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HYPERBOLICITY, INJECTIVE HULLS, AND HELLY GRAPHS (155 PP.)

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The δ-hyperbolicity of a graph can be viewed as a measure of how close a graph is to a tree metrically; the smaller the hyperbolicity of a graph, the closer it is metrically to a tree. For every graph G there exists a unique smallest Helly graph $\mathcal{H}(G)$ into which G isometrically embeds. $\mathcal{H}(G)$ is called the injective hull of G, and the hyperbolicity of $\mathcal{H}(G)$ is equal to the hyperbolicity of G. Moreover, the minimum distance between a vertex in $\mathcal{H}(G)$ and a vertex in G is directly proportional to the hyperbolicity of G, which is a small constant in many real-world networks. Motivated by this, we identify facility location and network analysis problems that are translatable to problems in a graph's underlying injective hull. Of particular interest for these problems is the eccentricity $e(v)$ of a vertex v, defined as the length of the longest shortest path from v to any other vertex.

We show that the injective hull $\mathcal{H}(G)$ of a graph G has many nice structural and metric properties that govern the hyperbolicity of G. We analyze the eccentricity function in distance-hereditary graphs (they are 1-hyperbolic) and fully characterize their centers (the set of vertices with minimum eccentricity). We describe the eccentricity terrain of a graph, that is, a topographic perspective of a graph with respect to vertex eccentricities, akin to geomorphometry. We identify terrain shapes such as hills, valleys, and plains, and discover that the length of some shapes depends only on the hyperbolicity. Additionally, we introduce a new graph parameter called the Helly-gap, a measure of a graph's metric deviation from its injective hull, and show that many well-known graph classes have a small Helly-gap. We present a new linear time algorithm for computing all vertex eccentricities in distance-hereditary graphs. For general graphs, we present various eccentricity approximation algorithms differing in quality in run time, parameterized by the δ-hyperbolicity or in part by the Helly-gap. We then identify several graph classes that are closed under Hellification, that is, $G \in \mathcal{C}$ implies $\mathcal{H}(G) \in \mathcal{C}$. We show that the injective hull of several graphs from some well-known classes of graphs are impossible to compute in subexponential time and provide a polynomial time algorithm to construct the injective hull of any distance-hereditary graph.

Hyperbolicity, injective hulls, and Helly graphs

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Chapter 1

Introduction

The δ-hyperbolicity of a graph can be viewed as a measure of how close a graph is to a tree metrically; the smaller the hyperbolicity of a graph, the closer it is metrically to a tree. Generally, it is known [110] that a graph $G = (V, E)$ is metrically a tree if and only if G is a block graph, i.e., each biconnected component of G is a complete graph. Metric tree-like structures are detected in a wide range of large-scale networks, from communication networks to various forms of social and biological networks [2, 3, 9, 29, 47, 84, 88, 99, 107, 111]. For example, preferential attachment networks were shown to be scaled hyperbolic [84], communication networks at the IP layer and at other levels were empirically observed to be hyperbolic [99], and the topology of the Internet efficiently maps to a hyperbolic space [111]. Moreover, extreme congestion at a very limited number of nodes in a very large traffic network was shown to be caused due to hyperbolicity of the network together with minimum length routing [18,37,85,109]. It has been suggested [99] that the property in which traffic between nodes tends to go through a relatively small core of the network (observed in real-world networks), as if the shortest path between them is curved inwards, may be due to global curvature of the network. This motivates much research to understand the structure and characteristics of hyperbolic graphs [2, 10, 23, 26, 27, 32, 37, 41, 91, 113, 114] as well as algorithmic implications of small hyperbolicity [27, 32, 35, 37, 38, 45, 70, 77, 94, 113]. One aims at developing approximation algorithms whose approximation factor depends only on the δ-hyperbolicity of the input graph or fixed parameter algorithms whose runtime is parameterized by δ.

A focal algorithmic interest of this dissertation resides in efficiently computing vertex eccentricities. The eccentricity $e_G(v)$ of a vertex v is the length of a longest shortest path from v to any other vertex, i.e., $e_G(v) = \max\{d(v,u) : u \in V\}$. The diameter $diam(G)$ (radius $rad(G)$) denotes the maximum (minimum) eccentricity of a vertex in G. The center $C(G)$ of a graph G is the set of all vertices with minimum eccentricity (i.e., $C(G) = \{v \in V : e_G(v) = rad(G)\}$). The eccentricity terrain illustrates the behavior of the eccentricity function along any shortest path: if a traveler begins at vertex y and ends at vertex x moving along $P(y,x)$, he may describe his journey as a combination of walking up-hill (to a vertex of higher eccentricity), down-hill (to a vertex of lower eccentricity), or along a plain (no change in eccentricity). Understanding the eccentricity function/terrain and being able to efficiently compute the diameter, radius, and all vertex eccentricities is of great importance. For example, in the analysis of social networks (e.g., citation networks or recommendation networks), biological systems (e.g., protein interaction networks), computer networks (e.g., the Internet or peer-to-peer networks), transportation networks (e.g., public transportation or road networks), etc., the eccentricity of a vertex v is used to measure the importance of v in the network: the eccentricity centrality index of v [92] is defined as $\frac{1}{e_G(v)}$. Vertex eccentricities also have other applications in network analysis and facility location problems. For example, the diameter of a social network represents its degrees of separation (i.e., the longest chain of friendships by which all people are connected). As another example, a central vertex $v \in C(G)$ represents an ideal location in a city network to build a hospital that is close to everyone. All eccentricities can be calculated in an unweighted undirected graph in $O(nm)$ time via a textbook algorithm to solve the All-Pairs Shortest Paths (APSP) problem, where n is the number of vertices and m is the number of edges. However, this becomes computationally impractical for graphs that are prohibitively large, necessitating an efficient approach to compute vertex eccentricities in large-scale networks.

As many real-world networks have a small hyperbolicity, we are interested in understanding what metric properties of graphs govern their hyperbolicity and in identifying structural obstructions to a small hyperbolicity. A graph G is called *Helly* if every system of pairwise intersecting disks of G has a non-empty intersection, where a disk $D(v,r)$ with radius r and centered at a vertex v is the set of all vertices with distance at most r from v. It is known [66, 83, 95] that for every graph G there exists a unique smallest Helly graph $\mathcal{H}(G)$ into which G isometrically embeds. $\mathcal{H}(G)$ is called the *injective hull* of G and the hyperbolicity of $\mathcal{H}(G)$ is equal to the hyperbolicity of G [83, 95]. Helly graphs are well investigated. They have several characterizations and important features as established in [15, 16, 49, 50, 100, 105]. They are exactly the so-called absolute retracts of reflexive graphs and possess a certain elimination scheme [15,16,49,50,100] which makes them recognizable in $O(n^2m)$ time [49]. The Helly property works as a compactness criterion on graphs [105]. Many of the nice properties of Helly graphs are based on the eccentricity of a vertex. For example, graph's diameter is tightly bounded in a Helly graph G as $2rad(G) \geq diam(G) \geq 2rad(G) - 1$ [49, 50]. Moreover, the eccentricity function in Helly graphs is unimodal [50], that is, any local minimum coincides with the global minimum; this is equivalent to the condition that, for any vertex $v \in V(G)$, $e_G(v) = d_G(v, C(G)) + rad(G)$ holds [49].

The rich theory behind Helly graphs and injective hulls makes them a useful underlying structure to prove properties about an arbitrary graph G and to solve problems on G. Recently, it was shown [95] that any vertex of $\mathcal{H}(G)$ is within 2δ to a vertex of G [95]. Problems such as finding the diameter or computing vertex eccentricities in G are translatable to finding the diameter of $\mathcal{H}(G)$ and computing eccentricities of vertices of the Helly graph $\mathcal{H}(G)$. The existence of $\mathcal{H}(G)$ alone can also be used to prove notable properties regarding the original graph G. For example, the existence of $\mathcal{H}(G)$ was used in [37] to prove a conjecture by Jonckheere et al. [85] that real-world networks with small hyperbolicity have a core congestion. It was shown [37] that any finite subset X of vertices in a locally finite δ-hyperbolic graph G admits a disk $D(m, 4\delta)$ centered at vertex m, which intercepts all shortest paths between at least one half of all pairs of vertices of X. The importance of $\mathcal{H}(G)$ as an underlying structure drives our interest in the hyperbolicity of Helly graphs in particular.

The results presented here establish strong metric and structural properties related to the δ-hyperbolicity in Helly graphs; this translates to interesting properties of any δ-hyperbolic graph G as G can be isometrically embedded into its hyperbolicity-preserving injective hull, the smallest unique Helly graph $\mathcal{H}(G)$. Consequently, we find several algorithmic implications for efficiently computing vertex eccentricities. We first focus on distance-hereditary graphs, i.e., the graphs in which every induced path is a shortest path, which are a subclass of 1-hyperbolic graphs. Later, we generalize our results to any δ-hyperbolic graph. We find that the injective hull can also be used to characterize a far-reaching superclass of δ-hyperbolic graphs, the so called α-weakly-Helly graphs. A graph G is called α-weakly-Helly if any system of pairwise intersecting disks in G has a nonempty common intersection when the radius of each disk is increased by an additive value α. The smallest value α for which a graph is α-weakly-Helly is called the Helly-gap, denoted $\alpha(G)$. We generalize many eccentricity-related results known for the Helly graphs, distance-hereditary graphs, and δ-hyperbolic graphs to all graphs, parameterized by $\alpha(G)$. As it turns out, many well-known graph classes have a small Helly-gap. Finally, we investigate the structural properties of injective hulls of several graph classes. For each graph class $\mathcal{C}$ considered, we specify whether $\mathcal{C}$ is closed under Hellification (i.e., $G \in \mathcal{C}$ implies $\mathcal{H}(G) \in \mathcal{C}$). Graphs that have an efficiently computable injective hull are of particular interest. A polynomial time algorithm to construct the injective hull of any distance-hereditary graph is provided and we show that the injective hull of several graphs from some other well-known classes of graphs are impossible to compute in subexponential time.

1.1 Outline

This dissertation is organized as follows, with relevant related works discussed in the appropriate chapters. Basic notations and a few useful auxiliary lemmas are provided in Chapter 2.

In Chapter 3, we show that all δ-hyperbolic Helly graphs contain three isometric subgraphs of size dependent on δ, and we characterize hyperbolicity by interval thinness as well as forbidden isometric subgraphs.

In Chapter 4, we show that the eccentricity function in any distance-hereditary graph G is almost unimodal, that is, every vertex v with $e(v) > rad(G) + 1$ has a neighbor with smaller eccentricity. Moreover, this result is used to fully characterize the centers of distance-hereditary graphs and to obtain several bounds on the eccentricity of a vertex. We introduce a new linear time algorithm to compute all eccentricities of a distance-hereditary graph.

In Chapter 5, we extend many of the eccentricity-related results obtained for the distance-hereditary graphs to δ-hyperbolic graphs. We also define a β-pseudoconvexity which implies the quasiconvexity found by Gromov in hyperbolic graphs, but additionally, is closed under intersection. Many sets of δ-hyperbolic graphs are shown to be $(2\delta - 1)$-pseudoconvex. We also identify several important features of the eccentricity terrain, which are insightful to the eccentricity bounds for a vertex. Three approximation algorithms for all eccentricities are given, including: (1) an $O(\delta m)$ time eccentricity approximation $\hat{e}(v)$ which satisfies $e_G(v) - 2\delta \leq \hat{e}(v) \leq e_G(v)$, (2) a spanning tree constructible in $O(\delta m)$ time which satisfies $e_G(v) \leq e_T(v) \leq e_G(v) + 4\delta + 1$, and (3) a spanning tree constructible in $O(m)$ time which satisfies $e_G(v) \leq e_T(v) \leq 6\delta$.

In Chapter 6, we introduce α-weakly-Helly graphs, a far reaching superclass of Helly graphs and δ-hyperbolic graphs. A graph G is called α-weakly-Helly if any system of pairwise intersecting disks in G has a nonempty common intersection when the radius of each disk is increased by an additive value α. The minimum α for which a graph G is α-weakly-Helly is called the Helly-gap of G and denoted by $\alpha(G)$. We characterize the Helly-gap of a graph by distances in its injective hull, which is then used as a tool to generalize many eccentricity related results known for Helly graphs ($\alpha(G) = 0$), as well as for chordal graphs ($\alpha(G) \leq 1$), distance-hereditary graphs ($\alpha(G) \leq 1$) and δ-hyperbolic graphs ($\alpha(G) \leq 2\delta$), to all graphs, parameterized by their Helly-gap $\alpha(G)$. Several additional graph classes are shown to have a bounded Helly-gap, including AT-free graphs and graphs with bounded tree-length, bounded chordality or bounded α_i-metric.

In Chapter 7, we investigate the structural properties of the injective hulls of various graph classes. We show that permutation graphs are not closed under Hellification, but chordal graphs, square-chordal graphs, and distance-hereditary graphs are. A polynomial time algorithm to construct the injective hull of any distance-hereditary graph is provided and we show that the injective hull of several graphs from some other well-known classes of graphs are impossible to compute in subexponential time. In particular, there are split graphs, cocomparability graphs, chordal bipartite graphs G such that $\mathcal{H}(G)$ contains $\Omega(a^n)$ vertices, where $a > 1$. Chapter 8 concludes.

1.2 Contribution

The results of this dissertation are based on the following papers:

1. Feodor F. Dragan and Heather M. Guarnera. Obstructions to a small hyperbolicity in helly graphs. *Discrete Mathematics*, 342(2):326 – 338, 2019 - Chapters 2 and 3.

