Geometric characteristics of real-world networks

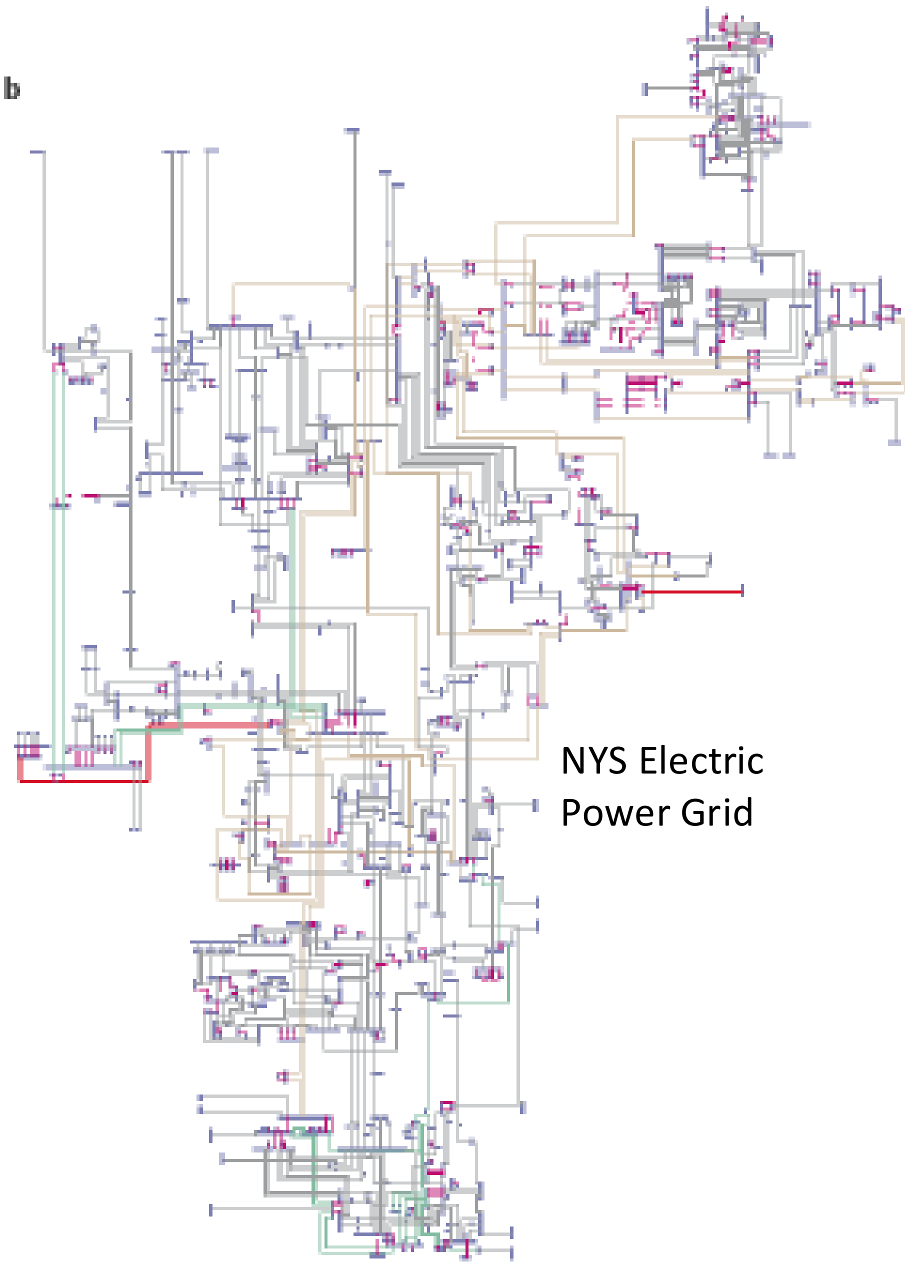
Heather M. Guarnera

The College of Wooster

Why graph networks?

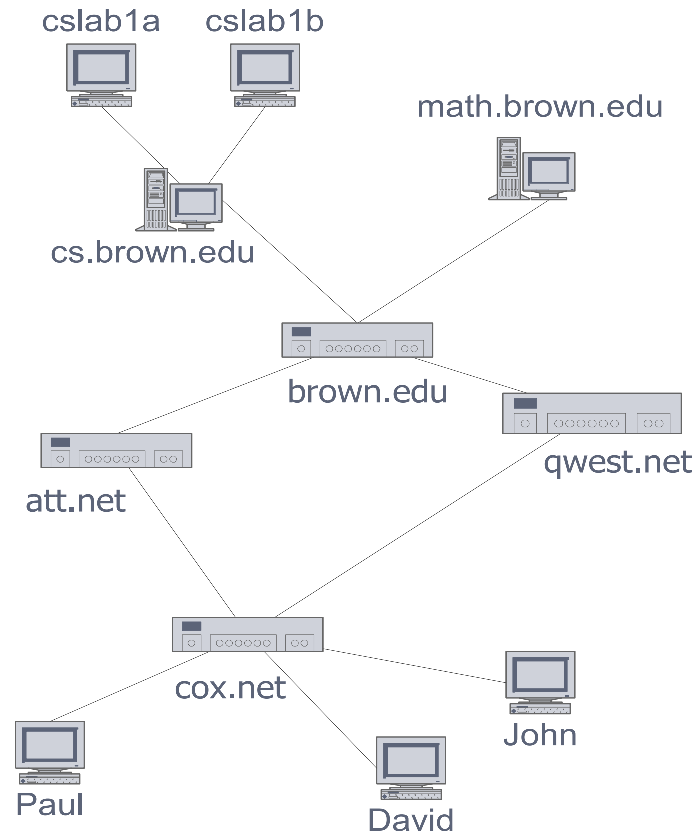
Graphs are everywhere

b



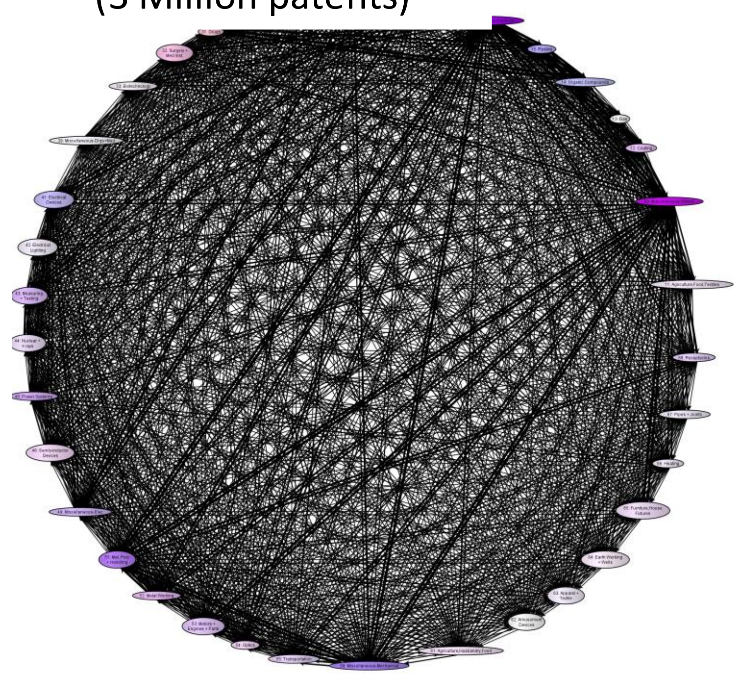
NYS Electric Power Grid

computer network



Graphs are everywhere

Utility Patent network
1972-1999
(3 Million patents)

