC}.\frac{LB}{CG} \Rightarrow \frac{AL}{LB}.\frac{BM}{MC}.\frac{NC}{NA} = 1$$

Substitute (– CN) for NC to obtain the result.

Assessment

It is import and necessary to pay close attention to the students and control their Activities to the desired direction. Specially, in geometry classes, the attention of the students is a very important matter.

Try to motivate them to do exercises repeatedly. Identify the slow and fast learners and handle them accordingly. Check the class work and homework you ordered to be attempted. You may give them a quiz to find out how they are following the lesson.

Answers to Exercise 6.1

1. Since $FC = \frac{2}{3}\ EC$ (by Theorem 6.1)

   and $EF + FC = EC \Rightarrow EF + \frac{2}{3}EC = EC$

   $$\Rightarrow\ EF = EC - \frac{2}{3}EC$$

   $$\therefore\ EF = \frac{1}{3}EC$$

2. Given $\angle APB \equiv \angle AQB$

   $\angle APR \equiv \angle RPB\ and\ \angle BQR \equiv \angle AQR$

   $RP = RQ \Rightarrow \overline{RP} \equiv \overline{RQ}$

   Required A, R, B lie on a straight line.

   Construct $\overline{PQ}$. Let E be the mid-point of $\overline{PQ}$. Join R to E.

   As ΔPRQ is isosceles, $\angle RPE \equiv \angle RQE$, $\overline{PR} \equiv \overline{QR}$ and $\overline{PE} \equiv \overline{QE}$. By SAS postulate $\Delta PRE \equiv \Delta\ QRE$

   Thus, $\angle PER \equiv \angle QER$. As a result m $(\angle PER) = 90^{\circ}$ and hence $\overline{RE}$ is perpendicular to $\overline{PQ}$.

   Now, join B to E.

   $\angle RPE \equiv \angle RQE$ and $\angle RQA \equiv \angle RQB$ (Given $\angle AQB$ and $\overline{RP}$, $\overline{RQ}$ are bisectors).

   Subtracting congruent angles, $\angle BPE \equiv \angle BQE$. It follows that ΔBPQ is isosceles and $\overline{BP} \equiv \overline{BQ}$ and $\overline{BE}$ perpendicular to $\overline{PQ}$, as both $\overline{RE}$ and $\overline{BE}$ are perpendicular to $\overline{PQ}$, $\overline{BE}$ lies on the line $\overline{RE}$.

   In a similar way since $\angle APB \equiv \angle AQB$ and $\angle BPE \equiv \angle BQE$ adding $\angle APE \equiv \angle AQE$. Thus ΔAPQ is isosceles and $\overline{AE}$ is a perpendicular bisector of $\overline{PQ}$. Hence A lies on $\overline{RE}$ $\therefore$ A, R, B lie on the same straight line.

3. Given ΔABC and medians $\overline{AD}$ and $\overline{BE}$ intersecting at a point O, such that $AD = BE$.

   We want to show ΔAOB is isosceles.

   Since $AO = \frac{2}{3}AD$ and $BO = \frac{2}{3}BE$,

   $AO = BO$ from $AD = BE$

   $\overline{AO} \equiv \overline{BO}$

   Hence ΔAOB is isosceles.

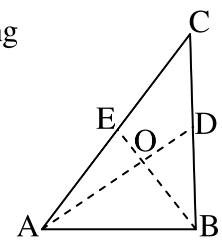
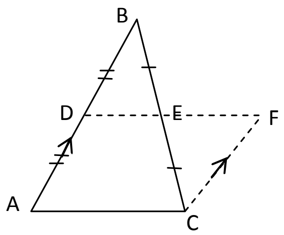


Figure 6.9

4. Given: ΔABC, $\overleftrightarrow{DE}$ a segment joining the mid-points of $\overline{BA}$ and $\overline{BC}$.

To prove: $\overline{DE} \,//\, \overline{AC}$ and $DE = \frac{1}{2}AC$



Proof:

Figure 6.10

Extend $\overline{DE}$ through E to F so that $\overline{DE} \equiv \overline{EF}$

i. Draw $\overline{CF}$ (through two points there is exactly one straight line)

ii. $\Delta DEB \equiv \Delta FEC$ (SAS)

$\therefore \angle BDE \equiv \angle CFE$ and $\overline{BA} \,//\, \overline{CF}$

$\therefore \overline{CF} \equiv \overline{AD}$ and $\overline{CF} \equiv \overline{DB}$ (D is the mid-point of $\overline{AB}$)

Therefore, $ACFD$ is a parallelogram

$\therefore \overline{DF} \equiv \overline{AC}$ and hence $\overline{DF} \,//\, \overline{AC}$

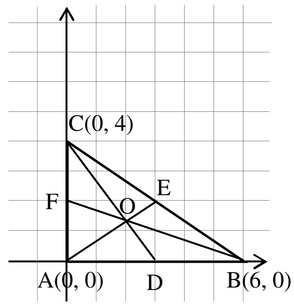
Hence, $DE = \frac{1}{2}(DF) = \frac{1}{2}\,(AC)$ (because $\overline{DF} \equiv \overline{AC}$)

5. a. i. Mid-points of the sides are D (3, 0), E (3, 2) and F(0, 2)

Equation of lines containing the medians are

$y = -\frac{4}{3}x + 4,\ \ y = \frac{2}{3}x$ and $y = -\frac{1}{3}x + 2$

The intersection point O of these line is $\left(2, \frac{4}{3}\right)$



This point is the point of intersection of the medians.

Figure 6.11

ii. From i, $AE = \sqrt{13},\ AO = \sqrt{\frac{52}{9}} = \frac{2}{3}\sqrt{13}$

So, $AO = \frac{2}{3}AE$

Similar for other medians.

b. i. Mid-points of the sides are: G (2, 0), H (3, 2) and I (1, 2)

The equations of the lines containing the medians are

$x = 2,\ y = \frac{2}{3}x$ and $y = -\frac{2}{3}x + \frac{8}{3}$

The intersection point of these lines O is $\left(2, \frac{4}{3}\right)$

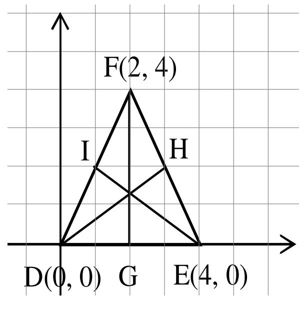


Figure 6.12

ii. For one of the medians $\overline{DH}$; $DO = \frac{2}{3}\sqrt{13} = \frac{2}{3}DH$

6. From Pythagoras Theorem $(CD)^2 = 5^2 - 4^2 = 9$

a. By the Altitude Theorem, $(CD)^2 = (AD)\,(BD)$

So $BD = \frac{9}{4}$ units

b. Since ΔBDC is right-angled, $BC = \sqrt{9 + \left(\frac{9}{4}\right)^2} = \frac{15}{4}$ units

7. Area of ΔABC = Area of ΔAPC + Area of ΔAPB + Area of ΔBPC

$\frac{1}{2}ah = \frac{1}{2}ah_1 + \frac{1}{2}ah_2 + \frac{1}{2}ah_3$

So, $h = h_1 + h_2 + h_3$ as required.

8. Given that D and D' are symmetric with respect to the midpoint M of $\overline{BC}$,

$DM = MD'$

Hence, $BD = D'C$ and $BD' = DC$

Similarly $CE = E'A$ and $CE' = EA$

$AF = F'B$ and $AF' = FB$

Suppose D, E and F are collinear. Then by Menelaus' Theorem, $\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = -1$

Substituting we have, $\frac{D'C}{BD'} \times \frac{E'A}{CE'} \times \frac{F'B}{AF'} = -1$

Equivalently, $\frac{BD'}{D'C} \times \frac{CE'}{E'A} \times \frac{AF'}{F'B} = -1$

Hence D', E' and F' are collinear

9. a. Given ΔABC, $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$

Hence, $\Delta ABC \sim \Delta EDC$

$\therefore \frac{BC}{DC} = \frac{AC}{EC}$

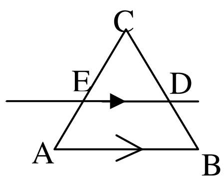


Figure 6.13

As $BC = BD + DC$ and $AC = AE + EC$, we have,

$\frac{BD + DC}{DC} = \frac{AE + EC}{EC} \Rightarrow \frac{BD}{DC} = \frac{AE}{EC}$

$\therefore rs = \frac{BD}{DC} \times \frac{CE}{EA} = \frac{AE}{EC} \times \frac{CE}{EA} = 1$, So, $rs = 1$

b. Suppose $rst = -1$ then by the converse of Menelaus' Theorem D, E and F are collinear.

If $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$, then $r.s = 1$

Hence, $\dfrac{AC}{FB} = -1$ i.e F bisects $\overline{AB}$ externally.

This is not possible as $\overleftrightarrow{DE}$ contains F and $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$

Thus, $\overleftrightarrow{DE} \nparallel \overleftrightarrow{AB}$ ($\overleftrightarrow{DE}$ is not parallel to $\overleftrightarrow{AB}$)

c. Suppose that $rs = 1$ then as shown in 7 a),

$rs = \dfrac{BD}{DC} \times \dfrac{EC}{EA} = 1$ gives $\dfrac{BD}{DC} \times \dfrac{EA}{CE} = \dfrac{EA}{EC} \times \dfrac{EC}{AE}$

$\therefore \dfrac{BD}{DC} + 1 = \dfrac{AE}{EC} + 1$, and hence $\dfrac{BD}{DC} + \dfrac{DC}{DC} = \dfrac{AE}{EC} + \dfrac{EC}{EC}$

Thus, $\dfrac{BC}{DC} = \dfrac{AC}{EC}$

As $\angle C$ is common, $\Delta ABC \sim \Delta EDC$, AAS similarity

$\therefore \angle ABC = \angle EDC$

It follows that $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$.

10. Given, $\dfrac{BD}{DC} = \mathrm{r}, \dfrac{CD'}{D'B} = r, \dfrac{CE}{EA} = 1, \dfrac{BD}{DC} = \dfrac{CE}{EA} \times \dfrac{AF}{FB} = -1$ (Since D, E and F are collinear)

And $\dfrac{BD'}{D'C} \times \dfrac{CE}{EA} \times \dfrac{AF'}{FB} = -1$ (Since D; E and F are collinear) where $E' = E$

To show $FA = BF$

Substituting r. 1 $\dfrac{AF}{FB} = -1$ and $\dfrac{1}{r} \times 1 \times \dfrac{AF'}{F'B} = -1$ since $\dfrac{D'B}{CD'} = \dfrac{BD'}{D'C} = \dfrac{1}{r}$

Thus, $\dfrac{AF}{FB} = -\dfrac{1}{r}$ and $\dfrac{\mathrm{AF'}}{\mathrm{F'B}} = -r$

$\therefore \dfrac{AF}{FB} = \dfrac{F'B}{AF'}$, and $\dfrac{FB}{AF} = \dfrac{AF'}{F'B}$

Since $F'A$, B and F are on a line

$FB = FA - BA$ and $F'A = F'B - AB$ i.e. $AF' = AB - F'B$

Substituting in $\dfrac{FB}{AF} = \dfrac{AF'}{F'B}$

$\dfrac{FA - BA}{AF} = \dfrac{AB - F'B}{F'B}$

$\dfrac{FA}{AF} - \dfrac{AB}{AF} = \dfrac{AB'}{F'B} - \dfrac{F'B}{F'B}$ and then $-1 - \dfrac{AB}{AF} = \dfrac{AB'}{F'B} - 1$

i.e. $\dfrac{AB}{AF} = \dfrac{AB}{F'B}$ as $BA = -AB$

Therefore $AF = F'B$, i.e $FA = BF'$

6.2 SPECIAL QUADRILATERALS

Periods allotted: 6 periods

Competency

At the end of this sub-unit, students will be able to:

- *apply theorems on special quadrilateral in solving related problems.*

Vocabulary: Trapezium, Parallelogram, Rectangle, Rhombus, Square.

Introduction

In this sub-unit, special quadrilaterals and their properties is studied. Students are required to identify trapezium, parallelogram, rectangle, rhombus and square. Theorems regarding these quadrilaterals are given.

Teaching Notes

The students are not new to the special quadrilaterals mentioned above. Discuss what each mean and focus on specific and common properties.

Here a trapezium is defined as a quadrilateral with only two of its sides parallel. Hence a trapezium is not a parallelogram and a parallelogram is not a trapezium.

A rectangle and rhombus are parallelograms where rectangle is equiangular but rhombus is equilateral. A square is a rectangle and a rhombus.

Activity 6.4 is useful in helping students revise basic concepts about parallel lines, properties of quadrilaterals and some special quadrilaterals. Give them time to relate and describe the basic terms and facts about quadrilaterals. Following the activity, help the students recognize some properties of trapezium, parallelogram, rectangle, rhombus and square.

Answers to Activity 6.4

1. For instance, for a rectangular blackboard, the upper and the lower edges extended indefinitely form parallel lines.
2. In a plane, through a point not on a given line, there is exactly one line parallel to the given line.
3. A polygon of four sides is called a quadrilateral.
   An equiangular quadrilateral is a quadrilateral whose all angles are equal and an equilateral quadrilateral is a quadrilateral whose all sides are equal.
4. A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.
   A rectangle is a parallelogram in which one of its angles is a right angle.
   A square is a rectangle which has congruent adjacent sides.
5. Altitude of a parallelogram is a perpendicular distance between two parallel sides of a parallelogram.
6. In **Figure 6.21** given in the student textbook,

i. $\overline{AB}$ and $\overline{BC}$; $\overline{AD}$ and $\overline{DC}$ are adjacent sides

ii. $\angle A$ and $\angle C$; $\angle D$ and $\angle B$ are opposite vertices of the quadrilateral.

iii. $\overline{AC}$ and $\overline{BD}$ are diagonals.

7. A line segment that joins two opposite vertices of a quadrilateral is called a diagonal of the quadrilateral. Two diagonals.

After discussion of definition of trapezium and parallelogram, give Activity 6.5 as a project work, make them give explanations and forward their perception about what they observe.

Answers to Activity 6.5

Give the following questions as a project work.

1. The quadrilateral $PQRS$ is parallelogram. i,e, quadrilateral formed by joining consecutive mid-points of a given quadrilateral is a parallelogram.
2. Base angles of an isosceles trapezium are congruent.
   The diagonals of an isosceles trapezium are congruent.
   Draw $\overline{DC}$. Using protractor, draw $\overline{AD}$ and $\overline{BC}$ of length 3 cm. Carefully construct $\overline{AB}$ of length 2 cm.
   m ($\angle$ADC) = m ($\angle$BCD) ≅ 71°
   AC = BD ≅ 4.2 cm

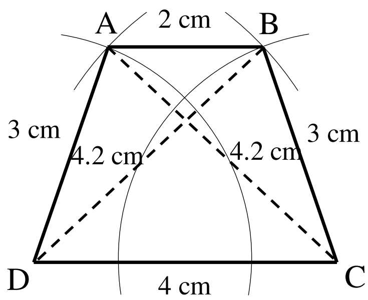


Figure 6.14

3. Guide students how to construct the required figure.

With the help of the discussion from Activity 6.4 and Activity 6.5, state Theorem 6.7 and let the students prove some of the statements. Ask them to define and identify a rectangle, rhombus and square.

To clearly identify the quadrilaterals mentioned, let the grouped students discuss on Group work 6.2.

Answers to Group work 6.2

1. A rectangle and a square are parallelograms and all are quadrilaterals. Each interior angle of a rectangle and a square is a right angle. A square is a rectangle. A square is an equilateral polygon.
2. Rhombus
3. Since the diagonals of a rhombus are perpendicular to each other, the four triangles formed by the diagonals of the rhombus are equal and they are isosceles triangles.

The following questions may be used for additional problems for fast learners.

1. Let $ABCD$ be an equilateral quadrilateral.
   a. Show that $ABCD$ is a rhombus.
   b. Show that $\overline{AC}$ bisects $\angle A$.
2. In the following figure $ABCD$ is a parallelogram. G is the mid-point of $\overline{AB}$ and C is the mid-point of $\overline{BE}$. Show that
   c. $\overline{AD} \equiv \overline{EC}$
   d. AE = 2 GC
   e. AECG is a trapezium

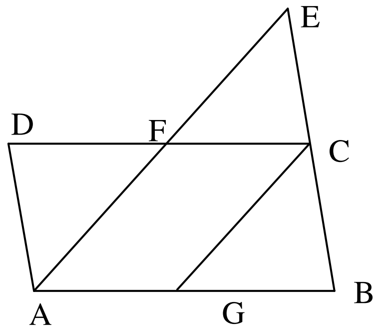


Figure 6.15

3. In the following figure ABCD is a parallelogram.

   Let $\overline{AB}$ be extended to E so that as $\overline{AD} \equiv \overline{BE}$ and $\overline{AD}$ to F so as $\overline{AD} \equiv \overline{BF}$. Show that
   f. BD = $\frac{1}{2}$FE
   g. E, C and F are collinear
   h. $\angle$ ABD $\equiv \angle$ AEF.

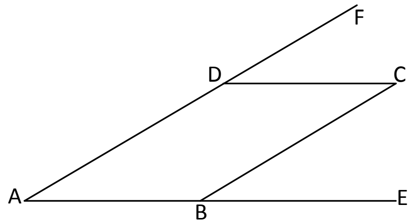


Figure 6.16

Here enough exercises are given so that students can practice on different types of problems. Use some of them to be done in class and some of the rest as stated in the assessment.

Assessment

It is hoped that almost all students will follow this lesson as fast learners. Collect the assignment you gave from Exercise 6.1 and check how they perform. Give them corrections and explanations if there are any. You may give some questions of Exercise 6.2 as home work and class work and check their answers.

Answers to Exercise 6.2

1. Given parallelogram $ABCD$
   Draw $\overline{PQ}$ (construction)
   $\overline{AP} \equiv \overline{PB} \equiv \overline{DQ} \equiv \overline{QC}$ (Given)
   $\overline{AD} \equiv \overline{BC}$ (Opposite sides of a parallelogram)
   $\angle D \equiv \angle B$ (definition of a parallelogram)
   $\therefore\ \Delta ADQ \equiv \Delta CBP\ (SAS)$
   So $\overline{AQ} \equiv \overline{CP}$ and $\angle QAD \equiv \angle PCB$
   Therefore $\overline{AQ} // \overline{PC}$. So $APCQ$ is a parallelogram.

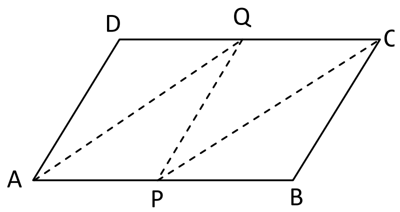


Figure 6.17

2. $ABCD$ is a rectangle E, F, G and H are Mid-points of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$ respectively. $EFGH$ is a rhombus.

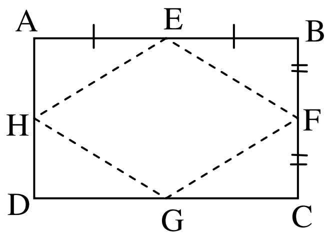


Figure 6.18

Proof:

Consider ΔHAE and ΔFBE

$\overline{HA} \equiv \overline{FB}$

$\overline{AE} \equiv \overline{BE}$

$\angle A \equiv \angle B$

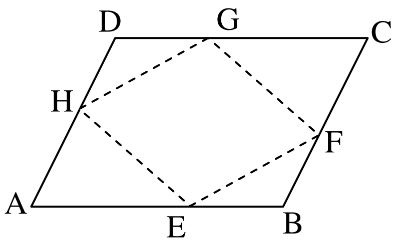
$\therefore \Delta HAE \equiv \Delta FBE$ (SAS)

$\therefore \overline{HE} \equiv \overline{FE}$

Similarly, considering ΔEBF and ΔGCF, we can prove that $\overline{EF} \equiv \overline{GF}$. Finally, considering ΔFCG and $\Delta HDG,$ we prove that $\overline{FG} \equiv \overline{HG}$.
Therefore $\overline{AE} \equiv \overline{EF} \equiv \overline{FG} \equiv \overline{GH}.$ thus, $EFGH$ is a rhombus.

3. $ABCD$ is a parallelogram E, F, G and H are the mid-points of the sides $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{DA}$ respectively.

We claim that GHEF is a parallelogram.



Proof: Consider ΔCGF and ΔAEH

$\angle A = \angle C$

$\overline{AE} \equiv \overline{CG}$

$\overline{AH} \equiv \overline{CF}$

$\therefore \Delta CGF \equiv \Delta AEH$ (SAS)

$\therefore \overline{GF} \equiv \overline{EH}$

Figure 6.19

Similarly, considering ΔDGH and $\Delta BEF,$ we can prove that $\overline{HG} \equiv \overline{EF}$.
Therefore, opposite sides of $GHEF$ are equal. Hence it is a parallelogram.

4. a. Given: Parallelogram $ABCD$ with $\overline{AC} \equiv \overline{BD}$

Proof:

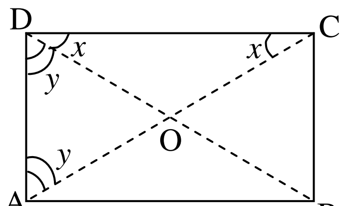


Figure 6.20

$\overline{AO} \equiv \overline{OC} \equiv \overline{OD} \equiv \overline{OB}$

Consider ΔOCD and ΔODA.

They are Isosceles with base angle each equal to x and y respectively.

Now $m\,(\angle DOA) = 2x$ and $m\,(\angle DOC) = 2y$

But $m\,(\angle DOA) + m\,(\angle DOC) = 180^{\circ}$

$\therefore 2x + y = 180°$

$\therefore x + y = 90°$

Therefore, $m\,(\angle D) = 90^{\circ}$

Since $ABCD$ is a parallelogram, we see that $m(\angle B) = 90^o$

Since $m(\angle C) + m(\angle B) = 180^o$, $m(\angle C) = 90^o$.

Obviously, $m(\angle A) = 90^o$

Therefore $ABCD$ is a rectangle.

b. Given quadrilateral $ABCD$ in which $\overline{AC}$ and $\overline{BD}$ bisect each other and one of its angle say $\angle D$ is a right angle.

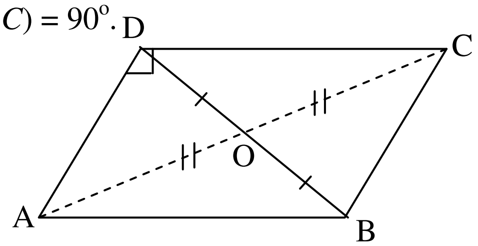


Figure 6.21

Since $\overline{AC}$ and $\overline{BD}$ bisect each other, we see that $\overline{AO} \equiv \overline{OC}$ and $\overline{OD} \equiv \overline{OB}$.

Consequently, $\Delta AOD \equiv \Delta COB$ and $\Delta COD \equiv \Delta AOB$

$\therefore \angle ADO \equiv \angle OBC$

But, since these are alternate interior angles, we conclude that $\overline{AD} // \overline{BC}$.

Similarly, since $\angle ODC \equiv \angle OBA$, it follows that $\overline{DC} // \overline{AB}$. Therefore, ABCD is a parallelogram. Thus since $\overline{DC} // \overline{AB}$, $m(\angle D) + m(\angle A) = 180^o$

But $m(\angle D) = 90^o$ (given)

$\therefore m(\angle A) = 90^o$

Moreover, $m(\angle B) = m(\angle D) = 90^o$

$m(\angle C) = m(\angle A) = 90^o$ (Opposite angles of a parallelogram are equal).

$\therefore ABCD$ is a rectangle.

c. Given $ABCD$ (quadrilateral) such that $\overline{AB} \equiv \overline{CD} \equiv \overline{DA} \equiv \overline{BC}$

Draw diagonal $\overline{DB}$

So, $\Delta ADB \equiv \Delta CDB$ (SSS)

$\Rightarrow \angle A \equiv \angle C$

Similarly $\angle B \equiv \angle D$

$\therefore$ $ADBC$ is a parallelogram

So, it is a rhombus since $\overline{AB} \equiv \overline{BC}$

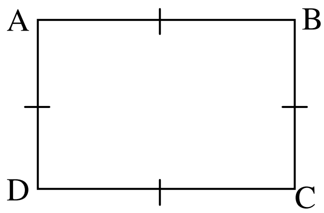


Figure 6.22

d. Given: A rhombus $ABCD$ with $\overline{AB} \equiv \overline{BC} \equiv \overline{CD} \equiv \overline{DA}$. Since $ABCD$ is a parallelogram the diagonals $\overline{AC}$ and $\overline{BD}$ bisect each other.

$\therefore \overline{AO} \equiv \overline{OC}$ and $\overline{DO} \equiv \overline{OB}$

Now consider ΔDOC and ΔBOC.

$\overline{DC} \equiv \overline{BC}$, $\overline{OC} \equiv \overline{OC}$

$\overline{DO} \equiv \overline{OB}$, see Group work 6.3 # 6

$\therefore \Delta DOC \equiv \Delta BOC$ (SSS)

But $m(\angle COD) + m(\angle COB) = 180^o$

Then, $m(\angle COD) = m(\angle COB)$

$\therefore \overline{AC} \perp \overline{BD}$

Thus, the diagonals are perpendicular.

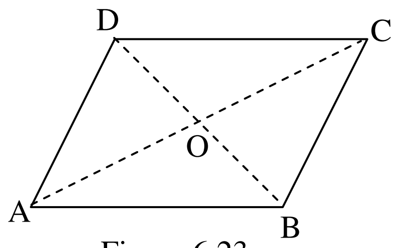


Figure 6.23

5. a. Given: Quadrilateral $ABCD$ in which $\overline{AB} \equiv \overline{CD}$ and $\overline{AD} \equiv \overline{BC}$

To prove: $ABCD$ is a parallelogram

Construction: Join $\overline{DB}$.

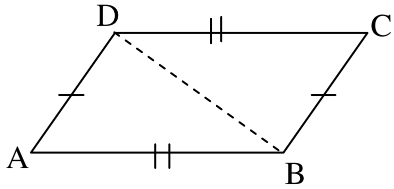


Figure 6.24

Proof: $\overline{AD} \equiv \overline{BC}, \overline{AB} \equiv \overline{DC}, \overline{DB} \equiv \overline{DB}$

$\therefore \Delta ADB \equiv \Delta CBD$ (SSS)

$\therefore m(\angle CDB) \equiv m(\angle ABD)$

But these angles are alternate interior angles.

$\therefore \overline{DC} // \overline{AB}$

Similarly, since $m(\angle ADB) \equiv m(\angle CBD)$, we see that $\overline{AD} // \overline{BC}$

$\therefore$Both pairs of opposite sides of $ABCD$ are equal and parallel. Hence, it is a parallelogram.

b. Given: $\overline{AB} \equiv \overline{DC}$ and $\overline{AB} // \overline{DC}$

To Prove: $ABCD$ is a parallelogram

Construction: Draw $\overline{BD}$.