2. Feodor F. Dragan and Heather M. Guarnera. Eccentricity function in distance-hereditary graphs. *Theoretical Computer Science*, 2020 - Chapter 4.

3. Feodor F. Dragan and Heather M. Guarnera. Eccentricity terrain of δ-hyperbolic graphs. *Journal of Computer and System Sciences*, 112:50–65, 2020 - Chapters 2 and 5.

4. Feodor F. Dragan and Heather M. Guarnera. Helly-gap of a graph and vertex eccentricities. *Manuscript, 2020* (under review by The Electronic Journal of Combinatorics) - Chapters 2

and 6.

5. Heather M. Guarnera, Feodor F. Dragan, and Arne Leitert. Injective hulls of various graph classes. *Manuscript, 2020* (in preparation) - Chapter 7.

Chapter 2

Preliminaries

All graphs $G = (V, E)$ appearing here are connected, finite, unweighted, undirected, loopless and without multiple edges. Let $n = |V|$ and $m = |E|$. We denote by $\langle S \rangle$ the subgraph of G induced by the vertices $S \subset V$. A *path* $P(v_0, v_k)$ is a sequence of vertices $v_0, ..., v_k$ such that $v_i v_{i+1} \in E$ for all $i \in [0, k-1]$; its *length* is k. A graph G is connected if there is a path between every pair of vertices. A *chord* of a path (cycle) $v_0, ..., v_k$ is an edge between two vertices of the path (cycle) that is not an edge of the path (cycle). The *distance* $d_G(u, v)$ between vertices u and v is the length of a shortest path connecting u and v in G. Let also $d_G(v, S) = \min\{d(v, u) : u \in S\}$. We omit the subscript when G is known by context. A subgraph G' of a graph G is called *isometric* (or a *distance preserving subgraph*) if for any two vertices x, y of G', $d_G(x, y) = d_{G'}(x, y)$ holds. By $G - \{x\}$ we denote an induced subgraph of G obtained from G by removing a vertex $x \in V$.

The *neighborhood of* v consists of all vertices adjacent to v, denoted by $N(v)$; the *closed neighborhood of* v is defined as $N[v] = N(v) \cup \{v\}$. The k^{th} *neighborhood* of v is the set of all vertices at distance k from v, denoted $N^k(v) = \{u \in V(G) : d(u, v) = k\}$. The degree $deg(v)$ of a vertex v is the number of neighbors it has, i.e., $deg(v) = |N(v)|$. Two vertex sets A and B of G said to be *joined* if each vertex of A is adjacent to every vertex of B. A vertex is *pendant* if $|N(v)| = 1$. Two vertices v and u are *twins* if they have the same neighborhood or the same closed neighborhood. *True twins* are adjacent; *false twins* are not. A set $M \subseteq V(G)$ is an *independent set* if for all $u, v \in V(G)$, $uv \notin E(G)$. A set $M \subseteq V(G)$ is a *clique* (or *complete subgraph*) if all distinct vertices $u, v \in M$ have $uv \in E(G)$. A set $M \subseteq V(G)$ is said to be a *2-set* if for every $x, y \in M$, $d(x, y) \leq 2$ holds. A 2-set M is *maximal* in G if it is maximal by inclusion. A vertex v is said to *suspend* a set

$M \subseteq V(G)$ if $vu \in E(G)$ for each $u \in M \setminus \{v\}$; v is also said to be *universal* to $M \setminus \{v\}$. We denote by C_k a cycle induced by k vertices, by W_k an induced wheel of size k, i.e., a cycle C_k with one additional vertex universal to C_k, and by K_n a clique of n vertices. Let $V = \{v_1, ..., v_n\}$. For an n-tuple of non-negative integers $(r(v_1), ..., r(v_n))$, a subset $M \subseteq V$ is an r-dominating set for a set $S \subseteq V$ in G if and only if for every $v \in S$ there is a vertex $u \in M$ with $d(u,v) \leq r(v)$. We also say that M *r-dominates* S in G. If $r(v_i) = 1$ for all i, then we say that M *dominates* S in G. If $S = V$ then we say that M r-dominates G. A graph B is *bipartite* if its vertex set can be partitioned into two independent sets X and Y, i.e., each edge $uv \in E(B)$ has one end in X and the other in Y.

The *k-th power* G^k of G is a graph that has the same set of vertices, but in which two distinct vertices are adjacent if and only if their distance in G is at most k, i.e., $G^k = (V, E')$ where $E' = \{uv : u, v \in V \text{ and } d_G(u,v) \leq k\}$. A *disk* $D_G(v,r)$ of a graph G centered at a vertex $v \in V$ and with radius r is the set of all vertices with distance no more than r from v (i.e., $D_G(v,r) = \{u \in V : d_G(v,u) \leq r\}$). We omit the subscript when G is known by context.

For any two vertices u, v of G, $I(u,v) = \{z \in V : d_G(u,v) = d_G(u,z) + d_G(z,v)\}$ is the (metric) *interval* between u and v, i.e., all vertices that lay on shortest paths between u and v. An interval *slice* is defined as $S_k(x,y) = \{v \in I(x,y) : d_G(x,v) = k\}$ for some non-negative integer k.

The eccentricity $e_G(v) = \max\{d_G(v,u) : u \in V\}$ of a vertex v is the length of a longest shortest path from v to any other vertex. The *diameter* of a graph G is defined as $diam(G) = max_{u,v \in V} d_G(u,v)$. A vertex with maximum eccentricity is called a *diametral vertex*. The set of all diametral vertices is $D(G) = \{v \in V : e(v) = diam(G)\}$. The *diameter of a set* $S \subseteq V$ is $diam(S) = \max_{x,y \in S} d_G(x,y)$. A vertex with minimum eccentricity is called a *central vertex*. The *center* is the set of vertices whose eccentricities are minimum: $C(G) = \{v \in V : e_G(v) = rad(G)\}$. Let $C^k(G) = \{v \in V : e_G(v) \leq rad(G) + k\}$; then $C(G) = C^0(G)$. The *radius* $rad(G)$ is the eccentricity of these central vertices. This is to say that $rad(G)$ is the smallest radius of a disk $D(v,r)$ centered at vertex $v \in C(G)$ which contains all vertices $u \in V$ at distance at most r. Thus, the diameter $diam(G)$ (radius $rad(G)$) denotes the maximum (minimum) eccentricity of a vertex in G. We denote the set of farthest vertices from v as $F_G(v) = \{u \in V : d_G(u,v) = e_G(v)\}$, omitting the subscript when G is known. Vertices x, y are considered to be *mutually distant* if $x \in F(y)$ and $y \in F(x)$; they are called *a pair of mutually distant vertices*. A pair $\{x, y\}$ is called *a diametral pair* if $d_G(x,y) = diam(G)$.

Let $M \subseteq V(G)$ be a subset of vertices of G. We distinguish the eccentricity with respect to M as follows: denote by $e_G^M(v)$ the maximum distance from vertex v to any vertex $u \in M$, i.e., $e_G^M(v) = \max_{u \in M} d_G(v,u)$. We then define $F_G^M(v) = \{u \in M : e_G^M(v) = d_G(v,u)\}$, $rad_G(M) = \min_{v \in V(G)} e_G^M(v)$, and $diam_G(M) = \max_{v \in M} e_G^M(v)$. Let $C_G^\ell(M) = \{v \in V(G) : e_G^M(v) \leq rad_G(M) + \ell\}$ and $C_G(M) = C_G^0(M)$. For simplicity, when $M = V$, we continue to use the notation $rad(G)$, $C(G)$, and $diam(G)$.

Let S be a set and let function $\hat{f} : S \to \mathbb{R}$ be an approximation of function $f : S \to \mathbb{R}$. We say that $\hat{f}$ is an *left-sided additive ϵ-approximation* of f if, for all $x \in S$, $f(x) - \epsilon \leq \hat{f}(x) \leq f(x)$. We say that $\hat{f}$ is an *right-sided additive ϵ-approximation* of f if, for all $x \in S$, $f(x) \leq \hat{f}(x) \leq f(x) + \epsilon$. The value ϵ is called *left-sided* (*right-sided*, respectively) *additive error*. For example, in a graph G, a left-sided error appears when a vertex is returned by an algorithm whose eccentricity is an approximation of the diameter of G (as its eccentricity cannot exceed $diam(G)$), whereas a right-sided error appears when a vertex is returned by an algorithm whose eccentricity is an approximation of the radius of G (as its eccentricity cannot be smaller than $rad(G)$).

2.1 Eccentricity terrain and terrain shapes

The eccentricity function partitions the vertex set of G into eccentricity layers, wherein each layer is defined as $C^{=k}(G) = \{v \in V(G) : e_G(v) = rad(G) + k\}$ for an integer $k \in [0, diam(G) - rad(G)]$. As the eccentricities of two neighboring vertices u and v can differ by at most one, if vertex u belongs to layer $C^{=k}(G)$, then any vertex v adjacent to u belongs to either $C^{=k-1}(G)$, $C^{=k}(G)$, or $C^{=k+1}(G)$. The first layer $C^{=0}(G)$ is exactly the center $C(G)$ (all vertices of G with minimum eccentricity). The last layer $C^{=p}(G)$, where $p = diam(G) - rad(G)$, consists of all diametral vertices v, i.e., with $e_G(v) = diam(G)$. Also of interest are the sets defined as $C^k(G) = \{v \in V : e_G(v) \leq rad(G) + k\}$, that is, the union of all eccentricity layers from $C^{=0}(G)$ to $C^{=k}(G)$. To describe the behavior of the eccentricity function, it is natural to define a locality of a vertex $v \notin C(G)$ as the minimum distance from v to a vertex with smaller eccentricity: $loc(v) = \min\{d_G(v,x) : x \in V,\ e_G(x) < e_G(v)\}$. By definition, the locality of a central vertex is 0.

The *eccentricity terrain* illustrates the behavior of the eccentricity function with respect to $M \subseteq V$ along any shortest path: if a traveler begins at vertex y and ends at vertex x moving along $P(y,x)$, he may describe his journey as a combination of walking up-hill (to a vertex of higher eccentricity), down-hill (to a vertex of lower eccentricity), or along a plain (no change in eccentricity). We identify such terrain features as hills, plains, valleys, terraces, and plateaus. First, we define an ordered pair of vertices uv, where $uv \in E$, as an *up-edge* if $e_G^M(u) < e_G^M(v)$, as a *down-edge* if $e_G^M(u) > e_G^M(v)$, and as a *horizontal-edge* if $e_G^M(u) = e_G^M(v)$. Thus, any path $P(y,x) = (y = v_0, v_1, ..., v_k = x)$ from vertex y to vertex x can be described by a series of k consecutive ordered pairs $v_i v_{i+1}$ which can be classified as either up-edges, down-edges, or horizontal-edges. We define an *m-segment* (m stands for monotonic) as a series of consecutive ordered pairs $(v_i, v_{i+1}, ..., v_{i+\ell-1}, v_{i+\ell})$ along a shortest path $P(y,x)$ in which each edge $v_j v_{j+1}$ has the same classification.

An *up-hill* (*down-hill*) on $P(y,x)$ is a maximal by inclusion m-segment $(v_i, \dots, v_{i+\ell})$ of $P(y,x)$ where $v_j v_{j+1}$ is an up-edge (down-edge) for each $j \in \{i, i+1, ..., i+\ell-1\}$. The value ℓ is called the *height* of the hill. A *plain* on $P(y,x)$ is a maximal by inclusion m-segment $(v_i, \dots, v_{i+\ell})$ of $P(y,x)$ where $v_j v_{j+1}$ is a horizontal-edge for each $j \in \{i, i+1, ..., i+\ell-1\}$. The value ℓ is called the *width* of the plain. A plain $(v_i, \dots, v_{i+\ell})$ of $P(y,x)$ with $i > 0$ and $i + \ell < k$ is called a *plateau* if $e_G^M(v_{i-1}) < e_G^M(v_i)$ and $e_G^M(v_{i+\ell+1}) < e_G^M(v_{i+\ell})$, is called a *valley* if $e_G^M(v_{i-1}) > e_G^M(v_i)$ and $e_G^M(v_{i+\ell+1}) > e_G^M(v_{i+\ell})$, is called a *terrace* if $e_G^M(v_{i-1}) < e_G^M(v_i)$ and $e_G^M(v_{i+\ell+1}) > e_G^M(v_{i+\ell})$ or $e_G^M(v_{i-1}) > e_G^M(v_i)$ and $e_G^M(v_{i+\ell+1}) < e_G^M(v_{i+\ell})$ (see Figure 2.1).

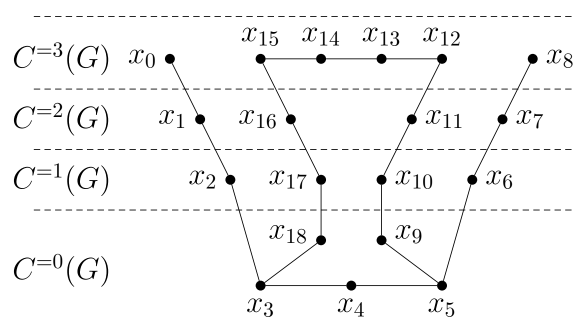


Figure 2.1: A graph G with $rad(G) = 6$ is shown with its eccentricity layers (for $M = V(G)$). Shortest path $P(x_0, x_{13})$ from x_0 to x_{13} consists of down-hill (x_0, x_1, x_2, x_3), valley (x_3, x_{18}), up-hill $(x_{18}, x_{17}, x_{16}, x_{15})$, and plain (x_{15}, x_{14}, x_{13}). Shortest path $P(x_{16}, x_{11})$ from x_{16} to x_{11} consists of up-hill (x_{16}, x_{15}), plateau (x_{15}, x_{12}), down-hill (x_{12}, x_{11}).

