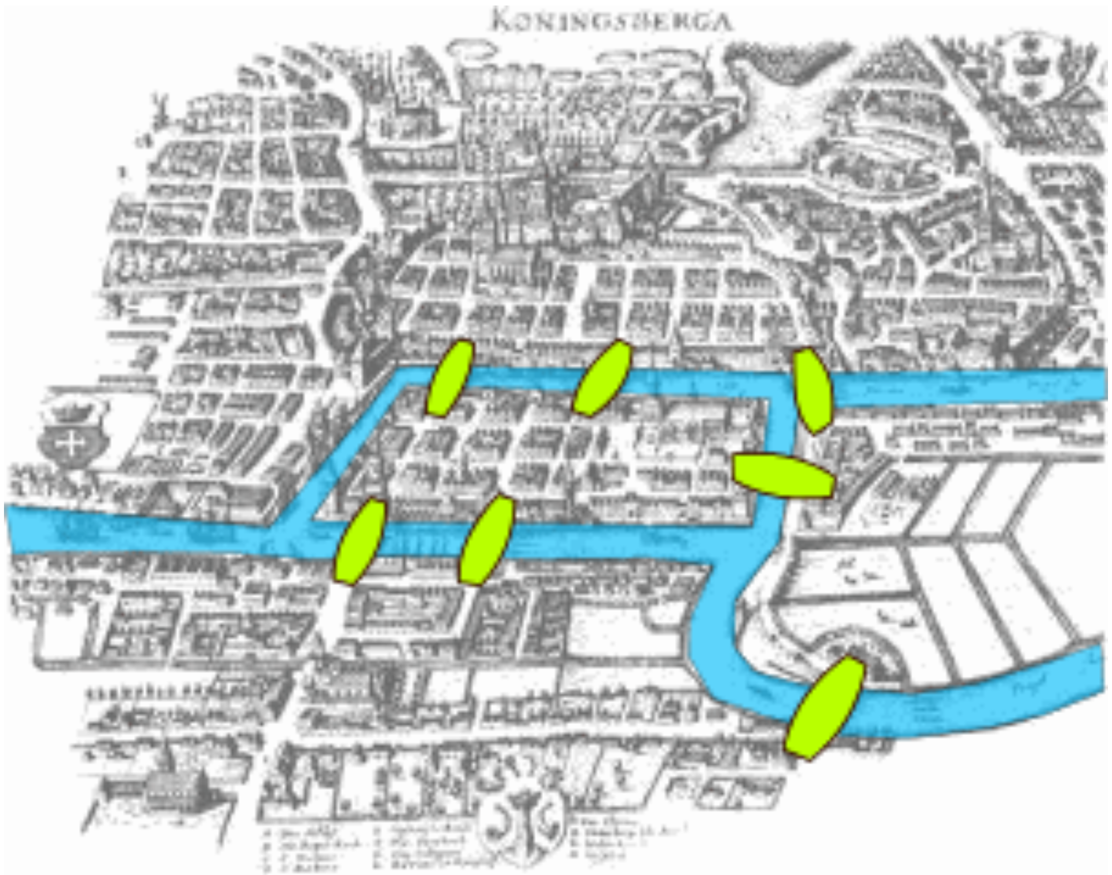
Introduction to Graphs



Slides by Lap Chi Lau

The Chinese University of Hong Kong

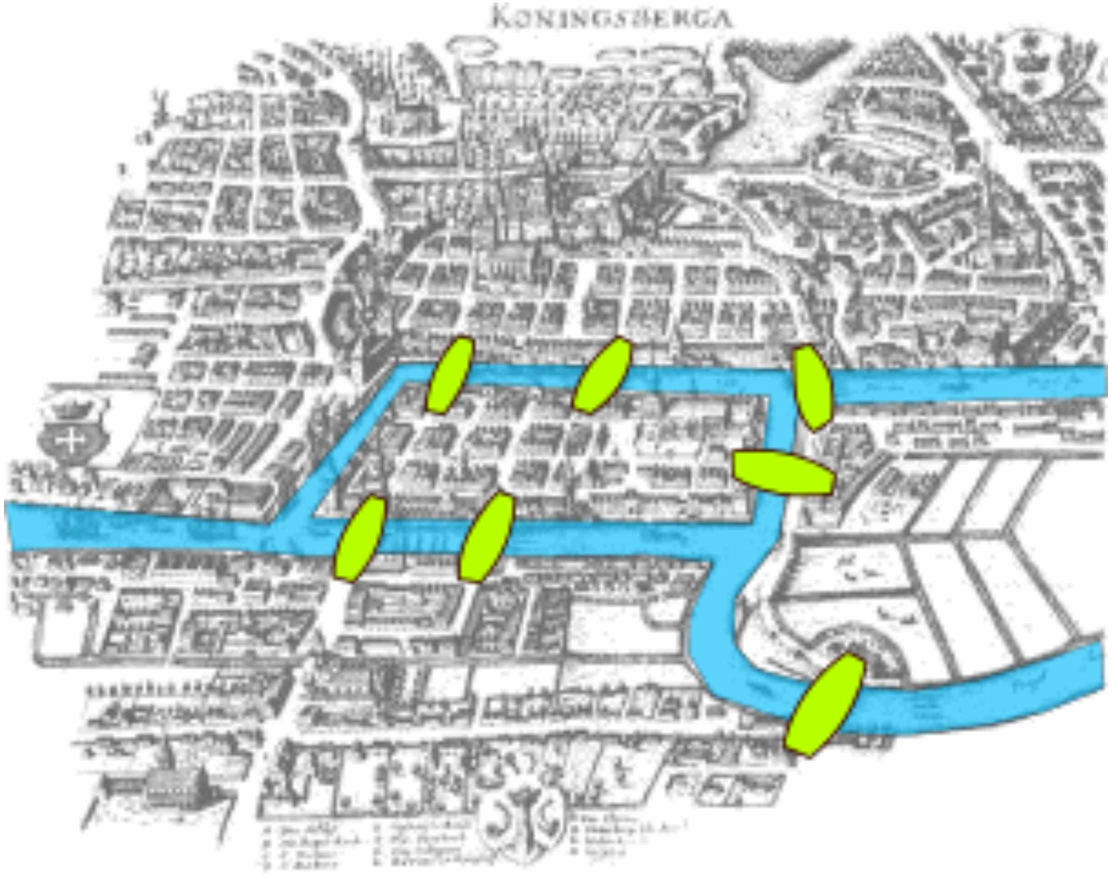
This Lecture

In this part we will study some basic graph theory.

Graph is a useful concept to model many problems in computer science.

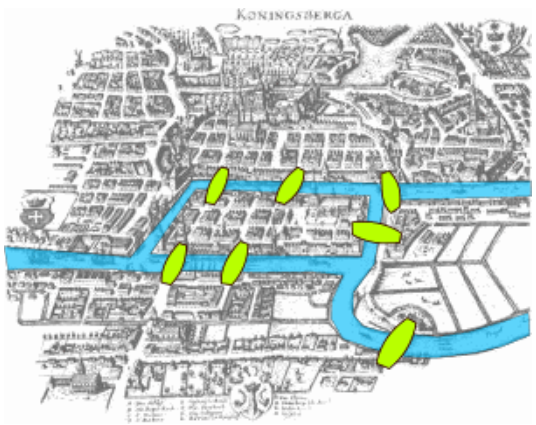
- Seven bridges of Konigsberg
- Graphs, degrees
- Isomorphism
- Path, cycle, connectedness
- Tree
- Eulerian cycle
- Graphs and networks
- Graph coloring

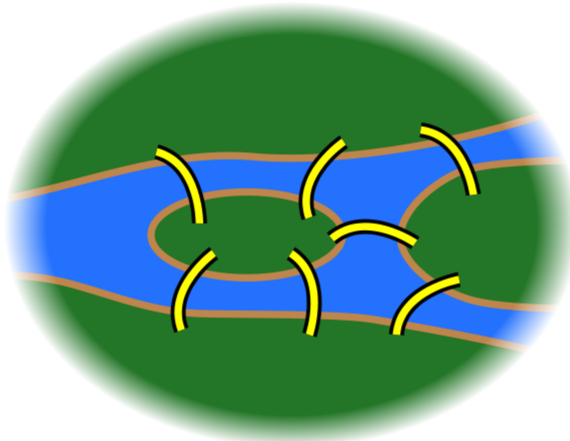
Seven Bridges of Königsberg

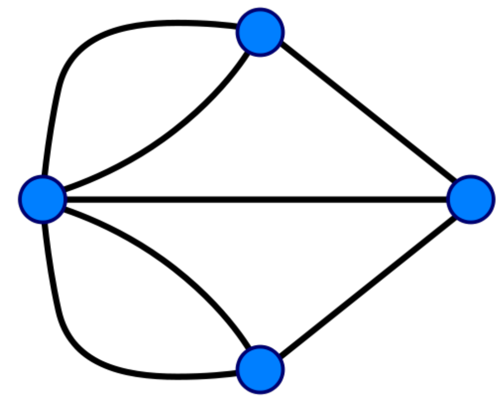


Is it possible to walk with a route that crosses each bridge exactly once?

Seven Bridges of Königsberg



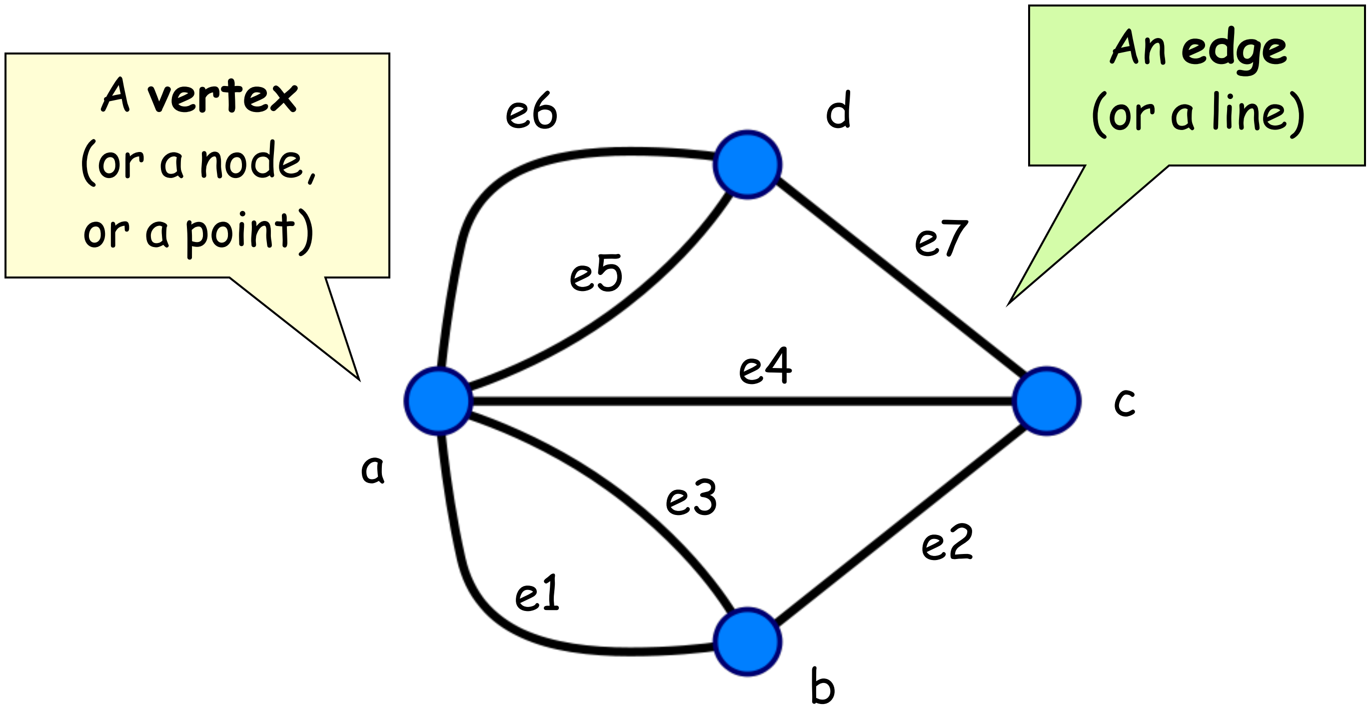




Forget unimportant details.

Forget even more.

A Graph

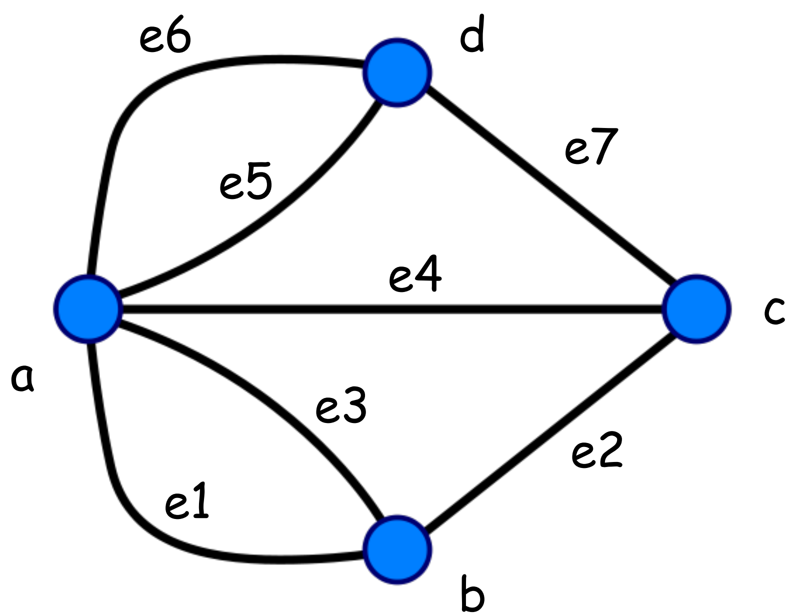


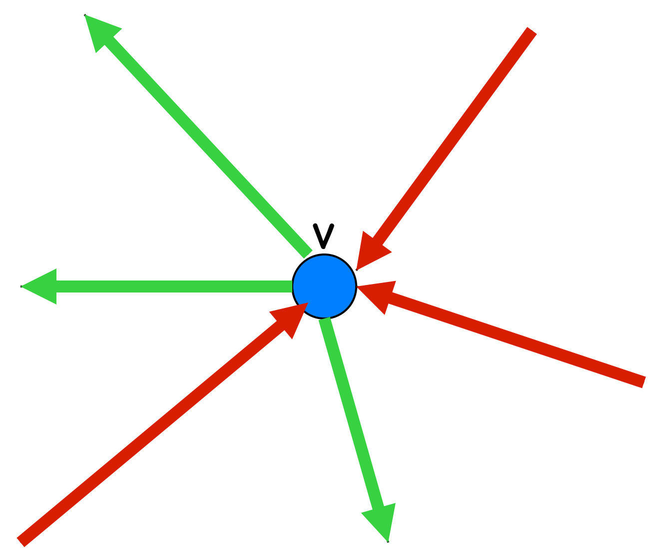
So, what is the **"Seven Bridges of Königsberg"** problem now?

