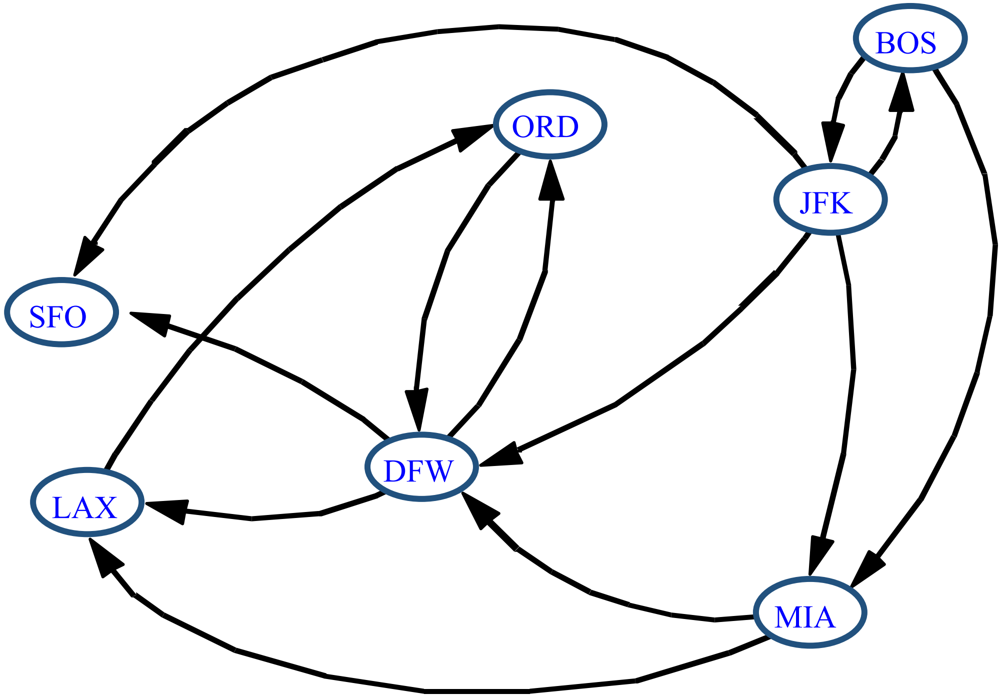
Directed Graphs



Outline and Reading

Reachability (6.4.1)

- Directed DFS
- Strong connectivity

Transitive closure (6.4.2)

- The Floyd-Warshall Algorithm

Directed Acyclic Graphs (DAGs) (6.4.4)

- Topological Sorting

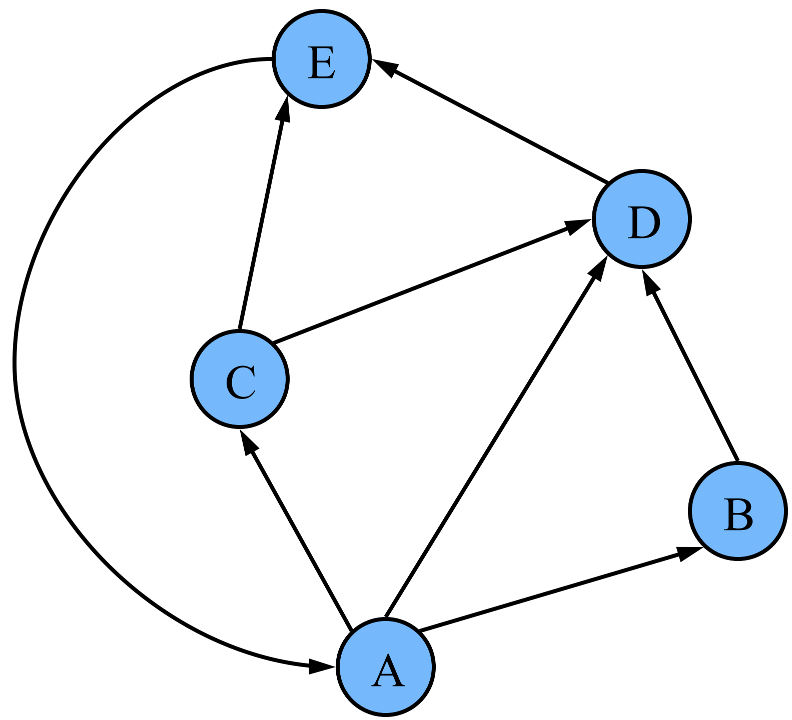
Digraphs

A **digraph** is a graph whose edges are all directed

- short for "directed graph"

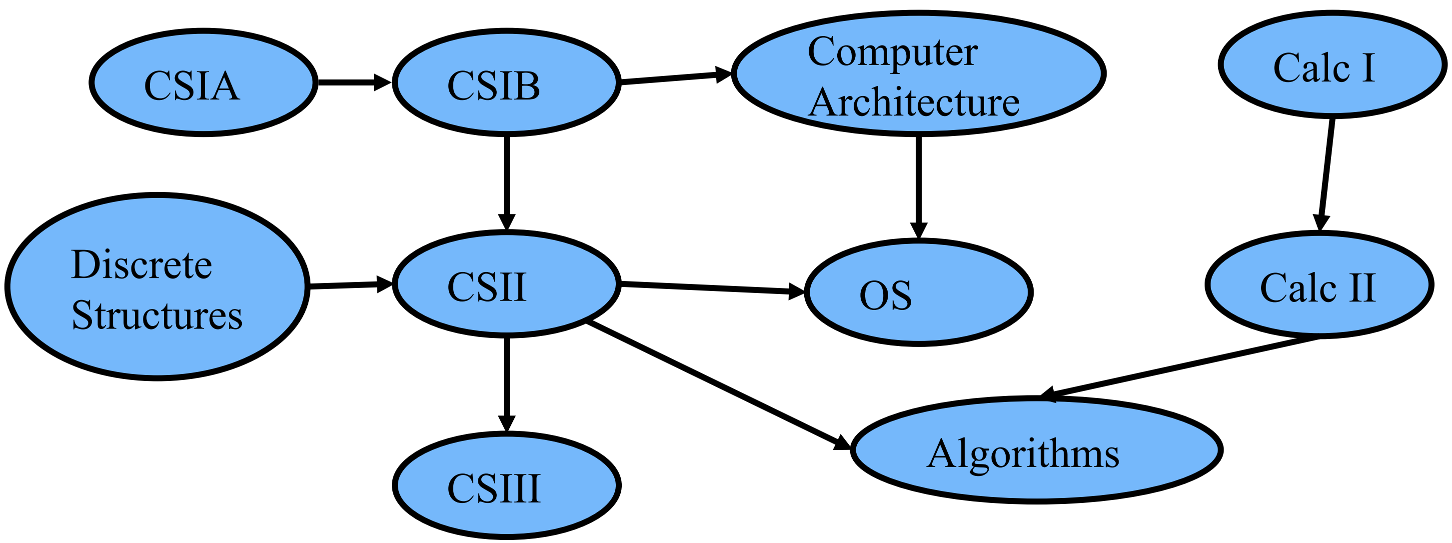
Applications

- one-way streets
- flights
- task scheduling



Digraph Application

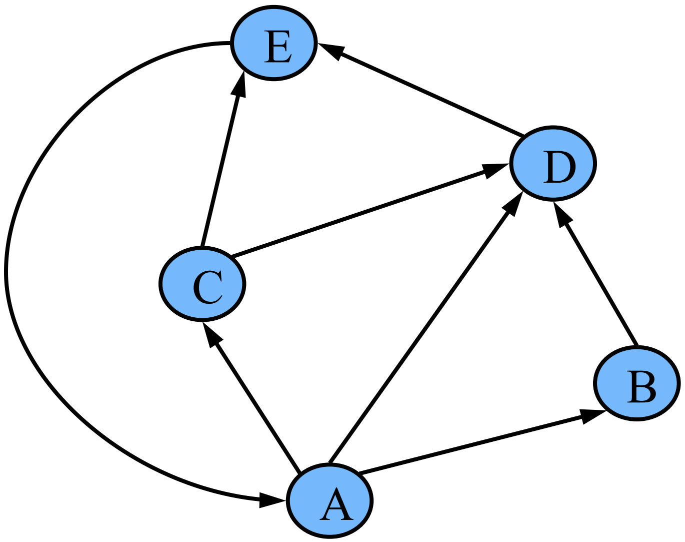
Scheduling: edge (a,b) means task a must be completed before b can be started.



Digraph Properties

A graph $G = (V, E)$ such that

- Each edge goes in one direction
- Ex: Edge (a,b) goes from a to b, but not b to a.



Properties:

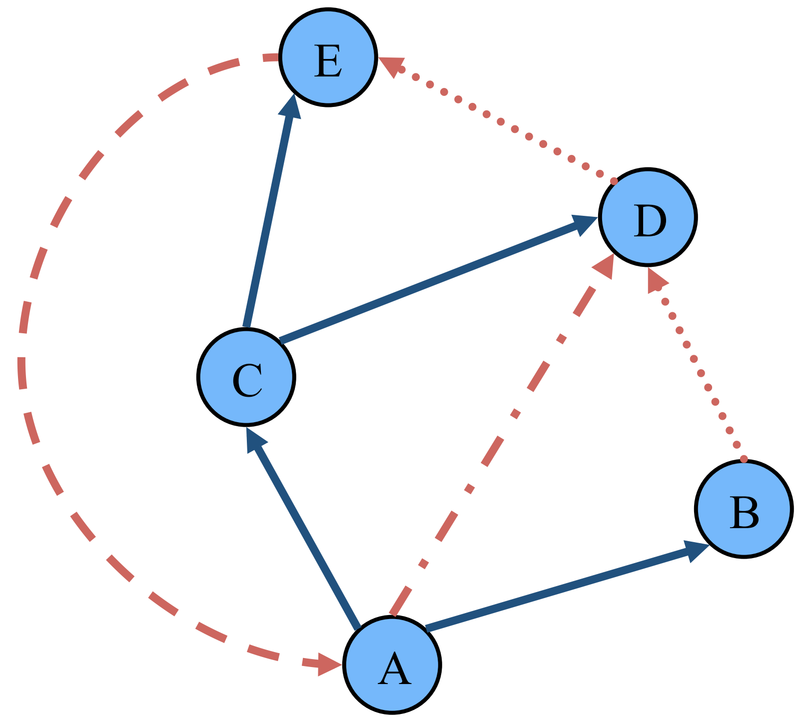
- If G is simple, $m \leq n(n-1)$.
- If we keep in-edges and out-edges in separate adjacency lists, we can perform listing of of the sets of in-edges and out-edges in time proportional to their size.

