# Learning in Path Planning Problem with Side-Observations - An Instance of Multi-armed Bandit Problems

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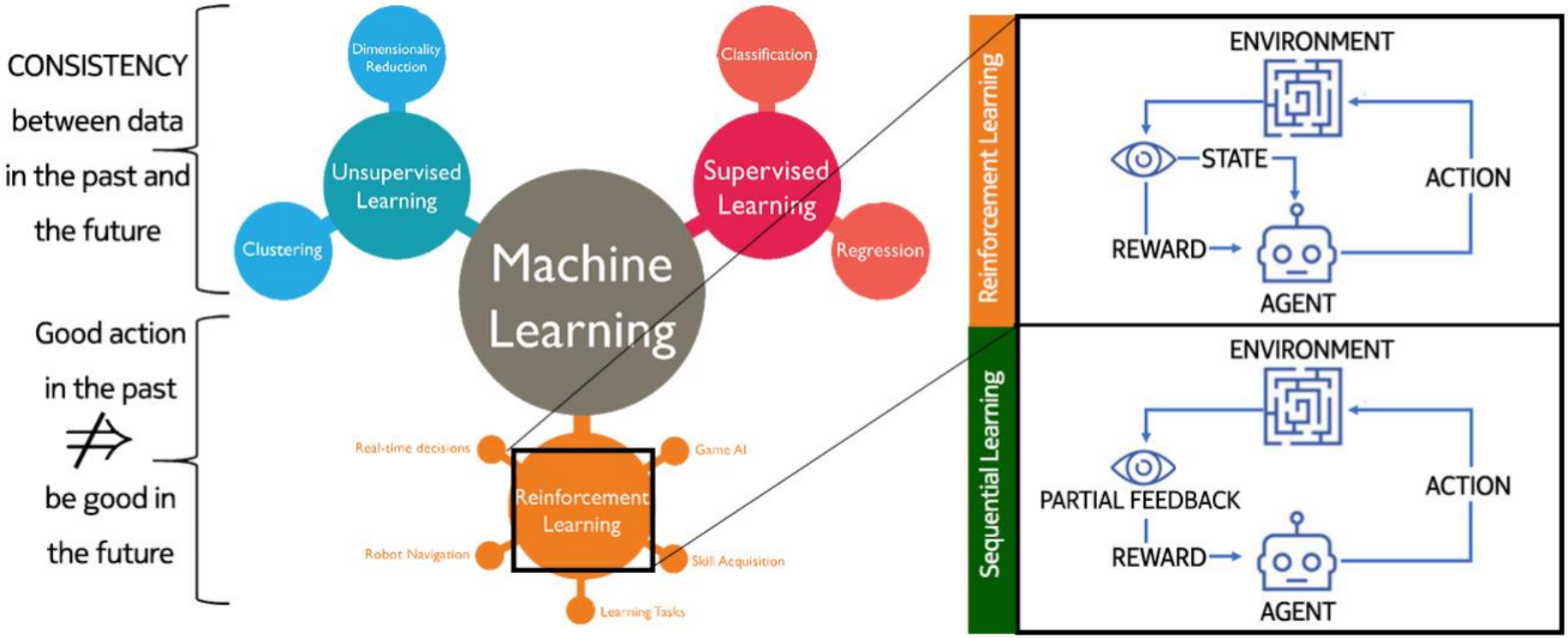
# Online/Sequential Learning




**Online Learning**

**for** $t = 1, 2, \ldots$
- receive question $\mathbf{x}_t \in \mathcal{X}$
- predict $p_t \in D$
- receive true answer $y_t \in \mathcal{Y}$
- suffer loss $l(p_t, y_t)$

- Data arrives one by one
- A decision has to be made for each new piece of data
- Get some feedback
- Update the decision rule
- Feedback may depend on decision

# Stochastic Multi-armed Bandit

A simple stochastic model:

$$\forall k = 1, \dots, K, \quad (X_{k,t})_{t \in \mathbb{N}} \quad \text{is} \quad \text{i.i.d.} \quad \text{with a distribution } \nu_k$$

$$K \text{ arms} \leftrightarrow K \text{ (unknown) probability distribution}$$


$\nu_1$ $\nu_2$ $\nu_3$ $\nu_4$ $\nu_5$

At round $t$, an agent:

