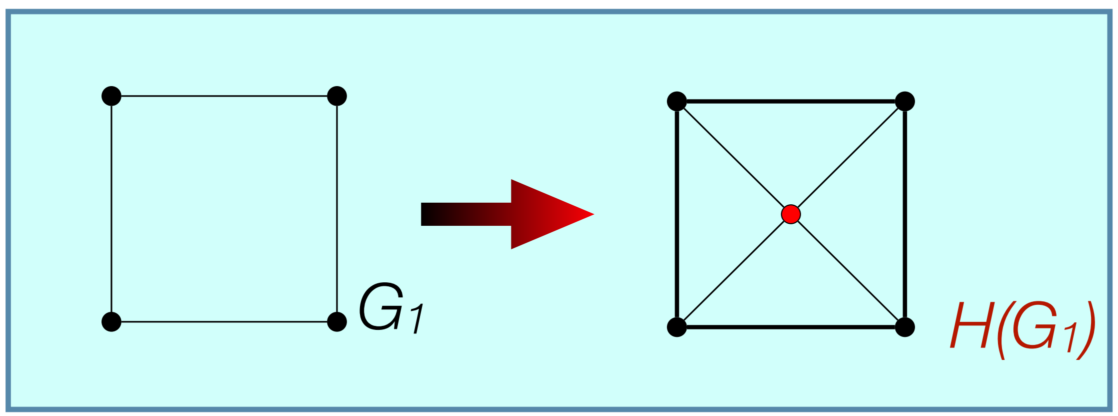
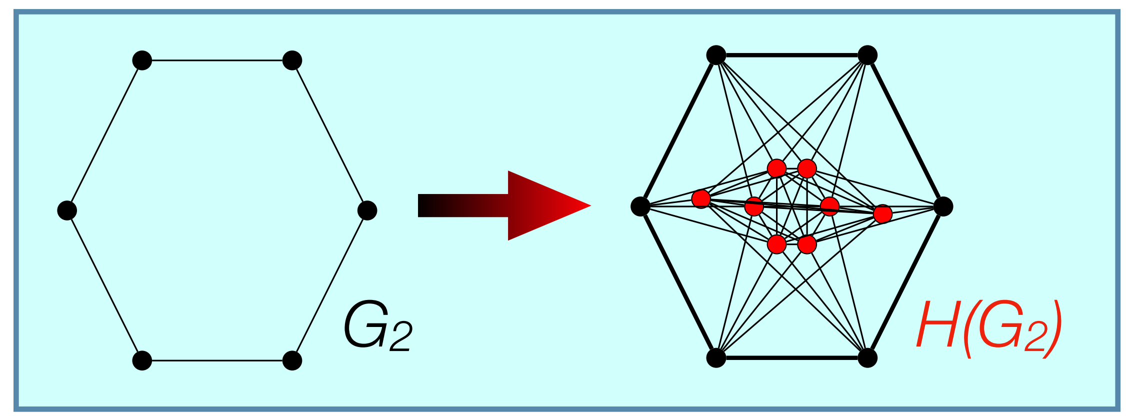
How hyperbolicity relates to Injective Hulls

Every graph G can be isometrically embedded into the smallest Helly graph $H(G)$ [1,2]

- $H(G)$ is called the injective hull of G
- $H(G)$ preserves hyperbolicity
- If G is δ-hyperbolic, any vertex in $H(G)$ is within 2δ to a vertex in G [3]

- A set S of sets S_i has the Helly property if for every subset T of S the following hold: if the elements of T pairwise intersect, then the intersection of all elements of T is also non-empty.
- A graph is called Helly if its family of disks satisfies the Helly property.