Directed DFS

- We can specialize the traversal algorithms (DFS and BFS) to digraphs by traversing edges only along their direction

- In the directed DFS algorithm, we have four types of edges
  - discovery edges
  - back edges
  - forward edges
  - cross edges

- A directed DFS starting at a vertex ***s*** determines the vertices reachable from ***s***

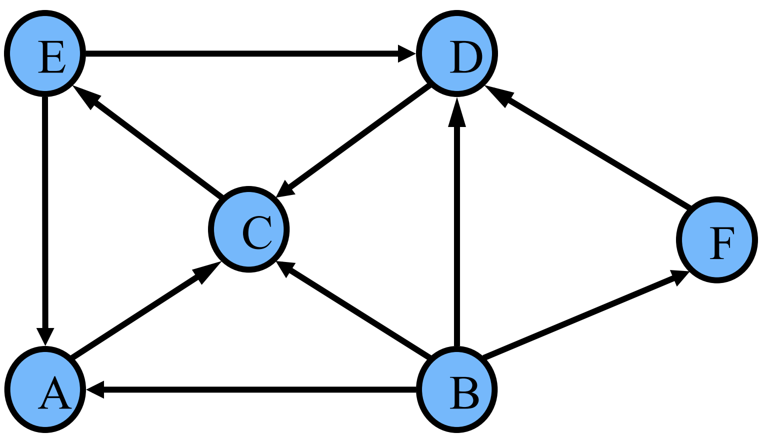


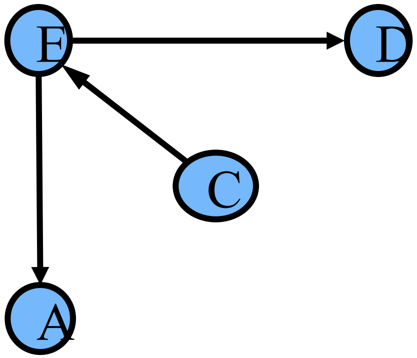
Reachability

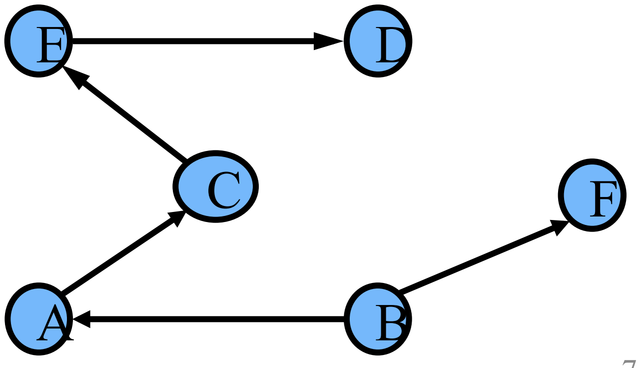
DFS tree rooted at *v*: vertices reachable from *v* via directed paths

Applications:

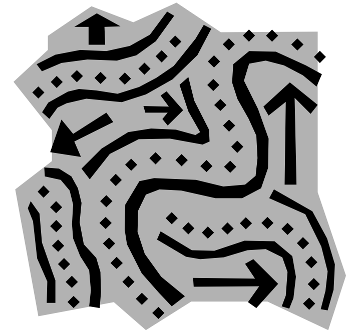
- Dead code detection/elimination
- Garbage collection



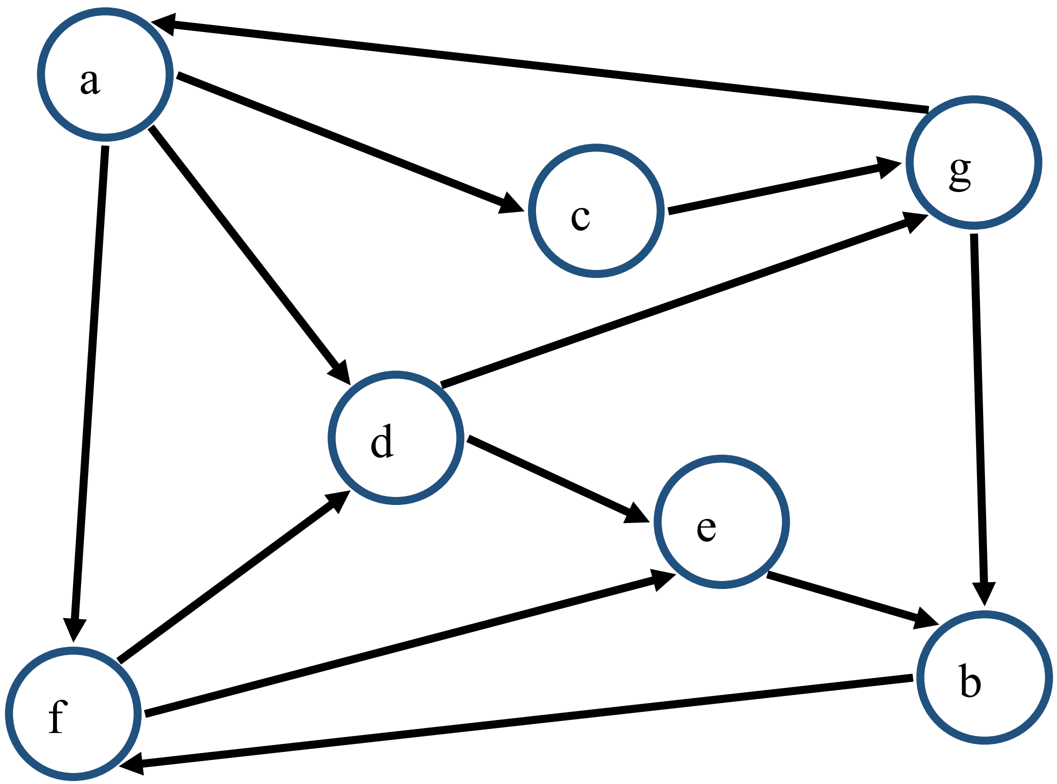




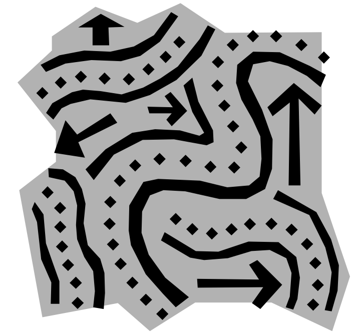
Strong Connectivity



Each vertex can reach all other vertices



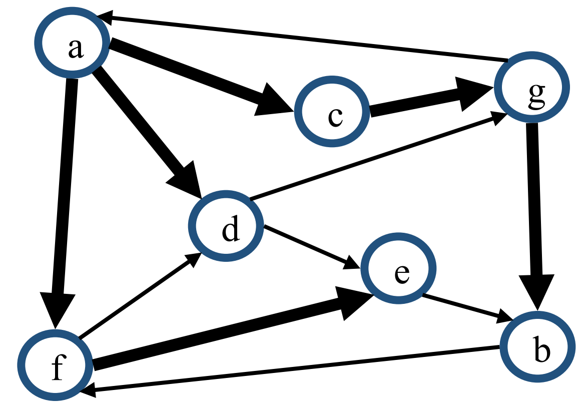
Strong Connectivity Algorithm



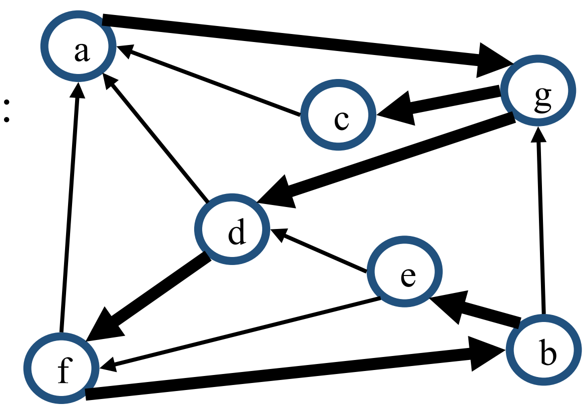
Determine if G is strongly connected

- Pick a vertex v in G
- Perform a DFS from v in G
  - If there's a w not visited, print "no"
- Let G' be G with edges reversed
- Perform a DFS from v in G'
  - If there's a w not visited, print "no"
  - Else, print "yes"

G:



G' :

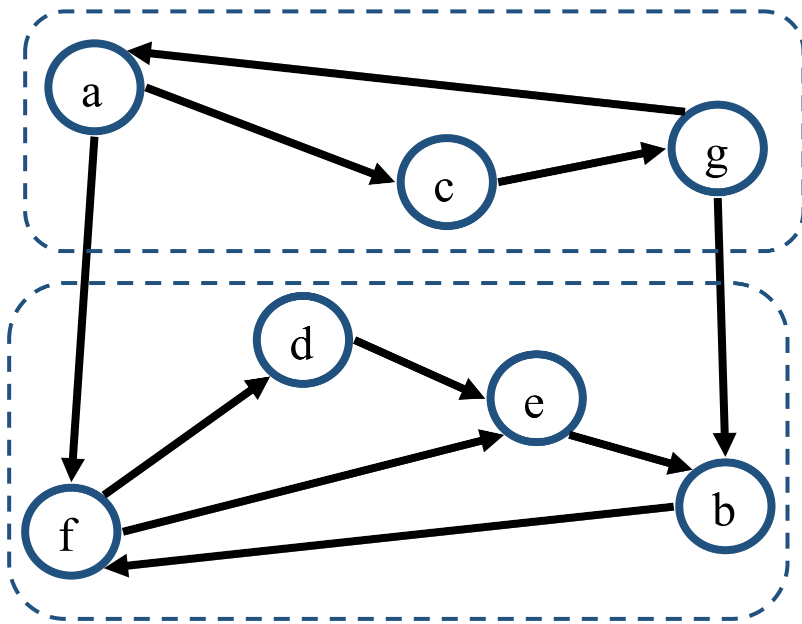


Running time: $O(n+m)$.

Strongly Connected Components

A strongly connected component is a maximal subgraph such that each vertex can reach all other vertices in the subgraph

- Can also be done in $O(n+m)$ time using DFS, but is more complicated (similar to biconnectivity).



{ a , c , g }

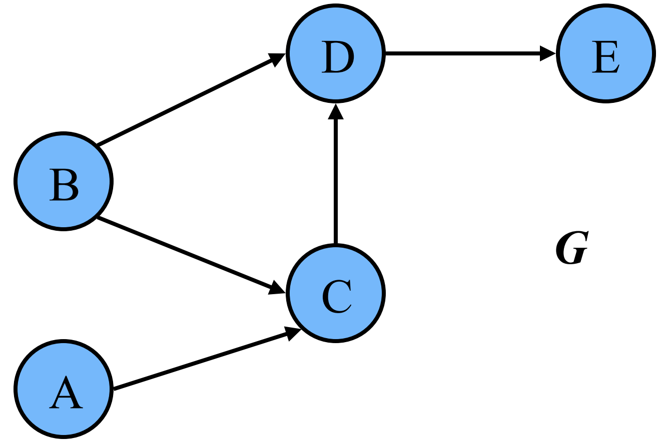
{ f , d , e , b }

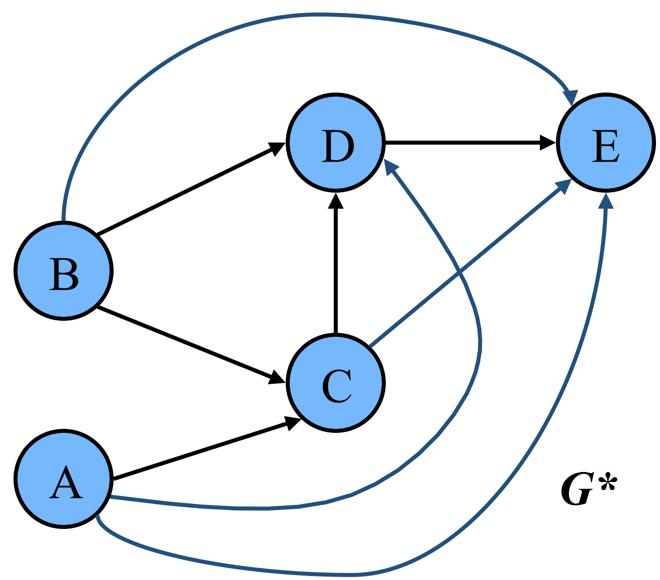
Transitive Closure

Given a digraph G, the transitive closure of G is the digraph G^* such that

- G^* has the same vertices as G
- if G has a directed path from u to v ($u \neq v$), G^* has a directed edge from u to v

The transitive closure provides reachability information about a digraph.



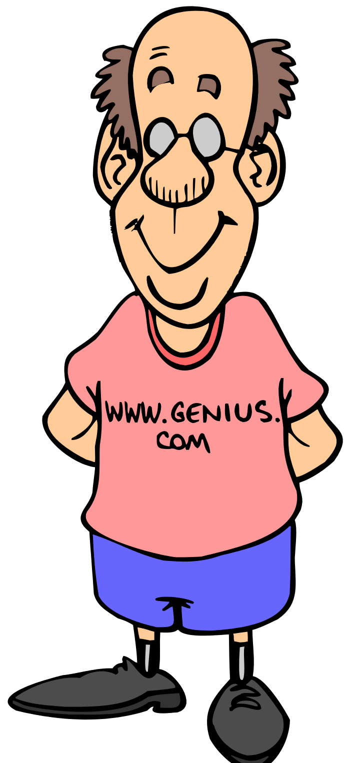


Computing the Transitive Closure

We can perform DFS starting at each vertex

- $O(n(n+m))$

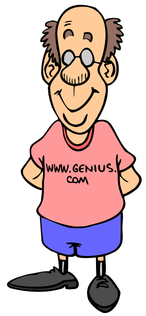
If there's a way to get from A to B and from B to C, then there's a way to get from A to C.



