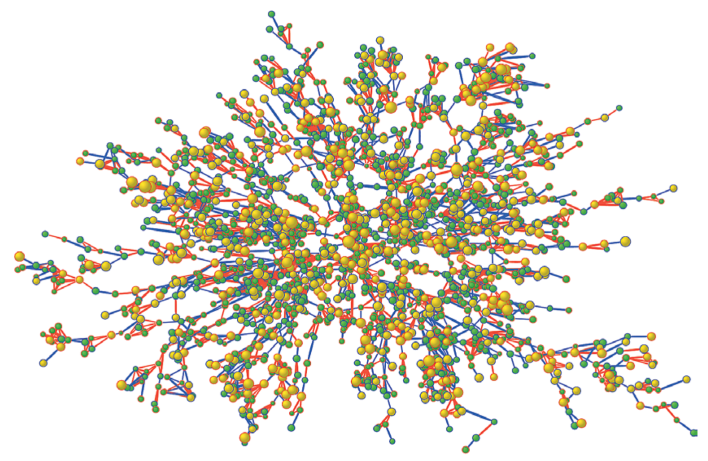

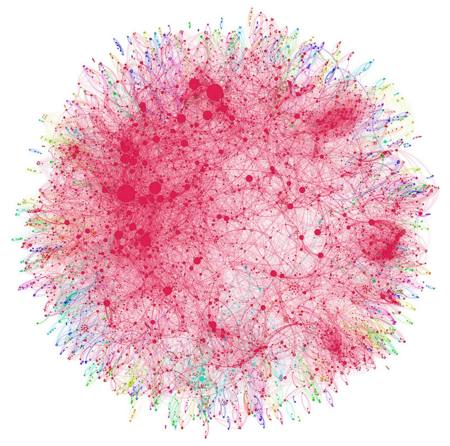

Hyperbolicity, injective hulls, and Helly graphs

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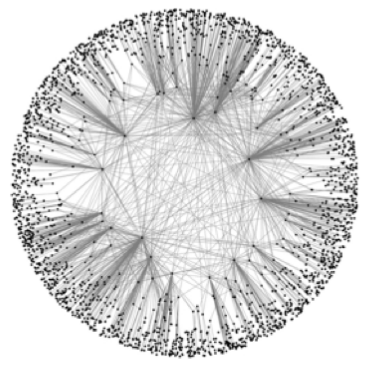
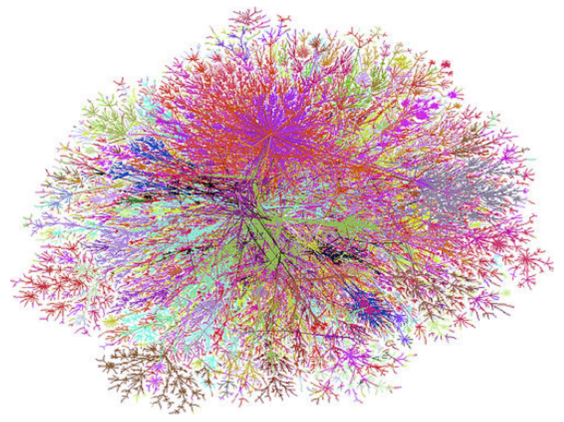
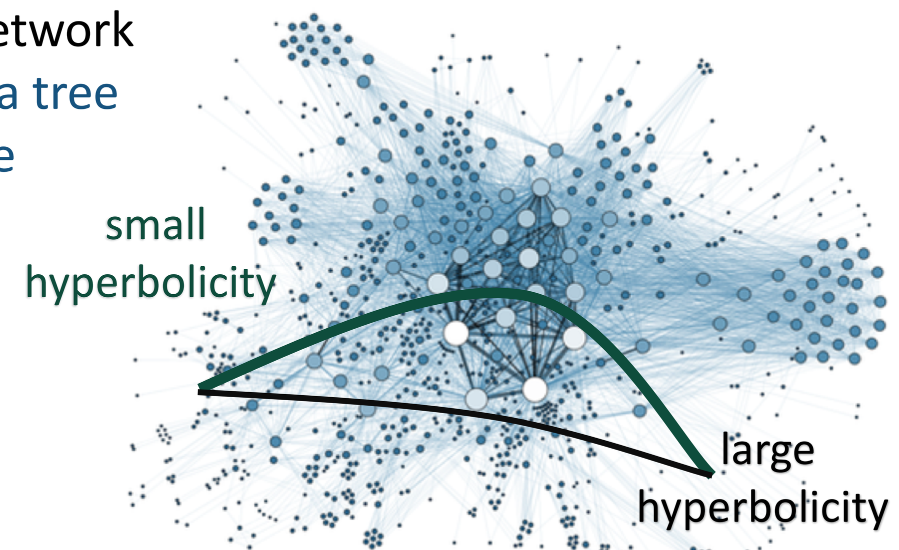




