

# An Introduction to Discrete Probability

Section 7.1

# Section Summary

- Finite Probability
- Probabilities of Complements and Unions of Events
- Probabilistic Reasoning



# Probability of an Event

Pierre-Simon Laplace  
(1749-1827)

- An *experiment* is a procedure that yields one of a given set of possible outcomes.
- The *sample space* of the experiment is the set of possible outcomes.
- An *event* is a subset of the sample space.

**Laplace's Definition:** If  $S$  is a finite sample space of *equally likely outcomes*, and  $E$  is an event (a subset of  $S$ ), then the *probability* of  $E$  is

$$p(E) = \frac{|E|}{|S|}$$

- For every event  $E$ ,  $0 \leq p(E) \leq 1$ .

# Applying Laplace's Definition

**Example:** An urn contains four blue balls and five red balls. What is the probability that a ball chosen from the urn is blue?

**Solution:**  $4/9$  (9 possible outcomes, and 4 of these produce a blue ball)

**Example:** What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

**Solution:**  $6/36 = 1/6$  (by the product rule there are  $6^2 = 36$  possible outcomes, and 6 of these outcomes have a sum of 7)

# Applying Laplace's Definition

**Example:** In a lottery, a player wins a large prize when they pick four digits that match, in correct order, four digits selected by a random mechanical process. What is the probability that a player wins the prize?

**Solution:**  $1/10,000 = 0.0001$

- There is only 1 way to pick the correct digits
- By the product rule there are  $10^4 = 10,000$  ways to pick four digits.

# Applying Laplace's Definition

**Example:** (*continued*) A smaller prize is won if only 3 digits are matched. What is the probability that a player wins the small prize?

**Solution:**  $36/10,000 = 9/2500 = 0.0036$

- If exactly 3 digits are matched, one of the four digits must be incorrect and the other 3 digits must be correct. For the digit that is incorrect, there are 9 possible choices. Hence, by the sum rule, there a total of 36 possible ways to choose four digits that match exactly 3 of the winning four digits.
- By the product rule there are  $10^4 = 10,000$  ways to pick four digits.

08

27

34

04

19

10

# Applying Laplace's Definition

**Example:** There are many lotteries that award prizes to people who correctly choose a set of six numbers out of the first  $n$  positive integers, where  $n$  is usually between 30 and 60. What is the probability that a person picks the correct six numbers out of 40?

**Solution:**  $1 / 3,838,380 \approx 0.00000026$

- The number of ways to choose six numbers out of 40 is  
$$C(40,6) = 40! / (34!6!) = 3,838,380.$$

*Can you work out the probability of winning the lottery with the biggest prize where you live?*

# Applying Laplace's Definition

**Example:** What is the probability that the numbers 11, 4, 17, 39, and 23 are drawn in that order from a bin with 50 balls labeled with the numbers 1,2, ..., 50 if

- a) The ball selected is not returned to the bin.
- b) The ball selected is returned to the bin before the next ball is selected.

**Solution:** Use the product rule in each case.

- a) *Sampling without replacement:* The probability is  $1/254,251,200$  since there are  $50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = 254,251,200$  ways to choose the five balls.
- b) *Sampling with replacement:* The probability is  $1/50^5 = 1/312,500,000$  since  $50^5 = 312,500,000$ .



# The Probability of Complements and Unions of Events

**Theorem 1:** Let  $E$  be an event in sample space  $S$ . The probability of the event  $\overline{E} = S - E$ , the complementary event of  $E$ , is given by

$$p(\overline{E}) = 1 - p(E).$$

**Proof:** Using the fact that  $|\overline{E}| = |S| - |E|$ ,

$$p(\overline{E}) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E).$$



# The Probability of Complements and Unions of Events

**Example:** Throw two dice. What's the probability that the two scores are **different**?

**Solution:**

- $E$  = the event that two scores are different
- $\overline{E}$  = the event that two scores are the same
- $|S| = 36$  ways for two dice to land

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{6}{36} = 5/6$$

# The Probability of Complements and Unions of Events

**Example:** A sequence of 10 bits is chosen randomly.  
What is the probability that at least one of these bits is 0?

**Solution:** Let  $E$  be the event that at least one of the 10 bits is 0. Then  $\overline{E}$  is the event that all of the bits are 1s.  
The size of the sample space  $S$  is  $2^{10}$ . Hence,