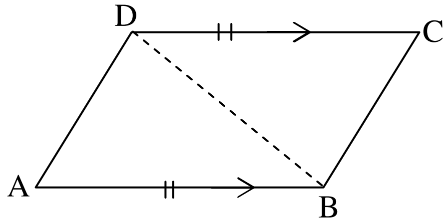


Figure 6.25

Proof i. $\angle CDB \equiv \angle ABD$ (Alternative interior angles and $\overline{AB} // \overline{DC}$)

ii. $\overline{DC} \equiv \overline{BA}$ (given)

iii. $\overline{DB} \equiv \overline{BD}$ (common)

$\therefore \quad \Delta CDB \equiv \Delta ABD \quad (SAS)$

Therefore, $\overline{AD} \equiv \overline{BC}$ and $\angle ADB \equiv \angle CBD$

But $\angle ADB$ and $\angle CBD$ are alternative interior angles.

$\therefore \overline{AD} // \overline{BC}$

$\therefore ABCD$ is a quadrilateral with opposite sides parallel and equal. Hence it is a parallelogram.

c. Given quadrilateral $ABCD$ with $\overline{AO} \equiv \overline{OC}$ and $\overline{DO} \equiv \overline{OB}$, see **Figure 6.26**.

To prove: $ABCD$ is parallelogram.

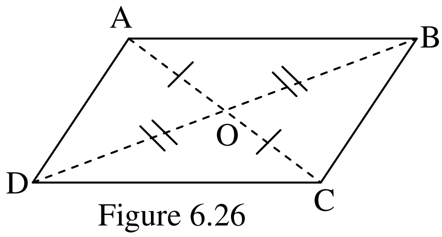


Figure 6.26

Proof:

i. $\overline{AO} \equiv \overline{CO}$ (given)

ii. $\overline{OB} \equiv \overline{DO}$ (given)

iii. $\angle AOB \equiv \angle COD$ (Vertically opposite angles)

$\therefore \Delta AOB \equiv \Delta COD$ (by SAS).

Hence $\overline{AB} = \overline{CD}$ and $\angle OAB \equiv \angle OCD$

$\therefore \overline{AB} // \overline{CD}$

Similarly, $\Delta AOD \equiv \Delta COB$ (by SAS)

$\overline{AD} \equiv \overline{CB}$ and $\angle ODA \equiv \angle OBC$

$\therefore \overline{AD} // \overline{CB}$

Therefore, $ABCD$ is a parallelogram.

6.

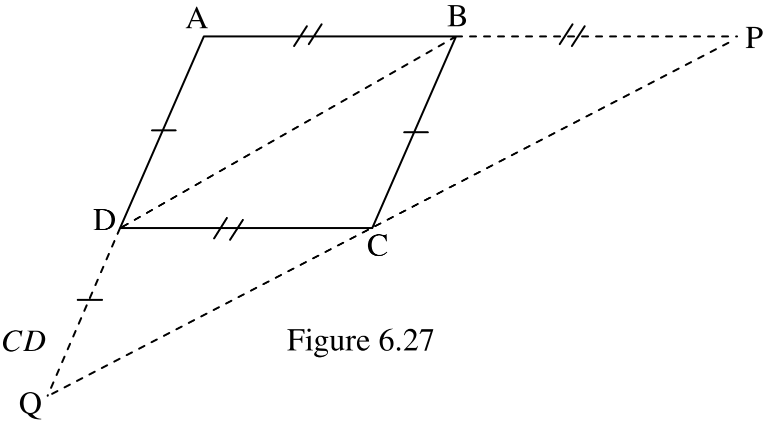


Figure 6.27

Proof:

i. Draw $\overline{DB}$

ii. Since $ABCD$ is a parallelogram,

$\overline{AD} \equiv \overline{BC}$

$\overline{AB} \equiv \overline{DC}$

Consider ΔQDC and ΔBCD

$\overline{QD} \equiv \overline{BC}$ (Given)

$\overline{DC} \equiv \overline{CD}$ (Common)

$\angle QDC \equiv \angle BCD$ (Since $ABCD$ is a parallelogram, $\overline{AD} // \overline{BC}$ and so $\overline{QD} // \overline{BC}$. They are alternative interior angles)

$\therefore \Delta QDC \equiv \Delta BCD$ (By SAS)

Hence, $\overline{QC} \equiv \overline{BD}$. Therefore, $DBCQ$ is a parallelogram. Thus, $\overline{DB} // \overline{QC}$ and similarly $\overline{DB} // \overline{CP}$, So $\overline{DB} // \overline{QP}$. Therefore, P, C and Q lie on one straight line.

7. Given: A parallelogram and M is the mid-point $\overline{BC}$.

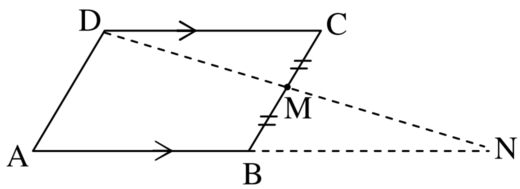


Figure 6.28

Proof:

i. $\angle DMC \equiv \angle NMB$ (vertically opposite angles)

ii. $\overline{BM} = \overline{CM}$ (M is mid-point)

iii. $\angle DCM \equiv \angle NBM$ (alternative interior angles)

$\therefore \Delta\, DMC \equiv \Delta NMB$ (by ASA)

Hence $\overline{DC} \equiv \overline{NB}$

But, $\overline{DC} \equiv \overline{AB}$ (opposite sides of a parallelogram)

Therefore $\overline{AB} \equiv \overline{BN}$ (as $\overline{BN}$ means $\overline{NB}$)

8. Given: A parallelogram $ABCD$ with M and N mid-points of $\overline{DC}$ and $\overline{AB}$ respectively.

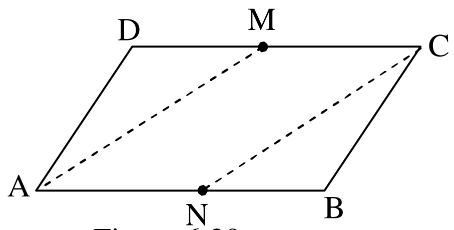


Figure 6.29

Proof

i. $\overline{DM} \equiv \overline{BN}$ (because $\overline{DC} // \overline{AB}$ and M and N are mid-points)

ii. $\overline{AD} \equiv \overline{CB}$ (opposite sides of the parallelogram)

iii. $\angle ADM \equiv \angle CBN$ (opposite angles of the parallelogram)

$\therefore\ \Delta ADM \equiv \Delta CBN$ (SAS)

Therefore, $\overline{AM} \equiv \overline{CN}$

9. Given: A parallelogram $ABCD$

Proof

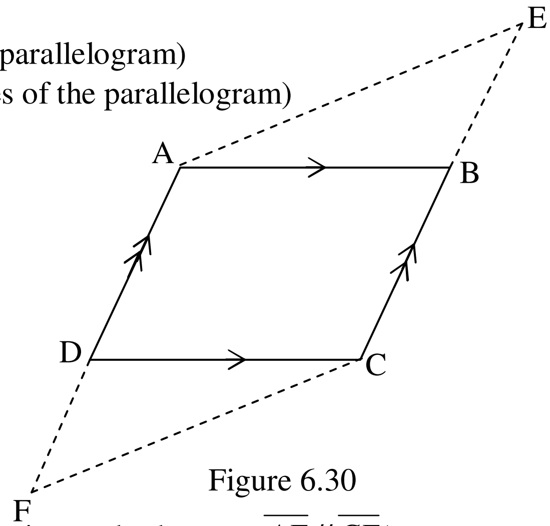


Figure 6.30

i. $\overline{DF} \equiv \overline{BE}$ (given)

ii. $\overline{AF} // \overline{CE}$ (because $\overline{AD} // \overline{CB}$)

iii. $\overline{AB} \equiv \overline{CD}$ (Opposite sides of a parallelogram)

iv. $\angle DAB \equiv \angle ABE$ (Alternative interior angles because $\overline{AF} // \overline{CE}$)

v. $\angle BCD \equiv \angle FDC$ (Alternative interior angles because $\overline{AF} // \overline{CE}$)

vi. But $\angle A \equiv \angle C$ (Opposite angles of a parallelogram).

Hence, $\angle \mathrm{ABE} \equiv \angle CDF$ (From steps iv and v)

Thus, $\Delta ABE \equiv \Delta CDF$.

Therefore $\overline{AE} \equiv \overline{CF}$ and $\angle F \equiv \angle E$.

Since $AECF$ is a quadrilateral with opposite angles equal and two opposite sides are parallel and equal, the quadrilateral $AECF$ is a parallelogram.

6.3 MORE ON CIRCLES

Periods allotted: 6 periods

Competency

At the end of this sub-unit, students will be able to:

- *apply the theorems on angles and arcs determined by lines intersecting inside, on and outside a circle to solve related problems.*

Vocabulary: Arc, Central angle, Inscribed angle, Secant line, Tangent line, Product property.

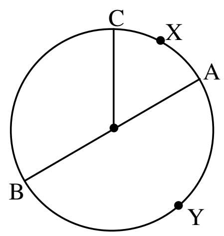
Introduction

Start the lesson by revising basic ideas on circle by asking your students what a circle is and what its parts are. Discuss with your students diameter, chords, tangent lines, secant lines, arcs and central angles of a circle. Activity 6.6 helps the students to revise what they did in the previous grades about circles and hence should be treated before starting sub-section 6.3.1. This is about angles and arcs determined by lines intersecting inside and on a circle.

You can state and prove theorems on angles and arcs determined by lines intersecting inside and on a circle. The next sub-section 6.3.2 is about angles and arcs determined by lines intersecting outside circle, treated in the same manner as sub-section 6.3.1.

Teaching Notes

In this topic the statements on circles, lines and angles are treated in two subsections 6.3.1 and 6.3.2. In 6.3.1, the main ideas are central angles and angles with vertices on a circle. Make your students relate the measures of these angles with the measures of arcs subtending them.
Consider the following figure:



O is the centre of the circle

i. $m(\angle AOC) = m(\overset{\frown}{AXC})$

ii. $m(\angle ABC) = \dfrac{1}{2} m(\overset{\frown}{AXC})$

Show them that $m\ (\angle AOC) = 2\ m\ (\angle ABC)$ from the triangle.

Figure 6.31

It follows that $m(\angle ACB) = \dfrac{1}{2} m(\overset{\frown}{AYB}) = \dfrac{1}{2} m(\overset{\frown}{AXB})$

$$\therefore m(\angle ACB) = \frac{1}{2}(180^{\circ}) = 90^{\circ}$$

To revise what has been discussed in the previous sections and clarify the terminologies they are going to use, let the students practice Activity 6.6. Using only a ruler and compass, let students construct an arc of 30º.

Answers to Activity 6.6

1. As shown in **Figure 6.32**
   Line ℓ_1 intersects the circle at one point P, ℓ_2 intersects the circle at two points A and B, but ℓ_3 does not intersect the circle.

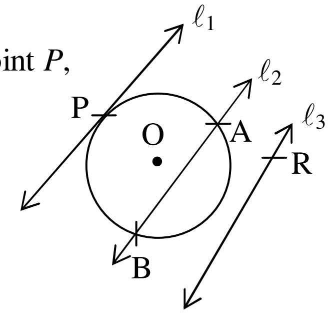


Figure 6.32

2. If the length of a radius is r, then the length of the diameter is $2r$.
3. a. $\overline{GA}, \overline{GH}, \overline{AC}$ and $\overline{AB}$ are chords; $\overline{PB}$ and $\overline{PD}$ are secants; and $\overline{QE}$ and $\overline{QA}$ are tangents.
   b. $\angle GFA$, $\angle GFC$
   c. $\angle AQE$ is an angle formed by two intersecting tangents.
   d. $\angle BPD$ is an angle formed by two intersecting secants.
4. You can use protractor to construct the given angles.
5. $\dfrac{\pi}{3}$
6. The degree measure of a semi-circle is 180°.
7. It is always true.

Angles and Arcs determined by lines intersecting inside and on a circle

Now, give descriptions of central angle and inscribed angle together with what a measure mean. Next discuss Theorem 6.9 which is fundamental for our study.

As a consequence of Theorem 6.9 which is: The measure of an angle inscribed in a circle is half the measure of the arc subtending it, three corollaries are presented. Corollary 6.9.1 called Thales Theorem is very important for different applications. Students should have a clear understanding of this idea. Ask them the following. In **Figure 6.40** of the text book, if $m(\angle BAC) = 70^\circ$, what is the measure of

i.	$\angle ABC$?	ii.	arc BC?, minor one	iii.	arc ADC?
iv.	$\angle$ACB?	v.	arc AB, minor one		

What is the sum of the measures of arcs BC and ADC?

After a brief discussion of the other corollaries, Theorems and examples of this subsection, let the students do Group work 6.3 and check their performance.

Answers to Group work 6.3

1. Two parallel lines intercept congruent arcs.
   So, $m(\overset{\frown}{AP}) = m(\overset{\frown}{BQ})$
   $m(\angle BOQ) = m(\overset{\frown}{BQ})$ ($\angle BOQ$ is central angle)
   $m(\overset{\frown}{BQ}) = 70^\circ$ and so $m(\overset{\frown}{AP}) = 70^\circ$
   Therefore, $m(\angle AOP) = m(\overset{\frown}{AP}) = 70^\circ$
2. $PO = 5$ unit, radius
   $m(\angle POQ) = 120^\circ$ and ΔPOQ is isosceles. From trigonometry
   $PQ = 2x = 2\times 5\sin 60^o = 10\left(\dfrac{\sqrt{3}}{2}\right) = 5\sqrt{3}$ units.

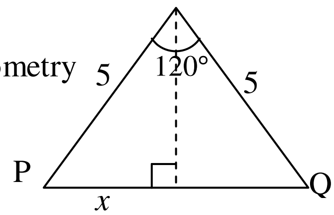


Figure 6.33

3. $m(\angle AOB) = m(\overset{\frown}{ADB}) = 90^{\circ}$

   a. $m(\angle ACB) = \frac{1}{2} m(\overset{\frown}{ADB}) = \frac{1}{2}(90^{\circ}) = 45^{\circ}$

   $m(\angle ADB) = \frac{1}{2} m(\overset{\frown}{ACB})$ (But $m(\overset{\frown}{ACB}) = 360^{\circ} - 90^{\circ} = 270^{\circ}$) $= \frac{1}{2}(270^{\circ}) = 135^{\circ}$

   b. Extend $\overline{AO}$ and $\overline{BO}$ to touch the circle at E and F respectively on the circle.
   $m(\overset{\frown}{FCE}) = 90^{\circ}$, $m(\overset{\frown}{CE}) = 40^{\circ}$. So $m(\overset{\frown}{FC}) = 90^{\circ} - 40^{\circ} = 50^{\circ}$
   $\therefore m(\angle CBO) = m(\angle FBC) = \frac{1}{2} m(\overset{\frown}{FC}) = 25^{\circ}$

Next, select some problems from Exercise 6.3 to be attempted in class and to be done as home work in order to deepen their knowledge of concepts. You can choose questions 6 – 10 for the home work purpose.

There is variety of problems for you to use for different assessments.

Answers to Exercise 6.3

1. Consider **Figure 6.51** of the textbook.
   Since, $\overline{BD} // \overline{OC}, m(\angle ABD) = m(\angle AOC)$ corresponding angles
   Hence, $m(\angle AOC) = 60^{\circ}$
   Since $m(\angle ABD) = 60^{\circ}$, $m(\overset{\frown}{ACD}) = 120^{\circ}$
   As $m(\overset{\frown}{AC}) = 60^{\circ}$, $m(\overset{\frown}{CD}) = m(\overset{\frown}{ACD}) - m(\overset{\frown}{AC})$
   $= 120^{\circ} - 60^{\circ} = 60^{\circ}$
   So $m(\angle COD) = 60^{\circ}$ and ΔCOD is isosceles
   $\therefore m(\angle OCD) = 60^{\circ}$.

2. Let **Figure 6.34** represent the given problem.
   Let $m(\angle ABC) = 90^{\circ}$
   By Theorem 6.9, $m(\angle ABC) = \frac{1}{2}(\overset{\frown}{ADC})$
   $90^{\circ} = \frac{1}{2} m(\overset{\frown}{ADC})$

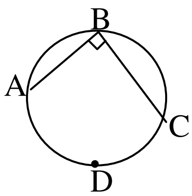


Figure 6.34

   $m(\overset{\frown}{ADC}) = 180^{\circ}$. Hence arc ADC is a semi-circle.

3. a. $m(\angle YLN) = \frac{1}{2} m(\overset{\frown}{MX}) + \frac{1}{2} m(\overset{\frown}{YN})$ (By Theorem 6.12)
   $= \frac{1}{2}(28^{\circ} + 50^{\circ}) = 39^{\circ}$

   b. Since $(ML)(LN) = (YL)(LX)$, $YL = \frac{(ML)(LN)}{LX} = \frac{4(7)}{5} = 5.6$ units

4. Yes, it is possible for $\angle MLX$ to be 30°. So

$$m(\angle MLX) = \frac{1}{2}m(\widehat{MX}) + \frac{1}{2}m(\widehat{YN})$$

$$30^\circ = \frac{1}{2}m(40^\circ) + \frac{1}{2}m(\widehat{YN})$$

$$m(\widehat{YN}) = 60^\circ - 40^\circ = 20^\circ$$

5. $m(\widehat{AB}) = 40^\circ$ and $m(\widehat{CD}) = 60^\circ$

i. $m(\angle AQB) = \frac{1}{2}(40^\circ + 60^\circ) = 50^\circ$

ii. $m(\angle BOC) = 180^\circ - (40^\circ + 60^\circ) = 80^\circ$

So, $m(\widehat{BC}) = 80^\circ$ and

m($\widehat{DE}$) $= 180^\circ - (80^\circ + 60^\circ) = 40^\circ = m(\angle DOE)$

So $m(\widehat{CDE}) = 60^\circ + 40^\circ = 100^\circ$

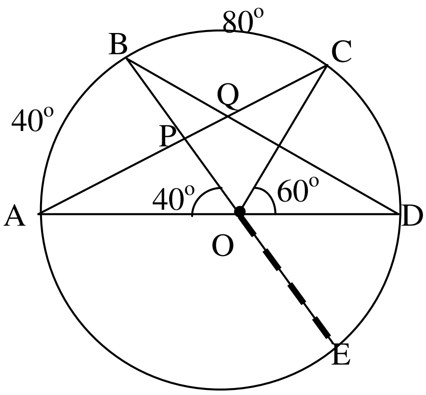


Figure 6.35

Hence, $m(\angle APB) = \frac{1}{2}(m(\widehat{AB}) + m(\widehat{CDE})) = \frac{1}{2}(40 + 100) = 70^\circ$

Therefore $m(\angle APB) = 70^\circ$

6. $m(\widehat{MF}) = 2m(\angle FAM) = 40^\circ$ and $m(\widehat{CE}) = 2m(\angle CPE) = 60^\circ$

$$m(\angle EYC) = \frac{1}{2}(m(\widehat{CE}) + m(\widehat{MF})) = \frac{1}{2}(40^\circ + 60^\circ) = 50^\circ \Rightarrow m(\angle EYC) = 50^\circ$$

7. a. i. $m(\widehat{ABC}) + m(\widehat{ADC}) = 360^\circ$

ii. $m(\angle ABC) = \frac{1}{2}m(\widehat{ADC})$ and $m(\angle ADC) = \frac{1}{2}m(\widehat{ABC})$

Thus, $m(\angle ABC) + m(\angle ADC) = \frac{1}{2}m(\widehat{ABC}) + \frac{1}{2}m(\widehat{ADC}) = \frac{1}{2}(360^\circ) = 180^\circ$

Therefore, $\angle ABC$ and $\angle ADC$ are supplementary.

b. $m(\angle P) + m(\angle R) = 60^\circ + 90^\circ = 150^\circ$. Thus, the two opposite angles of the quadrilateral are not supplementary. The sum of two opposite angles should be supplementary for the vertices of a quadrilateral to lie on a circle. Therefore, there is no circle containing *P*, *Q*, *R* and *S*.

8. $x = \frac{1}{2}m(\widehat{ABC}) = \frac{1}{2}(160^\circ) = 80^\circ \Rightarrow m(\widehat{ABC}) = m(\angle AOC) = 160^\circ$

$$m(\angle ABC) = y = \frac{1}{2}m(\widehat{ADC}) = \frac{1}{2}(360^\circ - 160^\circ) = \frac{1}{2}(200^\circ) = 100^\circ$$

9. $m(\angle EBD) = \frac{1}{2} m(\overset{\frown}{ED})$

$30^\circ = \frac{1}{2} m(\overset{\frown}{ED})$. Hence, $m(\overset{\frown}{ED}) = 60^\circ$

$p = \frac{1}{2} m(\overset{\frown}{ED}) = \frac{1}{2}(60^\circ) = 30^\circ$

$q = \frac{1}{2} m(\overset{\frown}{ED}) = \frac{1}{2}(60^\circ) = 30^\circ$

$m(\overset{\frown}{BC}) = 80^\circ$ because $m(\angle BEC) = 40^\circ$. So, $x = \frac{1}{2} m(\overset{\frown}{BC}) = \frac{1}{2}(80^\circ) = 40^\circ$

$m(\overset{\frown}{DC}) = 124^\circ$ because $m(\angle DBC) = 62^\circ$, So $y = \frac{1}{2} m(\overset{\frown}{DC}) = \frac{1}{2}(124^\circ) = 62^\circ$

10. $r = \frac{1}{2} m(\overset{\frown}{PQ}) = \frac{1}{2}(50^\circ) = 25^\circ$

$35^\circ = \frac{1}{2}(x + 50^\circ) \Rightarrow x + 50^\circ = 70^\circ$

$\Rightarrow x = 20^\circ$

Since $\frac{1}{2} m\left(\overset{\frown}{RQ}\right) = m(\angle RPQ) \Rightarrow m\left(\overset{\frown}{RQ}\right) = 80^\circ$, so $s = 40^\circ$

$y = \frac{1}{2} m\ (\overset{\frown}{PQ}) = \frac{1}{2}(50^\circ) = 25^\circ$

$m(\angle RUQ) = \frac{1}{2} m(\overset{\frown}{PS} + \overset{\frown}{RQ})$

$145^\circ = \frac{1}{2} t + \frac{1}{2}(80^\circ)$

$t + 80^\circ = 290^\circ \Rightarrow t = 210^\circ$

Angles and Arcs Determined by Lines Intersecting outside a Circle

This topic is a continuation of subsection 6.3.1, where the lines meet outside the circle. It refers to angles and arcs determined by lines intersecting outside a circle. Theorems 6.13, 6.14 and 6.15 can be summarized as:

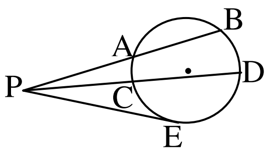


Figure 6.36

1. $m(\angle APC) = \frac{1}{2}\left[m(\overset{\frown}{BD}) - m(\overset{\frown}{AC})\right]$
2. $m(\angle CPE) = \frac{1}{2}\left[m(\overset{\frown}{DE}) - m(\overset{\frown}{CE})\right]$
3. $(PA)\ (PB) = (PC)\ (PD)$
4. $(PE)^2 = (PC)\ (PD)$

Help the students focus on the above relations that they can do similar and resulting questions as Example 5 given in the student text book. State the theorems and discuss the proofs. For further application, use Group work 6.4 as in the usual manner.

Answers to Group work 6.4

1. Refer to the following figure.

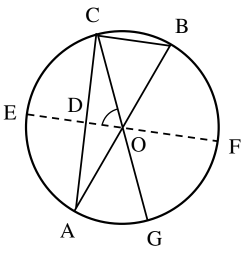


Figure 6.37

Given $\angle AOD \equiv \angle COD$, O is the centre of the circle.

To show $\overline{OD} // \overline{BC}$

$\Delta AOD \equiv \Delta COD$ by SAS congruency

$\therefore$ $\angle ADO \equiv \angle COD$. These angles are on $\overline{AC}$

so, m ($\angle ADO$) = 90°

$\angle ACB$ is inscribed in a semicircle, and hence m ($\angle ACB$) = 90°

It follows that $\angle ADO \equiv \angle ACB$. These angles are corresponding angles of $\overline{OD}$ and $\overline{BC}$ with $\overline{AC}$ a transversal.

$\therefore$ $\overline{OD} // \overline{BC}$

2. As given in the hint, draw a line through A parallel to $\overline{PX}$. Let this line meet the circle at D. Then m ($\angle APX$) = m ($\angle EAD$) corresponding angles, Where E is on $\overleftrightarrow{PA}$ with A between P and E.

$$m(\angle EAD) = \frac{1}{2}m(\overset{\frown}{AD}) = \frac{1}{2}m(\overset{\frown}{ACX}) - \frac{1}{2}m(\overset{\frown}{DCX})$$

From $\overline{AD} // \overline{PX}$, $m(\overset{\frown}{ABX}) = m(\overset{\frown}{DCX})$

$$\therefore m(\angle P) = m(\angle APX) = \frac{1}{2}m(\overset{\frown}{AD}) = \frac{1}{2}m(\overset{\frown}{ACX}) - \frac{1}{2}m(\overset{\frown}{ABX})$$

3. It is assumed that the earth is a sphere. In view of the satellite and the earth equator we have the following figure.

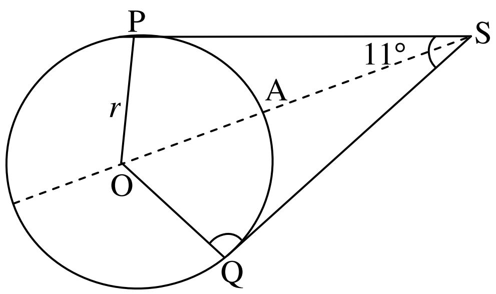


Figure 6.38

To find the length of arc $\overset{\frown}{PAQ}$.

r = OP = OQ, radius of the earth $\overline{OS}$ bisects $\angle PSQ$.

$\Rightarrow$ m ($\angle OSQ$) = 5.5° and ΔOSQ is a right triangle.

AS = 35,000

$$\therefore \sin 5.5^{\circ} = \frac{r}{r+35000}; \sin 5.5^{\circ} = 0.0958$$

$\Rightarrow$0.0958 (r + 35000) = r

$$\therefore r = \frac{0.0958(35000)}{1-0.0958} = 3708.25$$

From the relation $\frac{2\pi r}{2\pi} = \frac{\ell}{\theta}$, θ in radian ℓ = length of

$$\overset{\frown}{PAQ} = r\theta = 3708.25 \times 2(0.494)\pi \cong 10936.84\text{km, where } \theta = 2(90^{\circ} - 5.5^{\circ}).\frac{\pi}{180^{\circ}}$$

To apply the Theorems discussed above and expand their ability, let the students do some questions of Exercise 6.4 in class and check their performance. You can classify your students and select the questions based on their understanding. Give questions 7 – 9 as home take examination and mark it. Along your discussion, you may find fast/slow learning students. You can give to them questions of the following type as additional exercises.

Part *A*: for slow learners:

In the figure given below, *A*, *B* and *C* are on the circle.

If $m(\angle ABC) = 35°$ and $m(\angle BAC) = 70°$.

Find $m(\overset{\frown}{AXB})$

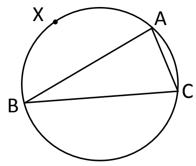


Figure 6.39

Part *B* for fast learners.