- chooses an arm $A_t$
- observes a sample $X_t = X_{A_t,t} \sim \nu_{A_t}$ (loss)

The sampling strategy (or bandit algorithm) $(A_t)$ is sequential:

$$A_{t+1} = F_t(A_1, X_1, \dots, A_t, X_t).$$

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**Several possible goals**:

- find quickly the arm with smallest mean (optimal exploration)
- Minimize cumulative losses $\mathbb{E}\left[\sum_{t=1}^{T} X_t\right]$ (exploration/exploitation tradeoff)
- (more general) learn quickly *something* about the distributions $\nu_k$

# Why Bandits ?


$\nu_1$ $\nu_2$ $\nu_3$ $\nu_4$ $\nu_5$

**Goal:** maximize ones' gains in a casino ?

(HOPELESS)

## Clinical trials

Historical motivation [Thompson 1933]





$\mathcal{B}(\mu_1)$ $\mathcal{B}(\mu_2)$ $\mathcal{B}(\mu_3)$ $\mathcal{B}(\mu_4)$ $\mathcal{B}(\mu_5)$

For the $t$-th patient in a clinical study,

- chooses a treatment $A_t$
- observes a response $X_t \in \{0, 1\} : \mathbb{P}(X_t = 1) = \mu_{A_t}$

**Goal:** identify the best treatment / maximize the number of patients healed

$$ Modern motivation
(recommender systems, online advertisement, A/B Testing...)


$\nu_1$ $\nu_2$ $\nu_3$ $\nu_4$ $\nu_5$

For the $t$-th visitor of a website,

- recommend a movie $A_t$
- observe a rating $X_t \sim \nu_{A_t}$ (e.g. $X_t \in \{1, \dots, 5\}$)

**Goal:** maximize the sum of ratings

## Cognitive radio transmission

**Agent**: a smart radio device
**Arms**: radio channels (frequency bands)

*streams indicating channel availabilities*

| | | | | | | |
|---|---|---|---|---|---|---|
| Channel 1 | $X_{1,1}$ | $X_{1,2}$ | $\dots$ | $X_{1,t}$ | $\dots$ | $X_{1,T}$ |
| Channel 2 | $X_{2,1}$ | $X_{2,2}$ | $\dots$ | $X_{2,t}$ | $\dots$ | $X_{2,T}$ |
| $\dots$ | $\dots$ | $\dots$ | $\dots$ | $\dots$ | $\dots$ | |
| Channel $K$ | $X_{K,1}$ | $X_{K,2}$ | $\dots$ | $X_{K,t}$ | $\dots$ | $X_{K,T}$ |

At round $t$, the device:

- selects channel $A_t$
- observes the channel availability $X_t = X_{A_t,t} = 0$ or 1

**Goal:** Maximize the number of sucessfull transmissions

# Adversarial Bandit

- Change notation: Actions $\boldsymbol{p} \in \mathcal{P}$ ($K := |\mathcal{P}|$). Loss $L_t(\boldsymbol{p})$ instead of Reward.
- No i.i.d assumption on the reward/loss generation.
- Oblivious adversary: the loss $L_t$ does not depends on actions $\tilde{\boldsymbol{p}}_s, s \leq t$ in previous stage i.e., the sequence of losses $L_1, L_2, \cdots, L_T$ is chosen in advance.
- Minimizing the expected regret:

$$R_T := \mathbb{E}\left[\sum\nolimits_{t\in[T]} L_t\left(\tilde{\boldsymbol{p}}_t\right)\right] - \min_{\boldsymbol{p}^*\in\mathcal{P}} \sum\nolimits_{t\in[T]} L_t\left(\boldsymbol{p}^*\right)$$

$$= \text{expected cumulative loss} - \text{cumulative loss of best action in hindsight}$$

