Propositional Logic

Section 1.1

Section Summary

- Propositions
- Connectives
  - Negation
  - Conjunction
  - Disjunction
  - Implication; contrapositive, inverse, converse
  - Biconditional
- Truth Tables

Propositions

- A *proposition* is a declarative sentence that is either true or false.
- Examples of propositions:
  - The Moon is made of green cheese.
  - Trenton is the capital of New Jersey.
  - Toronto is the capital of Canada.
  - $1 + 0 = 1$
  - $0 + 0 = 2$
- Examples that are not propositions.
  - Sit down!
  - What time is it?
  - $x + 1 = 2$
  - $x + y = z$

Examples: Is it a proposition?

- Do your homework.

  NO! It's a command, not a declarative sentence that's either true or false.

- Pigs can fly.

  YES! The proposition is false.

- x is greater than y.

  NO! Until we know the values of x and y, we can't say whether the statement is true or false.

Propositional Logic

- Constructing Propositions
  - Propositional variables: p, q, r, s, …
  - Compound Propositions: constructed from logical connectives and other propositions
    - Negation $\neg$
    - Conjunction $\wedge$
    - Disjunction $\vee$
    - Implication $\rightarrow$
    - Biconditional $\leftrightarrow$
- The proposition that is always true is denoted by **T** and the proposition that is always false is denoted by **F**.

Compound Propositions: Negation

- The *negation* of a proposition p is denoted by $\neg p$ and has this truth table:

p	$\neg p$
T	F
F	T

- **Example**:

p : "The earth is round."

$\neg p$ denotes "It is not the case that the earth is round," or more simply "The earth is not round."

Conjunction (AND)

- The *conjunction* of propositions p and q is denoted by $p \wedge q$ and has this truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- **Example**:

p : "I am at home."

q: "It is raining."

$p \wedge q$ denotes "I am at home and it is raining."

Disjunction (inclusive OR)

- The *disjunction* of propositions p and q is denoted by $p \vee q$ and has this truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- **Example**:

p: "I am at home."

q: "It is raining."

$p \vee q$: "I am at home or it is raining."

The Connective Or in English

- In English "or" has two distinct meanings.
  - Inclusive Or - "Students who have taken CS202 or Math120 may take this class". We assume that students need to have taken one of the prerequisites, but may have taken both. This is the meaning of disjunction. For $p \lor q$ to be true, either one or both of p and q must be true.
  - Exclusive Or - "Soup or salad comes with this entrée," we do not expect to be able to get both soup and salad. This is the meaning of Exclusive Or (Xor). In $p \oplus q$, one of p and q must be true, **but not both**. The truth table for $\oplus$ is:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication

- If p and q are propositions, then $p \rightarrow q$ is a *conditional statement* or *implication* which is read as "if p, then q" and has this truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Example**:

p : "I am at home." q: "It is raining."

$p \rightarrow q$: "If I am at home, then it is raining."

- In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

Understanding Implication

- In $p \rightarrow q$ there doesn't need to be any connection between the antecedent or the consequent. The "meaning" of $p \rightarrow q$ depends only on the truth values of p and q.
- These implications are valid (and true!):
  - "If the moon is made of green cheese, then I have more money than Bill Gates. "
  - "If the moon is made of green cheese, then I'm in Kent."
  - "If 1 + 1 = 3, then your grandma wears combat boots."

Understanding Implication (cont)

- One way to view the logical conditional is to think of an obligation or contract.
  - "If I am elected, then I will lower taxes."
  - "If you get 100% on the final, then you will get an A."
- If the politician is elected and does not lower taxes, then the voters can say that he or she has broken the campaign pledge. Something similar holds for the professor. This corresponds to the case where p is true and q is false.

Different Ways of Expressing $p \to q$

- **if p, then q**
- **if p, q**
- **q unless $\neg p$**
- **q if p**
- **q whenever p**
- **q follows from p**
- **p implies q**
- **p only if q**
- **q when p**
- **p is sufficient for q**
- **q is necessary for p**
- **It is necessary to q to p**

- **a necessary condition for p is q**
- **a sufficient condition for q is p**

Ex: Express in the form “if p, then q”

- It is necessary to do the boss’s laundry to get promoted.

  If you get promoted, then you’ve washed the boss’s laundry.