For a shortest path $P(y,x)$ from y to x in an arbitrary graph, we establish a relation between the number of up-edges, down-edges, and the eccentricities of x and y with respect to some $M \subseteq V$. Let $\mathbb{U}(P(y,x))$, $\mathbb{H}(P(y,x))$, and $\mathbb{D}(P(y,x))$ denote the number of up-edges, horizontal-edges, and down-edges, respectively, along a shortest path $P(y, x)$ from $y \in V$ to $x \in V$. Since each edge is classified as exactly one of the three categories, any shortest path $P(y, x)$ has $d_G(y, x) = \mathbb{U}(P(y, x)) + \mathbb{H}(P(y, x)) + \mathbb{D}(P(y, x))$.

Lemma 1. Let G be an arbitrary graph and $M \subseteq V$. For any shortest path $P(y, x)$ of G from a vertex y to a vertex x the following holds:

i) $\mathbb{D}(P(y, x)) - \mathbb{U}(P(y, x)) = e_G^M(y) - e_G^M(x)$, and

ii) $2\mathbb{U}(P(y, x)) + \mathbb{H}(P(y, x)) = d_G(y, x) - (e_G^M(y) - e_G^M(x))$.

Proof. We use an induction on $d_G(y, v)$ for any vertex $v \in P(y, x)$. First, assume that v is adjacent to y. If yv is an up-edge, then $e_G^M(y) - e_G^M(v) = -1$ and $\mathbb{D}(P(y, v)) - \mathbb{U}(P(y, v)) = -1$. If yv is a horizontal-edge, then $e_G^M(y) - e_G^M(v) = 0$ and $\mathbb{D}(P(y, v)) - \mathbb{U}(P(y, v)) = 0$. If yv is a down-edge, then $e_G^M(y) - e_G^M(v) = 1$ and $\mathbb{D}(P(y, v)) - \mathbb{U}(P(y, v)) = 1$. Now consider an arbitrary vertex $v \in P(y, x)$ and assume, by induction, that $e_G^M(y) - e_G^M(v) = \mathbb{D}(P(y, v)) - \mathbb{U}(P(y, v))$. Let vertex $u \in P(y, x)$ be adjacent to v with $d(y, u) = d(y, v) + 1$. By definition, $\mathbb{D}(P(y, u)) = \mathbb{D}(P(y, v)) + \mathbb{D}(P(v, u))$ and $\mathbb{U}(P(y, u)) = \mathbb{U}(P(y, v)) + \mathbb{U}(P(v, u))$. We consider three cases based on the classification of edge vu.

If vu is an up-edge, then $e_G^M(u) = e_G^M(v) + 1$, $\mathbb{U}(P(v, u)) = 1$, and $\mathbb{D}(P(v, u)) = 0 = \mathbb{U}(P(v, u)) - 1$. By the inductive hypothesis, $\mathbb{D}(P(y, u)) = \mathbb{D}(P(y, v)) + \mathbb{D}(P(v, u)) = \mathbb{U}(P(y, v)) + e_G^M(y) - e_G^M(v) + \mathbb{U}(P(v, u)) - 1 = \mathbb{U}(P(y, u)) + e_G^M(y) - e_G^M(v) - 1 = \mathbb{U}(P(y, u)) + e_G^M(y) - e_G^M(u)$.

If vu is a horizontal-edge, then $e_G^M(u) = e_G^M(v)$, $\mathbb{U}(P(v, u)) = 0 = \mathbb{D}(P(v, u))$. By the inductive hypothesis, $\mathbb{D}(P(y, u)) = \mathbb{D}(P(y, v)) + \mathbb{D}(P(v, u)) = \mathbb{U}(P(y, v)) + e_G^M(y) - e_G^M(v) + \mathbb{U}(P(v, u)) = \mathbb{U}(P(y, u)) + e_G^M(y) - e_G^M(v) = \mathbb{U}(P(y, u)) + e_G^M(y) - e_G^M(u)$.

If vu is a down-edge, then $e_G^M(u) = e_G^M(v) - 1$, $\mathbb{U}(P(v, u)) = 0$, and $\mathbb{D}(P(v, u)) = \mathbb{U}(P(v, u)) + 1$. By the inductive hypothesis, $\mathbb{D}(P(y, u)) = \mathbb{D}(P(y, v)) + \mathbb{D}(P(v, u)) = \mathbb{U}(P(y, v)) + e_G^M(y) - e_G^M(v) + \mathbb{U}(P(v, u)) + 1 = \mathbb{U}(P(y, u)) + e_G^M(y) - e_G^M(v) + 1 = \mathbb{U}(P(y, u)) + e_G^M(y) - e_G^M(u)$.

This completes the proof of (i) that $\mathbb{D}(P(y, x)) - \mathbb{U}(P(y, x)) = e_G^M(y) - e_G^M(x)$. To complete the proof of (ii), it now suffices to see that $\mathbb{U}(P(y, x)) + \mathbb{H}(P(y, x)) + (\mathbb{U}(P(y, x)) + e_G^M(y) - e_G^M(x)) = \mathbb{U}(P(y, x)) + \mathbb{H}(P(y, x)) + \mathbb{D}(P(y, x)) = d_G(y, x)$. Hence, $2\mathbb{U}(P(y, x)) + \mathbb{H}(P(y, x)) = d_G(y, x) - (e_G^M(y) - e_G^M(x))$. $\square$

The eccentricity function/terrain has been studied extensively in Helly graphs, chordal graphs, and $(\alpha_1, \triangle)$-metric graphs [36,49,53,63,102], among others. In [49], it is shown that the eccentricity function in Helly graphs exhibits unimodality: every vertex $v \notin C(G)$ has $loc(v) = 1$. In other words, any non-central vertex v has a shortest path P to a closest central vertex wherein any vertex on P appears in a strictly lower eccentricity layer than the previous vertex until $C(G)$ is reached. Thus, any local minimum of the eccentricity function $e_G(v)$ coincides with the global minimum on Helly graphs [49]. It is shown [49] that in such cases for any vertex $v \in V$, $e_G(v) = d_G(v, C(G)) + rad(G)$ holds. Additionally, $(\alpha_1, \triangle)$-metric graphs, which include chordal graphs and the underlying graphs of 7-systolic complexes, have a similar but slightly weaker property. In [63], it is shown that every vertex $v \notin C(G)$ of a $(\alpha_1, \triangle)$-metric graph G either has $loc(v) = 1$ or $e_G(v) = rad(G) + 1$, $diam(G) = 2rad(G)$, and $d(v, C(G)) = 2$. So, any non-central vertex v has a shortest path P to a closest central vertex upon which the eccentricity of each vertex $u \in P$ monotonically decreases until $C^{=1}(G)$ and, furthermore, $|P \cap C^{=1}(G)| \leq 2$. This leads to a linear time additive 2-approximation for all eccentricities in chordal graphs via careful construction of a spanning tree [53, 63].

Similar locality results have been established for δ-hyperbolic graphs [8] : any vertex v in a δ-hyperbolic graph has either $loc(v) \leq 2\delta + 1$ or $C(G)$ belongs to the set of vertices that are at most $4\delta + 1$ from v. A pioneering work [33] first showed that, in a δ-hyperbolic graph, $diam(G)$ and $2rad(G)$ are within $4\delta + 1$ from each other and that the diameter of $C(G)$ in G is at most $4\delta + 1$. It gave also fast approximation algorithms for computing the diameter and the radius of G and showed that there is a vertex c in G, computable in linear time, such that each central vertex of G is within distance at most $5\delta + 1$ from c. Later in [61], a better approximation algorithm for the radius was presented and a bound on the diameter of set $C^{2\delta}(G)$ was obtained, namely, $diam(C^{2\delta}(G)) \leq 8\delta + 1$. Recently, similar results were obtained in [36] for graphs with τ-thin geodesic triangles. Additionally to approximating the diameter and the radius, [36] gave efficient algorithms for approximating all eccentricities in such graphs via careful construction of a spanning tree. We will mention the relevant results from [33, 36, 61] in appropriate places later and compare them with our new results.

2.2 Helly graphs

A graph G is called *Helly* if every system of pairwise intersecting disks of G has a non-empty intersection.

Definition 1 (Helly graph)**.** A graph G is Helly if, for any system of disks $\mathcal{F} = \{D(v, r(v)) : v \in S \subseteq V\}$, the following Helly property holds: if $X \cap Y \neq \emptyset$ for every $X, Y \in \mathcal{F}$, then $\bigcap_{v \in S} D(v, r(v)) \neq \emptyset$.

Note that two disks $D(v, p)$ and $D(u, q)$ intersect each other if and only if $d(u, v) \leq p + q$. Two disks $D(v, p)$ and $D(u, q)$ of G are said to *see* each other, sometimes also referred to as *touching* each other, if they intersect or there is an edge in G with one end in $D(v, p)$ and other end in $D(u, q)$ (equivalently, if $d(u, v) \leq p + q + 1$). The *strong product* of a set of graphs G_i for $i = 1, 2, ..., k$ is the graph $\boxtimes_{i=1}^{k} G_i$ whose vertex set is the Cartesian product of the vertex sets V_i, and there is an edge between vertices $a = (a_1, a_2, ..., a_k)$ and $b = (b_1, b_2, ..., b_k)$ if and only if a_i is either equal or adjacent to b_i for $i = 1, 2, ..., k$. A *King-grid* is a strong product of two paths. King-grids are of interest because by construction they are Helly.

The following lemma will be frequently used. It is true for a larger family of pseudo-modular graphs but we formulate it in the context of Helly graphs.

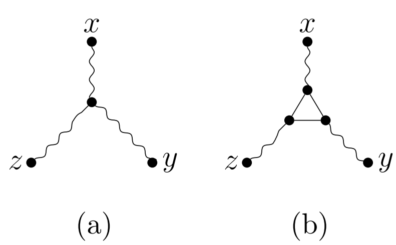


Figure 2.2: Vertices x, y, z and three shortest paths connecting them in Helly graphs.

Lemma 2. [14] For every three vertices x, y, z of a Helly graph G there exist three shortest paths $P(x, y)$, $P(x, z)$, $P(y, z)$ connecting them such that either (1) there is a common vertex v in $P(z, y) \cap P(x, z) \cap P(x, y)$ or (2) there is a triangle $\triangle(x', y', z')$ in G with edge $z'y'$ on $P(z, y)$, edge $x'z'$ on $P(x, z)$ and edge $x'y'$ on $P(x, y)$ (see Figure 2.2). Furthermore, (1) is true if and only if $d(x, y) = p + q$, $d(x, z) = p + k$ and $d(y, z) = q + k$, for some $k, p, q \in N$, and (2) is true if and only if $d(x, y) = p + q + 1$, $d(x, z) = p + k + 1$ and $d(y, z) = q + k + 1$, for some $k, p, q \in \mathbb{N}$.

Helly graphs have the following additional useful properties.

Lemma 3. **[49]** Let G be a Helly graph. For every $M \subseteq V$, the graph induced by the center $C_G(M)$ is Helly and is an isometric subgraph of G.

Given this lemma, it will be convenient to also denote by $C_G(M)$ a subgraph of G induced by $C_G(M)$.

Lemma 4. **[15, 49]** Let G be a Helly graph. If there are two distinct vertices $w, x \in V(G)$ such that $D(w, 1) \supseteq D(x, 1)$, then $G - \{x\}$ is Helly and an isometric subgraph of G.

Lemma 5. [49] Any power of a Helly graph is also a Helly graph.

For a graph G and a subset $M \subseteq V$, the eccentricity function $e_G^M(\cdot)$ is called *unimodal* if every vertex $v \in V \setminus C_G(M)$ has a neighbor u such that $e_G^M(u) < e_G^M(v)$.

Lemma 6. **[49, 50]** A graph G is Helly if and only if the eccentricity function $e_G^M(\cdot)$ is unimodal on G for every $M \subseteq V$.

The following two results were earlier proven in [49] only for $M = V$ and then later extended in [56] to all $M \subseteq V$.

Lemma 7. **[56]** Let G be a Helly graph. For any $M \subseteq V$, $2rad(M) - 1 \leq diam(M) \leq 2rad(M)$. Moreover, $rad(M) = \lceil diam(M)/2 \rceil$.

Lemma 8. **[56]** Let G be a Helly graph. For every $v \in V$ and $M \subseteq V$, $e_G^M(v) = d(v, C_G(M)) + rad(M)$ holds.

Corollary 1. For every Helly graph G, any subset $M \subseteq V$ and any integers $\ell \geq 0$ and $k \geq 0$, $C_G^{\ell+k}(M) = D(C_G^k(M) + \ell) = D(C_G(M) + k + \ell)$. Furthermore, $diam(C_G^{\ell+k}(M)) \leq diam(C_G^k(M)) + 2\ell$.

Proof. Consider a vertex v with $e_G^M(v) = k + \ell + rad(M)$ and a vertex c in $C_G(M)$ closest to v. By Lemma 8, $e_G^M(v) = d(v, C_G(M)) + rad(M) = d(v, c) + rad(M) = k + \ell + rad(M)$. Hence, for every vertex v, $d(v, C_G(M)) = k + \ell$ if and only if $e_G^M(v) = k + \ell + rad(M)$.

Let u be a vertex on a shortest path from v to c at distance k from c. By Lemma 8, $e_G^M(u) = d(u, c) + rad(M) = k + rad(M)$ and hence $e_G^M(v) = k + \ell + rad(M) = e_G^M(u) + \ell$. Therefore, $e_G^M(v) = k + \ell + rad(M)$ if and only if $d(v, C_G^k(M)) = d(v, u) = \ell$.