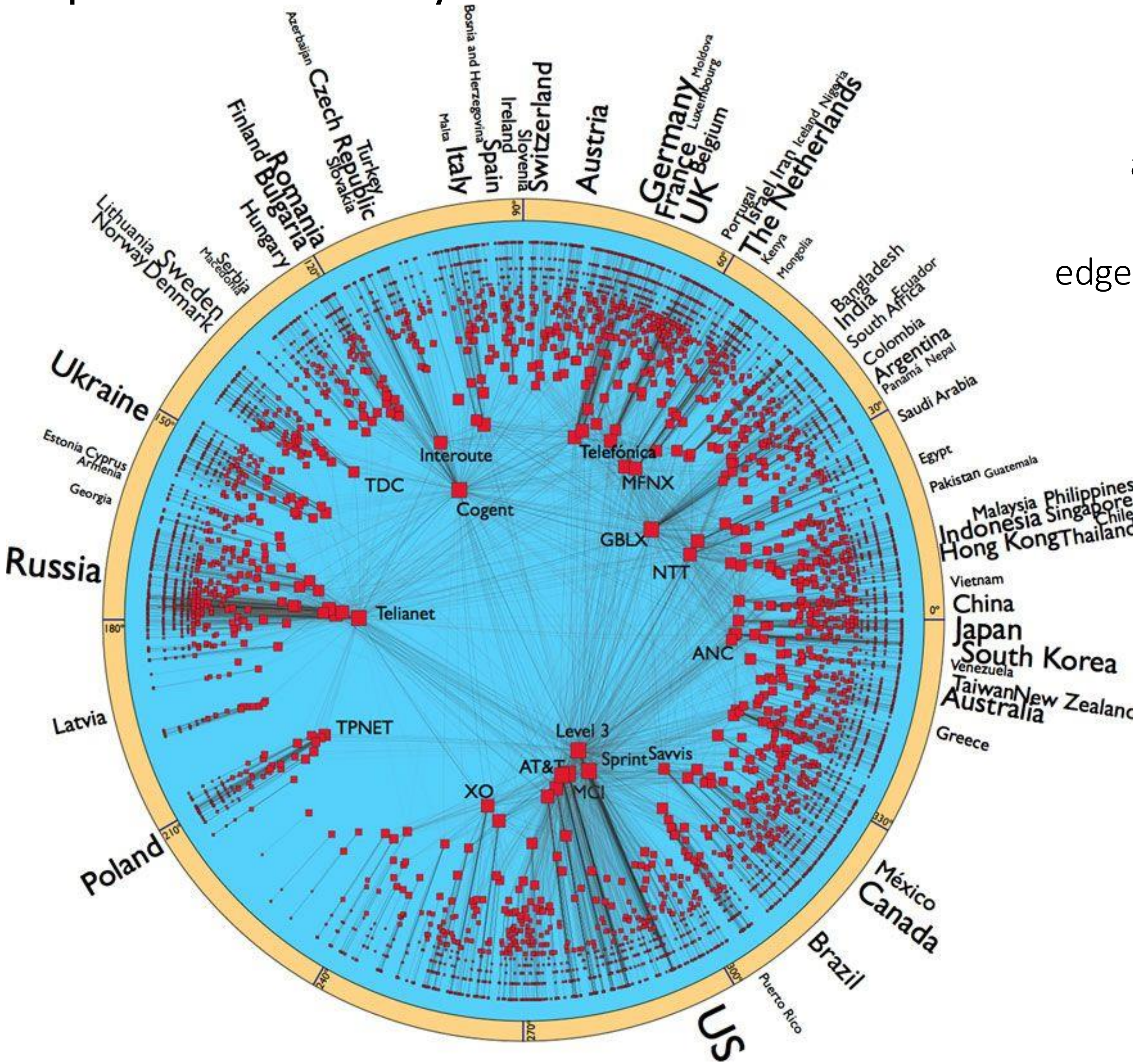


Graphs are everywhere

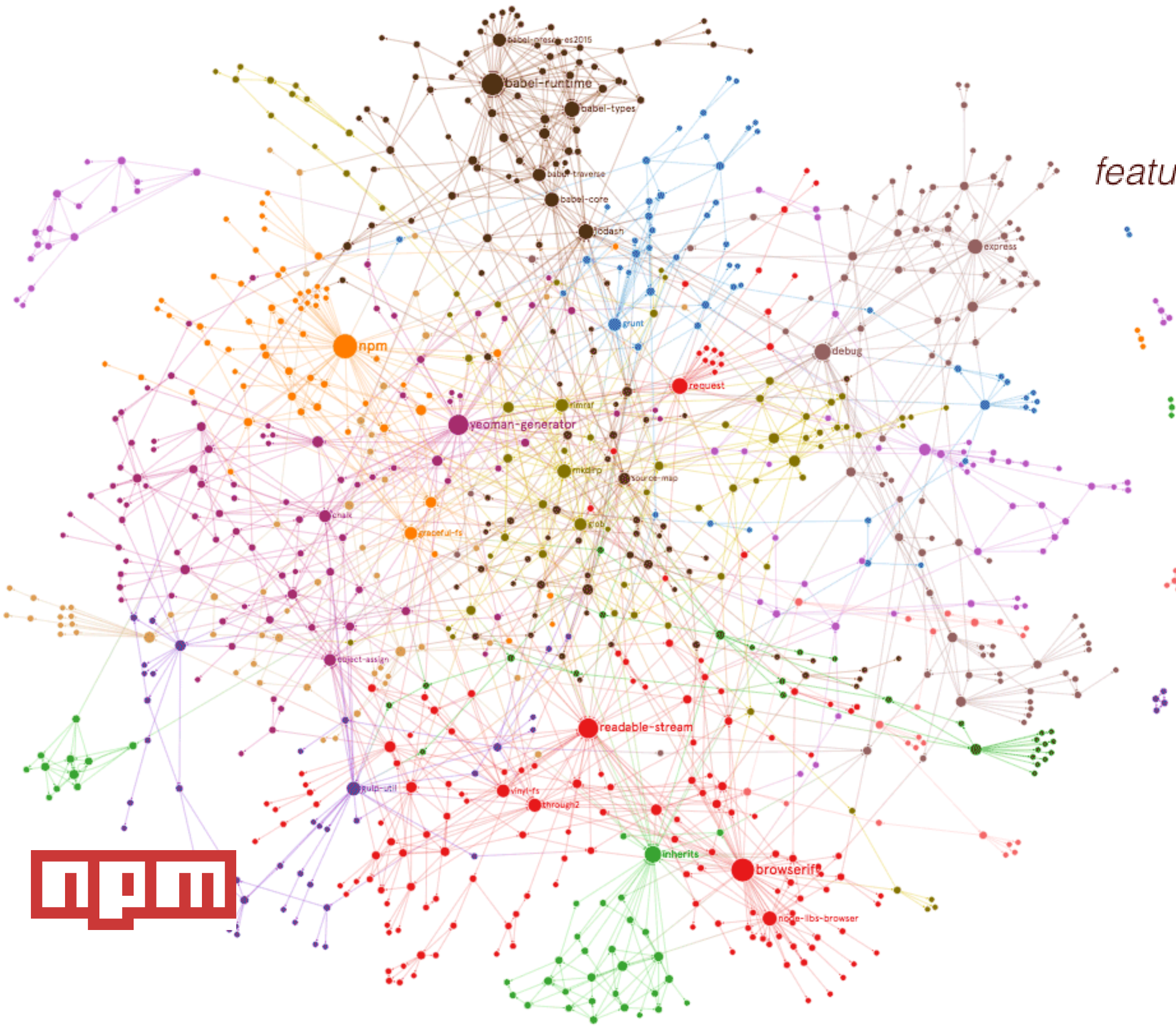
Internet (AS-level)