- John gets caught whenever he cheats.

  If John cheats, then he gets caught.

- The apple trees will bloom if it stays warm for a week.

  If it stays warm for a week, then the apple trees will bloom.

- To get tenure as a professor, it is sufficient to be world-famous.

  If you world-famous, then you will get tenure as a professor.

Converse, Contrapositive, and Inverse

- From $p \to q$ we can form new conditional statements .
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$
  - $q \to p$ is the **converse** of $p \to q$
  - $\neg q \to \neg p$ is the **contrapositive** of $p \to q$
  - **Example**: Find the converse, inverse, and contrapositive of "It is raining is a sufficient condition for me not going to town."

Solution:

inverse: If it is not raining, then I will go to town.

converse: If I do not go to town, then it is raining.

contrapositive: If I go to town, then it is not raining.

Biconditional

- If p and q are propositions, then we can form the *biconditional* proposition $p \leftrightarrow q$, read as "p if and only if q." The biconditional $p \leftrightarrow q$ denotes the proposition with this truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- If p denotes "I am at home." and q denotes "It is raining." then $p \leftrightarrow q$ denotes "I am at home if and only if it is raining."

Expressing the Biconditional

- Some alternative ways "p if and only if q" is expressed in English:

  - p **is necessary and sufficient for** q
  - **if** p **then** q , **and conversely**
  - p **iff** q

Examples

p: It's raining $\qquad$ q: I carry my umbrella

- What's the negation of p?
  $\neg p$: It's **not** raining.
- What's the disjunction?
  $p \vee q$: It's raining **or** I carry my umbrella.
- What's the conjunction?
  $p \wedge q$: It's raining **and** I carry my umbrella.
- What's the implication, using p as the hypothesis?
  $p \rightarrow q$: **If** it's raining, **then** I carry my umbrella.
- What's the biconditional?
  $p \leftrightarrow q$: It rains **if and only if** I carry my umbrella.

Examples

Use the implication $p \rightarrow q$, where

p: It's raining q: I carry my umbrella

- What's the converse?

  $q \rightarrow p$: If I carry my umbrella, then it's raining.
- What's the contrapositive?

  $\neg q \rightarrow \neg p$: If I don't carry my umbrella, then it's not raining.
- What's the inverse?

  $\neg p \rightarrow \neg q$: If it's not raining, then I don't carry my umbrella.

Truth Tables For Compound Propositions

- Construction of a truth table:
- Rows
  - Need a row for every possible combination of values for the atomic propositions.
- Columns
  - Need a column for the compound proposition (usually at far right)
  - Need a column for the truth value of each expression that occurs in the compound proposition as it is built up.

Example Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Equivalent Propositions

- Two propositions are *equivalent* if they always have the same truth value.
- **Example**: Show using a truth table that the implication is equivalent to the contrapositive.

Solution:

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Using a Truth Table to Show Non-Equivalence

Example: Show using truth tables that neither the converse nor inverse of an implication are equivalent to the implication. (i.e., neither $\neg p \rightarrow \neg q$ nor $q \rightarrow p$ is equivalent to $p \rightarrow q$)

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \rightarrow \neg q$	$q \rightarrow p$
T	T	F	F	T	T	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Problem

- How many rows are there in a truth table with n propositional variables?

Solution: 2^n We will see how to do this in Chapter 6.

- Note that this means that with n propositional variables, we can construct 2^n distinct (i.e., not equivalent) propositions.

Precedence of Logical Operators

Operator	Precedence
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

$p \vee q \rightarrow \neg r$ is equivalent to $(p \vee q) \rightarrow \neg r$
If the intended meaning is $p \vee (q \rightarrow \neg r)$ then parentheses must be used.

Example: Construct a truth table for $((p \wedge \neg q) \leftrightarrow (p \vee q) \oplus r)$

p	q	r	¬q	p∧¬q	p∨q	(p∧¬q)↔(p∨q)	((p∧¬q)↔(p∨q) ⊕ r)
F	F	F	T	F	F	T	T
F	F	T	T	F	F	T	F
F	T	F	F	F	T	F	F
F	T	T	F	F	T	F	T
T	F	F	T	T	T	T	T
T	F	T	T	T	T	T	F
T	T	F	F	F	T	F	F
T	T	T	F	F	T	F	T