1. In the **Figure 6.40,** $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are tangents to the circle centered at O. E is on the circle.
   Show that $(AD)(BC) = (AB)^2$.
   (Hint: $\angle AEB$ is a right angle and $\Delta ABC \approx \Delta DAB$)

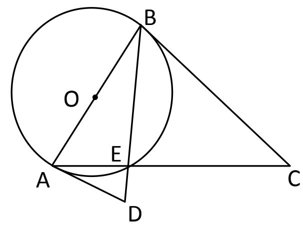


Figure 6.40

2. In the following figure $\overline{AB}$ and $\overline{BC}$ are secant and tangents to the circle respectively. If $AD = 6\ cm$ and $BC = 2\sqrt{10}$ cm, find the length of $\overline{AB}$.

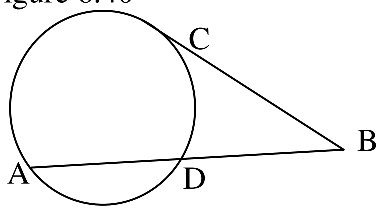


Figure 6.41

Assessment

Revise the main points of this section. Collect the home take assignment given questions 7 – 9 of Exercise 6.4. Correct and give comments on their work. Give them a chance to see their performance.

Answers to Exercise 6.4

1. $m(\angle P) = \frac{1}{2}[m(\overset{\frown}{BR}) - (\overset{\frown}{AQ})] = \frac{1}{2}(60^o - 30^o) = \frac{1}{2}(30^o) = 15^o$

2. $m(\angle CAP) \ = \ \frac{1}{2} m\ (\overset{\frown}{AC})$

   $m\ (\angle ABC) \ = \ \frac{1}{2} m(\overset{\frown}{AC})$

   $\therefore m\ (\angle CAP) \ = \ m(\angle ABC)$ and hence $\angle CAP \equiv \angle ABC$

3. Let A_s be the area of the square and let A_r be the area of the rectangle. $A_s = (AP)(AP) = (AP)^2$; and $A_r = (CP)(PD)$
   We know that $(AP)(PB) = (CP)(PD)$
   CD is the diameter and bisects chord AB. So, $AP = PB$
   So, $(AP)(PB) = (CP)(PD)$
   $(AP)(AP) = (CP)(PD)$, that is $AP^2 = (CP)(PD)$
   $\therefore A_s = A_r$
4. You can prove that two tangents intersect at a point equidistant from each tangent point. So $\overline{CB} \equiv \overline{CD}$. See solution of Group work 6.4 question 3.
   $\overline{ED}$ and $\overline{EF}$ are tangents to the circle. Hence, $\overline{EF} = \overline{DE}$.
   Therefore, $CB + EF = CD + DE = CE$.
5. $(DG)(GB) = (CG)(GA)$; $(PF)(PE)(PA)(PC)$
   $(3)(GB) = (4)(6)$; $(9)(PE) = (18)(8)$
   $GB = 8$ units ; $PE = PF + EF = 16$
   $EF = 16 - 9 = 7$ units
   $PT^2 = (PF)(PE) = (9)(16)$
   $PT = \sqrt{144} = 12$ units. So, the length of $\overline{PT}$ is 12 units.
6. $m(\angle BPC) = \frac{1}{2} m(\overset{\frown}{BC})$; $m(\angle BRC)) = \frac{1}{2} m[(\overset{\frown}{BC}) + (\overset{\frown}{PQ})]$

   $48^\circ = \frac{1}{2} m(\overset{\frown}{BC})$; $68^\circ = \frac{1}{2}(96^\circ) + \frac{1}{2} m(\overset{\frown}{PQ})$

   $96^\circ = m(\overset{\frown}{BC})$; $136^\circ = 96^\circ + m(\overset{\frown}{PQ})$

   $\Rightarrow m(\overset{\frown}{PQ}) = 40^\circ$

   $m(\angle BCR)) = \frac{1}{2} m(\overset{\frown}{BP})$

   $62^\circ = \frac{1}{2} m(\overset{\frown}{BP}) \Rightarrow m(\overset{\frown}{BP}) = 124^\circ$

   $m(\overset{\frown}{BC}) + m(\overset{\frown}{CQ}) + m(\overset{\frown}{QP}) + m(\overset{\frown}{PB}) = 360^\circ$

   $96^\circ + m(\overset{\frown}{CQ}) + 40^\circ + 124^\circ = 360^\circ$. So $m(\overset{\frown}{CQ}) = 100$

   $m(\angle CBA)) = \frac{1}{2}(m(\overset{\frown}{CQ}) + m(\overset{\frown}{QP}))$

   $m(\angle CBA)) = \frac{1}{2}(100^\circ + 40^\circ) = 70^\circ$

   $m(\angle BCA) = \frac{1}{2}(m(\overset{\frown}{PB}) + m(\overset{\frown}{QP})) = \frac{1}{2}(124^\circ + 40^\circ) = 82^\circ$

   $m(\angle BAC)) = \frac{1}{2}(96^\circ - 40^\circ) = 28^\circ$, or $m(\angle BAC) = 180^\circ - (70^\circ + 82^\circ) = 28^\circ$

7. As shown in **Figure 6.42**, $\overline{FC}$ and $\overline{DB}$ are chords of the circle intersecting at E.
So $(DE)(EB) = (CE)(EF)$
$(6)(6) = (10)(EF)$ (diagonals of a parallelogram bisect each other)

$EF = \frac{36}{10} = 3.6$

$AF = AE - EF = 10 - 3.6 = 6.4$ cm

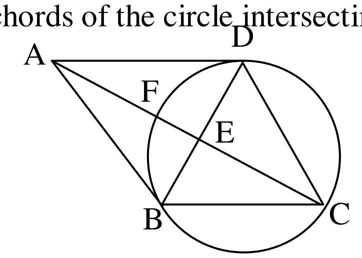


Figure 6.42

8. $(PC)(PD) = (PB)(PA)$
$8(18) = 6(AB + 6)$
$24 = AB + 6 \Rightarrow AB = 18$ cm
$PT^2 = (PB)(PA)$
$PT^2 = 6 \times 24 = 144 \Rightarrow PT = 12$ cm

9. Since $(XW)^2 = (XY)(XZ)$ and $XZ = 2XY$ as Y is the mid-point of $\overline{XZ}$
$(XW)^2 = 2(XY)(XY)$.
Thus, $WX = \sqrt{2}(XY)$

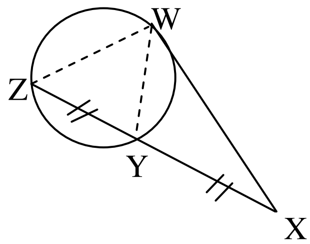


Figure 6.43

6.4 REGULAR POLYGONS

Periods allotted: 5 periods

Competencies

At the end of this sub-unit, students will be able to:

- *calculate the perimeters of regular polygons.*
- *calculate the areas of regular polygons.*

Vocabulary: Regular polygon, Inscribed polygon, Radius, Perimeter, Area of a regular polygon.

Introduction

In this sub-unit students are required to find perimeter and area of a regular polygon. They should be assisted to explain a circumscribed polygon about a circle and inscribed polygon in a circle to start the lesson.

Teaching Notes

You may start the lesson by guiding the students to describe what is meant by a polygon is inscribed in a circle and a polygon is circumscribed about a circle. For this purpose, discuss the questions in Activity 6.7. Then formulate the perimeter and area of a regular polygon with active participation of your students.

For an n sided regular polygon inscribed in a circle of radius r,

i. the length of a side s of a regular polygon is $s = 2r \sin \dfrac{180^\circ}{n}$

ii. the perimeter of the polygon is $P = ns$

iii. the area of the polygon is :

$$A = \frac{1}{2} nr^2 \sin \frac{360^\circ}{n}.$$

Give time for the students to practice using the examples and Activity 6.7.
This activity strengthens their background capability to work on Theorem 6.16. There are enough questions to give for both slow and fast learning students.

Answers to Activity 6.7

1. A regular polygon is a polygon whose angles are equal and whose sides are equal.
2. Polygons of any number of sides can be circumscribed about or inscribed in a circle. So, by construction, you can circumscribe quadrilateral, triangle and 7-sided polygon about a circle of radius 5 cm.
3. Using problem 2, a circle is circumscribed about a square. Use the point of intersection of the diagonals.
4. A circle is not actually inscribed unless each side of the polygon is tangent to the circle. A rectangle can be inscribed in a circle, but does not always circumscribe a circle. By construction, you can see that a circle can be tangent to three sides of the four sides of a rectangle if the rectangle is not a square.
5. A circle can always be circumscribed about a quadrilateral whose opposite angles have sum two right angles because two opposite angles are supplementary.
   In **Figure 6.44,**

$$m(\angle ABC) + m(\angle ADC) = \frac{1}{2} m(\overset{\frown}{ADC}) + \frac{1}{2} m(\overset{\frown}{ABC})$$
$$= \frac{1}{2}(360^\circ) = 180^\circ.$$

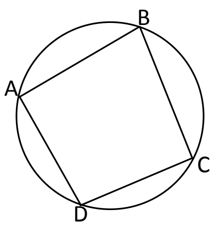


Figure 6.44

6. By problem 5 above and since the quadrilateral is a parallelogram, the quadrilateral is a rectangle.
7. $120^\circ, 72^\circ, 36^\circ, \dfrac{360^\circ}{n}$, respectively
8. $120^\circ, \dfrac{360^\circ}{7}, 36^\circ$ and $\dfrac{360^\circ}{n}$, respectively.
9.

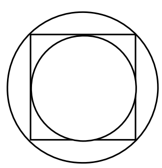


Figure 6.45

Perimeter of a Regular Polygon

Let the students state the formula for finding perimeter of a regular polygon. Make them try the examples given.

Area of a regular polygon

Help them to prove Theorem 6.16.

Activity 6.8 is presented here so that students can apply Theorem 6.16 immediately by themselves.

Answers to Activity 6.8

a. $\angle AOB$ has degree measure $\frac{360^{\circ}}{4} = 90^{\circ}$

b. $A = \frac{1}{2} nr^2 \sin\frac{360^{o}}{n} \Rightarrow A = \frac{1}{2}(4)r^2 \sin 90^{\circ} = 2r^2$

c. From b) we have $A = 2r^2$, given $r = 10$ cm
Hence, $A = 2\ (10 \text{ cm})^2 = 200 \text{ cm}^2$

Let the students do example 3 in the textbook to practice as a continuation of Activity 6.8. Now, select some questions from Exercise 6.5 to be done in class and some for a home work.

For fast learning students you can give questions of the following type:

1. Given a regular pentagon with radius 5 units, find
   a. the perimeter
   b. the area, of the polygon.
2. A square is inscribed in a circle of radius 6 cm. Find the area of the circle inscribed in the square.
3. If the perimeter of a regular hexagon is 10 cm, what is the radius of the circle circumscribing the polygon?
4. An interior angle of a regular polygon is 120°. If its radius in 5 units, find the area of the polygon.

Answers to Exercise 6.5

1. $$A = \frac{1}{2} nr^2 \sin\frac{360^{\circ}}{n}$$
$$= \frac{1}{2} \times 9 \times 5^2 \sin\frac{360^{\circ}}{9}$$
$$= \frac{225}{2} \sin 40^{o}$$
$$\approx 72.3 \text{ sq.units}$$

2. $$A = \frac{1}{2} nr^2 \sin\frac{360^{\circ}}{n}$$
$$= \frac{1}{2} \times 12 \times 3^2 \sin\frac{360^{\circ}}{12}$$
$$= 6 \times 9 \times \sin 30^{\circ}$$
$$= 6 \times 9 \times \frac{1}{2} = 27 \text{ sq.units}$$

3. $A = \frac{1}{2}\ nr^2 \sin\frac{360^o}{n}$

$= \frac{1}{2} \times 3 \times r^2 \sin\frac{360^0}{3}$

$= \frac{1}{2}\ \times 3 \times r^2 \sin 120^o$

$= \frac{3}{2}\ r^2\ \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}r^2}{4}$

a. $3\sqrt{3}\ \text{cm}^2$

b. $\frac{27}{4}\sqrt{3}\ \text{cm}^2$

c. $\frac{3}{2}\sqrt{3}\,\text{cm}^2$

d. $9\sqrt{3}\ \text{cm}^2$

4. $A = \frac{1}{2}\ nr^2 \sin\frac{360^o}{n}$

$= \frac{1}{2} \times 4 \times r^2 \sin\frac{360^o}{4}$

$= 2r^2 \sin 90^o = 2r^2$

a. 18 cm^2

b. 8 cm^2

c. 6 cm^2

d. 32 cm^2

5. In ΔOAB, in each case, $r^2 = \left(\frac{s}{2}\right)^2 + a^2$; in ΔOCD, $r^2 = a^2 + \left(\frac{s}{2}\right)^2$

$a^2 = r^2 - \frac{s^2}{4} \Rightarrow a = \sqrt{r^2 - \frac{s^2}{4}}$ as $a > 0$.

Hence, the distance of side $\overline{AB}$ from O is the same as the distance of $\overline{CD}$ from O.

6. $m\ (\angle\ AOP) = \frac{1}{2} m\ (\angle AOB) = \frac{1}{2}\left(\frac{360^o}{n}\right) = \frac{180^o}{n}$

$\cos\ (\angle AOP) = \frac{PO}{AO} \Rightarrow \cos(\frac{180^o}{n}) = \frac{a}{r}; a = r\cos\frac{180^o}{n}$

7. a. $a = 12 \cos\ \frac{180^o}{3}$

$= 12 \cos 60^o = 6$ cm

b. $a = 12 \cos\ \frac{180^o}{4}$

$= 12 \cos 45^o = 6\sqrt{2}$ cm

c. $a = 12 \cos\ \frac{180^o}{6}$

$= 12 \cos 30^o = 6\sqrt{3}$ cm

d. $a = 12 \cos\ \frac{180^o}{9}$

$= 12 \cos 20^o \approx 11.28$ cm

8. Consider **Figure 6.46**

$a(\Delta COD) = \frac{1}{2} \times a \times CD$

Similarly, $a(\Delta AOB) = \frac{1}{2} \times a \times AB$

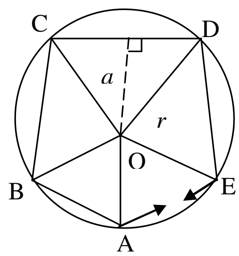


Figure 6.46

Hence, area of the polygon is

$$A = \frac{1}{2} \times a \times BC + \frac{1}{2} \times a \times CD + \ldots + \frac{1}{2} \times a \times AB$$

$$= \frac{1}{2} a (BC + CD + \ldots + AB)$$

$$= \frac{1}{2} aP, \textit{Where Perimeter}: P = (BC + CD + \ldots + AB)$$

a. $P = 6r\sin 60^\circ$ and $a = r\cos 60^\circ$

$= 3\sqrt{3}r$ and $a = \frac{1}{2}r$

$$\therefore A = \frac{1}{2}pa = \frac{1}{2}(3\sqrt{3}r) \times \frac{1}{2}r = \frac{3\sqrt{3}r^2}{4}$$

b. $P = 8r\sin 45^\circ$ and $a = r\cos 45^\circ$, $P = 4\sqrt{2}\,r$ and $a = \frac{\sqrt{2}}{2}r$

Area: $A = \frac{1}{2}pa = \frac{1}{2} \times 4\sqrt{2}r \times \frac{\sqrt{2}}{2}r = 2r^2$

c. $P = 12r\sin 30^\circ$ and $a = r\cos 30^\circ$. $P = 6\,r$ and $a = \frac{\sqrt{3}}{2}r$

Area: $A = \frac{1}{2}pa = \frac{1}{2}(6r \times \frac{\sqrt{3}}{2}r) = \frac{3\sqrt{3}r^2}{2}$

d. $P = 16r\sin 22.5^\circ$ and $a = r\cos 22.5^\circ$

Area: $A = \frac{1}{2}pa = \frac{1}{2} \times 16r \sin 22.5^\circ \times r\cos 22.5^\circ$

$= 8\,r^2 \sin 22.5^\circ \cos 22.5^\circ = 4r^2 \sin 45^\circ = 2\sqrt{2}r^2$

9. a. $A = \frac{1}{2}Pa$, where $a = r\cos\frac{180^\circ}{n}$

Thus, $A = \frac{1}{2}P\left(r\cos\frac{180^\circ}{n}\right) = \frac{1}{2}Pr\cos\frac{180^\circ}{n}$

b. $A_1 = \frac{1}{2}nr_1^2 \sin\frac{360^o}{n}$ and $A_2 = \frac{1}{2}nr_2^2 \sin\frac{360^o}{n}$

$$\frac{A_1}{A_2} = \frac{\frac{1}{2}nr_1^2 \sin\frac{360^o}{n}}{\frac{1}{2}nr_2^2 \sin\frac{360^o}{n}} = \frac{r_1^2}{r_2^2} = \left(\frac{r_1}{r_2}\right)^2$$

Thus, $\frac{A_1}{A_2} = \left(\frac{r_1}{r_2}\right)^2$

c. $A_1 = \frac{1}{2}a_1ns_1$ and $A_2 = \frac{1}{2}a_2ns_2$ as $P = ns$

$$= \frac{1}{2}\left(r_1\cos\frac{180^o}{n}\right) \times ns_1 \text{ and } A_2 = \frac{1}{2}\left(r_2\cos\frac{180^o}{n}\right) \times ns_2$$

$$\frac{A_1}{A_2} = \frac{\frac{1}{2}r_1ns_1\cos\frac{180^o}{n}}{\frac{1}{2}r_2ns_2\cos\frac{180^o}{n}} = \frac{r_1s_1}{r_2s_2}$$

Since $s_1 = 2r_1 \sin\frac{180^o}{n}$ and $s_2 = 2r_2 \sin\frac{180^o}{n}$ So, $\frac{s_1}{s_2} = \frac{r_1}{r_2}$

$$\frac{A_1}{A_2} = \frac{s_1}{s_2}.\frac{r_1}{r_2} = \frac{(s_1)^2}{(s_2)^2} = \left(\frac{s_1}{s_2}\right)^2$$

d. $\tan\frac{180^o}{n} = \frac{s_1}{2a}$ and $\tan\frac{180^o}{n} = \frac{s_2}{2a}$

$$a_1 = \frac{s_1}{2\tan\frac{180^o}{n}} \quad \text{and} \quad a_2 = \frac{s_2}{2\tan\frac{180^o}{n}}$$

$$\frac{A_1}{A_2} = \frac{a_1s_1}{a_2s_2} = \frac{\frac{s_1}{2\tan\frac{180^o}{n}} \times s_1}{\frac{s_2}{2\tan\frac{180^o}{n}} \times s_2} = \frac{(s_1)^2}{(s_2)^2}$$

Thus, $\frac{A_1}{A_2} = \left(\frac{s_1}{s_2}\right)^2$

10. Consider **Figure 6.47**

Area of a square = $s^2 = d^2$

$= (2r)^2 = 4r^2$

Area of a circle = πr^2

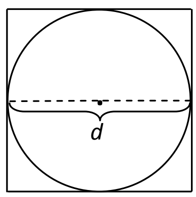


Figure 6.47

Area of the uncovered = $A_{square} - A_{circle}$

$= 4r^2 - \pi r^2 = r^2(4 - \pi)$

Now, $\frac{r^2(4 - \pi)}{4r^2} \times 100\% = \frac{4-\pi}{4} \times 100\% \approx 21.46\%$

Therefore, the percentage of the square which remains uncovered is 21.46%.

Make a short review using the summary of the unit and give some questions of the Review Exercise on unit 6 to be done in class. Select also some for an assignment.

Answers to Review Exercises on Unit 6

1. Given: A parallelogram $ABCD$ with E and F are mid points of sides $\overline{AB}$ and $\overline{AD}$ respectively.

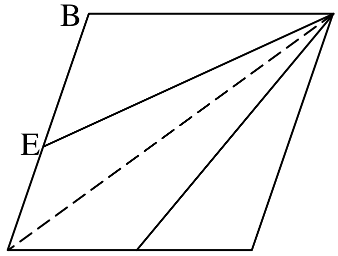


Figure 6.48

Proof:

Draw the diagonal $\overline{AC}$

$a\ (\Delta ABC) = a\ (\Delta AEC) + a\ (\Delta EBC) = 2a\ (\Delta AEC)$

$a(\Delta AEC) = a\ (\Delta EBC)$

Similarly, $a(\Delta ADC) = 2a\ (\Delta AFC)$

$\therefore\ a(ABCD) = a\ (\Delta ABC) + a\ (\Delta ADC)$

$= 2a\ (\Delta AEC) + 2a(AFC) = 2a\ (AECF)$

Therefore, $a\ (AECF) = \frac{1}{2}\ a\ (ABCD)$

2. Consider **Figure 6.49**

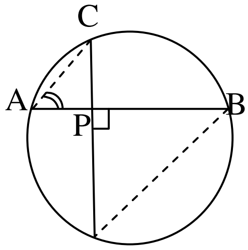


Figure 6.49

$m\ (\angle APC) = 90^{\circ}$

$m(\angle CAB) = \frac{1}{2} m(\widehat{CB})$

$35^{\circ} = \frac{1}{2} m(\widehat{CB})$

$m(\widehat{CB}) = 70^{\circ}$

$m(\angle APD) = \frac{1}{2} m(\widehat{CB} + \widehat{AD})$

$90^{\circ} = \frac{1}{2}(70^{\circ} + m(\widehat{AD}))$

$180^{\circ} = 70^{\circ} + m(\widehat{AD})$

$m(\widehat{AD}) = 110^{\circ}$

$m(\angle ABD) = \frac{1}{2} m((\widehat{AD})) = \frac{1}{2}(110^{\circ}) = 55^{\circ}$

3. $m\ (\angle P) = \frac{1}{2}\ (m\ (\widehat{ABC})\ ;\ m\ (\angle ABC) = \frac{1}{2}\ m\ (\widehat{APC}))$

$= \frac{1}{2}(130^{\circ})$ $\qquad = \frac{1}{2}(360^{\circ} - 130^{\circ})$

$\therefore\ y = 65^{\circ}$ $\qquad = \frac{1}{2}\left(230^{\circ}\right) = 115^{\circ}$

$\therefore\ x = 115^{\circ}$

4. $m(\angle A) = \frac{1}{2}\left[m(\overset{\frown}{XY}) - m(\overset{\frown}{EF})\right]$

   So, $10 = \frac{1}{2}\left[m(\overset{\frown}{XY}) - 15^\circ\right]$

   $m(\overset{\frown}{XY}) = 20^\circ + 15^\circ = 35^\circ$

   $m(\angle B) = \frac{1}{2}\left[m(\overset{\frown}{CD}) - m(\overset{\frown}{XY})\right]$

   $= \frac{1}{2}\left[95^\circ - 35^\circ\right] = 30^\circ$

5. Apply Theorem 6.12 and 6.14. Extend $\overline{PO}$ to meet the circle at Q as shown in **Figure 6.50.**

   We know that $(PB)(PA) = (PR)(PQ)$
   $= (PO - r)(2r + PO - r)$
   $= (PO - r)(r + PO)$
   $= PO^2 - r^2$

   $\therefore (PB)(PA) = PO^2 - r^2$

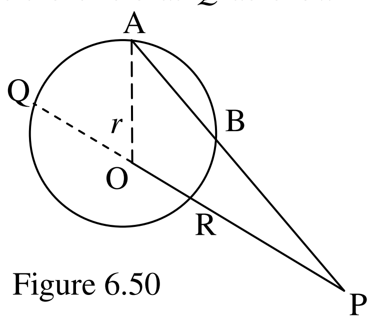


Figure 6.50

6. Consider **Figure 6.51**

   Draw $\overline{CB}$, $\overline{AD}$, $\overline{AT}$ and $\overline{TB}$

   $\Delta PBC \sim \Delta PDA$ (AA similarily theorem)

   $\frac{PB}{PD} = \frac{BC}{DA} = \frac{PC}{PA}$

   $\frac{PB}{PD} = \frac{PC}{PA}$

   $(PB)(PA) = (PD)(PC)$

   $\Delta ATP = \Delta TBP$ (AA Similarity theorem)

   $\frac{AT}{TB} = \frac{TP}{BP} = \frac{AP}{TP}$

   $\frac{TP}{BP} = \frac{AP}{TP} \Rightarrow (TP)(TP) = (PA)(PB)$

   $(TP)^2 = (PA)(PB)$.

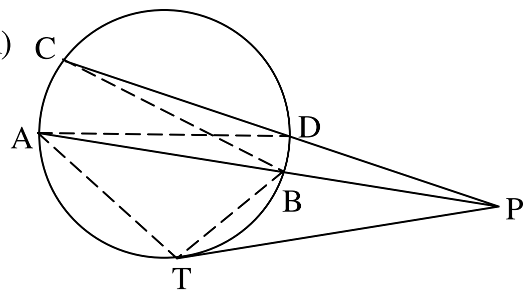


Figure 6.51

7. Consider **Figure 6.52**

   $x^2 + 4^2 = 6^2$

   $x^2 + 16 = 36$

   $x^2 = 20$

   $x = 2\sqrt{5}$

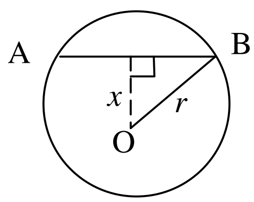


Figure 6.52

8. Since MN is diameter, it bisects the chord QR such that $QL = LR$.
   $(QL)(LR) = (ML)(LN)$
   $(QL)(QL) = (ML)(LN)$
   $(QL)^2 = (ML)(LN)$
9. $(CB)(CA) = (CD)(CE)$
   $(6)(8) = (CD)(13 + CD)$; Let $CD = x$
   $48 = x(13 + x)$
   $x^2 + 13x - 48 = 0$
   $(x - 3)(x + 16) = 0$
   $x = 3$ or $x = -16$. But distance cannot be negative. Hence $CD = 3$
10. Consider the following figure

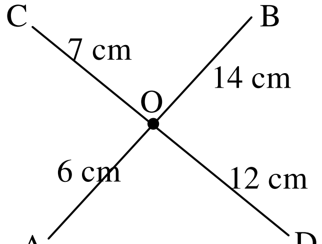


Figure 6.53

Now, since $AB = 20$ cm and $AO = 6$ cm, we have that $OB = 14$ cm and since $CD = 19$ cm and $CO = 7$ cm, we have that $OD = 12$ cm
Next, since $(AO)(OB) = (14)(6) = (12)(7) = (CO)(OD) = 84$, we can conclude that $ACBD$ is inscribed in a circle and hence, $ACBD$ is a cyclic quadrilateral.

11. Let **Figure 6.54** represent the given problem.

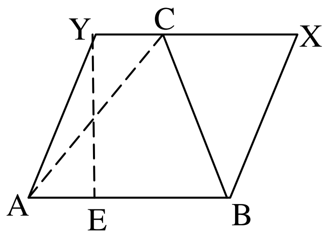


Figure 6.54

   a. $a(ABXY) = (EY)(AB)$
   $18 \text{ cm}^2 = (EY)(6 \text{ cm})$
   $EY = 3$ cm
   Therefore, the altitude of the parallelogram is 3 *cm*.