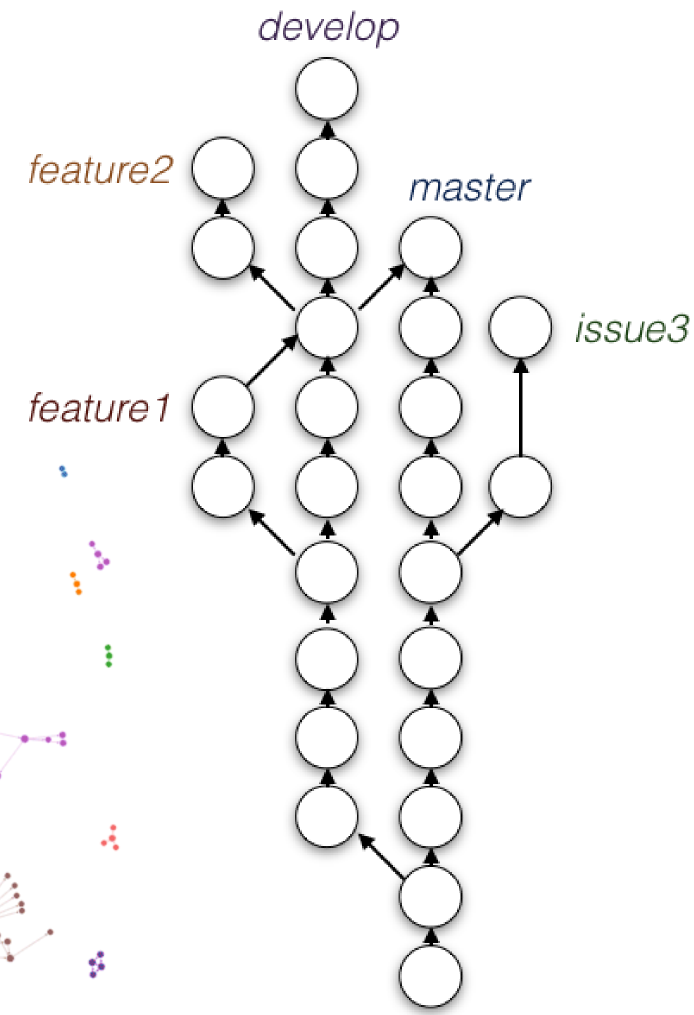
nodes n = 23,752 autonomous systems

edges m = 58,416 AS links



Graphs are everywhere





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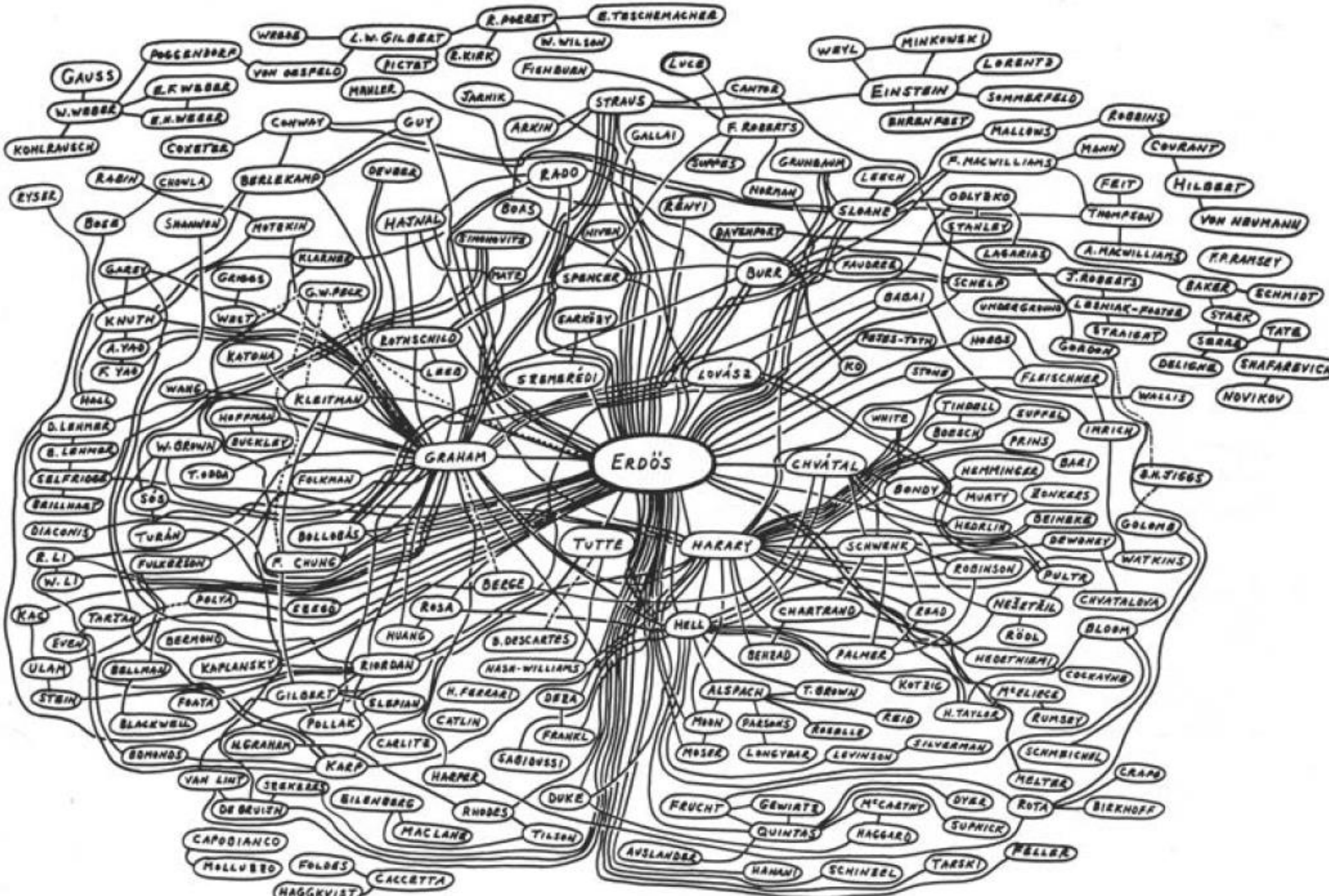
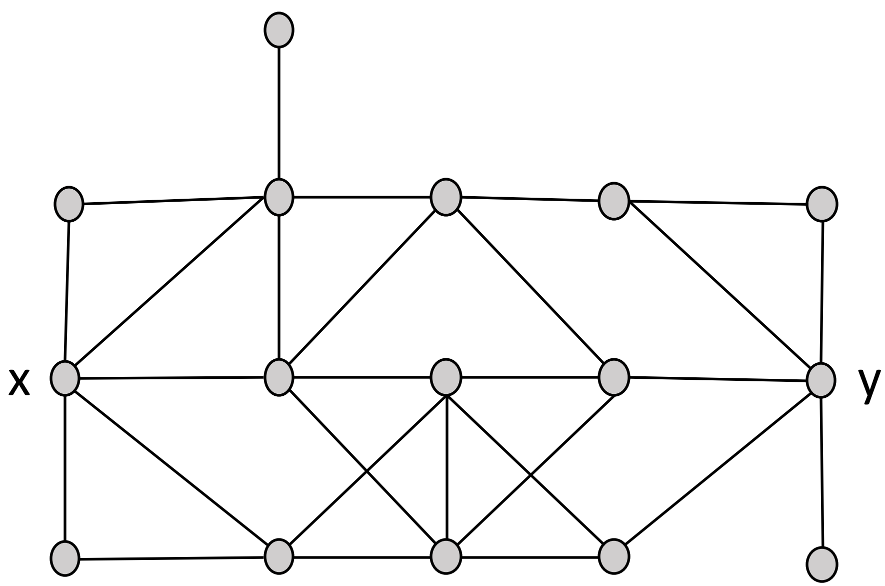


What is Fellow Travelers Phenomenon?

(Interval) Thinness of graphs

For any two *x,y* vertices on a graph $I(x,y) = \{z \in V : d(x,y) = d(x,z) + d(z,y)\}$ denotes the (metric) **interval**, i.e., all vertices that lay on a shortest path between x and y.



