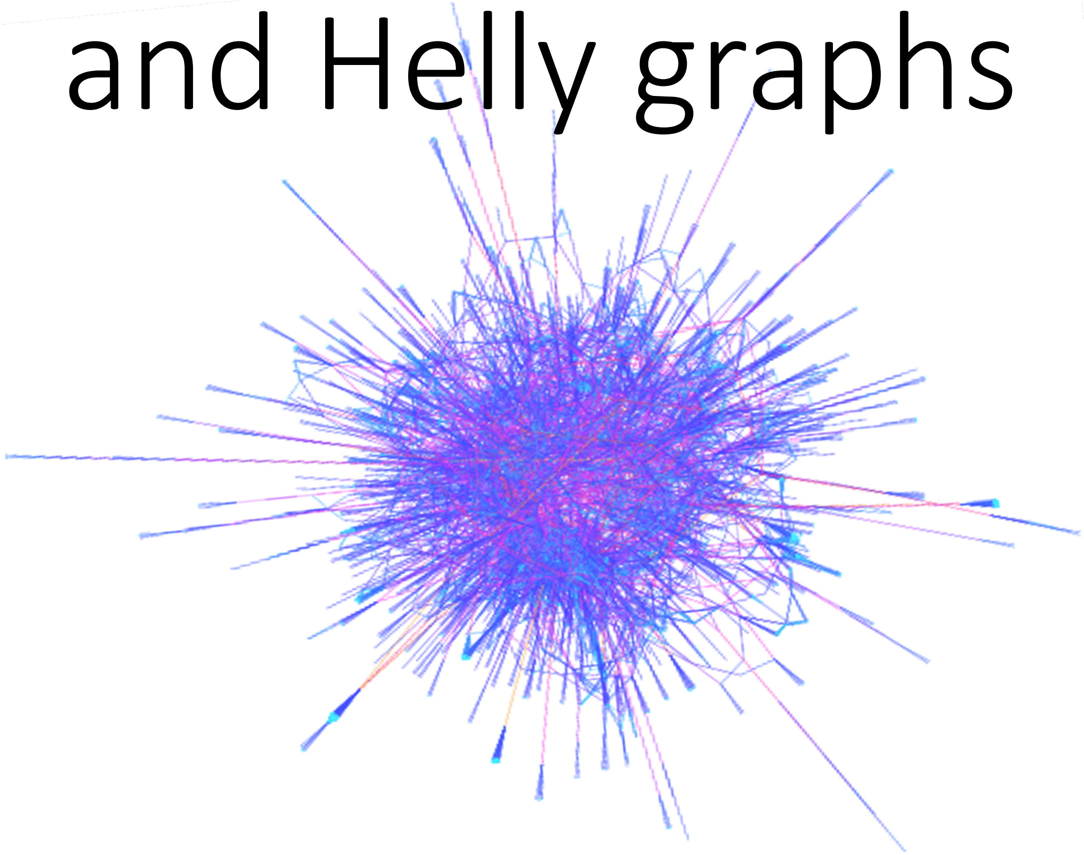
Hyperbolicity, injective hulls, and Helly graphs



Heather M. Guarnera

Motivation

Every graph G can be isometrically embedded into the smallest Helly graph $\mathcal{H}$(G).

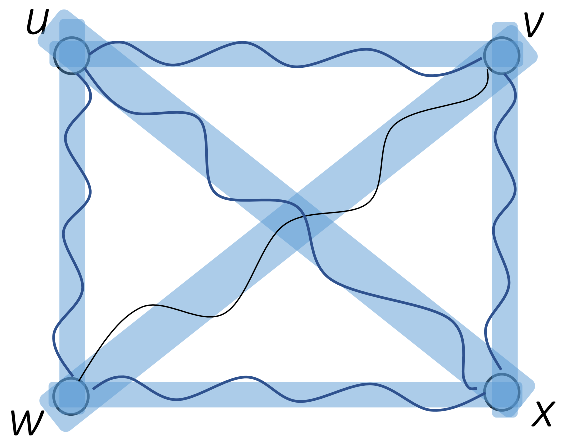
$\mathcal{H}(G)$ is called the injective hull of G [Isbell 1964, Dress 1984].

- $\mathcal{H}(G)$ preserves hyperbolicity
- If G is δ-hyperbolic, then any vertex of $\mathcal{H}(G)$ is within 2δ to a vertex of G [Lang 2013]

This motivates finding solutions to problems in $\mathcal{H}(G)$, which can lead to approximate solutions in G.

δ-Hyperbolicity: Meaning and Prevalence

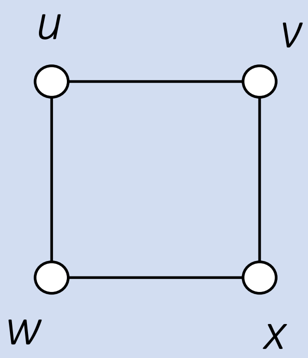
δ-Hyperbolicity



Definition (Gromov's 4-point condition)

For any four points u,v,w,x, the two larger of the distance sums $d(u,v)+d(w,x)$, $d(u,w)+d(v,x)$, $d(u,x)+d(v,w)$ differ by at most $2\delta \geq 0$.

Example:



$d(u,v) + d(w,x) = 2$

$d(u,w) + d(v,x) = 2$

$d(u,x) + d(v,w) = 4$

So, $\delta = \frac{4-2}{2} = 1$

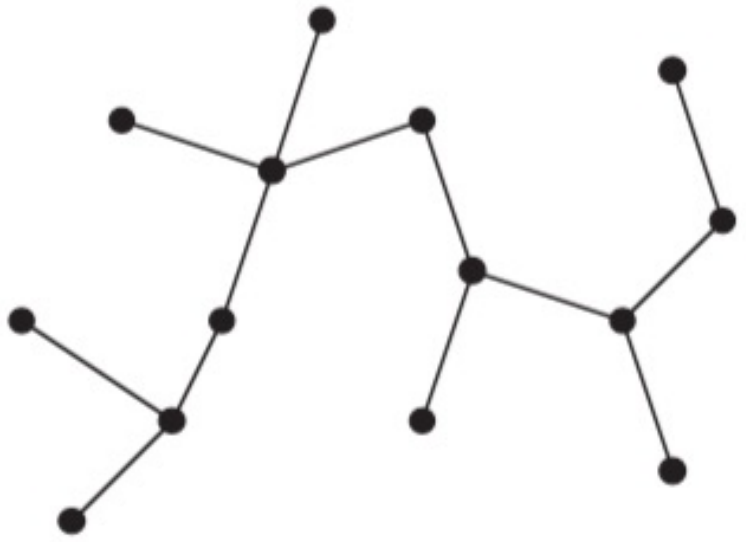
Complexity to calculate:

Naïve	$O(n^4)$
(Fournier et al. 2015)	$O(n^{3.69})$
(Borassi et al. 2015)	$O(n^4)$

δ–Hyperbolicity measures metric tree-likeness

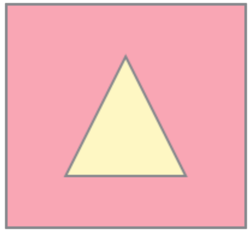
Smaller value of δ = {0, 1/2, 1, ...} indicates the graph

- is metrically closer to a tree (δ=0 in a tree)

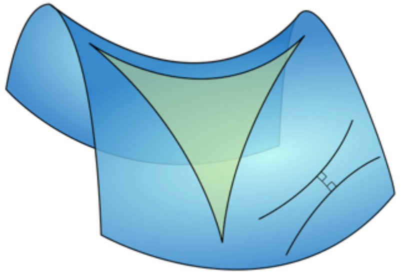


δ–Hyperbolicity measures metric tree-likeness

Zero Curvature



Negative Curvature

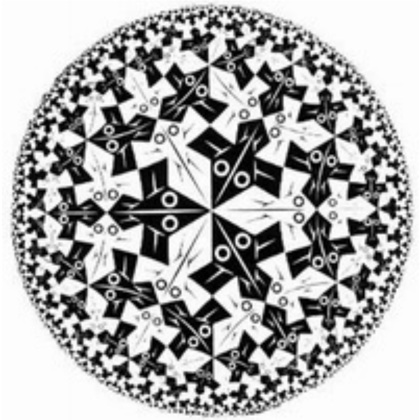


Smaller value of $\delta = \{0, 1/2, 1, \ldots\}$ indicates the graph

- is metrically closer to a tree (δ=0 in a tree)
- has global negative curvature

Many real-world networks have small hyperbolicity (biological, social, collaboration, communication, etc.)

- Often δ is a small constant < 4.

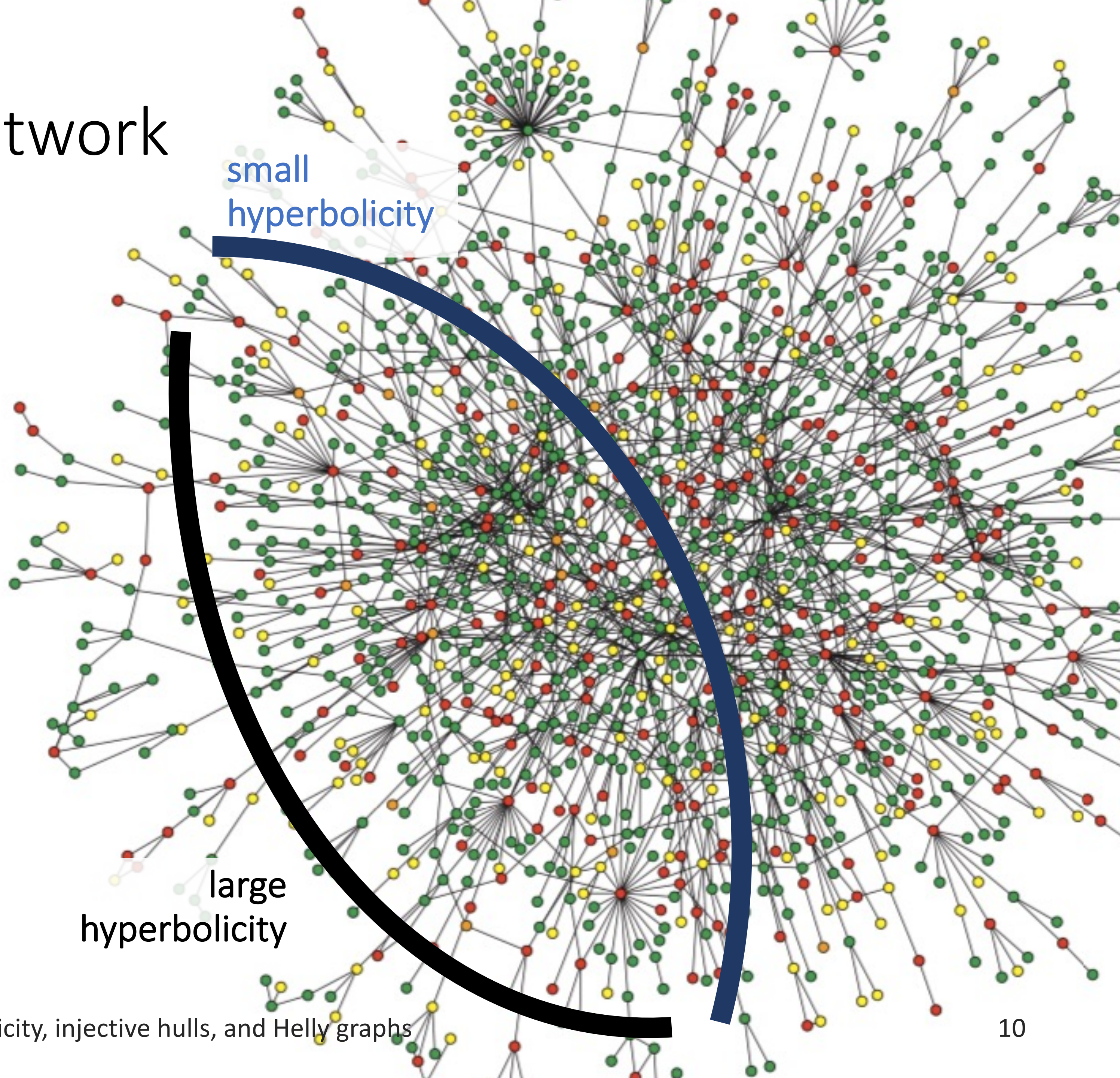


Ex: Protein interaction network

nodes n = 1,870 proteins

edges m = 2240 direct physical interactions between proteins

$\delta = 3.5$



Ex: Other real-world networks with small hyperbolicity

- Social networks (subset of Facebook)
  - nodes n = 293,501 users
  - edges m = 5,589,802 friendships between users