Let x, y be vertices of $C_G^{\ell+k}(M)$ with $d(x,y) = diam(C_G^{\ell+k}(M))$. Since $d(v, C_G^k(M)) \leq \ell$ for each $v \in \{x, y\}$, by the triangle inequality, we have $d(x,y) \leq d(x, C_G^k(M)) + diam(C_G^k(M)) + d(y, C_G^k(M)) \leq diam(C_G^k(M)) + 2\ell$. □

2.3 Hyperbolicity and variants

We are also interested in hyperbolic graphs (sometimes referred to as graphs with a negative curvature). For metric spaces (X, d), there are several equivalent definitions of δ-hyperbolicity with different but comparable values of δ [7, 22, 78, 79]. This dissertation uses Gromov's 4-point condition [79].

2.3.1 δ-Hyperbolicity (Gromov's 4-point condition)

Definition 2 (Hyperbolicity)**.** A metric space (X, d) is δ-hyperbolic if for any four points $u, v, w, x \in X$, the two larger of the distance sums $d(u,v) + d(w,x), d(u,w) + d(v,x), d(u,x) + d(v,w)$ differ by at most $2\delta \geq 0$.

δ-Hyperbolic graphs play an important role in geometric group theory and in the geometry of negatively curved spaces, and have recently become of interest in several domains of computer science, including algorithms and networking. The hyperbolicity of a graph can be viewed as a measure of how close it is to a tree metrically; the smaller the hyperbolicity of a graph, the closer it is metrically to a tree.

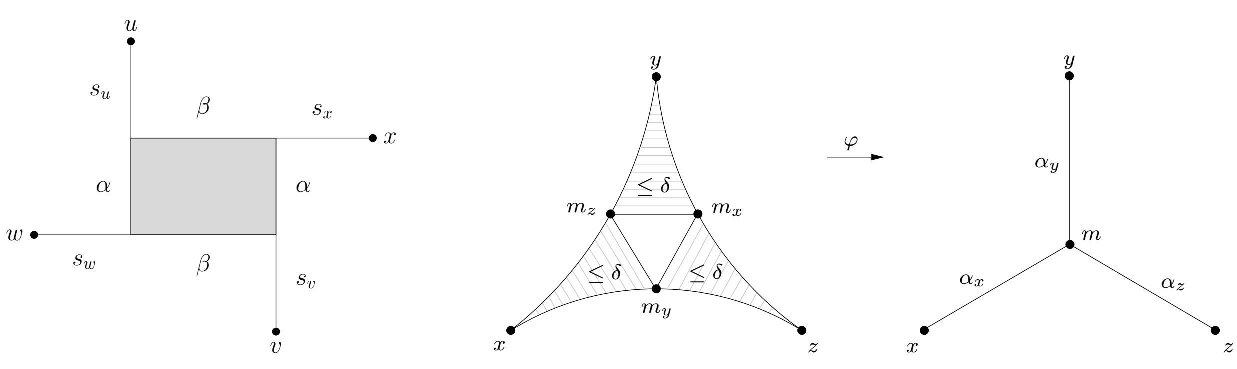


Figure 2.3: Illustration to the definitions of δ: realization of the 4-point condition in the rectilinear plane (left), and a geodesic triangle $\Delta(x, y, z)$, the points m_x, m_y, m_z, and the tripod $T(x, y, z)$ (right).

A connected graph equipped with the standard graph metric d_G is δ-hyperbolic if the metric space (V, d_G) is δ-hyperbolic. The smallest value δ for which G is δ-hyperbolic is called the *hyperbolicity* of G and is denoted by $hb(G)$ or $\delta(G)$. Note that $\delta(G)$ is an integer or a half-integer. Let also $hb(u, v, w, x)$ $(u, v, w, x \in V)$ denote one half of the difference between the two larger distance sums from $d(u, v) + d(w, x), d(u, w) + d(v, x), d(u, x) + d(v, w)$. Every 4-point metric d has a canonical representation in the rectilinear plane. In Figure 2.3, the three distance sums are ordered from large to small, implying that $\alpha \leq \beta$. Then β is half the difference of the largest minus the smallest sum, while α is half the difference of the largest minus the medium sum. Hence, a metric space (X, d) is δ-hyperbolic if $\alpha \leq \delta$ for any four points $u, v, w, x \in X$. Notice that any graph is δ-hyperbolic for some hyperbolicity $\delta \leq diam(G)/2$.

There is a simple and naive $O(n^4)$ time method to calculate hyperbolicity via checking all possible quadruplets to find the maximum hyperbolicity, i.e., $\delta(G) = \max hb(u, v, w, x)$ for all $u, v, w, x \in V$. Fournier et. al [75] present the best known algorithm to exactly compute hyperbolicity in $O(n^{3.69})$ time via fast matrix multiplication, as well as a $O(n^{2.69})$ time 2-approximation algorithm. Borassi et. al [19] implement an efficient algorithm whose worst-case time complexity is $O(n^4)$, but in practice is much faster and allows for the computation of hyperbolicity on a graph as large as 200,000 vertices (20x more than the previous exact solution).

2.3.2 Thin geodesic triangles

At times we will compare our results to those known from literature, including those known for graphs defined by thin geodesic triangles as follows. Let (X, d) be a metric space. An (x, y)-geodesic is a (continuous) map γ from the segment $[a, b]$ of $\mathbb{R}^1$ to X such that $\gamma(a) = x$, $\gamma(b) = y$, and $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [a, b]$. A metric space (X, d) is geodesic if every pair of points in X can be joined by a geodesic. A geodesic triangle $\Delta(x, y, z)$ with $x, y, z \in X$ is defined on a geodesic metric space as the union $[x, y] \cup [x, z] \cup [y, z]$ of three geodesic segments connecting x, y, z. Let m_x be the point of the geodesic segment $[y, z]$ located at distance $\alpha_y = (x|z)_y$ from y. Then, m_x is located at distance $\alpha_z = (x|y)_z$ from z because $\alpha_y + \alpha_z = d(y, z)$. Analogously, define the points $m_y \in [x, z]$ and $m_z \in [x, y]$ both located at distance $\alpha_x = (y|z)_x$ from x; see Figure 2.3 for an illustration. There is a unique isometry φ which maps $\Delta(x, y, z)$ to a tripod $T(x, y, z)$ consisting of three solid segments $[x, m]$, $[y, m]$, and $[z, m]$ of lengths α_x, α_y, and α_z, respectively.

This function maps the vertices x, y, z of $\Delta(x, y, z)$ to the respective leaves of $T(x, y, z)$ and the points m_x, m_y, and m_z to the center m of $T(x, y, z)$. Any other point of $T(x, y, z)$ is the image of exactly two points of $\Delta(x, y, z)$. A geodesic triangle is called δ-thin if for all points $u, v \in \Delta(x, y, z)$, $\varphi(u) = \varphi(v)$ implies $d(u, v) \leq \delta$. A graph G is δ-thin if all geodesic triangles in G are δ-thin. The smallest value δ for which G is δ-thin is called the *thinness* of G and is denoted by $\tau(G)$.

The thinness and hyperbolicity of a graph are comparable as follows (similar inequalities are known for general geodesic metric spaces).

Proposition 1. [7,22,78,79] For a graph G, $\delta(G) \leq \tau(G) \leq 4\delta(G)$, and the inequalities are sharp.

2.3.3 Interval thinness

The diameter of a slice $S_k(x, y)$ is the maximum distance in G between any two vertices of $S_k(x, y)$. An interval $I(x, y)$ is said to be κ-thin if diameters of all slices $S_k(x, y)$, $k \in N$, of it are at most κ. A graph G is said to have κ-thin intervals if all intervals of G are κ-thin. The smallest κ for which all intervals of G are κ-thin is called the *interval thinness* of G and denoted by $\kappa(G)$.

That is,

$$\kappa(G) = \max\{d(u, v) : u, v \in S_k(x, y), x, y \in V, k \in N\}.$$

The following lemma is a folklore and easy to show using the definition of hyperbolicity. It shows that hyperbolic graphs have the so-called fellow travelers property: any two travelers following a shortest path from the same starting vertex to the same destination vertex at the same speed have always a bounded distance between them.

Lemma 9. For any graph G, $\kappa(G) \leq 2\delta(G)$.

Proof. Consider any interval $I(x, y)$ in G and arbitrary two vertices $u, v \in S_k(x, y)$. Consider the three distance sums $S_1 = d(x, y) + d(u, v)$, $S_2 = d(x, u) + d(y, v)$, $S_3 = d(x, v) + d(y, u)$. As $u, v \in S_k(x, y)$, we have $S_2 = S_3 = d(x, y) \leq S_1$. Hence, $2\delta(G) \geq S_1 - S_2 = d(x, y) + d(u, v) - d(x, y) = d(u, v)$ for any two vertices from the same slice of G, i.e., $2\delta(G) \geq \kappa(G)$. □

2.3.4 Gromov Product

The Gromov product of two vertices $x, y \in V$ with respect to a third vertex $z \in V$ is defined as $(x|y)_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y))$. Using the Gromov product, we can reformulate Lemma 2.

Since $d(x,y) = (x|z)_y + (z|y)_x$, it is easy to check that for any three vertices x, y, z of an arbitrary graph, either all products $(y|z)_x$, $(y|x)_z$, $(x|z)_y$ are integers or all are half-integers.

Lemma 10. For every three vertices x, y, z of a Helly graph G there exist three shortest paths $P(z,y)$, $P(x,z)$, $P(x,y)$ connecting them such that either (1) there is a common vertex v in $P(z,y) \cap P(x,z) \cap P(x,y)$ or (2) there is a triangle $\triangle(x', y', z')$ in G with edge $z'y'$ on $P(z,y)$, edge $x'z'$ on $P(x,z)$ and edge $x'y'$ on $P(x,y)$ (see Figure 2.2). Furthermore, (1) is true if and only if $(x|y)_z$ is an integer and $(x|y)_z = d(z,v)$, and (2) is true if and only if $(x|y)_z$ is a half-integer and $\lfloor (x|y)_z \rfloor = d(z,z')$.

Proof. Let $\alpha_z = (t|y)_z$. We have $d(z,x) - \alpha_z = d(z,x) - \frac{1}{2}(d(x,z) + d(y,z) - d(x,y)) = \frac{1}{2}(d(x,z) + d(x,y) - d(y,z)) = (z|y)_x$. Similarly, $(z|x)_y = d(z,y) - \alpha_z$. So, $\alpha_x := (z|y)_x$ and $\alpha_y := (z|x)_y$ are integers if and only if α_z is an integer.

Let α_z be an integer. Consider three disks $D(z,\alpha_z)$, $D(x,\alpha_x)$, $D(y,\alpha_y)$. As $(z|y)_x + (z|x)_y = d(x,y)$, they pairwise intersect. By the Helly property, there must exist a vertex v with $d(v,i) = \alpha_i$ for each $i \in \{x,y,z\}$. The converse follows from Lemma 2 for $p = \alpha_x$, $q = \alpha_y$ and $k = \alpha_z$.

Let α_z be a half integer. Consider the three disks $D(z, \lfloor \alpha_z \rfloor)$, $D(x, \lceil \alpha_x \rceil)$, $D(y, \lceil \alpha_y \rceil)$. They pairwise intersect. Hence, by the Helly property, there must exist a vertex z' with $d(z',z) = \lfloor \alpha_z \rfloor$, $d(z',x) = \lceil \alpha_x \rceil$, $d(z',y) = \lceil \alpha_y \rceil$. Considering now three disks $D(z',1)$, $D(x, \lfloor \alpha_x \rfloor)$, $D(y, \lceil \alpha_y \rceil)$, we get a vertex x' adjacent to z', at distance $\lfloor \alpha_x \rfloor$ from x and at distance $\lceil \alpha_y \rceil$ from y. Finally, pairwise intersecting disks $D(z',1), d(x',1), D(y, \lfloor \alpha_y \rfloor)$, guarantee the existence of a vertex y' which is adjacent to z', x' and at distance $\lfloor \alpha_y \rfloor$ from y. The converse follows from Lemma 2 for $p = \lfloor \alpha_x \rfloor$, $q = \lfloor \alpha_y \rfloor$ and $k = \lfloor \alpha_z \rfloor$. □

We will frequently use also the following lemma which is known from [79].

Lemma 11. [79] For every four vertices x, y, z, t of a graph G, there exist $\beta, \delta \in \{0, \frac{1}{2}, 1, \frac{3}{2}, ...\}$ such that

$$d(x,y) = (y|t)_x + \beta + (x|z)_y, \qquad d(t,z) = (x|z)_t + \beta + (y|t)_z,$$

$$d(y,z) = (x|z)_y + \delta + (t|y)_z, \qquad d(x,t) = (y|t)_x + \delta + (x|z)_t,$$

$$d(t,y) = (x|z)_t + \beta + \delta + (x|z)_y, \qquad d(x,z) = (y|t)_x + \beta + \delta + (y|t)_z,$$

and $hb(x, y, z, t) = min(\delta, \beta)$ (see Figure 2.4). Moreover, if $i((y|t)_x)$ indicates whether $(y|t)_x$ is an integer or a half-integer, then $i((y|t)_x) = i((y|t)_z))$ if and only if $i((x|z)_y) = i((x|z)_t)$.

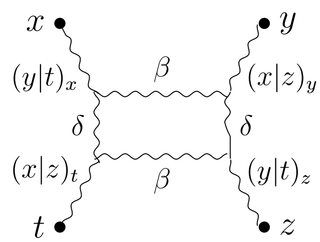


Figure 2.4: Distance realizations (see Lemma 11).

