Logic

- **Negation** of *p*:

  'not'

p	$\neg p$
T	F
F	T

- **Conjunction** of *p* and *q*:

  'and'

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Disjunction** of *p* and *q*:

  'or'

P	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- **Exclusive or** (XOR):

  $p \oplus q$

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- **Implication**: $p \Rightarrow q$

  "if p then q", "q if p"

  Contrapositive of $p \Rightarrow q$: $\neg q \Rightarrow \neg p$

  logically equivalent

  Converse of $p \Rightarrow q$: $q \Rightarrow p$

  not equivalent

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Biconditional**: $p \Leftrightarrow q$

  "p if, and only if q"

  $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$

p	q	$p \Leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- **Tautology**: a compound proposition that is always true.

  e.g. $p \vee \neg p$ $\quad$ $False \Rightarrow p$

- **Contradiction**: a compound proposition that is always false.

  e.g. $p \wedge \neg p$

- Logical equivalence

  $p \equiv q$ when p and q have the same truth table

- De Morgan's Laws

  $\neg(p \vee q) \equiv \neg p \wedge \neg q$

  $\neg(p \wedge q) \equiv \neg p \vee \neg q$

- Other logical equivalences

- Show/prove logical equivalence
  - Compare truth tables
  - Use known logical equivalences

Commutative	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity	$p \wedge T \equiv p$	$p \vee F \equiv p$
Negation	$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$
Idempotent	$p \wedge p \equiv p$	$p \vee p \equiv p$
De Morgan	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Double negation	$\neg(\neg p) \equiv p$	

Sets

- Definitions

  $x \in S, x \notin S,$

  $A \subseteq B, A \subset B, A = B$

  $|S|$, $\{\}$ or $\emptyset$, $\mathbb{U}$

- How to write sets
  - Enumeration
  - Set builder notation

- Special sets

  $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and their relationships

- Operations on sets

  Union of A and B $\quad A \cup B$

  elements are either in A or in B or in both

  Intersection of A and B $\quad A \cap B$

  elements are in both A and B

  Difference of A minus B $\quad A - B$

  elements in A but not in B

  Complement of A $\quad \overline{A} = \mathbb{U} - A$

  Power set of S: $\mathcal{P}(S)$

  elements are subsets of S

  $\{\, subset\ with\ size\ 0,\ subsets\ with\ size\ 1,$
  $subsets\ with\ size\ 2, \ldots, subset\ with\ size\ |S|\,\}$

  Cartesian Product of A and B $\quad A \times B$

  $\{\,(a, b) \colon a \in A\ and\ b \in B\}$

- A partition of a set A

  $$P = \{A_1, A_2, \cdots, A_n\}$$

  satisfying three conditions

- Inclusion-exclusion principle

  For two finite sets:

  $$|A \cup B| = |A| + |B| - |A \cap B|$$

  For three finite sets:

  $$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| \\ -|B \cap C| + |A \cap B \cap C|$$