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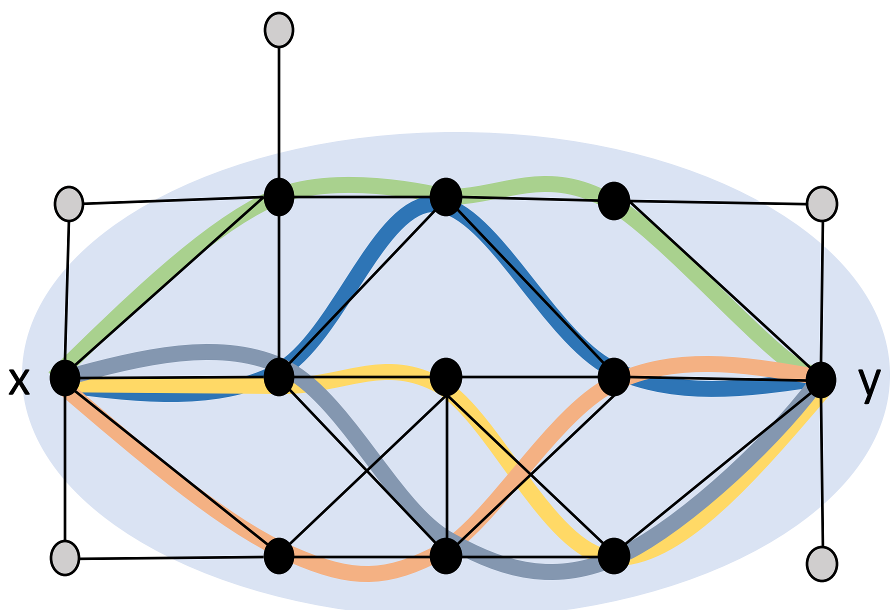

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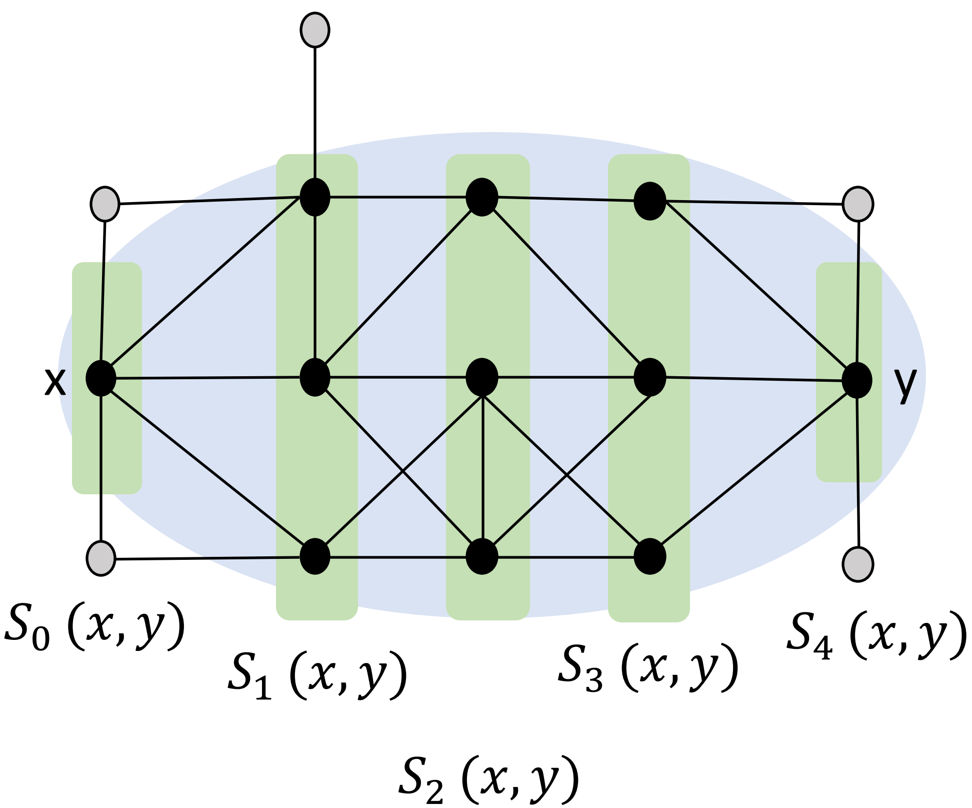
The set $S_p(x,y) = \{z \in I(x,y) : d(z,x) = p\}$ is called a **slice** of the interval from x to y.



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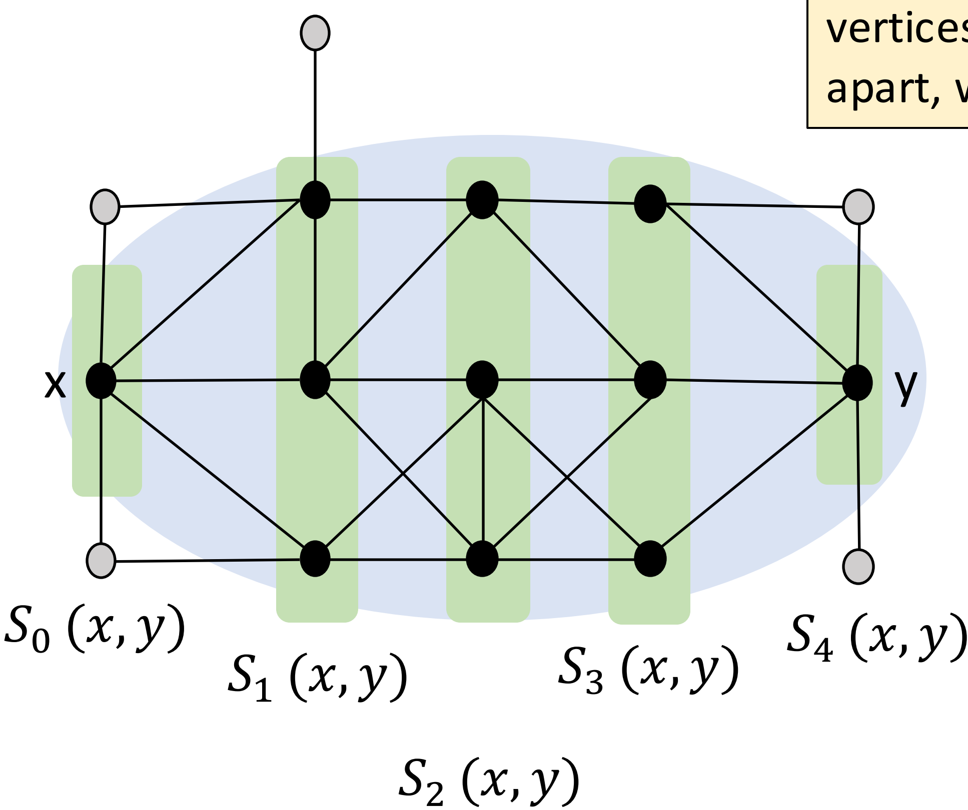
An interval $I(x, y)$ is said to be **κ-thin** if any two vertices u, v of the slice $S_p(x, y)$ are at most κ apart, where integer p satisfies $0 \leq p \leq d(x, y)$.



Ex: $I(x, y)$ is 2-thin.

The smallest value κ for which all intervals of G are κ-thin is the **thinness of the graph**, denoted $\boldsymbol{\kappa(G)}$.