$\delta = 2$

- Web networks (from Google)
  - nodes n = 855,802 websites
  - edges m = 4,291,352 hyperlinks connecting sites

$\delta = 2$

- Peer-to-peer networks (Gnutella)
  - nodes n = 62,561 hosts
  - edges m = 147,878 connections between hosts

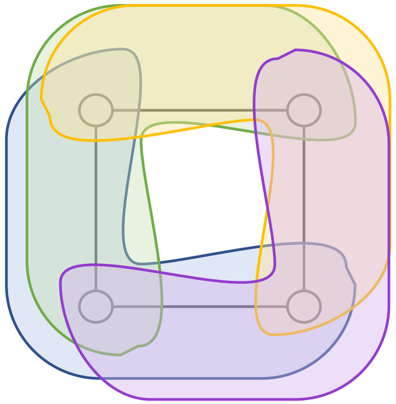
$\delta = 2.5$

Helly graphs and injective hulls

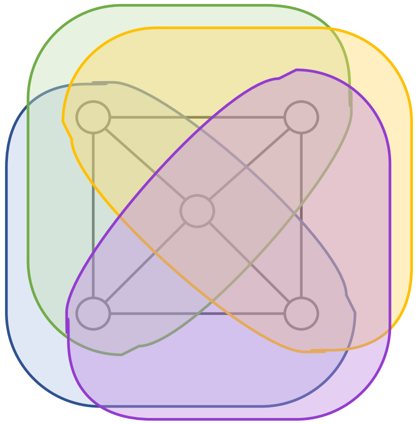
Helly property and Helly graphs

A family F of sets has the Helly property if for every subfamily S of F the following hold: if the elements of S pairwise intersect, then the intersection of all elements of S is also non-empty.

A graph is called Helly if its family of disks satisfies the Helly property.



Not Helly



Helly

Injective hull $\mathcal{H}(G)$ of graph G

The unique smallest Helly graph into which G isometrically embeds [Isbell '64]

[Dress '84] Each vertex of $\mathcal{H}(G)$ can be represented as a vector with nonnegative integer values *f(x)* for each vertex *x* of *V(G)* such that the following two properties hold:

$$\forall x, y \in V(G) \;\; f(x) + f(y) \geq d_G(x, y) \qquad (1.1)$$

$$\forall x \in V(G) \;\; \exists y \in V(G) \;\; f(x) + f(y) = d_G(x, y) \qquad (1.2)$$

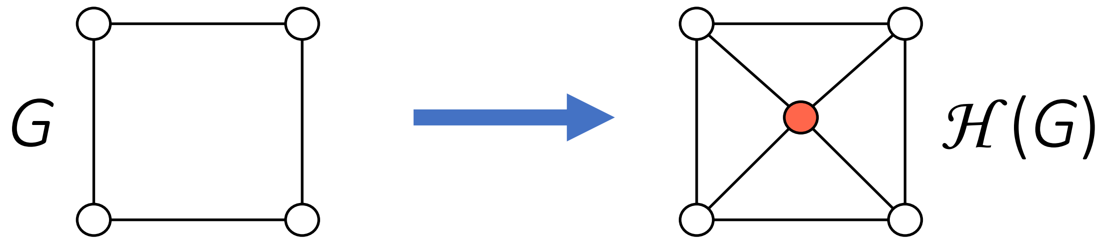
There is an edge between two vertices *f* and *g* of $V(\mathcal{H}(G))$ if and only if their Chebyshev distance is 1, that is, $\max_{x \in V(G)} |f(x) - g(x)| = 1$

An isometric embedding of G into $\mathcal{H}(G)$ is obtained by mapping each vertex *z* in *V(G)* to its distance vector d_z, where $d_z(x) = d_G(z, x)$ for all vertices *x* in *V(G)*

Injective hull $\mathcal{H}(G)$ of graph G

The unique smallest Helly graph into which G isometrically embeds [Isbell '64]

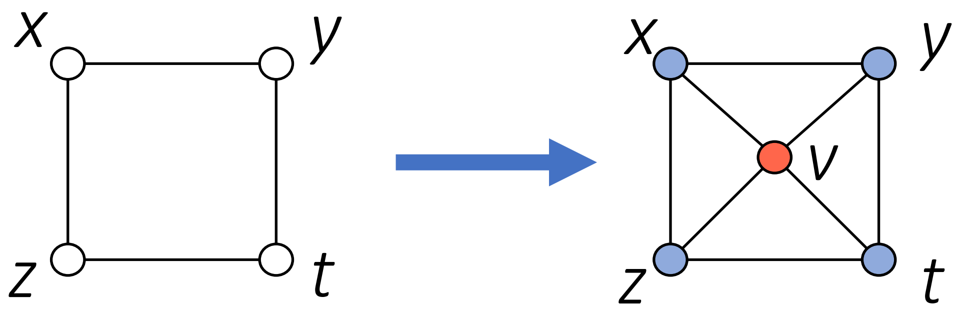
We use the terms Hellify (verb) and Hellification (noun) to describe the process by which edges and Helly vertices are added to G to construct $\mathcal{H}(G)$



Vertices of $\mathcal{H}(G)$ are described as either real vertices or Helly vertices.

An isometric embedding of G into $\mathcal{H}(G)$ is obtained by mapping each vertex *z* in *V(G)* to its distance vector d_z, where $d_z(x) = d_G(z,x)$ for all vertices *x* in *V(G)*

Algorithmically computing the injective hull for an arbitrary graph



1. Generate the set of n-tuples S = $\{0, 1, \ldots, diam(G)\}^n$
2. Remove from S the n-tuples which do not satisfy conditions (1.1) & (1.2)
3. Remaining n-tuples are vertices of the injective hull
4. Add an edge between vertices of $\mathcal{H}(G)$ if their Chebyshev distance is 1

Total run time: $O((diam(G) + 1)^n)$

xytz
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0001
0002
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0121 x
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1012 y
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1210 t
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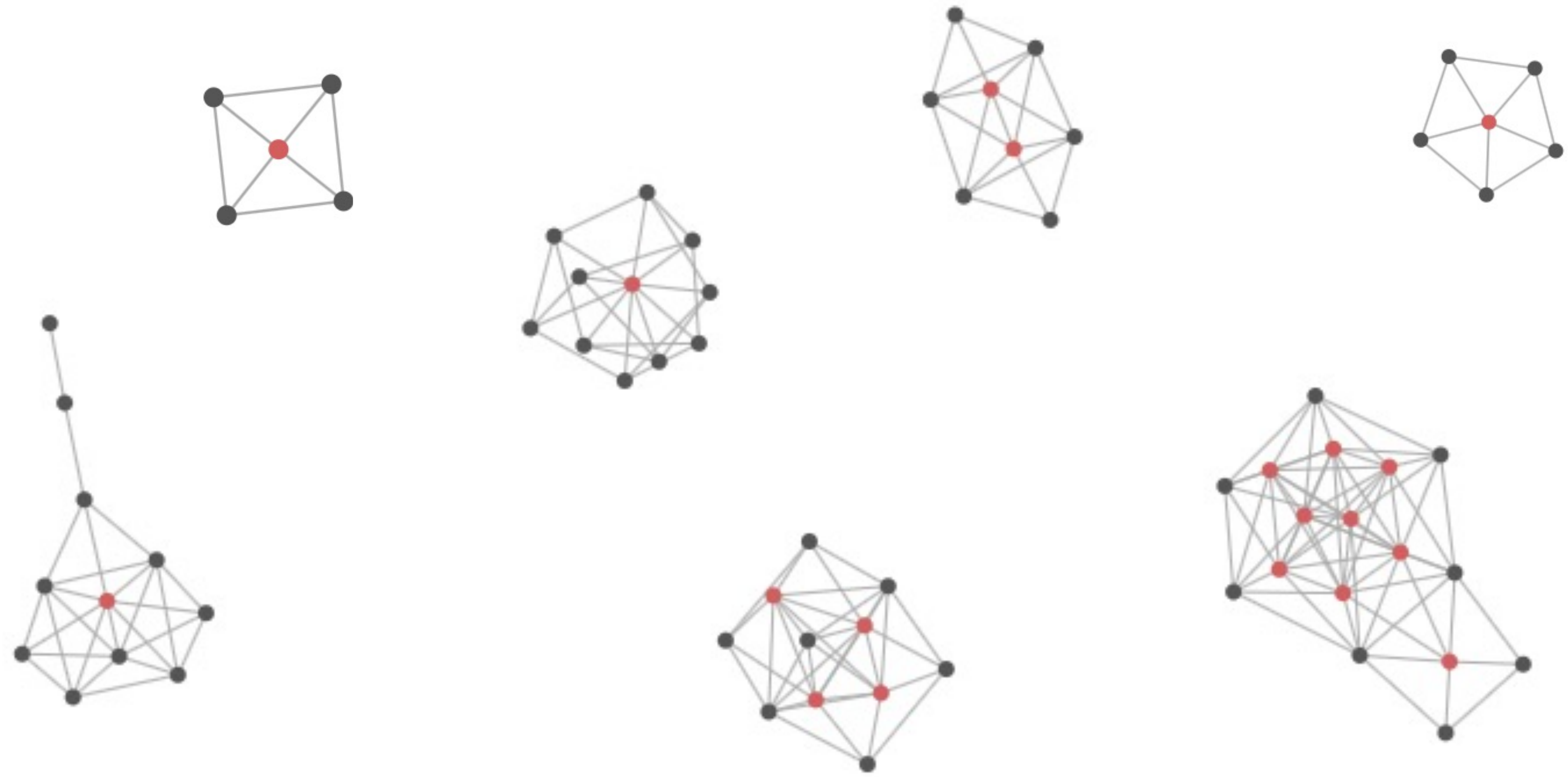
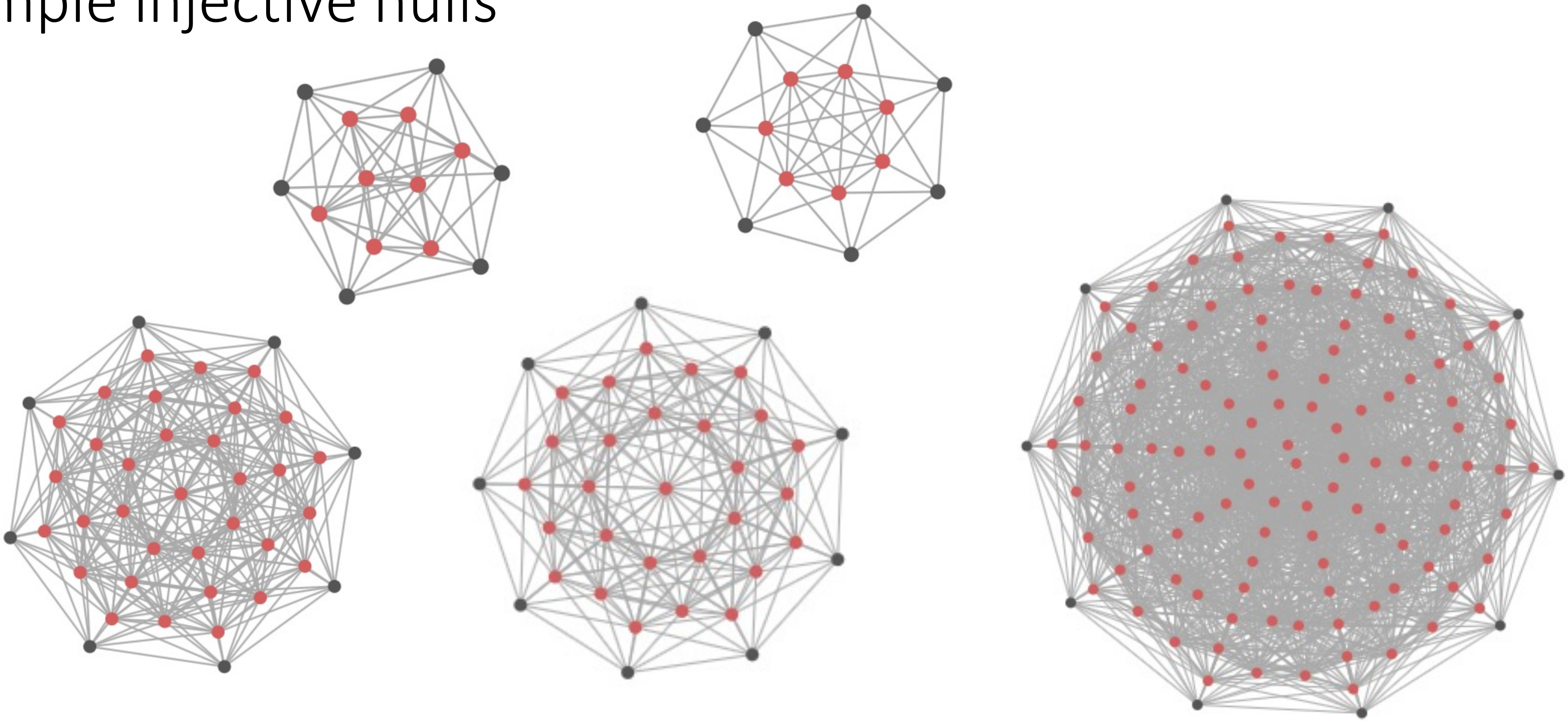
Example injective hulls