Applications of Hyperbolicity

- Many real world networks have small hyperbolicity (biological, social, collaboration, communication, etc.)
- Smaller value means the network
  - is metrically closer to a tree
  - has negative curvature
- Small hyperbolicity implies that the shortest path between two points curves inward towards the core of the network.



- A graph is δ-hyperbolic provided for any vertices x, y, u, v in it, the two larger of the three sums $d(u,v) + d(x,y)$, $d(u,x) + d(v,y)$, and $d(u,y) + d(v,x)$ differ by at most 2δ.

How Hyperbolicity Relates to Injective Hulls

- Every graph G can be isometrically embedded into the smallest Helly graph $H(G)$ [3,4]
  - A set S of sets S_i has the Helly property if for every subset T of S the following hold: if the elements of T pairwise intersect, then the intersection of all elements of T is non-empty.
  - A graph is Helly if its family of disks satisfies the Helly property. For example, king grids are Helly.
- $H(G)$ is called the injective hull of G
  - $H(G)$ preserves hyperbolicity
  - If G is δ-hyperbolic, any vertex in $H(G)$ is within 2δ to a vertex in G [5]

We want to understand:
- **Q1**: What governs hyperbolicity in Helly graphs? This will help to understand what governs hyperbolicity in regular graphs.
- **Q2**: How does the injective hull grow for various graph classes? Finding efficient solutions to problems in $H(G)$ can lead to approximate solutions in G. However, this approach requires an efficient calculation of $H(G)$.

Open Questions and Future Work

- What other graph classes require exponentially many Helly vertices?
- What other graph classes can be Hellified efficiently?
- What kind of problems can use H(G) to solve problems efficiently on G?

References

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[4] J. Isbell. Six theorems about injective metric spaces. *Commentarii mathematici Helvetici*, 39:65–76, 1964.
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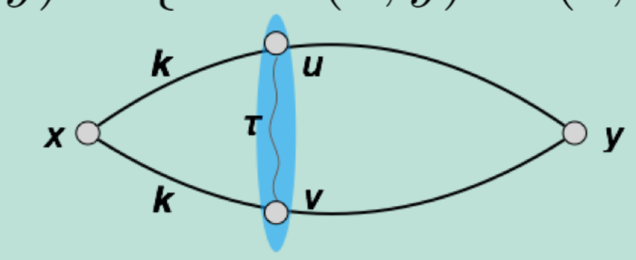
Q1: What governs hyperbolicity in Helly graphs?

Interval thinness

- An interval $I(x,y)$ is the set of all vertices from shortest (x,y)-paths
- A slice of an interval at distance k is defined as:

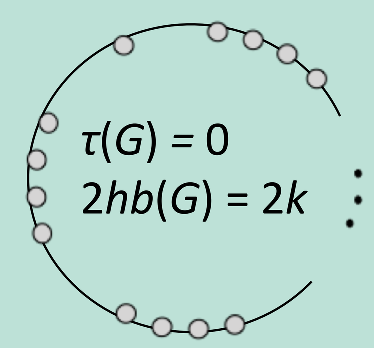
$$S_k(x,y) = \{z \in I(x,y) : d(z,x) = k\}$$



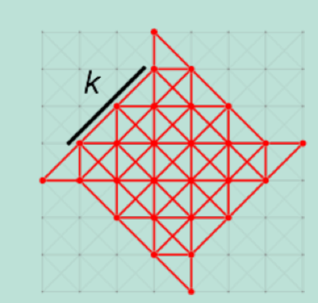
- An interval is τ-thin if for any natural number k and any two u,v vertices of $S_k(x,y)$ are at most τ apart.
- A graph is τ-thin if all of its intervals are at most τ-thin.

For general graphs $\tau(G) \le 2hb(G)$, but $\tau(G)$ and $hb(G)$ can be far apart.

Theorem [1]: We found that for Helly graphs, $\tau(G) \le 2hb(G) \le \tau(G) + 1$.



