Sequences and Summations

Section 2.4

Section Summary

- Sequences
  - Ex: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Ex: Fibonacci Sequence
- Summations

Introduction

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, …….
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

Sequences

Definition: A *sequence* is a function from a subset of the integers (usually either the set {0, 1, 2, 3, 4, …..} or {1, 2, 3, 4, ….}) to a set *S*.

- The notation a_n is used to denote the image of the integer *n*. We can think of a_n as the equivalent of $f(n)$ where f is a function from {0,1,2,…..} to *S*.
- We call a_n a *term* of the sequence.

Sequences

Example: Consider the sequence $\{a_n\}$ where $a_n = \frac{1}{n}$

$$\{a_n\} = a_1, a_2, a_3, a_4, \ldots$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

Geometric Progression

Definition: A *geometric progression* is a sequence of the form: $a, ar, ar^2, \ldots, ar^n, \ldots$
where the *initial term* a and *the common ratio* r are real numbers.

Ex:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = b_0, b_1, b_2, b_3, b_4, \ldots = 1, -1, 1, -1, 1, \ldots$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = c_0, c_1, c_2, c_3, c_4, \ldots = 2, 10, 50, 250, 1250, \ldots$$

3. Let $a = 6$ and $r = 1/3$. Then:

$$\{d_n\} = d_0, d_1, d_2, d_3, d_4, \ldots = 6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots$$

Arithmetic Progression

Definition: An *arithmetic progression* is a sequence of the form: $a, a+d, a+2d, \ldots, a+nd, \ldots$

where the *initial term* a and the *common difference* d are real numbers.

Ex:

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = s_0, s_1, s_2, s_3, s_4, \ldots = -1, 3, 7, 11, 15, \ldots$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = t_0, t_1, t_2, t_3, t_4, \ldots = 7, 4, 1, -2, -5, \ldots$$

3. Let $a = 1$ and d = 2:

$$\{u_n\} = u_0, u_1, u_2, u_3, u_4, \ldots = 1, 3, 5, 7, 9, \ldots$$

Strings

Definition: A *string* is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by λ.
- The string *abcde* has *length* 5.

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

Questions about Recurrence Relations

Example 1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots.$ and suppose that $a_0 = 2$. What are a_1, a_2 and a_3?

[Here $a_o = 2$ is the initial condition.]

Solution: We see from the recurrence relation that

$$a_1 = a_o + 3 = 2 + 3 = 5$$
$$a_2 = 5 + 3 = 8$$
$$a_3 = 8 + 3 = 11$$

Questions about Recurrence Relations

Example 2: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2,3,4,\ldots.$ and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?

[Here the initial conditions are $a_0 = 3$ and $a_1 = 5$.]

Solution: We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, by:

- Initial Conditions: $f_0 = 0$, $f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6 .

Answer:

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$

Solving Recurrence Relations

- Finding a formula for the *n*th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.
- Many methods for solving recurrence relations (Ch. 8)
- Here we illustrate by example the method of *iteration* in which we need to guess the formula. The guess can be proved correct by the method of induction (Ch. 5).

Iterative Solution Example

Method 1: Working upward, forward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 2,3,4,....$ and suppose that $a_1 = 2$.

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

.

.

.

$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

Iterative Solution Example

Method 2: Working downward, backward substitution

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n\text{-}1} + 3$ for $n = 2,3,4,\ldots.$ and suppose that $a_1 = 2$.

$$
\begin{aligned}
a_n &= a_{n\text{-}1} + 3 \\
&= (a_{n\text{-}2} + 3) + 3 = a_{n\text{-}2} + 3 \cdot 2 \\
&= (a_{n\text{-}3} + 3\,) + 3 \cdot 2 \;= a_{n\text{-}3} + 3 \cdot 3 \\
&\quad . \\
&\quad . \\
&\quad . \\
&= a_2 + 3(n - 2) \;\; = (a_1 + 3) + 3(n - 2) \;= 2 + 3(n - 1)
\end{aligned}
$$

Financial Application

Example: Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let P_n denote the amount in the account after 30 years. P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Continued on next slide →

Financial Application

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\,P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Solution: Forward Substitution

$P_1 = (1.11)P_0$

$P_2 = (1.11)P_1 = (1.11)^2P_0$

$P_3 = (1.11)P_2 = (1.11)^3P_0$

$\vdots$

$P_n = (1.11)P_{n-1} = (1.11)^n\mathrm{P}_0 = (1.11)^n\,10{,}000$

$P_n = (1.11)^n\,10{,}000$ (Can prove by induction, covered in Ch. 5)

$P_{30} = (1.11)^{30}\,10{,}000 = \$228{,}992.97$

Useful Sequences

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, . . .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, . . .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, . . .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, . . .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, . . .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, . . .
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Example: Conjecture a formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047

Solution: $a_n = 3^n - 2$

Summations

- Sum of terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$
- The notation:

$$\sum_{j=m}^{n} a_j \qquad \textstyle\sum_{j=m}^{n} a_j \qquad \sum_{m \le j \le n} a_j$$

represents

$$a_m + a_{m+1} + \cdots + a_n$$

- The variable *j* is called the *index of summation*. It runs through all the integers starting with its *lower limit* *m* and ending with its *upper limit* *n*.

Summations

- More generally for a set *S*: $$\sum_{s \in S} f(s)$$

- **Examples**:
  - $$If \;\; S=\{2,5,7,10\}, \;\; then \;\; \sum_{j \in S} j = 2+5+7+10$$
  - $$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \; ... \; = \sum_{i=0}^{\infty} \frac{1}{i}$$
  - $$r^0 + r^1 + r^2 + r^3 + \; ... \; + r^n = \sum_{j=0}^{n} r^j$$

Geometric Series

Sums of terms of geometric progressions

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

Proof: Let $S_n = \sum_{j=0}^{n} ar^j$

To compute S_n, first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$rS_n = r\sum_{j=0}^{n} ar^j$$

$$= \sum_{j=0}^{n} ar^{j+1}$$ By the distributive property

Continued on next slide →

Geometric Series

$$= \sum_{j=0}^{n} ar^{j+1}$$ From previous slide.

$$= \sum_{k=1}^{n+1} ar^{k}$$ Shifting the index of summation with $k = j + 1$.

$$= \left(\sum_{k=0}^{n} ar^{k} \right) + (ar^{n+1} - a)$$ Removing $k = n + 1$ term and adding $k = 0$ term.

$$= S_n + (ar^{n+1} - a)$$ Substituting S for summation formula

$$\therefore \quad rS_n = S_n + (ar^{n+1} - a)$$

Continued on next slide →

Geometric Series

$\therefore \quad rS_n = S_n + (ar^{n+1} - a)$ From previous slide.

$rS_n - S_n = (ar^{n+1} - a)$ Solving for S_n

$S_n(r-1) = (ar^{n+1} - a)$

if $r \neq 1$ $$S_n = \frac{ar^{n+1} - a}{r-1}$$

if $r = 1$ $$S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$$

QED

Some Useful Summation Formulae

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)