Example injective hulls



$\mathcal{H}(G)$ can grow exponentially for arbitrary graphs.

Goal of dissertation

- Understand structural and metric characteristics of $\mathcal{H}(G)$ that govern hyperbolicity
- Use $\mathcal{H}(G)$ to efficiently solve facility location problems on G
  - Distance-hereditary graphs (they are 1-hyperbolic)
  - δ-Hyperbolic graphs
  - Weakly-Helly graphs
- Identify graph classes for which $\mathcal{H}(G)$ can be found efficiently

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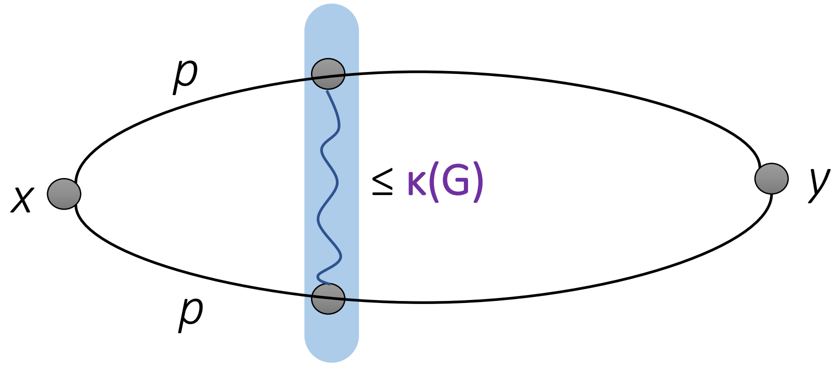
Interval thinness

- For any two x,y vertices, $I(x,y)=\{ z \in V : d(x,y)=d(x,z)+d(z,y) \}$ denotes the (metric) interval, i.e., all vertices that lay on shortest paths between x and y.
- The set $S_p(x,y) = \{z \in I(x,y) : d(z,x) = p\}$ is called a slice of the interval from x to y.



- An interval I(x,y) is said to be κ-thin if for any natural number p, any two vertices u,v of the slice $S_p(x,y)$ are at most κ apart.
- The smallest value κ for which all intervals of G are κ-thin is denoted κ(G).

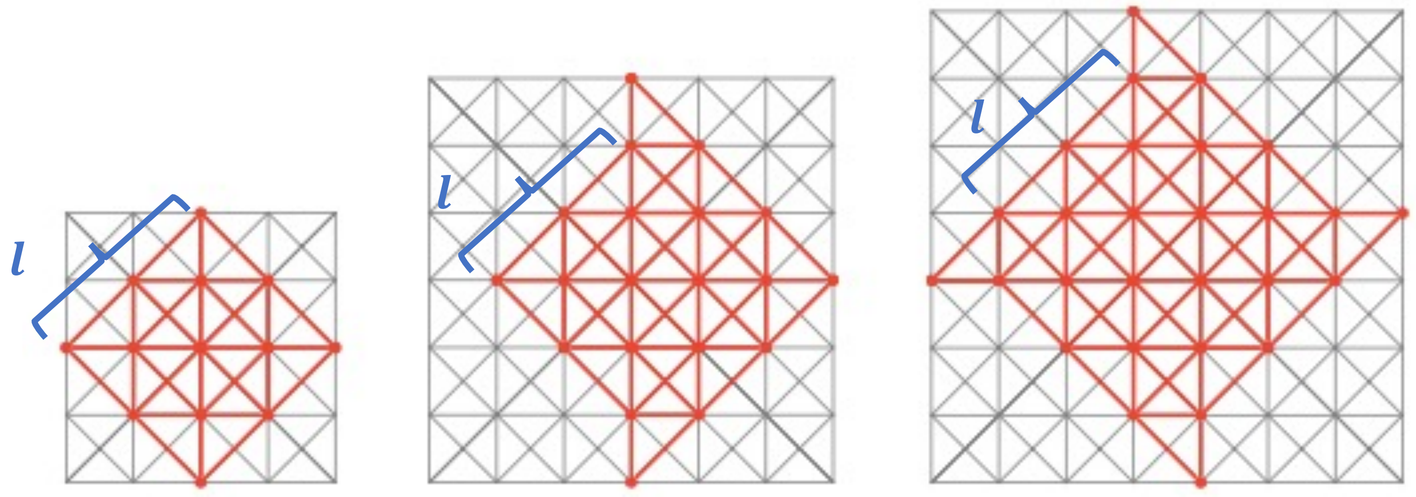
Lemma (**Fellow travelers property**): For any graph G, $\kappa(G) \le 2\delta(G)$

Theorem 2. For every Helly graph G, $\kappa(G) \le 2\delta(G) \le \kappa(G)+1$.

Select frames present in δ-hyperbolic Helly graphs

Lemma 16. Let G be a Helly graph with $\delta(G) = k > 0$.

- If $\kappa(G) = 2k$ and k is an integer, then G contains H_1^k as an isometric subgraph.
- If $\kappa(G) = 2k$ and k is a half-integer, then G contains $H_2^{k-1/2}$ as an isometric subgraph.
- *If* $\kappa(G) = 2k - 1$, then k is an integer and G contains H_3^{k-1} as an isometric subgraph.



H_1^l H_2^l H_3^l

Characterization by forbidden subgraphs

Theorem 3. Let *G* be a Helly graph and *k* be a non-negative integer.

- $\delta(G) \leq k$ if and only if *G* contains no H_2^k as an isometric subgraph.
- $\delta(G) \leq k+1$ if and only if *G* contains neither H_1^{k+1} nor H_3^k as an isometric subgraph.

A Helly graph is δ-hyperbolic if and only if it contains none of the following as isometric subgraph(s).

δ=0

H_2^0

δ=1/2

H_1^1

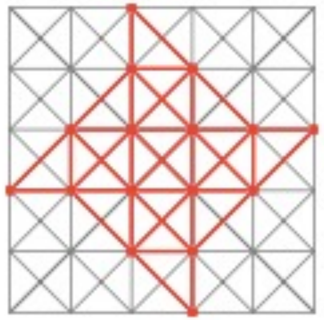
H_3^0

δ=1

H_2^1

δ=3/2

H_1^2



H_3^1

δ=...

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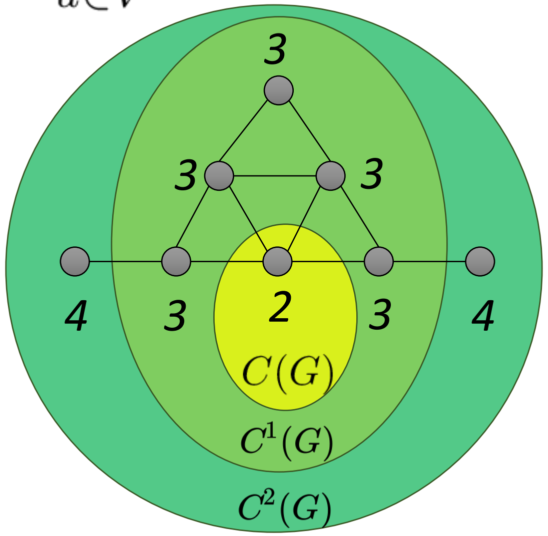
Eccentricity function and centers

The eccentricity *e(x)* of a vertex *x* is the distance to a furthest *u* vertex to *x*

$$e(x) = \max_{u \in V} d(x, u)$$



The minimum and maximum eccentricities are called the radius *rad(G)* and diameter *diam(G)* of the graph, respectively

