③ Given:

K - any field

K[x] - ring of polynomials over K

Prove:

K[x] is an euclidean domain

<u>Proof</u>:

K[x] is euclidean if $\exists$ function

$$d: K[x] \setminus \{0\} \longmapsto \mathbb{N} \cup \{0\}$$

Satisfying: $\forall a, b \in K[x]$, $b \neq 0$; $\exists q, r \in K[x]$

$$a = qb + r$$

$$\deg(r) < \deg(q) \text{ or } r = 0$$

From long division of polynomials we have:

$$\forall f, g \in K[x], g \neq 0, \exists q, r \in K[x]$$

$$f = qg + r$$

with $\deg(r) < \deg(q)$ or $r = 0$

We can then define the size function of K[x] as

$\underline{d: p \to \deg(p)}$, with $p \neq 0$ since $\deg(0)$ is not defined.

K[x] is therefore euclidean with an <u>eucliden function</u>

$$d: p \mapsto \deg(p)$$

4 a)

Given $k$ divisions to find $\gcd(a, b)$, $a > b > 0$

Fib. seq $f_1, f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$ for $n \geq 2$

Prove: $a \geq f_{k+2}$

Proof:

Statement is true when $k = 1$, since

$$b \geq 1 = f_2$$
$$a \geq 2 = f_3$$

Assuming $a \geq f_{k+2}$ when $k-1$ divisions are made

Show $a \geq f_{k+2}$ for $k$ divisions

Since $k > 0$, $b > 0$. After 1 division $(a, b) := (b, a \bmod b)$

By induction, we have:

$$b \geq f_{k+1}$$
$$a \bmod b \geq f_k$$

Since $a > b$, $\lfloor a/b \rfloor \geq 1$

So we have

$$a \geq b + (a - \lfloor a/b \rfloor b) = b + a \bmod b$$

We can write $b$ as $f_{k+1}$ and $a \bmod b$ as $f_k$, so we have:

$$\underline{a \geq f_{k+1} + f_k = f_{k+2}}$$

(As shown, for $k = 1$, the inequality is not always strict.