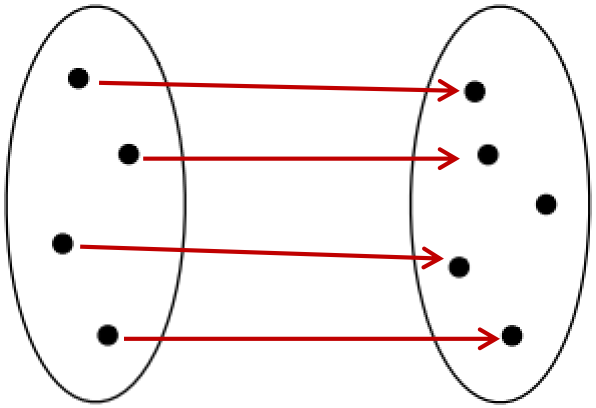
- Venn diagram

- Set properties

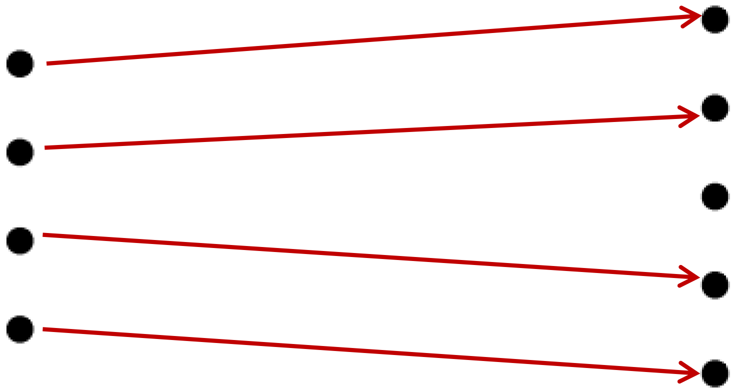
Commutative	$A \cap B = B \cap A$	$A \cup B = B \cup A$
Associative	$(A \cap B) \cap C = A \cap (B \cap C)$	$(A \cup B) \cup C = A \cup (B \cup C)$
Distributive	$A \cap (B \cup C)$ $= (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C)$ $= (A \cup B) \cap (A \cup C)$
Identity	$A \cap \mathbb{U} = A$	$A \cup \emptyset = A$
Negation	$A \cup \overline{A} = \mathbb{U}$	$A \cap \overline{A} = \emptyset$
Idempotent	$A \cap A = A$	$A \cup A = A$
De Morgan	$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$	$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
Complement	$\overline{\mathbb{U}} = \emptyset$	$A - B = A \cap \overline{B}$

- Functions:

A mapping from A to B, satisfying that each element in A maps to one element in B

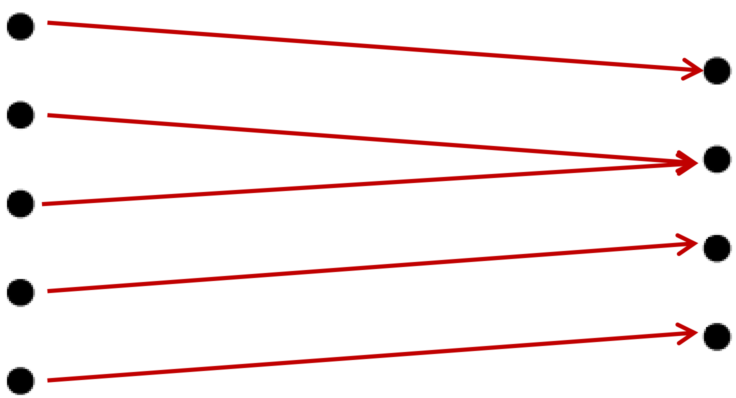


- Injective function/injection



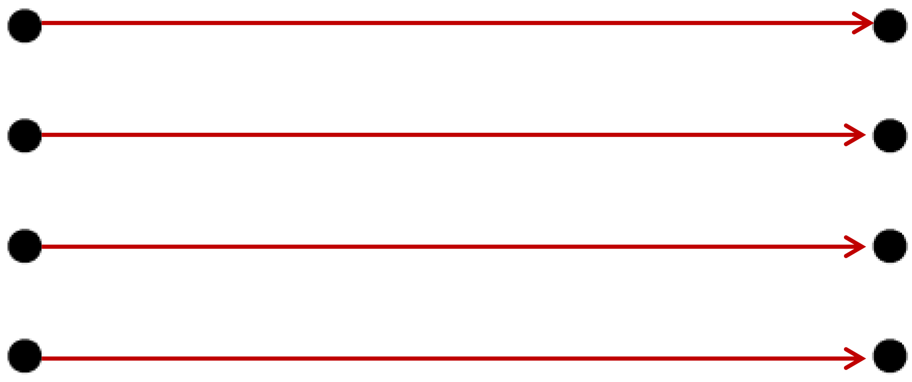
$|A| \leq |B|$

- Surjective function/surjection



$|A| \geq |B|$

- Bijective function: both injective and surjective



$|A| = |B|$

- |A|=|B| if and only if there is a bijection *f*: A->B
- $|\mathbb{Q}|=|\mathbb{Z}|=|\mathbb{N}|=\aleph_0$ ⇒ The smallest cardinality that an infinite set can have
- $|\mathbb{R}|>|\mathbb{N}|$
- A set is countable when $|S| \leq |\mathbb{N}|$

