

Rules of Inference

Section 1.6

Section Summary

- Valid Arguments
- Inference Rules for Propositional Logic
 - Building Arguments
- Inference Rules for Quantified Statements
 - Building Arguments

Revisiting the Socrates Example

- We have the two premises:
 - “All men are mortal.”
 - “Socrates is a man.”
- And the conclusion:
 - “Socrates is mortal.”
- How do we get the conclusion from the premises?

The Argument

- We can express the **premises** (above the line) and the **conclusion** (below the line) in predicate logic as an **argument**:

$\forall x(Man(x) \rightarrow Mortal(x))$ **premise**

$Man(Socrates)$ **premise**

$\therefore Mortal(Socrates)$ **conclusion**

- We will see shortly that this is a valid argument.

Arguments in Propositional Logic

- A *argument* in propositional logic is a sequence of propositions. All but the final proposition are called *premises*. The last statement is the *conclusion*.
- The argument is *valid* if the premises imply the conclusion.
 - If the premises are p_1, p_2, \dots, p_n and the conclusion is q then $(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$ is a tautology.
- An *argument form* is an argument that is valid no matter what propositions are substituted into its propositional variables.
- Inference rules are all simple argument forms that will be used to construct more complex argument forms.

Rules of Inference for Propositional Logic:

Modus Ponens

Corresponding Tautology:

$$(p \wedge (p \rightarrow q)) \rightarrow q$$

$$\frac{p \rightarrow q \quad p}{\therefore q}$$

Example:

Let p be “It is snowing.”

Let q be “I will study discrete math.”

“If it is snowing, then I will study discrete math.”

“It is snowing.”

“Therefore, I will study discrete math.”

Modus Tollens

Corresponding Tautology:

$$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$$

$$\frac{p \rightarrow q \quad \neg q}{\therefore \neg p}$$

Example:

Let p be “It is snowing.”

Let q be “I will study discrete math.”

“If it is snowing, then I will study discrete math.”

“I will not study discrete math.”

“Therefore, it is not snowing.”

Hypothetical Syllogism

Corresponding Tautology:

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$$

Example:

Let p be “it snows.”

Let q be “I will study discrete math.”

Let r be “I will get an A.”

“If it snows, then I will study discrete math.”

“If I study discrete math, I will get an A.”

“Therefore, If it snows, I will get an A.”

$$p \rightarrow q$$

$$q \rightarrow r$$

$$\therefore p \rightarrow r$$

Disjunctive Syllogism

Corresponding Tautology:

$$(\neg p \wedge (p \vee q)) \rightarrow q$$

$$p \vee q$$

$$\neg p$$

$$\therefore q$$

Example:

Let p be “I will study discrete math.”

Let q be “I will study English literature.”

“I will study discrete math or I will study English literature.”

“I will not study discrete math.”

“Therefore , I will study English literature.”

Addition

Corresponding Tautology:

$$p \rightarrow (p \vee q)$$

$$\frac{p}{\therefore p \vee q}$$

Example:

Let p be “I will study discrete math.”

Let q be “I will visit Las Vegas.”

“I will study discrete math.”

“Therefore, I will study discrete math or I will visit Las Vegas.”

Simplification

Corresponding Tautology:

$$(p \wedge q) \rightarrow q$$

$$\frac{p \wedge q}{\therefore q}$$

Example:

Let p be “I will study discrete math.”

Let q be “I will study English literature.”

“I will study discrete math and English literature”

“Therefore, I will study discrete math.”

Conjunction

Corresponding Tautology:

$$((p) \wedge (q)) \rightarrow (p \wedge q)$$

Example:

Let p be “I will study discrete math.”

Let q be “I will study English literature.”

“I will study discrete math.”

“I will study English literature.”

“Therefore, I will study discrete math and I will study English literature.”

$$\frac{p \quad q}{\therefore p \wedge q}$$

Resolution plays an important role in AI and is used in Prolog.

Resolution

Corresponding Tautology:

$$((\neg p \vee r) \wedge (p \vee q)) \rightarrow (q \vee r)$$

Example:

Let p be “I will study discrete math.”

Let r be “I will study English literature.”

Let q be “I will study databases.”

“I will not study discrete math or I will study English literature.”

“I will study discrete math or I will study databases.”