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

# The Probability of Complements and Unions of Events

**Theorem 2:** Let  $E_1$  and  $E_2$  be events in the sample space  $S$ . Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

**Proof:** Given the *inclusion-exclusion* formula

$|A \cup B| = |A| + |B| - |A \cap B|$ , it follows that

$$\begin{aligned} p(E_1 \cup E_2) &= \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|} \\ &= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|} \\ &= p(E_1) + p(E_2) - p(E_1 \cap E_2). \end{aligned}$$



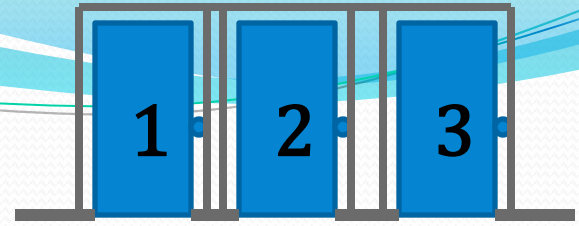
# The Probability of Complements and Unions of Events

**Example:** What is the probability that a randomly selected positive integer between 1 and 100 is divisible by either 2 **or** 5?

**Solution:**

- Let  $E_1$  be the event that the integer is divisible by 2
- Let  $E_2$  be the event that it is divisible 5
- Then,  $E_1 \cup E_2$  is the event that the integer is divisible by 2 **or** 5
- And,  $E_1 \cap E_2$  is the event that it is divisible by 2 **and** 5.

$$\begin{aligned} p(E_1 \cup E_2) &= p(E_1) + p(E_2) - p(E_1 \cap E_2) \\ &= 50/100 + 20/100 - 10/100 = 3/5. \end{aligned}$$



# Monty Hall Puzzle

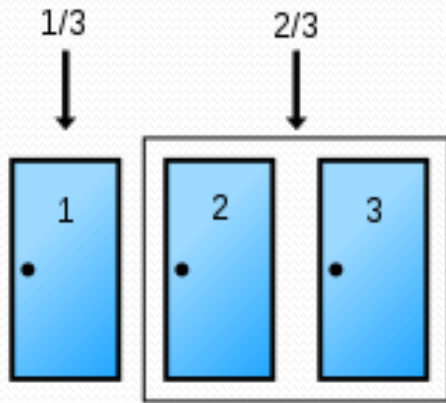
**Example:** You are asked to **select one of three doors to open**. Behind one of the doors is a car; behind the others, goats. If you select the door with a car, you win the car. After you select a door, the game show host opens one of the other doors (**which he knows is not the winning door**). The prize is not behind the door and he gives you the opportunity to switch your selection. Should you switch?

*(This is a notoriously confusing problem that has been the subject of much discussion. Do a web search to see why!)*

**Solution:** You should switch. The probability that your initial pick is correct is  $1/3$ . This is the same whether or not you switch doors. But since **the game show host always opens a door that does not have the prize**, **if you switch the probability of winning will be  $2/3$** , because you win if your initial pick was not the correct door and the probability your initial pick was wrong is  $2/3$ .

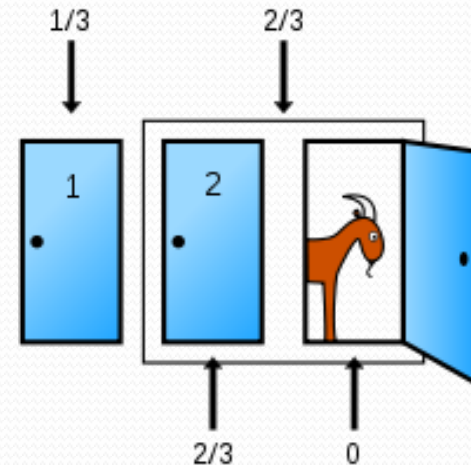
# Monty Hall Puzzle

Before the host picks:



Car has  $1/3$  chance of being behind the player's pick and a  $2/3$  chance of being behind one of the other two doors.

After the host picks a door with a goat:



The odds for the two sets don't change, but the odds move to 0 for the open door and  $2/3$  for the closed door.