To find a walk that visits each edge exactly once.

Euler's Solution

Question: Is it possible to find a walk that visits each edge exactly once.



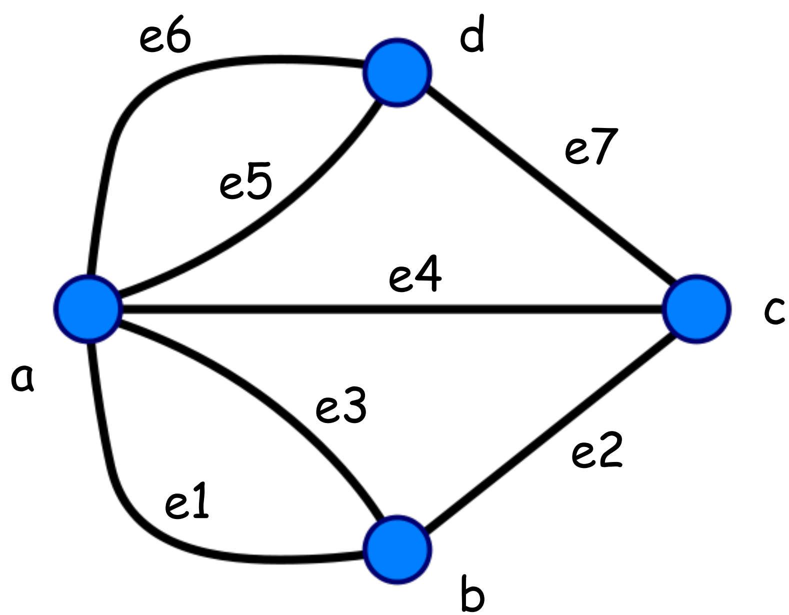


Suppose there is such a walk, there is a starting point and an endpoint point.

For every "intermediate" point v, there must be the same number of incoming and outgoing edges, and so v must have an **even number of edges**.

Euler's Solution

Question: Is it possible to find a walk that visits each edge exactly once.



So, at most **two** vertices can have odd number of edges.

In this graph, every vertex has only an odd number of edges, and so there is no walk which visits each edge exactly one.

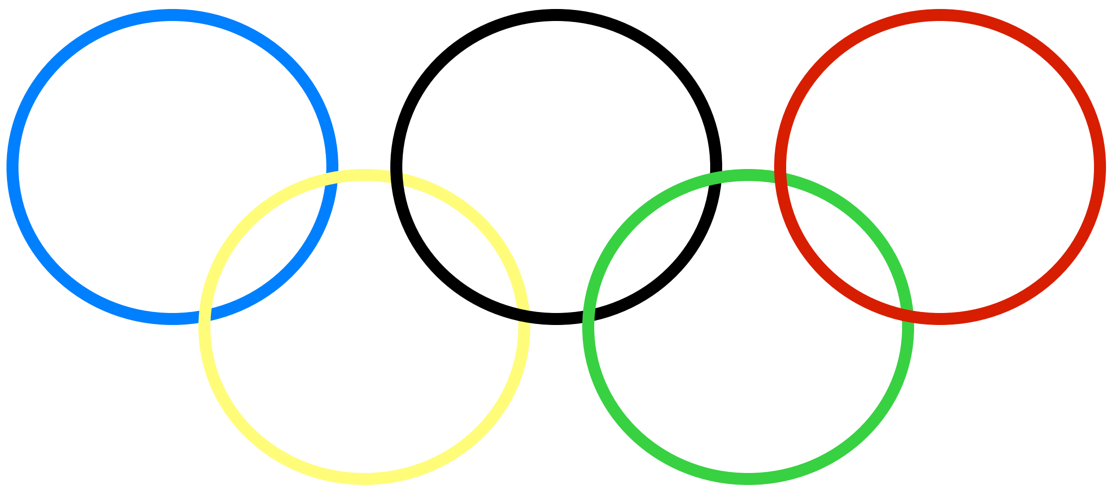
Suppose there is such a walk, there is a starting point and an endpoint point.

For every "intermediate" point v, there must be the same number of incoming and outgoing edges, and so v must have an **even number of edges**.

Euler's Solution

So Euler showed that the **"Seven Bridges of Königsberg"** is unsolvable.

When is it possible to have a walk that visits every edge exactly once?

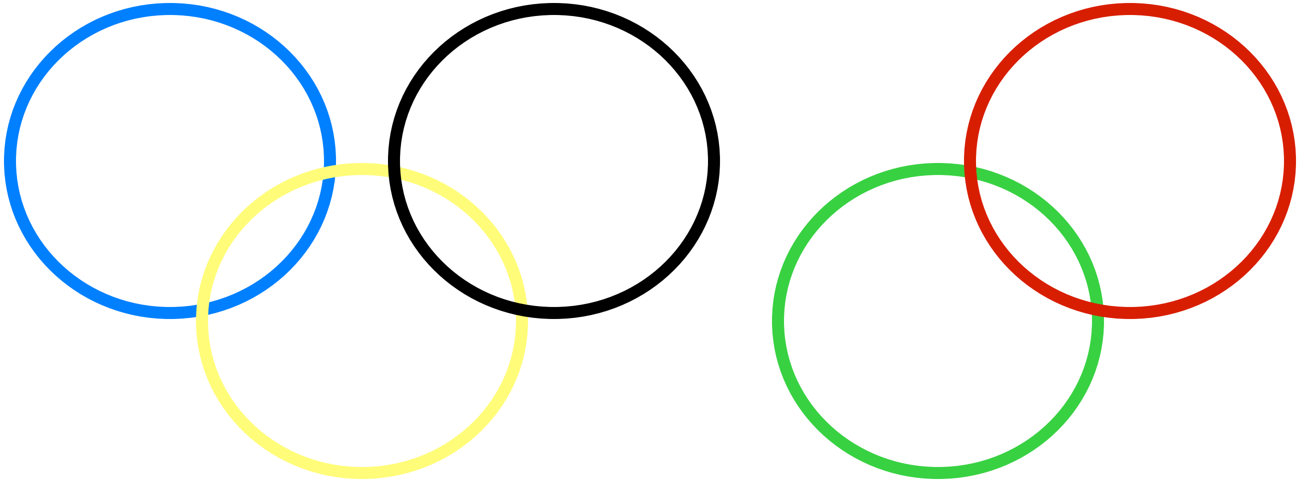


Is it always possible to find such a walk if there is **at most two** vertices with odd number of edges?

Euler's Solution

So Euler showed that the **"Seven Bridges of Königsberg"** is unsolvable.

When is it possible to have a walk that visits every edge exactly once?



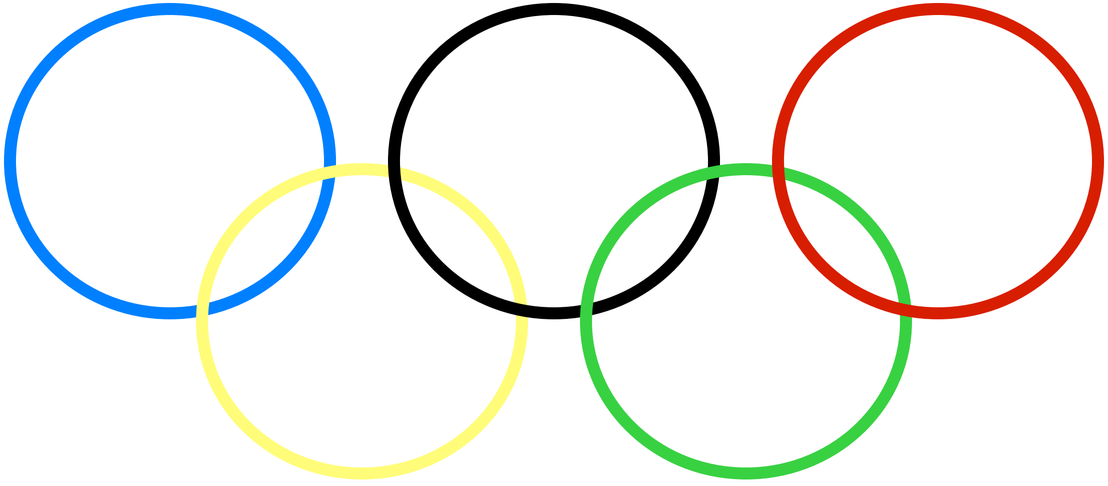
Is it always possible to find such a walk if there is **at most two** vertices with odd number of edges?

NO!

Euler's Solution

So Euler showed that the **"Seven Bridges of Königsberg"** is unsolvable.

When is it possible to have a walk that visits every edge exactly once?



Is it always possible to find such a walk if the graph is **"connected"** and there are **at most two** vertices with odd number of edges?

YES!

Euler's Solution

So Euler showed that the **"Seven Bridges of Königsberg"** is unsolvable.

When is it possible to have a walk that visits every edge exactly once?

Eulerian path

Euler's theorem: A graph has an Eulerian path if and only if it is "connected" and has at most two vertices with an odd number of edges.

This theorem was proved in 1736,
and was regarded as the starting point of graph theory.

This Lecture

- Seven bridges of Konigsberg
- **Graphs, degrees**
- Isomorphism
- Path, cycle, connectedness
- Tree
- Eulerian cycle
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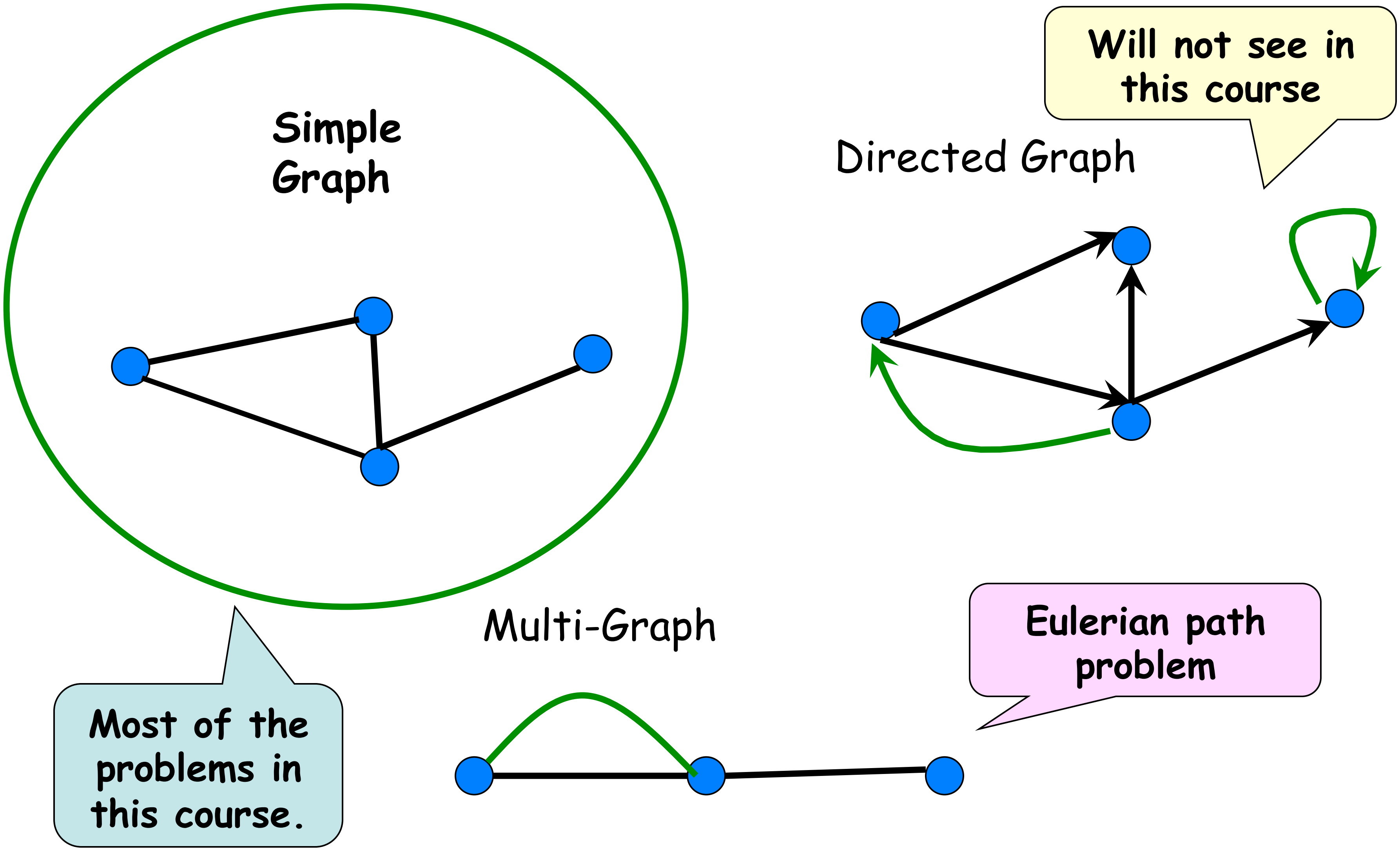
Types of Graphs



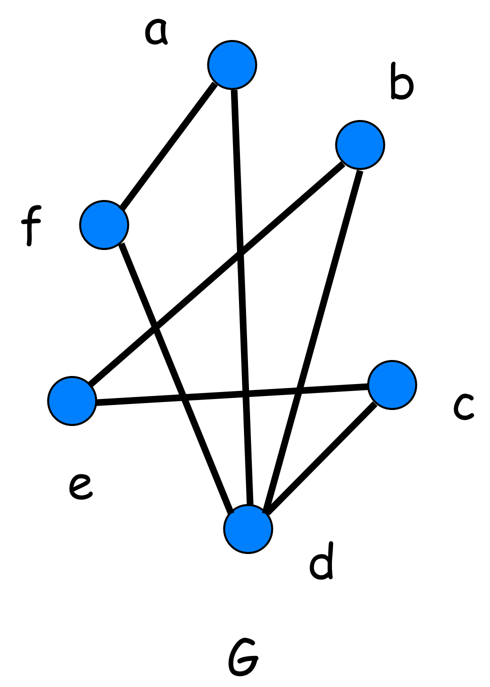
Simple Graphs

A graph G=(V,E) consists of:

A set of vertices, *V*

A set of *undirected* edges, *E*

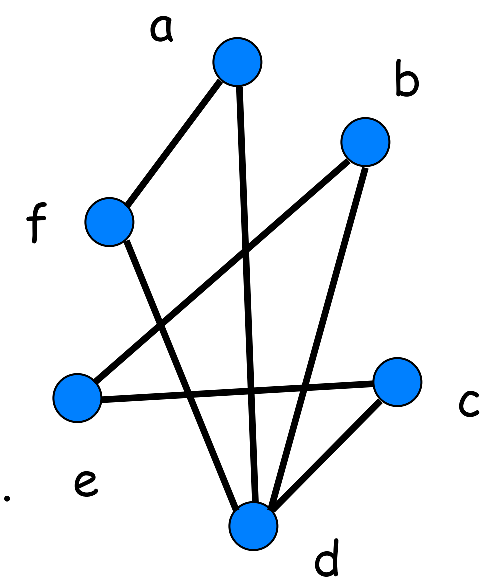
- V(G) = {a,b,c,d,e,f}
- E(G) = {ad,af,bd,be,cd,ce,df}



Two vertices u,v are ***adjacent*** (***neighbours***) if the edge uv is present.