Counting

- What are we counting?

	r-permutations	r-combinations
Without repetition	$P(n,r) = \frac{n!}{(n-r)!}$	$C(n,r) = \frac{n!}{r!\,(n-r)!}$
With repetition	n^r	C(n+r-1,r)

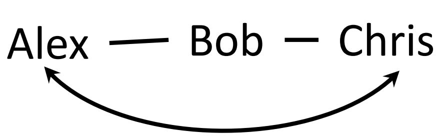
Permutations with indistinguishable objects
$\frac{n!}{n_1!\,n_2!\cdots n_k!},\qquad n = n_1 + n_2 + \cdots + n_k$

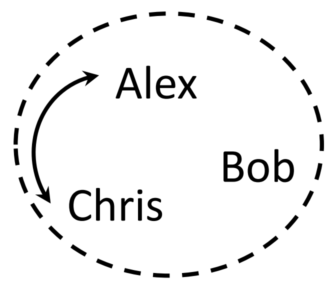
- When does order matter?

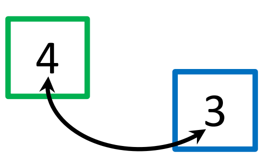
Different positions:	Line-up of cricket player	Selection of football players
Different colours:	Roll two dice with different colours	Roll two dice with the same colour
Different time:	Toss one coin twice	Toss two coins at the same time

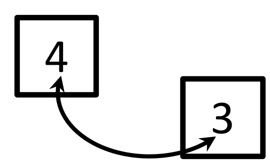
Imagine a result, and then swap two objects in the outcome.

Is the result different?









- Counting rules
  - Solve problem step by step

    The product rule:

    $|A_1 \times A_2 \times \ldots \times A_n| = |A_1| \times |A_2| \times \ldots \times |A_n|$
  - Divide a problem into different cases

    The sums rule:

    If $\{S_1, S_2, \ldots, S_n\}$ is a partition of finite set S, then,

    $$|S| = |S_1| + |S_2| + \ldots + |S_n|$$
- The itertools module in Python

  https://docs.python.org/3/library/itertools.html

Probability

- Probability $$P(E) = \frac{|E|}{|S|}$$

S: sample space with equally likely outcomes

E: event, a subset of S

Example: Flip two coins.

How many possible outcomes?

{H,H}, {H,T}, {T,T}

The outcomes are not equally likely!