Example: odd cycle with $4k$+1 vertices



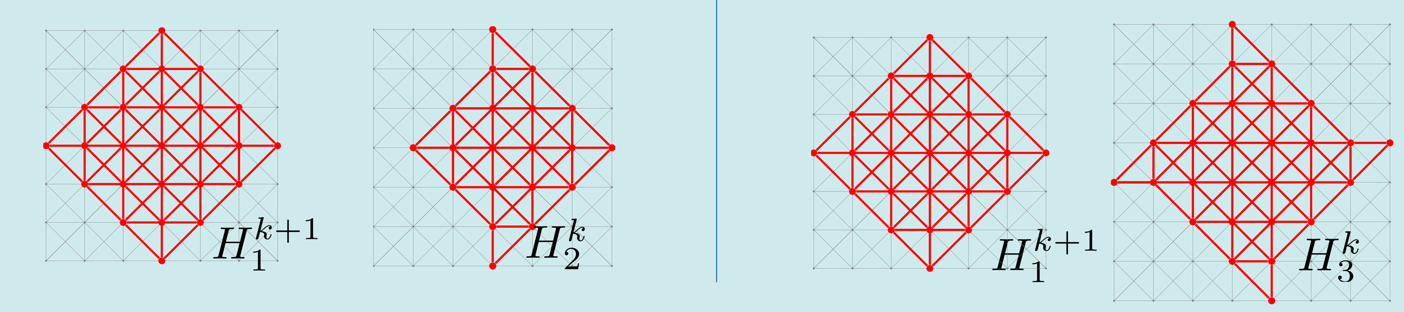
A Helly graph example when $2hb(G) = \tau(G) + 1$.

Special subgraphs of a king grid

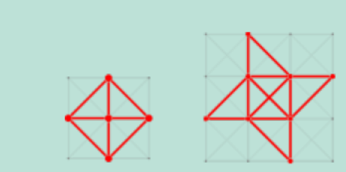
Theorem [1]: We show that for Helly graphs and any integer k,

$hb(G) \le k$ if and only if G has neither isometric H_1^{k+1} nor H_2^k

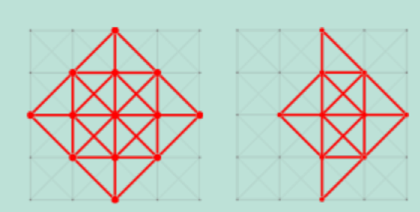
$hb(G) \le k + ½$ if and only if G has neither isometric H_1^{k+1} nor H_3^k



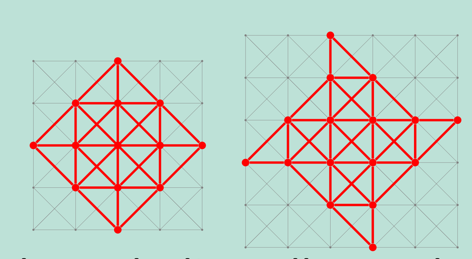
For example, here are the forbidden isometric subgraphs for….