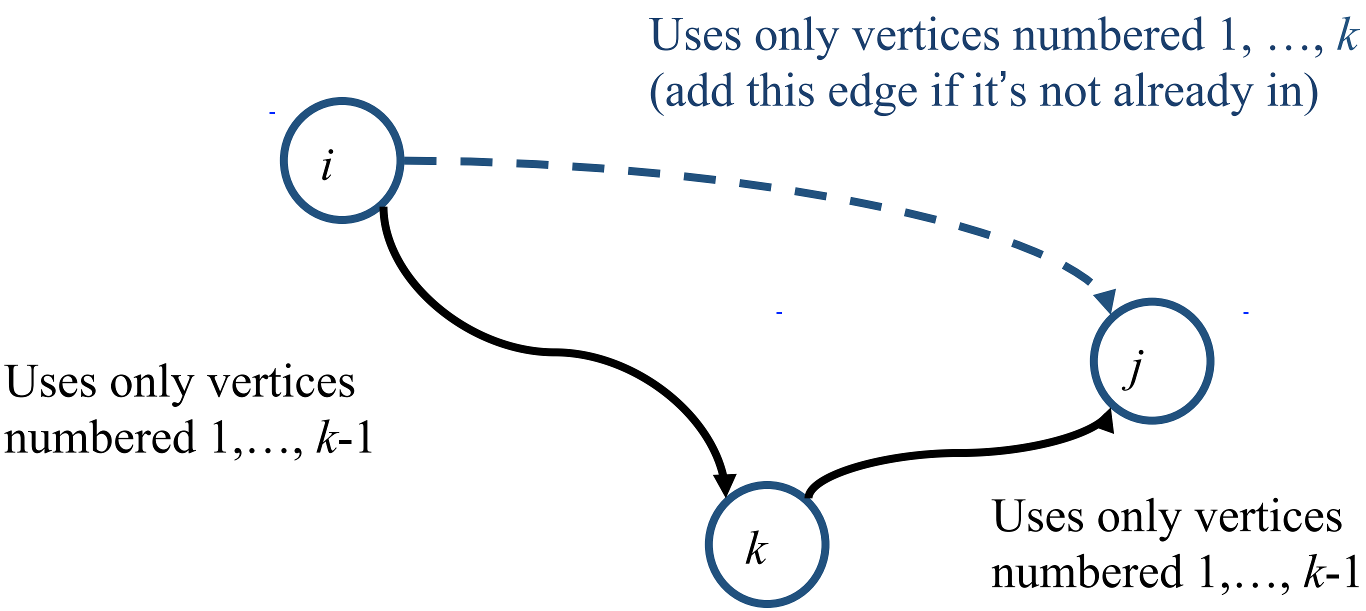
Alternatively ... Use dynamic programming:

Floyd-Warshall Algorithm

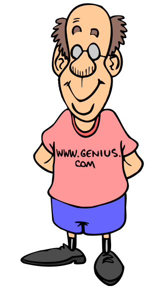
Floyd-Warshall Transitive Closure



- Idea #1: Number the vertices 1, 2, …, n.
- Idea #2: Consider paths that use only vertices numbered 1, 2, …, k, as intermediate vertices:



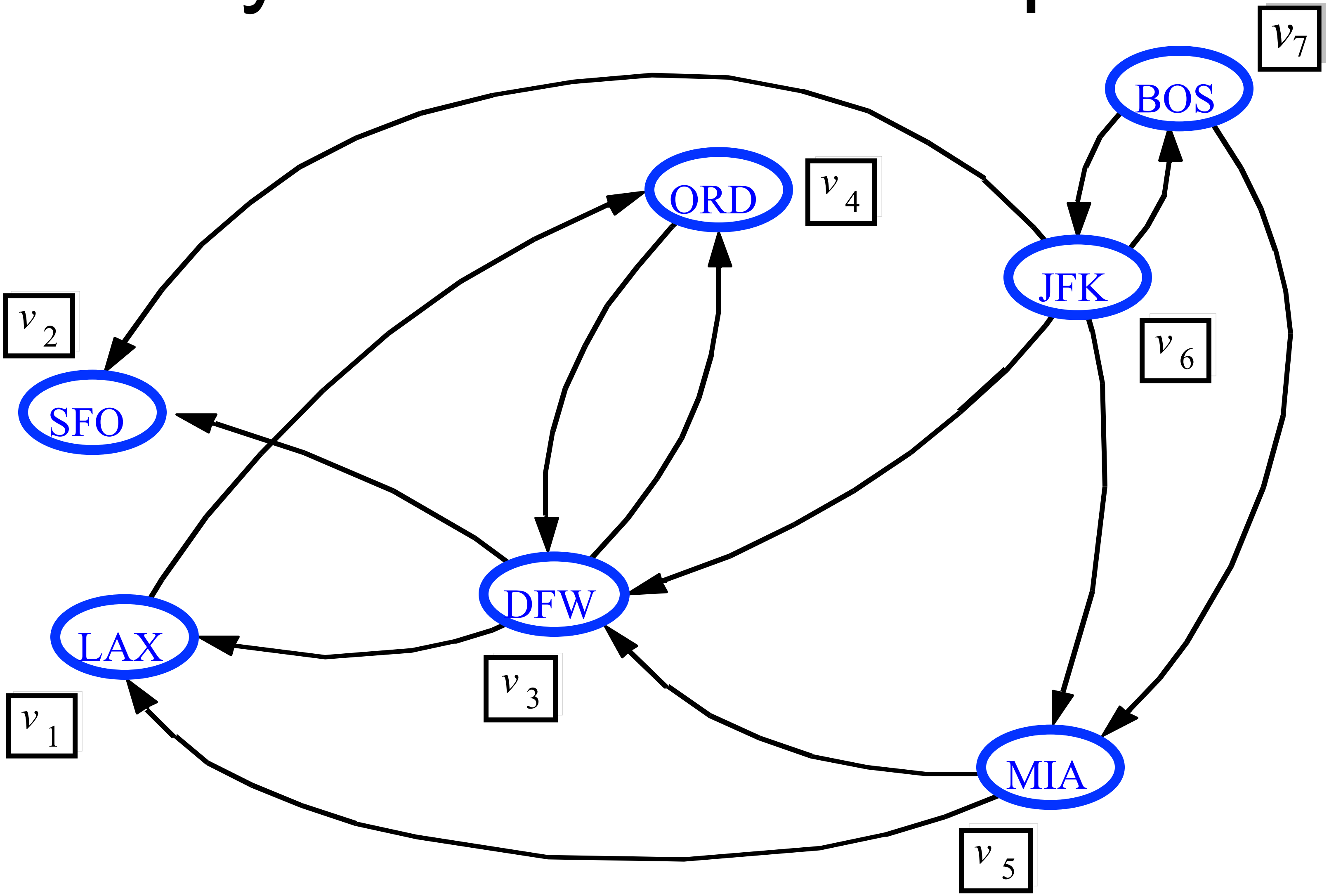
Floyd-Warshall's Algorithm



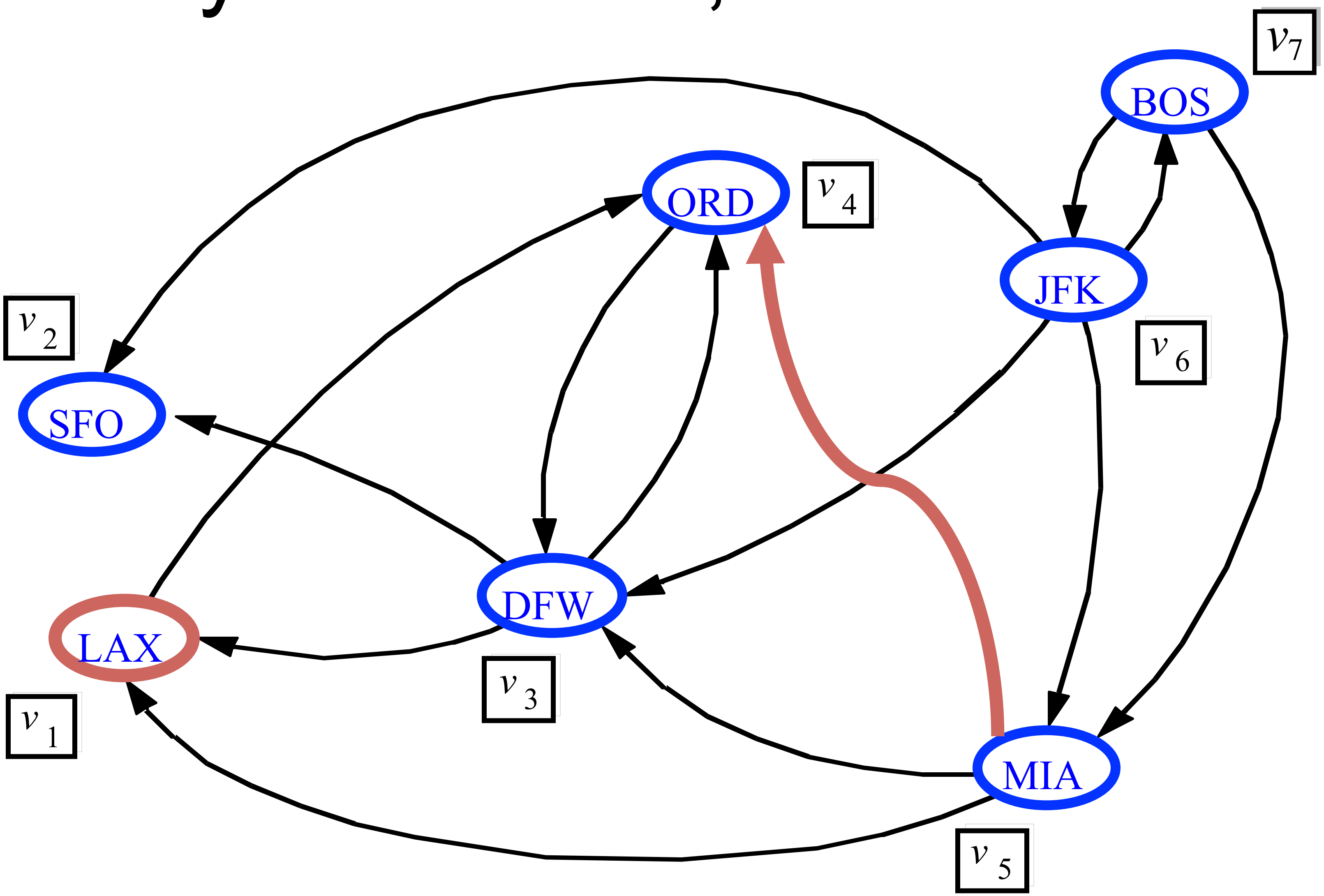
- Numbers the vertices of G as $v_1, \ldots, v_n$ and computes a series of digraphs $G_0, \ldots, G_n$
  - $G_0 = G$
  - G_k has a directed edge (v_i, v_j) if G has a directed path from v_i to v_j with intermediate vertices in the set $\{v_1, \ldots, v_k\}$
- We have that $G_n = G^*$
- In phase k, digraph G_k is computed from G_{k-1}
- Running time: $O(n^3)$, assuming areAdjacent is $O(1)$ (e.g., adjacency matrix)

Algorithm *FloydWarshall*(G)
  Input digraph G
  Output transitive closure G^* of G
  $i \leftarrow 1$
  for all $v \in G.vertices()$
    denote v as v_i
    $i \leftarrow i + 1$
  $G_0 \leftarrow G$
  for $k \leftarrow 1$ **to** n **do**
    $G_k \leftarrow G_{k-1}$
    for $i \leftarrow 1$ **to** n $(i \neq k)$ **do**
      for $j \leftarrow 1$ **to** n $(j \neq i, k)$ **do**
        if $G_{k-1}.areAdjacent(v_i, v_k) \wedge$
            $G_{k-1}.areAdjacent(v_k, v_j)$
          if $\neg\, G_k.areAdjacent(v_i, v_j)$
            $G_k.insertDirectedEdge(v_i, v_j, k)$
    return G_n