* Deterministic policy is doom in worst-case scenario $\Rightarrow$ Randomized policy = trade-off between exploration and exploitation. Target: sub-linear regret

$$u_1 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_5 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \cdots$$

$$R_T \sim \frac{T}{2}$$

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Expected Regret's Bounds

| | $K$-Arm | |
|---|---|---|
| | Full | Partial |
| Lower Bound | $\sqrt{T \ln K}$ | $\sqrt{TK}$ |
| Upper Bound | $\sqrt{T \ln K}$ | $\sqrt{TK}$ |

# Combinatorial Bandit

At each time $t$,

- A loss vector $\boldsymbol{\ell}_t \in [0,1]^n$ is adversarially chosen.
- Unknowing this, the learner chooses a vector $\tilde{\boldsymbol{p}}_t \in \mathcal{P} \subset \{0,1\}^n$ (note that $|\mathcal{P}| = \Omega(2^n)$).
- The learner suffers the loss $L_t := \langle \tilde{\boldsymbol{p}}_t, \boldsymbol{\ell}_t \rangle = \sum_{i \in [n]} \boldsymbol{\ell}_t(\tilde{\boldsymbol{p}}_t)[i] \cdot \tilde{\boldsymbol{p}}_t[i] \in \mathbb{R}$.
- Feedback on losses are observed:
  - Full information: Observe the whole vector $\boldsymbol{\ell}_t \Rightarrow$ know losses of all actions even the non-chosen ones.
  - Bandit feedback: Knows only the scalar $L_t \in \mathcal{R}$.
  - Semi-bandit feedback: Knows the products $\boldsymbol{\ell}_t(\tilde{\boldsymbol{p}}_t)[i] \cdot \tilde{\boldsymbol{p}}_t[i]$, i.e., knows the losses corresponding to the element where $\tilde{\boldsymbol{p}}_t[i] = 1$.

Motivation: Routing/Path Planning Problems, Multi-task Learning, Perfect Matching, Balanced Cuts, etc.,

Naive idea: applying MAB algorithms (say, Exp2) we obtain:

- $R_T \leq \mathcal{O}(\sqrt{T|\mathcal{P}|})$ (Very Large Bound).
- Issues with the complexity of the algorithms.

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# The Path Planning Problem

**Path planning problem (PPP):**

- Directed acyclic graph $G = (V, \mathcal{E})$; $\mathcal{P}$ set of paths from source to sink.
- At each stage $t \in [T]$:
  - Edge $e \in \mathcal{E}$ is embedded with a loss $\ell_t(e) \in [0, 1]$ (unknown to learner),
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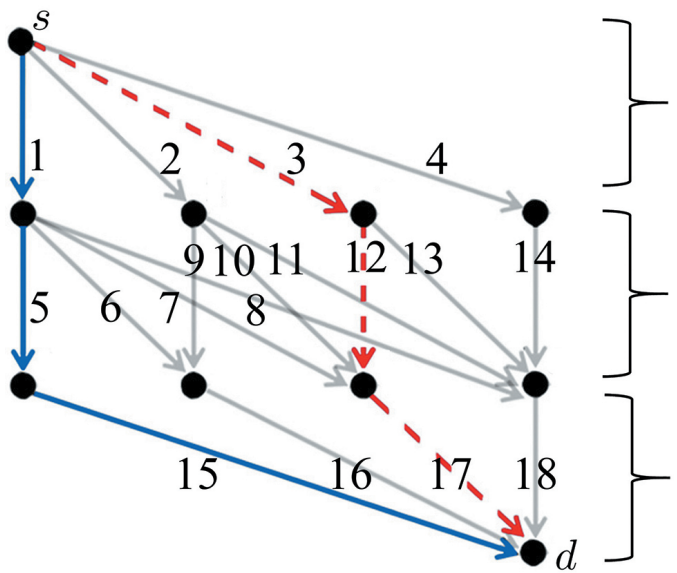
  - Feedback setting: Full-info, semi-bandit, bandit.