The center of a graph *C(G)* is the set of vertices with minimum eccentricity

$$C(G) = \{v \in V : e(v) = rad(G)\}$$

$$C^{\ell}(G) = \{v \in V : e(v) \leq rad(G) + \ell\}$$

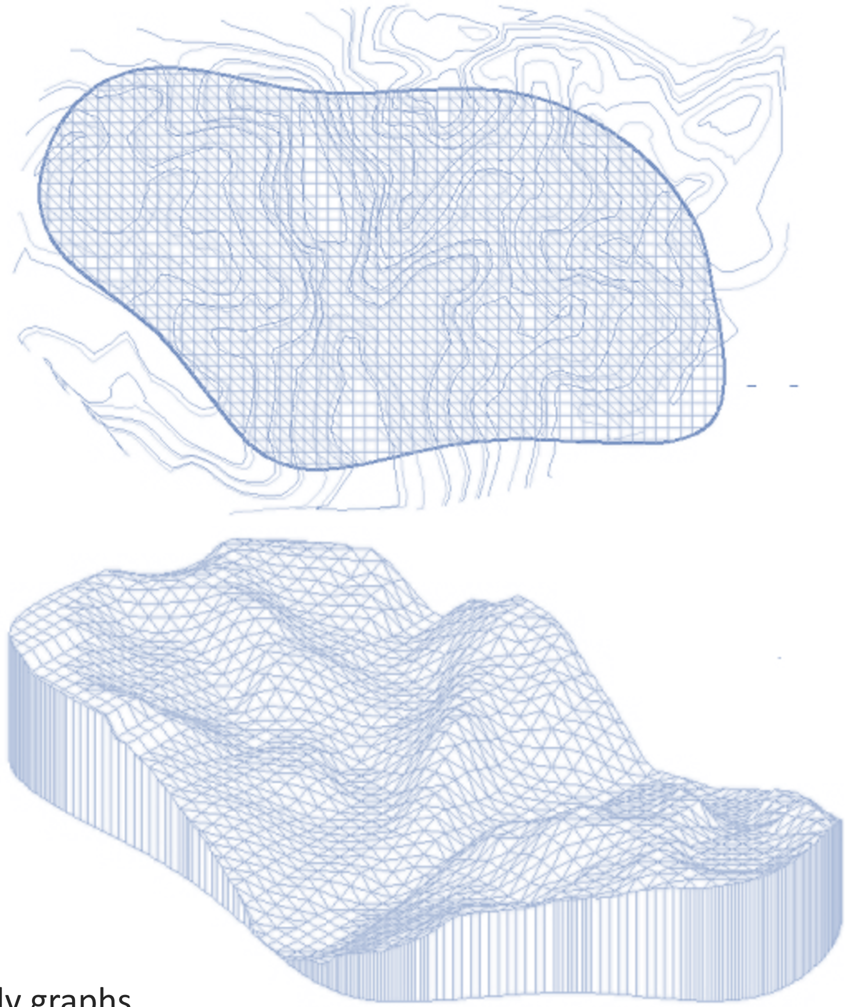
Applications:

- Measure the importance of a node (centrality indices)
- Facility location problems
- Detecting small-world networks (degrees of freedom)

Eccentricity terrain

A graph's eccentricity terrain can be described by how a traveler would walk along a shortest path

- Up hill (to a vertex of higher eccentricity)
- Down hill (to a vertex of lower eccentricity)
- Plateau (to a vertex of the same eccentricity)

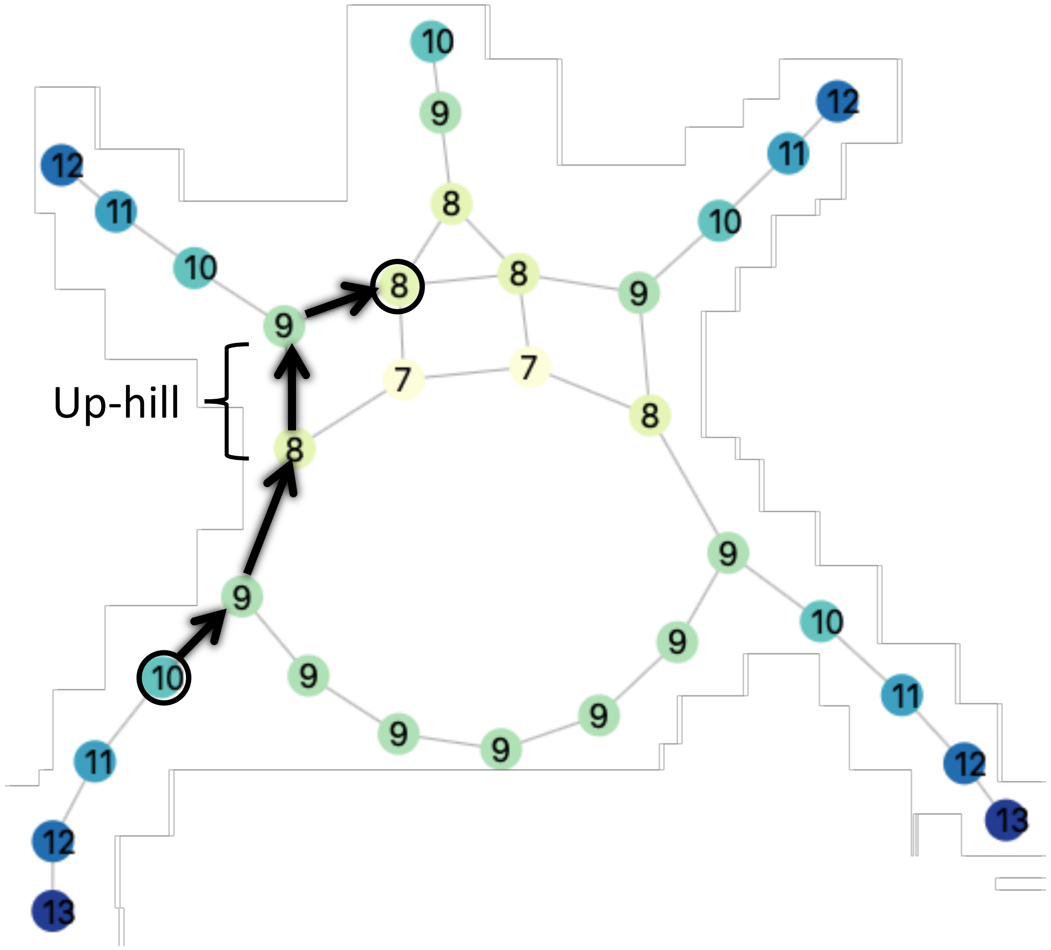


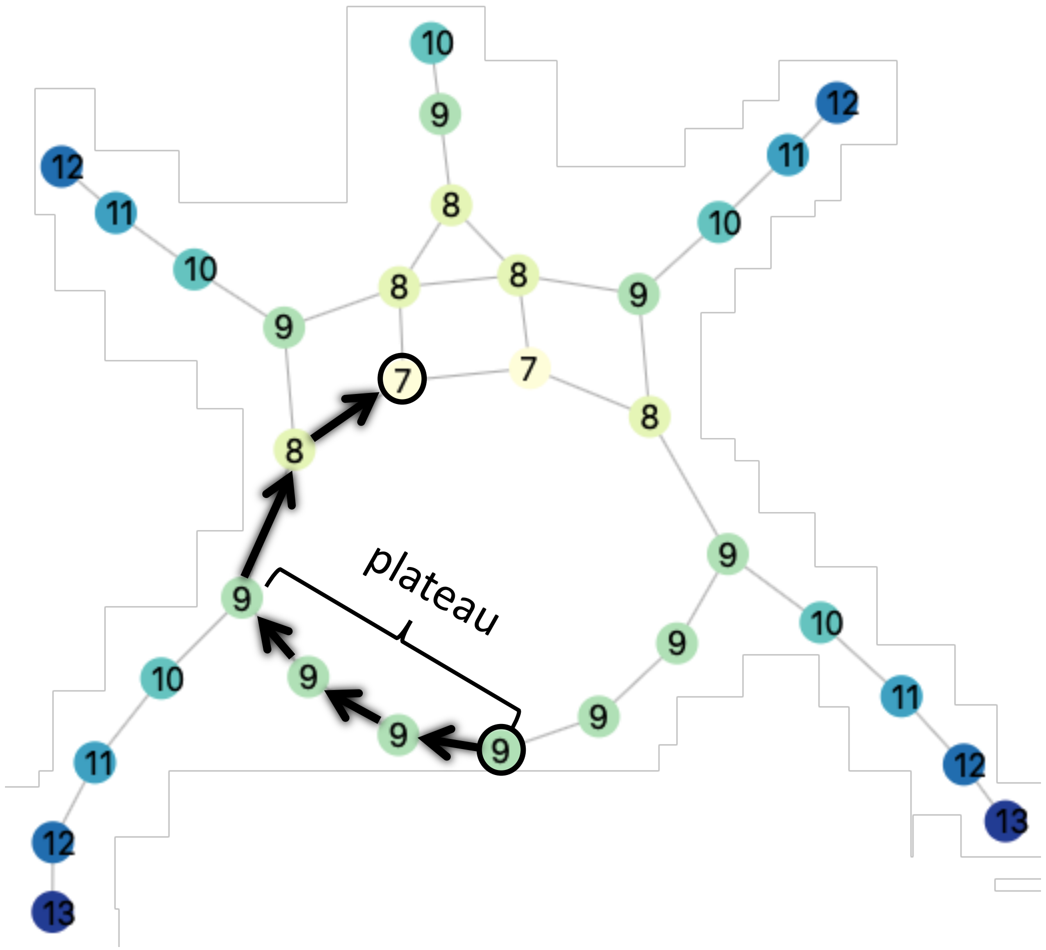
Along a shortest path *P(y,x)* from *y* to *x*, we define

$\mathbb{D}(P(y,x))$, $\mathbb{H}(P(y,x))$, and $\mathbb{U}(P(y,x))$

as the number of down-edges, up-edges, and horizontal edges, respectively.

Example: Eccentricity terrain of a graph





Computing eccentricity

- $O(nm)$ – Run breadth-first search (BFS) from *each* vertex
  - This is not suitable for large graphs

- Current research focuses on efficient approximations
  - Carefully construct a spanning tree T of the original graph G
  - Show that eccentricities in T are close to what they were in G

$$e_T(v) \leq e_G(v) \leq e_T(v) + k$$

  - T is called an eccentricity k-approximating tree

- Recent work: [Chepoi et al. 2019]
  - eccentricity (24δ+ 1)-approximation in O(m) time
  - eccentricity (8δ)-approximation in O(δm) time

Eccentricity terrain in Helly graphs [Dragan ’89]

- **There is** a shortest path to the center upon which vertex eccentricities monotonically descend
  - No up-hills
  - No plateaus

- The eccentricity function is unimodal
  - Local minimum is global minimum
  - Every vertex $v \notin C(G)$ has an adjacent vertex *u* with *e(u) < e(v).*
  - Equivalently,

$$e_G(v) = d(v, C(G)) + rad(G)$$

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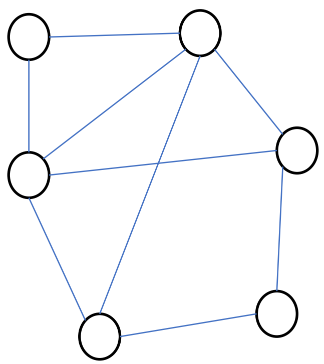
Distance-hereditary graphs (subclass of 1-hyperbolic graphs)

- Vertex v is a pendant to u if N(v)={u}
- Vertex v is a twin to u if N(v) \ {u} = N(u) \ {v}
  - True twins are adjacent
  - False twins are not



G is a distance-hereditary graph if and only if G can be reduced to one vertex graph by a pruning sequence of one-vertex deletions: removing a pendant vertex or a single vertex from a pair of twin vertices.