“Therefore, I will study databases or I will English literature.”

$$\frac{\neg p \vee r \quad p \vee q}{\therefore q \vee r}$$

Using the Rules of Inference to Build Valid Arguments

- A *valid argument* is a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference. The last statement is called conclusion.
- A valid argument takes the following form:

$$\begin{array}{c} S_1 \\ S_2 \\ \cdot \\ \cdot \\ \cdot \\ S_n \\ \therefore C \end{array}$$

Valid Arguments: Example 1

Example: From the single proposition

$$p \wedge (p \rightarrow q)$$

Show that q is a conclusion.

Solution:

Step

1. $p \wedge (p \rightarrow q)$

2. p

3. $p \rightarrow q$

4. q

Reason

Premise

Simplification using (1)

Simplification using (1)

Modus Ponens using (2) and (3)

Valid Arguments: Example 2

Example: With these hypotheses:

- “It is not sunny this afternoon and it is colder than yesterday.” $p \wedge q$
- “We will go swimming only if it is sunny.” $r \rightarrow p$
- “If we do not go swimming, then we will take a canoe trip.” $\neg r \rightarrow s$
- “If we take a canoe trip, then we will be home by sunset.” $s \rightarrow t$

Using the inference rules, construct a valid argument for the conclusion: “**We will be home by sunset.**” t

Solution:

p : “It is sunny this afternoon.”

q : “It is colder than yesterday.”

t : “We will be home by sunset.”

r : “We will go swimming.”

s : “We will take a canoe trip.”

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Valid Arguments: Example 2

Hypotheses: $\neg p \wedge q, r \rightarrow p, \neg r \rightarrow s, s \rightarrow t$

Conclusion: t

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Handling Quantified Statements

- Valid arguments for quantified statements are a sequence of statements. Each statement is either a premise or follows from previous statements by rules of inference which include:
 - Rules of Inference for Propositional Logic
 - Rules of Inference for Quantified Statements
- The rules of inference for quantified statements are introduced in the next several slides.

Rules of Inference for Quantified Statements: Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

Example:

Our domain consists of all dogs and Fido is a dog.

“All dogs are cuddly.”

“Therefore, Fido is cuddly.”

Rules of Inference for Quantified Statements: Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Example:

“If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$ ”

“For all positive real numbers x and y , if $x > y$, then $x^2 > y^2$ ”

Rules of Inference for Quantified Statements: Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

“There is someone who got an A in the course.”

“Let’s call her a and say that a got an A”

Rules of Inference for Quantified Statements: Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

“Michelle got an A in the class.”

“Therefore, someone got an A in the class.”

Using Rules of Inference: Example 1

Ex: Using the rules of inference, construct a valid argument to show that “**John Smith has two legs**” is a consequence of the premises:

“**Every man has two legs.**” and “**John Smith is a man.**”

Solution: Let $M(x)$ denote “ x is a man” and $L(x)$ “ x has two legs” and let John Smith be a member of the domain.

Step	Reason
1. $\forall x(M(x) \rightarrow L(x))$	Premise
2. $M(J) \rightarrow L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Using Rules of Inference: Example 2

Ex: Use the rules of inference to construct a valid argument showing that the conclusion “**Someone who passed the first exam has not read the book.**” follows from the premises

“A student in this class has not read the book.”

“Everyone in this class passed the first exam.”

Solution: Let $C(x)$ denote “ x is in this class,” $B(x)$ denote “ x has read the book,” and $P(x)$ denote “ x passed the first exam.”

$$\frac{\begin{array}{l} \exists x(C(x) \wedge \neg B(x)) \\ \forall x(C(x) \rightarrow P(x)) \end{array}}{\therefore \exists x(P(x) \wedge \neg B(x))}$$

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Using Rules of Inference: Example 2

Valid Argument:

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	EG from (8)

Returning to the Socrates Example

$$\frac{\forall x(Man(x) \rightarrow Mortal(x)) \quad Man(Socrates)}{\therefore Mortal(Socrates)}$$

Valid Argument Step

1. $\forall x(Man(x) \rightarrow Mortal(x))$
2. $Man(Socrates) \rightarrow Mortal(Socrates)$
3. $Man(Socrates)$
4. $Mortal(Socrates)$

Reason

Premise
UI from (1)
Premise
MP from (2)
and (3)

Universal Modus Ponens

Universal Modus Ponens combines universal instantiation and modus ponens into one rule.

$$\frac{\begin{array}{l} \forall x(P(x) \rightarrow Q(x)) \\ P(a), \text{ where } a \text{ is a particular} \\ \text{element in the domain} \end{array}}{\therefore Q(a)}$$

This rule could be used in the Socrates example.