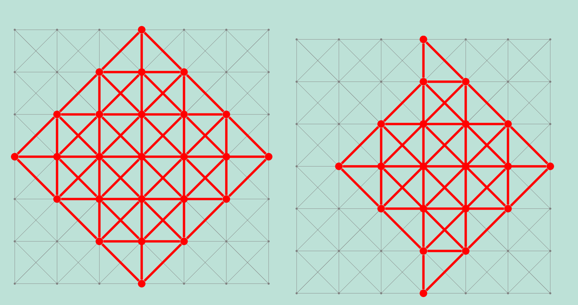
1/2-hyperbolic Helly graphs



1-hyperbolic Helly graphs



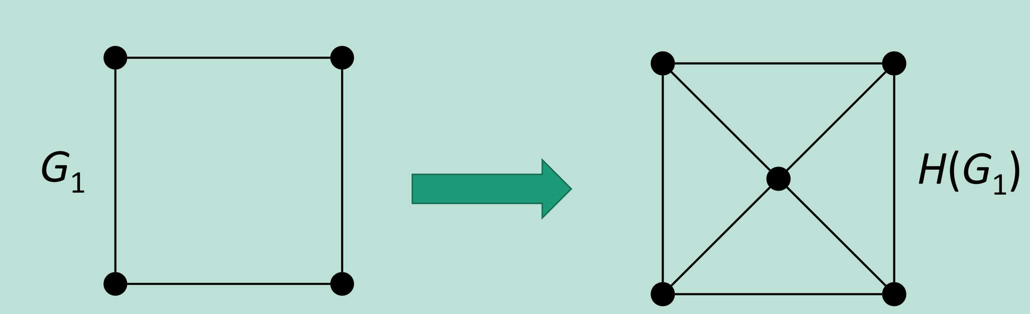
3/2-hyperbolic Helly graphs

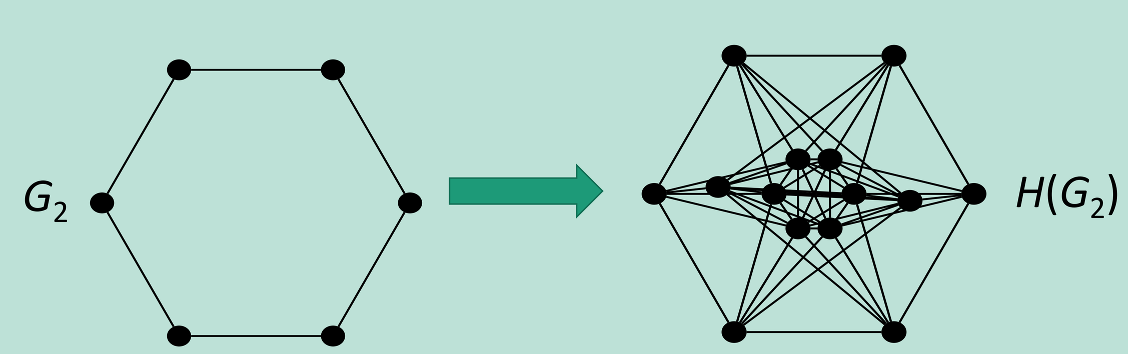


2-hyperbolic Helly graphs

Q2: How big is *H(G)* with respect to *G*?

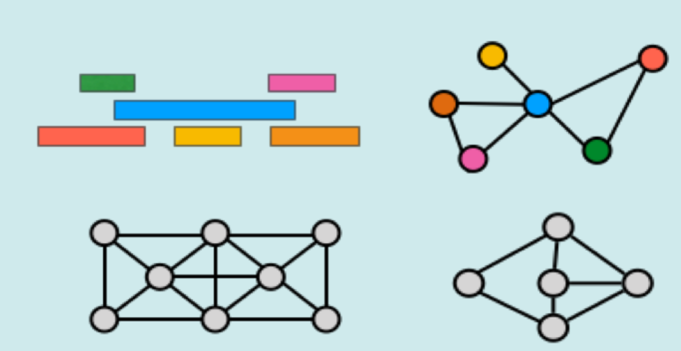
Hellification *(noun)*
1. the process of finding the smallest Helly graph into which G isometrically embeds, i.e., finding the injective hull $H(G)$



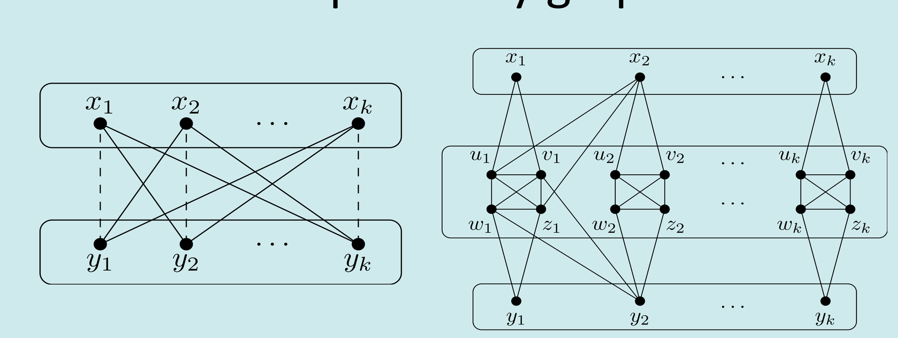


Theorem [2]: We show that there are some graph classes for which $H(G)$ …

contains at most $2n$ vertices.
- Interval graphs
- Helly graphs
- Distance hereditary graphs



can contain at least 2^n vertices.
- Chordal bipartite graphs
- Chordal graphs
- Cocomparability graphs



We developed an algorithm to find the injective hull of these graph classes in linear time [2].

We also produced a method to create graphs whose injective hull cannot be found in less than exponential time [2].