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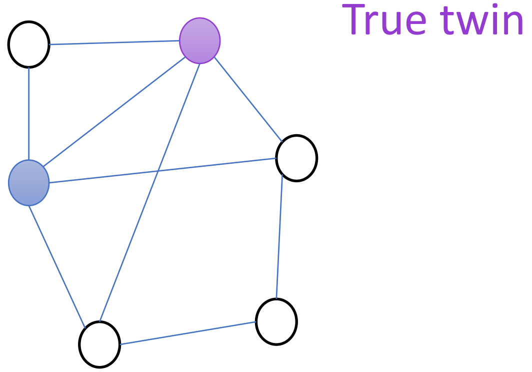

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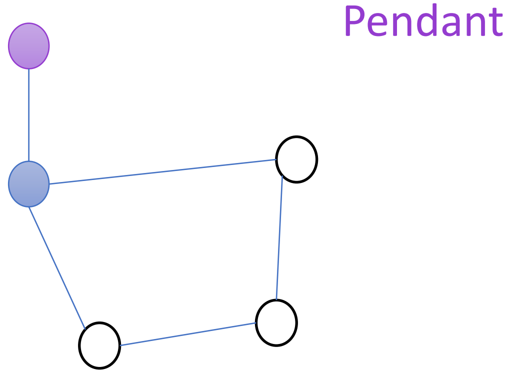

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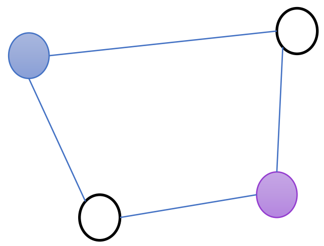

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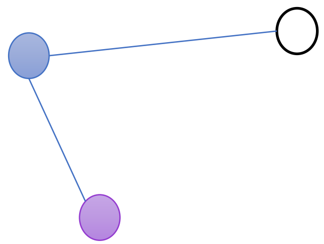
Pendant



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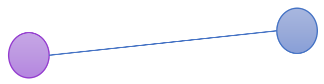

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G is a distance-hereditary graph if and only if G can be reduced to one vertex graph by a pruning sequence of one-vertex deletions: removing a pendant vertex or a single vertex from a pair of twin vertices.

Eccentricity function/terrain in distance-hereditary graphs

- [Lemma 31] All shortest paths to the center monotonically descend until $C^1(G)$
  - No up-hills
  - Plateaus can be at most length 1
  - [Theorem 5]. Unimodality only breaks at a vertex *v* under specific conditions: when e(v) = rad(G)+1, diam(G) = 2rad(G), and d(v, C(G)) = 2

- [Lemma 31]. All vertices $v \notin C(G)$ satisfy $e(v) = d(v, C^1(G)) + rad(G) + 1$

- [Theorem 8]. There is a linear time algorithm to compute all eccentricities in a distance-hereditary graph.

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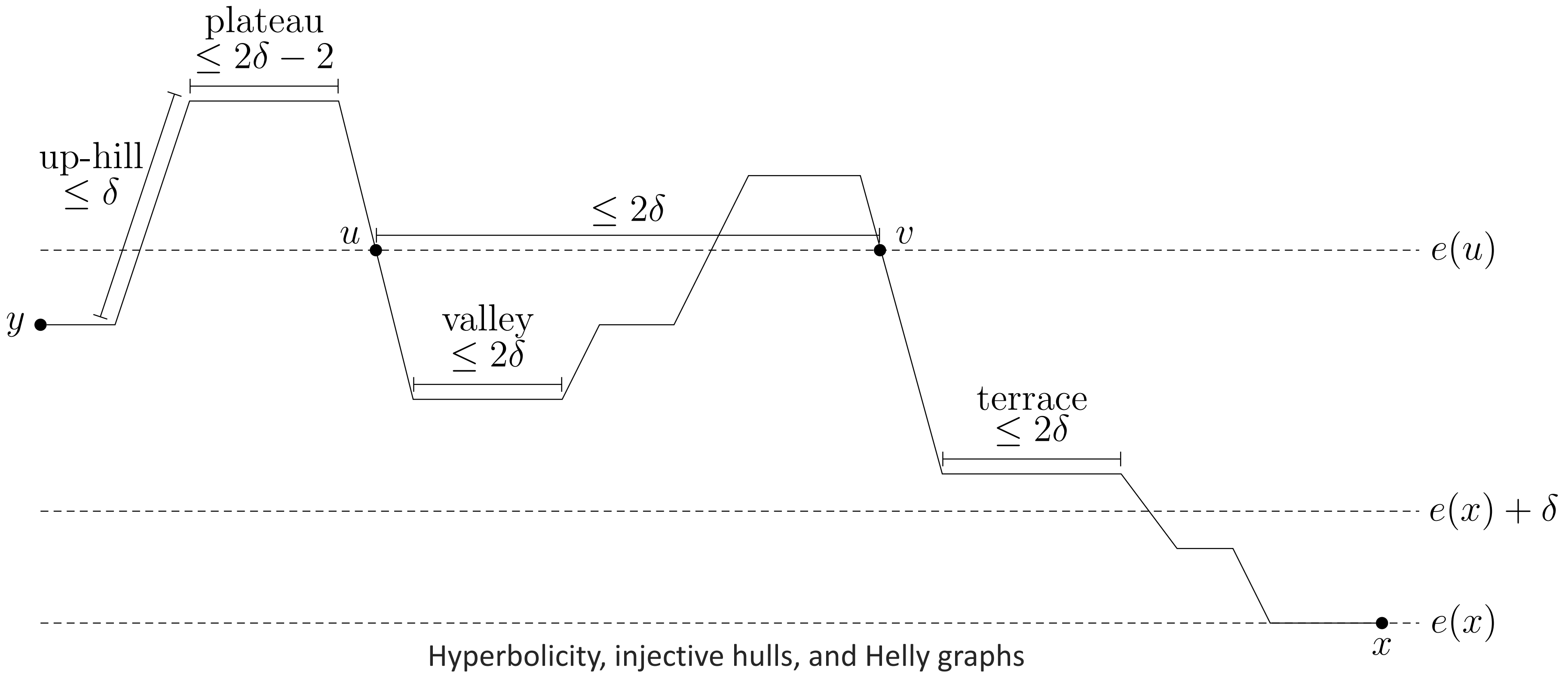
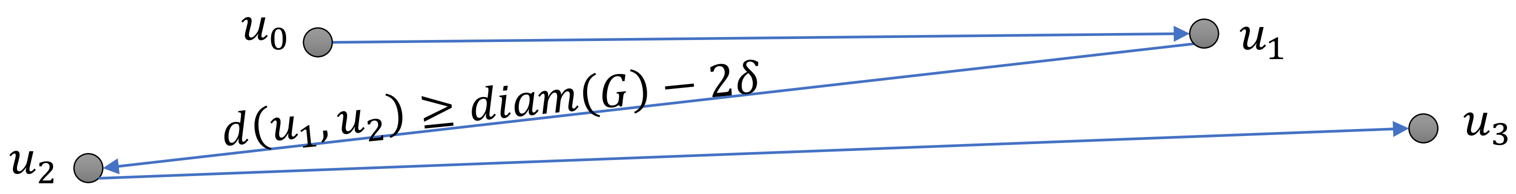
Eccentricity terrain in δ-hyperbolic graphs

- [Theorem 10 & Theorem 11] On a shortest path to C(G)
  - Up-hills can occur anywhere but have height at most δ
  - Plains (plateaus, valleys, terraces) can occur anywhere but have width at most 2δ if sufficiently large eccentricity



Efficient eccentricity approximation via eccentricity approximating spanning tree

- Find a long path in $O(m)$ time



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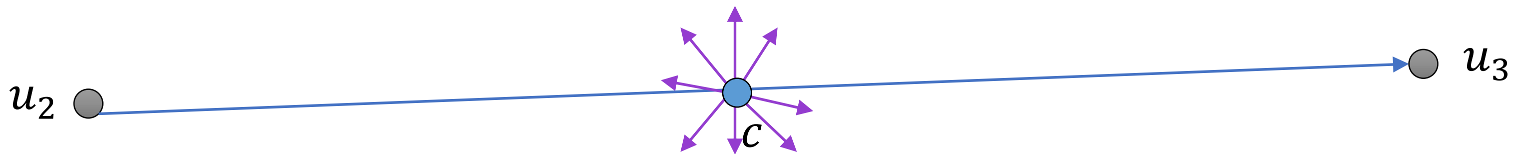
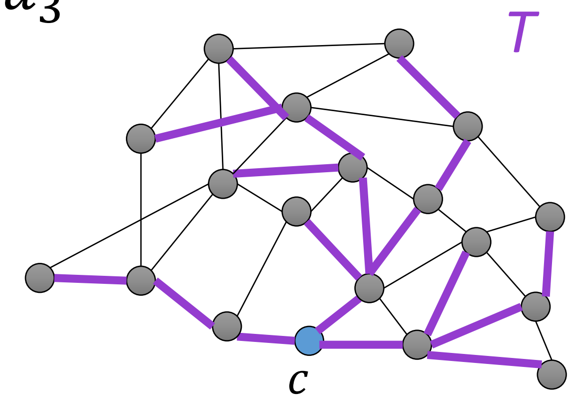


- Run breadth-first search (BFS) from the middle vertex c between $u_2 u_3$
- We show $e_T(v) \leq e_G(v) \leq e_T(v) + 6\delta$



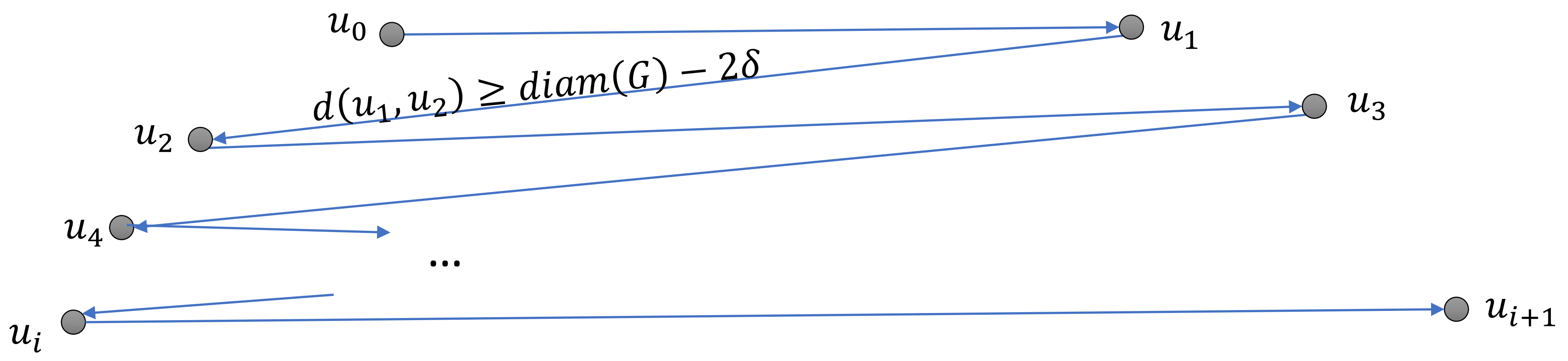
Ours: 6δ approximation of all eccentricities in total $O(m)$ time

[Chepoi et al. ‘19]: (24δ+ 1)-approximation

Efficient eccentricity approximation via eccentricity approximating spanning tree

- Find a mutually distant pair of vertices x,y in $O(\delta m)$ time

$$e(x) = d(x,y) = e(y)$$

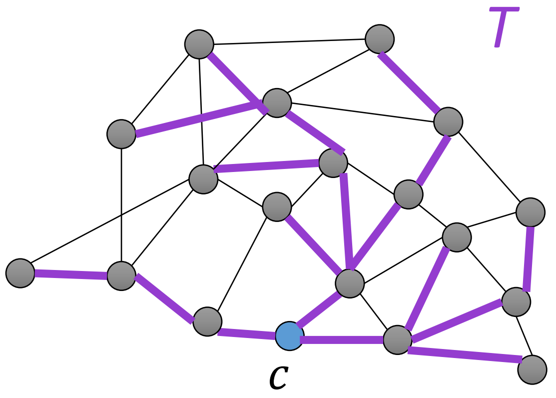


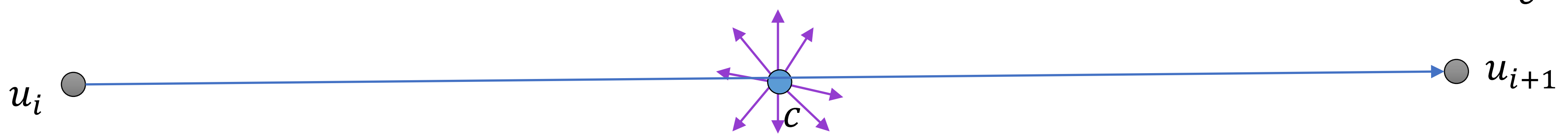
Efficient eccentricity approximation via eccentricity approximating spanning tree