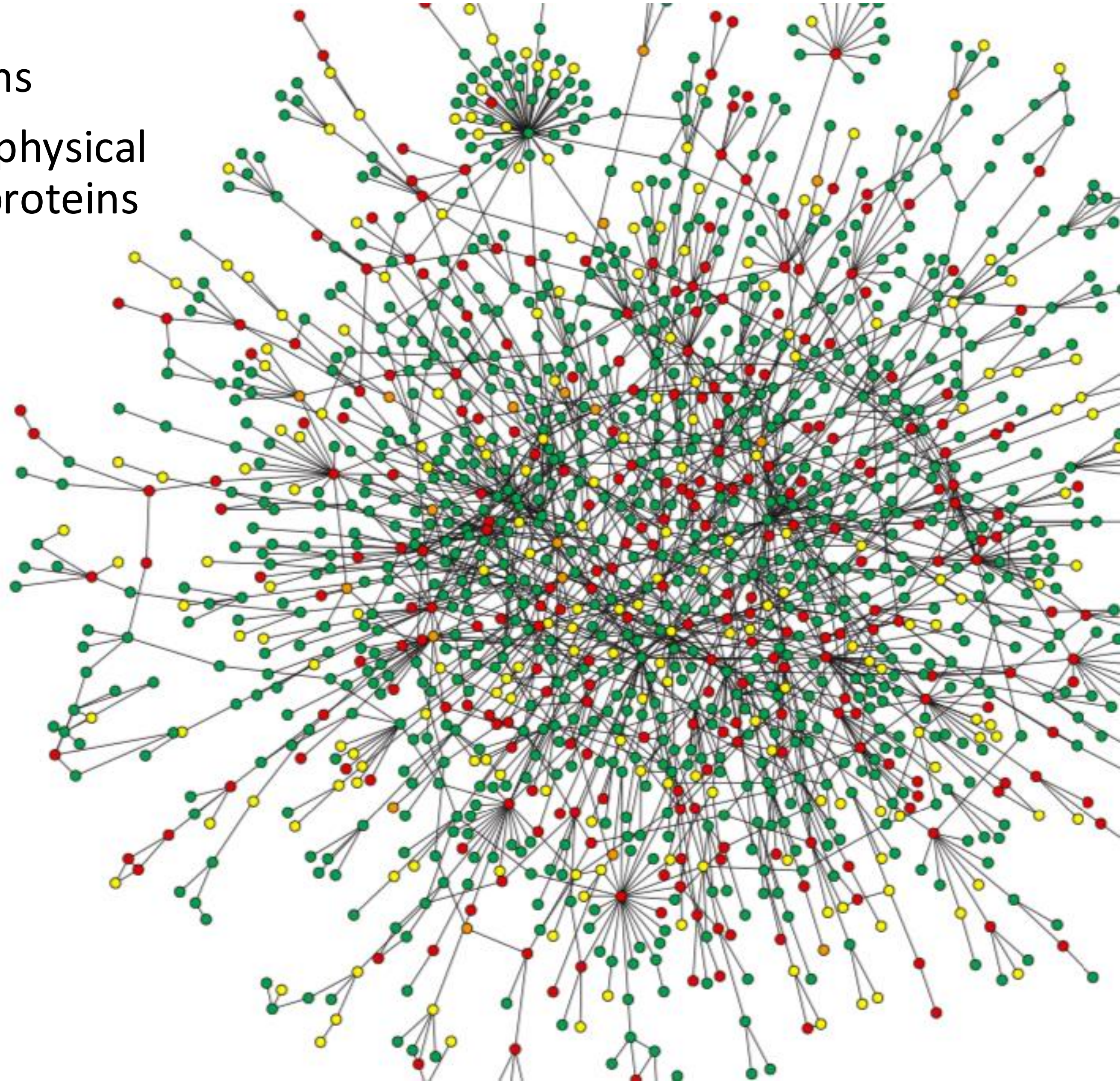
$\boldsymbol{\kappa(G)}$ is a small constant in many real-world networks!

Ex: Protein Interaction Network

nodes n = 1,870 proteins

edges m = 2240 direct physical interactions between proteins

$$\boldsymbol{\kappa(G) \leq 7}$$



Ex: Other real-world networks with small thinness

- Social networks (subset of Facebook)
  - nodes n = 293,501 users
  - edges m = 5,589,802 friendships between users

$$\boldsymbol{\kappa(G) \leq 7}$$

- Web networks (from Google)
  - nodes n = 855,802 websites
  - edges m = 4,291,352 hyperlinks connecting sites

$$\boldsymbol{\kappa(G) \leq 4}$$

- Peer-to-peer networks (Gnutella)
  - nodes n = 62,561 hosts
  - edges m = 147,878 connections between hosts

$$\boldsymbol{\kappa(G) \leq 5}$$

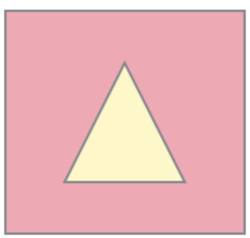
Fellow travelers phenomenon is attributed to the negative curvature of the graph

Geometric characteristics of real-world networks

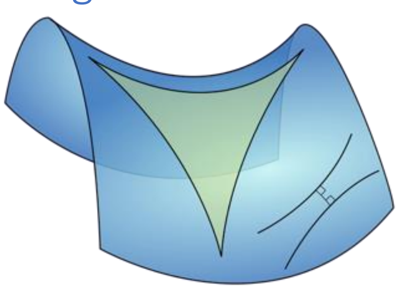
- Surge of recent empirical and theoretical work analyzes geometric characteristics

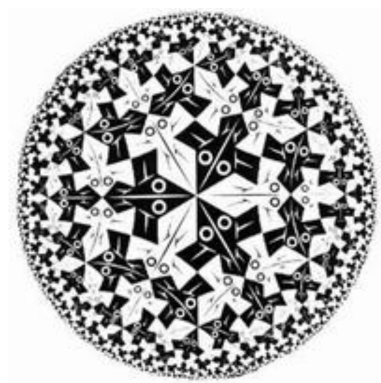
- One important property: **negative curvature**
  - causes traffic between vertices to pass through a relatively small core of the network – as if the shortest paths between them were curved inwards
  - measured in many different (somewhat equivalent) ways

Zero Curvature



Negative Curvature





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- Measures of negative curvature
  - κ Interval thinness
  - τ Geodesic triangle thinness
  - **δ Gromov Hyperbolicity**
  - ς Slimness
  - ι Rooted Insize

How can this geometric information be applied?

Parameterized complexity/approximation factor

- **Goal**: create algorithms which solve problems utilizing these geometric properties

- Example: Consider δ hyperbolicity, which is known to be small in many real-world networks.
  - Solve a problem in $O(f(\delta)\, m)$ time
  - Compute a $f(\delta)$ approximation

- Some problems this has been applied to:
  - Covering/packing problems
  - Computing the diameter/radius
  - Facility location problems
  - Network analysis
  - Vertex pursuit games on graphs
  - Traveling salesman problem

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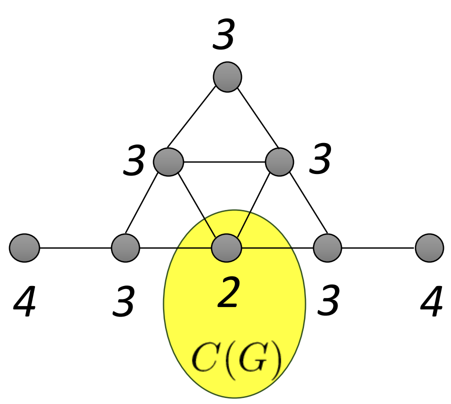
1. F. Dragan and **H. Guarnera**. Helly-gap of a graph and vertex eccentricities. Theoretical Computer Science, 867:68-84, 2021.
2. F. Dragan and **H. Guarnera**. Eccentricity function in distance-hereditary graphs. Theoretical Computer Science, 833: 26-40, 2020.
3. F. Dragan and **H. Guarnera**. Eccentricity terrain of δ-hyperbolic graphs. Journal of Computer and System Sciences, 112: 50-56, 2020.
4. F. Dragan, G. Ducoffe, **H. Guarnera**. Fast deterministic algorithms for computing all eccentricities in (hyperbolic) Helly graphs, the 17th Algorithms and Data Structures Symposium (WADS'21), 2021.
5. Mohammed, F. Dragan, **H. Guarnera**. Fellow Travelers Phenomenon Present in Real-World Networks, Complex Networks & Their Applications, 2022.