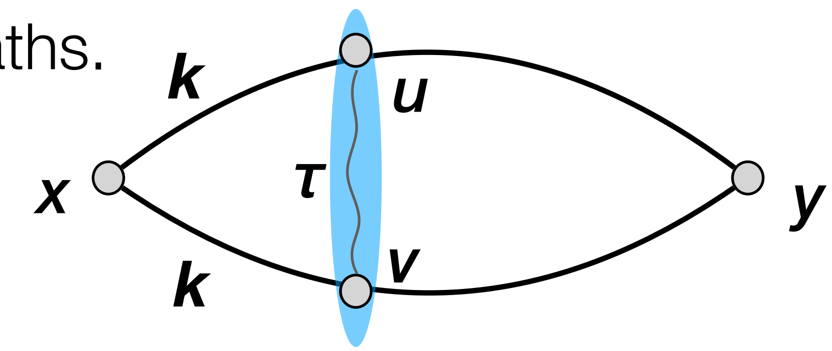
We want to understand:

- **(Q1) what governs hyperbolicity in Helly graphs in order to understand what governs hyperbolicity in regular graphs, and**
- **(Q2) how does the injective hull grow for various graph classes?**

[1] J. Isbell. *Six theorems about injective metric spaces*, Comment. Math. Helv (1964).
[2] A. Dress. *Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups,* Adv. in Math (1984).
[3] U. Lang, *Injective hulls of certain discrete metric spaces and groups*, J. Topol. Anal. (2013)

(**Q1**) Interval thinness governs hyperbolicity in Helly graphs

- An interval $I(x,y)$ is the set of all vertices from shortest (x,y)-paths.



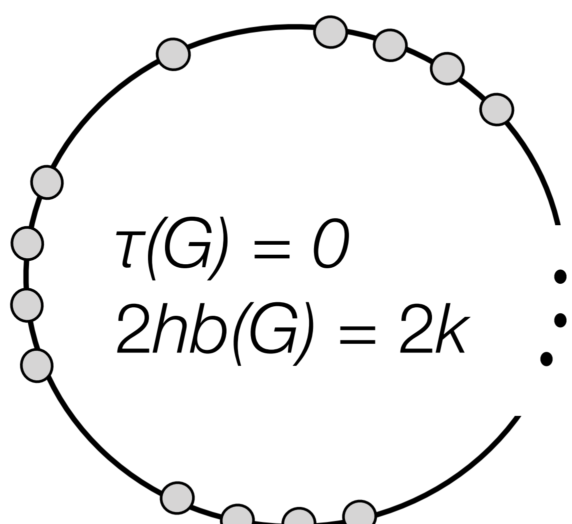
- A slice of an interval at distance k is defined as:

$$S_k(x,y) = \{z \in I(x,y) : d(z,x) = k\}$$

- An interval is τ-thin if for any natural number k and any two u,v vertices of $S_k(x,y)$ are at most τ apart.

- A graph is τ-thin if all of its intervals are at most τ-thin.

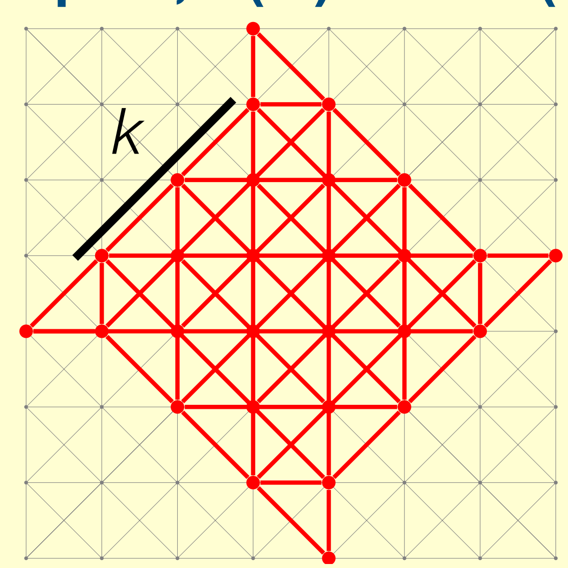
For general graphs τ(*G*) ≤ 2*hb*(*G*), but τ(*G*) and *hb*(*G*) can be far apart.



example: odd cycle with 4k+1 vertices

Our Contribution

Theorem [4]:
For Helly graphs, τ(G) ≤ 2hb(G) ≤ τ(G) + 1.



Example when $2hb(G) = \tau(G) + 1$

[4] F. Dragan and H. Guarnera, *Obstructions to a small hyperbolicity in Helly graphs*, under review.