- Find a mutually distant pair of vertices *x,y* in $O(\delta m)$ time

$$e(x) = d(x, y) = e(y)$$

- Run breadth-first search (BFS) from the middle vertex *c* of the mutually distant pair
- We show $e_T(v) \leq e_G(v) \leq e_T(v) + 4\delta$



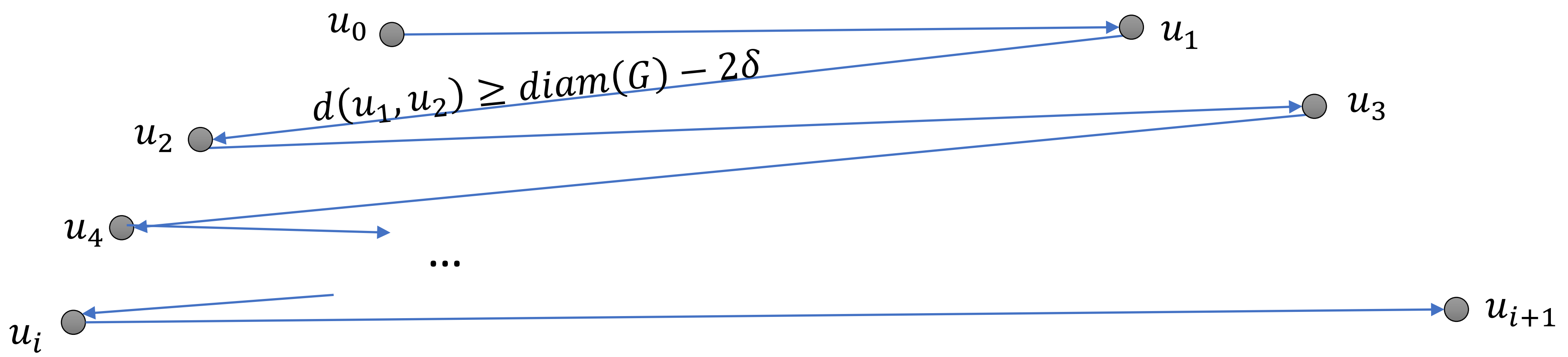


Ours: 4δ approximation of all eccentricities in total $O(\delta m)$ time

[Chepoi et al. '19]: (8δ+ 1)-approximation

Efficient eccentricity approximation via distances to the ends of a long path

- Find a mutually distant pair of vertices *x*,*y* in $O(\delta m)$ time



Efficient eccentricity approximation via distances to the ends of a long path

- Find a mutually distant pair of vertices *x*,*y* in $O(\delta m)$ time
- Find distances from *x* and *y* to all other vertices in $O(m)$ time
- Set approximate eccentricity for each vertex *v* as $\hat{e}(v) = \max\{d(v,x), d(v,y)\}$

- We show that $e(v) - 2\delta \leq \hat{e}(v) \leq e(v)$

2δ approximation of all eccentricities in total $O(\delta m)$ time

Goal of dissertation

- Understand structural and metric characteristics of $\mathcal{H}(G)$ that govern hyperbolicity
- Use $\mathcal{H}(G)$ to efficiently solve facility location problems on G
    - Distance-hereditary graphs (they are 1-hyperbolic)
    - δ-Hyperbolic graphs
    - Weakly-Helly graphs
- Identify graph classes for which $\mathcal{H}(G)$ can be found efficiently

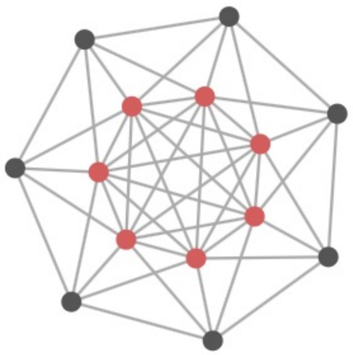
Weakly-Helly graphs

A graph G is α-weakly-Helly if for any system of disks $\mathcal{F} = \{D_G(v, r(v)) : v \in S \subseteq V(G)\}$ the following α-weakly-Helly property holds:

- if $X \cap Y \neq \emptyset$ for every $X, Y \in \mathcal{F}$, then $\bigcap_{v \in S} D_G(v, r(v) + \alpha) \neq \emptyset$

The smallest value α for which a graph *G* is α-weakly-Helly is called the Helly-gap of *G*, denoted α(G).

Theorem 17. For any vertex $h \in V(\mathcal{H}(G))$ there is a real vertex $v \in V(G)$ such that $d_{\mathcal{H}(G)}(h, v) \leq \alpha$ if and only if G is an α-weakly-Helly graph.



α(G)=1

Many classes of graphs have a small Helly-gap

Graph Class C	Helly-gap
Helly	$\alpha(G) = 0$
Distance-hereditary	$\alpha(G) \leq 1$
k-Chordal	$\alpha(G) \leq \lfloor k/2 \rfloor$
Chordal	$\alpha(G) \leq 1$
AT-Free	$\alpha(G) \leq 2$
n x n Rectilinear grid	$\alpha(G) = 1$
δ-Hyperbolic	$\alpha(G) \leq 2\delta$
Cycle C_n	$\alpha(G) \leq \lfloor n/4 \rfloor$
α_i-metric	$\alpha(G) = \lceil i/2 \rceil$
Tree-breadth tb(G)	$\alpha(G) \leq tb(G)$
Tree-length tl(G)	$\alpha(G) \leq tl(G)$

We obtain similar results on the eccentricity of weakly-Helly graphs, parameterized in part by α(G).

Theorem. On a shortest path from $y \notin C^{\alpha}(G)$ to closest vertex $x \in C^{\alpha}(G)$, the following holds:

$$2\mathbb{U}(P(y,x)) + \mathbb{H}(P(y,x)) \leq 2\alpha$$

Theorem. There is an eccentricity spanning tree dependent on the diameter of center and Helly-gap.

Goal of dissertation

- Understand structural and metric characteristics of $\mathcal{H}(G)$ that govern hyperbolicity
- Use $\mathcal{H}(G)$ to efficiently solve facility location problems on G
  - Distance-hereditary graphs (they are 1-hyperbolic)
  - δ-Hyperbolic graphs
  - Weakly-Helly graphs
- Identify graph classes for which $\mathcal{H}(G)$ can be found efficiently

Injective hulls of a graph $G \in C$

We are interested in two questions:

- Is G closed under Hellification, i.e., is $\mathcal{H}(G) \in C$?
- How efficiently can $\mathcal{H}(G)$ be computed?

Graph Class *C*	**Closed under Hellification**	**Hardness to compute H(G) for any *G* ∈ *C***
δ-Hyperbolic	Yes	$\Omega(a^n)$
Chordal	Yes	$\Omega(a^n)$
Square-chordal	Yes	?
Distance-hereditary	Yes	$O(n^3)$
Permutation	No	?
Cocomparability	?	$\Omega(a^n)$
AT-Free	?	$\Omega(a^n)$
Chordal Bipartite (or any triangle free)	No	$\Omega(a^n)$

Conclusions

Main contributions vis-à-vis eccentricity terrain

- Fully describe the eccentricity terrain of
  - Distance-hereditary graphs (and their centers)
  - δ-Hyperbolic graphs
  - α-Weakly-Helly graphs

- Compute all eccentricities of distance-hereditary graphs in O(m) time
- Approximate eccentricities of δ-hyperbolic graphs
  - 6δ approximation in O(m) time
  - 4δ (right-sided) approximation in O(δm) time
  - 2δ (left-sided) approximation in O(δm) time
- There is an $\lfloor diam(C^{\alpha}(G))/2 \rfloor + 3\alpha$ approximation of eccentricities in α-weakly-Helly graphs

Other contributions

- F. Dragan, H. Guarnera. “**Obstructions to a small hyperbolicity in Helly graphs**,” Discrete Mathematics, 342(2):326 – 338, 2019.
- F. Dragan, H. Guarnera, “**Eccentricity terrain of δ-hyperbolic graphs**”, Journal of Computer and System Sciences, 112:50-65, 2020.
- F. Dragan, H. Guarnera. “**Eccentricity function in distance-hereditary graphs**,” Theoretical Computer Science, 2020.
- F. Dragan, H. Guarnera. “**Helly-gap of a graph and vertex eccentricities**,” under review by The Electronic Journal of Combinatorics.
- H. Guarnera, F. Dragan, A. Leitert. “**Injective hulls of various graph classes**,” *in preparation*.
- F. Dragan, G. Ducoffe, H. Guarnera. “**Fast deterministic algorithms for computing eccentricities in (hyperbolic) Helly graphs**,” *in preparation*.

Conclusion

- $\mathcal{H}(G)$ can be constructed efficiently for some graph classes (e.g., distance-hereditary graphs).
  - It is computationally difficult to compute for other (even basic) graph classes

- The existence of $\mathcal{H}(G)$ is a powerful tool to gain insight into a graph class from various perspectives
  - structurally
  - metrically
  - algorithmically

- It lends itself nicely to approximation algorithms dependent on $\alpha(G)$

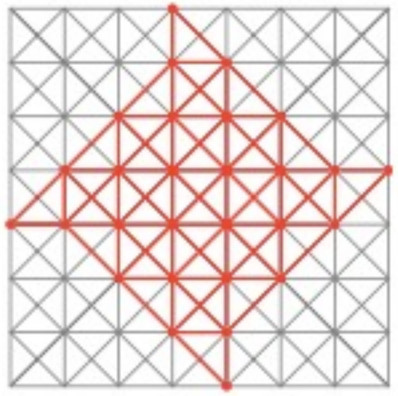
Future work

- Understand the impact of β-pseudoconvexity in hyperbolic graphs
- Identify features of the injective hull of various other graph classes
  - Permutation graphs
  - AT-free graphs
  - Cocomparability graphs
- Identify other characteristics that are possibly closed under Hellification, e.g.,
  - Slimness?
  - Triangle thinness?
  - β-Pseudoconvexity?
- Metric characterization of Helly-gap
- Cops-and-robbers on weakly-Helly graphs

Questions?

Interval thinness

H_3^k



Theorem 2. For every Helly graph G, $\kappa(G) \leq 2\delta(G) \leq \kappa(G)+1$.

Furthermore, $\delta(G) = 2\kappa(G)$ if and only if $\kappa(G)$ is odd and G contains graph $\lfloor \kappa(G) \rfloor$ as an isometric subgraph.

Corollary 2. For every Helly graph G, if $\kappa(G)$ is even, then $\delta(G)$ is an integer and equal to $2\kappa(G)$.

Centers of distance-hereditary graphs

[Yeh and Chang 2003]

- C(G) is either a cograph of a connected graph with diam(C(G)) = 3
- Any cograph is the center of a distance-hereditary graph

[Theorem 7]. Let H be a subgraph of a distance-hereditary graph G induced by C(G). Either

- H is a cograph, or
- H is a connected distance-hereditary graph with diam(H) = 3 and C(H) is a connected cograph with rad(C(H)) = 2.