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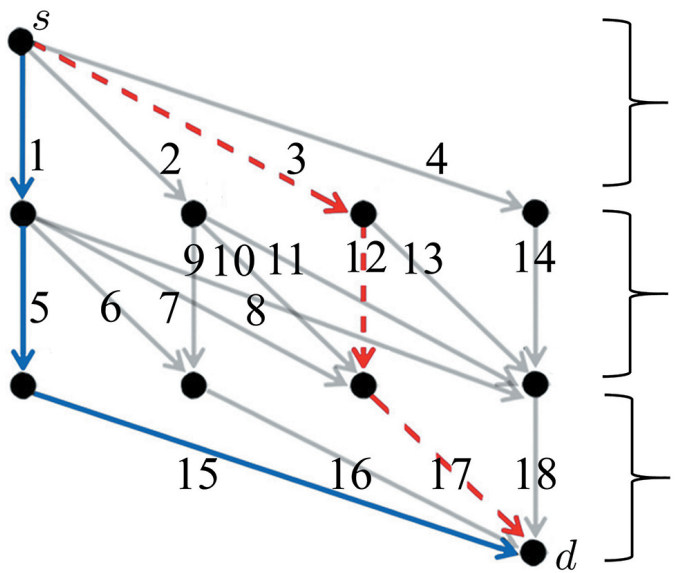

# Bandit Feedback: the ComBand Benchmark

**State-of-the-art**: ComBand algorithm[1] can be applied to PPP: at stage $t$,

- Keep weight for each path and update according to *exponential multiplicative* rule;
- Sample a path based on normalized weights mixed with *exploration distribution* $\mu$;
- Estimate the loss for each path (*unbiased estimator*) based on the *co-occurrence matrix* $C_t$. Use this loss estimator to update weights.
  - $C \in \mathbb{M}_{E \times E}$ and $C_{i,j}$ is the probability that the chosen path contains edge $i$ and $j$

**Regret guarantee**: $R_T \leq \mathcal{O}(\sqrt{TE \ln(|\mathcal{P}|) + nT \ln(|\mathcal{P}|)/\lambda_{\min}})$

- $n$: length of the longest path
- $\lambda_{\min}$: smallest non-zero eigenvalue of co-occurrence matrix generated from $\mu$

**Main issues**:

1. **Running time is exponential** in $n$ (polynomial in $|\mathcal{P}|$ and $T$).
2. **Which exploration distribution** $\mu$ to optimizes $\lambda_{\min}$?
   (Standard uniform distribution can have $\lambda_{\min}$, of the order of $1/|\mathcal{P}|$).

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# Bandit Feedback: Our Algorithm EdgeCB

We propose **EdgeCB**, a novel variant of ComBand with two updates based on weight pushing (a dynamic programming technique).

Key idea: maintain weights on edges only (not paths)

- Solving Issue 1.: apply weight pushing to efficiently sample the path and compute the co-occurrence matrix (similar to CombD3[2] but with easier representation)
  $\Rightarrow$ Running time is in $\mathcal{O}(E^2T)$.
- Solving Issue 2.: propose an efficient derivative-free (black-box) optimization procedure on exploration distributions defined on the edges' weights
  $\Rightarrow$ Heuristically improves the exploration distribution.

$\Rightarrow$ Efficient algorithm
$\Rightarrow$ Same regret guarantee as ComBand but with better $\lambda_{\min}$

$$R_T \leq \mathcal{O}(\sqrt{TE\ln(|\mathcal{P}|) + nT\ln(|\mathcal{P}|)/\lambda_{\min}})$$