   So, $a(\Delta ABC) = \frac{1}{2}(AB)(EY) = \frac{1}{2} \times 6 \text{ cm} \times 3 \text{ cm} = 9 \text{ cm}^2$

   b. Consider ΔABY having equal area with ΔABC as they have equal base and altitude.
   $a(\Delta AYB) = \frac{1}{2}(AY)$ (length of altitude from B to $\overline{AY}$)
   $9 \text{ cm}^2 = \frac{1}{2}(4)$(length of altitude from B to $\overline{AY}$)
   4.5 cm =length of altitude from B to $\overset{\frown}{AY}$.
   Therefore, the distance from B to $\overline{AY}$ is 4.5 cm.

   c. $a(\Delta ABC) = \frac{1}{2}(BC)$ (length of altitude from A to $\overline{CB}$)
   $9 \text{ cm}^2 = \frac{1}{2}(5\text{cm})$ (length of altitude from A to $\overline{CB}$)
   3.6 cm = length of altitude from A to $\overline{CB}$.

Therefore, the distance from A to $\overline{CB}$ is 3.6 cm.

Assessment

At this time, you have identified the strengths and weaknesses of the students. Mark the assignment given. It is good if a test is given here to encourage them to study more. Some additional problems are given below as options to be used if necessary.

Additional problems

1. In **Figure 6.55** below $\overleftrightarrow{AC}$ is tangent to the circle at C and

   $m\,(\angle CAB) = m(\angle BCD)$. Show that $\overline{BC}$ is a diameter of the circle.

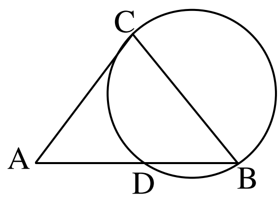


Figure 6.55

2. A chord of a circle 8 cm long is 3 cm far from the center of the circle. Another chord $\overline{AB}$ with A and B on the circle is 2 cm far from O. Find the area of ΔAOB.

Questions for fast learners

1. What kind of triangle has three angle bisectors that are also altitudes and medians?
2. Five ways to prove that a quadrilateral is a parallelogram
   a. Show that both pairs of opposite sides are parallel.
   b. Show that both pairs of opposite sides are congruent.
   c. Show that one pair of opposite sides are congruent and parallel.
   d. Show that both pairs of opposite angles are congruent.
   e. Show that the diagonals bisect each other.
3. A kite is a quadrilateral that has two pairs of congruent sides, its opposite sides are not congruent. Draw a convex kite. Discover, state and prove whatever you can about the diagonals and angles of a kite.
4. Find the perimeter and area of a regular decagon (12 sides) inscribed in a circle with radius 1.

UNIT 7 MEASUREMENT

INTRODUCTION

In this unit, the lessons the students learned in the lower grades about solid figures will be revised systematically. On the other hand, students should be made to use their knowledge and skill for calculating the volume and the surface area of prisms, cylinders, pyramids, cones and spheres dealt with so far by defining "**frustum of a pyramid**" and "**frustum of a cone**". Using the knowledge the students already gained on volume and surface areas of pyramids and cones, you should help them how to calculate the volume and surface areas of solid figures. To this end, you should give examples corresponding to each subsection to apply and use the basic concepts which the students are expected to know.

Unit Outcomes

At the end of this unit, students will be able to:

- *solve problems involving surface area and volume.*
- *know basic facts about frustums of cones and pyramids.*

Suggested Teaching Aids in Unit 7

So as to make the lesson interesting and easily understandable, you are advised to make the following materials ready and demonstrate while you are teaching:

- ✓ Objects which have prismatic or cylindrical shape such as boxes, their text books, cans, etc, and objects with shapes of pyramid, cone and sphere.
- ✓ Models of these shapes made of drawing paper.
- ✓ Look for teaching aids such as cones, prisms and pyramids from school store or mathematics club. Or, you can improvise these models together with students.

7.1 REVISION ON SURFACE AREAS AND VOLUMES OF PRISMS AND CYLINDERS

Periods allotted: 3 periods

Competency

After completing this sub-unit, students will be able to:

- *apply the formula for calculating surface area and volume of prism and cylinder.*

Vocabulary: Surface area, Volume, Prism, Cylinder, Lower base, Upper base, Lateral edge, Lateral face.

Introduction

In this sub-unit, a revision on surface area and volume of prisms and cylinders is discussed. For such surfaces, explanation is given to lateral surface, total surface, lower base, upper base, different types of prisms and cylinders such as triangular prism, right prism, parallelepiped and cube.

Teaching Notes

The students were introduced to solid figures such as prisms and cylinders in lower grades. Introduce some important terms such as right cylinder and right circular cylinder. The following is a description of parts of a cylinder.

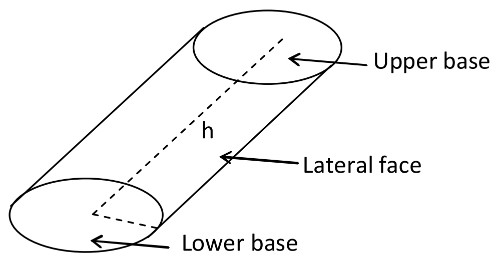


Figure 7.1

Definition 1.1 gives the most common types of prisms. Discuss this definition and continue to **Figure 7.5** to explain different parts of a prism. Strengthen your explanation by giving Activity 7.1.

The totality of lateral face, upper base and lower base gives the total surface of a cylinder.

The following is a review of the formulae for lateral surface area and volume of

a. **a prism**

   i. Lateral surface area, $A_L = ph$

   ii. Total surface area, $A_T = 2A_B + A_L$

   iii. Volume, $V = A_B h$

b. **a cylinder**

   i. Lateral surface area, $A_L = 2\pi rh$

   ii. Total surface area, $A_T = 2\pi r\,(r + h)$

   iii. Volume, $V = A_B h = \pi r^2 h,$

Show models when appropriate to help them understand more.

Alternating Teaching Approach

You can teach your students here by defining a prism first. You then continue to define a circular cylinder as a special type of prism, a prism with circular bases.

In order to identify common solid figures like prisms and cones, and their parts, let students practice on Activity 7.1 in class. Encourage fast learners to do question 5 by themselves.

Answers to Activity 7.1

1. Taking the lower base,

   a. it has 5 edges. These are $\overline{A'B'}$, $\overline{B'C'}$, $\overline{C'D'}$, $\overline{D'E'}$ and $\overline{A'E'}$.

   b. $AEE'A'$, $EDD'E'$, $BCC'B'$, $DCC'D'$ and $ABB'A'$ are lateral faces of the given prism.

2. a. prism b. Triangular right prism

   c. Cylinder d. Rectangular prism, parallelepiped

   e. prism (parallelepiped), rectangular

   f. prism (cube) g. prism (rectangular prism), parallelepiped

3. The lateral edges of a prism are equal and parallel.

4. a. Prism b. Upper base c. Lateral edges

   d. Lateral face e. PP' f. Parallelepiped

   g. Right prism

5. a. $A_B = \ell w$

   b. $A_L = 2\,\ell h + 2wh = (2\ell + 2wh)$

   $= Ph$, where P is the perimeter of the base.

   c. $A_T = 2A_B + A_L$

Discuss the formulae to find lateral surface and total surface area of a prism and a cone are derived. Then, continue to the volume of a prism and a cone. Practice with the help of the examples and select some problems from Exercise 7.1, say question 1- 8 for class exercises and the rest can be given as homework for further practice and understanding. You may select questions of the following type for fast learning students.

1. A right triangular prism has a lateral edge 12 cm long. If the base triangle has sides 4, 5 and 3 cm long, find
   a. The lateral surface area
   b. The volume, of the prism
2. A right rectangular prism has base 6 cm and 4 cm, and height 10 cm. At one corner of the prism, a piece with one cm by one cm, and length 10 cm is cut off. What is the lateral surface area of the remaining part?
3. A right circular cylinder whose diameter is 6 cm has total surface area 42 π square cm. What is the length of its altitude?

Assessment

After you complete this sub-topic, you can assess how much your students have understood the subject matter by either of the following methods:

- ✓ Asking oral questions
- ✓ Collect homework and marking their work.
- ✓ Giving home-take test (may be questions 9,11,12,13 of Exercise 7.1)

Answers to Exercise 7.1

1. a. $A_L = Ph = 2(2 + 3) \times 4 = 40$ sq. units.
   b. $A_T = 2A_B + A_L = 2(2 \times 3) + 40\text{cm}^2 = 52$ sq. units
   c. $V = A_Bh = (2 \times 3) \times 4 = 24$ cub. units
2. $A_L = Ph = (3 + 4 + 6 + 5 + 4) \times 5 = 110$ sq. units
3. $A_L = Ph = 20 \times 6 = 120$ sq. units
4. a. $A_L = Ph = (2\pi r) \times h = (2\pi \times 2) \times 10 = 40\pi\,\text{cm}^2$
   b. $A_L = Ph = (4 + 5 + 5 + 7) \times 10 = 210\text{ cm}^2$
5. $A_L = Ph$
   180 units $= P \times 4$ units
   $P = 45$ units
6. a. $A_T = 2A_B + A_L$

$$= \frac{2 \times a^2\sqrt{3}}{4} + Ph = \frac{2 \times 3^2\sqrt{3}}{4} + 3(3)\times 8 = \left(\frac{9}{2}\sqrt{3} + 72\right) \text{ sq cm.}$$

b. $V = A_B h$

$= \dfrac{3^2\sqrt{3}}{4} \times 8 = \dfrac{9\sqrt{3}}{4} \times 8 = 18\sqrt{3}$ cubic. cm

7. a. $A_T = 2A_B + Ph$

$= 2 \times 63 + 2(7 + 9) \times 3$

$= 222\ \text{cm}^2$

b. $V = (7 \times 9 \times 3)\ \text{cm}^3 = 189\ \text{cm}^3$

c. $\sqrt{139}\,\text{cm}$

8. a. $A_T = 2A_B + Ph$; $V = A_B h$

$= 2(\pi r^2) + (2\pi r)\,h$; $= \pi r^2 h$

$= 2(\pi \times 3^2) + 2\pi(3) \times 6$; $= \pi(3^2 \times 6) = 54\pi\ \text{cm}^3$

$= 54\pi\,\text{cm}^2$

b. $A_T = 2A_B + Ph$; $V = \ell \times w \times h$

$= 2(6 \times 2) + 2\,(6 + 2) \times 7$; $= 6 \times 2 \times 7 = 84\ \text{cm}^3$

$= 136\ \text{cm}^2$

c. $A_T = 2A_B + Ph$; $V = A_B h$

$= 2\left(\dfrac{1}{2} \times 8 \times 6\right) + (8 + 6 + 10) \times 5$; $= \left(\dfrac{1}{2} \times 8 \times 6\right)5$

$= 168\ \text{cm}^2$; $= 120\ \text{cm}^3$

9. 16 sq. units

10. a. $A_L = Ph$

$= (2\pi r)\,h$

$= (2\pi \times 2) \times 3$

$= 12\pi\,\text{cm}^2$

b. $A_T = 2A_B + A_L$

$= 2\pi r^2 + 2\pi rh$

$= 2\pi(2)^2 + 2\pi \times 2 \times 3$

$= 8\pi + 12\pi = 20\pi\,\text{cm}^2$

c. $V = A_B h$

$= \pi r^2 h = \pi(2)^2 \times 3 = 12\pi\,\text{cm}^3$

11. Let **Figure 7.2** represent the given problem.

$A_L = Ph$, where

P is perimeter of the base

and $P = 2\pi r$

Therefore, $A_L = Ph = (2\pi r)\,h = 2\pi rh$.

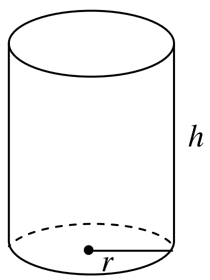


Figure 7.2

12. Let **Figure 7.3** represent the given problem.

a. $A_T = 2A_B + A_L$

$= 2\pi r^2 + 2\pi rh$

$= 2\pi r^2 + 2\pi r(2r)$

$= 2\pi r^2 + 4\pi r^2 = 6\pi r^2$ sq. unit

b. $V = A_B\ h$

$= \pi r^2 h = \pi r^2\ (2r) = 2\pi r^3$ unit^3

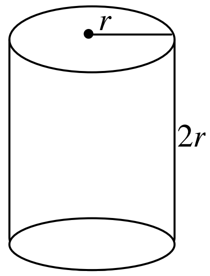


Figure 7.3

13. Let **Figure 7.4** represent the given problem.

$A_T = 2\pi Rh + 2\pi rh + 2(\pi R^2 - \pi r^2)$

$= 2\pi \times 6 \times 8 + 2\pi \times 5 \times 8 + 2\pi\,(6^2 - 5^2)$

$= 96\pi + 80\pi + 22\pi$

$= 198\pi\,\text{cm}^2$

Volume of the resulting solid is:

V_R = Volume of larger cylinder – volume of smaller cylinder

$= \pi R^2 h - \pi\, r^2 h$

$= \pi\,(6)^2 \times 8 - \pi\,(5)^2 \times 8$

$= 288\pi - 200\pi = 88\pi\,\text{cm}^3$

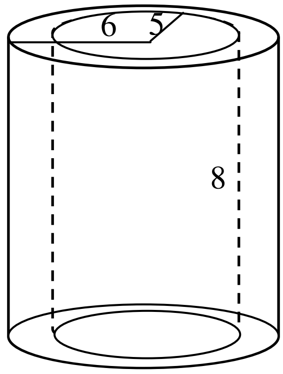
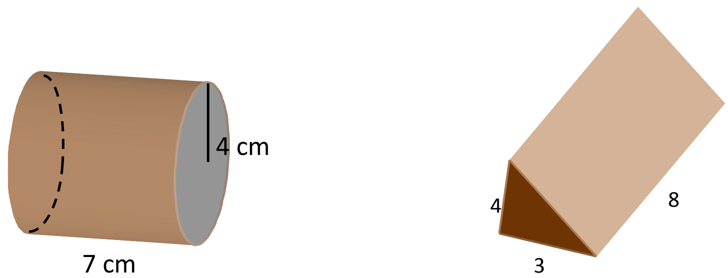


Figure 7.4

Additional exercises for fast learners

1. Find the total surface area and the volume of the following solid figures.



a b

Figure 7.5

2. The most common shape for a cylindrical container is one in which the height and the diameter are equal. Find its

 a. total surface area b. volume

3. Five mitotic cubes whose base edges are 1 cm, 2 cm, 2 cm, 3 cm, and 4 cm are melted down and formed into a single right rectangular prism whose bases have dimensions 3 cm by 4 cm. Find the surface area and volume of the new rectangular prism.

4. A block of lead 1.5 m × 2 m × 6 m is hammered out to make a square sheet of thickness 20 m. Find the dimensions of the square and its surface area.

7.2 PYRAMIDS, CONES AND SPHERES

Periods Allotted: 8 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *calculate the surface area of a given pyramid or a cone.*
- *calculate the volume of a given pyramid or a cone.*
- *calculate the surface area of a given sphere.*
- *calculate the volume of a given sphere.*

Vocabulary: Pyramid, Cone, Slant height, Altitude (height), Sphere.

Introduction

In this sub-unit, pyramids, cones and spheres are presented. In the pyramid presentation, different pyramids are indicated on the basis of their base (triangular, quadrilateral, pentagonal, hexagonal, etc,). The presentation also indicates how the pyramids are erected (inclined or axis perpendicular to the base). The base could be regular or non regular. There is also a similar presentation of cones. At least an introduction is given to spheres.

Teaching Notes

The main focus of this sub-unit is computation of lateral surface area, total surface area and volume. After you give formal definitions of pyramid, cone and sphere as stated in definitions 7.2, 7.3 and 7.4, discuss how to obtain the formulae for their surface areas and volumes.

1. Pyramid

   Surface area: Lateral $A_L = \frac{1}{2}P\ell$

   Total $A_T = A_B + A_L$

   Volume: $V = \frac{1}{3}A_B h$

2. Cone

   Surface area:

   Lateral $A_L = \frac{1}{2}P\ell = \pi r\ell,\ \ell = \sqrt{h^2 + r^2}$

   Total $A_T = A_B + A_L = \pi r\,(r + \ell)$

   Volume: $V = \frac{1}{3}A_B h = \frac{1}{3}\pi r^2 h$

3. Sphere

   Surface area $A = 4\pi r^2$

Volume $V = \frac{4}{3}\pi r^3$

In each part, Activities are given to be done by the students in class. Ask students on individual basis to give answers and assist them as needed. Practice and apply the formulae using examples 1 – 6.

Now, after defining what a pyramid is, let the students have clear understanding that a pyramid is named in relation to its base. To further characterize pyramids and cones, discuss Activity 7.2 in class.

Answers to Activity 7.2

1. A regular pyramid is a pyramid whose base is a regular polygon and whose altitude pass through the center of the base.
2. If the base of the pyramid is a triangular region, it is called a tetrahedron.
3. a. True b. True c. True
4. a. Pyramid d. $\overline{VP}$
   b. Lateral face e. Lateral edges
   c. Base of the pyramid f. Hexagonal pyramid
5. **Figure 7.5** represents a cone and its parts.

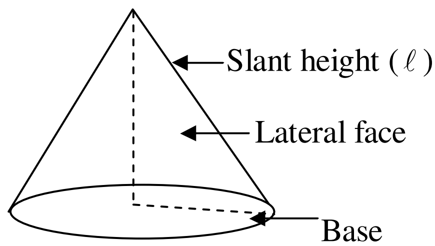


Figure 7.6

Based on Activity 7.2, give the definition of a cone and let the students describe its parts. For more exercise do Activity 7.3 in class. This is important to understand surface area and volume of a pyramid and a cone.

Answers to Activity 7.3

1. Consider **Figure 7.7**

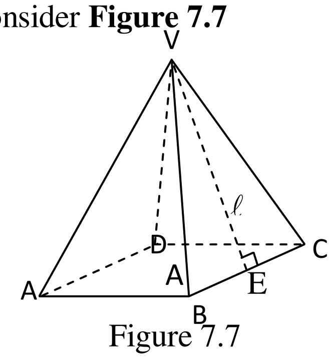


Figure 7.7

a. Four lateral faces.

b. $A_L = \frac{1}{2}(BC)\ell = \frac{1}{2}(6 \text{ cm} \times 5 \text{ cm}) = 15 \text{ cm}^2$

Since the given figure is a regular square pyramid,

c. $A_{L1} = A_{L2} = A_{L3} = A_{L4} = 15\text{cm}^2$

There are 4 isosceles triangles

So, $A_L = 4\left(\frac{1}{2}BC \times VE\right) = 4\left(\frac{1}{2} \times 6\text{ cm} \times 5\text{ cm}\right) = 60\text{ cm}^2$

2. $A_T = A_B + A_L = 6\text{ cm} \times 6\text{ cm} + 60\text{ cm}^2 = 96\text{ cm}^2$ $A_T = A_B + A_L$, where A_B is area of base and A_L is the lateral surface area.

Practical Work

After defining the terms pyramid, cone and sphere, ask them to compare and contrast

- ✓ Prism and pyramid
- ✓ Cylinder and cone
- ✓ Circle and sphere

Furthermore , you can carry out the following investigation:

Required Materials :

1. Models of prism and pyramid with the same height and base
2. Models of cylinder and cone with the same height and base
3. Sugar or sand or powder

Task: Compare the volume (capacity) of the prism and the pyramid, the cylinder and the cone.

Expected Generalization (outcome):

The volume of a prism (cylinder) is three-times the volume of a pyramid (cone)

provided that they have equal heights and congruent bases.

Assist the students to state and prove the surface area and volume of a cone and pyramid. Let them do examples given in their text book.

Next, describe a sphere and discuss how to find the surface area and volume of a sphere by using Example 6.

You can give questions of the following type for fast learning students.

1. A regular pyramid has a square base of side 5 cm long. If its lateral surface area is 110 cm^2,
   a. What is the total surface area of the pyramid?
   b. Find the length of its slanted height.
2. A right circular cone has radius r and altitude h. What must be the length of the altitude of a right circular cylinder if it has the same base as the cone?
3. A cylindrical container of base radius 8 cm has enough water in it. An iron ball of radius 3 cm is inserted in the cylinder. Assuming that the ball is completely immersed, how high does the water level rise?

Assessment

Select some questions from Exercise 7.2 for class work and encourage some students to do questions on the board. Give some questions of Exercise 7.2 as home work. if time allows, you can use questions 10, 11 and 12 as group work.

Answers to Exercise 7.2

1. a. $V = \frac{1}{3}A_B h$

$= \frac{1}{3}\left(\frac{1}{2} \times 3 \times 5\right) \times 6$

$= 15$ cubic.units

b. $V = \frac{1}{3}A_B h$

$= \frac{1}{3}(10 \times 10) \times \sqrt{350}$

$= \frac{500}{3}\sqrt{14}$ cubic.units

c. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(9^2\right) \times 12 = 324\pi$ cubic units

2. a. $A_T = \frac{1}{2}p\ell + A_B$

$= \frac{1}{2}(4 \times 6) \times \sqrt{55} + 6 \times 6$

$= \left(12\sqrt{55} + 36\right)\text{cm}^2$

b. $V = \frac{1}{3}A_B h = \frac{1}{3}(36) \times \sqrt{46}$

$= 12\sqrt{46}\ \text{cm}^3$

3. In a regular polygon with n sides, you recall that:

$$s = 2r\sin\left(\frac{180^\circ}{n}\right)$$

$$6\text{cm} = 2r\sin\left(\frac{180^\circ}{3}\right)$$

$$\Rightarrow 6 = 2r\sin 60^\circ$$

$$\Rightarrow 3 = \frac{\sqrt{3}}{2}r$$

$$\Rightarrow \frac{6}{\sqrt{3}} = r \Rightarrow r = 2\sqrt{3}$$

a. let e be the length of an edge.

$e^2 = r^2 + 6^2 = 12 + 36 = 48$

$$\ell^2 = e^2 - \left(\frac{6}{2}\right)^2 = 48 - 9 = 39$$

$$\Rightarrow \ell = \sqrt{39}$$

$$A_L = \frac{1}{2}P\ell$$

$$= \frac{1}{2}(3 \times 6)\sqrt{39} = 9\sqrt{39}$$

$$A_T = \frac{1}{2}P\ell + A_B$$

$$= \frac{1}{2}(3\times 6)\sqrt{39} + \frac{\sqrt{3}}{4}(36)$$

$$= 9\sqrt{39} + 9\sqrt{3}$$

b. $V = \frac{1}{3}A_B h$

$$= \frac{1}{3}\left(\frac{\sqrt{3}}{4}(36)\right)6 = 18\sqrt{3}\,\text{cm}^2$$

4. a. $A_T = \frac{1}{2}p\ell + A_B$

$$= \frac{1}{2}(4\times 7)\frac{7\sqrt{3}}{2} + 7\times 7$$

$$= 7\sqrt{147} + 49$$

$$= 7\left(\sqrt{147} + 7\right)\text{cm}^2$$

$$= 49\ (1 + \sqrt{3}\ )\ \text{cm}^2$$

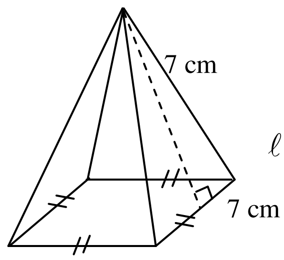


Figure 7.8

$$\ell = \frac{\sqrt{147}}{2} = \frac{7\sqrt{3}}{2}$$

b. $V = \frac{1}{3}A_B h$

$$= \frac{1}{3}(7\times 7)\times\frac{7\sqrt{2}}{2} = \frac{1}{6}\left(343\sqrt{2}\right) = \frac{343}{6}\sqrt{2}\,\text{cm}^3$$

5. **Figure 7.9** represents the given problem

$\ell^2 = r^2 + h^2$

$\ell^2 = 144 + 25 = 169 \Rightarrow \ell = 13$

a. $A_T = A_B + A_L = \pi r^2 + \pi r\ell$

$$= \pi(25) + \pi(5)(13) = 25\pi + 65\pi = 90\pi\ \text{cm}^2.$$

b. $V = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi(25)12 = 100\pi\ \text{cm}^3$$

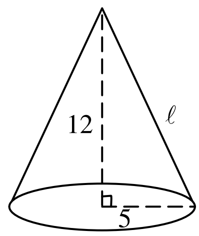


Figure 7.9

6. **Figure 7.10** represent the given problem.

$V = \frac{1}{3}A_B\,h$

$240\,\text{cm}^3 = \frac{1}{3}(6\,\text{cm}\times 4\,\text{cm})\times h$

$240\ \text{cm}^3 = 8\ \text{cm}^2 \times h$

$\therefore\ h = 30$ cm is the altitude.