Vertex Degrees

An edge uv is ***incident*** on the vertex u and the vertex v.



The ***neighbour set*** N(v) of a vertex v is the set of vertices adjacent to it.

e.g. N(a) = {d,f}, N(d) = {a,b,c,f}, N(e) = {b,c}.

degree of a vertex = # of ***incident*** edges

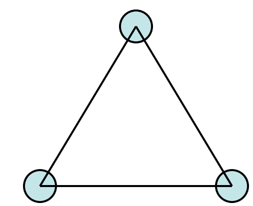
e.g. deg(d) = 4, deg(a)=deg(b)=deg(c)=deg(e)=deg(f)=2.

the degree of a vertex v = the number of neighbours of v?

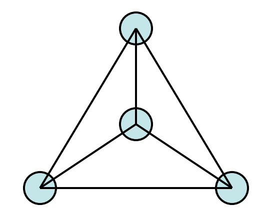
For multigraphs, NO. For simple graphs, YES.

Degree Sequence

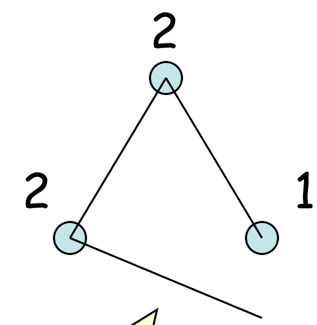
Is there a graph with degree sequence (2,2,2)? YES.



Is there a graph with degree sequence (3,3,3,3)? YES.



Is there a graph with degree sequence (2,2,1)? NO.



Is there a graph with degree sequence (2,2,2,2,1)?

Where to go?

NO.

What's wrong with these sequences?

Handshaking Lemma

For any graph, sum of degrees = twice # edges

Lemma.

$$2|E| = \sum_{v \in V} \deg(v)$$

Corollary.

1. Sum of degree is an even number.
2. Number of odd degree vertices is even.

Examples. 2+2+1 = odd, so impossible.

2+2+2+2+1 = odd, so impossible.

Handshaking Lemma

Lemma. $$\boxed{2|E| = \sum_{v \in V} \deg(v)}$$

Proof. Each edge contributes 2 to the sum on the right. Q.E.D.

Question. Given a degree sequence, if the sum of degree is even, is it true that there is a graph with such a degree sequence?

For simple graphs, NO, consider the degree sequence (3,3,3,1).

For multigraphs (with self loops), YES!

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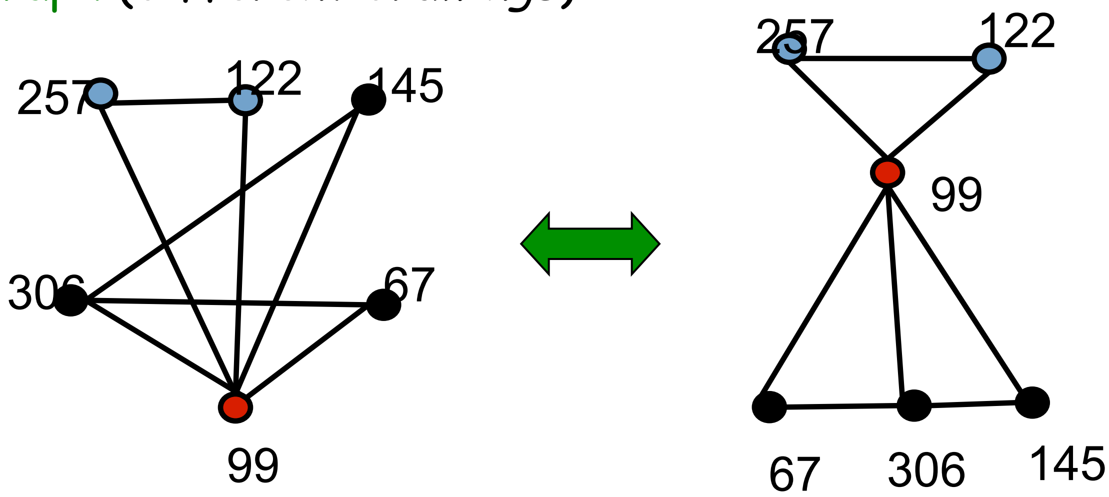
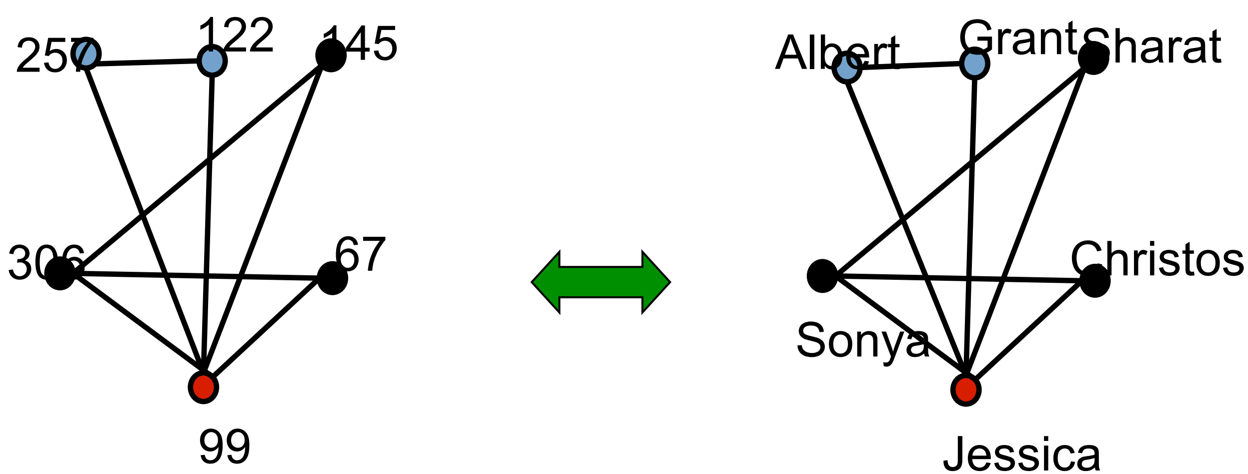
Same Graphs?

Same graph (different *drawings*)



Same graph (different *labels*)



Graph Isomorphism

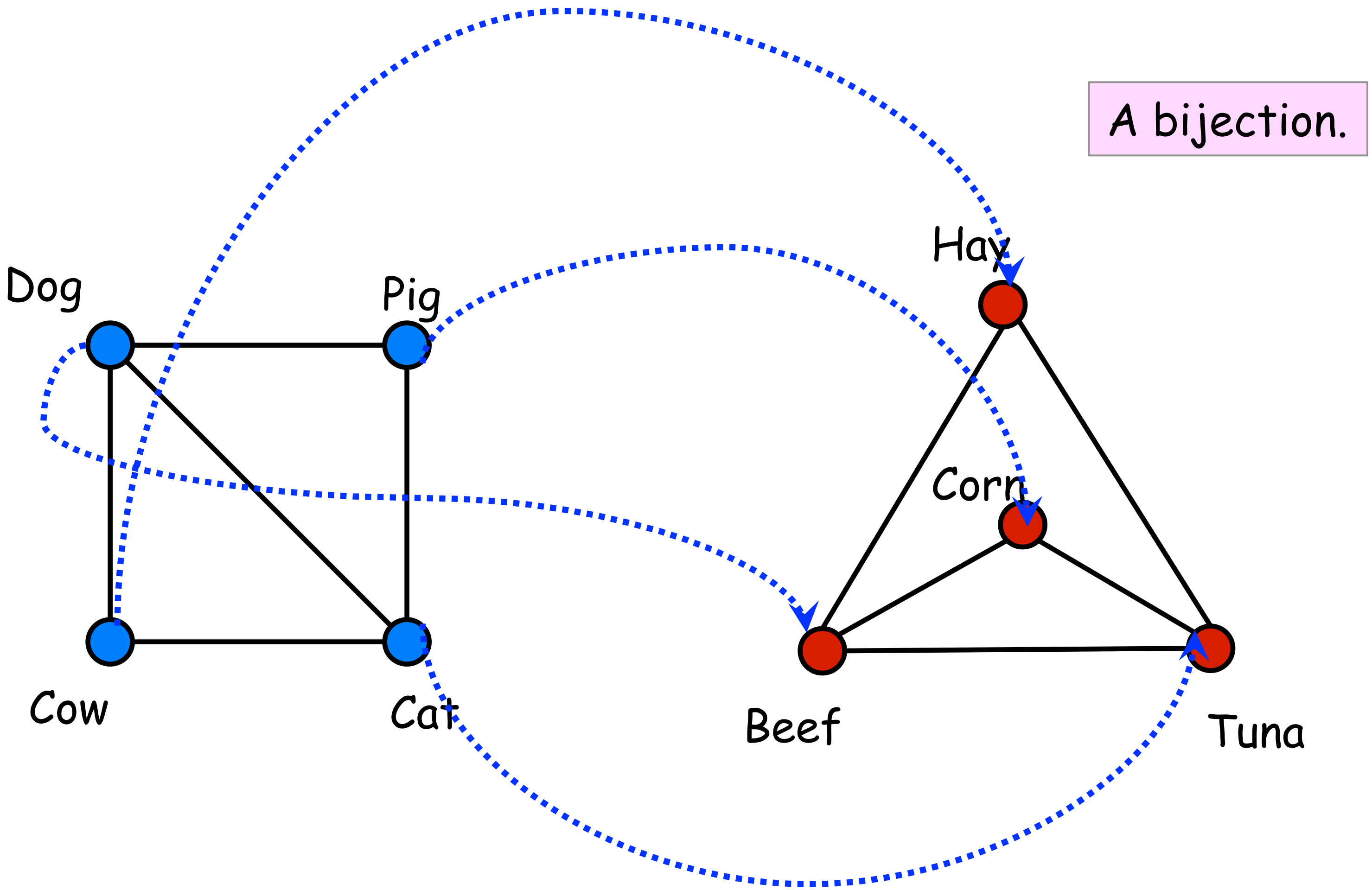
All that matters is the *connections.*

Graphs with the same connections are *isomorphic*.

Informally, two graphs are isomorphic if they are the same after *renaming*.

Graph isomorphism has applications like checking fingerprint, testing molecules…

Are These Isomorphic?



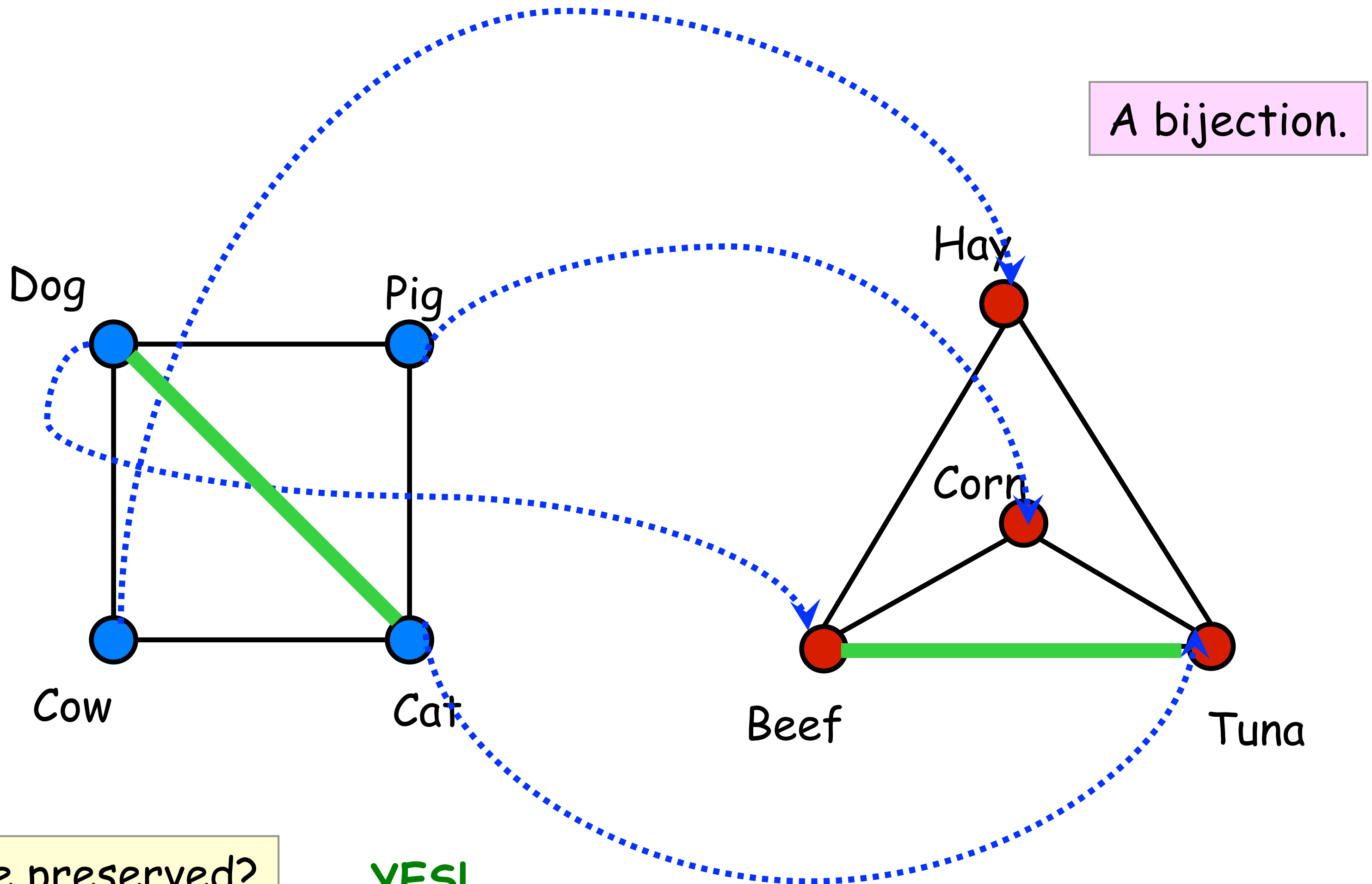
f (Dog) = Beef

f (Cat) = Tuna

f (Cow) = Hay

f (Pig) = Corn

Are These Isomorphic?