2.4 Injective hull

We say that a metric space (X, d) is *injective* (or *hyperconvex*) if any family of disks $D(x_i, r_i)$ with centers x_i and radii r_i, $i \in I$, satisfying $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$ has a nonempty intersection [11]. In a graph-theoretic sense, consider a graph G and let the metric space be the vertex set V equipped with the standard distance metric d_G. Then $(V(G), d_G)$ is injective when all disks of G satisfy the Helly property (i.e., when G is Helly).

We formulate the following works on injective hulls of metric spaces in the context of graphs. In the 1960s, Isbell [83] showed that for every graph G there exists a unique smallest Helly graph $\mathcal{H}(G)$ into which G isometrically embeds; $\mathcal{H}(G)$ is called the *injective hull* of G. The concept of a unique, smallest $\mathcal{H}(G)$ for any G was rediscovered by Dress two decades later [66] under the terminology of a *tight span*. Moreover, if G is δ-hyperbolic, then $\mathcal{H}(G)$ is also δ-hyperbolic [83, 95].

By an equivalent definition of an injective hull [66] (also called a tight span), each vertex $f \in V(\mathcal{H}(G))$ can be represented as a vector with nonnegative integer values $f(x)$ for each $x \in V(G)$, such that the following two properties hold:

$$\forall x, y \in V(G)\ f(x) + f(y) \geq d_G(x, y) \tag{2.1}$$

$$\forall x \in V(G)\ \exists y \in V(G)\ f(x) + f(y) = d_G(x, y) \tag{2.2}$$

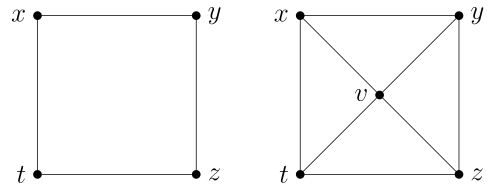
Additionally, there is an edge between two vertices $f, g \in V(\mathcal{H}(G))$ if and only if their Chebyshev distance is 1, i.e., $\max_{x \in V(G)} |f(x) - g(x)| = 1$. Thus, $d_H(f, g) = max_{x \in V(G)} |f(x) - g(x)|$. Notice that if $f \in V(\mathcal{H}(G))$, then $\{D(x, f(x)) : x \in V(G)\}$ is a family of pairwise intersecting disks. For a vertex $z \in V(G)$, define the distance function d_z by setting $d_z(x) = d_G(z, x)$ for any $x \in V(G)$. By the triangle inequality, each d_z belongs to $V(\mathcal{H}(G))$. An isometric embedding of G into $\mathcal{H}(G)$ is obtained by mapping each vertex z of G to its distance vector d_z.

We classify every vertex v in $V(\mathcal{H}(G))$ as either a real vertex or a Helly vertex. A vertex $f \in V(\mathcal{H}(G))$ is a *real vertex* provided $f = d_z$ for some $z \in V(G)$, i.e., there is a one-to-one correspondence between $z \in V(G)$ and its representative real vertex $f \in V(\mathcal{H}(G))$ which uniquely satisfies $f(z) = 0$ and $f(x) = d_G(z, x)$ for all $x \in V(G)$. When working with $\mathcal{H}(G)$, we will use interchangeably the notation $V(G)$ to represent the vertex set in G as well as the vertex subset of $\mathcal{H}(G)$ which uniquely corresponds to the vertex set of G. Then, a vertex $v \in V(\mathcal{H}(G))$ is a real vertex if it belongs to $V(G)$ and a *Helly vertex* otherwise. Equivalently, a vertex $h \in V(\mathcal{H}(G))$ is a Helly vertex provided that $h(x) \geq 1$ for all $x \in V(G)$, that is, a Helly vertex exists only in the injective hull $\mathcal{H}(G)$ and not in G. A path $P(x, y)$ in $\mathcal{H}(G)$ connecting vertices $x, y \in V(G)$ is said to be a *real path* if each vertex $u \in P(x, y)$ is real.

The distance condition (2.1) and the extremal condition (2.2) from the definition of an injective hull directly yield the following algorithmic construction of a graph G's injective hull. First, generate the finite set of n-tuples $S := \{0, 1, ..., diam(G)\}^n$, where $n = |V(G)|$. Next, remove from S any function f that does not satisfy the condition (2.1) or condition (2.2). The set of functions that remain form the vertex set of $\mathcal{H}(G)$. Note that the vertices of $\mathcal{H}(G)$ can have exponentially many vertices, i.e., $V(\mathcal{H}(G)) \in O((diam(G) + 1)^n)$.

We often use the terms *Hellify* (verb) and *Hellification* (noun) to describe the process by which edges and Helly vertices are added to G to construct $\mathcal{H}(G)$.

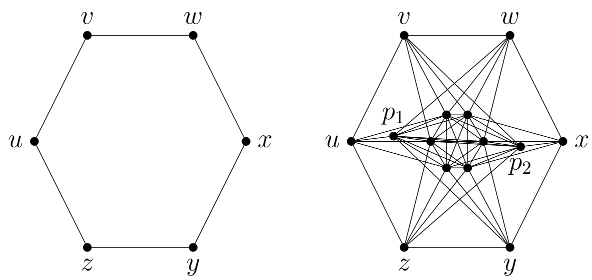
Example 2.4.1. Consider an example of Hellification on a small graph $G = C_4$ shown in Figure 2.5(a). First, we construct a finite function set $S = \{(0, 0, 0, 0), (0, 0, 0, 1), (0, 0, 0, 2), (0, 0, 1, 0),$ $..., (2, 2, 2, 1), (2, 2, 2, 2)\}$. Then, any functions that do not meet the inequality $f(x) + f(y) \geq d_G(x, y)$ and which are not extremal are removed from the set to form $V(\mathcal{H}(G)) = \{(0, 1, 2, 1),$ $(1, 0, 1, 2), (2, 1, 0, 1), (1, 2, 1, 0), (1, 1, 1, 1)\}$. Only one Helly vertex is created - $(1, 1, 1, 1)$. The other vertices correspond to the distance function for each vertex in V. Original edges are preserved, and new edges are added from $f = (1,1,1,1)$ to each $g \in V(\mathcal{H}(G)) \setminus \{f\}$ given that $||f-g||_\infty = 1$. The resulting graph $\mathcal{H}(G)$ is shown in Figure 2.5(b).



(a) (b)

Figure 2.5: A graph $G = C_4$ (a) and its injective hull $\mathcal{H}(G)$ (b).

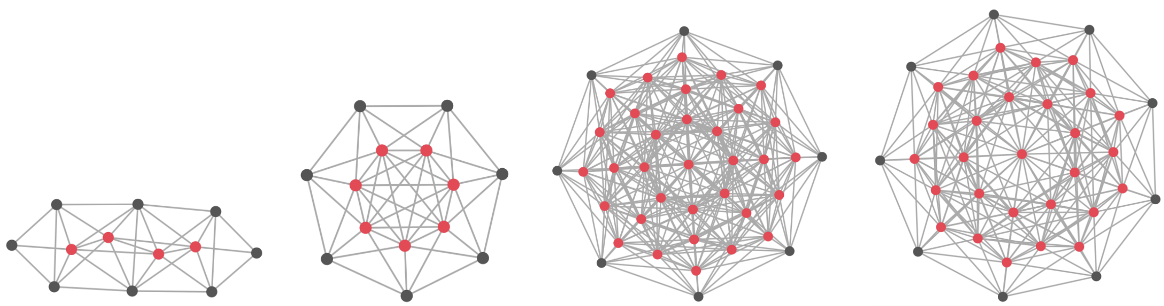
Example 2.4.2. Consider an example of Hellification on a small graph $G = C_6$ shown in Figure 2.6(a). First, we construct a finite function set $S = \{(0,0,0,0),\ (0,0,0,1),\ (0,0,0,2),\ (0,0,0,3),\ ...,\ (3,3,3,2),\ (3,3,3,3)\}$. Then, any functions that do not meet the inequality $f(x) + f(y) \geq d_G(x,y)$ and which are not extremal are removed from the set to form $V(\mathcal{H}(G)) = \{(0,1,1,2,2,3),\ (1,0,2,1,3,2),\ (1,1,1,2,2,2),\ (1,1,2,1,2,2),\ (1,2,0,3,1,2),\ (1,2,1,2,1,2),\ (1,2,2,1,1,2),\ (2,1,1,2,2,1),\ (2,1,2,1,2,1),\ (2,1,3,0,2,1),\ (2,2,1,2,1,1),\ (2,2,2,1,1,1),\ (2,3,1,2,0,1),\ (3,2,2,1,1,0)\}$. Note that the six functions which contain a zero correspond to the distance function for each vertex in $V(G)$. The remaining eight functions are Helly vertices in $\mathcal{H}(G)$. Original edges are preserved, and new edges are added from a vertex $f \in V(\mathcal{H}(G))$ to $g \in V(\mathcal{H}(G))$ if and only if $||f-g||_\infty = 1$. For example, $(2,2,1,2,1,1),(2,2,2,1,1,1)$ is an edge, while $(3,2,2,1,1,0),(1,1,1,2,2,2)$ is not an edge. The resulting graph $\mathcal{H}(G)$ is shown in Figure 2.6(b). We remark that the six vertices in the center form a clique and both p_1 and p_2 are adjacent to every vertex in the clique. Each vertex of the original C_6 is adjacent to either p_1 or p_2, and the vertices alternate between p_1 and p_2. This is to say that p_1 is adjacent to every other vertex in C_6, and p_2 is adjacent to the remaining vertices in C_6. Lastly, each vertex in C_6 is adjacent to three vertices in the center clique.



(a) (b)

Figure 2.6: A graph $G = C_6$ (a) and its injective hull $\mathcal{H}(G)$ (b).

Several other examples of the injective hull of various graphs are shown in Figure 2.7. Additional examples may be found at `https://algorithmic-lab.github.io/helly/`.



(a) (b) C_7 (c) C_8 (d) C_9

Figure 2.7: Examples of Hellification, where the injective hull of each graph is shown with $V(G)$ in black and Helly vertices in red.

2.5 Special graph classes

Definition 3 (Distance-hereditary graph [82])**.** A graph is *distance-hereditary* if every induced path is a shortest path.

The following propositions provide basic information on distance-hereditary graphs necessary for the next sections.

Proposition 2. [13, 46] For a graph G, the following conditions are equivalent:

(i) G is distance-hereditary;

(ii) The house, domino, gem, and the cycles C_k of length $k \geq 5$ are not induced subgraphs of G (see Figure 2.8);

(iii) For an arbitrary vertex x of G and every pair of vertices $v, u \in N^k(x)$, that are connected in the same component of the graph $\langle V \setminus N^{k-1}(x) \rangle$, we have $N(v) \cap N^{k-1}(x) = N(u) \cap N^{k-1}(x)$;

(iv) (4-point condition) For any four vertices u, v, w, x of G at least two of the following distance sums are equal: $d(u, v) + d(w, x)$, $d(u, w) + d(v, x)$, and $d(u, x) + d(w, v)$. If the smaller sums are equal, then the largest one exceeds the smaller ones by at most 2.

(v) G can be reduced to one vertex graph by a pruning sequence of one-vertex deletions: removing a pendant vertex or a single vertex from a pair of twin vertices.

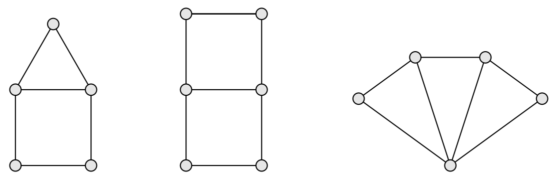


Figure 2.8: Forbidden induced subgraphs in a distance-hereditary graph.

Proposition 3. [52] Let G be a distance-hereditary graph with n-tuple $(r(v_1), ..., r(v_n))$ of non-negative integers and $M \subseteq V$. If every vertex pair $u, v \in M$ satisfies $d(u, v) \leq r(u) + r(v) + 1$ then M has an r-dominating clique C. If every vertex pair $u, v \in M$ satisfies $d(u, v) \leq r(u) + r(v)$ then there exists either a single vertex or a pair of adjacent vertices which r-dominates M.

Proposition 4. [52] For every vertex u of a distance-hereditary graph G, a furthest from u vertex $v \in F(u)$ satisfies $e(v) \geq 2rad(G) - 3$.

Proposition 5. Let G be a distance-hereditary graph and $x, y \in V$. Any vertex $v \in S_k(x, y)$ has $S_{k+1}(x, y) \subseteq N(v)$, i.e., neighboring interval slices are joined.

Proof. Consider a vertex $u \in S_{k+1}(x, y) \cap N(v)$ and any other vertex $w \in S_{k+1}(x, y)$. Then u and w are connected in $< V \setminus N^{k+1}(x) >$ via shortest paths $P(u, y)$ and $P(w, y)$. By Proposition 2(iii), they share neighboring vertices in $S_k(x, y)$. Hence, $w \in N(v)$. □

Pseudo-modular graphs are a far-reaching superclass of Helly graphs. By definition, a graph G is *pseudo-modular* if every triple x, y, z of its vertices admits either a 'median' vertex or a 'median' triangle, i.e., either there is a vertex v such that $d(x, y) = d(x, v) + d(v, y), d(x, z) = d(x, v) + d(v, z), d(z, y) = d(z, v) + d(v, y)$ or there is a triangle (three pairwise adjacent vertices) v, u, w such that $d(x,y) = d(x,v) + 1 + d(u,y), d(x,z) = d(x,v) + 1 + d(w,z), d(z,y) = d(z,w) + 1 + d(u,y)$. Pseudo-modular graphs are characterized as follows.