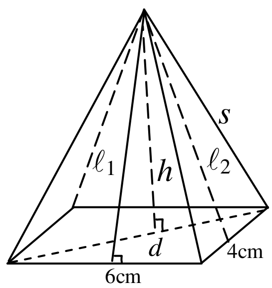


Figure 7.10

$$d^2 = (6\text{ cm})^2 + (4\text{ cm})^2 = 36 + 16 = 52\text{ cm}^2$$

$$d = \sqrt{52} = 2\sqrt{13}\text{ cm}$$

$$h^2 + \left(\frac{d}{2}\right)^2 = s^2 \quad ; \quad \ell_1^{\,2} + 3^2 = s^2 \quad ; \quad \text{and } \ell_2^{\,2} + 2^2 = s^2$$

$$\sqrt{913} = s$$

$$30^2 + \left(\sqrt{13}\right)^2 = s^2 ; \quad \ell_1^{\,2} + 9 = 913 \quad , \ell_2^{\,2} + 4 = 913$$

$$\ell_1 = \sqrt{904} \quad , \quad \ell_2 = \sqrt{909}$$

$$A_L = 2\left(\frac{1}{2} \times 6 \times \ell_1\right) + 2\left(\frac{1}{2} \times 4 \times \ell_2\right)$$

$$= 2\left(\frac{1}{2} \times 6 \times \sqrt{904}\right) + 2\left(\frac{1}{2} \times 4 \times \sqrt{909}\right)$$

$$= 6\sqrt{904} + 4\sqrt{909} \approx 301\text{ cm}^2 \text{ is the lateral sarface area.}$$

7. $$\ell^2 + \left(\frac{s}{2}\right)^2 = \mathrm{s}^2 \quad ; \quad h^2 + \left(\frac{s}{2}\right)^2 = \ell^2$$

$$\ell^2 + \frac{\mathrm{s}^2}{4} = \mathrm{s}^2 \quad ; \quad h^2 + \frac{s^2}{4} = \left(\frac{s\sqrt{3}}{2}\right)^2$$

$$\ell^2 = \frac{3}{4}s^2 \quad ; \quad h^2 + \frac{s^2}{4} = \frac{3}{4}s^2$$

$$\ell = \frac{\mathrm{s}\sqrt{3}}{2} \qquad h^2 = \frac{1}{2}s^2$$

$$h = \frac{s}{\sqrt{2}} = \frac{s\sqrt{2}}{2}$$

$$V = \frac{1}{3}A_B\, h = \frac{1}{3}(s \times s) \times \frac{s\sqrt{2}}{2} = \frac{s^3\sqrt{2}}{6}$$

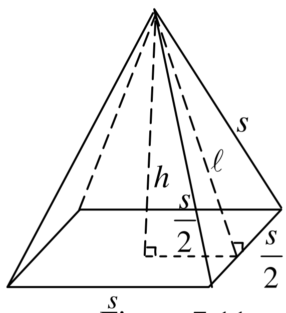


Figure 7.11

8. A regular tetrahedron is a regular triangular pyramid whose lateral edge and base edge are equal.

$$s = 2r\sin\left(\frac{180^o}{n}\right)$$

$$8 = 2r \sin 60^\circ$$

$$8 = 2r \times \frac{\sqrt{3}}{2}$$

$$r = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3}$$

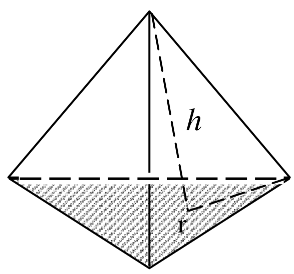


Figure 7.12

Now, $h^2 + r^2 = 8^2$

$$h^2 + \left(\frac{8\sqrt{3}}{3}\right)^2 = 64$$

$$h^2 + \frac{64}{3} = 64 \Rightarrow h = \sqrt{\frac{128}{3}} = 8\sqrt{\frac{2}{3}}\text{ cm}$$

9. The altitude h of the cone is given by:

$$r^2 = \sqrt{\ell^2 - h^2} = \sqrt{13^2 - 12^2} = \sqrt{25} = 5\text{ cm}$$

$$V = \frac{1}{3}A_B h = \frac{1}{3}(\pi r^2)\ h = \frac{1}{3}\pi\ 5^2 \times 12 = 100\pi\text{ cm}^3$$

10. $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(10)^3 = \frac{4000}{3}\pi\text{ cm}^3,$

Surface area of the sphere $= 4\pi r^2 = 4\pi(10)^2 = 400\pi\text{cm}^2$

11. Since $d = 2r$, $r = 10$ cm

$V = 0.1\ \ell t = 100\text{ cm}^3$

$$V = \frac{1}{3}\pi r^2 h$$

$$100 = \frac{1}{3}\pi \times 10^2 \times h \Rightarrow \frac{3}{\pi} = h \Rightarrow h = \frac{21}{22}\text{cm}$$

12. $r = \sqrt[3]{72} = 2\sqrt[3]{9}$ cm

7.3 FRUSTUMS OF PYRAMIDS AND CONES

Periods Allotted: 7 Periods

Competencies

At the end of the sub-unit, students will be able to:

- *define frustums of a pyramid and of a cone.*
- *calculate the surface areas of frustums of pyramids or of cones.*
- *calculate the volumes of frustum of pyramids or of cones.*

Vocabulary: Cross-section, Horizontal cross section, Frustum, Height (Altitude) of a frustum.

Introduction

This sub-unit is a continuation of sub-unit 7.2, where pyramids or cones are cut by a plane parallel to the base. The solid between the base and this plane is our interest called **frustum**. The main focus here is investigating lateral surface area, total surface area and volume of the frustum. Students will be exposed to applying the concepts in searching surface area and volume of objects of the shape of frustums.

Teaching Notes

In the preceding sub-unit, the students have reviewed pyramids and cones. This sub-unit is devoted to the new notion frustum of pyramids and cones. After a brief discussion of Definition 7.5 and Theorems 7.1 and 7.2, Example 1 could be used to apply the results. Then, the notion of frustum of a pyramid should be discussed. Explanations of frustums of pyramids and cones should be carried out with the help of Activity 7.4 and Activity 7.5. Here the students should observe that:

1. The lateral faces of a frustum of a pyramid are trapeziums.
2. The lateral faces of a frustum of a regular pyramid are congruent isosceles trapeziums.

The theorem on lateral surface area of frustum of a regular pyramid (Theorem 7.2) is given without proof, but should be attempted by students. It is given as exercise in question 9 of Exercise 7.3. The application of the theorem could be demonstrated by using Examples 2, 3 and 6. The formula for the volume of a frustum of a pyramid has to be observed as the difference between the original pyramid and the volume of the pyramid that has been cut off to form the frustum. In short, the lateral surface area (A_L) of a frustum of a pyramid (regular pyramid) is:

$A_L = \frac{1}{2}\ell(P + P')$ where P and P′ are perimeters of the lower and upper bases respectively.

The Volume (V_f) of frustum of a pyramid is:

$V_f = \frac{1}{3}h'\left(A + A' + \sqrt{AA'}\right)$ where A and A' are lower and upper base areas respectively and h' is the height of the frustum.

The notion frustum of a cone could be introduced in the same way as frustum of a pyramid was introduced. The formula for lateral surface area and volume of a frustum and a cone are also adapted from the formula for surface area and volume of frustum of a pyramid.

To help them understand frustum of a cone, the following is the lateral surface area (A_L) and volume (V_f):

$$A_L = \frac{1}{2}\ell(c + c') = \frac{1}{2}\ell(2\pi r + 2\pi r')$$

$$\therefore \quad A_L = \ell\pi(r + r')$$

The total surface area is then $A_T = A_b + A_c + A_L = \pi r^2 + \pi(r')^2 + \ell\pi(r + r')$

Volume of this frustum is: $V_f = \frac{1}{3}\pi h'(r^2 + (r')^2 + rr')$,

where r is the radius of the lower (larger) base and r is the radius of the upper (smaller) base and h' is the height of the frustum of the cone.

There are many physical objects in a form of a frustum of a cone. As a model, a bucket can be used. Order some students to bring a bucket and measure its surface area and volume. One can also roll a paper in a form of a cone. Then, cut it by a plane parallel to a base to form a frustum. Find the lateral surface area of the frustum.

Since the concepts in this sub-unit are not easy, you should assist and follow the students. Give the definition of a horizontal cross-section of a cone and a pyramid. Guide them to conclude that the ratio of the sides are equal for a frustum formed as a horizontal cross- section. Let the students do Activity 7.4 that will help them to understand Theorem 7.1

Answers to Activity 7.4

Note that the plane containing $\Delta\, DEF$ is parallel to the base $\Delta\, ABC$. Thus,

$\Delta\, VEF \sim \Delta\, VBC$ (by AA similarity theorem)

So, $\frac{VE}{VB} = \frac{EF}{BC} = \frac{VF}{VC}$

a. $\frac{VE}{VB} = \frac{VF}{VC}$ (But, $VB = VE + EB$)

$$\frac{VE}{VE+EB} = \frac{VF}{VC}$$

$$\frac{VE}{VE+3VE} = \frac{VF}{VC}\ (As\ EB = 3VE)$$

$$\frac{VE}{4VE} = \frac{VF}{VC}$$

$$\frac{1}{4} = \frac{VF}{VC}$$

b. Since $\frac{EF}{BC} = \frac{VF}{VC}$,

and $\frac{VF}{VC} = \frac{1}{4}$,

$$\frac{EF}{BC} = \frac{1}{4}$$

c. $\frac{a(\Delta VEF)}{a(\Delta VBC)} = \left(\frac{VF}{VC}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

Therefore, $a\,(\Delta VEF) = \frac{1}{16}\, a\,(\Delta VBC)$

Similarly, $a\,(\Delta\, DEF) = \frac{1}{16}\, a\,(\Delta\, ABC)$

Discuss Theorem 7.1 and let the students do Example 1 to apply the theorem. Next let them work on Activity 7.5 so that students can try the questions for next study on frustum of a pyramid or a cone.

You may use the following question for fast learners.

A cone of altitude 10 cm has base area 48 cm^2. If the radius of a horizontal cross-section is $\sqrt{3}$ cm, what is the altitude of the frustum?

Answers to Activity 7.5

1. Let A_c be the area of the horizontal cross-section. Thus, $A_c = \frac{144}{25}$ sq. units.
2. Let A' and A be the area of the horizontal cross-section parallel to the base and the area of the base respectively.

$\frac{A'}{A}=\frac{k^2}{h^2}$, where k is the distance of the cross-section from the vertex and h is the altitude of the pyramid.

$\frac{A'}{64}=\frac{2^2}{8^2} \Rightarrow A' = \frac{64\times 4}{64} = 4 \text{ cm}^2$

3. Let A' be the area of the horizontal cross-section.
A be the area of the base
$A' = \pi r^2 = \pi(2\text{cm})^2 = 4\pi \text{ cm}^2$
$A = \pi R^2 = \pi(3\text{cm})^2 = 9\pi \text{ cm}^2$

$$\frac{A'}{A} = \frac{k^2}{h^2}$$

$$\frac{4\pi}{9\pi}=\frac{k^2}{(k+5)^2} \Rightarrow \frac{k}{k+5}=\frac{2}{3}$$

$$3k = 2k + 10 \Rightarrow k = 10 \text{ cm}$$

Hence, the altitude of the cone is $h = 10 + 5 = 15$ cm

Now discuss what frustum of pyramids and cones are. Let students state their parts and explain how to obtain the formula to find the lateral surface area of a frustum of pyramid. Arrange the groups and let them discuss on Group work 7.1.

Group Work 7.1 will help them to understand Theorem 7.3.

Answers to Group Work 7.1

1. $A_b = \pi r_1^2$ where A_b is area of the lower base $A_c = \pi r_2^2$, where A_c is area of the upper base.
2. $c_1 = 2\pi r_1$ and $c_2 = 2\pi r_2$
where c_1 is the circumference of the base and
c_2 is the circumference of the cross-section
3. The lateral surface area of the bigger cone is
$A_L = \pi r_1 \ell_1 = \pi r_1\sqrt{r_1^2+k_1^2}$, where ℓ is the slant height of the bigger cone
4. The lateral surface area of the smaller cone is
$A_L = \pi r_2 \ell' = \pi r_2\sqrt{r_2^2+k_2^2}$ where ℓ' is the slant height of the smaller cone.
5. The lateral surface area of the frustum =
the lateral surface area of the bigger cone – the lateral surface area of the smaller cone
$$A_L = \pi r_1\ell - \pi r_2\ell' = \pi r_1\sqrt{r_1^2+k_1^2} - \pi r_2\sqrt{r_2^2+k_2^2}$$
$$= \pi\left(r_1\sqrt{r_1^2+k_1^2} - r_2\sqrt{r_2^2+k_2^2}\right)$$
6. $V_f = \frac{1}{3}\pi(k_1 r_1^2 - k_2 r_2^2)$

Next, let the students do Example 4 and then state Theorem 7.3. Introducing required representation, Give the volume of a frustum of a cone and a pyramid. Let students practice using the examples presented in the student textbook.
Now select some of the questions from Exercise 7.3 to be done in class and some other can be assigned as assignment. For fast learning students you can give questions of the following type.

1. A right circular cone with altitude h and radius r is cut at a height of $\frac{1}{3}$ from the base to form a frustum. What proportion is the volume of this frustum to the volume of the cone?
2. The altitude of a pyramid with base area of 45cm^2 is 10cm. If a frustum of this pyramid having the same lower base has an upper base with area of 20cm^2, then find the volume of the frustum.
3. A cup is in a form of a frustum of a right circular cone has base radii 2 cm and 5 cm. If its lateral surface area is 56πcm^2, find the altitude of the cup.

Answers to Exercise 7.3

1. a. $A_L = \frac{1}{2}\,\ell(P+P') = \frac{8}{2}\,(4(6+3)) = 144 \text{ cm}^2$

   b. $A_T = A_L + A_{bases} = 144 \text{ cm}^2 + 36 \text{ cm}^2 + 9 \text{ cm}^2 = 189 \text{ cm}^2$

2. Let **Figure 7.13** represent the given problem.
   Altitude of the smaller cone is

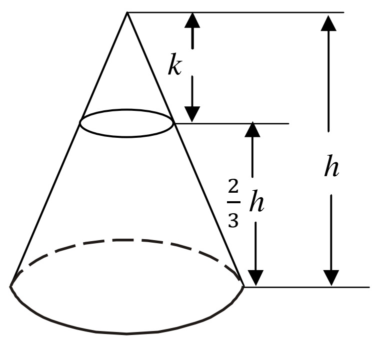


Figure 7.13

$$k = h - \frac{2}{3}h = \frac{1}{3}h \Rightarrow \frac{A'}{A} = \frac{k^2}{h^2}$$

$$\frac{A'}{A} = \left(\frac{\frac{1}{3}h}{h}\right)^2 \quad \frac{A'}{A} = \frac{1}{9}$$

$$\Rightarrow A' = \frac{1}{9}A$$

$$V_{frustum} = V_{largercone} - V_{smaller\ cone}$$

$$= \frac{1}{3}A_B h - \frac{1}{3}A_B' k$$

$$= \frac{1}{3}\pi r^2 h - \frac{1}{3}\times\frac{1}{9}\pi r^2\left(\frac{1}{3}h\right) = \frac{1}{3}\pi r^2 h - \frac{1}{81}\pi r^2 h = \frac{26}{81}\pi r^2 h$$

or from similarity of triangles, $\frac{r'}{r} = \frac{\frac{1}{3}h}{h} \Rightarrow r' = \frac{1}{3}r$

So, $V_f = \frac{1}{3}\pi h'(r^2 + (r')^2 + r.r'),\ h' = \frac{2}{3}h$

$$= \frac{1}{3}\pi.\frac{2}{3}h\left(r^2 + \frac{1}{9}r^2 + \frac{1}{3}r^2\right) = \frac{26}{81}\pi r^2 h$$

3. $A = 49$, $A' = 25$ and $h' = 3$.

So, $V = \frac{1}{3}\times 3\times\left(25+49+\sqrt{25\times 49}\right) = 109\ \text{cm}^3$

4. a. $A_L = \frac{\ell}{2}(c+c')$

$= \frac{10}{2}(2\pi(6)+2\pi(3)) = 5(12\pi+6\pi) = 90\pi\,\text{cm}^2$

b. $A_T = A_L + \pi r^2 + \pi R^2 = 90\pi\,\text{cm}^2 + \pi(36)\,\text{cm}^2 + \pi(9)\,\text{cm}^2 = 135\pi\,\text{cm}^2$

c. $V_f = \frac{\pi h'}{3}(r^2+(r')^2+r'r) = \frac{\pi}{3}.\sqrt{91}\ (36+9+18) = 21\sqrt{91}\,\pi\,\text{cm}^3$

5. $V_f = \frac{5}{3}\left(250-100\sqrt{2}+\sqrt{15,000-10,000\sqrt{2}}\right)cm^3$

Hint: $A' = A\frac{k^2}{h^2}$, $h = 5\sqrt{2}$ and $V_f = \frac{h}{3}\left(A+A'+\sqrt{AA'}\right)$, See question 11.

6. $\frac{A'}{A} = \frac{k^2}{h^2} \Rightarrow \frac{A'}{36} = \frac{25}{100} \Rightarrow A' = 9\text{cm}^2$

$V = \frac{1}{3}\ h'\left(A+A'+\sqrt{AA'}\right) = \frac{5}{3}(9+36+18) = 105\ \text{cm}^3$

7. a. $V_f = \frac{\pi}{3}h'(r^2+(r')^2+rr')$

$6000\ \text{cm}^3 = \frac{\pi}{3}h'(144+400+240)$

$\Rightarrow 18,000 = h'\pi(784) \approx \frac{22}{7}(784)h' = 2,464h'$

$\Rightarrow h' \approx \frac{18,000}{2,464} \approx 7.3\text{cm}$

b. $\ell^2 = 8^2 + (7.3)^2$

$= 64 + 53.29 \approx 117.29$

$\Rightarrow \ell =$ slant height ≈ 10.8 cm.

8. $\ell_1^2 = 16^2 + 8^2$

$\Rightarrow \ell_1 = \sqrt{320} = 8\sqrt{5}$, slant height of the cone

Then, $\frac{16}{12} = \frac{8\sqrt{5}}{\ell_2} \Rightarrow \ell_2 = \frac{96\sqrt{5}}{16} = 6\sqrt{5}$, slant height of the frustum and

$\frac{16}{4} = \frac{8}{r'} \Rightarrow r' = 2$

a . $A_L = \frac{1}{2}\ell_2(c+c') = \ell_2\pi(r+r') = 6\sqrt{5}\pi(2+8)$

$\Rightarrow A_L = 60\sqrt{5}\ \pi\,\text{cm}^2$

b. $A_T = 60\sqrt{5}\pi\text{cm}^2 + 64\pi\text{cm}^2 + 4\pi\text{cm}^2 = \left(60\sqrt{5} + 68\right)\pi\text{cm}^2$

c. $V_f = \dfrac{h'\pi}{3}(r^2 + (r)^2 + rr')$

$= \dfrac{12}{3}\pi(64 + 4 + 16) = 336\ \pi\text{cm}^3$

Therefore, $V_f = 336\ \pi\text{cm}^3$

9. Let the number of sides of the base be n. Since the bases are regular polygons, we have

i. length of one side of the lower base is $\dfrac{P}{n}$

ii. length of one side of the upper base is $\dfrac{P'}{n}$

Thus, area of a lateral face is $\dfrac{\ell}{2}\left(\dfrac{P}{n} + \dfrac{P'}{n}\right) = \dfrac{\ell}{2}\dfrac{(P+P')}{n}$

Since the n lateral faces of the frustum are congruent, we have

$A_L = n \times \dfrac{1}{2}\ \ell\ \dfrac{(P+P')}{n} = \dfrac{\ell}{2}(P+P')$

10. $\dfrac{A'}{A} = \dfrac{k^2}{h^2}$

$\dfrac{A'}{16\pi} = \dfrac{5^2}{10^2} = A' = 16\pi \times \dfrac{25}{100}$ $\quad A' = 4\pi = \pi r^2 \Rightarrow r = 2$ cm

$\ell^2 = R^2 + h^2$; $\quad (\ell')^2 = r^2 + k^2$

$\ell^2 = 10^2 + 4^2 = 116$ $\quad (\ell')^2 = 4 + 25$

$\ell = \sqrt{116} = 2\sqrt{29}$ cm $\quad \ell' = \sqrt{29}$

Slant height ℓ_1 of the frustum is $\ell_1 = \ell - \ell'$

$= 2\sqrt{29} - \sqrt{29} = \sqrt{29}$

a. $A_L = \dfrac{\ell_1}{2}(2\pi R + 2\pi r)$

$= \dfrac{\sqrt{29}}{2}(2\pi\times 4 + 2\pi\times 2) = \dfrac{\sqrt{29}}{2}(12\pi) = 6\pi\sqrt{29}\text{ cm}^2$

b. $V_f = \dfrac{h}{3}\left(A + A' + \sqrt{AA'}\right)$ or $V_f = \dfrac{h'}{3}\pi(R^2 + r^2 + Rr)$

$= \dfrac{5}{3}(16\pi + 4\pi + 8\pi) = \dfrac{5}{3}(28\pi) = \dfrac{140}{3}\ \pi\text{cm}^3$

11. $AC^2 = AB^2 + BC^2$

$= (3\sqrt{2})^2 + (3\sqrt{2})^2$

$AC^2 = 36$

$AC = 6\text{cm}$

$AV^2 = VO^2 + AO^2$

$\left(3\sqrt{2}\right)^2 = VO^2 + (3)^2$

$18 - 9 = VO^2$

$h = VO = 3$ cm

$h = k + 2 = k \Rightarrow 3 - 2 = 1$ cm

$$\frac{A'}{A} = \left(\frac{k}{h}\right)^2 \Rightarrow \frac{A'}{18} = \left(\frac{1}{3}\right)^2 \Rightarrow A' = 2 \text{ cm}^2$$

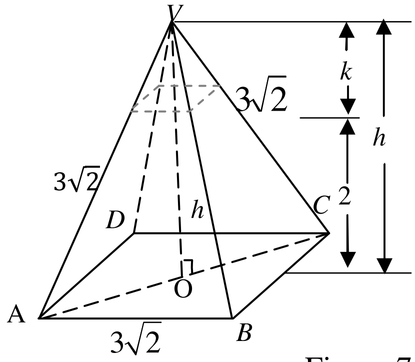


Figure.7.14

$$V = \frac{h}{3}\left(A + A' + \sqrt{AA'}\right)$$

$$= \frac{2}{3}\left(18 + 2 + 6\right) = \frac{52}{3}\text{cm}^3$$

12. Area A' of the lower base:

$A' = \pi r^2 = \pi\ (20 \text{ cm})^2 = 400\ \pi \text{cm}^2$

Area A of the upper base:

$A = \pi R^2 = \pi(60)^2 = 3600\pi \text{cm}^2$

$$V_f = \frac{h}{3}\left(A + A' + \sqrt{AA'}\right)$$

$$= \frac{40}{3}(3600\pi + 400\pi + 1200\pi) = \frac{40}{3}(5200\pi) \approx 69{,}333.33\pi \text{cm}^3$$

Assessment

At this level, the students are aware of pyramids, cones and their frustums. It is hoped that you know how far they are following. Collect the assignment given and mark it appropriately.

7.4 SURFACE AREAS AND VOLUMES OF COMPOSED SOLIDS

Periods Allotted: 7 Periods

Competencies

At the end of this sub-unit, students will be able to:

- *determine the surface areas of simple composed solids.*
- *calculate volumes of simple composed solids.*

Vocabulary: Composed solid

Introduction

In this sub-unit, surface areas and volumes of solids composed of cylinders, cones, prisms and spheres are investigated. There are many physical objects that are combinations of cylinders, cones, prisms and spheres we usually use. Specially if one goes to a chemistry or physics laboratory, such objects can be observed.

Teaching Notes

The students are already familiar with prisms, cylinders, cones, pyramids, sphere, frustums of cones and pyramids. In this section, they will deal with the problem of finding the surface areas and volumes of solid figures that are formed by combining these solids. The students need to be guided to try Examples 1, 2, and 3 to see and practice how to determine surface area and volume of composed solids. For further practice Exercise 7.4 is useful. Finally, after you have briefly reviewed the unit by using the summary part, you can make the students do the review Exercises on unit 7 independently.

Students can easily do problems in this sub-section if revisions are made using Activity 7.6. Let the student do this activity. Use the following additional problems for fast learners.

1. Given a cone of base radius 3 units and height 4 units and a cylinder of base radius 3 units and half the height of the cone. Which one has a larger lateral surface area?
2. A spherical container of radius 3 units is full of water. If the water is poured into a cylinder of the same base radius, what will be the height of the level of the water?

Answers to Activity 7.6

1. a. i. $2\pi rh$
   ii. Ph, where P is perimeter of the base
   iii. $\pi r\ell$, where h is slant height of the cone
   iv. $\frac{1}{2}P\ell$, where P is perimeter and ℓ is slant height
   v. $4\pi r^2$ vi. $\frac{1}{2}\ell(P+P')$ vii. $\ell\pi(r+r')$

   b. i. $\pi r^2 h$ ii. $A_B h$ iii. $\frac{1}{3}\pi r^2 h$
   iv. $\frac{1}{3}A_B h$ v. $\frac{4}{3}\pi r^3$
   vi. $\frac{1}{3}h'\left(A_B + A'_B + \sqrt{A_B A'_B}\right)$ vii. $\frac{1}{3}h'\pi(r^2 + r'^2 + rr')$

2. $V_s = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = \frac{4}{3}\pi\frac{d^3}{8} = \frac{\pi}{6}d^3$ (Volume of the original one)

If the diameter d is halfed it will be $\frac{d}{2}$ and

$$V=\left(\frac{4}{3}\pi\right)\left(\frac{d}{4}\right)^3=\left(\frac{4}{3}\pi\right)\frac{d^3}{64}=\frac{1}{16}\left(\frac{\pi}{3}d^3\right)=\frac{1}{8}\left(\frac{\pi}{6}d^3\right)$$

The new volume will be $\frac{1}{8}$ of the original is:

Surface area of the original $4\pi\left(\frac{d}{2}\right)^2=\pi d^2$

If the diameter is halved, then

$$A=4\pi\left(\frac{d}{4}\right)^2=4\pi\,\frac{d^2}{16}=\frac{\pi 6^2}{4}=\frac{1}{4}(\pi d^2)$$

Thus, its surface area will be one fourth of the original.

3. $V_{sphere}=\frac{4}{3}\pi r^3$ and $V_{cone}=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi r^2(2r)=\frac{2}{3}\pi r^3$

The ratio is $2:1$

With the help of Activity 7.6 and what has been discussed in sub-unit 7.3, discuss Examples 1, 2 and 3. Ask some students to do the problems on the board for others. Let others comment what is done on the board.

Next, form the usual group and let them discuss on Group work 7.2. Check the progress of each group and tell them to do all the problems.

Answer to Group Work 7.2

1. Volume of the road half submerged is

$$V_R=\frac{1}{2}\pi r^2 h_R$$

$$=\frac{1}{2}\pi(2)^2\times 6=12\pi\ \text{cm}^2$$

The volume of water that rises is

$$V_{w'}=\pi r^2 h' \quad =\pi(4)^2 h'=16\ \pi h'$$

But, $V_R=V_{w'}$

$12\pi = 16\pi h'$

$\frac{3}{4}\text{cm}=h' = 0.75$ cm.

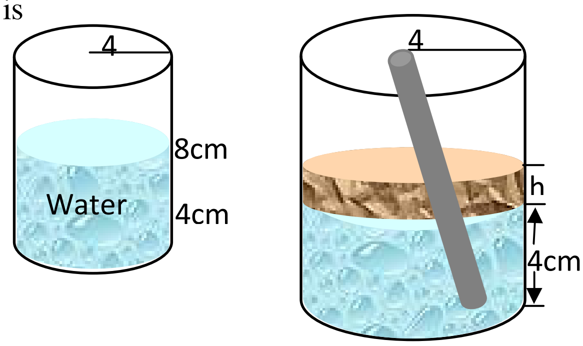


Figure 7.15

The new depth of the water level is h_w = 4 cm + 0.75 cm = 4.75 cm

2. $\left(84+4\sqrt{5}\right)\pi\ \text{cm}^2$, $\left(80+\frac{16}{3}\right)\pi\ \text{cm}^3$

3. $\frac{2048}{3}\pi\ \text{cm}^3$, 4096 cm^3

4. $\frac{4000}{3}\pi\, cm^3$, $100\pi\left(2+\sqrt{5}\right)cm^2$

5. $\left(240\pi+\frac{380}{3}\pi\right)\text{cm}^3$

Additional problems for fast learners

1. Find the total surface area and volume of the following solid.

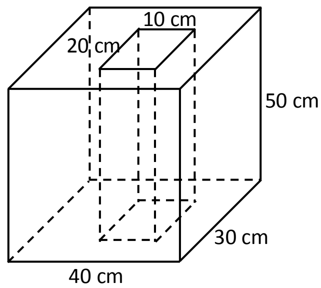


Figure 7.16

2. A sphere is inscribed in a cylinder. Show that the area of the sphere equals the lateral surface area of the cylinder.
3. The container shown below has the shape of a rectangular solid. When a rock is submerged, the water level rises 0.3 cm. Find the volume of the rock.