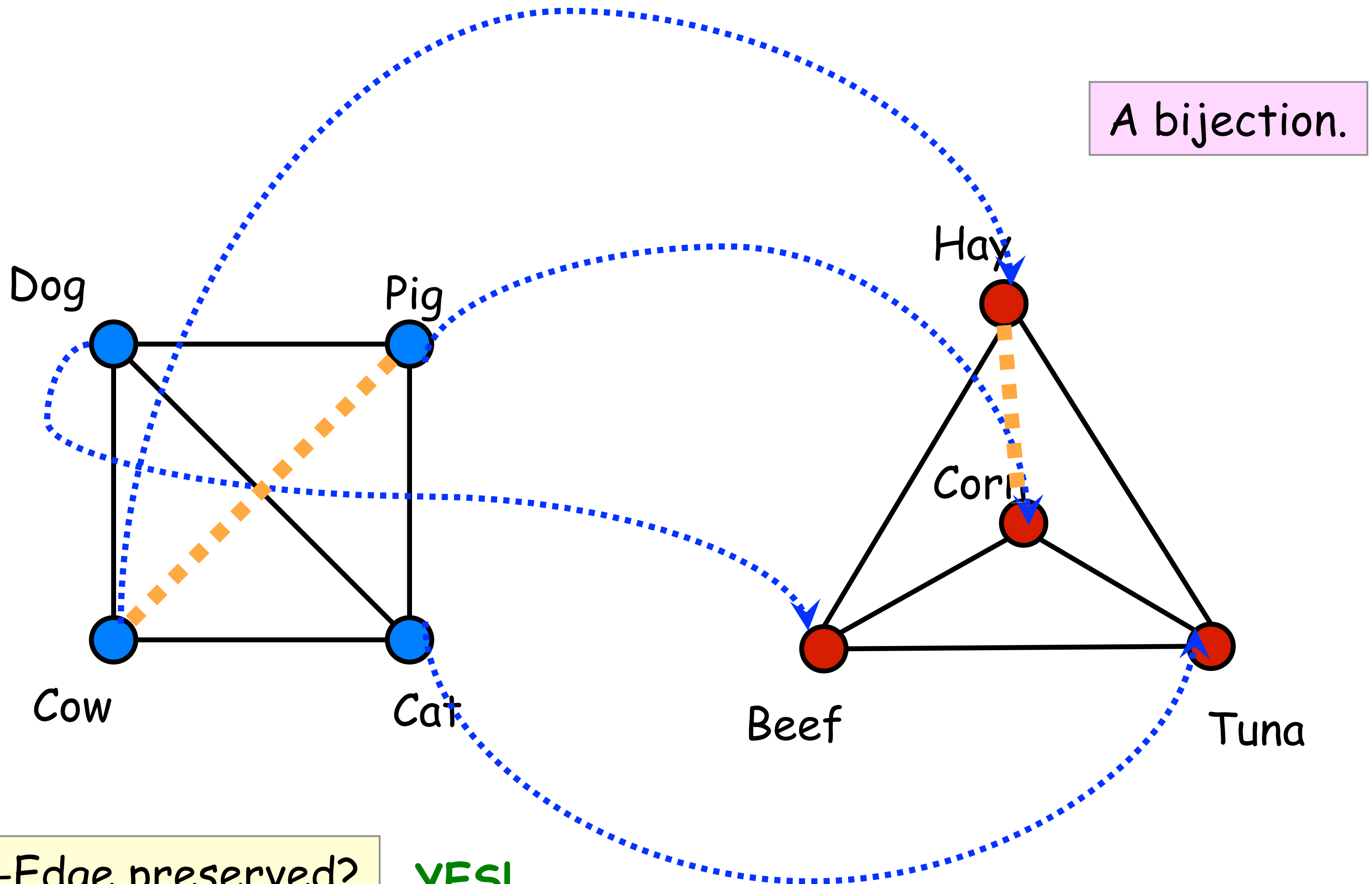


Edge preserved?

YES!

If there is an edge in the original graph, there is an edge after the mapping.

Are These Isomorphic?



Non-Edge preserved?

YES!

If there is no edge in the original graph, there is no edge after the mapping.

Graph Isomorphism

G_1 *isomorphic* to G_2 means there is an *edge-preserving vertex matching.*

bijection $f: V_1 \to V_2$

$u - v$ in E_1 iff $f(u) - f(v)$ in E_2

uv is an edge in G1

f(u)f(v) is an edge in G2

- If G1 and G2 are isomorphic, do they have the same number of vertices? YES
- If G1 and G2 are isomorphic, do they have the same number of edges? YES
- If G1 and G2 are isomorphic, do they have the same degree sequence? YES
- If G1 and G2 have the same degree sequence, are they isomorphic? NO

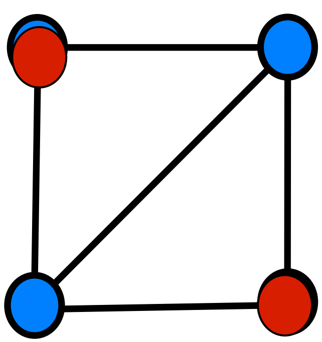
Checking Graph Isomorphism

How to show two graphs are isomorphic?

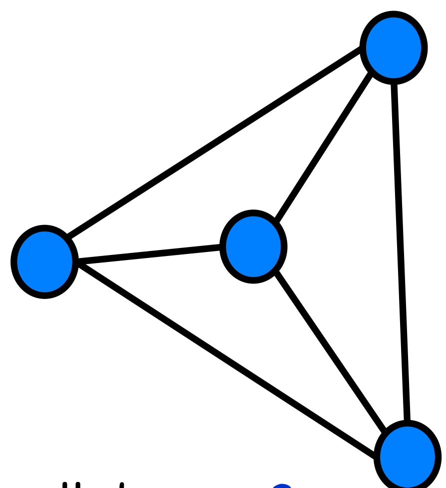
Find a mapping and show that it is edge-preserving.

How to show two graphs are non-isomorphic?

Find some isomorphic-preserving properties which is satisfied in one graph but not the other.



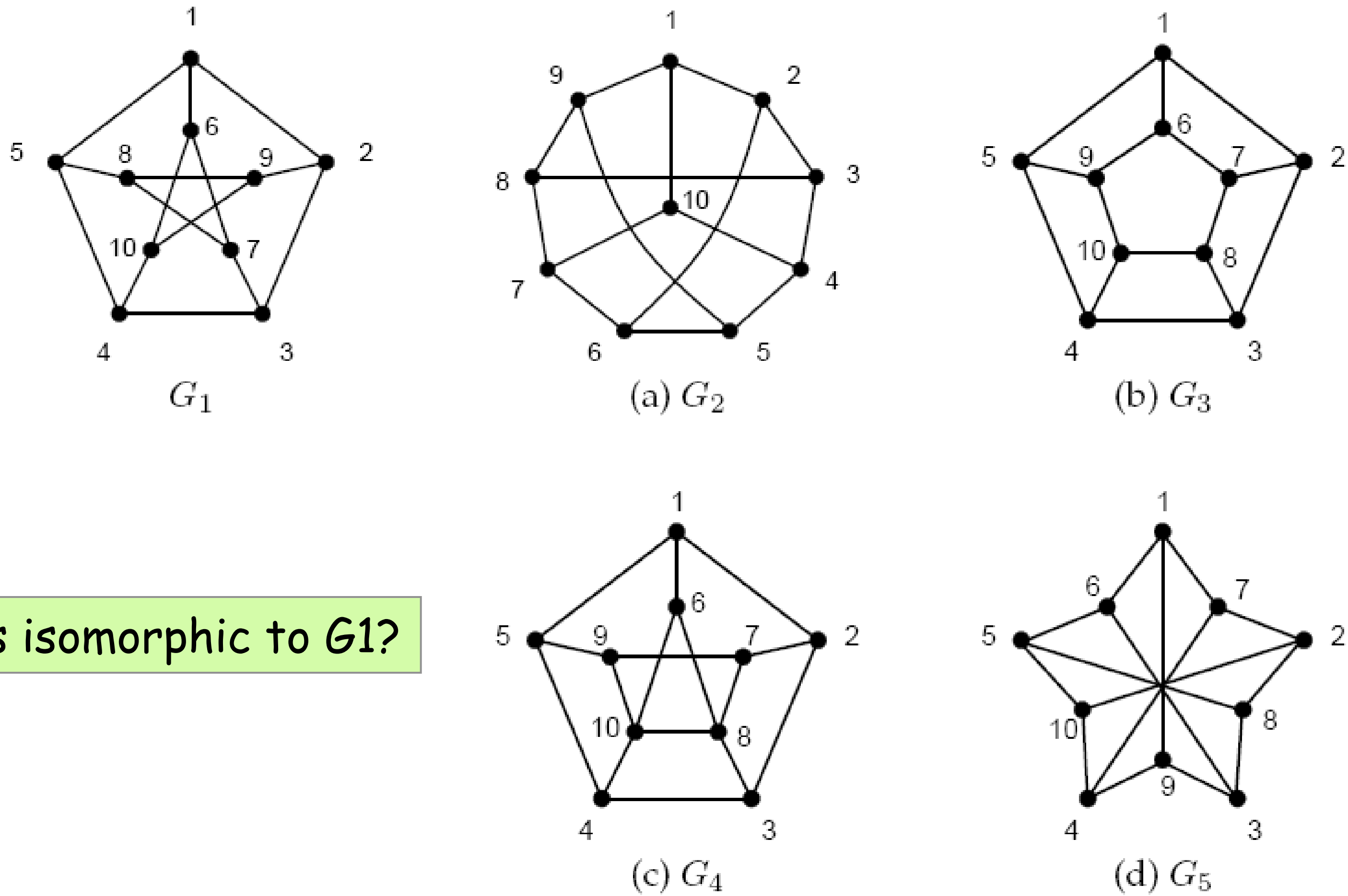
degree 2



all degree 3

Non-isomorphic

Exercise


G_1 (a) G_2 (b) G_3 (c) G_4 (d) G_5

Which is isomorphic to G1?

Testing graph isomorphism is not easy –

No **known*** general method to test graph ismorphism much more efficient than checking all possibilities.

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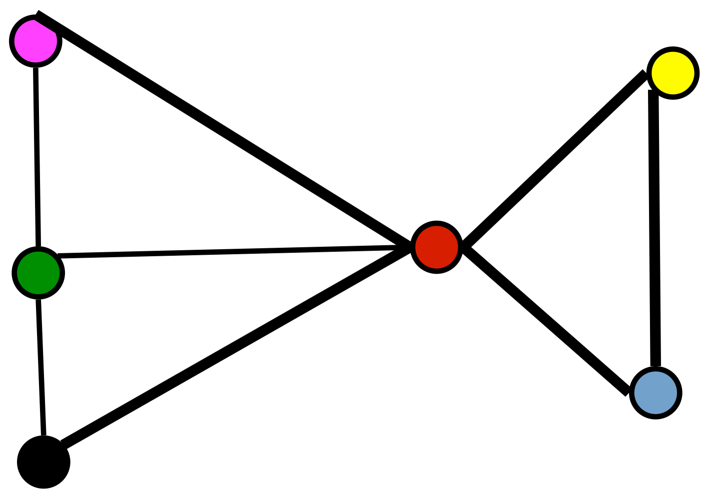
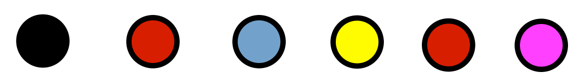
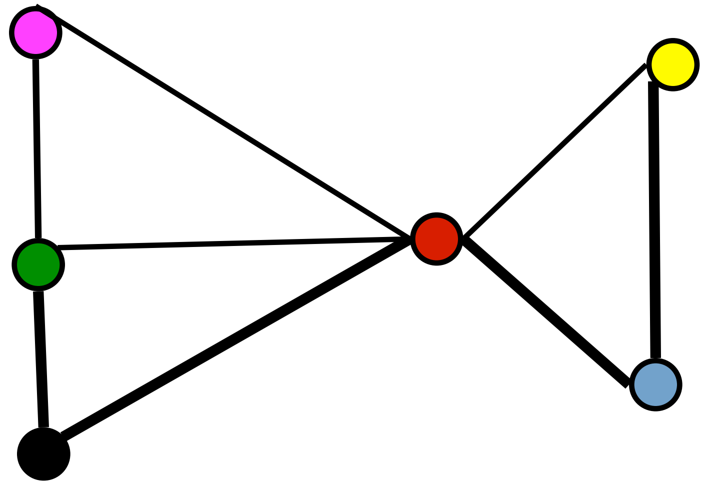
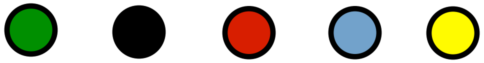
Paths