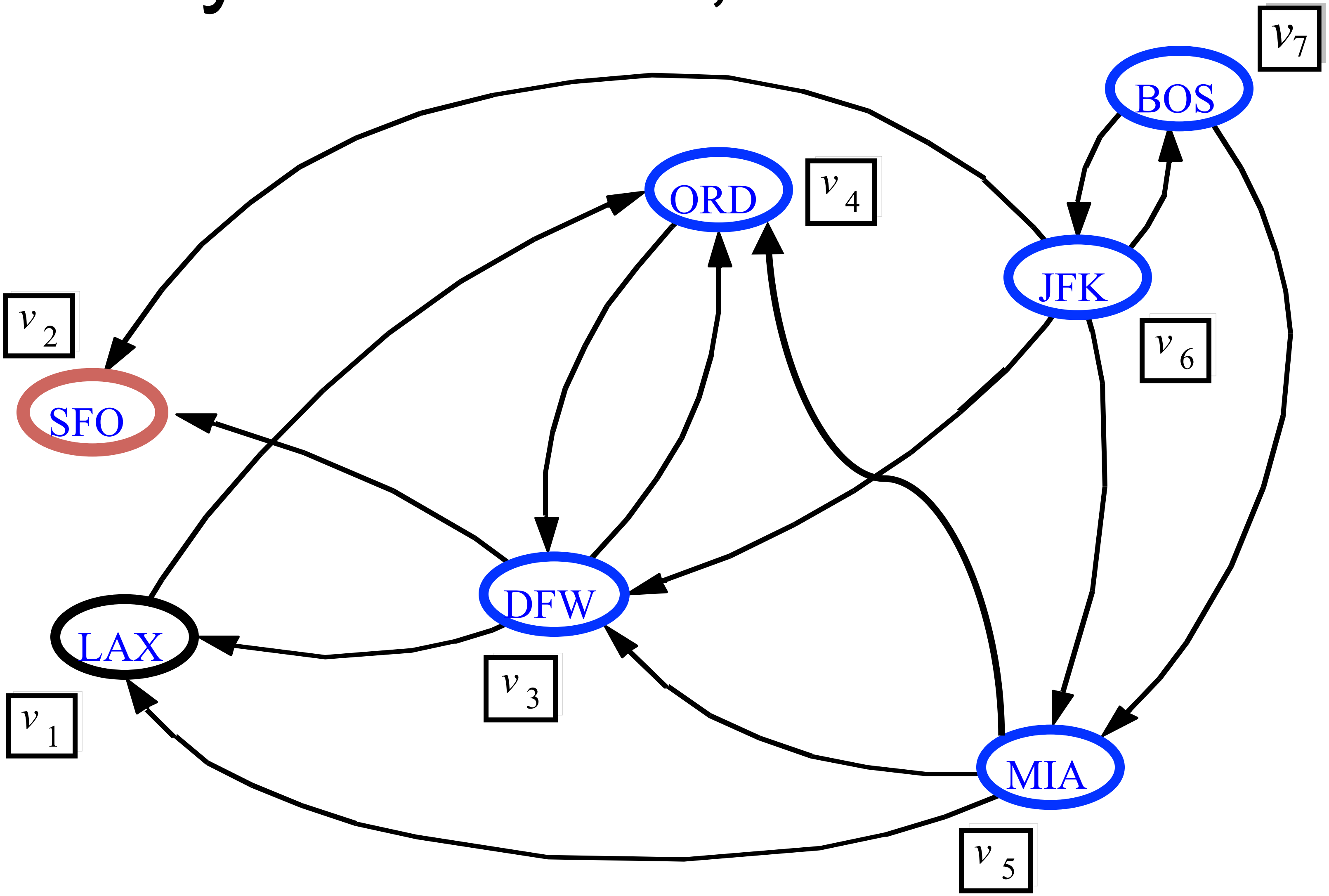
Floyd-Warshall Example



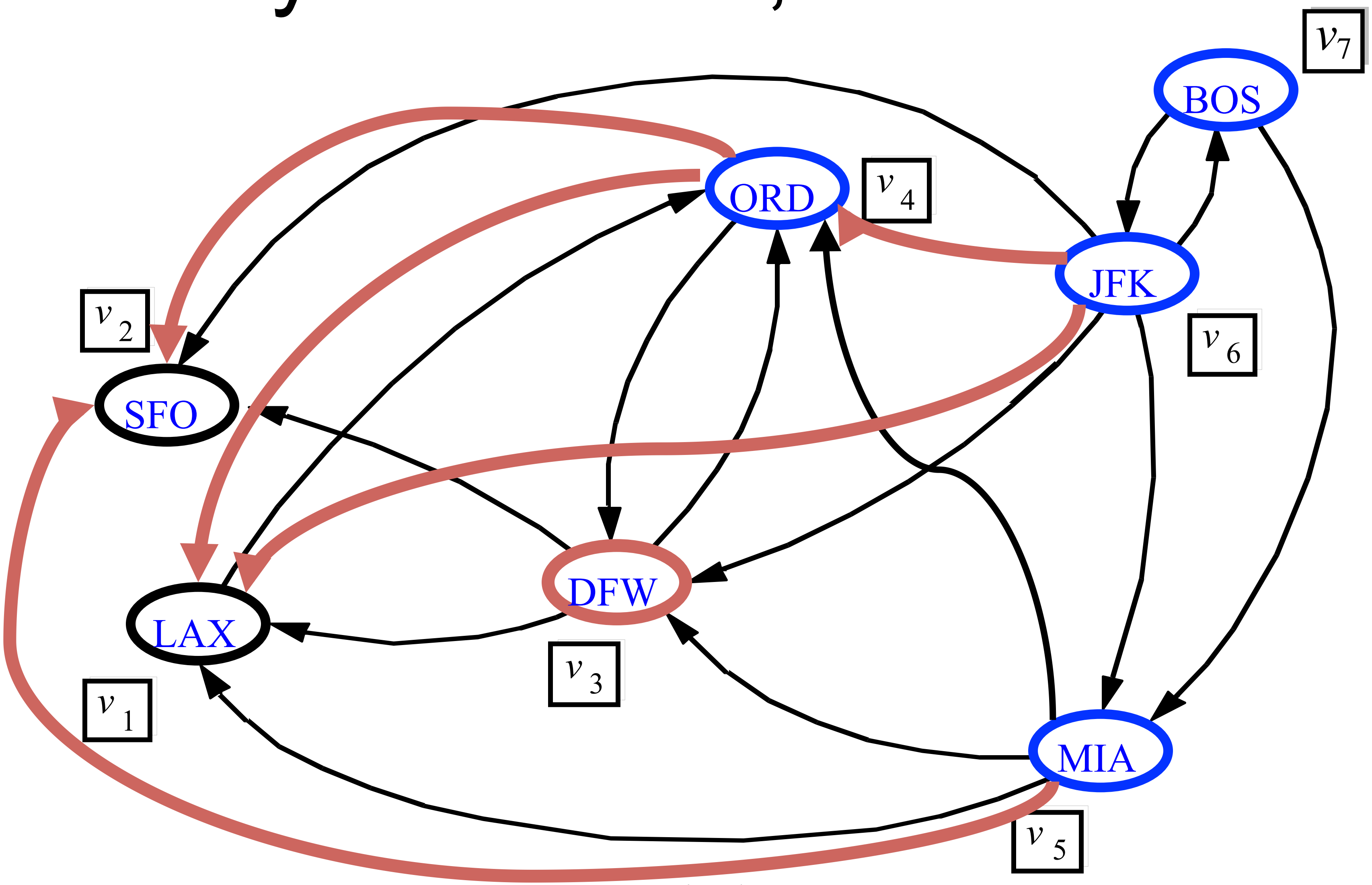
Floyd-Warshall, Iteration 1



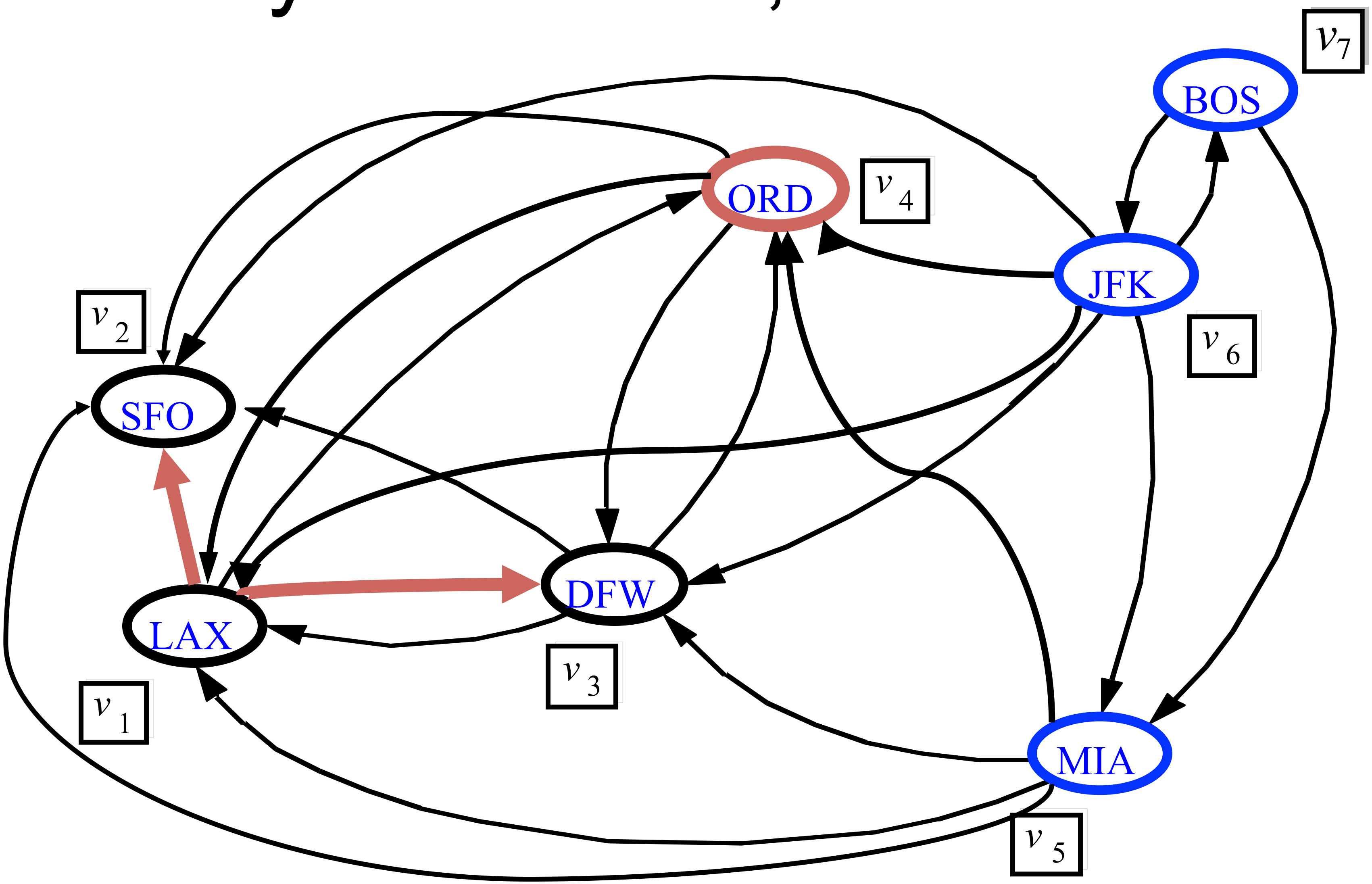
Floyd-Warshall, Iteration 2



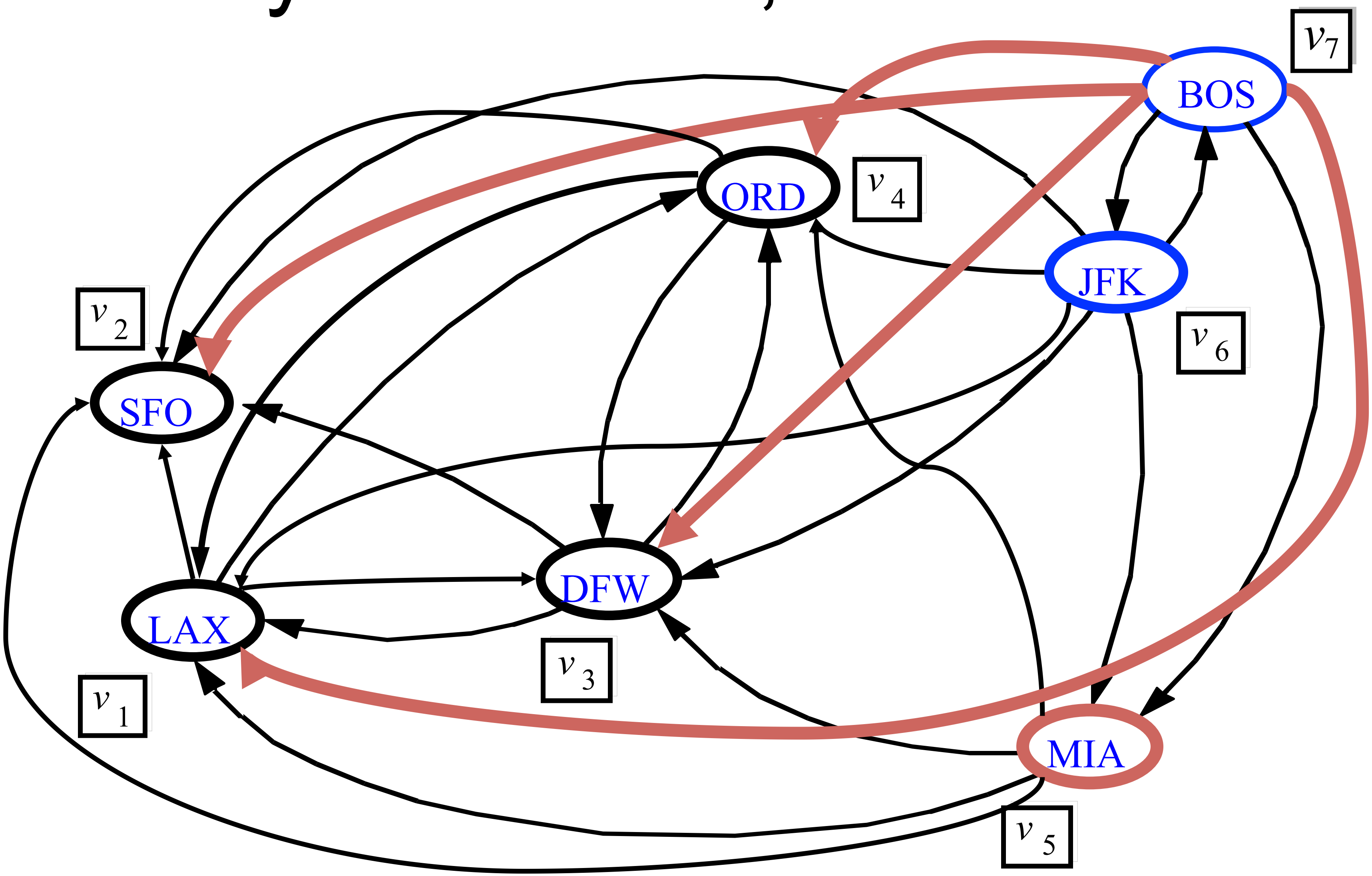
Floyd-Warshall, Iteration 3



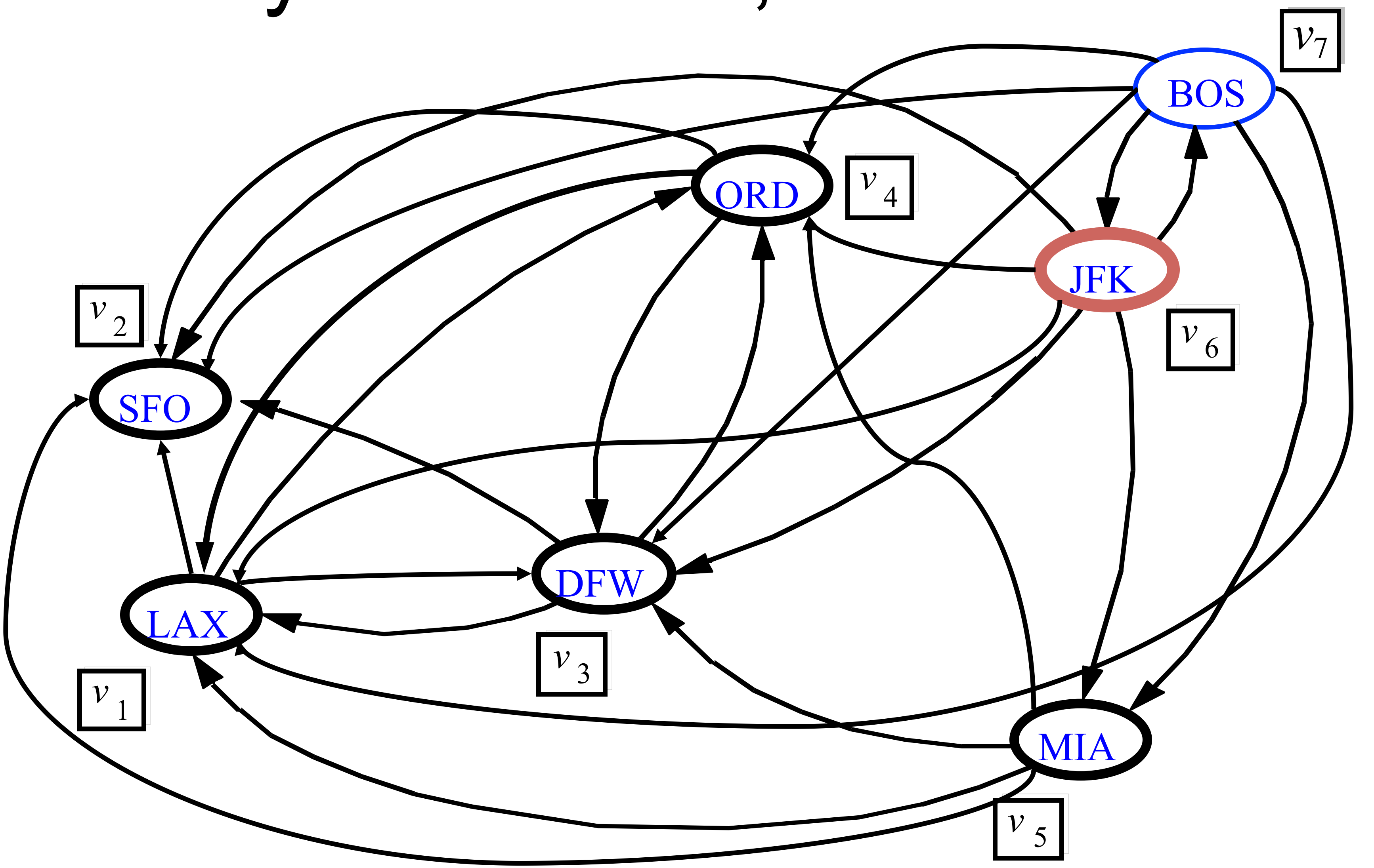
Floyd-Warshall, Iteration 4



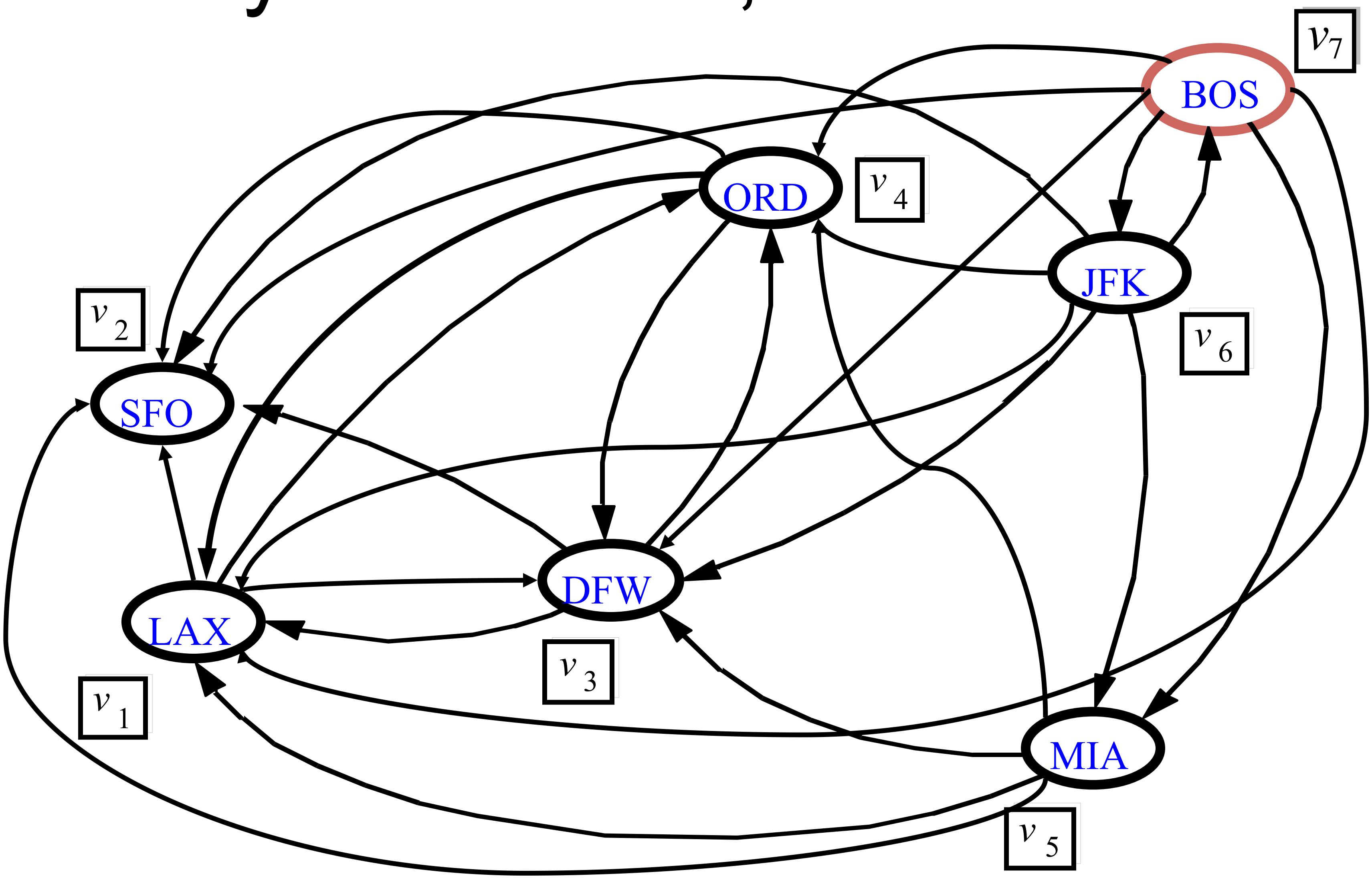
Floyd-Warshall, Iteration 5



Floyd-Warshall, Iteration 6