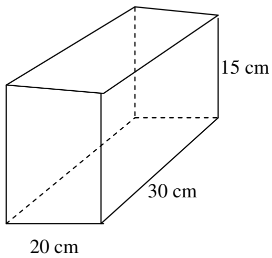


Figure 7.17

4. i. Guess which contains more, the can or the bottle.(Assume that the top part of the bottle is a complete cone).
   ii. See if your guess is right by finding the volumes of both.

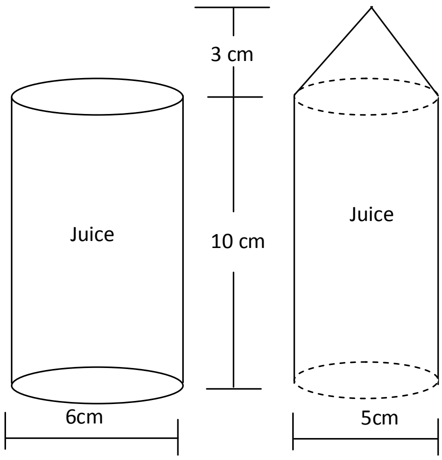


Figure 7.18

Assessment

This is the end of the text book and the year. Collect the assignment given and give corrections if there are. You may make the students study this unit and give them a test by selecting questions from the Review Exercises. Some additional questions are given below.

Answers to Exercise 7.4

1. a. $V_T = V_{frustum} + V_{cylinder}$

$$= \frac{\pi}{3}h(R^2 + r^2 + Rr) + \pi R^2 h = \frac{5}{3}\pi(9+4+6) + \pi(9)(8) = \left(72 + \frac{95}{3}\right)\pi \text{cm}^3$$

b. $V = (4\times4\times10) + \frac{1}{2}\times4\times2\times10 = (160+40)\text{cm}^3 = 200 \text{ cm}^3$

2. Let **Figure 7.19** represent the given problem.

$V_T = V_{cylinder} + V_{hemisphere}$

$$= \pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right); \text{ since } h = 9-2 = 7\,\text{cm},\ r = 2\,\text{cm}$$

$$= \pi(2)^2\times7 + \frac{1}{2}\left(\frac{4}{3}\pi(2)^3\right) = 28\pi + \frac{16}{3}\pi = \frac{100}{3}\pi\,\text{cm}^3$$

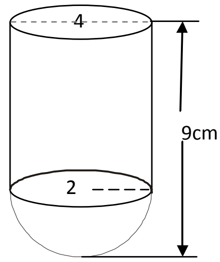


Figure 7.19

3. **Figure 7.20** represent the given problem.

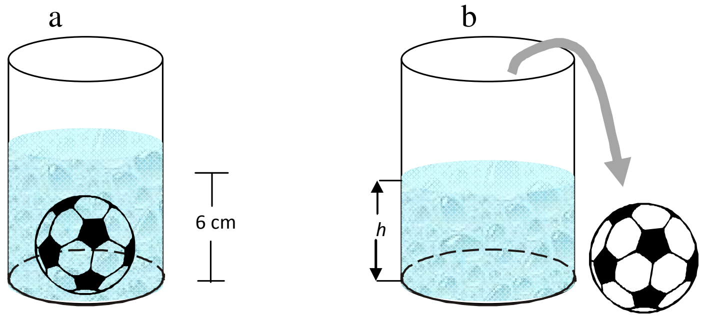


Figure 7.20

$$V_{ball} = \frac{4}{3}\pi\left(\frac{5}{2}\right)^3 = \frac{125}{6}\pi\text{cm}^3$$

$$V_{ball} + V_{water} = \frac{125}{6}\pi\text{cm}^3 + \pi(5)^2\times h$$

But $V_{ball} + V_{water} = \pi(5)^2\times6 = 150\pi\,\text{cm}^3$

$$\therefore\ \frac{125}{6}\pi + 25\pi h = 150\pi$$

$$\Rightarrow 25h = \left(150 - \frac{125}{6}\right) \Rightarrow h = \frac{775}{6\times25}\text{ cm} = \frac{31}{6}\text{ cm}$$

Therefore, the level of the water after the ball is removed drops $\left(6 - \frac{31}{6}\right)\text{cm} = \frac{5}{6}\text{cm}.$

4. $V = V_{hem} - V_{cone}$

$$= \frac{2}{3}\pi r^3 - \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi[2r^3 - r^3], \text{ because } r = h \text{ for the cone}$$

$$= \frac{1}{3}\pi r^3 = \frac{1}{3}\pi(8^3) = \frac{512}{3}\pi \text{ cm}^3$$

Therefore, the volume of the resulting solid is $\frac{512}{3}\pi$ cm^3

$$A_T = 2\pi r^2 + \pi r\ell, \ell = 8\sqrt{2}$$

$$= 2\pi(8 \text{ cm})^2 + \pi(8 \text{ cm})\left(8\sqrt{2} \text{ cm}\right)$$

$$= 128\pi \text{ cm}^2 + 64\sqrt{2}\ \pi \text{cm}^2 = 64\left(2+\sqrt{2}\right)\pi \text{ cm}^2$$

Therefore, the total surface area of the resulting solid is $= 64\left(2+\sqrt{2}\right)\pi \text{ cm}^2$

5. Let V_f =volume of the frustum = $\frac{h'}{3}\left(A + A' + \sqrt{AA'}\right)$

$$= \frac{h'}{3}\left(\pi(4) + \pi(36) + \sqrt{4\pi \times 36\pi}\right)$$

$$= \frac{20}{3}(4\pi + 36\pi + 12\pi) = \frac{1040}{3}\pi \text{cm}^3$$

V_t =Volume of the cylinder $= \pi r^2 h' = \pi(4)(20) = 80\pi \text{cm}^3$

Therefore, volume of the resulting solid = $\left(\frac{1040}{3}\pi - 80\pi\right)\text{cm}^3 = \frac{800}{3}\pi \text{ cm}^3$.

$$\ell = \sqrt{20^2 + 4^2} = 4\sqrt{26}$$

$$A_{frustum} = \pi\ell(r + r_1) = 32\pi\sqrt{26} \text{ cm}^2$$

$$A_{cylinder} = 2\pi r_1 h = 80\pi \text{ cm}^2$$

$$A_{base} = \pi r^2 - \pi r_1^2 = 32\pi \text{ cm}^2$$

$$\therefore\ A_T = A_{frustum} + A_{cylinder} + A_{base} = 16\pi\left(7 + 2\sqrt{26}\right) \text{ cm}^2$$

6. Let volume of the hemispherical shell with radius 4 and 6 be V_4 and V_6 respectively, then

Volume of the resulting solid = $V_6 - V_4 = \frac{1}{2}\left(\frac{4}{3}\pi(6)^3 - \frac{4}{3}\pi(4)^3\right)$

$$= \frac{1}{2}\left(288\pi - \frac{256}{3}\pi\right) = \frac{1}{2}\left(\frac{608}{3}\pi\right) = \frac{304}{3}\pi \text{ unit}^3$$

$$A_6 = \frac{1}{2}(4\pi 6^2) = 72\pi$$

$$A_4 = \frac{1}{2}4\pi(4^2) = 32\pi$$

$$A_{\text{base}} = \pi(6^2) - \pi(4^2) = 20\pi$$

Therefore, $A_T = 124\pi \text{unit}^2$

7. $V_{cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(8)^2 \times 9\text{cm} = 192\pi\,\text{cm}^3$

$V_{cylinder} = \pi r^2 h = \pi(8)^2 \times 18 = 1152\pi\text{cm}^3$

$V_{resulting\ solid} = 1152\pi - 192\pi = 960\pi\,\text{cm}^3$

$\text{Ratio}: \frac{192\ \pi\,\text{cm}^3}{960\ \pi\,\text{cm}^3} = \frac{1}{5} = 1:5$

This is the end of the year. Revise the unit to remind them the major formulae. Then select some questions for a class exercise and some as assignment. You may use some of the questions for a test.

Answers to Review Exercises on Unit 7

1. a. The apothem a is $a = 2\sqrt{3}$, from equilateral triangle, $\ell = 6\sqrt{3}$

$$A_L = \frac{1}{2}P\ell = \frac{1}{2}(3\times 12)6\sqrt{3} = 108\sqrt{3}\,\pi\text{cm}^2$$

$$h^2 + a^2 = \ell^2$$

$$h^2 = \ell^2 - a^2 = \left(6\sqrt{3}\right)^2 - \left(2\sqrt{3}\right)^2$$

$$=108 - 12 = 96$$

$$h = \sqrt{96} = 4\sqrt{6}$$

$$V = \frac{1}{3}A_b.h = \frac{1}{3}\times 36\sqrt{3}\times 4\sqrt{6} = 144\sqrt{2}\ \text{cm}^3$$

b. $A_L = 2\pi r^2 = 2\pi(5)^2 = 50\pi\,cm^2$

$$V = \frac{2}{3}\pi r^3 = \frac{2}{3}\pi\left(5\right)^3 = \frac{250}{3}\pi\,\text{cm}^3$$

c. $\ell = \sqrt{h^2 + r^2}$

$$= \sqrt{12^2 + 6^2} = 6\sqrt{5}\,cm$$

$$A_L = \pi r\ell = \pi(6)\left(6\sqrt{5}\right) = 36\sqrt{5}\pi\ \text{cm}^2$$

$$V = \frac{1}{3}\pi(36)12 = 144\pi\,\text{cm}^3$$

d. $A_L = 2\pi rh = 2\pi(2)(6) = 24\pi\text{cm}^2$

$$V = \pi r^2 h = \pi(4)(6) = 24\pi\,\text{cm}^3$$

2. $A_L = 216\ \text{cm}^2$

3. $A_T = 2\pi r^2 + 2\pi rh = 2\pi r(r+h) = 2\pi r(2r) = 4\pi r^2$

$V = \pi r^2 h = \pi r^3$

4. Let **Figure 7.21** represent the given problem.

$\ell^2 = h^2 + OE^2$

$\left(50\sqrt{2}\right)^2 = h^2 + (50)^2$

$5000 = h^2 + 2500$

$\Rightarrow h = 50$ m

$V = \frac{1}{3} A_B \ h$

$= \frac{1}{3}(100\,\text{m})^2 \times 50\,\text{m}$

$= \frac{500000}{3}\ \text{m}^3 \approx 166,666.7\ \text{m}^3$

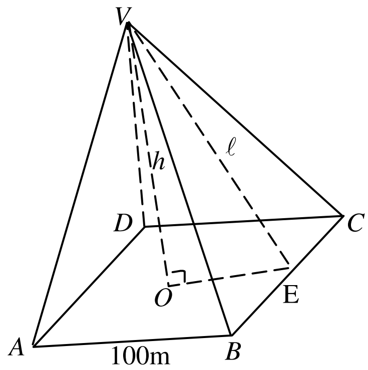


Figure 7.21

5. The base of the regular pyramid is hexagonal region and its area is

$$A_B = \frac{1}{2} nr^2 \sin\left(\frac{360^o}{n}\right)$$

$$= \frac{1}{2}\times 6\times 8^2 \sin\frac{360^\text{o}}{6} \quad = 96\sqrt{3}\,\text{cm}^2$$

Apothem: $a = r\cos 30^\text{o}$ or from trigonometry ; $A_T = A_L + A_B$

$a = 8\times\frac{\sqrt{3}}{2} = 4\sqrt{3}$

$\ell = \sqrt{12^2 + \left(4\sqrt{3}\right)^2}$

$\ell = \sqrt{192} = 8\sqrt{3}$

$= \frac{1}{2}P\ell + A_B$

$= \frac{1}{2}(6\times 8)\times 8\sqrt{3} + 96\sqrt{3}$

$= 192\sqrt{3} + 96\sqrt{3} = 288\sqrt{3}\ \text{cm}^2$

6. $\ell = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$

$A_L = \pi r\ell = \pi\times 6\times 10 = 60\pi\,\text{cm}^2$

7. Let **Figure 7.22** represents the given problem

$A_T = A_B + A_L$

$= \pi r^2 + \pi r\ell$, use $\ell = \sqrt{r^2 + h^2}$

$= \pi r^2 + \pi r\sqrt{r^2 + h^2} = \pi r\left(r + \sqrt{r^2 + h^2}\right)$

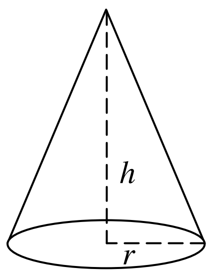


Figure 7.22

8. Let **Figure 7.23** represents the given problem.

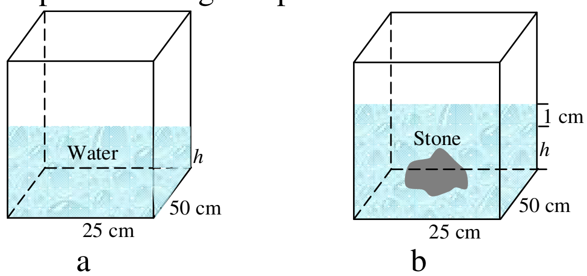


Figure 7.23

Volume of water after a lump of stone is submerged is

$$V_w = 25 \times 50 \times 1 = 1250 \text{ cm}^3$$

$$V_w = V_{stone} = 1250 \text{ cm}^3 = \text{the volume of the submerged stone}$$

9. $$V_f = \frac{\pi h}{3}(R^2 + r^2 + Rr)$$

$$= \frac{25\pi}{3}(8^2 + 6^2 + 48)$$

$$= \frac{25\pi}{3}(148) = \frac{3700\pi}{3}\text{cm}^3$$

10. Let **Figure 7.24** represents the given problem.
The diameter of the hole is 10 – 4 = 6 cm

$$V_{material} = V_{outer} - V_{hole}$$

$$= \pi R^2 h - \pi r^2 h$$

$$= \pi \times 30 \times (5^2 - 3^2)$$

$$= 30\pi \times 16$$

$$= 480\pi \text{cm}^3$$

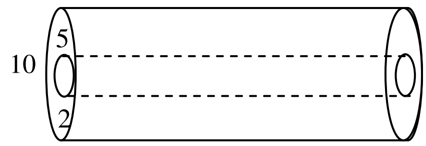


Figure 7.24

11. $$A' = \pi r^2 = \pi(2)^2 = 4\pi cm^2$$

$$A = \pi R^2 = \pi(3)^2 = 9\pi cm^2$$

$$\frac{A'}{A} = \frac{k^2}{h^2}$$

$$\Rightarrow \frac{k}{h} = \frac{2}{3} \Rightarrow k = \frac{2}{3}h$$

$$V_{small\ cone} = \frac{1}{3}\pi r^2 k = \frac{1}{3}\pi(2)^2 \times \frac{2}{3}h = \frac{8}{9}\pi h$$

$$V_{large\ cone} = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(9) \times h = 3\pi h$$

$$V_{frustum} = V_{large\ cone} - V_{smaller\ cone}$$

$$= 3\pi h - \frac{8}{9}\pi h = \frac{19}{9}\pi h$$

$$\Rightarrow 80 = \frac{19}{9}\pi h$$

$$\Rightarrow h = \frac{720}{19\pi}\text{cm} \text{ but } k = \frac{2}{3}h = \frac{2}{3}\left(\frac{720}{19\pi}\right)\text{cm} = \frac{480}{19\pi}\text{cm}$$

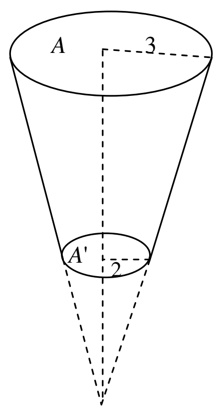


Figure 7.25

Therefore, the height of the cup is $h' = h - k = \frac{720}{19\pi}\text{cm} - \frac{480}{19\pi}\text{cm} = \frac{240}{19\pi}\text{cm}.$

Or $V = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$

$\Rightarrow 80 = \frac{1}{3}\pi h(9+4+6) \quad \Rightarrow h = \frac{240}{19\pi}\text{cm}$

12. $A_L = \pi r\ell$; $\ell^2 - r^2 = h^2$

$= \pi \times (4\text{cm}) \times 16\text{cm}$ $\quad 16^2 - 4^2 = h^2$

$= 64\pi\ \text{cm}^2$

So, $V = \frac{1}{3}\pi r^2 h$ $\quad \therefore\ h = \sqrt{240} = 4\sqrt{15}$ cm

$= \frac{1}{3}\pi(4)^2 \times 4\sqrt{15} \quad = \frac{64\pi}{3}\sqrt{15}\ \text{cm}^3$

13. $V = \frac{1}{3}\pi r^2 h$ $\quad \ell^2 = r^2 + h^2$

$720\pi = \frac{1}{3}\pi \times (12)^2 \times h$ $\quad = 12^2 + 15^2$

$h = \frac{720\pi}{48\pi} = 15\,cm$ $\quad \ell = \sqrt{369} = 3\sqrt{41}$

$A_L = \pi r\ell$

$= \pi \times 12 \times 3\sqrt{41} = 36\pi\sqrt{41}\,\text{cm}^2$

14. The new volume is 8 times the original as $V_N = \frac{4}{3}\pi(2r)^3 = 8V_o$

The new lateral surface area is 4 times the original.

15. Given: Radius of the hemisphere is r

Altitude of the cone is $2r$

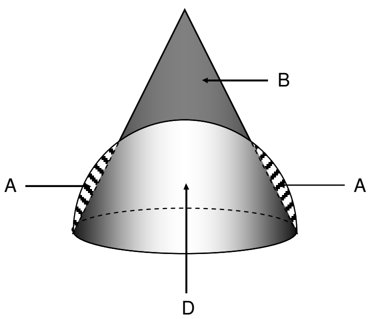


Figure 7.26

Let V_D = the volume of the solid D

Volume of A = Volume of the hemisphere – V_D = $\frac{1}{2}.\frac{4}{3}\pi(r)^3 - V_D = \frac{2}{3}\pi r^3 - V_D$

Volume of B = Volume of the cone – V_D = $\frac{1}{3}\pi r^2 h - V_D = \frac{1}{3}\pi r^2(2r) - V_D = \frac{2}{3}\pi r^3 - V_D$

Therefore, Volume of A = Volume of B.

Table of Trigonometric Functions

↓→	sin	cos	tan	cot	sec	csc	
0°	0.0000	1.0000	0.0000	.....	1.000	.....	**90°**
1°	0.0175	0.9998	0.0175	57.29	1.000	57.30	**89°**
2°	0.0349	0.9994	0.0349	28.64	1.001	28.65	**88°**
3°	0.0523	0.9986	0.0524	19.08	1.001	19.11	**87°**
4°	0.0698	0.9976	0.0699	14.30	1.002	14.34	**86°**
5°	0.0872	0.9962	0.0875	11.43	1.004	11.47	**85°**
6°	0.1045	0.9945	0.1051	9.514	1.006	9.567	**84°**
7°	0.1219	0.9925	0.1228	8.144	1.008	8.206	**83°**
8°	0.1392	0.9903	0.1405	7.115	1.010	7.185	**82°**
9°	0.1564	0.9877	0.1584	6.314	1.012	6.392	**81°**
10°	0.1736	0.9848	0.1763	5.671	1.015	5.759	**80°**
11°	0.1908	0.9816	0.1944	5.145	1.019	5.241	**79°**
12°	0.2079	0.9781	0.2126	4.705	1.022	4.810	**78°**
13°	0.2250	0.9744	0.2309	4.331	1.026	4.445	**77°**
14°	0.2419	0.9703	0.2493	4.011	1.031	4.134	**76°**
15°	0.2588	0.9659	0.2679	3.732	1.035	3.864	**75°**
16°	0.2756	0.9613	0.2867	3.487	1.040	3.628	**74°**
17°	0.2924	0.9563	0.3057	3.271	1.046	3.420	**73°**
18°	0.3090	0.9511	0.3249	3.078	1.051	3.236	**72°**
19°	0.3256	0.9455	0.3443	2.904	1.058	3.072	**71°**
20°	0.3420	0.9397	0.3640	2.747	1.064	2.924	**70°**
21°	0.3584	0.9336	0.3839	2.605	1.071	2.790	**69°**
22°	0.3746	0.9272	0.4040	2.475	1.079	2.669	**68°**
23°	0.3907	0.9205	0.4245	2.356	1.086	2.559	**67°**
24°	0.4067	0.9135	0.4452	2.246	1.095	2.459	**66°**
25°	0.4226	0.9063	0.4663	2.145	1.103	2.366	**65°**
26°	0.4384	0.8988	0.4877	2.050	1.113	2.281	**64°**
27°	0.4540	0.8910	0.5095	1.963	1.122	2.203	**63°**
28°	0.4695	0.8829	0.5317	1.881	1.133	2.130	**62°**
29°	0.4848	0.8746	0.5543	1.804	1.143	2.063	**61°**
30°	0.5000	0.8660	0.5774	1.732	1.155	2.000	**60°**
31°	0.5150	0.8572	0.6009	1.664	1.167	1.942	**59°**
32°	0.5299	0.8480	0.6249	1.600	1.179	1.887	**58°**
33°	0.5446	0.8387	0.6494	1.540	1.192	1.836	**57°**
34°	0.5592	0.8290	0.6745	1.483	1.206	1.788	**56°**
35°	0.5736	0.8192	0.7002	1.428	1.221	1.743	**55°**
36°	0.5878	0.8090	0.7265	1.376	1.236	1.701	**54°**
37°	0.6018	0.7986	0.7536	1.327	1.252	1.662	**53°**
38°	0.6157	0.7880	0.7813	1.280	1.269	1.624	**52°**
39°	0.6293	0.7771	0.8098	1.235	1.287	1.589	**51°**
40°	0.6428	0.7660	0.8391	1.192	1.305	1.556	**50°**
41°	0.6561	0.7547	0.8693	1.150	1.325	1.524	**49°**
42°	0.6691	0.7431	0.9004	1.111	1.346	1.494	**48°**
43°	0.6820	0.7314	0.9325	1.072	1.367	1.466	**47°**
44°	0.6947	0.7193	0.9667	1.036	1.390	1.440	**46°**
45°	0.7071	0.7071	1 .0000	1.000	1.414	1.414	**45°**
	cos	sin	cot	tan	csc	sec	←↑

Table of Common Logarithms

n	0	1	2	3	4	5	6	7	8	9
1.0	0.0000	0.0043	0.0086	0.0128	0.0170	0.0212	0.0253	0.0294	0.0334	0.0374
1.1	0.0414	0.0453	0.0492	0.0531	0.0569	0.0607	0.0645	0.0682	0.0719	0.0755
1.2	0.0792	0.0828	0.0864	0.0899	0.0934	0.0969	0.1004	0.1038	0.1072	0.1106
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430
1.4	0.1461	0.1492	0.1523	0.1553	0.1584	0.1614	0.1644	0.1673	0.1703	0.1732
1.5	0.1761	0.1790	0.1818	0.1847	0.1875	0.1903	0.1931	0.1959	0.1987	0.2014
1.6	0.2041	0.2068	0.2095	0.2122	0.2148	0.2175	0.2201	0.2227	0.2253	0.2279
1.7	0.2304	0.2330	0.2355	0.2380	0.2405	0.2430	0.2455	0.2480	0.2504	0.2529
1.8	0.2553	0.2577	0.2601	0.2625	0.2648	0.2672	0.2695	0.2718	0.2742	0.2765
1.9	0.2788	0.2810	0.2833	0.2856	0.2878	0.2900	0.2923	0.2945	0.2967	0.2989
2.0	0.3010	0.3032	0.3054	0.3075	0.3096	0.3118	0.3139	0.3160	0.3181	0.3201
2.1	0.3222	0.3243	0.3263	0.3284	0.3304	0.3324	0.3345	0.3365	0.3385	0.3404
2.2	0.3424	0.3444	0.3464	0.3483	0.3502	0.3522	0.3541	0.3560	0.3579	0.3598
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784
2.4	0.3802	0.3820	0.3838	0.3856	0.3874	0.3892	0.3909	0.3927	0.3945	0.3962
2.5	0.3979	0.3997	0.4014	0.4031	0.4048	0.4065	0.4082	0.4099	0.4116	0.4133
2.6	0.4150	0.4166	0.4183	0.4200	0.4216	0.4232	0.4249	0.4265	0.4281	0.4298
2.7	0.4314	0.4330	0.4346	0.4362	0.4378	0.4393	0.4409	0.4425	0.4440	0.4456
2.8	0.4472	0.4487	0.4502	0.4518	0.4533	0.4548	0.4564	0.4579	0.4594	0.4609
2.9	0.4624	0.4639	0.4654	0.4669	0.4683	0.4698	0.4713	0.4728	0.4742	0.4757
3.0	0.4771	0.4786	0.4800	0.4814	0.4829	0.4843	0.4857	0.4871	0.4886	0.4900
3.1	0.4914	0.4928	0.4942	0.4955	0.4969	0.4983	0.4997	0.5011	0.5024	0.5038
3.2	0.5051	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172
3.3	0.5185	0.5198	0.5211	0.5224	0.5237	0.5250	0.5263	0.5276	0.5289	0.5302
3.4	0.5315	0.5328	0.5340	0.5353	0.5366	0.5378	0.5391	0.5403	0.5416	0.5428
3.5	0.5441	0.5453	0.5465	0.5478	0.5490	0.5502	0.5514	0.5527	0.5539	0.5551
3.6	0.5563	0.5575	0.5587	0.5599	0.5611	0.5623	0.5635	0.5647	0.5658	0.5670
3.7	0.5682	0.5694	0.5705	0.5717	0.5729	0.5740	0.5752	0.5763	0.5775	0.5786
3.8	0.5798	0.5809	0.5821	0.5832	0.5843	0.5855	0.5866	0.5877	0.5888	0.5899
3.9	0.5911	0.5922	0.5933	0.5944	0.5955	0.5966	0.5977	0.5988	0.5999	0.6010
4.0	0.6021	0.6031	0.6042	0.6053	0.6064	0.6075	0.6085	0.6096	0.6107	0.6117
4.1	0.6128	0.6138	0.6149	0.6160	0.6170	0.6180	0.6191	0.6201	0.6212	0.6222
4.2	0.6232	0.6243	0.6253	0.6263	0.6274	0.6284	0.6294	0.6304	0.6314	0.6325
4.3	0.6335	0.6345	0.6355	0.6365	0.6375	0.6385	0.6395	0.6405	0.6415	0.6425
4.4	0.6435	0.6444	0.6454	0.6464	0.6474	0.6484	0.6493	0.6503	0.6513	0.6522
4.5	0.6532	0.6542	0.6551	0.6561	0.6571	0.6580	0.6590	0.6599	0.6609	0.6618
4.6	0.6628	0.6637	0.6646	0.6656	0.6665	0.6675	0.6684	0.6693	0.6702	0.6712
4.7	0.6721	0.6730	0.6739	0.6749	0.6758	0.6767	0.6776	0.6785	0.6794	0.6803
4.8	0.6812	0.6821	0.6830	0.6839	0.6848	0.6857	0.6866	0.6875	0.6884	0.6893
4.9	0.6902	0.6911	0.6920	0.6928	0.6937	0.6946	0.6955	0.6964	0.6972	0.6981
5.0	0.6990	0.6998	0.7007	0.7016	0.7024	0.7033	0.7042	0.7050	0.7059	0.7067
5.1	0.7076	0.7084	0.7093	0.7101	0.7110	0.7118	0.7126	0.7135	0.7143	0.7152
5.2	0.7160	0.7168	0.7177	0.7185	0.7193	0.7202	0.7210	0.7218	0.7226	0.7235
5.3	0.7243	0.7251	0.7259	0.7267	0.7275	0.7284	0.7292	0.7300	0.7308	0.7316
5.4	0.7324	0.7332	0.7340	0.7348	0.7356	0.7364	0.7372	0.7380	0.7388	0.7396