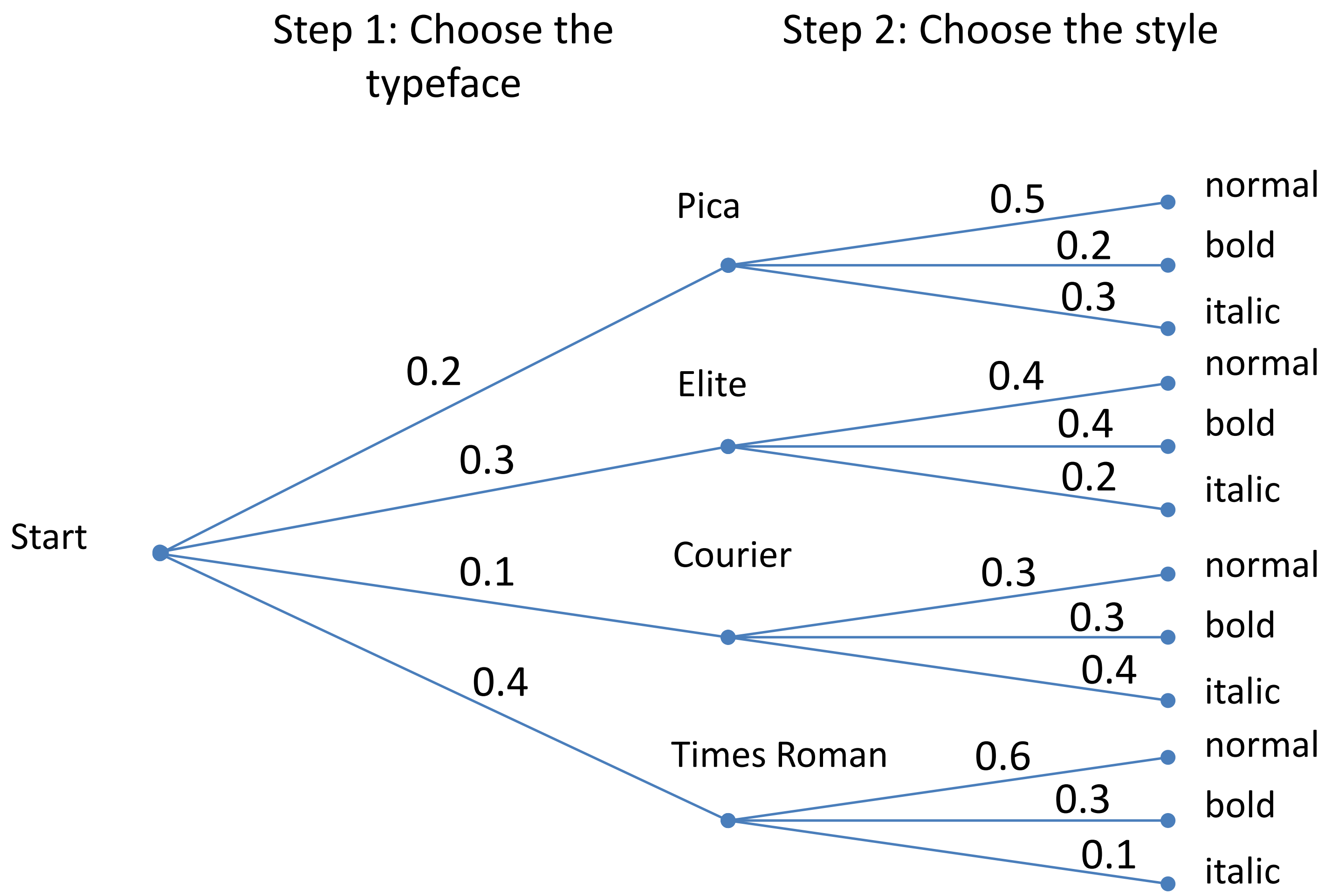
What's the probability of….?

Write the outcomes as ordered sequence:

(H,H), (H,T), (T,H), (T,T)

- Combine probabilities
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ sum rule
  - When A, B disjoint (mutually exclusive) $P(A \cup B) = P(A) + P(B)$
  - $P(\bar{E}) = 1 - P(E)$
  - $P(A|B) = \frac{P(A \cap B)}{P(B)}$
  - $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$ product rule
  - When A, B independent $P(A \cap B) = P(A)P(B)$
- Possibility Trees

  To track outcomes of a sequence of events



- What's the probability of choosing Pica with normal style?

  $P(\text{Pica} \cap \text{normal}) = P(\text{Pica})P(\text{normal}|\text{Pica}) = 0.2 \times 0.5 = 0.1$

- The random module in Python

https://docs.python.org/3/library/random.html

- Simulate random processes: flip a coin, roll a die, shuffle
  - Generate all possible outcomes
  - Unbiased: ensure equal probability for every outcome
  - Fisher Yates shuffle vs. the alternative shuffle

- Pigeonhole principle

  If n+1 objects are put into n boxes,

  then at least one box must contain more than one objects.

Proof

- Direct proof
- Proof by cases
- Proof by counter-examples
- Proof by contradiction
- Proof by contraposition
- Proof by induction
- The format