Path: sequence of *adjacent* vertices ()

Simple Paths

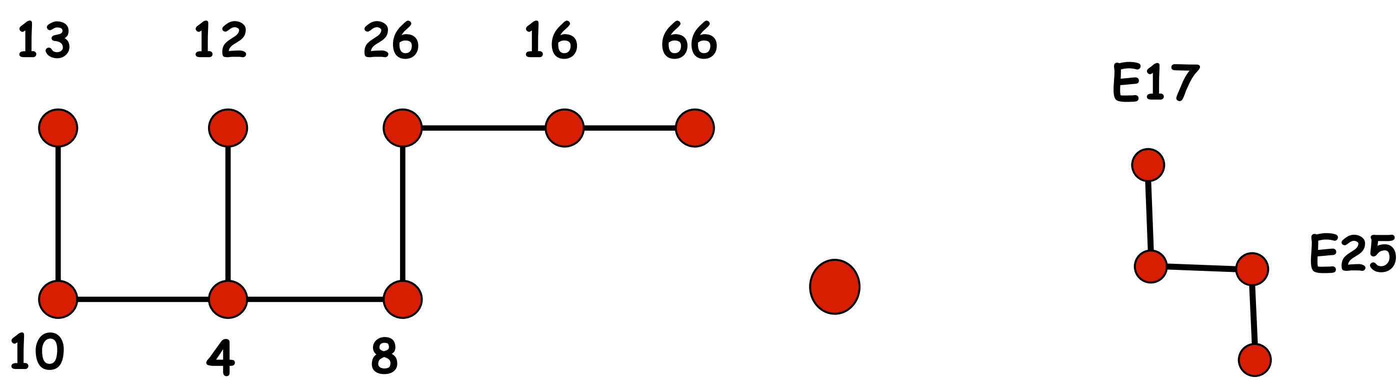


Simple Path: all vertices different ()

Connectedness

- Vertices *v*, *w* are *connected* if and only if there is a path starting at *v* and ending at *w*.
- A *graph* is *connected* iff every pair of vertices are connected.

Every graph consists of separate connected pieces called *connected components*

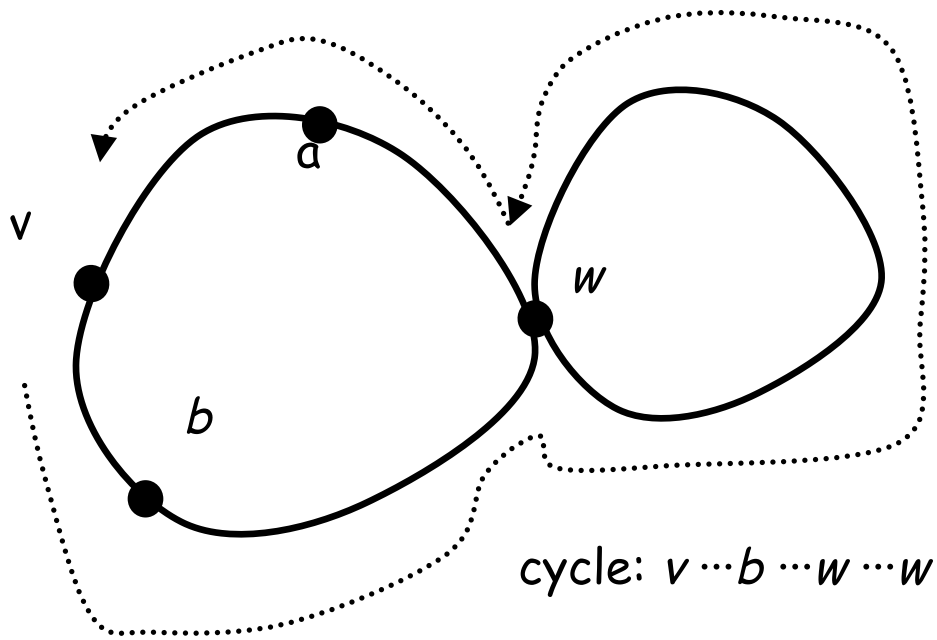


3 connected components

So a graph is connected if and only if it has only 1 connected component.

Cycles

A *cycle* is a path that begins and ends with same vertex.



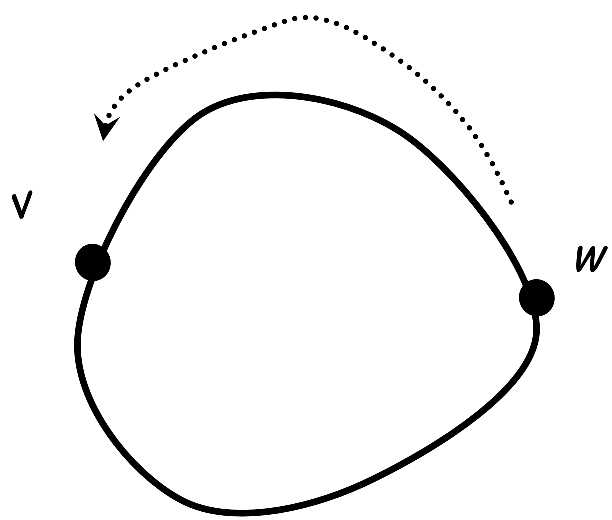
cycle: $v \cdots b \cdots w \cdots w \cdots a \cdots v$

also: $a \cdots v \cdots b \cdots w \cdots w \cdots a$

Simple Cycles

A simple *cycle* is a cycle that doesn't cross itself

In a simple cycle, every vertex is of degree exactly 2.



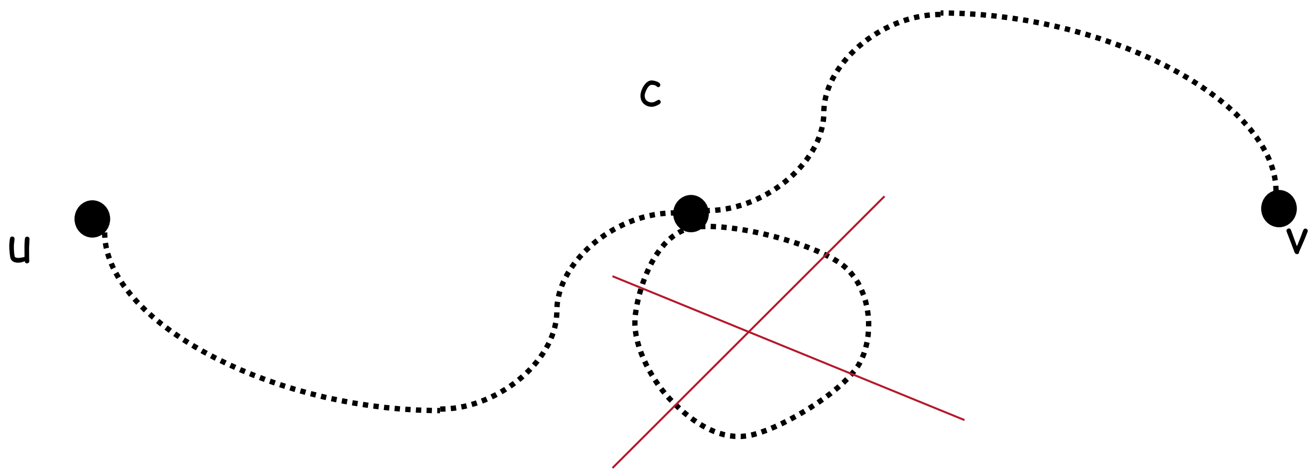
cycle: $v \cdots w \cdots v$ also: $w \cdots v \cdots w$

Shortest Paths

A path between u and v is a ***shortest path***
if among all u-v paths it uses the minimum number of edges.

Is a shortest path between two vertices always simple?

Idea: remove the cycle will make the path shorter.



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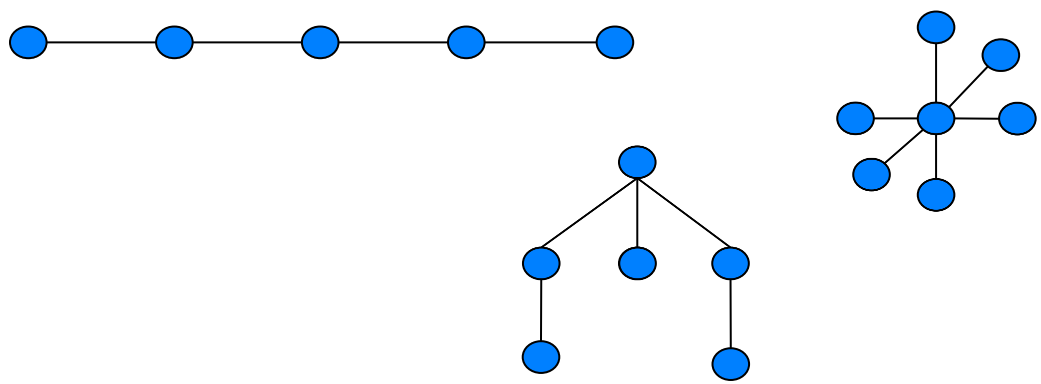
Tree

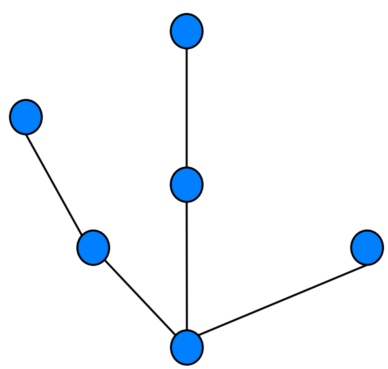
Graphs with no cycles?

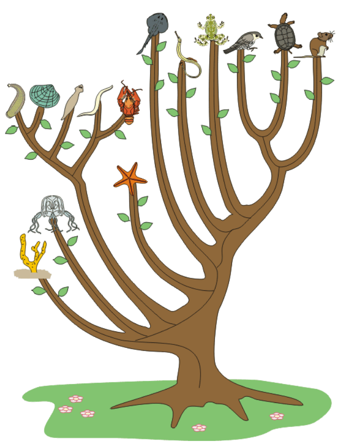
A forest.



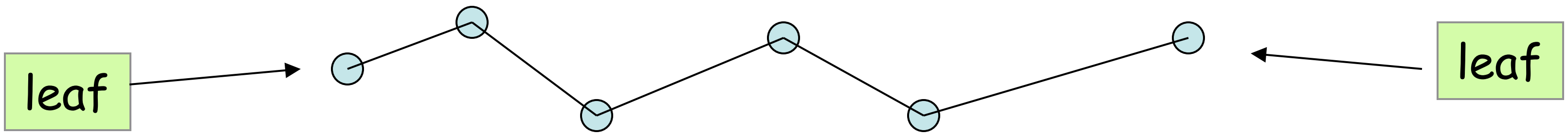
Connected graphs with no cycles?

A tree.

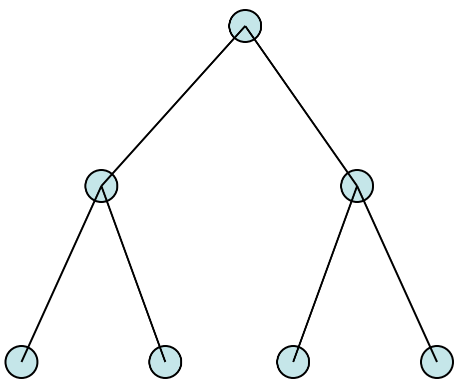




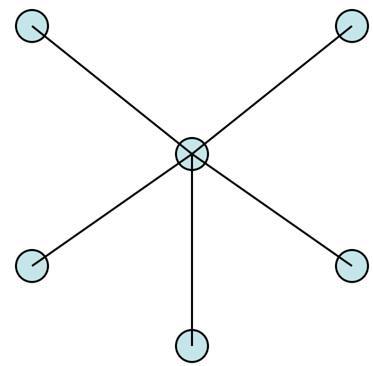
More Trees



A leaf is a vertex of degree 1.



More leaves.



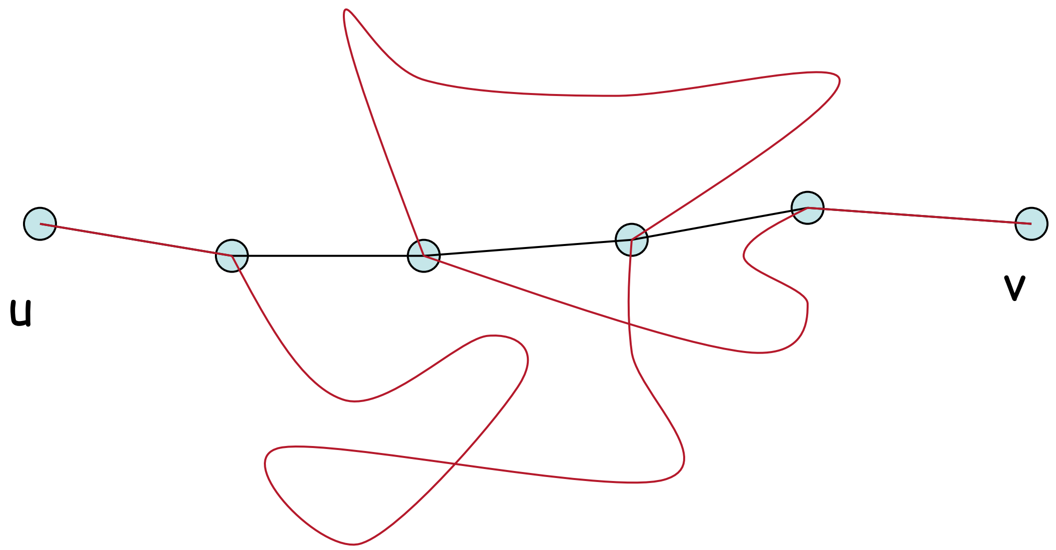
Even more leaves.

Tree Characterization by Path

Definition. A tree is a connected graph with no cycles.