Example: eccentricity function and centers

The eccentricity *e(x)* of a vertex *x* is the distance to a furthest *u* vertex to *x*

$$e(x) = \max_{u \in V} d(x, u)$$



The minimum and maximum eccentricities are called the radius *rad(G)* and diameter *diam(G)* of the graph, respectively

The center of a graph *C(G)* is the set of vertices with minimum eccentricity

$$C(G) = \{v \in V : e(v) = rad(G)\}$$

Applications:

- Measure the importance of a node (centrality indices)
- Facility location problems
- Detecting small-world networks (degrees of freedom)

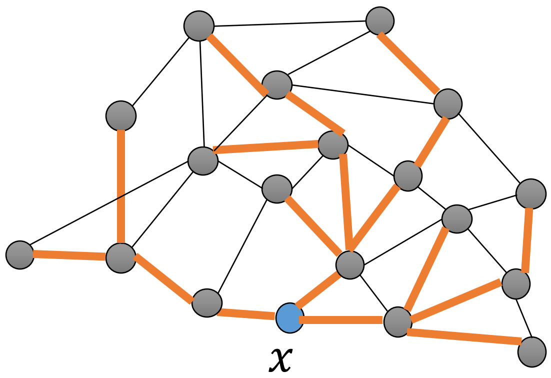
Computing vertex eccentricities straightforwardly.

The eccentricity *e(x)* of a vertex *x* is the distance to a furthest *u* vertex to *x*

$$e(x) = \max_{u \in V} d(x, u)$$

Take a connected graph with n vertices and m edges.

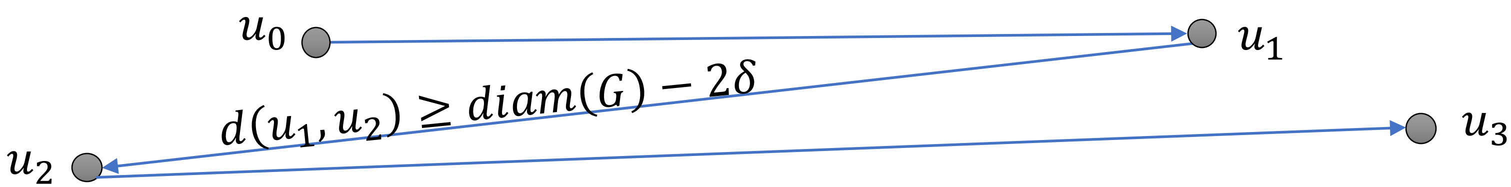
- A single Breadth-First Search (BFS) from a vertex x
  - runs in *O(m)* time
  - yields *e(x)*
- Call BFS for each of the n vertices
- Total *O(nm)* runtime



This is prohibitively expensive on many real-world networks, as they are huge!

Efficient eccentricity approximation via eccentricity approximating spanning tree

- Find a long path in $O(m)$ time



Efficient eccentricity approximation via eccentricity approximating spanning tree

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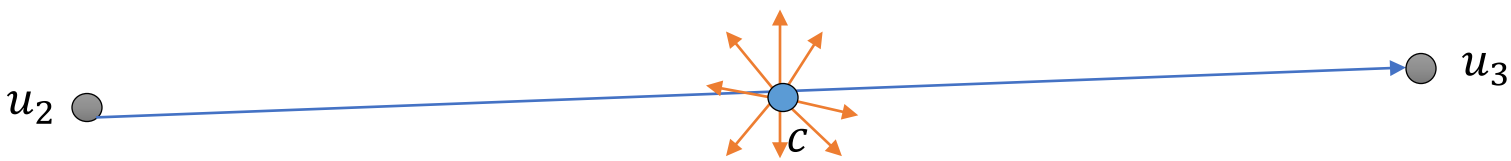
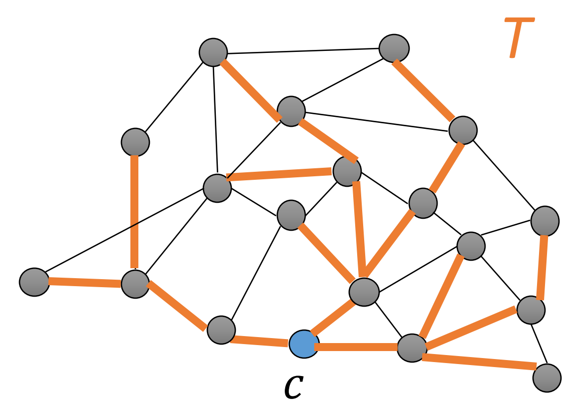


- Run breadth-first search (BFS) from the middle vertex *c* between $u_2 u_3$
- We show $e_T(v) \leq e_G(v) \leq e_T(v) + 6\delta$



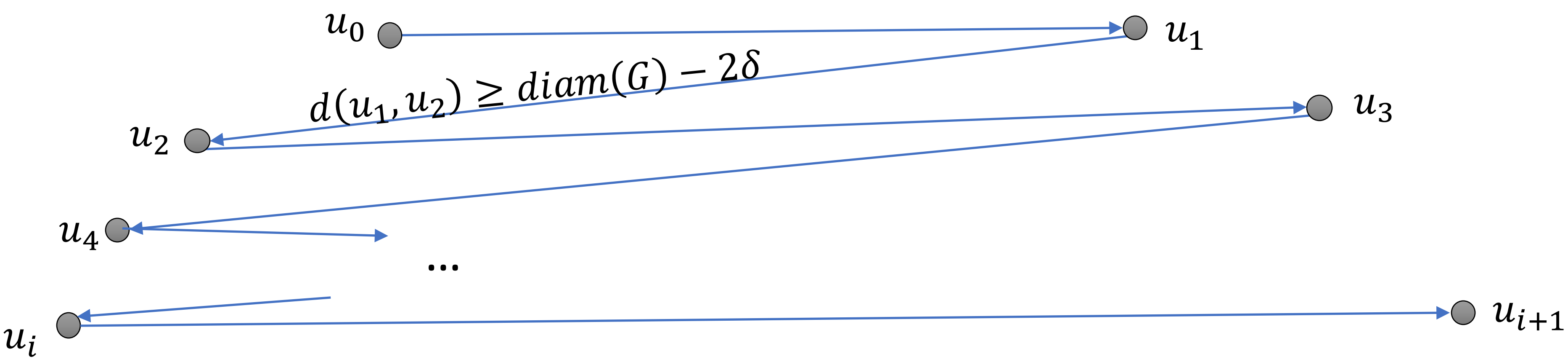
Theorem [2]: There is a 6δ approximation of all eccentricities in total $O(m)$ time

[2] F. Dragan and **H. Guarnera**. Eccentricity terrain of δ-hyperbolic graphs. Journal of Computer and System Sciences, 112: 50-56, 2020.