Proposition 6. [14] For a connected graph G the following are equivalent:

i) G is pseudo-modular.

ii) Any three pairwise intersecting disks of G have a nonempty intersection.

iii) If $1 \leq d(v,w) \leq 2$ and $d(u,v) = d(u,w) = k \geq 2$ for vertices u, v, w of G, then there exists a vertex x such that $d(v,x) = d(w,x) = 1$ and $d(u,x) = k-1$.

The presence of pseudo-modularity in a graph G is of algorithmic interest because it limits the number of disk families which must satisfy the Helly property for G to be considered Helly. Specifically, a pseudo-modular graph is Helly if and only if it is *neighborhood-Helly*, i.e., if the family of its all unit disks (all closed neighborhoods) $\{D(v,1) : v \in V(G)\}$ satisfies the Helly property.

Proposition 7. [15] G is Helly if and only if it is pseudo-modular and neighborhood-Helly.

It is clear that G is neighborhood-Helly if and only if all maximal 2-sets of G are suspended.

Definition 4 (k-Chordal graphs)**.** A graph is *k-chordal* provided it has no induced cycle of length greater than k. A graph is *chordal* if it is 3-chordal.

Definition 5 (Bridged graphs [72])**.** A graph is *bridged* if it contains no isometric cycles of length greater than 3.

As any isometric subgraph is an induced subgraph, bridged graphs form a superclass of chordal graphs.

Definition 6 (Tree decomposition [107])**.** A *tree-decomposition* $(\mathcal{T}, T)$ for a graph $G = (V, E)$ is a family $\mathcal{T} = \{B_1, B_2, ...\}$ of subsets of V, called *bags*, such that $\mathcal{T}$ forms a tree T with the bags in $\mathcal{T}$ as nodes which satisfy the following conditions:

(i) each vertex is contained in a bag,

(ii) for each edge $(u,v) \in E$, $\mathcal{T}$ has a bag B with $u, v \in B$, and

(iii) for each vertex $v \in V$, the bags containing v induce a subtree of T.

The width of a tree decomposition is the size of its largest bag minus one. A tree decomposition has breadth ρ if, for each bag B, there is a vertex v in G such that $B \subseteq D_G(v, \rho)$. A tree decomposition has length λ if the diameter in G of each bag B is at most λ. The *tree-width* $tw(G)$ [107], *tree-breadth* $tb(G)$ [62] and *tree-length* $tl(G)$ [48] are the minimum width, breadth, and length, respectively, among all possible tree decompositions of G. By definition, $tb(G) \leq tl(G) \leq 2tb(G)$, as for any graph G and any set $M \subseteq V(G)$, $rad_G(M) \leq diam_G(M) \leq 2rad_G(M)$ holds.

The definitions of graph classes not given here can be found in [21].

2.6 List of symbols

For easy reference, we provide below a list of frequently used symbols. Note that G is often omitted as a subindex when the graph is known by context.

Graph terminology and manipulations

$G = (V, E)$	a graph with vertex set V and edge set E
$V(G)$	vertex set of G
$E(G)$	edge set of G
$G - \{x\}$	an induced subgraph of G obtained by removing vertex $x \in V$
$\mathcal{H}(G)$	injective hull of graph G
$\langle S \rangle$	subgraph induced by $S \subseteq V$
G^k	kth power of G

Graph parameters

$\delta(G)$, $hb(G)$	hyperbolicity
$\tau(G)$	thinness of geodesic triangles
$\kappa(G)$	thinness of intervals
$\Delta(G)$	maximum vertex degree
$tw(G)$	tree-width
$tb(G)$	tree-breadth
$tl(G)$	tree-length
$\alpha(G)$	Helly-gap

Distance related notations

$d_G(u,v)$	distance from u to v
$d_G(u,S)$	smallest distance from u to a vertex of set S
$D_G(v,r)$	disk centered at vertex v
$D_G(S,r)$	disk centered at set S
$N(v)$	open neighborhood
$N[v]$	closed neighborhood
$N^k(v)$	kth neighborhood
$I(u,v)$	interval
$S_k(u,v)$	interval slice
$loc(v)$	locality of a vertex v
$(x\|y)_z$	Gromov product of x, y with respect to z
$hb(x,y,z,t)$	half the difference of the largest two pairwise distance sums among x, y, z, t

Notations used for eccentricity and centers

$F_G(v)$	set of vertices in G furthest from v
$F_G^M(v)$	set of vertices in $M \subseteq V$ furthest from v
$e_G(v)$	eccentricity of v
$e_G^M(v)$	eccentricity of v with respect to M
$rad(G)$	radius of G
$diam(G)$	diameter of G
$rad_G(M)$	radius of $M \subseteq V$
$diam_G(M)$	diameter of $M \subseteq V$
$C(G)$	center of G
$C_G(M)$	center of G with respect to $M \subseteq V$
$C^k(G)$	set of vertices with eccentricity at most $rad(G)+k$
$C_G^k(M)$	set of vertices with eccentricity with respect to $M \subseteq V$ at most $rad(G)+k$
$C^{=k}(G)$	set of vertices with eccentricity equal to $rad(G)+k$
$C_G^{=k}(M)$	set of vertices with eccentricity with respect to $M \subseteq V$ equal to $rad(G)+k$

$D(G)$	set of diametral vertices
$P(y, x)$	shortest path from y to x
$\mathbb{U}(P(y, x))$	number of up-edges on $P(y, x)$
$\mathbb{H}(P(y, x))$	number of horizontal-edges on $P(y, x)$
$\mathbb{D}(P(y, x))$	number of down-edges on $P(y, x)$

Common graphs

C_n	cycle of length n
S_n	complete n-sun graph
K_n	complete graph of n vertices
P_n	path graph of n vertices

Set notations

$\|S\|$	cardinality of the set S
$\cup$	set union
$\cap$	set intersection
$\setminus$	set subtraction
$\subseteq$	subset
$\subset$	proper subset

Math notations

$\mathbb{N}$	natural numbers $\{0, 1, 2, ...\}$
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\lfloor x \rfloor$	floor function
$\lceil x \rceil$	ceiling function
$\boxtimes$	strong product

Chapter 3

Hyperbolicity in Helly graphs*

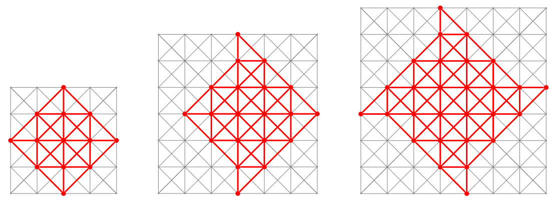
In this chapter, we are interested in understanding what structural properties of graphs govern their hyperbolicity and in identifying in graphs structural obstructions to a small hyperbolicity. It is a well-known fact that the tree-width of a graph G is always greater than or equal to the size of the largest square grid minor of G. Furthermore, in the other direction, the celebrated grid minor theorem by Robertson and Seymour [107] says that there exists a function f such that the tree-width is at most $f(r)$ where r is the size of the largest square grid minor. To the date the best bound on $f(r)$ is $O(r^{98+o(1)})$: every graph of tree-width larger than $f(r)$ contains an $(r \times r)$ grid as a minor [28]. We aim to prove similar "obstruction" results for the hyperbolicity parameter.

We show that the thinness of metric intervals governs the hyperbolicity of a Helly graph and that *three isometric subgraphs of the King-grid* are the only obstructions to a small hyperbolicity in Helly graphs. From our main results for Helly graphs, one can state the following.

Theorem 1. An arbitrary graph G has hyperbolicity at most δ if and only if its injective hull $\mathcal{H}(G)$ contains
- neither $H_1^{\delta+1}$ nor H_2^{δ}, when δ is an integer,
- neither $H_1^{\delta+\frac{1}{2}}$ nor $H_3^{\delta-\frac{1}{2}}$, when δ is a half-integer,
from Figure 3.1 as an isometric subgraph.

The injective hull here can be viewed as playing a similar role as the minors in the grid minor theorem for tree-width by Robertson and Seymour. Note that each of the graphs H_1^k, H_2^k, H_3^k contains a square grid of side k (see Figure 3.1) as an isometric subgraph.

*Results from this chapter have been published in *Discrete Mathematics* [58]



(a) H_1^k (b) H_2^k (c) H_3^k

Figure 3.1: Examples of H_1^k, H_2^k, and H_3^k where $k = 2$. Isometric embeddings of those graphs into the King-grid are shown.

Previously, it was known that the hyperbolicity of median graphs is controlled by the size of isometrically embedded square grids (see [11, 26]), and recently [26] showed that the hyperbolicity of weakly modular graphs (a far reaching superclass of the Helly graphs) is controlled by the sizes of metric triangles and isometric square grids: if G is a weakly modular graph in which any metric triangle is of side at most μ and any isometric square grid contained in G is of side at most ν, then G is $O(\nu + \mu)$-hyperbolic. Recall that three vertices x, y, z of a graph form a metric triangle if for each vertex $v \in \{x, y, z\}$, any two shortest paths connecting it with the two other vertices from $\{w, y, z\}$ have only v in common. Projecting this general result to Helly graphs (where $\mu \leq 1$) one gets only that every Helly graph with isometric grids of side at most ν is $O(\nu)$-hyperbolic with a constant larger than 1.

There has also been much related work on the characterization of δ-hyperbolic graphs via forbidden isometric subgraphs - particularly, when $\delta = \frac{1}{2}$. Koolen and Moulton [91] provide such a characterization for $\frac{1}{2}$-hyperbolic bridged graphs via six forbidden isometric subgraphs. Bandelt and Chepoi [10] generalize these results for $\frac{1}{2}$-hyperbolic graphs via the same forbidden isometric subgraphs and the property that all disks of G are convex. Additionally, Coudert and Ducoffe [41] prove that a graph is $\frac{1}{2}$-hyperbolic if and only if every graph power G^i is C_4 free for $i \geq 1$, and one additional graph is C_4 free. Brinkmann et al. [23] characterize $\frac{1}{2}$-hyperbolic chordal graphs via two forbidden isometric subgraphs. Wu and Zhang [114] prove that a 5-chordal graph is $\frac{1}{2}$-hyperbolic if and only if it does not contain six isometric subgraphs. Cohen et al. [39] prove that a biconnected outerplanar graph is $\frac{1}{2}$-hyperbolic if and only if either it is isomorphic to C_5 or it is chordal and does not contain a forbidden subgraph. We give a detailed characterization of $\frac{1}{2}$-hyperbolic Helly graphs, as well as a characterization of any δ-hyperbolic Helly graph via three forbidden isometric subgraphs.

3.1 Thinness of intervals governs the hyperbolicity of a Helly graph

We focus now on demonstrating that the converse of Lemma 9 for Helly graphs is also true, up to some small fraction. Note that, for general graphs G, the values of $\kappa(G)$ and $2\delta(G)$ can be very far from each other. Consider an odd cycle with $4k+1$ vertices; each pair of vertices has a unique shortest path, so no two vertices are in the same slice. Thus $\kappa(G) = 0$ and $2\delta(G) = 2k$.

We say that a graph $G' = (V', E')$ with $\{a, b, c, d\} \subset V'$ is an $\{a, b, c, d\}$-*distance-preserving subgraph* of a graph G if $d_G(x, y) = d_{G'}(x, y)$ for every pair of vertices x, y from $\{a, b, c, d\}$. On Figure 3.7(c), an $\{a, b, c, d\}$-distance-preserving subgraph of a graph G with $d_G(a, c) = d_G(b, d) = k+l+3$, $d_G(a, b) = d_G(c, d) = l + 2$, and $d_G(b, c) = d_G(d, a) = k + 2$ is shown.

Lemma 12. For every Helly graph G, $\delta(G) \leq \frac{\kappa(G)+1}{2}$. Furthermore, $\delta(G) = \frac{\kappa(G)+1}{2}$ if and only if $\kappa(G)$ is odd and there exists in G an $\{a, b, c, d\}$-distance-preserving subgraph depicted on Figure 3.7(c) with $k = l = \lfloor \frac{\kappa(G)}{2} \rfloor$.

Proof. Consider arbitrary four vertices a, b, c, d with $\delta(G) = hb(a, b, c, d) =: \delta$ and let $d_G(a, c) + d_G(b, d) \geq d_G(a, b) + d_G(c, d) \geq d_G(a, d) + d_G(b, c)$. Let also $\kappa := \kappa(G)$. Based on Lemma 2, there are three cases, up-to symmetry, to consider.

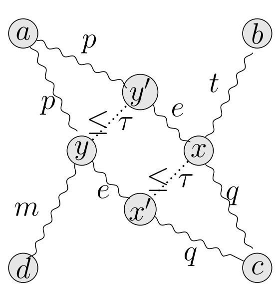


Figure 3.2: Illustration for Case 1 for the proof of Lemma 12.

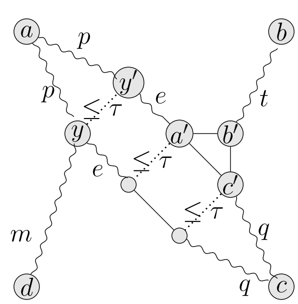


Figure 3.3: Illustration for Case 2 for the proof of Lemma 12.

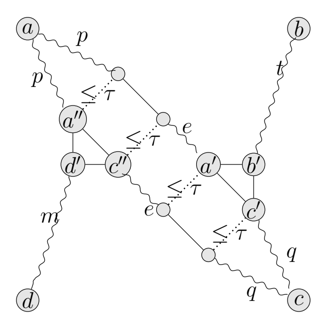


Figure 3.4: Illustration for Case 3 for the proof of Lemma 12.