Can there be no path between u and v? NO

Can there be more than one simple path between u and v? NO



This will create cycles.

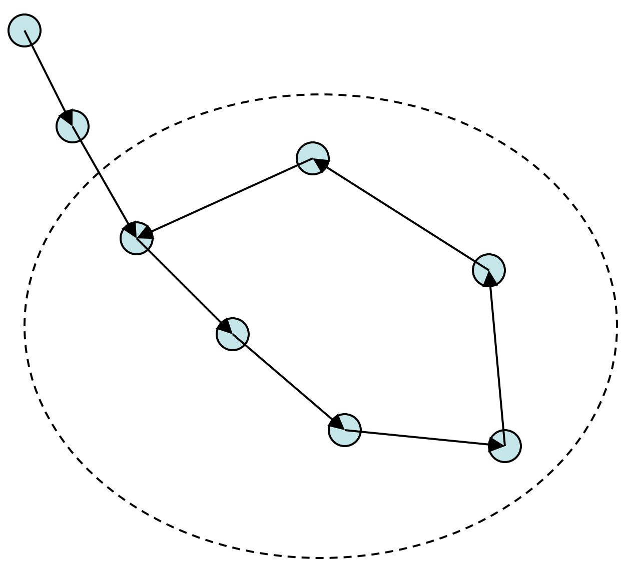
Claim. In a tree, there is a unique simple path between every pair of vertices.

Tree Characterization by Number of Edges

Definition. A tree is a connected graph with no cycles.

Can a tree have no leaves? NO

Then every vertex has degree at least 2.



Go to unvisited edges as long as possible.

Cannot get stuck,

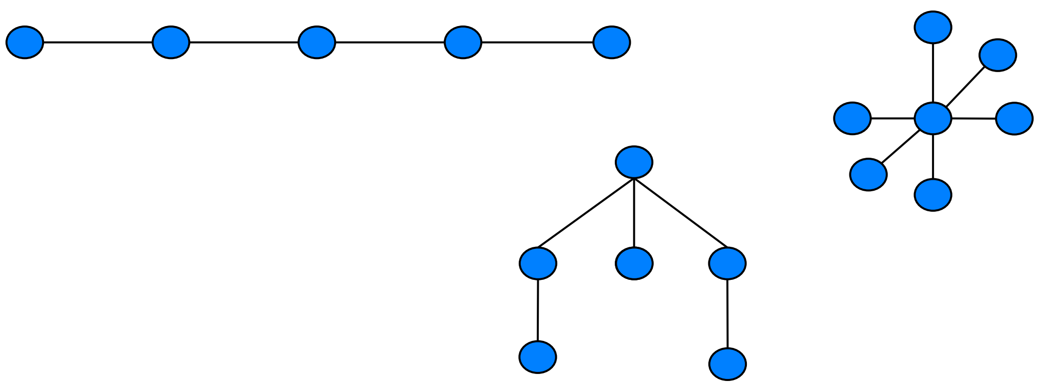
unless there is a cycle.

Tree Characterization by Number of Edges

Definition. A tree is a connected graph with no cycles.

Can a tree have no leaves? NO

How many edges does a tree have? n-1?



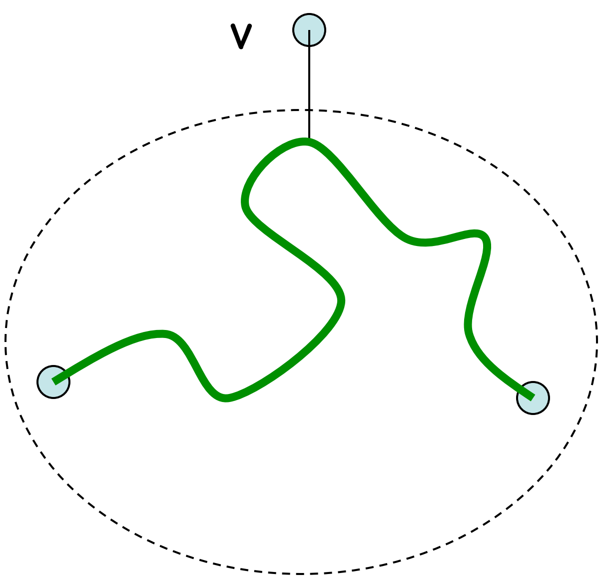
We usually use n to denote the number of vertices,
and use m to denote the number of edges in a graph.

Tree Characterization by Number of Edges

Definition. A tree is a connected graph with no cycles.

Can a tree have no leaves? NO

How many edges does a tree have? n-1?



Look at a leaf v.

Is T-v a tree? YES

1. Can T-v have a cycle? NO
2. Is T-v connected? YES

By induction, T-v has (n-1)-1=n-2 edges.

So T has n-1 edges.

Tree Characterizations

Definition. A tree is a connected graph with no cycles.

Characterization by paths:

A graph is a tree if and only if

there is a unique simple path between every pair of vertices.

Characterization by number of edges:

A graph is a tree if and only if it is connected and has n-1 edges.

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Eulerian Graphs

Euler's theorem: A graph has an Eulerian path if and only if it is connected and has at most two vertices with an odd number of edges.

Can a graph have only 1 odd degree vertex?

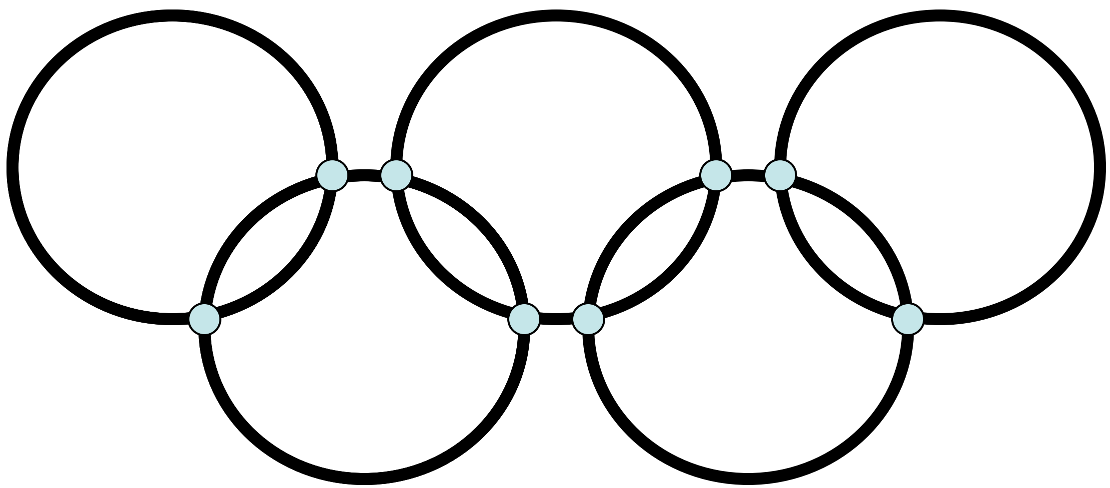
Odd degree vertices.

Euler's theorem: A connected graph has an Eulerian path if and only if it has zero or two vertices with odd degrees.

Eulerian Cycle

Euler's theorem: A connected graph has an Eulerian cycle if and only if every vertex is of even degree.

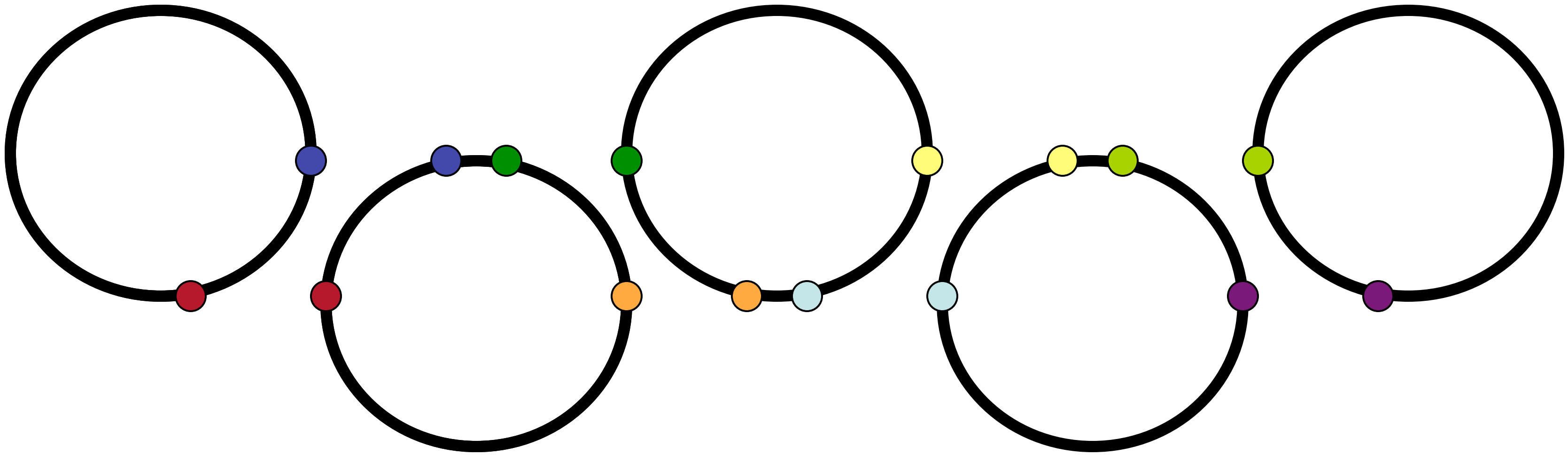
First we find an Eulerian cycle in the below example.



Eulerian Cycle

Euler's theorem: A connected graph has an Eulerian cycle if and only if every vertex is of even degree.

Note that the edges can be partitioned into five simple cycles.

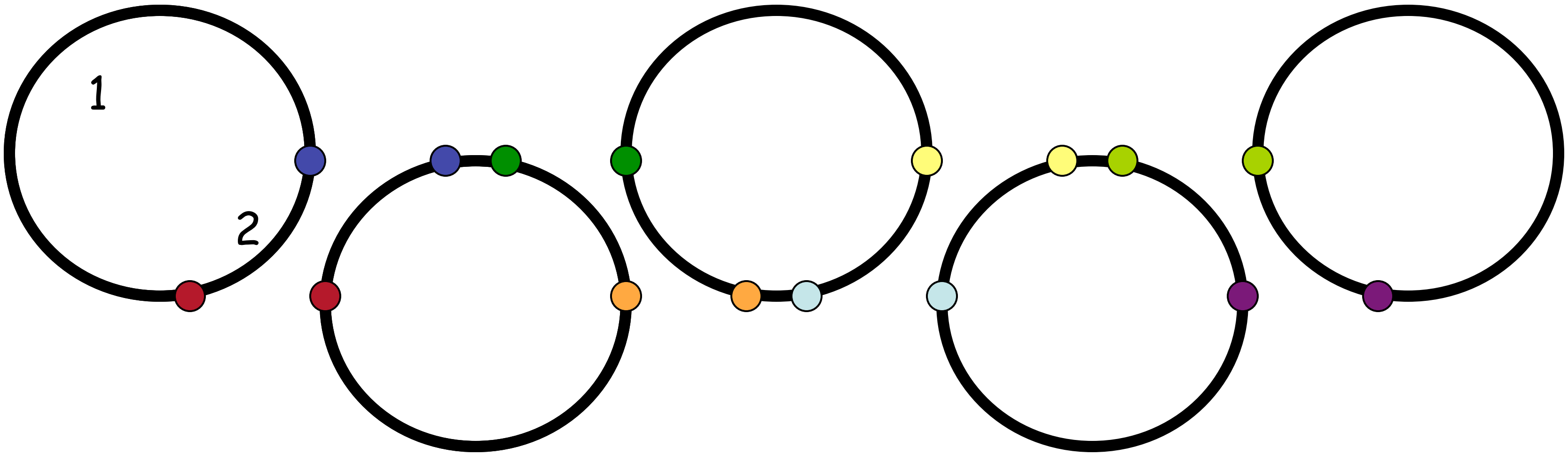


Vertices of the same color represent the same vertices.

Eulerian Cycle

Euler's theorem: A connected graph has an Eulerian cycle if and only if every vertex is of even degree.

The idea is that we can construct an Eulerian cycle by adding cycle one by one.

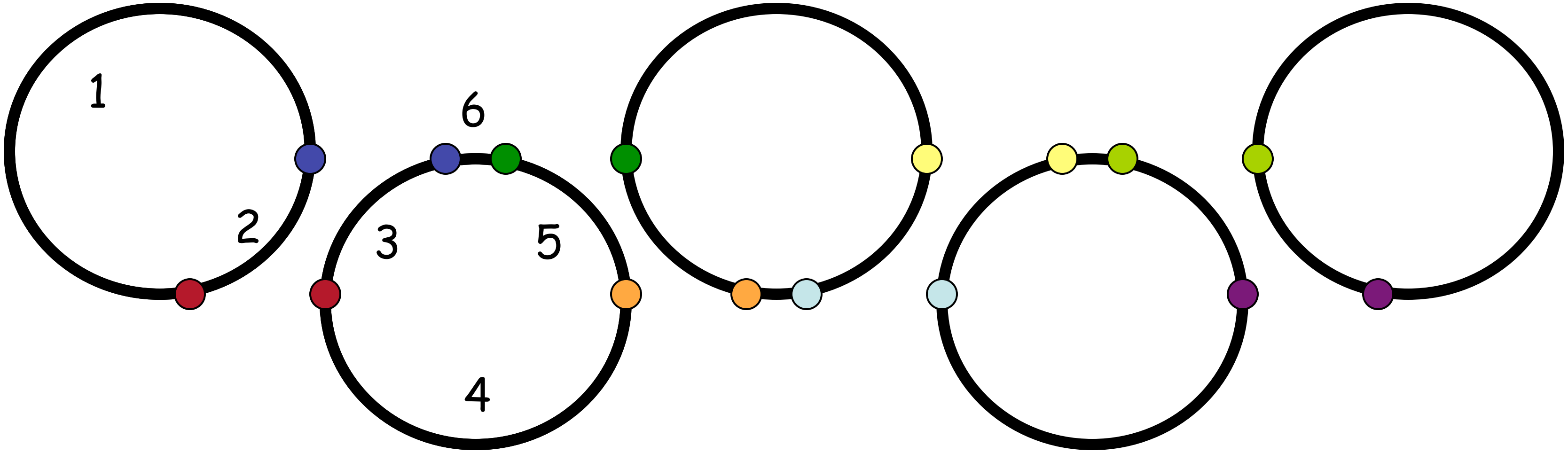


First traverse the first cycle.