Efficient eccentricity approximation via eccentricity approximating spanning tree

- Find a **mutually distant** pair of vertices x,y in $O(\delta m)$ time

$$e(x) = d(x, y) = e(y)$$

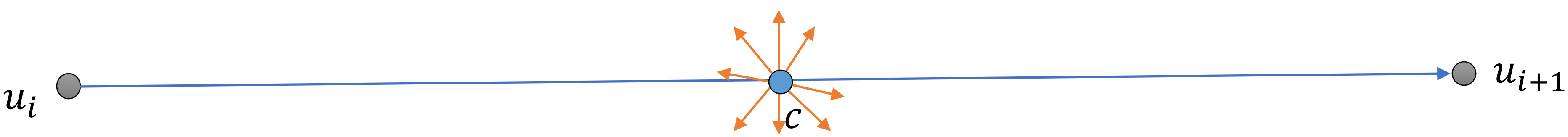


Efficient eccentricity approximation via eccentricity approximating spanning tree

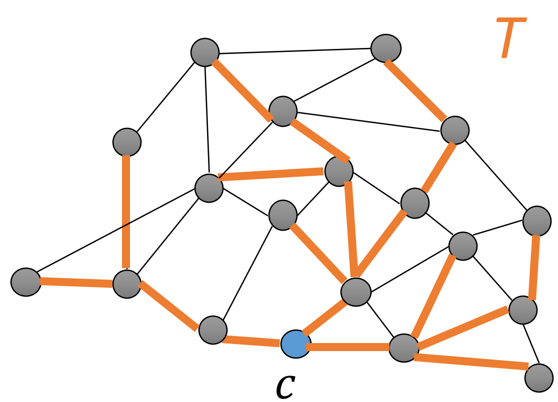
- Find a **mutually distant** pair of vertices x,y in $O(\delta m)$ time

$$e(x) = d(x, y) = e(y)$$

- Run breadth-first search (BFS) from the middle vertex *c* of the mutually distant pair



- We show $e_T(v) \le e_G(v) \le e_T(v) + 4\delta$



Theorem [2]: There is a 4δ approximation of all eccentricities in total $O(\delta m)$ time

[2] F. Dragan and **H. Guarnera**. Eccentricity terrain of δ-hyperbolic graphs. Journal of Computer and System Sciences, 112: 50-56, 2020.

Conclusion

- Many real world networks exhibit the fellow travelers property
  - Biological networks
  - Communication networks
  - Social networks
  - Software ecosystems

- We can take advantage of this nice geometric property to solve problems faster on these networks
  - Ex: computing vertex eccentricities

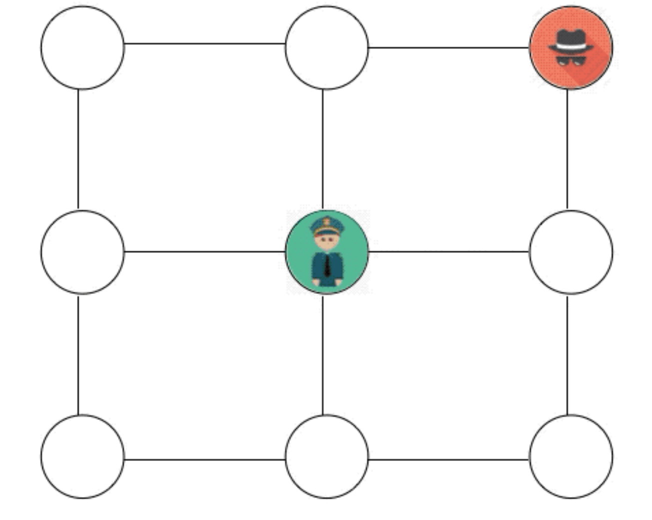
Conclusion and future work

- Many real world networks exhibit the fellow travelers property
  - Biological networks
  - Communication networks
  - Social networks
  - Software ecosystems
  - **What else?**

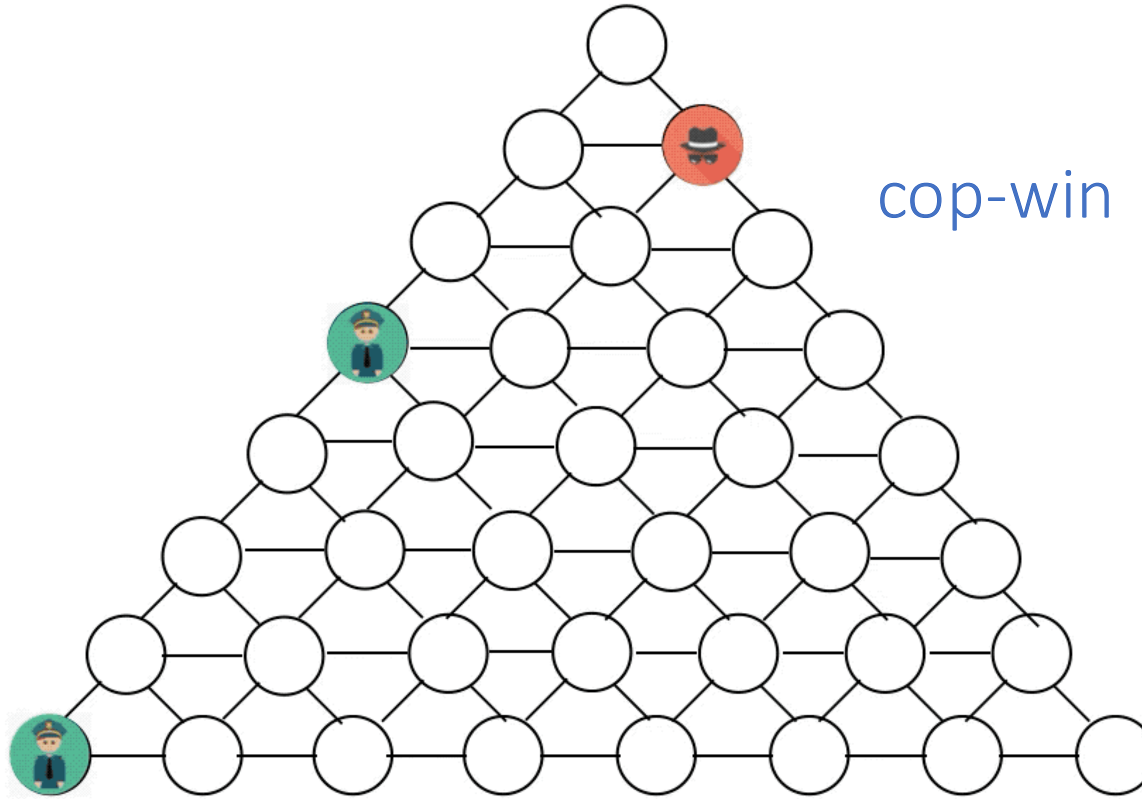
- We can take advantage of this nice geometric property to solve problems faster on these networks
  - Ex: computing vertex eccentricities
  - **What else? Ex: vertex pursuit games**

- **Routes:**
  - **Theoretical**
  - **Applied**

Games on graphs: cops vs. robbers



robber-win



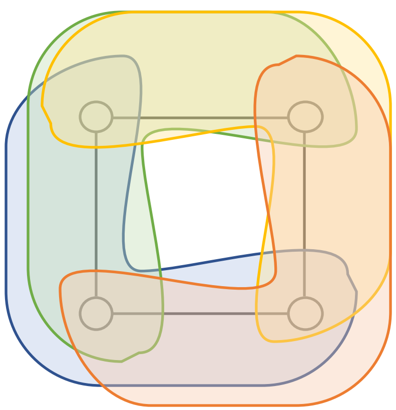
cop-win

Paths for Future Work (Theoretical)

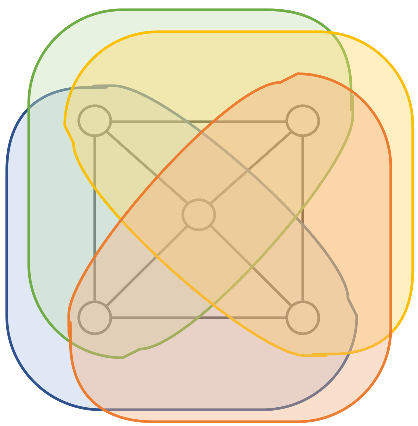
Every graph G can be isometrically embedded into the smallest Helly graph $\mathcal{H}$ (G).

A family F of sets has the **Helly property** if for every subfamily S of F the following hold: if the elements of S pairwise intersect, then the intersection of all elements of S is also non-empty.

A graph is called **Helly** if its family of disks satisfies the Helly property.



Not Helly



Helly

Paths for Future Work (Theoretical)

Every graph *G* can be isometrically embedded into the smallest Helly graph $\mathcal{H}$ (G).