Case 1. *For vertices a, b, c there are three shortest paths $P(a, b)$, $P(b, c)$, $P_b(a, c)$ that share a common vertex x. For vertices a, d, c there are three shortest paths $P(a, d)$, $P(d, c)$, $P_d(a, c)$ that share a common vertex y.*

This situation is shown on Figure 3.2. It is unknown if x and y are on the same slice of $I(a,c)$ or not, so we consider vertices $y' \in P_b(a,c)$ and $x' \in P_d(a,c)$ with $d_G(a,y) = d_G(a,y') =: p$ and $d_G(c,x) = d_G(c,x') =: q$. Set also $e := d_G(y',x) = d_G(y,x')$, $m := d_G(d,y)$, $t := d_G(b,x)$ (see Figure 3.2). Vertices x, x' lie on the same slice of $I(a,c)$, as do y, y'. Given that intervals of G are κ-thin, we get $2\delta = d_G(a,c)+d_G(b,d)-(d_G(a,b)+d_G(c,d)) \leq p+e+q+t+\kappa+e+m-(p+e+t+m+e+q) = \kappa$, i.e., $\delta \leq \frac{\kappa}{2}$.

Case 2. For vertices a,b,c there are three shortest paths $P(a,b)$, $P(b,c)$, $P_b(a,c)$ and a triangle $\triangle(b',c',a')$ in G with edge $a'b'$ on $P(a,b)$, edge $b'c'$ on $P(b,c)$ and edge $a'c'$ on $P_b(a,c)$. For vertices a,d,c there are three shortest paths $P(a,d)$, $P(d,c)$, $P_b(a,c)$ that share a common vertex y.

This situation is shown on Figure 3.3. Since intervals of G are κ-thin, we get $2\delta = d_G(a,c) + d_G(b,d) - (d_G(a,b)+d_G(c,d)) \leq p+e+1+q+t+1+\kappa+e+m-(p+e+1+t+m+e+1+q) = \kappa$, i.e., $\delta \leq \frac{\kappa}{2}$.

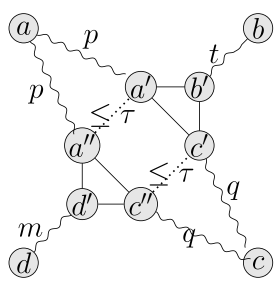


Figure 3.5: A special subcase of Case 3.

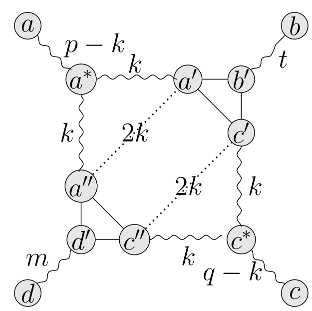


Figure 3.6: When $\kappa(G) = 2k$, $\delta(a,b,c,d) = k$.

Case 3. For vertices a,b,c there are three shortest paths $P(a,b)$, $P(b,c)$, $P_b(a,c)$ and a triangle $\triangle(b',c',a')$ in G with edge $a'b'$ on $P(a,b)$, edge $b'c'$ on $P(b,c)$ and edge $a'c'$ on $P_b(a,c)$. For vertices a,d,c there are three shortest paths $P(a,d)$, $P(d,c)$, $P_d(a,c)$ and a triangle $\triangle(d',c'',a'')$ in G with edge $a''d'$ on $P(a,d)$, edge $d'c''$ on $P(d,c)$ and edge $a''c''$ on $P_d(a,c)$.

This situation is shown on Figure 3.4. If vertices a', a'' are not in the same slice of $I(a,c)$ then, as before (see Case 1), it is easy to conclude that $\delta = \frac{d_G(a,c)+d_G(b,d)-(d_G(a,b)+d_G(c,d))}{2} \leq \frac{\kappa}{2}$.

If vertices a', a'' are in the same slice of $I(a,c)$ (see Figure 3.5 for this special subcase; only in this subcase we may have $\delta = \frac{\kappa+1}{2}$) then, using notations from Figure 3.5, $\delta = \frac{d_G(a,c)+d_G(b,d)-(d_G(a,b)+d_G(c,d))}{2} \leq \frac{p+1+q+t+1+\kappa+1+m-(p+1+t+m+1+q)}{2} = \frac{\kappa+1}{2}$. Furthermore, if $\delta = \frac{\kappa+1}{2}$, then $d_G(a', a'') = d_G(c', c'') = \kappa$.

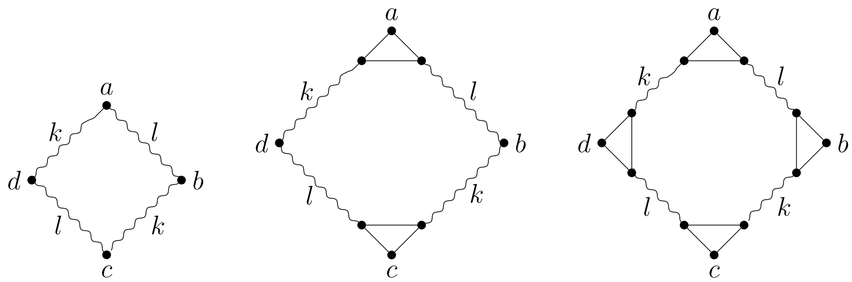
Assume that $\delta = \frac{\kappa+1}{2}$ and κ is even (see Figure 3.6). Let $\kappa = 2k$. Consider disks $D(a, p-k), D(a'', k), D(a', k)$ in G. These disks pairwise intersect. Hence, there must exist a vertex a^* at distance $p-k$ from a and at distance k from both a' and a''. Similarly, there is a vertex c^* in G at distance $q-k$ from c and at distance k from both c' and c''. These vertices a^* and c^* belong to slice $S_{t+1+k}(b, d)$ of $I(b, d)$. Hence, $d_G(a^*, c^*) \leq \kappa = 2k$ must hold. On the other hand, $p+1+q = d_G(a, c) \leq d_G(a, a^*) + d_G(a^*, c^*) + d_G(c^*, c) \leq p-k+2k+q-k = p+q$, a contradiction. Thus, when κ is even, δ must be equal to $\frac{\kappa}{2}$.

Assume now that $\delta = \frac{\kappa+1}{2}$ and κ is odd. Let $\kappa = 2k+1$. As $d_G(a', a'') = 2k+1$ and $d_G(a, a') = d_G(a, a'') = p$, by Lemma 2, there must exist three shortest paths $P(a, a')$, $P(a, a'')$, $P(a', a'')$ and a triangle $\triangle(x, y, z)$ in G with edge xy on $P(a, a')$, edge xz on $P(a, a'')$ and edge zy on $P(a', a'')$ (note that $P(a, a')$, $P(a, a'')$, $P(a', a'')$ cannot have a common vertex because of distance requirements). Similarly, there must exist three shortest paths $P(c, c')$, $P(c, c'')$, $P(c', c'')$ and a triangle $\triangle(u, v, w)$ in G with edge uv on $P(c, c')$, edge uw on $P(c, c'')$ and edge vw on $P(c', c'')$. Thus, by distance requirements, four triangles $\triangle(x, y, z)$, $\triangle(a', b', c')$, $\triangle(u, v, w)$, $\triangle(d', a'', c'')$ with corresponding shortest paths $P(y, a') \subseteq P(a, a')$, $P(a'', z) \subseteq P(a'', a)$, $P(c'', w) \subseteq P(c'', c)$, $P(c', v) \subseteq P(c', c)$ of length $k = \lfloor \frac{\kappa(G)}{2} \rfloor$ each form in G an $\{x, b', u, d'\}$-distance-preserving subgraph isomorphic to the one depicted on Figure 3.7(c) with $k = l$.

To complete the proof, it is enough to verify that if $\kappa(G)$ is odd and there exists in G an $\{a, b, c, d\}$-distance-preserving subgraph depicted on Figure 3.7(c) with $k = l = \lfloor \frac{\kappa(G)}{2} \rfloor$, then $hb(a, b, c, d) = \frac{\kappa(G)+1}{2}$.

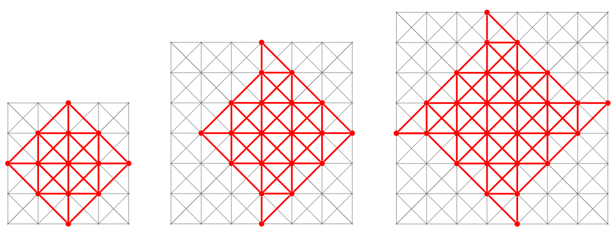
□

The following lemmas prove that the three $\{a, b, c, d\}$-distance preserving subgraphs shown in Figure 3.7 can be isometrically embedded into three Helly graphs termed $H_1^{k,l}$, $H_2^{k,l}$, and $H_3^{k,l}$, respectively. Each of $H_1^{k,l}$, $H_2^{k,l}$, and $H_3^{k,l}$ is an isometric subgraph of a King-grid (Figure 3.8 gives small examples for $k = l = 2$). We will show in Section 3.2 that any Helly graph G has an isometric H_1^k, H_2^k, or H_3^k, where k is a function of $\delta(G)$. These isometric subgraphs will be equally important as forbidden subgraphs for $hb(G) \leq \delta$ in Section 3.2. We provide the Hellification of all three graphs here for completeness, however, the remainder of this section will use only the graph in Figure 3.7(c) and its Hellification $H_3^{k,l}$ in order to refine result of Lemma 12 in the special case when $\delta(G) = \frac{\kappa(G)+1}{2}$.



(a) (b) (c)

Figure 3.7: $\{a, b, c, d\}$-distance preserving subgraphs.



(a) $H_1^{k,l}$ (b) $H_2^{k,l}$ (c) $H_3^{k,l}$

Figure 3.8: Examples of H_1^k, H_2^k, and H_3^k where $k = l = 2$, based on respective inputs from Figure 3.7. We omit the second superscript and use the notation H_i^k when $k = l$. Isometric embeddings of those graphs into the King-grid are shown.

Lemma 13. If a Helly graph G has an $\{a, b, c, d\}$-distance preserving subgraph depicted on Figure 3.7(a), then G has an isometric subgraph $H_1^{k,l}$ with a, b, c, d as corner points (see Figure 3.8(a)).

Proof. Let a, b, c, d be four vertices of a Helly graph G such that $d_G(a, c) = d_G(b, d) = k + l$ and $d_G(a, b) = d_G(c, d) = l$ and $d_G(b, c) = d_G(d, a) = k$. Let $P(d, a) = (d, v_1, v_2, \dots, v_k = a)$, $P(d, c) = (d, u_1, u_2, \dots, u_l = c)$ be two shortest paths connecting the appropriate vertices. Consider disks $D(v_1, 1)$, $D(u_1, 1)$, $D(b, k + l - 2)$ in G. These disks pairwise intersect. Hence, by the Helly property, there is a vertex d' which is adjacent to both v_1 and u_1 and at distance $k + l - 1$ from b. Since disks $D(a, 1), D(b, l - 1), D(d', k - 1)$ pairwise intersect, there must exist a vertex a' such that a' is adjacent to a and at distance $k - 1$ from d' and distance $l - 1$ from b. Similarly, considering pairwise intersecting disks $D(c, 1), D(b, k - 1), D(d', l - 1)$, there exists a vertex c' which is adjacent to c and at distance $l - 1$ from d' and distance $k - 1$ from b. For vertices a', b, c', d' we have $d_G(a', c') = d_G(b, d') = l + k - 2$ and $d_G(a', b) = d_G(c', d') = l - 1$ and $d_G(b, c') = d_G(d', a') = k - 1$.

Hence, by induction, we may assume that in G there is an isometric subgraph $H_1^{k-1,l-1}$ with a', b, c', d' as corner points. In what follows, using the Helly property, we extend this $H_1^{k-1,l-1}$ to isometric $H_1^{k,l}$ with a, b, c, d as corner points (see Figure 3.9 for an illustration).

Let $P(d', a') = (d' = v'_1, v'_2, \ldots, v'_k = a')$ be the shortest path of $H_1^{k-1,l-1}$ connecting d' with a'. For each edge $v'_i v'_{i+1}$ of this path, denote by w_i a vertex of $H_1^{k-1,l-1}$ which forms a triangle with $v'_i v'_{i+1}$. First we show that path $P(d, a) = (d, v_1, v_2, \ldots, v_k = a)$ can be chosen in such a way that $v_i v'_i$ is an edge of G for each i. Let $i \geq 2$ be the smallest index such that $v_i v'_i \notin E$. Consider pairwise intersecting disks $D(v_{i-1}, 1), D(v'_i, 1), D(a, d_G(a, v_i))$. By the Helly property, there is a vertex v^*_i in G which is adjacent to both v_{i-1} and v'_i and at distance $d_G(a, v_i)$ from a. Hence, we can replace part of $P(d, a)$ from v_i to a with a new shortest path from v^*_i to a. So, we can assume that $v_i v'_i \in E$ for each i. Since vertices $a, v'_k, w_{k-1}, v'_{k-1}, v_{k-1}$ are pairwise at distance at most 2, by the Helly property, there must exist a vertex w'_{k-1} which is adjacent to all $a, v'_k, w_{k-1}, v'_{k-1}, v_{k-1}$. Having vertex w'_{k-1}, we can use the Helly property to impose a new vertex w'_{k-2} adjacent to all $v_{k-1}, v'_{k-1}, w'_{k-1}, w_{k-2}, v'_{k-2}, v_{k-2}$. Continuing this way, we obtain a new vertex w'_i (for $i = k-3, k-4, \ldots, 1$) which is adjacent to all $v_{i+1}, v'_{i+1}, w'_{i+1}, w_i, v'_i, v_i$. This completes the addition to $H_1^{k-1,l-1}$ along the path $P(d, a) = (d, v_1, v_2, \ldots, v_k = a)$. Similarly, the addition along the path $P(d, c) = (d, u_1, u_2, \ldots, u_l = c)$ can be done completing the extension of $H_1^{k-1,l-1}$ to $H_1^{k,l}$ which is clearly an isometric subgraph of G. □

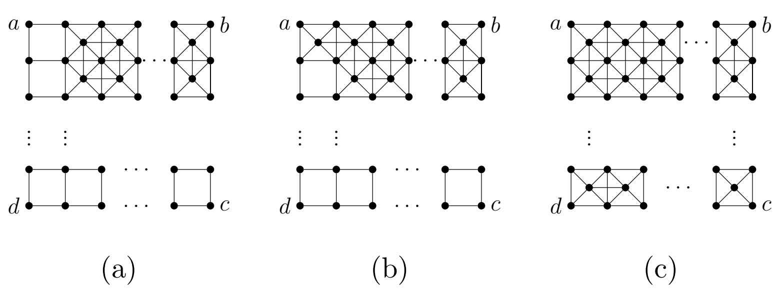


Figure 3.9: Proof of Lemma 13. Extension of an isometric subgraph $H_1^{k-1,l-1}$ to an isometric subgraph $H_1^{k,l}$.