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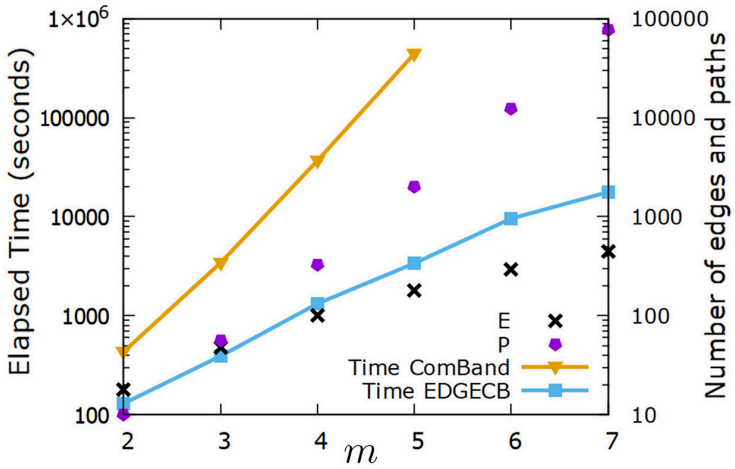
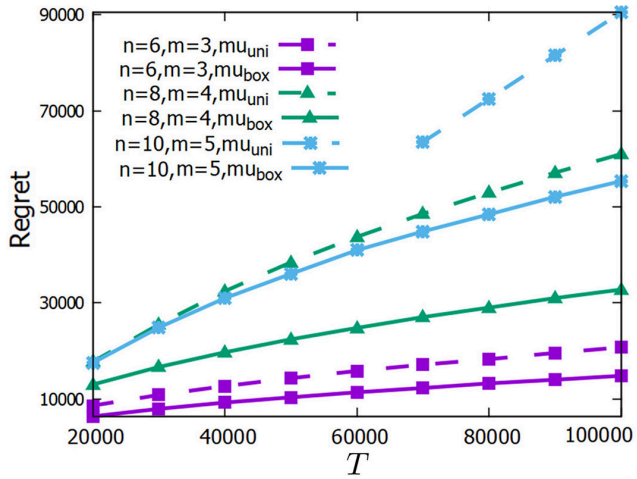
# Numerical Experiments

- Efficiency improvement: Running time of ComBand and EdgeCB on instances where $m$ increases and $n = 2m$; $y$-axis is drawn with log-scale:




- Performance guarantee: Actual regrets of EdgeCB $(\mu_{\text{uni}})$ and EdgeCB$(\mu_{\text{box}})$.



# Semi-bandit feedback: Side Observations

**Recall**: Semi-bandit feedback: at the end of each stage, the learner observes losses of edges belonging to chosen path.

**Side observations**: from semi-bandit feedback, the learner can deduce additional feedback (at no cost)

- By observing an edge she can deduce losses of some other edges
- Observation graph $G_O^t$ (depending on $t$ and unknown at decision time):
  - Each vertex $v_e$ of $G_O^t$ corresponds to an edge $e$ of the graph $G$.
  - There is an edge from vertex $v_e$ to vertex $v_{e'}$ if in $G$, by observing the loss on edge $e$, the learner can deduce the loss on $e'$.

$\Rightarrow$ Path planning problem with semi-bandit feedback plus side-observations = SOPPP.

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# Semi bandit feedback: The FPL-IX benchmark

**State-of-the-art**: FPL-IX[3] (Follow-The-Perturbed-Leader with Implicit Exploration)

- FPL: find action that minimizing cumulative past loss estimate, perturbed
- IX: construct optimistic estimation of loss using geometric resampling

**Regret guarantee**: $R_T \leq \tilde{\mathcal{O}}(n^{3/2}\sqrt{T\alpha})$

- $\alpha$: upper bound of the independent number[4] of the observation graphs $G_O^t, t \in [T]$ (note that $\alpha \leq E$).

**Main issues**:

1. the running time of FPL-IX is only guaranteed either with high-probability or in expectation (it *can be inefficient in worst-case scenarios*).
2. FPL-IX *requires to have an efficient oracle* for solving optimization problems at each stage.

[3] Tomáš Kocák, Gergely Neu, Michal Valko, and Rémi Munos. Efficient learning by implicit exploration in bandit problems with side observations. In NIPS 2014

[4] The size of the largest independent sets that are the sets whose vertices are not connected.

# Semi-bandit Feedback: Our Algorithm Exp3-OE

We propose **Exp3-OE**, an Exp3-type algorithm solving these issues while improving the bound of FPL-IX in the CB game with semi-bandit feedback.

- Novel loss estimator (biased) based on implicit exploration with *fixed parameter* $\beta$: $\hat{\boldsymbol{\ell}}_t(e) = \boldsymbol{\ell}_t(e)/(q_t(e) + \beta)$ where $q_t(e)$ is the probability that edge $e$ is revealed at $t$.
- Weight pushing to efficiently sample a path and compute this novel loss estimator.