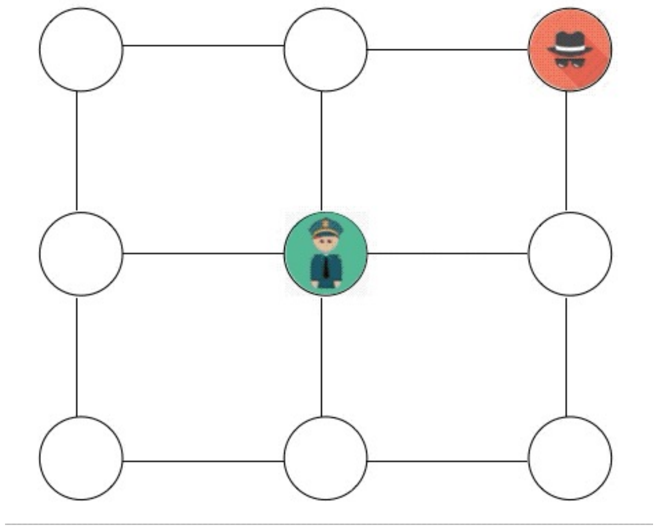
Furthermore, any such graph H is the center of some distance-hereditary graph.

Eccentricity terrain in δ-hyperbolic graphs

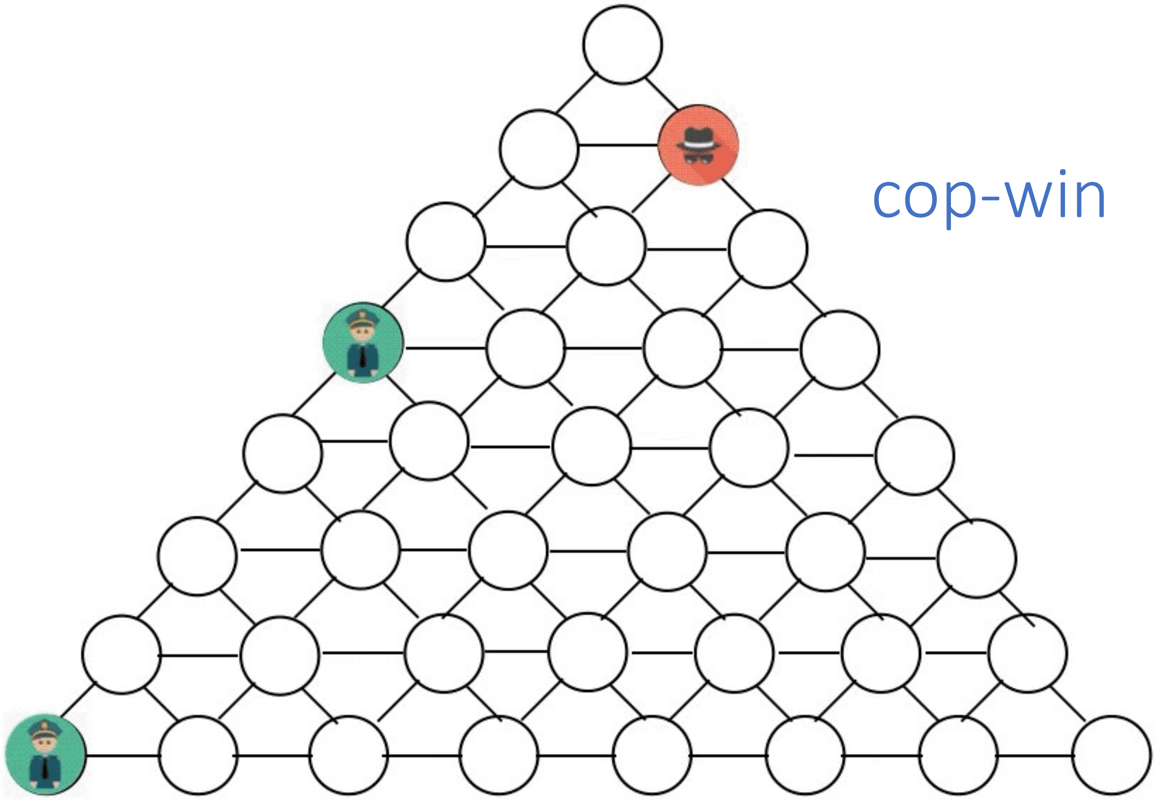
- [Theorem 12]. If G is δ-hyperbolic, then for any shortest path P(y,x) from vertex y to vertex x with minimal eccentricity the following holds:

$$2\mathbb{U}(P(y,x)) + \mathbb{H}(P(y,x)) \leq 4\delta + 1$$

Games on Graphs:
Cop vs. Robbers

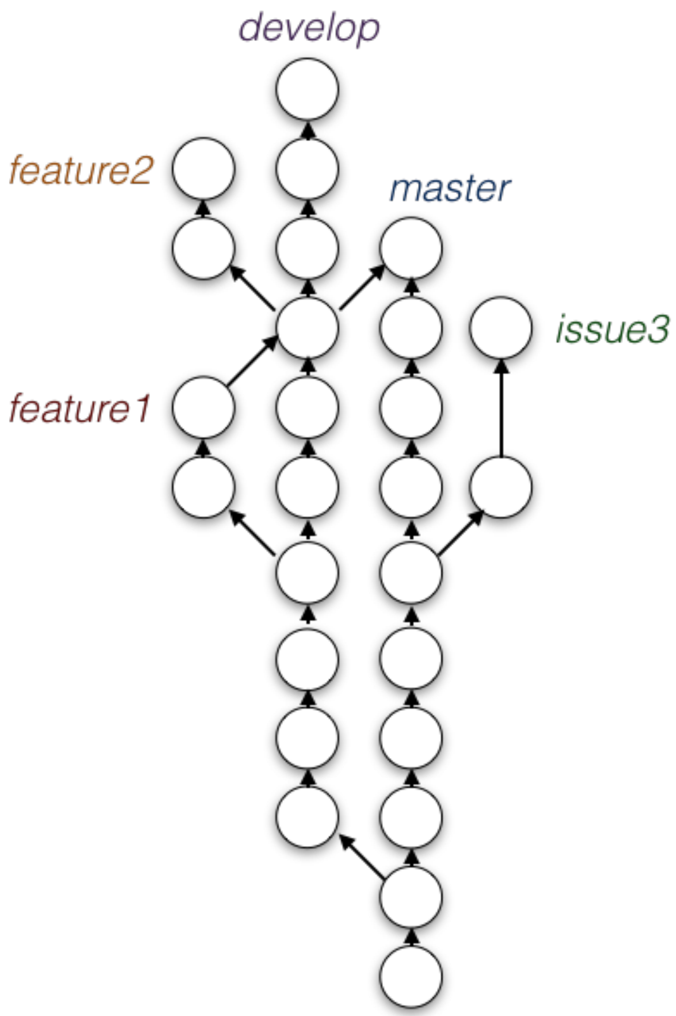


robber-win

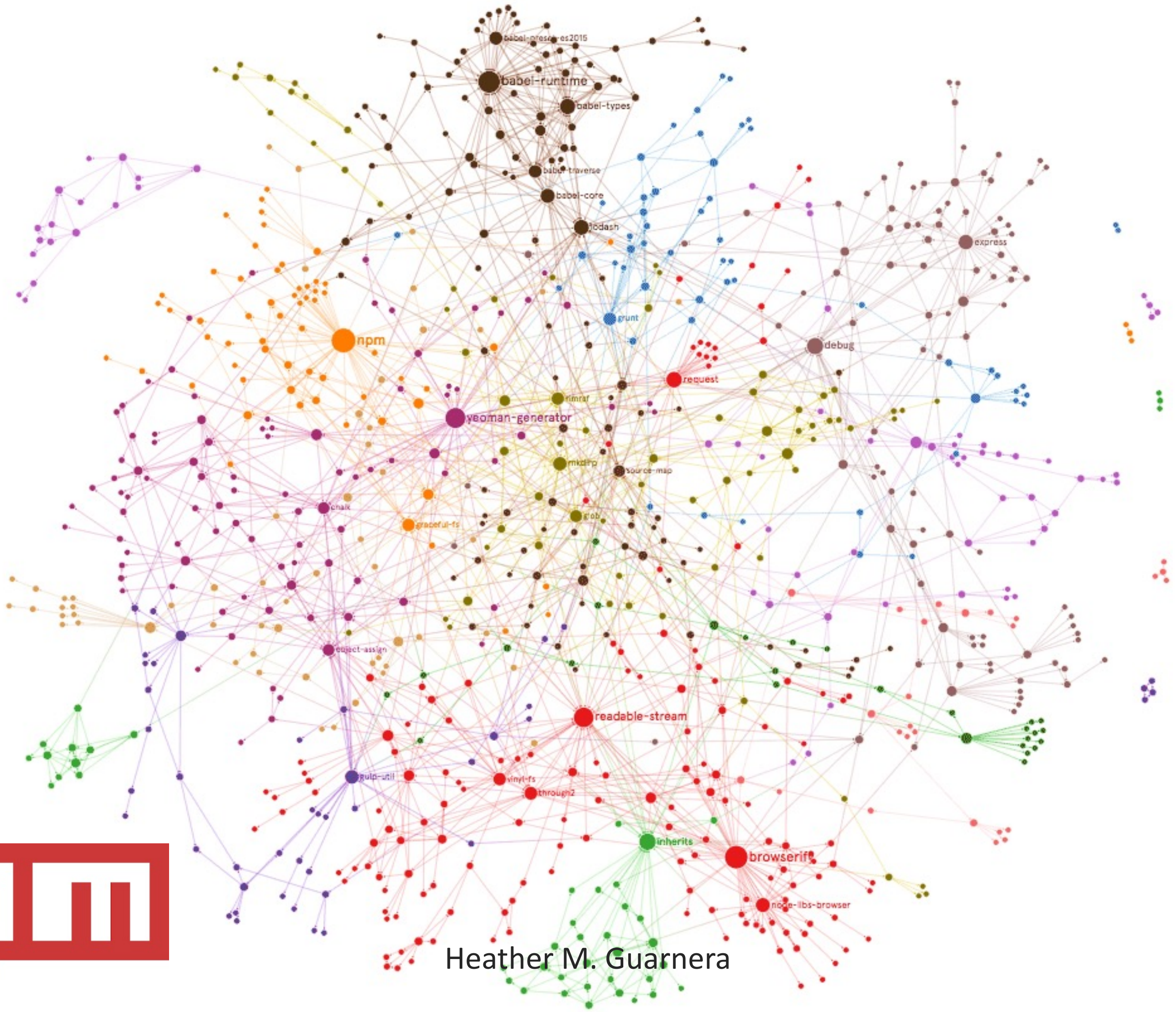


cop-win

Software Engineering: Version Control History



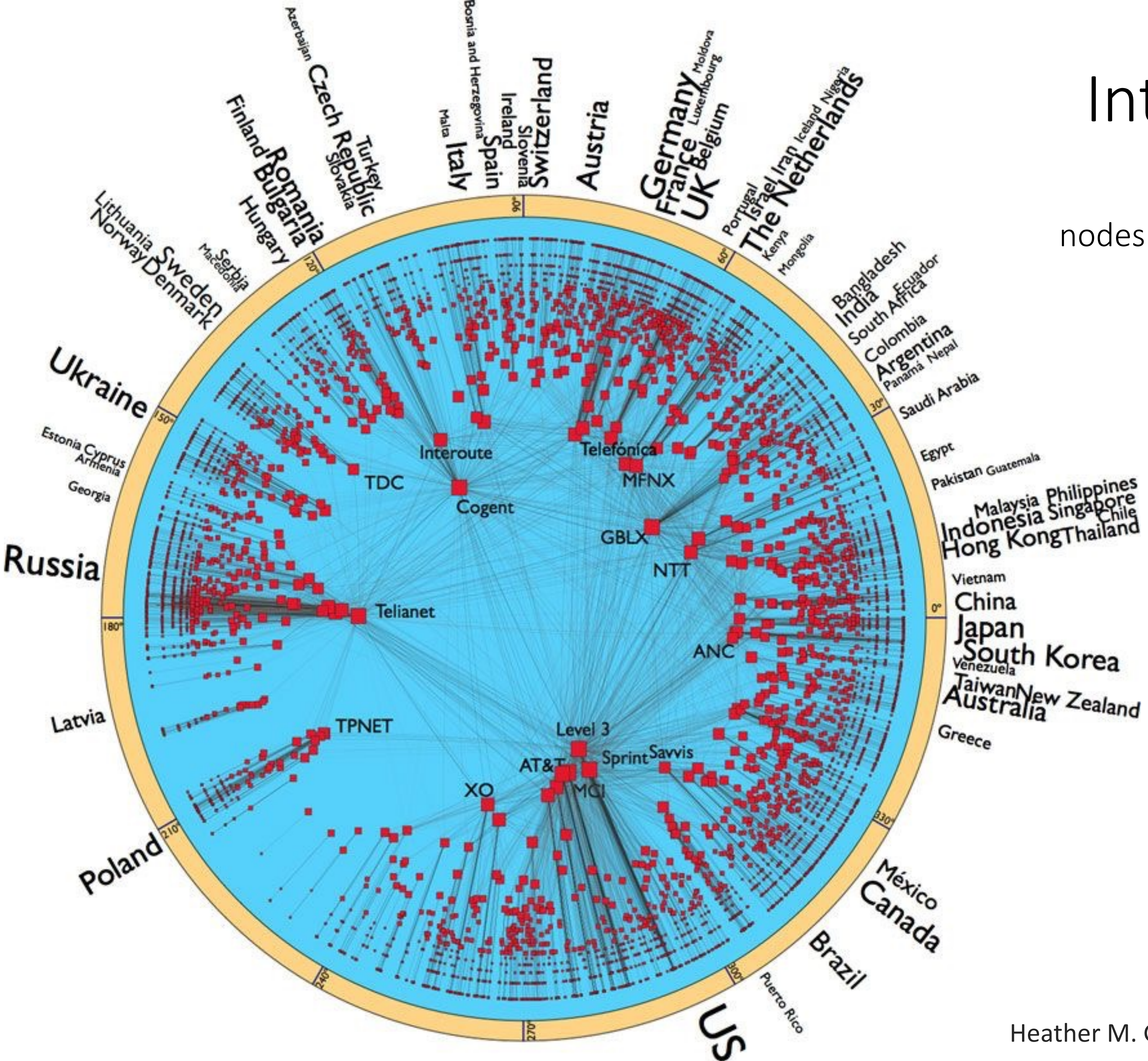
Software Engineering: Software Ecosystems



Internet (AS-Level)

nodes n = 23,752 autonomous systems

edges m = 58,416 AS links



δ = 2.5

Heather M. Guarnera

Hyperbolicity values in real-networks

Network Name	n_{lbc}	m_{lbc}	D_{lbc}	$Vis.Pairs$	IsD	$T.Triplets$	$Vis.ISides$	ς	δ	$\check{\tau}$
CA-AstroPh	15929	193922	10	12	Y	6.7×10^{11}	5.6×10^{4}	4	3	6
CA-CondMat	17234	84595	12	6	Y	8.5×10^{11}	1.4×10^{4}	5	3.5	7
CA-GrQc	2651	10480	11	5	Y	3.1×10^{9}	4.6×10^{3}	5	3.5	7
CA-HepPh	9025	114046	11	201	N*	1.2×10^{11}	6.6×10^{5}	5	3	7
CA-HepTh	5898	20983	11	4	Y	3.4×10^{10}	7.6×10^{3}	5	4	8
Erdos	2145	7067	4	2	Y	1.6×10^{9}	2	2	2	4
GEOM	1901	6816	10	7	Y	1.1×10^{9}	3.1×10^{3}	4	3	7
DIMES-AUG052009	18344	76557	6	42	Y	1.0×10^{12}	6.1×10^{5}	3	2	4
DIMES-AUG052010	18840	85488	6	161	Y	1.1×10^{12}	2.2×10^{6}	3	2	4
DIMES-AUG052011	18439	89859	7	26	N	1.0×10^{12}	3.1×10^{5}	3	2	3
DIMES-AUG042012	16907	66489	7	13	N*	8.0×10^{11}	1.9×10^{5}	3	2	4
CAIDA-20130101	27454	124672	10	$1.5\ 10^{4}$	Y	3.4×10^{12}	1.5×10^{8}	3	2.5	5
CAIDA-20140401	30027	151354	9	$7.6\ 10^{3}$	Y	4.5×10^{12}	1.7×10^{7}	3	2	4
CAIDA-20150101	30421	161153	8	$2.4\ 10^{2}$	Y	4.6×10^{12}	1.6×10^{6}	3	2	4
CAIDA-20160101	33939	191467	10	$1.8\ 10^{4}$	N*	6.5×10^{12}	3.2×10^{8}	3	2	4
CAIDA-20170101	35907	219377	10	$1.4\ 10^{3}$	N*	7.7×10^{12}	8.2×10^{8}	3	2	4
CAIDA-20170201	36296	218906	10	$1.9\ 10^{3}$	N	7.9×10^{12}	3.7×10^{7}	4	2.5	5
CAIDA-20170301	36649	224860	9	121	N*	8.2×10^{12}	9.4×10^{5}	4	2.5	5

Abdulhakeem Mohammed (2019), Slimness, thinness, and other negative curvature parameters of graphs (Dissertation), Kent State University

Hyperbolicity values in real-networks

(a) Social Networks

Network	$\|V\|$	$\|E\|$	diam.	$\mathbf{E}d(x,y)$	*mean* c_v	med c_v	$\delta_{\max}$	$\mathbf{E}\delta(x,y)$
sn-Facebook	63392	816886	15	4.322	0.223	0.167	2.0	0.229
sn-Small	6116	31374	15	5.096	0.243	0.179	3.0	0.304
sn-Medium	26567	226566	14	4.258	0.223	0.167	2.0	0.230
sn-Large	293501	5589802	13	4.213	0.181	0.136	2.0	0.224
wiki-Vote	7066	100736	7	3.248	0.142	0.1	1.5	0.174

(b) Signed Networks

Network	$\|V\|$	$\|E\|$	diam.	$\mathbf{E}d(x,y)$	*mean* c_v	med c_v	$\delta_{\max}$	$\mathbf{E}\delta(x,y)$
soc-sign-Epinions	63192	648634	14	3.623	0.312	0.167	1.5	0.365
Slashdot081106	49166	440370	11	3.584	0.0903	0.00735	1.5	0.386
Slashdot090216	53405	469210	12	3.646	0.0937	0.00647	1.5	0.386
Slashdot090221	53592	471933	12	3.645	0.0942	0.00654	1.5	0.385
soc-Slashdot0811	77360	546487	12	4.024	0.0555	0.0	1.5	0.191
soc-Slashdot0922	82168	582533	13	4.069	0.0603	0.0	1.5	0.191

W. Sean Kennedy, Onuttom Narayan, Iraj Saniee, On the Hyperbolicity of Large-Scale Networks, arXiv:1307.0031

Hyperbolicity values in real-networks

(c) RocketFuel Networks

Network	$\|V\|$	$\|E\|$	diam.	$\mathbf{E}d(x,y)$	$mean\ c_v$	med c_v	$\delta_{\max}$	$\mathbf{E}\delta(x,y)$
1221 Telstra	2998	3806	12	5.525	0.0141	0.0	2.0	0.0987
1239 Sprintlink	8341	14025	13	5.184	0.0247	0.0	2.0	0.236
1755 Ebone	605	1035	13	5.960	0.0510	0.0	2.5	0.276
2914 Verio	7102	12291	13	6.0422	0.0691	0.0	2.5	0.258
3257 Tiscali	855	1173	14	5.300	0.0133	0.0	2.0	0.139
3356 Level3	3447	9390	11	5.0693	0.0854	0.0	2.0	0.158
3967 Exodus	895	2070	13	5.944	0.177	0.0	2.5	0.320
4755 VSNL	121	228	6	3.196	0.0372	0.0	2.0	0.071
7018 AT&T	10152	14319	12	6.947	0.00580	0.0	2.5	0.251

W. Sean Kennedy, Onuttom Narayan, Iraj Saniee, On the Hyperbolicity of Large-Scale Networks, arXiv:1307.0031

Hyperbolicity values in real-networks

(d) Peer-to-peer Networks

Network	$\lvert V \rvert$	$\lvert E \rvert$	diam.	$\mathbf{E}d(x,y)$	$mean\ c_v$	med c_v	$\delta_{\max}$	$\mathbf{E}\delta(x,y)$
p2p-Gnutella04	10876	39994	10	4.636	0.00622	0.0	2.5	0.288
p2p-Gnutella05	8842	31837	9	4.596	0.00720	0.0	2.0	0.292
p2p-Gnutella06	8717	31525	10	4.572	0.00668	0.0	2.5	0.290
p2p-Gnutella08	6299	20776	9	4.643	0.0109	0.0	2.0	0.296
p2p-Gnutella09	8104	26008	10	4.767	0.00954	0.0	2.5	0.298
p2p-Gnutella24	26498	65359	11	5.418	0.00551	0.0	2.5	0.294
p2p-Gnutella25	22663	54693	11	5.545	0.00531	0.0	2.5	0.325
p2p-Gnutella30	36646	88303	11	5.750	0.00630	0.0	2.5	0.319
p2p-Gnutella31	62561	147878	11	5.936	0.00547	0.0	2.5	0.314

W. Sean Kennedy, Onuttom Narayan, Iraj Saniee, On the Hyperbolicity of Large-Scale Networks, arXiv:1307.0031

Hyperbolicity values in real-networks

(a) Collaboration Networks

Network	$\|V\|$	$\|E\|$	diam.	$\mathbf{E}d(x,y)$	*mean* c_v	med c_v	$\delta_{\max}$	$\mathbf{E}\delta(x,y)$
CA-AstroPh	17903	197031	14	4.194	0.633	0.667	2.0	0.235
CA-CondMat	21363	91342	15	5.352	0.642	0.700	2.5	0.286
CA-GrQc	4158	13428	17	6.049	0.557	0.533	3.0	0.377
CA-HepPh	11204	117649	13	4.673	0.622	0.673	2.0	0.252
CA-HepTh	8638	24827	18	5.945	0.482	0.333	3.0	0.338

(b) Web Networks

Network	$\|V\|$	$\|E\|$	diam.	$\mathbf{E}^{\star}d(x,y)$	*mean* c_v	med c_v	$\delta_{\max}$	$\mathbf{E}\delta(x,y)$
web-Google	855802	4291352	24	6.334	0.605	0.519	2.0	0.234
web-BerkStan	654782	6581871	208	7.106	0.615	0.607	2.0	0.306
web-Stanford	255265	1941926	164	6.815	0.619	0.667	1.5	0.198

W. Sean Kennedy, Onuttom Narayan, Iraj Saniee, On the Hyperbolicity of Large-Scale Networks, arXiv:1307.0031

Center C(G) and median M(G) do not always coincide