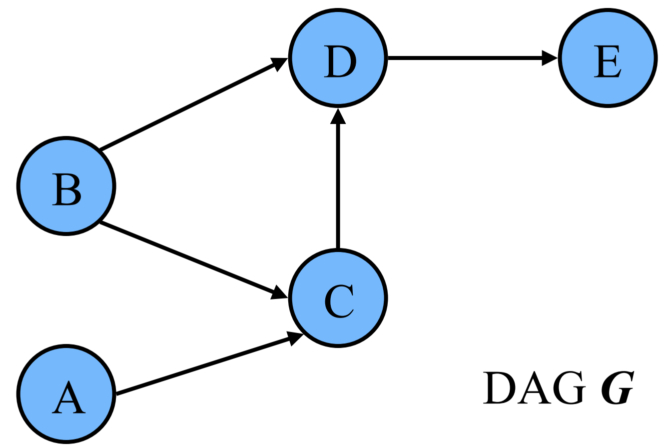


Floyd-Warshall, Conclusion



DAGs and Topological Ordering

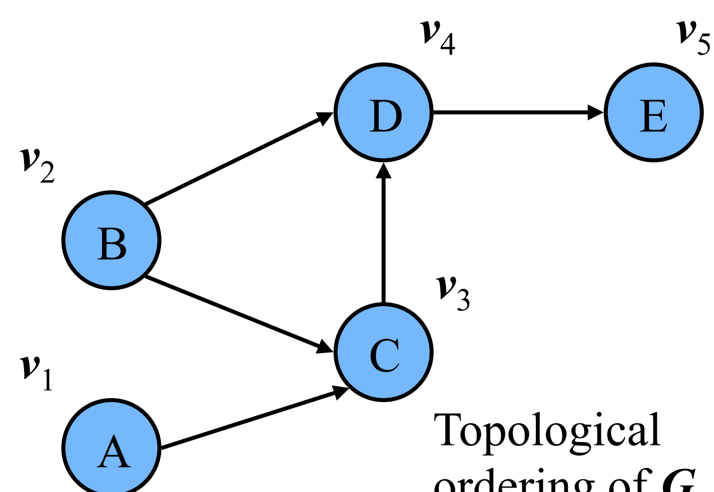
- A directed acyclic graph (DAG) is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering $v_1, \ldots, v_n$ of the vertices such that for every edge (v_i, v_j), we have $i < j$
- Ex: in a task scheduling digraph, a topological ordering a task sequence that satisfies the precedence constraints