## Results for arbitrary SOPPP

- Exp3-OE *always* runs efficiently in $\mathcal{O}(E^3T)$ *without auxiliary oracle*.
- Exp3-OE guarantees $R_T \leq \tilde{\mathcal{O}}(n\sqrt{T\alpha \ln(|\mathcal{P}|)})$ ($\sim$ FPL-IX for general case).

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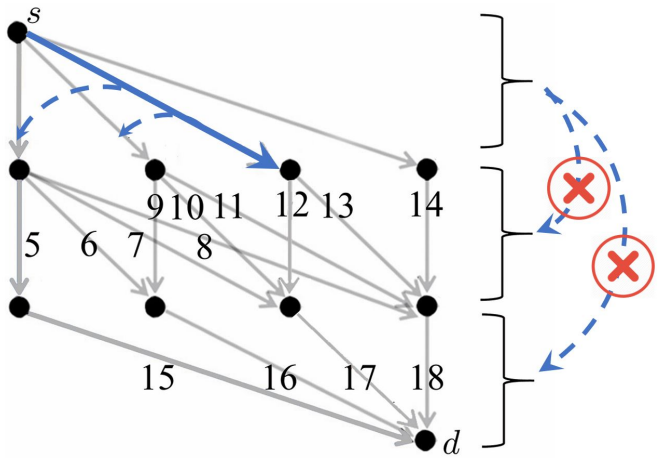
# Improved Exp3-OE bounds for the Colonel Blotto Game

**Assumption** $(A0)$: if two edges belong to a path in $G$, then they cannot simultaneously reveal the loss of another edge.

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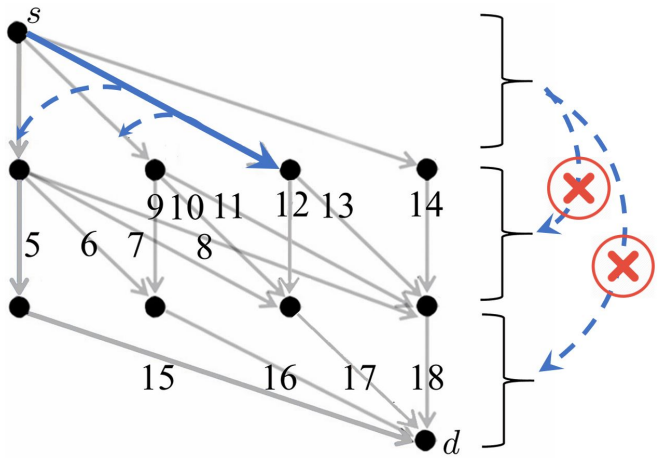


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# Summary

**Bandit feedback**:

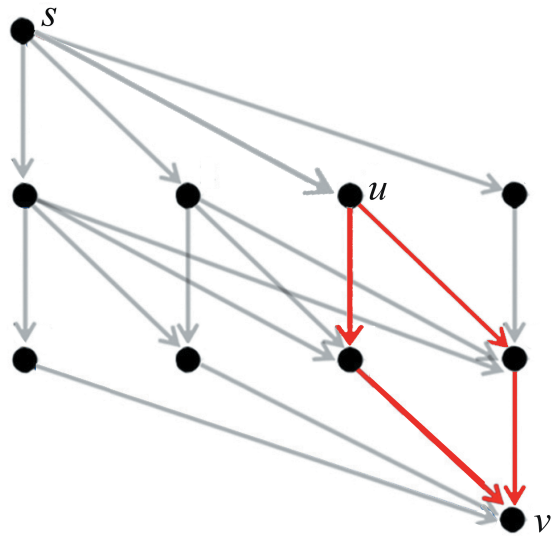
| Algorithm | Running time | Regret guarantee | Notes |
|---|---|---|---|
| ComBand | $\mathcal{O}(\exp(n)T)$ | $R_T \leq \mathcal{O}(\sqrt{T\ln(\lvert\mathcal{P}\rvert)(E+n/\lambda_{\mu_{\text{uni}}}})$ | |
| EdgeCB | $\mathcal{O}(n^2T)$ | $R_T \leq \mathcal{O}(\sqrt{T\ln(\lvert\mathcal{P}\rvert)(E+n/\lambda_{\mu_{\text{box}}}})$ | $\lambda_{\mu_{\text{box}}} \gg \lambda_{\mu_{\text{uni}}}$ |