Eulerian Cycle

Euler's theorem: A connected graph has an Eulerian cycle if and only if every vertex is of even degree.

The idea is that we can construct an Eulerian cycle by adding cycle one by one.

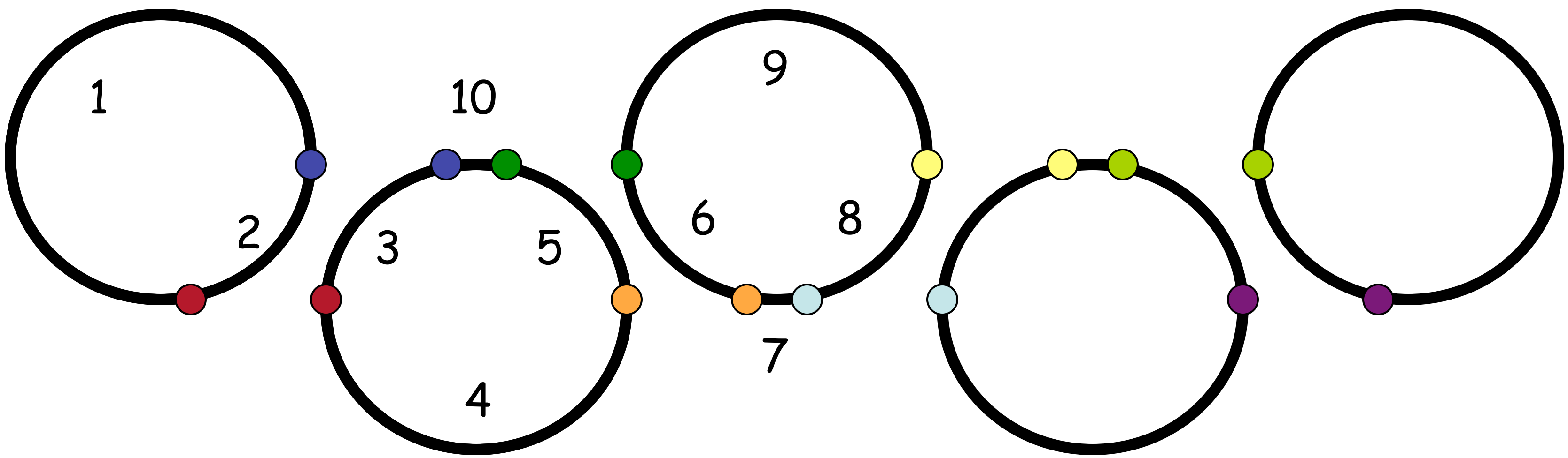


Then traverse the second cycle.

Eulerian Cycle

Euler's theorem: A connected graph has an Eulerian cycle if and only if every vertex is of even degree.

How to deal with the third cycle?

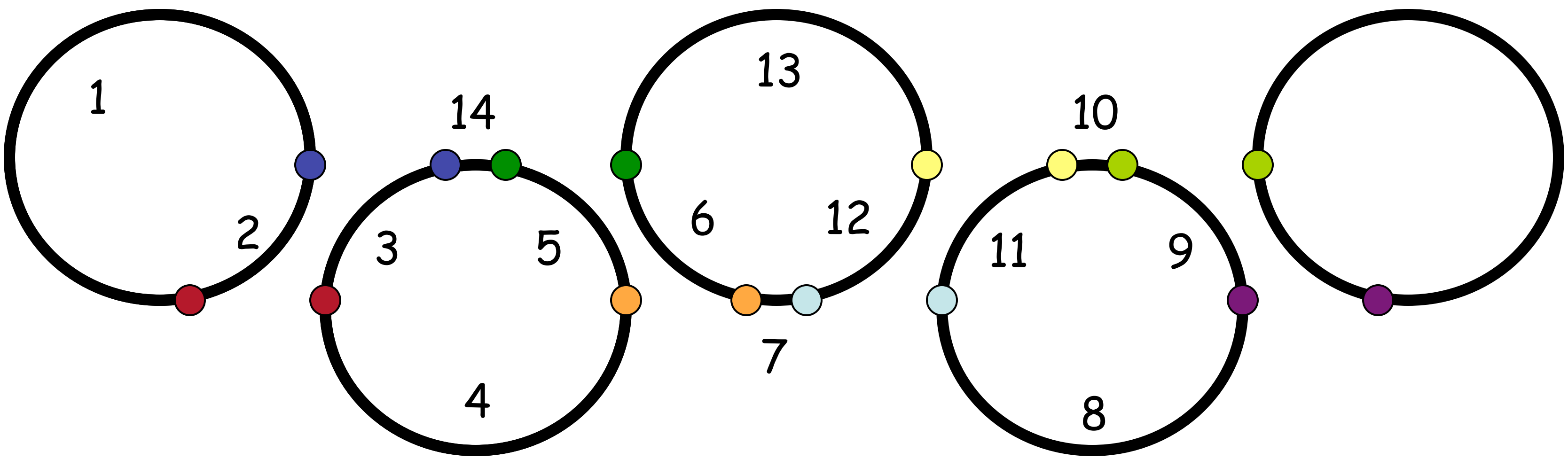


We can "detour" to the third cycle before finishing the second cycle.

Eulerian Cycle

Euler's theorem: A connected graph has an Eulerian cycle if and only if every vertex is of even degree.

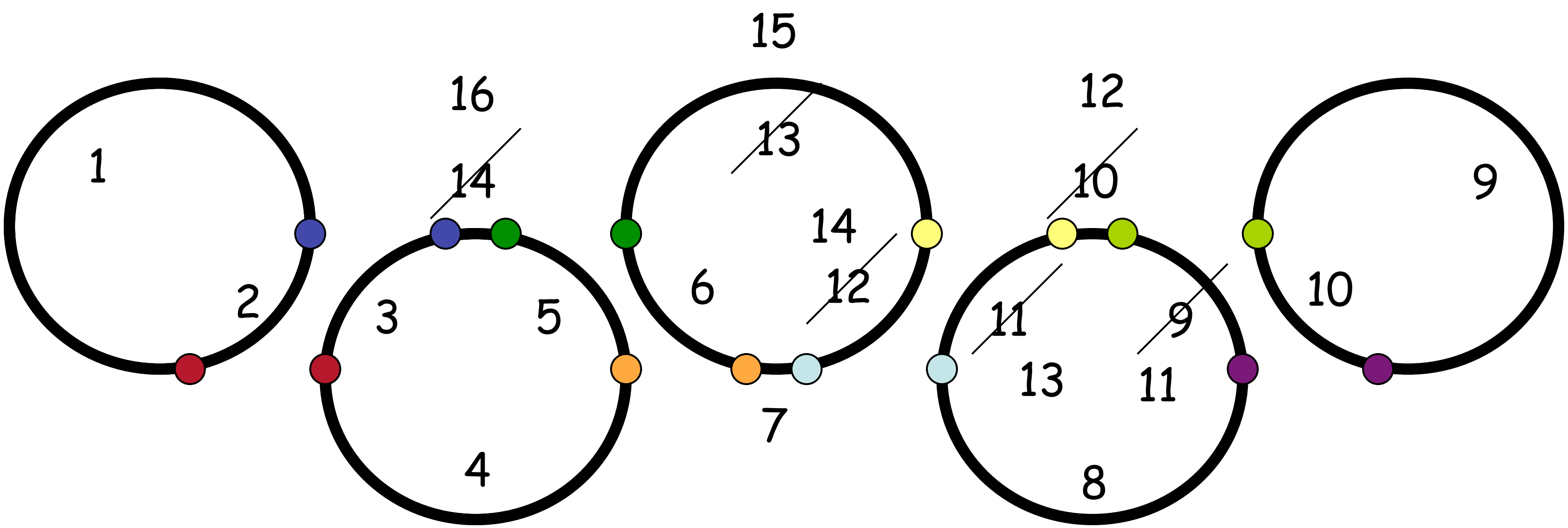
We use the same idea to deal with the fourth cycle



We can "detour" to the fourth cycle at an "intersection point".

Eulerian Cycle

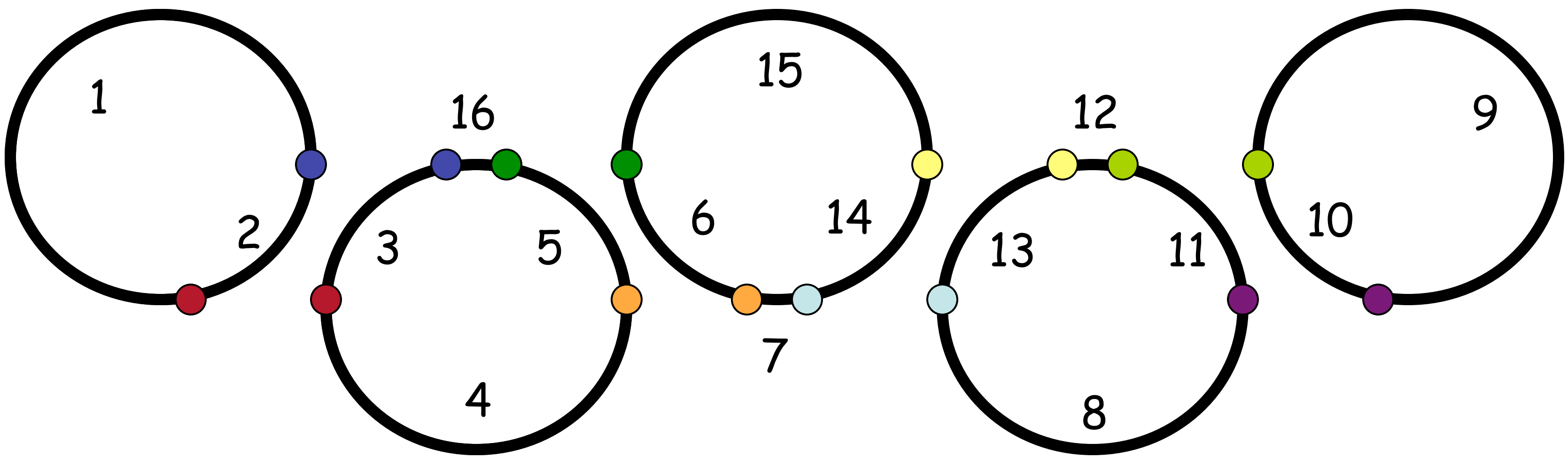
Euler's theorem: A connected graph has an Eulerian cycle if and only if every vertex is of even degree.



We can "insert" the fifth cycle at an "intersection point".

Eulerian Cycle

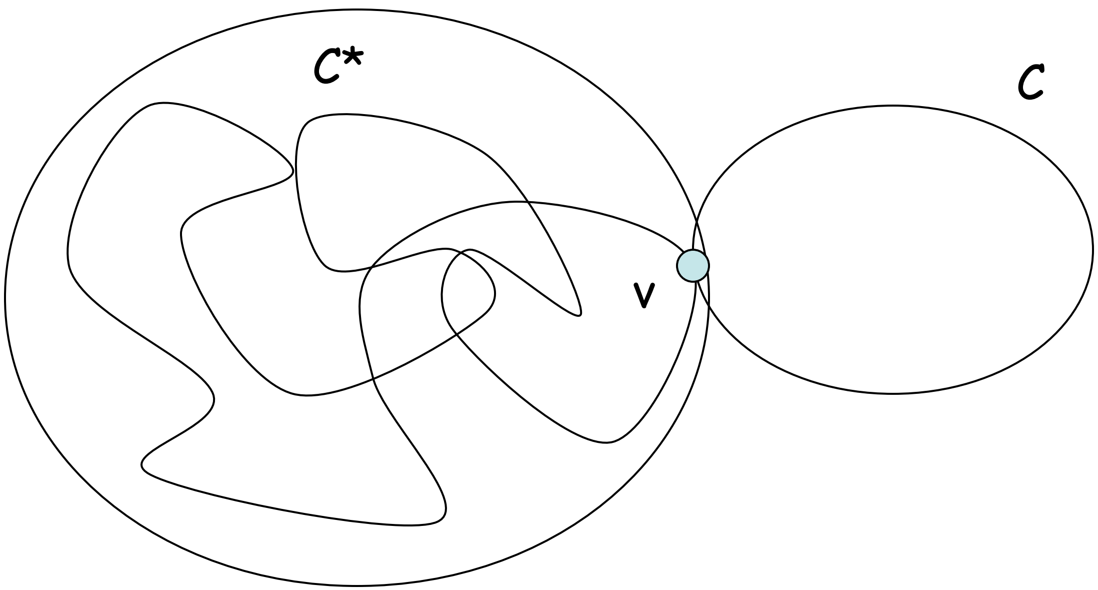
Euler's theorem: A connected graph has an Eulerian cycle if and only if every vertex is of even degree.



So we have an Eulerian cycle of this example

Idea

In general, if we have a "partial" Eulerian cycle C^*, and it intersects with a cycle C on a vertex v, then we can extend the Eulerian cycle C^* to include C.



First follow C^* until we visit v, then follow C until we go back to v, and then follow C^* from v to the end.

Proof

We have informally proved the following claim in the previous slides.

Claim 1. If the edges of a connected graph can be partitioned into simple cycles, then we can construct an Eulerian cycle.

Euler's theorem: A connected graph has an Eulerian cycle if and only if every vertex is of even degree.

We can prove Euler's theorem if we can prove the following claim.

Claim 2. If every vertex is of even degree, then the edges can be partitioned into simple cycles.