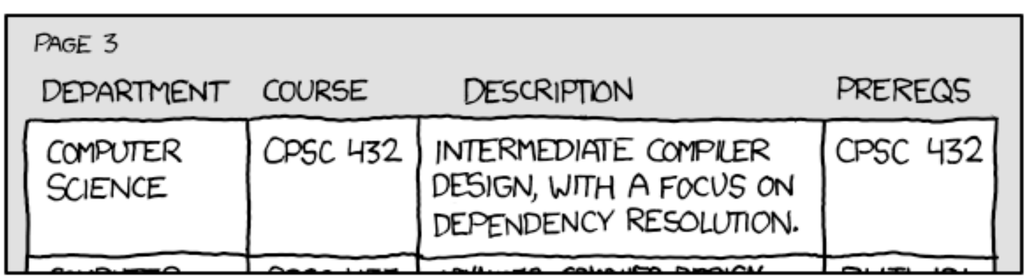
DAG G

Theorem

A digraph admits a topological ordering if and only if it is a DAG



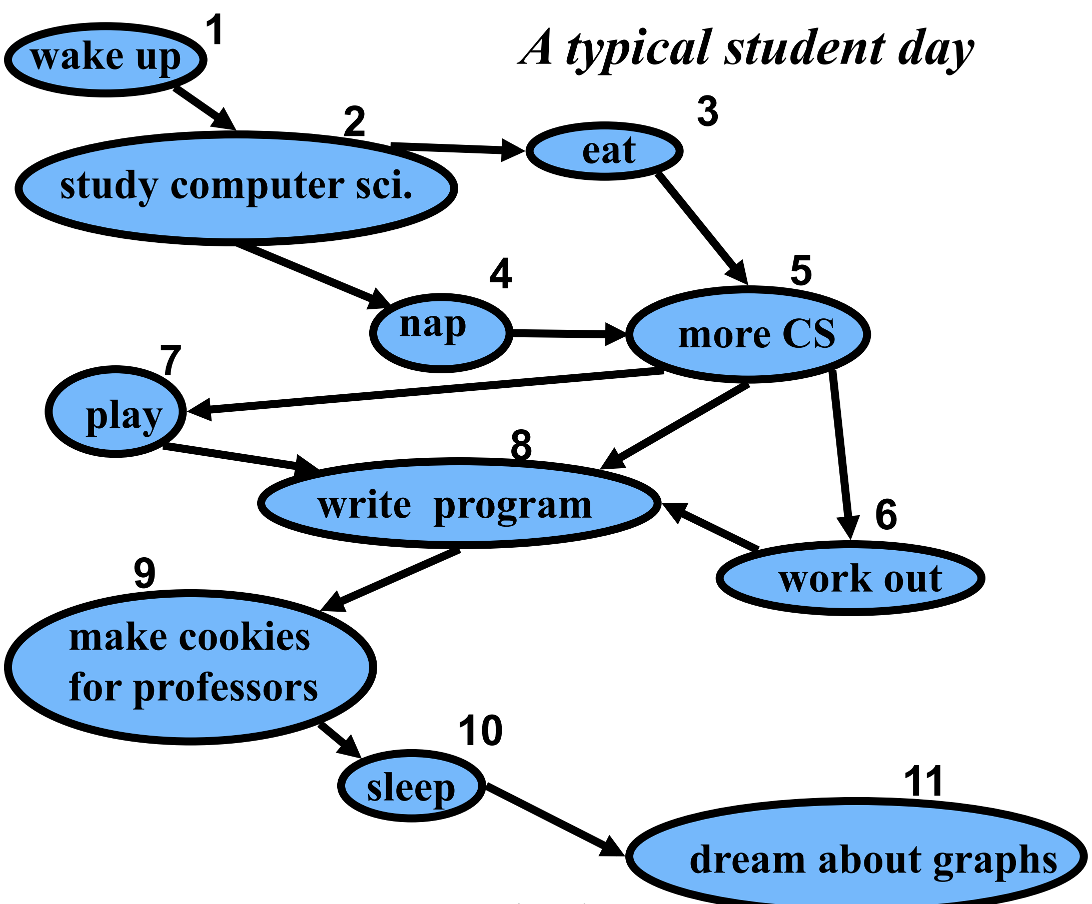
Topological ordering of G



xkcd #754

Topological Sorting

Number vertices, so that (u,v) in E implies $u < v$



Algorithm for Topological Sorting

- Note: This algorithm is different than the one in Goodrich-Tamassia

```
Method TopologicalSort(G)
    H ← G   // Temporary copy of G
    n ← G.numVertices()
    while H is not empty do
     Let v be a vertex with no outgoing edges
     Label v ← n
     n ← n - 1
     Remove v from H
```

- Running time: O(n + m). How…?

Topological Sorting Algorithm using DFS

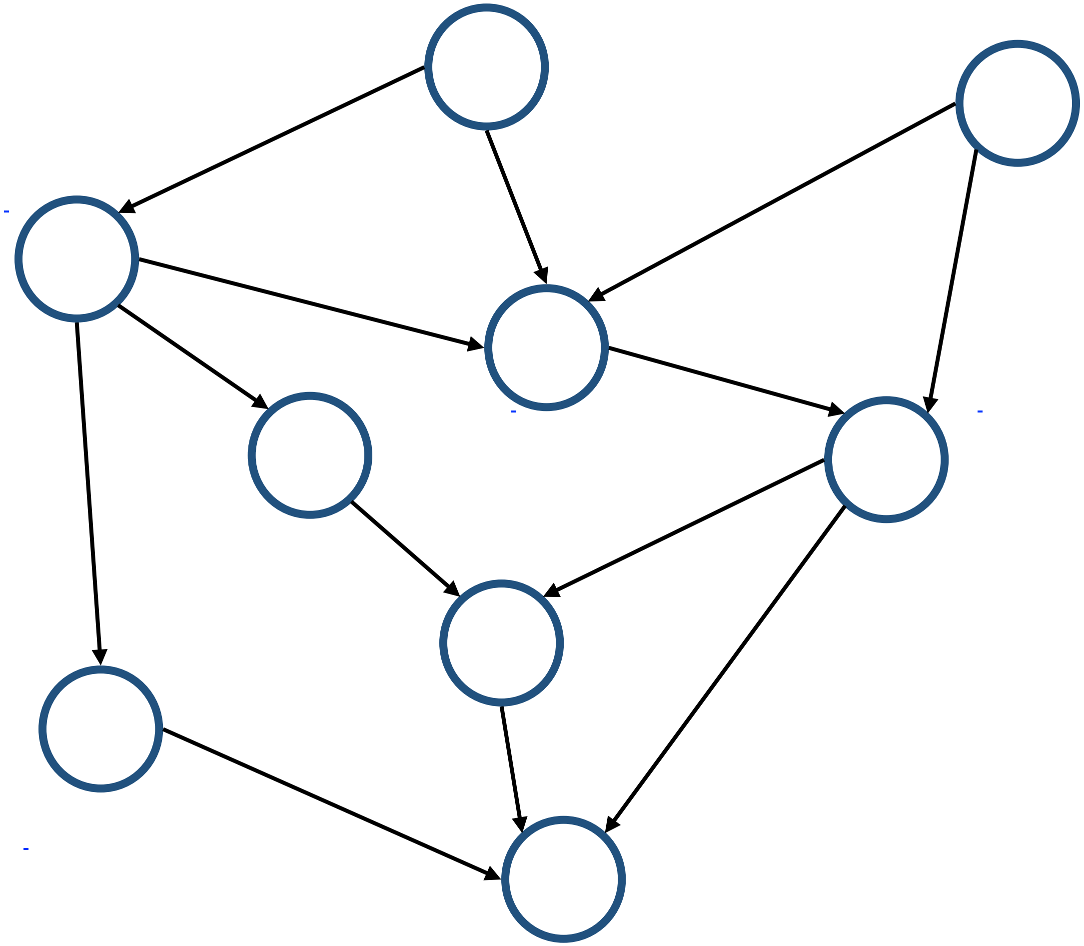
Simulate the algorithm by using DFS

Algorithm *topologicalDFS*(*G*)
  Input dag *G*
  Output topological ordering of *G*
  $n \leftarrow$ *G.numVertices*()
  for all $u \in$ *G.vertices*()
    setLabel(*u, UNEXPLORED*)
  for all $e \in$ *G.edges*()
    setLabel(*e, UNEXPLORED*)
  for all $v \in$ *G.vertices*()
    if *getLabel*(*v*) = *UNEXPLORED*
      topologicalDFS(*G, v*)

- O(n+m) time.

Algorithm *topologicalDFS*(*G, v*)
  Input graph *G* and a start vertex *v* of *G*
  Output labeling of the vertices of *G* in the connected component of *v*
  setLabel(*v, VISITED*)
  for all $e \in$ *G.incidentEdges*(*v*)
    if *getLabel*(*e*) = *UNEXPLORED*
      $w \leftarrow$ *opposite*(*v,e*)
      if *getLabel*(*w*) = *UNEXPLORED*
        setLabel(*e, DISCOVERY*)
        topologicalDFS(*G, w*)
      else
        {*e* is a forward or cross edge}
  Label *v* with topological number *n*
  $n \leftarrow n - 1$

Topological Sorting Example



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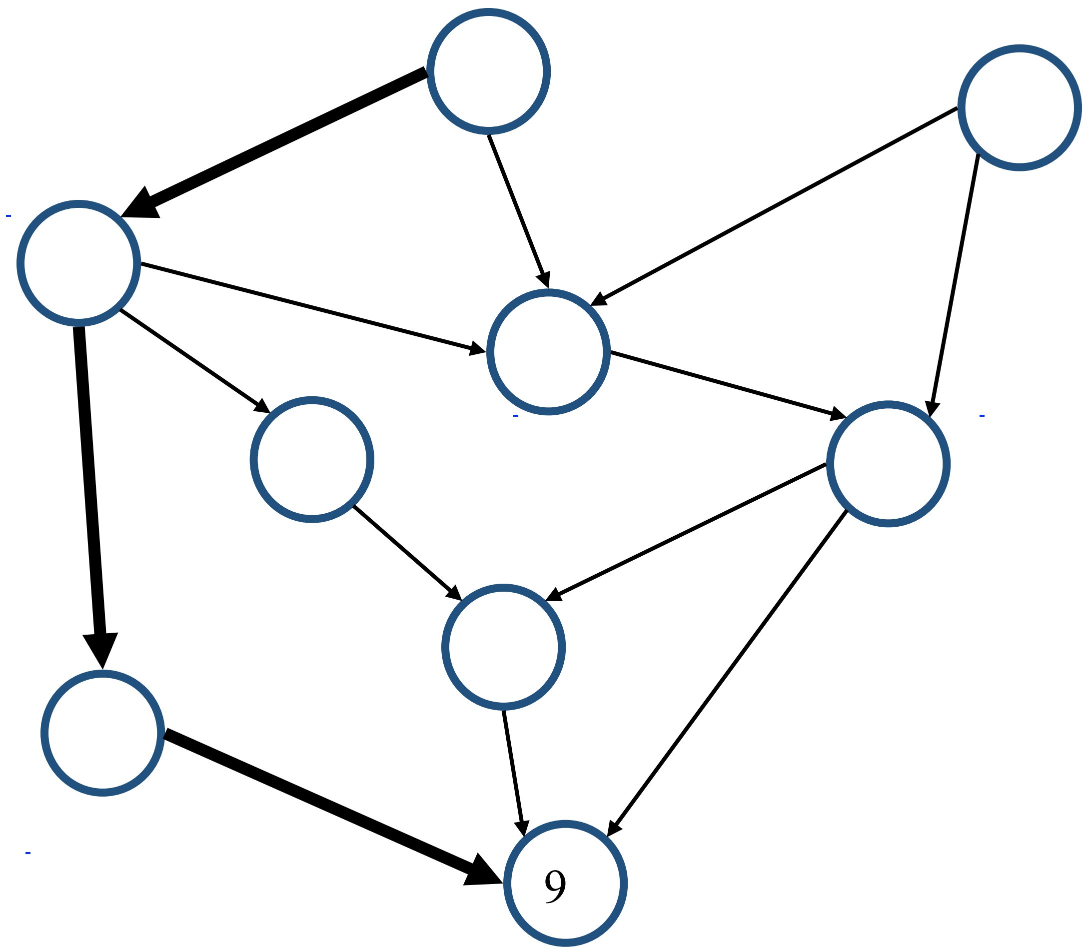