$\mathcal{H}$ (G) is called the injective hull of *G* [Isbell 1964, Dress 1984].

- $\mathcal{H}$ (G) preserves hyperbolicity
- If *G* is δ-hyperbolic, then any vertex of $\mathcal{H}$ (G) is within 2δ to a vertex of G [Lang 2013]
- **[1]** Any vertex of $\mathcal{H}$ (G) is within α(G) to a vertex of G, where α(G) is the Helly-gap.

This motivates finding solutions to problems in $\mathcal{H}$ (G) which can lead to approximate solutions in G.

[1] F. Dragan and **H. Guarnera**. Helly-gap of a graph and vertex eccentricities. Theoretical Computer Science, 867:68-84, 2021.

Paths for Future Work (Theoretical)

$\mathcal{H}(G)$ can be constructed efficiently [6] for some graph classes (e.g., distance-hereditary graphs).

- It is computationally difficult to compute for other (even basic) graph classes [6]
- Investigate other graph classes (alpha-i metric graphs, chordal graphs, etc.)

The existence of $\mathcal{H}(G)$ is a powerful tool to gain insight into a graph class from various perspectives

- structurally — what other classes are closed under Hellification?
- metrically — what parameters are preserved?
- algorithmically — what else can be solved?

It lends itself nicely to approximation algorithms dependent on $\alpha(G)$.

- Apply those algorithms to specific graph classes

[6] **Heather M. Guarnera**, Feodor F. Dragan, and Arne Leitert. Injective hulls of various graph classes. Graphs and Combinatorics 38, 112 (2022).

Paths for Future Work (Applied)

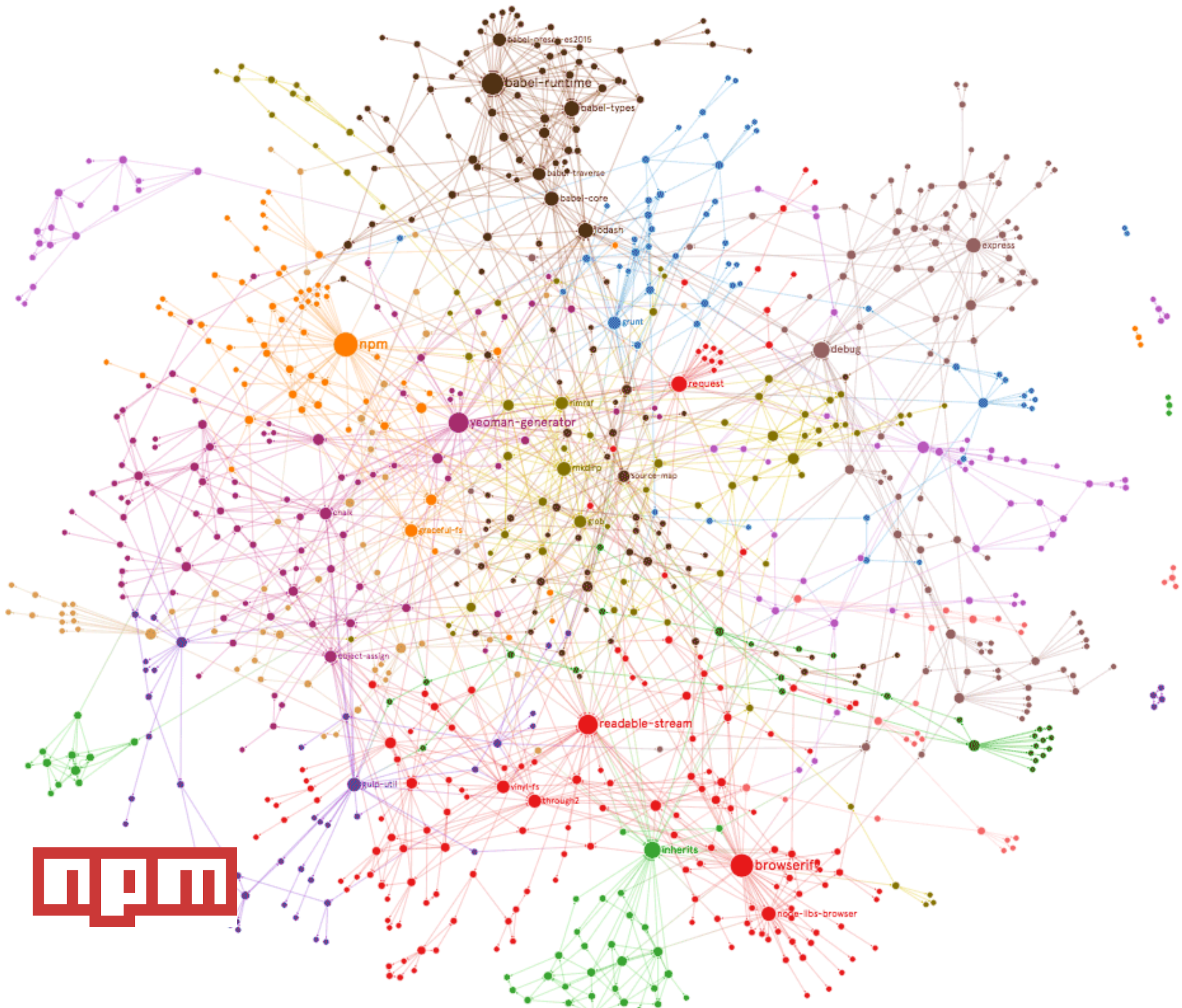
- Analyze existing real-world networks
  - Biological networks
  - Communication networks
  - Social networks
  - **Software ecosystems**

- Compute values of graph parameters on existing networks
  - Form conjectures for theoretical work

- Optimize algorithms to run efficiently on enormous networks

- Mining software repositories

Network analysis: software ecosystems

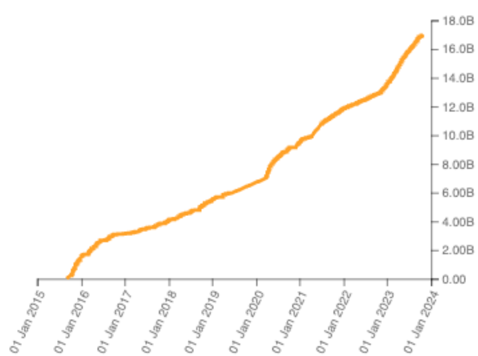


Software Heritage Project

GITORIOUS GitLab GitHub Packagist The PHP Package Repository npm GO Maven git git R heptapod Bitbucket Gogs launchpad SOURCEFORGE pub.dev fedora PAGURE debian Phabricator python Package Index Google code GNU opam

Source files

17,001,055,063



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Commits

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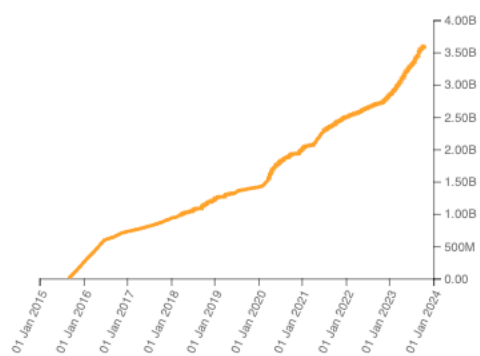
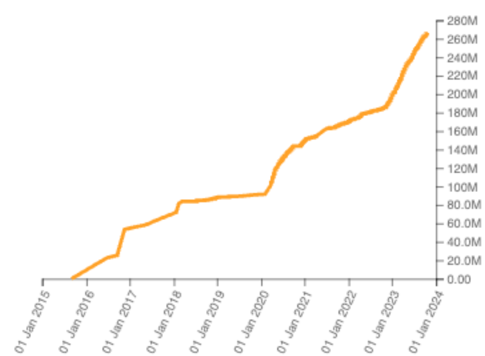


Authors

66,817,476

Projects

266,259,421



Releases

77,841,991

Thank you! Questions?