5.5	0.7404	0.7412	0.7419	0.7427	0.7435		0.7443	0.7451	0.7459	0.7466	0.7474
5.6	0.7482	0.7490	0.7497	0.7505	0.7513		0.7520	0.7528	0.7536	0.7543	0.7551
5.7	0.7559	0.7566	0.7574	0.7582	0.7589		0.7597	0.7604	0.7612	0.7619	0.7627
5.8	0.7634	0.7642	0.7649	0.7657	0.7664		0.7672	0.7679	0.7686	0.7694	0.7701
5.9	0.7709	0.7716	0.7723	0.7731	0.7738		0.7745	0.7752	0.7760	0.7767	0.7774
6.0	0.7782	0.7789	0.7796	0.7803	0.7810		0.7818	0.7825	0.7832	0.7839	0.7846
6.1	0.7853	0.7860	0.7868	0.7875	0.7882		0.7889	0.7896	0.7903	0.7910	0.7917
6.2	0.7924	0.7931	0.7938	0.7945	0.7952		0.7959	0.7966	0.7973	0.7980	0.7987
6.3	0.7993	0.8000	0.8007	0.8014	0.8021		0.8028	0.8035	0.8041	0.8048	0.8055
6.4	0.8062	0.8069	0.8075	0.8082	0.8089		0.8096	0.8102	0.8109	0.8116	0.8122
6.5	0.8129	0.8136	0.8142	0.8149	0.8156		0.8162	0.8169	0.8176	0.8182	0.8189
6.6	0.8195	0.8202	0.8209	0.8215	0.8222		0.8228	0.8235	0.8241	0.8248	0.8254
6.7	0.8261	0.8267	0.8274	0.8280	0.8287		0.8293	0.8299	0.8306	0.8312	0.8319
6.8	0.8325	0.8331	0.8338	0.8344	0.8351		0.8357	0.8363	0.8370	0.8376	0.8382
6.9	0.8388	0.8395	0.8401	0.8407	0.8414		0.8420	0.8426	0.8432	0.8439	0.8445
7.0	0.8451	0.8457	0.8463	0.8470	0.8476		0.8482	0.8488	0.8494	0.8500	0.8506
7.1	0.8513	0.8519	0.8525	0.8531	0.8537		0.8543	0.8549	0.8555	0.8561	0.8567
7.2	0.8573	0.8579	0.8585	0.8591	0.8597		0.8603	0.8609	0.8615	0.8621	0.8627
7.3	0.8633	0.8639	0.8645	0.8651	0.8657		0.8663	0.8669	0.8675	0.8681	0.8686
7.4	0.8692	0.8698	0.8704	0.8710	0.8716		0.8722	0.8727	0.8733	0.8739	0.8745
7.5	0.8751	0.8756	0.8762	0.8768	0.8774		0.8779	0.8785	0.8791	0.8797	0.8802
7.6	0.8808	0.8814	0.8820	0.8825	0.8831		0.8837	0.8842	0.8848	0.8854	0.8859
7.7	0.8865	0.8871	0.8876	0.8882	0.8887		0.8893	0.8899	0.8904	0.8910	0.8915
7.8	0.8921	0.8927	0.8932	0.8938	0.8943		0.8949	0.8954	0.8960	0.8965	0.8971
7.9	0.8976	0.8982	0.8987	0.8993	0.8998		0.9004	0.9009	0.9015	0.9020	0.9025
8.0	0.9031	0.9036	0.9042	0.9047	0.9053		0.9058	0.9063	0.9069	0.9074	0.9079
8.1	0.9085	0.9090	0.9096	0.9101	0.9106		0.9112	0.9117	0.9122	0.9128	0.9133
8.2	0.9138	0.9143	0.9149	0.9154	0.9159		0.9165	0.9170	0.9175	0.9180	0.9186
8.3	0.9191	0.9196	0.9201	0.9206	0.9212		0.9217	0.9222	0.9227	0.9232	0.9238
8.4	0.9243	0.9248	0.9253	0.9258	0.9263		0.9269	0.9274	0.9279	0.9284	0.9289
8.5	0.9294	0.9299	0.9304	0.9309	0.9315		0.9320	0.9325	0.9330	0.9335	0.9340
8.6	0.9345	0.9350	0.9355	0.9360	0.9365		0.9370	0.9375	0.9380	0.9385	0.9390
8.7	0.9395	0.9400	0.9405	0.9410	0.9415		0.9420	0.9425	0.9430	0.9435	0.9440
8.8	0.9445	0.9450	0.9455	0.9460	0.9465		0.9469	0.9474	0.9479	0.9484	0.9489
8.9	0.9494	0.9499	0.9504	0.9509	0.9513		0.9518	0.9523	0.9528	0.9533	0.9538
9.0	0.9542	0.9547	0.9552	0.9557	0.9562		0.9566	0.9571	0.9576	0.9581	0.9586
9.1	0.9590	0.9595	0.9600	0.9605	0.9609		0.9614	0.9619	0.9624	0.9628	0.9633
9.2	0.9638	0.9643	0.9647	0.9652	0.9657		0.9661	0.9666	0.9671	0.9675	0.9680
9.3	0.9685	0.9689	0.9694	0.9699	0.9703		0.9708	0.9713	0.9717	0.9722	0.9727
9.4	0.9731	0.9736	0.9741	0.9745	0.9750		0.9754	0.9759	0.9763	0.9768	0.9773
9.5	0.9777	0.9782	0.9786	0.9791	0.9795		0.9800	0.9805	0.9809	0.9814	0.9818
9.6	0.9823	0.9827	0.9832	0.9836	0.9841		0.9845	0.9850	0.9854	0.9859	0.9863
9.7	0.9868	0.9872	0.9877	0.9881	0.9886		0.9890	0.9894	0.9899	0.9903	0.9908
9.8	0.9912	0.9917	0.9921	0.9926	0.9930		0.9934	0.9939	0.9943	0.9948	0.9952
9.9	0.9956	0.9961	0.9965	0.9969	0.9974		0.9978	0.9983	0.9987	0.9991	0.9996

REFERENCE MATERIALS

These days search for a reference is at forefront with authentic supply of electronic references. However, with the assumption that there will be limitations in some parts to over utilize ICT, some hard copy reference materials are listed here that can help develop better learning and teaching of mathematics and these units. These books are selected assuming that they are available in many schools. For those who have access to the internet, e-resources are offered as a supplement to those hard copies, if not essentially preferred. You can also access additional reference materials that are available in you school library. These are simply guides to help you use them as references. However, they are not the only to be prescribed. You can also use the web sites given here for reference and demonstration.

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Federal Democratic Republic of Ethiopia

Ministry of Education

Mathematics Syllabus

Grade 10

2009

General Introduction

Mathematics learning at this cycle, should contribute towards educating students to be ready to take part in constructing the future society. At this level everything has to be done to develop personalities characterized by a scientific view of life, high moral qualities and readiness to take part in social activities. Each student should acquire a solid, applicable and extendable mathematical knowledge and develop the appropriate mathematical skills either to pursue with his/her study of preparatory school (Grades 11 and 12) mathematics or join the technical and vocational trainings after which he/she is able to participate in activities of shaping a new society. By including historical facts and real life applications from different fields of social life (agriculture, industry, trade, investment, etc) in word problems, students shall recognize that mathematics is playing an important role in the development of the country.

At this cycle, students should gain solid knowledge of fundamental mathematical notions, theorems, rules and procedures and develop reliable competencies in using this knowledge for solving problems independently.

It is important to identify and realize problems that cause challenging situations to the students and support them in formulating and solving the problems. Formulating and solving problems must be part of a methodical strategy. The task of the teacher is to facilitate in selecting and arranging the order of the problems, as well as helping and motivating students to solve the problems by themselves in a planned and organized way.

Stabilization must have a central place within mathematics learning. It begins with motivation and orientation, by selecting appropriate problems that were already discussed. Concepts that have not been mastered up to now have to be stabilized. A precondition for dealing with new content is always to ensure the necessary level of ability for solving problems. In mathematics learning as a whole, special emphasis has to be put on committing essential facts, notions, definitions, theorems and formulae to the students' memory as well as enabling students to reproduce and interpret what they have learnt in their own words. The main instruments used for stabilization in mathematics learning are activities and exercises.

General Objectives

At this cycle students acquire and develop solid mathematics knowledge, skills and attitudes that significantly contribute to the creation of citizens who are conscious of the social, economic, political and cultural realities of Ethiopia and that can actively and effectively participate in the ongoing process of development of the country. To this end, the following are the objectives of mathematics learning at this cycle. Students will be able to:

- appreciate the power, elegance and structure of mathematics.
- use mathematics in their environment and social needs.
- understand the essential contribution of mathematics to Engineering, Science, Economics, Agriculture, etc.
- mathematical knowledge and skills to enable them pursue with their further education or future vocational trainings.
- gain satisfaction and enjoyment from learning and applying mathematics.
- develop their cognitive, creative and appreciative potential by relating mathematics with societal need.

Allotment of Periods for Units and Sub-units of Mathematics Grade10

Grade	*Unit*	*Sub-unit*	*Number of Periods*	
			Sub-unit	*Total*
10	**Unit 1: Polynomial functions**	1.1 Introduction to polynomial functions	5	20
		1.2 Theorems on polynomials	6	
		1.3 Zeros of a polynomial function	4	
		1.4 Graphs of polynomial functions	5	
	Unit 2: Exponential and logarithmic functions	2.1 Exponents and logarithms	6	30
		2.1.1 Exponents		
		2.1.2 Logarithms		
		2.2 The exponential functions and their	5	
		graphs	6	
		2.3 The logarithmic functions and their	7	
		graphs	6	
		2.4 Equations involving exponents and logarithms		
		2.5 Applications of exponential and logarithmic functions		
	Unit 3: Solving inequalities	3.1 Systems of linear inequalities involving absolute value	4	20
		3.2 Systems of linear inequalities in two variables	5	
		3.3 Quadratic inequalities	11	
	Unit 4: Coordinate geometry	4.1 Distance between two points	2	15
		4.2 Division of a line segment	2	
		4.3 Equation of a line	8	
		4.4 Parallel and perpendicular lines	3	
	Unit 5: Trigonometric functions	5.1 Basic trigonometric functions	15	30
		5.1.1 The sine, cosine and tangent functions		
		5.1.2 Trigonometric values of angles		
		5.1.3 Graphs of the sine, cosine and tangent functions.		
		5.2 The reciprocal functions of the basic trigonometric functions	7	
		5.3 Simple trigonometric identities	3	
		5.4 Real life application problems	5	
	Unit 6: Plane geometry	6.1 Theorems on triangles	5	22
		6.2 Special quadrilaterals	6	
		6.3 More on circles	6	
		6.4 Regular polygons	5	
	Unit 7: Measurement	7.1 Revision on surface areas and volumes of prisms and cylinders	3	25
		7.2 Pyramids, cones and spheres	8	
		7.3 Frustums of pyramids and cones	7	
		7.4 Surface area and volumes of composed solids.	7	

Introduction

Mathematics learning in grade ten has to significantly contribute towards the fulfillment of the main objectives of learning mathematics. In this respect, developing high level abilities and competencies in calculating is one of the main tasks of learning Grade 10 mathematics. Students should be in a position to tackle problems in an effective way, including estimations, rough calculations and checking the exactness of the result. The use of polynomial, exponential, logarithmic and trigonometric functions for describing phenomena in nature and society and solving respective real life problems should be realized and appreciated by students.

Since improving the linguistic abilities of students and the development of their communication skills serves in the development of their mathematical understanding, emphasis has to be given to group activities and the correct use of mathematical symbolism and language. Students should be in a position to recognize that the use of mathematical symbols make it easier to identify the structure of complicated phenomenon and to make mental work more effective and rational.

Objectives

After studying grade 10 mathematics, students should be able to:

- Define polynomial functions and apply the theorems on polynomials to solve related problems.
- Sketch the graphs of various polynomial functions.
- Apply basic concepts of exponential and logarithmic functions to solve related problems including real life problems.
- Solve linear in equalities involving absolute value, and having two variables.
- Solve quadratic inequalities.
- Apply distance formula to solve related problems
- Determine the slope (gradient) of a given line
- Determine the equation of a given line.
- Apply the properties of the slopes of parallel and perpendicular lines to solve related problems.
- Determine the equation and sketch the graph of a circle given appropriate information.
- Sketch the graphs of basic trigonometric functions.
- Identify and use trigonometric identities.
- Solve real -life problems involving trigonometric functions.
- Define and deal with the reciprocals of the basic trigonometric functions.

1: Polynomial Functions (*20 periods)*

Unit outcomes: Students will be able to:

- define polynomial functions
- perform the four fundamental operations on polynomials
- apply the theorems on polynomials to solve related problems
- sketch the graphs of polynomial functions

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
Student will be able to: • define the polynomial function of one variable • identify the degree, leading coefficient and constant terms of a given polynomial functions. • give different forms of polynomial functions • perform the four fundamental operation on polynomials	***1. Polynomial Functions*** **1.1 Introduction to polynomial functions** *(5 periods)* • Definition of Polynomial function • Operations with polynomial functions.	• start the lesson by discussing about identification of some types of functions from a list of functions like: linear, constant, quadratic functions. • State the formal definition of polynomial function and discuss related terms (like: degree, leading coefficient and constant term) by using several examples. • Assist students in adding, subtracting multiplying and dividing polynomial functions • Guide students to determine whether the resulting expression (function) is a polynomial or not and to determine its domain and range.	• After giving a list of several functions, ask students to identify the type of a function by giving good reason for their answer. • You can give exercises on identifying the degree, the coefficients, the leading coefficient the constant terms of several polynomial functions. • Ask students to find the sums, differences, products and quotients of polynomial functions • You can ask students about their generalization about the degree of the sum, product, difference and quotient of two polynomial functions.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• State and prove the polynomial division theorem • apply the polynomial division theorem	**1.2 Theorems on polynomials** *(6 periods)* • Polynomial division theorem.	• You may start the lesson by revising the division algorithm for real numbers and then analogously after stating the Polynomial Division Theorem and discuss its proof. • Guide the students to practice the application of the Polynomial Division Theorem using examples and exercises. • Let the student conclude the relationship between the degree of the dividend, the divisor and the remainder expressions.	• Ask students to divide one polynomial by another and see how they apply the theorem property.
• State and prove the Factor Theorem • apply the Factor Theorem	• The Remainder Theorem	• Let the students divide a polynomial p(x) by "$x - a$" and guide them to compare the remainder with P(a). • State "The Remainder Theorem" and encourage the students to prove it. • Assist Students to apply "The Remainder Theorem"	• Give exercise problems on the application of the remainder theorem and check their works.
• State and prove the Factor Theorem • apply the Factor Theorem	• The Factor Theorem	• You can start the lesson by considering the division of a polynomial P(x) by "$x–a$" which results in zero remainder, and then guide the students to write the result as • P(x) = ($x - a$) Q(x) where Q(x) is the quotient. • State "The Factor Theorem" and encourage the students to prove it. • Assist the students to apply "The Factor Theorem" such as in factorizing a given polynomial.	• Ask students to factorize a given polynomial (which is factor able) by using the factor theorem.
• determine the zero(s) of a given polynomial function.	**1.3 Zeros of a Polynomial Function** *(4 periods)*	• You can start the lesson by revising how to find solutions of linear and quadratic equations, and in relation to this introduce what is meant by	• Ask students to find zero(s) of a given polynomial function (the given exercise

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		the "Zero(s) of a linear function or a quadratic function". • State the formal definition of "Zero(s) of a Polynomial Function" and help students to determine zero(s) of some polynomial function.	problem may suggest the use of appropriate theorem stated above)
• State the Location Theorem • apply the Location theorem to approximate the zero(s) of a given polynomial function	• Location Theorem	• State and discuss the Location theorem by using several examples. • encourage the students to approximate the zero(s) of some polynomial functions by using the location theorem.	• Give students exercise problem on the application of the location theorem and check their performance.
• apply the rational root test to determine the zero(s) of a given polynomial function.	• Rational root test	• State the rational root test and discuss on how to use it in determining the zero(s) of a given polynomial functions. • assist the students to practise the application of rational root test.	• Give exercise problems on the application of the rational root test.
• Sketch the graph of a given polynomial function. • describe the properties of the graphs of a given polynomial function.	**1.4 Graphs of polynomial functions.** *(5 periods)*	• You may start the lesson by setting an activity that allows students to construct table of values for a given polynomial function. • Assist students to sketch the graph of a given polynomial function using tables of values they make above. • Let students practise sketching the graphs of different polynomial functions. • Assist students to describe some of the properties of the graphs of some polynomials.	• You can ask students to describe the properties of a given polynomial function using its graph.

Unit 2: Exponential and Logarithmic Functions *(30 periods)*

Unit outcomes: Students will be able to:

- understand that the laws of exponents are valid for real exponents
- know specific facts about logarithms
- know basic concepts about exponential and logarithmic functions
- solve mathematical problems involving exponents and logarithms

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
Students will be able to: • Explain what is meant by exponential expression • State and apply the properties of exponents (where the exponents are real numbers)	***2. Exponential and Logarithmic Functions*** **2.1 Exponents and logarithms** *(6 periods)* **2.1.1 Exponents** • Revision on the properties of exponents	• Revise the notion "power" in a form of class discussion. Students should be assisted to recall power, base and exponent. • Assist students to show the validity of the properties of exponents for real exponents such as: 1. $a^x \cdot a^y = a^{x+y}$ 2. $\frac{a^x}{a^y} = a^{x-y}$ 3. $(a^x)^y = a^{xy}$ 4. $a^x b^x = (ab)^x$ 5. $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$ 6. $a^{-x} = \frac{1}{a^x}$ using different examples • Encourage students to apply the above properties and simplify exponential expressions.	• Ask students to apply the laws of exponents in simplifying exponential expressions.
• Express what is meant by logarithmic expression by using the concept of exponential expression. • Solve simple logarithmic equation by using the properties of logarithm	**2.1.2 Logarithms** • Properties of logarithms	• Define the notion logarithm with the active participation of students that is $\log_a^b = c$ if and only if $a^c = b$. • Assist students to determine the logarithm of a given number to a given base by using different examples • State the properties of logarithms. i.e. (i) $\log_c ab = \log_c a + \log_c b$ (ii) $\log_c \frac{a}{b} = \log_c a - \log_c b$ (iii) $\log_c a^b = b\log_c a$ by using (iv) $\log_c c = 1$	• Ask students to change a given exponential expression to logarithmic expression. • Ask students to simplify logarithmic expression by using the properties. • Give exercise problems on

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
		(v) $a^{\log_a c} = c$ and encourage the students to justify these properties by giving their own examples.	finding logarithm of a number (whose logarithm is simple to determine)
• recognize the advantage of using logarithm to the base 10 in calculation • identify the "characteristics" and "mantissa" of a given common logarithm • use the table for finding logarithm of a given positive number and antilogarithm of a number. • Compute using logarithm	• Common logarithms	• Discuss with students the possibility of using different bases in computing with logarithms and facilitate for students to realize the advantage of using logarithms to the base 10. • After introducing what is meant by "common logarithm" use several examples in order to describe related terms such as "mantissa" and "characteristics" of a given logarithm. • Assist students to read the mantissa of a logarithm from the table of logarithm • After explaining what is meant by "Antilogarithm" of a number, guide students how the table is used in finding antilogarithm of a given number • Encourage students to evaluate (compute) expressions like: $(0.52)^8$ using common logarithm.	• Give exercise problems on determining the mantissa and characteristics of a given logarithm • Ask students to find the antilogarithm of a given number. • Let the students compute or evaluate the value of a mathematical expression like a) $\sqrt[5]{76.98}$ b) $(0.4873)^{\frac{2}{3}}$ by using logarithm
• Define an exponential function. • Draw the graph of a given exponential function.	**2.2 The exponential functions and their graphs** *(5 periods)* • Exponential functions	• Revise the notion "function" and types of functions with appropriate examples.	• Ask students to list the properties of an exponential function by examining its graph and check their answer/ feedback.
• Describe the graphical	• Graphs of exponential	• State the formal definition of an exponential function and discuss	

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
relationship of exponential functions having bases reciprocal to each other.	functions.	on the restriction of the base. (i.e. define an exponential function as $f(x) = a^x$ (where $a > 0$ and $a \neq 1$) • After demonstrating a sample graph, students should be encouraged and assisted in drawing and enlisting the properties of graphs of the functions. $f(x) = 2^x$ and $f(x) = (½)^x$ $y = 10^x$ and $y = (1/10)^x$ as representative of the functions. $y = f(x) = z^x$ ($z \in \mathbb{R}$, $a>0$ and $a \neq 1$)	
• describe the properties of an exponential function by using its graph.		• Assist the students to give different examples of exponential functions and then to discuss the respective properties of the functions by sketching the graph of each (i.e. domain, range, y - intercept, and the behavior of the graph based on the base). • After introducing the irrational number "**e**", (including its historical background) discuss the properties of the function f given by $f(x) = e^x$ and then sketch the graph. • encourage and assist students to apply the graph of $y = e^x$ to approximate the value of e^k where K is any real number.	• Give exercise problems from real life which involves the application of natural logarithm (e.g. multiplication or reproduction of bacteria)
• define a logarithmic function • draw the graph of a given logarithmic function	**2.3 The Logarithmic Functions and their Graphs** *(6 periods)* • Logarithmic Functions	• Revise the relationship of exponential equation and its corresponding logarithmic equation by using several examples.	• Give exercise problems on logarithmic function
• describe the properties of a	• Graphs of Logarithmic	• State the formal definition of a logarithmic function and discuss	• Ask students to some

Competencies	Contents	Teaching / Learning Activities and Resources	Assessment
logarithmic function by using its graph • describe the graphical relationship of logarithmic function having bases reciprocal to each other.	Functions	the restriction on the domain and the base (i.e., define a logarithmic function as $f(x) = \log_a x$ where $x > 0$, $a > 0$, $a \neq 1$) • After demonstrating a sample graph, assist and encourage your students to draw and enlist the properties of graphs of functions like $f(x) = \log_2 x$, $f(x) = \log_{½} x$, $y = \log_{10}$ x and $y = \log_{1/10}$ x as representative functions for $y = f(x) = \log_a x$ ($x > 0$, $a > 0$ and $a \neq 1$) • Assist the students to give different examples of logarithmic functions and also to discuss the respective properties (i.e., domain, range y - interest and the behaviour of the graph in relation to reciprocal bases and for $x \to \infty$ and $x \to 0$) • discuss the properties of the function given by f $(x) = \log_e x$ (or $y = \ln x$) using its graph.	properties of logarithmic functions by using (students my use graph) and analyse their feedback.
• describe how the domains and ranges of $y = a^x$ and $y = \log_a^x$ are related	• Relation between the functions • $y = a^x$ and $y = \log_a^x$ ($a > 0$ and $a \neq 1$)	• You may start the lesson by considering the functions $y = 2^x$ and $y = \log_2^x$ and assist the students to construct table of values for each of these functions and let them conclude what they observe from these tables (how the values of x and y are interchanged)	• Give exercise problems on the relation of these functions (exponential and logarithmic) having the same base, you can set the problems.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• explain the relationship of the graphs of $y = a^x$ and $y = \log_a^x$		• Assist students to sketch the graphs of the above functions on the same coordinate plane and ask them to write a statement on how the graphs are related (i.e. one of the graphs can be obtain by reflecting the other along the line $y = x$) • Help the students in solving several exercises on sketching graphs of exponential and logarithmic functions (pair wise) having the same bases on the same coordinate plane and guide the students to realize that the relationship of these functions hold true for any pair of functions given by $y = a^x$ and $y = \log_a^x$ $(a > 0, a \neq 1)$.	
• solve equations involving exponents • solve equation involving logarithms	**2.4 Equations involving exponents and logarithms** (7 *periods)* • Exponential equations • Logarithmic equations	• You may start the lesson by revising the properties of exponents, which were discussed in the first section of this chapter and introduce the property "if $\mathbf{b > 0, b \neq 1}$ and m and n are real numbers, then $\mathbf{b^n = b^m}$ if and only if $\mathbf{n = m}$ "	

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
                       • Solve problems, involving exponential and logarithmic functions, from real life.	 **2.5 Application of Exponential and Logarithmic functions** *(6 periods)*	• Consider simple exponential equation and discuss the technique of solving it you may take examples like: e.g solve for x, if (a) $e^x = 81$ (b) = $(2/3)^x = 3/2$ So/n :- (a) $3^x = 81$ $3^x = 3^4$ ----- $(81 = 3^4)$ $\therefore\ x = 5$ ----- (if $b^n = b^m$, then $n = m$) (b) $(2/3)^x = (3/2)$ $(2/3)^x = (2/3)^{-1}$ -------- $[\,3/2 = (2/3)^{-1}]$ $\therefore\ x = -1$ • After revising the properties of logarithms, consider some simple logarithmic equations and discuss the techniques in solving it you may take examples like: e.g. $\log 5 + \log x = 1$ • Assist and encourage students to solve problems on practical applications of exponential and logarithmic function from different fields such as population growth, compound interest, etc.	 • Give several exercise problems and analyse the feed back (the problems can be taken from appropriate field of studies)

Unit 3: Solving Inequalities *(20 periods)*

Unit outcomes: Students will be able to:

- know and apply methods and procedures in solving problems on inequalities involving absolute value.
- know and apply methods in solving system of linear inequalities
- apply different techniques of solving quadratic inequalities.