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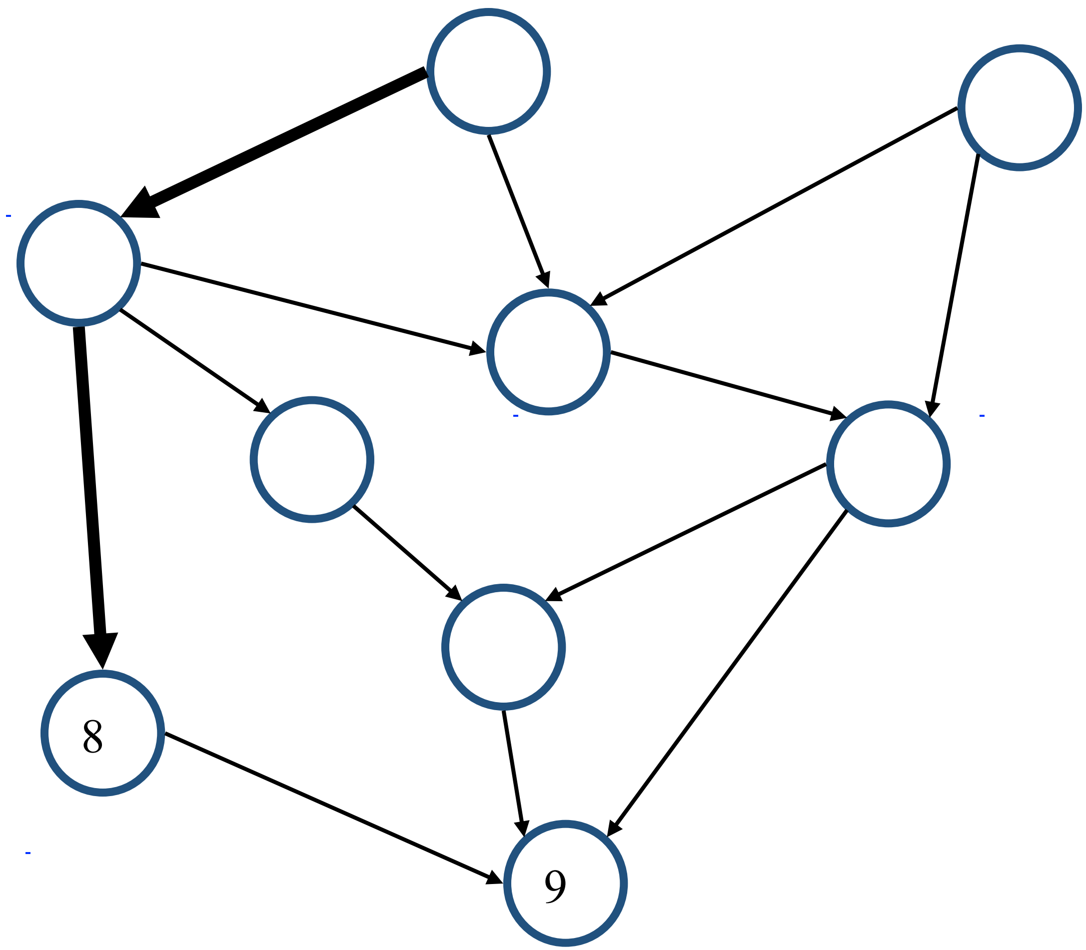

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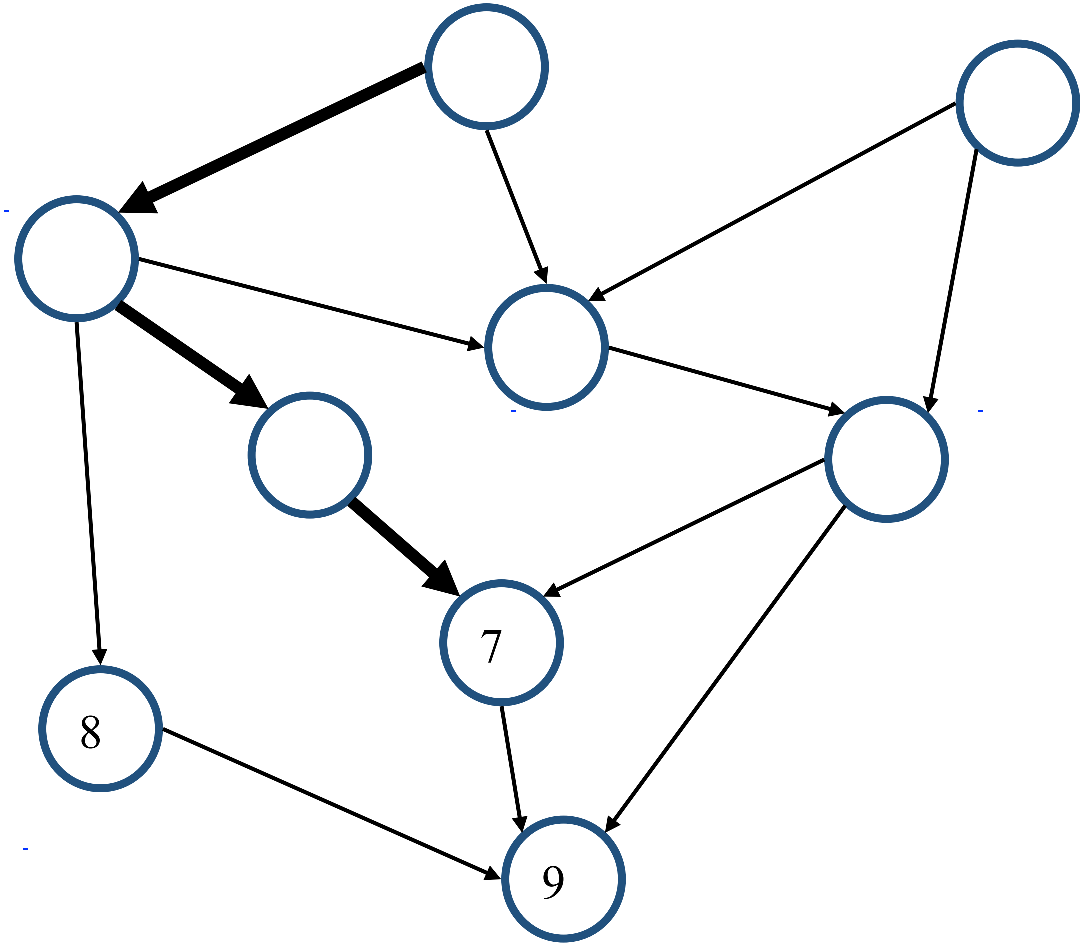

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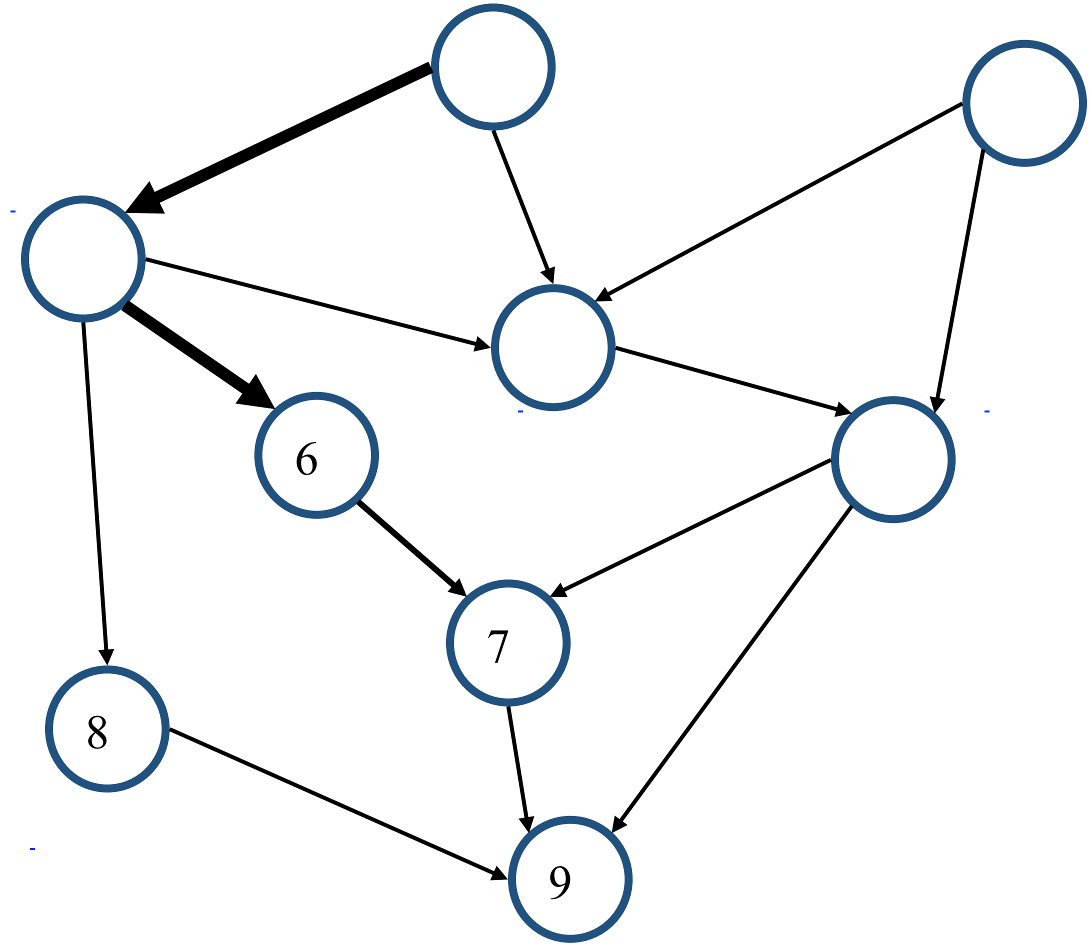