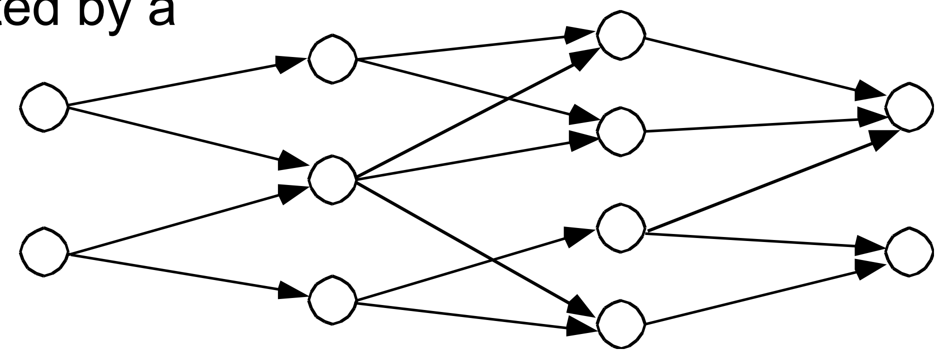
Proof is not considered here.

This Lecture

- Seven bridges of Konigsberg
- Graphs, degrees
- Isomorphism
- Path, cycle, connectedness
- Tree
- Eulerian cycle
- **Graphs and networks**
- Graph coloring

Graphs and Networks

•Many problems can be represented by a graphical network representation.

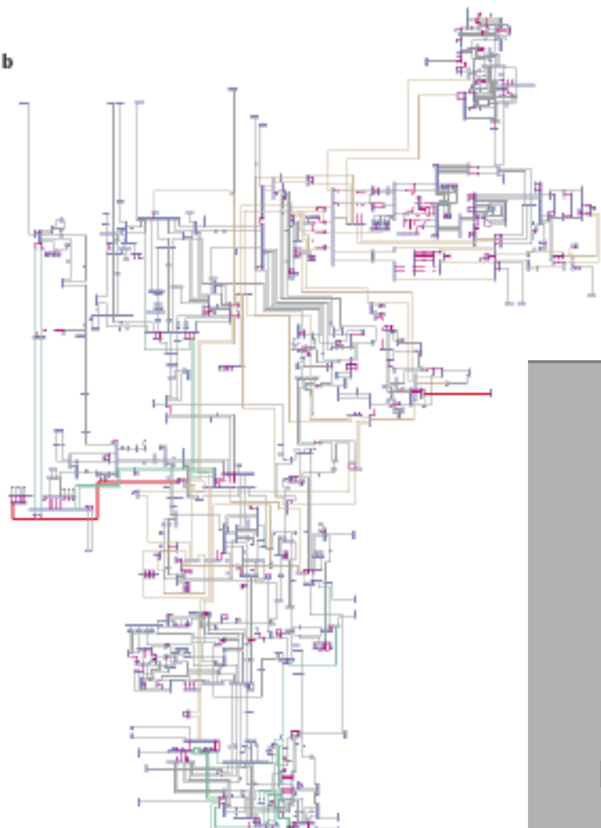


•Examples:
- Distribution problems
- Routing problems
- Maximum flow problems
- Designing computer / phone / road networks
- Equipment replacement
- And of course the Internet

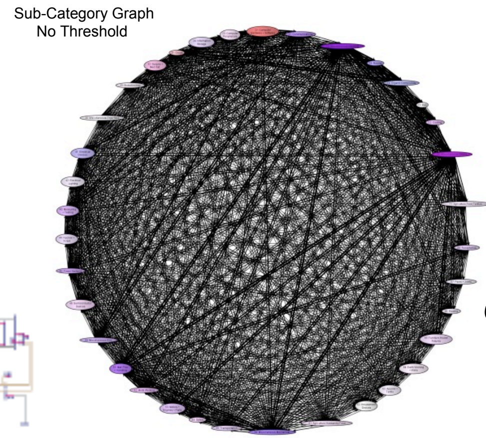
Aside: finding the right problem representation is one of the key issues.

New Science of Networks

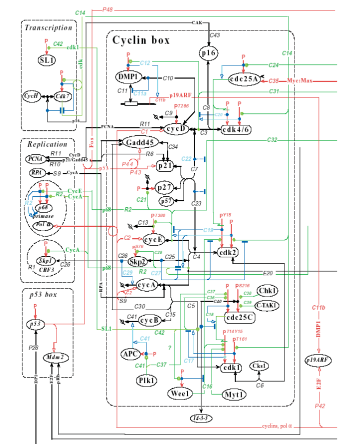
Networks are pervasive



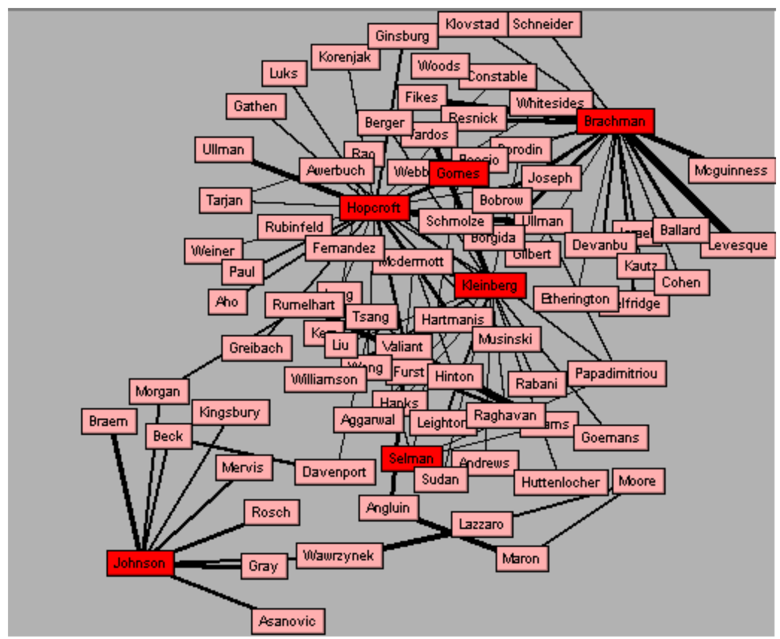
NYS Electric
Power Grid
(Thorp,Strogatz,Watts)



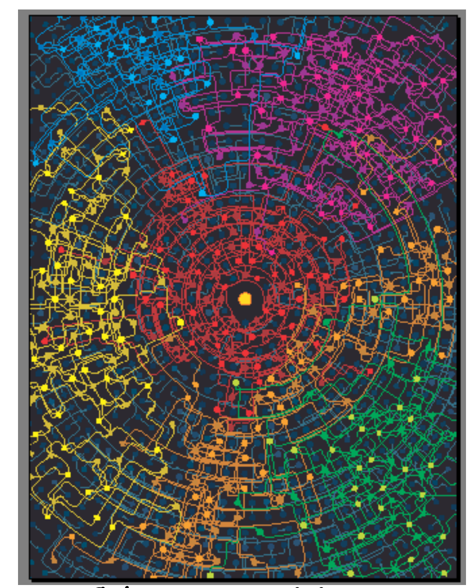
Utility Patent network
1972-1999
(3 Million patents)
Gomes,Hopcroft,Lesser,Sel
man



Neural network of the
nematode worm C- elegans
(Strogatz, Watts)



Network of computer scientists
ReferralWeb System
(Kautz and Selman)



Cybercommunities
(Automatically discovered)
Kleinberg et al

Applications of Networks

Applications	Physical analog of nodes	Physical analog of arcs	Flow
Communication systems	phone exchanges, computers, transmission facilities, satellites	Cables, fiber optic links, microwave relay links	Voice messages, Data, Video transmissions
Hydraulic systems	Pumping stations Reservoirs, Lakes	Pipelines	Water, Gas, Oil, Hydraulic fluids
Integrated computer circuits	Gates, registers, processors	Wires	Electrical current
Mechanical systems	Joints	Rods, Beams, Springs	Heat, Energy
Transportation systems	Intersections, Airports, Rail yards	Highways, Airline routes Railbeds	Passengers, freight, vehicles, operators

This Lecture

- Seven bridges of Konigsberg
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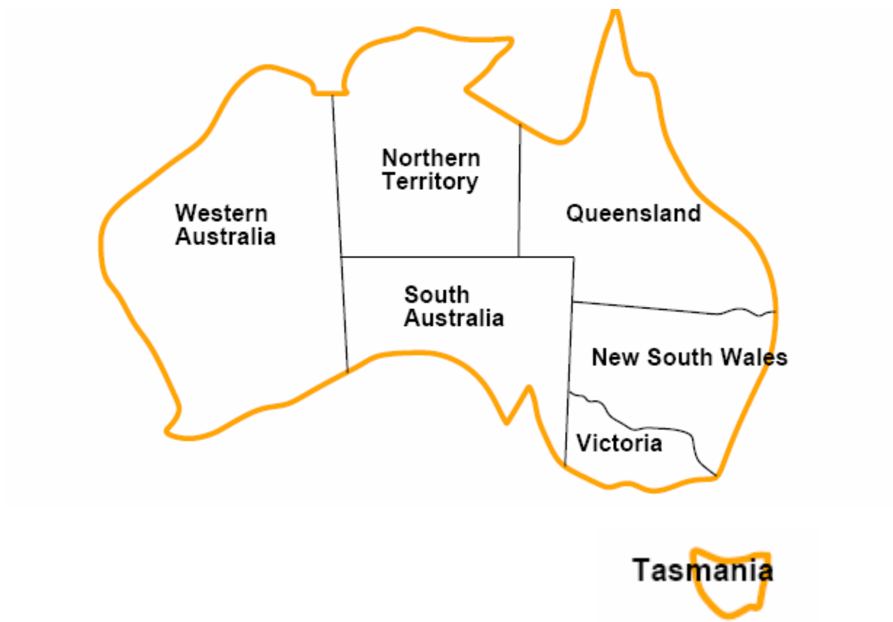
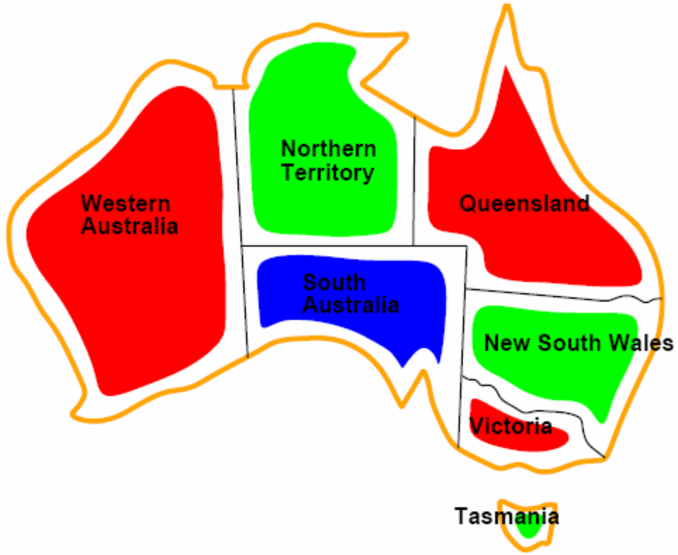
- Graphs and networks
- **Graph coloring**

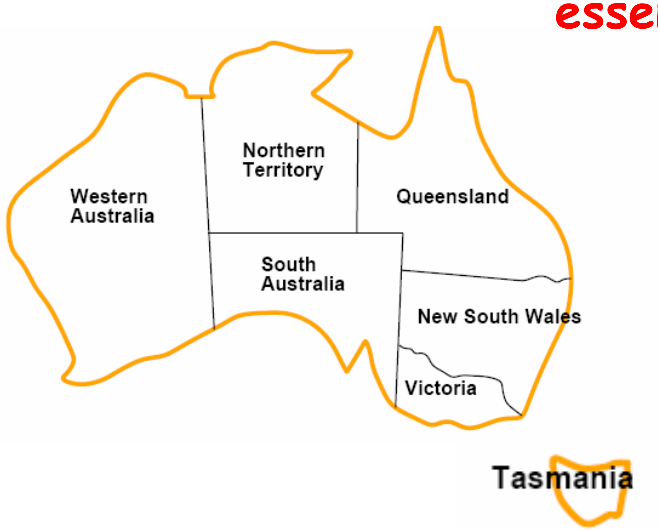
Example: Coloring a Map



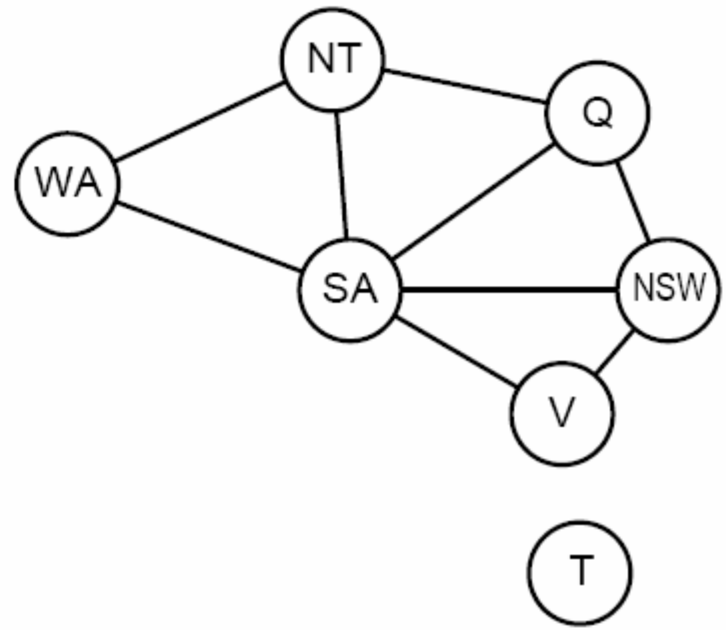


How to color this map so that no two adjacent regions have the same color?

Graph representation

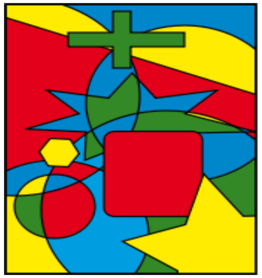


Abstract the essential info:



Coloring the nodes of the graph:
What's the minimum number of colors such that any two nodes connected by an edge have different colors?

Four Color Theorem



Four color map.

- The chromatic number of a graph is the least number of colors that are required to color a graph.

- The Four Color Theorem – ***the chromatic number of a planar graph is no greater than four. (quite surprising!)***

- Proof by Appel and Haken 1976;
- careful case analysis performed by computer;
- Proof reduced the infinitude of possible maps to 1,936 reducible configurations (later reduced to 1,476) which had to be checked one by one by computer.
- The computer program ran for hundreds of hours. The first significant *computer-assisted* mathematical proof. *Write-up was hundreds of pages including code!*