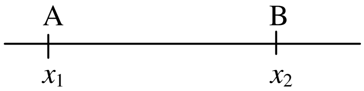
Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • describe sets using interval notation. • Solve inequalities involving absolute value of linear expression	***3. Solving Inequalities*** **3.1 Inequalities involving absolute values** *(4 periods)*	• Revise different ways of describing sets that the students had learnt in Grade 8 and introduce the interval notation (i.e. open, half closed and closed intervals) using several examples. • You may start the lesson by revising the definition of absolute value and how to solve equation involving absolute value of linear expression like $\lvert 3x - 1 \rvert = 5$ • Assist students to solve inequalities of the form $\lvert ax + b \rvert \le c$, $\lvert ax + b \rvert \ge c$, $\lvert ax + b \rvert < c$ and $\lvert ax + b \rvert > c$ where c is a non-negative real number by giving them several examples and exercises.	• Ask students to solve inequalities involving absolute value of different linear expression and observe their feed back • Let the students give their answer in interval notation
• Solve system of linear inequalities in two variables by using graphical method	**3.2 Systems of linear inequalities in two variables** *(5 periods)*	• Let the students revise sketching of the graphs of relations like R = $\{(x, y) : y \le x + 1$ and $y > 1 - x\}$ • Guide students to find the solution for system of linear inequalities graphically. You may consider examples like: **e.g** Solve the following system of inequalities. $\begin{cases} y + x > 0 \\ y - x \le 1 \\ x \le 2 \end{cases}$ **Note:** The system should not have more than three linear inequalities	• Give different exercises problems on system of linear inequalities and check students work (let students to give their answers in interval notation as well)

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• Solve quadratic inequalities by using product properties	**3.3 Quadratic Inequalities** *(11 periods)* • Solving quadratic inequalities using the product properties	• You can start the lesson by defining quadratic inequalities where the quadratic expression is given in its general form and as a product of two linear expressions, i.e. (i) $ax^2 + bx + c \geq 0$ (ii) $(ax + b)\ (x - c) \leq 0$ • Discuss the product property i.e. (i) M.N > 0 means M > 0 and N > 0 or M < 0 and N < 0 (ii) M.N < 0 means M > 0 and N < 0 or M < 0 and N > 0 • encourage your students to apply the above product property in solving quadratic inequalities.	• Ask students to solve different quadratic inequalities using product property and check whether they use (apply) this property correctly.
• Solve quadratic inequalities using the sign chart method. • Solve quadratic inequalities using graphs	• Solving quadratic inequalities using the sign chart method • Solving quadratic inequalities graphically	• Show the students how the sign chart can be prepared and discuss its application by considering simple examples such as e.g $(x + 3)\ (x - 2) > 0$ e.g. $x^2 + 1 > 0$ $x^2 + 3x + 5 < 0$	• Give several exercise problems on the application of the two methods discussed in the lesson.

Unit 4: Coordinate Geometry *(15 Periods)*

Unit outcomes: Students will be able to:

- apply distance formula to find distance between any two given points in the coordinate plane.
- formulate and apply section formula to find a point that divides a given line segment in a given ratio
- write different forms of equations of a line and understand related terms.
- describe parallel or perpendicular lines in terms of their slopes
- write equation of a circle

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
Students will be able to: • derive the distance formula (to find distance between two points in the coordinate plane) • Apply the distance formula to solve related problems in the coordinates plane	***4. Coordinate Geometry*** **4.1 Distance between two points** *(2 periods)*	• You can start the lesson by discussing how to find the distance between two points on a number line.  i.e. AB = $\lvert x_2 - x_1 \rvert$ gives the distance between points A and B. • encourage students to derive and apply the distance formula which is used to find distance between any two points in the coordinate plane.	• Give exercise problems on computing distance between two points, i.e., by applying distance formula.
• determine the coordinates of points that divide a given line segment in a given ratio.	**4.2 Division of a line segment** *(2 periods)*	• You may start the lesson by considering a horizontal line segment with end points A(x_1, y) and B (x_2, y) and discuss how to determine the coordinate of a point P that divides AB in a given ration m:n internally • Write the statement about a point "P(x, y)" that divides any given line segment (say AB in the ratio m:n and derive the "section formula". • Let students infer that the "mid point formula" is a special case of the section formula.	• Give several exercises and problems on the application of section formula. • Ask students to justify that "mid point formula" is a special case of section formula.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• define the gradient of a given line • determine the gradient of a given line (given two points on the line) • express the slope of a line in terms of the angle formed by the line and the x-axis	**4.3 Equation of a line** *(8 periods)* • Gradient (slope) of a line • Slope of a line in terms of angle of inclination	• Assist students to practice the section formula in finding the coordinates of a point that divides a given line segment in a given ratio by using several examples (exercises) • You may start the lesson by defining what is meant by "Gradient (slope) of a line", and discuss how the slope of a line is expressed in terms of the tangent of a positive angle,θ, formed by the line and the x-axis (where θ is measured from the x-axis) • By considering arbitrary line on the coordinates, plane and by active participation of the students derive and state the formula for slope of the line.	• Ask students to find the slope of a line by taking several pairs of points from it and ask what they can infer form their work. • Ask students how to determine whether a given point is on a given line or not.
• determine the equation of a given line	• Different forms of Equations of lines	• Assist students to conclude that the slope a line does not depend on the choice of coordinates of points on the line (i.e., the uniqueness of the slope). • Guide and encourage students to determine slopes of different lines and let them observe the nature of the line in relation to the slope i.e. 1) if $m \in \mathbb{R}$and • $m > 0$, then the line rises from left to right • $m < 0$, then the line goes down ward from left to right • $m = 0$, then the line is horizontal	• Give several exercises and problems on equations of a line (such as changing one form of the equations in the other form)

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
		2) a vertical line has no slope • Discuss on the different forms of equations of a line, i.e., two point form, slope intercept form and point slope forms of equations. • Assist students to practice writing these forms of equations of a line through sufficient exercises.	
• identify whether to lines are parallel or not.	**4.4 Parallel and perpendicular lines** *(3 periods)* • Slopes of parallel and perpendicular lines.	• You can start the lesson by taking two parallel lines on the coordinates plane and writing their equations, discuss with students how their slopes are related.	• Give exercise problems on determining whether a given pair of lines are parallel or perpendicular or neither of them.
• identify whether two lines are perpendicular or not. • apply the properties of the slopes of parallel and perpendicular lines to solve related problems		• As suggested above you may do the same thing for perpendicular lines. • State and prove theorems on the slopes of parallel and perpendicular lines. • Allow students to solve problems involving the properties of the slopes of parallel and perpendicular lines.	• Ask students to state and prove the converse of the two theorems.

Unit 5: Trigonometric Functions *(30 periods)*

Unit outcomes: Students will be able to:

- know principles and methods in sketching graphs of basic trigonometric functions.
- understand important facts about reciprocals of basic trigonometric functions.
- identify trigonometric identities
- solve real life problems involving trigonometric functions.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • define the sine, cosine and tangent functions of an angle in the standard position. • determine the values of the functions for an angle in the standard position, given the terminal side of that angle. • determine the values of the sine, cosine and tangent functions for quadrantal angles	***5. Trigonometric Functions*** **5.1 Basic trigonometric functions** *(15 periods)* **5.1.1 The sine, cosine and tangent functions** • In standard position • Using the unit circle • Trigonometric values of positive and negative angles	• You can start the lesson by defining the sine, cosine and tangent functions by considering an angle θ in the standard position and a point P(x, y) on its terminal side on the coordinate plane, in doing so, remind them the ratios they had seen in grade 8, so that they can relate those ratios with the definitions given here. • Using some examples let the students evaluate the values of the functions for a given angle θ in the standard position and with its terminal side containing a certain coordinate, you may take examples like: e.g Given the terminal side of θ containing (3,4) then determine sin θ, cos θ and tan θ. • Guide students to contract a unit circle and using it let them determine the values of the sine, cosine and tangent functions for quadrantal angles (i.e., 0°, 90°, 180^0, 270°, 360°) and also assist them to approximate the values of these functions for special angles, i.e. 30°, 45° and 60°.	• Ask students to use the definition given in grade 8 and describe the sine, cosine and tangent of an angle, θ, given in standard position • Give exercise problems on evaluating (or approximating) the triogonometrical values of angles by using the unit circle on the coordinates plane and check their answers.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
• locate negative and positive angles by identifying the direction of rotation		• Discuss with your students how to locate negative angles in relation to the corresponding positive angles.	
• determine the values of trigonometric functions for some negative angles.		• Allow students to determine the trignometrical values of -30°, -45°, -60°, -90°, -180°, -270° and -360° • By using the unit circle guide the students to reach on cos (-θ) = cos θ, sin(-θ) = -sin θ and tan(-θ) = -tan θ.	• Ask students to give their own conclusion about the trigonometrical values of angles having equal measures but opposite in signs.
• determine the algebraic signs of the sine, cosine and tangent functions of angles in different quadrants.	**5.1.2 Values of Trigonometric functions for related angles** • Algebraic signs of sine, cosine and tangent	• You may start the lesson by considering different angles in each quadrant, and helping the students to determine the trigonometircal values of these angles based on the definition. • Based on the results obtained in the above activities, allow the students to put the algebraic sign for value of each function in the respective quadrants (i.e. in quadrants I, II, III and IV).	• Give enough exercise problems to students and after checking their work ask them to write a summary about the algebraic signs of the values of the trigonometric functions in each quadrant.
• describe the relationship between trigonometrical values of complementary angles.	• Complementary angles.	• After a brief revision of complementary angles let the students discuss on the relationship between the trigonometrical values of complementary angles, first by considering 30^0 and 60^0 angles and then by taking any pair of complementary angles.	• Give several exercise problems on the application of the relation between the trigonometrical values of complementary angles as well as supplementary angles.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		(**Note**: In this case they can use, table of trigonometrical values). ▪ Using the above activities guide students to generalize, that "if θ and β are complementary angles, then sin θ = cos β, cos θ = sin β and tan θ = $\frac{1}{\tan \beta}$	
• describe the relationship between trignonometrical values of supplementary angles.	• Supplementary angles	• After a brief revision of supplementary angles let the students discuss on the relationship between the trigonometrical values of supplementary angles and then encourage them to conclude how the trigonometrical values of supplementary angles are related. (i.e. $\sin\theta = \sin(180° - \theta)$, $\cos \theta = -\cos(180° - \theta)$ and $\tan \theta = -\tan(180° - \theta)$). • You may start the lesson by defining what is meant by "coterminal angles" and elaborate the definition by taking different examples	• Give enough exercise problems on finding trigonometrical values of co-terminal angles as well as of large angles.
• determine the relationship between trigonometrical values of coterminal angles. • determine the trigonometrical values of large angles	• Co-terminal angles • Large angles	• By considering different pairs of coterminal angles whose measures are between -360^0 and 360^0 assist students to generalize the relationship between their trigonometrical values. • Let the students observe that trigonometrical values for angles larger than 360^0 (2π rad.) can be obtained from trigonometrical values of their coterminal acute angles.	

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• construct a table of values for y = sin θ where -2π ≤ θ ≤ 2π. • draw the graph of y = sin θ • determine the domain range and period of the sine function.	**5.1.3 Graphs of the Sine, Cosine and Tangent functions** • The Graph of sine function; y = sin θ	• You can start the lesson by guiding the students to construct a table of values for y = sin θ where -2π ≤ θ ≤ 2π in doing so you may use the unit circle to read values for quadrantal and/or special angles. • Assist students to sketch the graph of *y* = sin θ where -2π ≤ θ ≤ 2π and let them use this graph to determine the period of the sine function and encourage them to extend the graph in both direction based on the period of the function. • With the help of the graph encourage students to determine the domain and range of the sine function	• Ask students to list some properties of the sine function by observing the graph they draw, such as - its domain and range - its continuity in the domain - its usage in reading values of angles discussed in section 5.1.3 above
• Construct a table of values for y = cos θ, where -2π ≤ θ≤ 2π. • draw the graph of y = cos θ • determine the domain, range and period of the cosine function.	• The Graph of Cosine function y = cos θ.	• All students to repeat the steps followed earlier in sketching y = sin θ to sketch the graph of y = cos θ and assist them to use this graph in determining the period of cosine function. At the end encourage them to extend the graph in both direction based on the period of the function. • With the help of the graph of *y* = cos θ, encourage the students to determine the domain and range of the cosine function.	• Ask students to list some important facts from the graph about the cosine function such as: - its domain and range - its continuity in its domain - its use in finding values for angles discussed in sections 5.1.3 e.g. cos θ and cos(-θ) • Give other exercise problems

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• Construct a table of values for $y = \tan\theta$ where $-2\pi \le \theta \le 2\pi$. • draw the graph the tangent function $y = \tan\theta$. • determine the domain, range and period of the tangent function. • discuss the behavior of the graph of tangent function.	• The Graph of Tangent function, $y = \tan\theta$	• You may start the lesson by guiding the students to construct a table of values for $y = \tan\theta$, where $-2\pi \le \theta \le 2\pi$ in doing so give emphasis to values of θ for which the function is undefined (i.e., for $\frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}$ and $\frac{3\pi}{2}$ in the interval from -2π to 2π) • Assist students to sketch the graph of $y = \tan\theta$ where $-2\pi \le \theta \le 2\pi$ using the table they construct and let them use this graph to determine the period of the tangent function. • Based on the period encourage the students to extend the graph of $y = \tan\theta$ in both direction. • Guide the students to determine the domain and range of the tangent function (to do so you may use the graph of the function) and let them discuss the behaviour of the function using its graph.	• Ask students to give: - the domain and range of $y = \tan\theta$ - where this function is undefined or for what value of θ does the graph discontinuous (Let the students justify their answer either using the unit circle or the ratio) • Give some more exercise problems and check their answers
• define the cosecant function • determine the values of cosecant function for some angles.	**5.2 The reciprocals of the basic trigonometrical functions** *(7 periods)* • The cosecant function. $y = \csc\theta$	• You may start the lesson by defining the cosecant function, i.e. using the relationship between sine of θ and cosecant of θ and introduce the notation "$\csc\theta$" • Discuss with your students about the values of θ for which $\csc\theta$ is undefined based on the definition.	• Ask students to restate the definition for each of the reciprocal of the basic trigonometric functions.

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
		• Assist students in determining the values of cosecant function for some angles.	
• define the secant function. • determine the values of secant function for some angles.	• The secant function y = sec θ	• You can start the lesson by defining the secant function using the relationship between cos θ and second of θ for the same value of θ after introducing the notation sec θ. • Assist students in evaluating the values of secant function for some angles, in doing so don't forget give emphasis to angles where *y* = sec is θ undefined.	• Ask students to explain the properties of each reciprocal of basic trigonometric functions.
• define the cotangent function • determine the values of cotangent function for some angles.	• The cotangent function. y = cot θ	• After defining the cotangent function using the tangent function introduce the notation "cot θ". • Guide the students in determining the values of cotangent function for some angles and encourage them in differentiating angles whose cotangent is undefined.	• Make students in group to draw rough sketch of each reciprocal trigonometric functions.
• explain the concept of co-functions.	• Co-functions	• By using the idea of reciprocal of trigonometric functions and trigonometrical values of complementary angles introduce the concept of co-function. i.e. $\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta$ $\sec\left(\frac{\pi}{2}-\theta\right)=\csc\theta$ $\tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta$	• Ask students about the functional relationship of the co-functions by giving their own examples

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
• derive some of the trignometric identities. • Identify the quotient identities • Solve problems related to trigonometrical identities.	**5.3 Simple trigonometrical identities.** *(3 periods)* • Quotient identities	• Assist students to derive trigonometrical identify $\sin^2\theta + \cos^2\theta = 1$ using unit circle and then encourage them to drive $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \csc^2\theta$ from $\sin^2\theta + \cos^2\theta = 1$. • Let students differentiate trigonometrical identity from trigonometrical equation by using different examples. • With active participation of the students discuss about the quotient identities, namely. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	• Give exercise problems on the application of the trigonometric identities in finding value of one of the function while the other is known (given) and check students' work.
• Solve real life problems using trigonometircal ratios	**5.4 Real life application problems** *(5 periods)* • Angle of elevation and angle of depression	• Allow students to apply trigonometrical ratios and trigonometrical identities in solving related problems. • You can start the lesson by reminding the students about the properties of right angled triangles and the trigonometric ratios on right angled triangles and allow students to practice solving right angled triangles through different exercises. • Introduce the concepts "angle of elevation" and "angle of depression" and give some practical examples on applying these concepts through solving right angled triangles.	Give exercise problems on the application of trigonometrical ratios in solving real life problems and check students' work.

Unit 6: Plane Geometry (*22 periods)*

Unit outcomes: Students will be able to:

- know more theorems special to triangles.
- Know basic theorems specific to quadrilaterals
- Know theorems on circles and angles inside, on and out side a circle
- Solve geometrical problems on quadrilaterals, circles and regular polygons

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
Students will be able to: • apply the incidence theorems to solve related problems.	***6. Plane Geometry*** **6.1 Theorems on Triangles** *(5 periods)*	• You may start the lesson by revising concepts about triangles that the students had learnt in previous grades (concepts like: angle bisector, bisector and perpendicular bisector of side of a triangle, altitude of a triangle and other) • State and verify by construction theorems on concurrency of altitudes, medians, angle bisectors and perpendicular bisectors of sides of a triangle, the Altitude Theorem and the Menelaus' Theorem with active participation of students. • Assist students to practice the application of these theorems.	• You can ask to apply coordinate geometry to find the point of concurrency and check their answers. • Give exercise problems on verification of the theorems on concurrency geometrically.
• Apply theorems on special quadrilateral in solving related problems	**6.2 Special quadrilateral** *(6 periods)* • Theorem on special quadrilaterals (Trapezium parallelogram , rectangle, rhombus and square)	• You may start the lesson by revising the definitions and some properties of special quadrilaterals (Trapezium, parallelogram, rectangle, Rhombus and square) • State and prove (by construction and measurement) theorem on these special quadrilaterals with active participation of students. • Help students to practice the application of these theorems through examples and exercises.	• Give exercise problems on construction to prove the theorems and check their answers
• Apply the theorems on angles and arcs determined by	**6.3 More on Circles** *(6 periods)* • Theorems on angles and	• You may start the lesson by discussing about angles and arcs that are formed by two intersecting chords, by two intersecting secants, by two	• Give exercise problems on proving the theorems and their

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
lines intersecting inside, on and outside a circle to solve related problems	arcs determined by lines intersecting inside, on and outside a circle.	intersecting tangents and by intersecting secants and tangent. • State and prove theorems on angles and arcs determined by lines intersecting inside, on and outside a circle with active participation of the students. • Help students to practice the application of these theorem through examples and exercises.	application in find the degree measures of angles and arcs and check their work.
• Calculate the perimeters of regular polygons	**6.4 Regular Polygons** *(5 periods)*	• Assist the students to drive the formula $S = 2r\sin\left(\frac{180^{\circ}}{n}\right)$	▪ Give exercise problems on the application of the formulae.
	• Perimeter	for side(s) of regular polygon with radius **r** and number of side **n** by reducing the problem back to right angled triangles. Then guide them to drive the formula for perimeter (P) of n sided regular polygon as: $P = ns$ or $P = 2nr\sin\left(\frac{180^{\circ}}{n}\right)$ • Motivating and assisting students in solving practicable problems demanding the application of these formulas.	▪ Ask students to find their own ways of computed either the side or the perimeters of a given regular polygon with out using the formulas and encourage them to do this.
• Calculate the areas of regular polygons	• Area	• First assist the students to drive the formula A = $ab\sin\hat{C}$ for the area of a triangle by dividing (resolving) the given triangle into two right angled triangles, and expressing the height with the trignometric ratios. Then	• Ask students to compute by themselves the area of a give regular polygon by using their previous knowledge about areas of different plane figures.

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		• Guide the students in driving the area formula of a regular polygon by using the area formula of a triangle derived above. That is: $A = \frac{1}{2} nr^2 \sin\left(\frac{360^{\circ}}{n}\right)$	
		• Assisting students in applying the formula $A = \frac{1}{2} nr^2 \sin\left(\frac{360^{\circ}}{n}\right)$ of a regular polygon in solving related problems.	• Give exercise problems on the application of the formulas and check the works of the students.

Unit 7: Measurement *(25 periods)*

Unit outcomes: Students will be able to:

- solve problems involving surface area and volume
- know basic facts about frustums of cones and pyramids

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
Students will be able to: • apply the formulae for calculating surface area and volume of prism and cylinder	***7. Measurement*** **7.1 Revision on surface area and volume of prisms and cylinders.** *(3 periods)*	• You may begin the lesson with a brief revision on what the students had learnt about prism and cylinder in earlier grades and then state the formulae for surface area of prism and cylinder. • Assist students to apply the formulae in order to calculate the surface area of prism and cylinder. • After discussing the formulae for calculating the volume of prism and cylinder, assist students to apply these formulae to find the volume of prism and cylinder.	• Ask students oral question about what they can remember from their earlier grades. • Give students exercise problems on the application of the formulae and check their work.
• Calculate surface areas of a given pyramid or a cone • calculate the volumes of a given pyramid or a cone.	**7.2 Pyramids, Cones and Sphere** *(8 periods)* ▪ Pyramids and cones	• You may start the lesson by revising important points about pyramids and cones that the students had learnt in earlier grade, in doing so, it is better to use models of the solid figures. • By using several examples discuss with students the derivation of the formulae that is used to calculate the surface areas of pyramids and cones and assist students to use these formulae in order to find surface areas of the respective solid figures.	• Give students exercise problems on the application of the formulae for calculating surface area and volumes of pyramids and cones. • After analyzing their feed back or check their work try to give more concept building exercises by reducing that requires more computation

Competencies	*Contents*	*Teaching / Learning Activities and Resources*	*Assessment*
		• Similarly discuss with the students the derivation of the formulae for calculating volumes of pyramids and cones encourage students to apply these formulae to calculate the volumes of these solid figures.	
• Calculate the surface area of a given sphere • Calculate the volume of a given sphere	• Spheres	• By using the model of sphere revise main ideas about sphere that the students had learnt in the earlier grades. • After stating the formulae for calculating surface area and volume of spheres encourage the students to calculate the surface area and volume of a given sphere by using the respective formulae.	• Ask oral question about sphere that the students can remember from their earlier grades. • Give exercise problems on the application of the formulae.
• define frustums of a pyramid and of a cone. • calculate the surface areas of frustums of pyramids of cones. • calculate the volumes of pyramids or of cones.	**7.3 Frustums of pyramids and cones** *(7 periods)*	• You may start the lesson by discussing how frustums of pyramids and cones are generated and then define what is meant by "frustum of a pyramid and frustum of a cone" and use model of frustum. • Discuss with students the derivation of the formulae for calculating the surface areas of frustum of pyramids and frustum of cones and then state the formulae as a result of the discussion. • Encourage the students to apply these formulae to calculate surface areas of a given frustum (either of a pyramid or of a cone).	• Ask students some oral questions on important ideas during the class discussion and analyse their feedback. • Give exercise problems on the application the formulae for surface area and volume of

Competencies	***Contents***	***Teaching / Learning Activities and Resources***	***Assessment***
		• You may follow the same steps, as you did for formulae of surface areas of frustums, to derive the formulae for volumes of frustums of pyramids and of cones. • State the formulae for the volume of frustums as a result of the discussion and encourage students to carry out calculation on volumes of frustums (either of pyramids or of cones) in doing so don't forget to relate the exercises with real life problems.	
• determine the surface area of simple composed solids. • Calculate volumes of simple composed solids	**7.4 Surface areas and volumes of composed solids** *(7 periods)*	• You may start the lesson by introducing solid figures formed by two solids having different shapes. • Assist students to compute the surface areas and volumes of some simple composed solid figures, in doing so it is better to produce a model of some of these solids.	• Ask students to mention some composed solid figures from their environment. • Give students exercise problems on the computation of surface area and volume of different composed solid figures and check their work. • Let the students use their own ways of computation and ask why they use the method.

MINIMUM LEARNING COMPETENCIES (MLCs)

No	Content	Minimum Learning Competencies (MLCs)
1	**ALGEBRA** Solving Equations and Inequalities	• describe sets using interval notation • solve inequalities involving absolute value of linear expression • solve system of linear inequalities in two variables by using graphical method • solve quadratic inequalities by using product properties • solve quadratic inequalities using the sign chart method • solve quadratic inequalities using graphs
2	**RELATIONS AND FUNCTIONS**	• define the polynomial function of one variable • identify the degree, leading coefficient and constant terms of a given polynomial functions • give different forms of polynomial functions • perform the four fundamental operation on polynomials • state and apply the Polynomial Division Theorem • state and apply the Factor Theorem • determine the zero(s) of a given polynomial function • state and apply the Location Theorem to approximate the zero(s) of a given polynomial function • apply the rational root test to determine the zero(s) of a given polynomial function • sketch the graph of a given polynomial function • describe the properties of the graphs of a given polynomial function • explain what is meant by exponential expression • state and apply the properties of exponents (where the exponents are real numbers) • express what is meant by logarithmic expression by using the concept of exponential expression • solve simple logarithmic equation by using the properties of logarithm • recognize the advantage of using logarithm to the base 10 in calculation • identify the "characteristics" and "mantissa" of a given common logarithm • use the table for finding logarithm of a given positive number and antilogarithm of a number • compute using logarithm • define an exponential function. • draw the graph of a given exponential function • describe the graphical relationship of exponential functions having bases reciprocal to each other • describe the properties of an exponential function by using its graph

		• define a logarithmic function • draw the graph of a given logarithmic function • describe the properties of a logarithmic function by using its graph • describe the graphical relationship of logarithmic function having bases reciprocal to each other • describe how the domains and ranges of $y = a^x$ and $y = \log_a{}^x$ are related • explain the relationship of the graphs of $y = a^x$ and $y = \log_a{}^x$ • solve equations involving exponents logarithms as well • Solve problems, involving exponential and logarithmic functions, from real life • define the sine, cosine and tangent functions of an angle in the standard position • determine the values of the functions for an angle in the standard position, given the terminal side of that angle • determine the values of the sine, cosine and tangent functions for quadrantal angles • locate negative and positive angles • determine the values of trigonometric functions for some negative angles • determine the algebraic signs of the sine, cosine and tangent functions of angles in different quadrants • describe the relationship between trigonometrical values of complementary angles • describe the relationship between trignonometrical values of supplementary angles • determine the relationship between trigonometrical values of co-terminal angles • determine the trigonometrical values of large angles • construct a table of values for $y = \sin\theta$ where $-2\pi \leq \theta \leq 2\pi$ • draw the graph of $y = \sin\theta$ • determine the domain range and period of the sine function. • construct a table of values for $y = \cos\theta$, where $-2\pi \leq \theta \leq 2\pi$ • draw the graph of $y = \cos\theta$ • determine the domain, range and period of the cosine function. • construct a table of values for $y = \tan\theta$ where $-2\pi \leq \theta \leq 2\pi$ • draw the graph the tangent function $y = \tan\theta$ • determine the domain, range and period of the tangent function • discuss the behavior of the graph of tangent function • define the cosecant function • determine the values of cosecant function for some angles • define the secant function • determine the values of secant function for some angles • define the cotangent function

3		• determine the values of cotangent function for some angles • explain the concept of co-functions • derive some of the trignometric identities • identify the quotient identities • solve problems related to trigonometrical identities • solve real life problems using trigonometircal ratios
3	**PLANE GEOMETRY AND MEASUREMENT**	• derive the distance formula (to find distance between two points in the coordinate plane) • apply the distance formula to solve related problems in the coordinates plane • determine the coordinates of points that divide a given line segment in a given ratio • define the gradient of a given line • determine the gradient of a given line (given two points on the line) • determine the equation of a given line • identify whether two lines are parallel or not • identify whether two lines are perpendicular or not • apply the properties of the slopes of parallel and perpendicular lines to solve related problems • apply the incidence theorems to solve related problems • apply theorems on special quadrilateral in solving related problems • apply the theorems on angles and arcs determined by lines intersecting inside, on and outside a circle to solve related problems • calculate the perimeters of regular polygons • calculate the areas of regular polygons • apply the formulae for calculating surface area and volume of prism and cylinder • calculate surface areas of a given pyramid or a cone • calculate the volumes of a given pyramid or a cone • calculate the surface area of a given sphere • calculate the volume of a given sphere • define frustums of a pyramid and of a cone • calculate the surface areas of frustums of pyramids of cones • calculate the volumes of pyramids or of cones. • determine the surface area of simple composed solids • calculate volumes of simple composed solids