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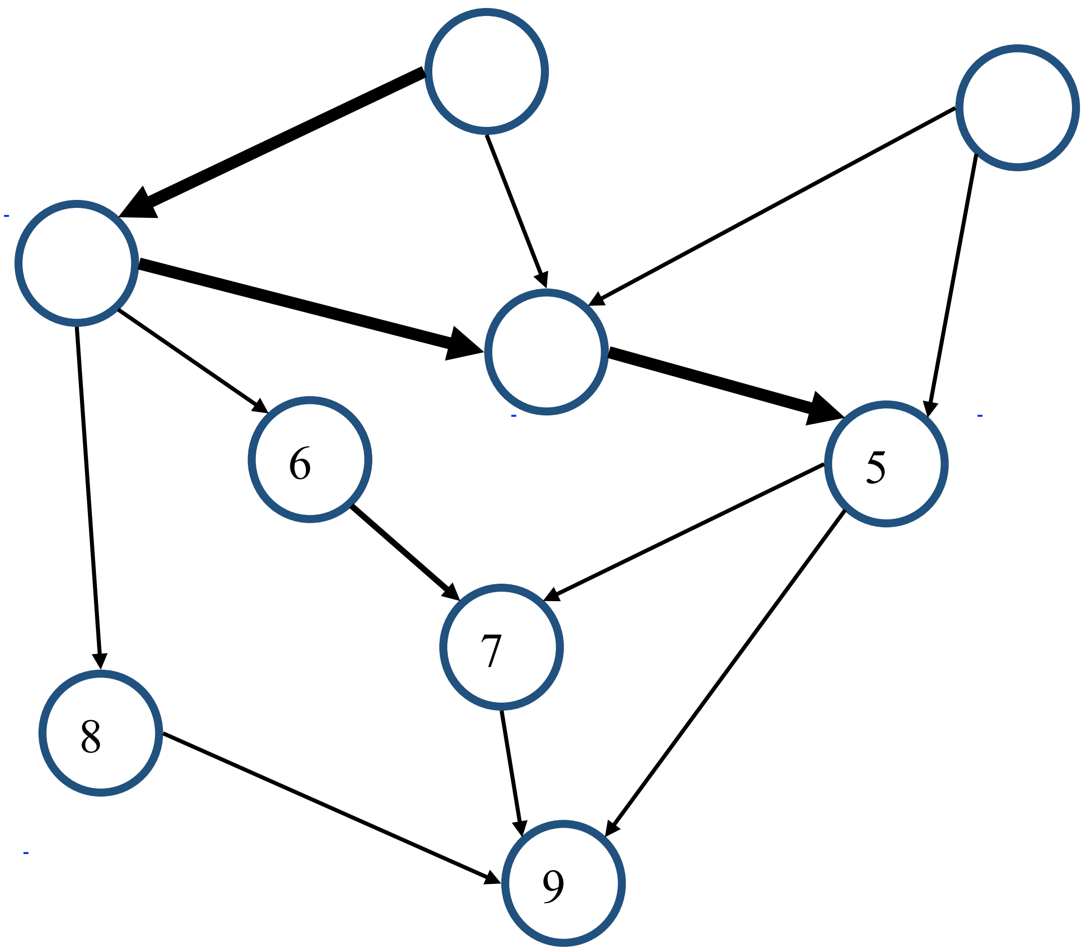
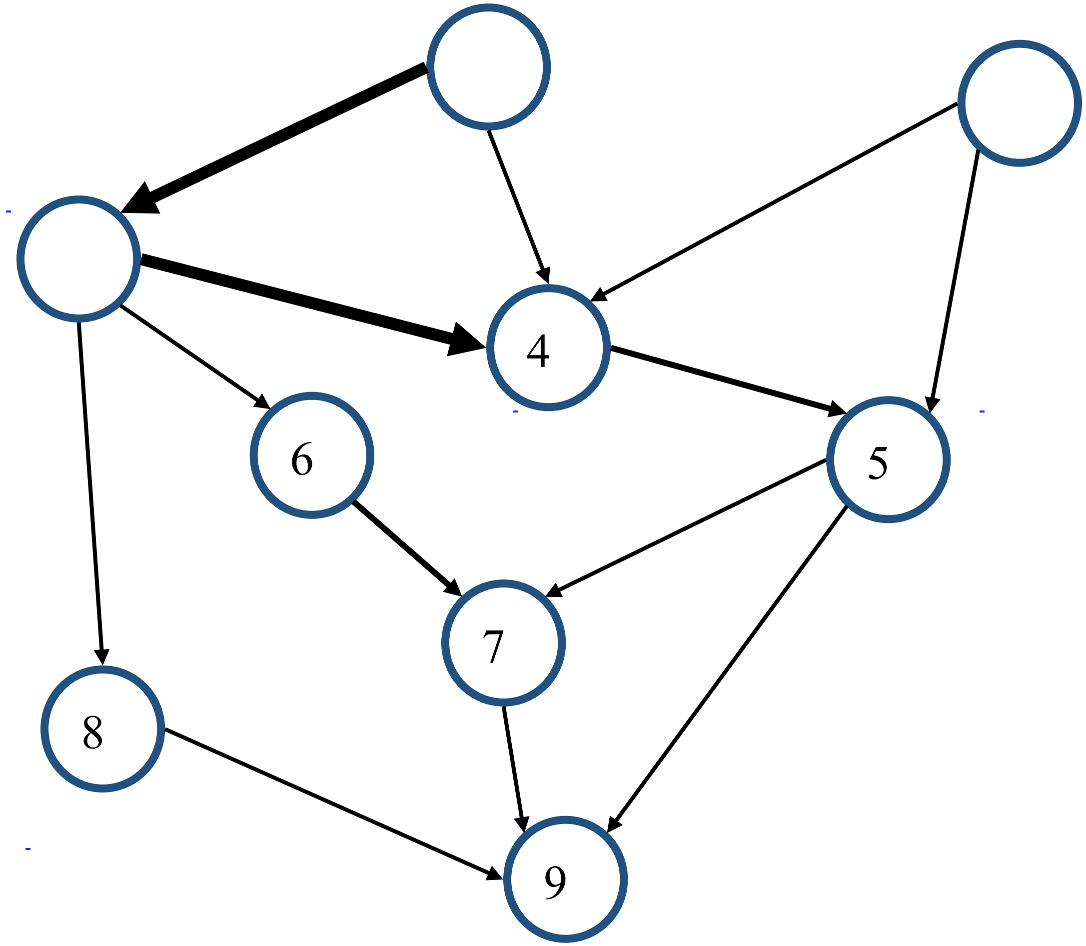


Topological Sorting Example



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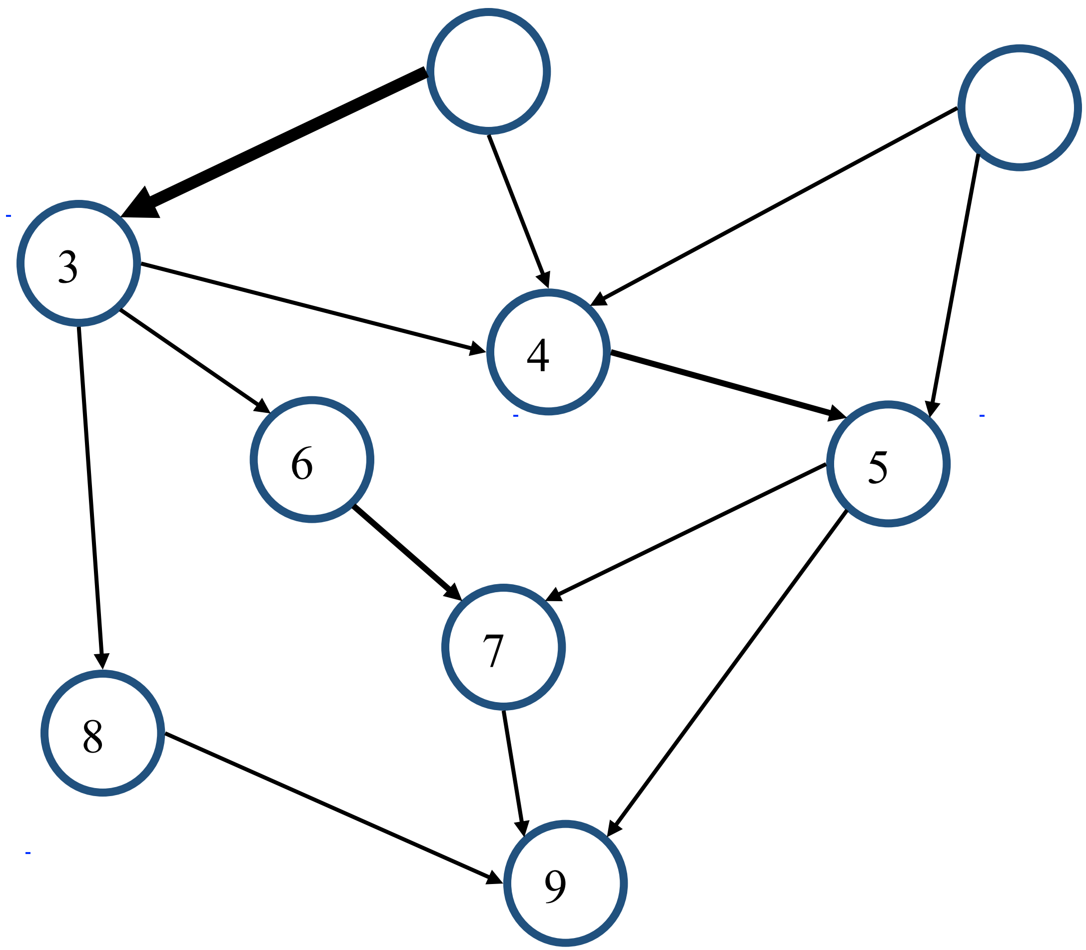

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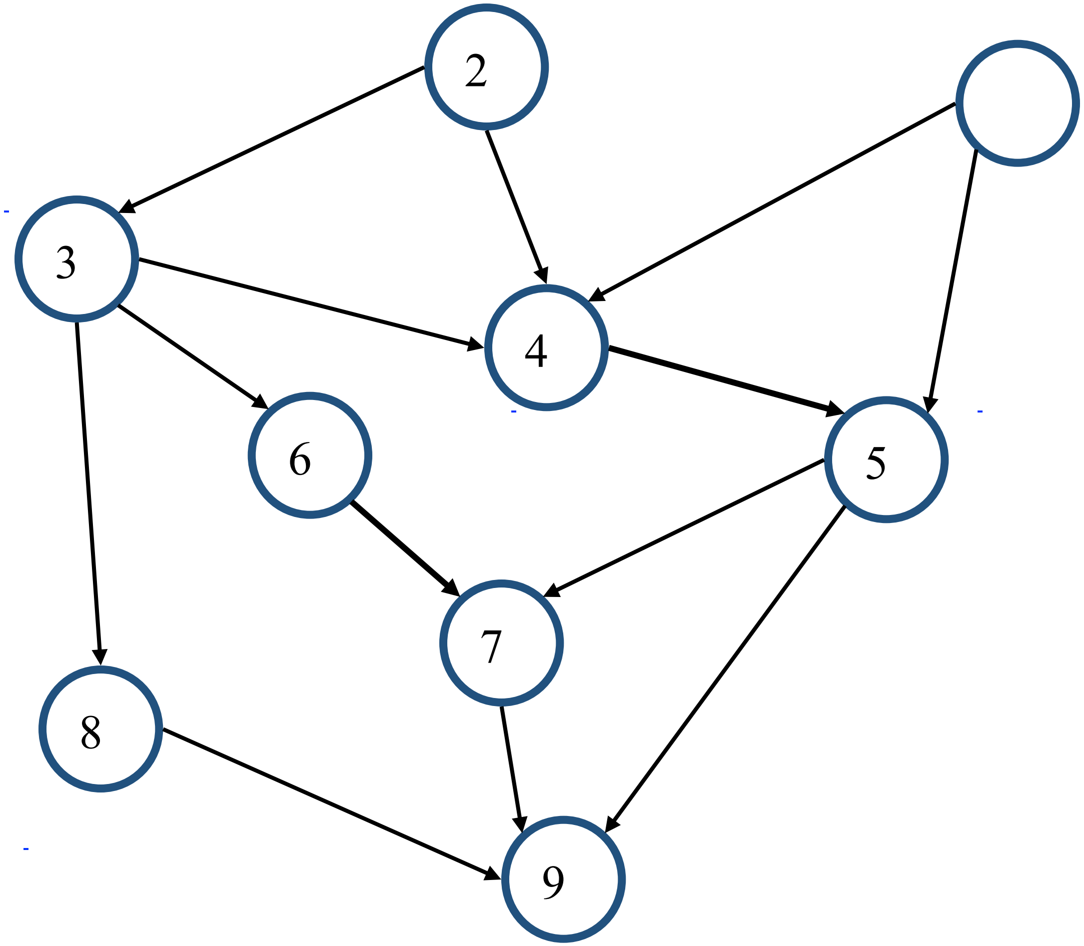

Topological Sorting Example