**Semi-bandit feedback**:

| Algorithm | Running time | Regret guarantee | Notes |
|---|---|---|---|
| FPL-IX | $\text{Poly}(n)T^{3/2}$ with high-prob | $R_T \leq \tilde{\mathcal{O}}\sqrt{n^3\alpha T}$ | Require oracles |
| OSMD | $\mathcal{O}(\exp(n)T)$ | $R_T \leq \mathcal{O}\sqrt{nET}$ | Require oracles<br>Ignores side-obs |
| Exp3-OE | $\mathcal{O}(n^3T)$ | $R_T \leq \tilde{\mathcal{O}}(\sqrt{n^3\alpha T})$ | Improved if satisfy (A0) |

# Weight Pushing Technique

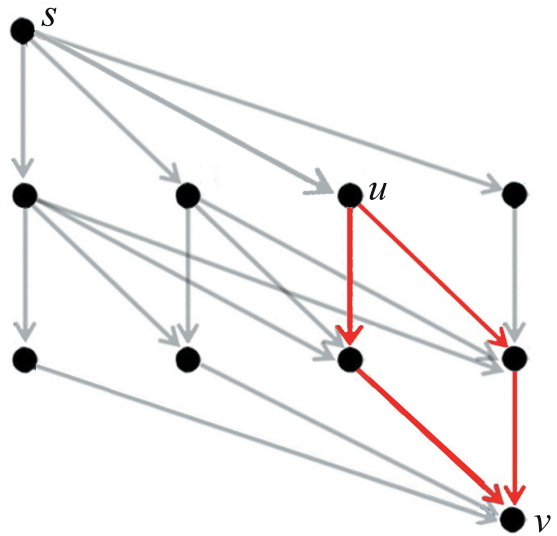
- Purpose: keeping track and updating path weights via edges ($E \ll |\mathcal{P}|$):
  - Create edges weights $w_t(e), e \in \mathcal{E}$ such that path weight = product of corresponding edges' weights.
  - Denote $H(u, v)$ the sum of paths weights from vertex $u$ to $v$.
  - Compute recursively $H(v_1, v_2), \forall v_1, v_2$ (e.g., compute $H(u, d)$ based on $H(v, d)$ for any successor vertex $v$ of $u$).
  - Computation of $H(u, v)$ for any vertex $u, v$ is done in $O(E^2)$.




- Sample a path:
  - Starting from the source $s$, sample vertex $v$ with probability $H(v, d)/H(p(v), d)$ where $p(u)$ is the predecessor of $v$; stop when reaching $d \Rightarrow$ A path is sampled out with the necessary probability distribution.
- Compute the co-occurrence matrix:
  - Entry $C_{e_1,e_2}$ of the co-occurrence matrix $C$ is the probability that $e_1, e_2$ belong to the chosen path $=$ sum of weights of paths passing through $e_1, e_2$.
- Compute the loss estimator in Exp3-OE:
  - Compute $q_t(e) :=$ probability that an edge $e$ is revealed $=$ sum of weights of paths passing through edges revealing $e$.

# Weight Pushing Technique

- Purpose: keeping track and updating path weights via edges ($E \ll |\mathcal{P}|$):
  - Create edges weights $w_t(e), e \in \mathcal{E}$ such that path weight = product of corresponding edges' weights.
  - Denote $H(u, v)$ the sum of paths weights from vertex $u$ to $v$.
  - Compute recursively $H(v_1, v_2), \forall v_1, v_2$ (e.g., compute $H(u, d)$ based on $H(v, d)$ for any successor vertex $v$ of $u$).
  - Computation of $H(u, v)$ for any vertex $u, v$ is done in $O(E^2)$.

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