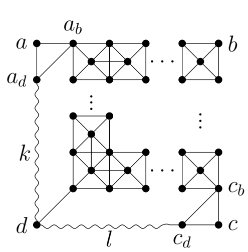
Lemma 14. If a Helly graph G has an $\{a, b, c, d\}$-distance preserving subgraph depicted on Figure 3.7(b), then G has an isometric subgraph $H_2^{k,l}$ with a, b, c, d as corner points (see Figure 3.8(b)).

Proof. Let a, b, c, d be vertices of a Helly graph G, with $\triangle(a, a_b, a_d)$ and $\triangle(c, c_b, c_d)$ such that $d_G(a, c) = k + l + 2 = d_G(a, a_b) + d_G(a_b, c_b) + d_G(c_b, c) = d_G(a, a_d) + d_G(a_d, c_d) + d_G(c_d, c)$, $d_G(b, d) =$

$k+l+1 = d_G(d, a_d) + 1 + d_G(a_b, b) = d_G(d, c_d) + 1 + d_G(c_b, b)$, and $d_G(b, c) = d_G(d, a) = k+1$ and $d_G(a, b) = d_G(c, d) = l+1$. Consider disks $D(a_b, k)$, $D(c_b, l)$, and $D(d, 1)$ in G. These disks pairwise intersect. Hence, by the Helly property, there is a vertex d' which is adjacent to d and at distance k from a_b and at distance l from c_b. For vertices a_b, b, c_b, d', we have $d_G(a_b, b) = d_G(c_b, d') = l$, $d_G(b, c_b) = d_G(d', a_b) = k$, and $d_G(a_b, c_b) = d_G(d', b) = k + l$. By Lemma 13, there is an isometric subgraph $H_1^{k,l}$ with a_b, b, c_b, d' as corner points (see Figure 3.10).

Let $P(a_b, d') = (a_b = v'_0, v'_1, v'_2, ..., v'_k = d')$ be the shortest path of $H_1^{k,l}$ connecting d' with a_b. For each edge $v'_i v'_{i+1}$ of this path, denote by w'_{i+1} a vertex of $H_1^{k,l}$ which forms a triangle with $v'_i v'_{i+1}$. Since vertices d, v'_k, v'_{k-1}, w'_k are pairwise distant at most 2 and distant from a at most $k+1$, by the Helly property there must exist a vertex v^*_k adjacent to d, v'_k, v'_{k-1}, w'_k and at distance k from a. Having vertex v^*_k, we can use the Helly property to impose a new vertex v^*_{k-1} which is adjacent to all $v^*_k, v'_{k-1}, v'_{k-2}, w'_{k-1}$ and at distance $k-1$ from a. Continuing this way, we obtain a new vertex v^*_i which is adjacent to all $v^*_{i+1}, v'_i, v'_{i-1}, w'_i$ and at distance i from a (for $i = k-2, k-3, ..., 1$). This completes the addition to $H_1^{k,l}$ along the path $P(a_b, d')$. Similarly, the addition along the path $P(c_b, d') = (c_b = u'_0, u'_1, u'_2, ..., u'_l = d')$ can be done. This completes the extension of $H_1^{k,l}$ to $H_2^{k,l}$.

Clearly, $H_2^{k,l}$ obtained from $H_1^{k,l}$ is an isometric subgraph of G. Recall that $H_2^{k,l}$ is a $\{a, b, c, d\}$-distance preserving subgraph of G. We know from Lemma 13 that $H_1^{k,l}$-part of $H_2^{k,l}$ is an isometric subgraph of G. We know also that every pair $x, y \in H_2^{k,l} \setminus H_1^{k,l}$ belongs to a shortest path of G from a to c passing through d. Finally, every pair x, y with $x \in H_2^{k,l} \setminus H_1^{k,l}$ and $y \in H_1^{k,l}$ belongs to a shortest path of G connecting a with c or b with d. □



(a)

Figure 3.10: Proof of Lemma 14. Using the Helly property, the graph from Figure 3.7(b) is shown to have $H_2^{k,l}$ as an isometric subgraph.

Lemma 15. If a Helly graph G has an $\{a, b, c, d\}$-distance preserving subgraph depicted on Figure 3.7(c), then G has an isometric subgraph $H_3^{k,l}$ with a, b, c, d as corner points (see Figure 3.8(c)).

Proof. Let a, b, c, d be vertices of a Helly graph G such that $d_G(a,c) = d_G(b,d) = l+k+3$ and $d_G(a,b) = d_G(c,d) = l+2$ and $d_G(b,c) = d_G(d,a) = k+2$ (see Figure 3.11(a)). Since $d(a_b, c_b) = k+l+1$, $d(a_b, d) = k+1+1$ and $d(c_b, d) = 1+l+1$, by Lemma 2, there is a triangle $\triangle(d', d'_a, d'_c)$ such that d is adjacent to d' and $d_G(d'_c, c_b) = l$, $d_G(d'_a, a_b) = k$. For vertices a_b, b, c_b, d', we have $d_G(a_b, b) = d_G(c_b, d') = l+1$, $d_G(b, c_b) = d_G(d', a_b) = k+1$, and $d_G(d', b) = k+l+2$, as well as $d_G(a_b, c_b) = k+l+1$. By Lemma 14, there is in G an isometric $H_2^{k,l}$ with a_b, b, c_b, d' as corner points (see Figure 3.11(b)).

Let $P(a_b, d') = (a_b = v'_0, v'_1, v'_2, ..., v'_k, d')$ be the shortest path of $H_2^{k,l}$ connecting d' with a_b, and let $P(c_b, d') = (c_b = u'_0, u'_1, u'_2, ..., u'_l, d')$ be the shortest path of $H_2^{k,l}$ connecting d' with c_b. For each edge $v'_i v'_{i+1}$, denote by w_{i+1} a vertex of $H_2^{k,l}$ which forms a triangle with $v'_i v'_{i+1}$. Since vertices d', d, v'_k, u'_l are pairwise at distance at most 2 and at distance at most $k+2$ from a, by the Helly property, there must exist a vertex v^*_{k+1} adjacent to d', d, v'_k, u'_l and at distance $k+1$ from a. Having vertex v^*_{k+1}, we can use the Helly property to impose a new vertex v^*_k which is adjacent to all $v^*_{k+1}, w_k, v'_k, v'_{k-1}$ and at distance k from a. Continuing this way, we obtain a new vertex v^*_i which is adjacent to all $v^*_{i+1}, w_i, v'_i, v'_{i-1}$ and at distance i from a (for $i = k-1, k-2, ..., 1$).

For each edge $u'_i u'_{i+1}$ denote by y_{i+1} a vertex of $H_2^{k,l}$ which forms a triangle with $u'_i u'_{i+1}$. Since vertices d, v^*_{k+1}, u'_l, v'_k are pairwise at distance at most 2 and at distance at most $l+2$ from c, by the Helly property, there must exist a vertex u^*_{l+1} adjacent to d, v^*_{k+1}, u'_l, v'_k and at distance $l+1$ from c. Having vertex u^*_{l+1}, we can use the Helly property to impose a new vertex u^*_l which is adjacent to all $u^*_{l+1}, u'_l, u'_{l-1}, y_l$ and at distance l from c. Continuing this way, we obtain a new vertex u^*_i which is adjacent to all $u^*_{i+1}, u'_i, u'_{i-1}, y_i$ and is at distance i from c (for $i = l-1, l-2, ..., 1$). This completes the extension of $H_2^{k,l}$ to $H_3^{k,l}$.

Clearly, $H_3^{k,l}$ obtained from $H_2^{k,l}$ is an isometric subgraph of G. Recall that $H_3^{k,l}$ is a $\{a,b,c,d\}$-distance preserving subgraph of G. We know from Lemma 14 that $H_2^{k,l}$-part of $H_3^{k,l}$ is an isometric subgraph of G. We know also that every pair $x, y \in H_3^{k,l} \setminus H_2^{k,l}$ belongs to a shortest path of G from a to d or from d to c or from a to c passing through a neighbor of d. Finally, every pair x, y with $x \in H_3^{k,l} \setminus H_2^{k,l}$ and $y \in H_2^{k,l}$ belongs to a shortest path of G connecting s with t where $s, t \in \{a,b,c,d\}$. □

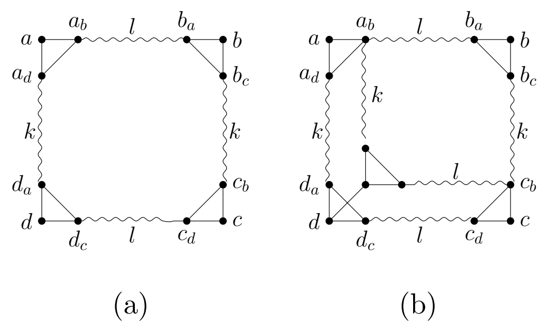


Figure 3.11: Proof of Lemma 15. Using the Helly property, the graph from Figure 3.7(c) is shown to have $H_3^{k,l}$ as an isometric subgraph.

Combining Lemma 9, Lemma 12 and Lemma 15, we conclude with a tight bound on hyperbolicity with respect to interval thinness in Helly graphs, as well as with a characterization of the case in which the hyperbolicity of a Helly graph realizes the upper bound.

Theorem 2. For every Helly graph G, $\frac{\kappa(G)}{2} \leq \delta(G) \leq \frac{\kappa(G)+1}{2}$. Furthermore, $\delta(G) = \frac{\kappa(G)+1}{2}$ if and only if $\kappa(G)$ is odd and G contains graph H_3^k with $k = \lfloor \frac{\kappa(G)}{2} \rfloor$ as an isometric subgraph.

Corollary 2. For every Helly graph G, if $\kappa(G)$ is even, then $\delta(G)$ is an integer and equal to $\frac{\kappa(G)}{2}$.

As stated previously, for general graphs, $\kappa(G) \leq 2\delta(G)$ and $\kappa(G)$ and $\delta(G)$ can be far apart. From Theorem 2 we know that in Helly graphs either $\kappa(G) = 2\delta(G)$ or $\kappa(G) = 2hb(G) + 1$. Recall that the injective hull $\mathcal{H}(G)$, the minimum Helly graph into which G isometrically embeds, preserves hyperbolicity. An interesting consequence of Theorem 2 is that interval thinness may increase from G to $\mathcal{H}(G)$. Therefore $\kappa(G) \leq \kappa(\mathcal{H}(G))$. This is a result of new shortest paths which are created in $\mathcal{H}(G)$.

3.2 Three isometric subgraphs of the King-grid are the only obstructions to a small hyperbolicity in Helly graphs

In this section, we will identify three isometric subgraphs of the King-grid that are responsible for the hyperbolicity of a Helly graph G.

These are named H_1^k, H_2^k, H_3^k, and are shown in Figure 3.8. We may assume that $\delta(G) > 0$ as the structure of any graph with hyperbolicity 0 is well-known; recall that they are exactly the block graphs, i.e., graphs where each biconnected component is a complete graph [79].

The following lemma shows the existence of one of the three isometric subgraphs in a Helly graph G with $\delta(G) = k > 0$.

Lemma 16. Let G be a Helly graph with $\delta(G) = k > 0$.
If $\kappa(G) = 2k$ and k is an integer, then G contains H_1^k as an isometric subgraph.
If $\kappa(G) = 2k$ and k is a half-integer, then G contains $H_2^{k-\frac{1}{2}}$ as an isometric subgraph.
If $\kappa(G) = 2k - 1$, then k is an integer and G contains H_3^{k-1} as an isometric subgraph.

Proof. Let $\delta(G) = k > 0$, and let interval $I(x, y)$ realize the maximum thinness, that is there are vertices $z, t \in S_\alpha(x, y)$, for some integer α, such that $d(z, t) = \kappa(G)$. By Theorem 2, either $\kappa(G) = 2k$ or $\kappa(G) = 2k - 1$. If $\kappa(G) = 2k - 1$, then by Theorem 2,$\kappa(G)$ is odd (thus k is an integer) and G contains H_3^{k-1} as an isometric subgraph. If $\kappa(G) = 2k$, then $\kappa(G)$ can be even or odd (since k can be a half-integer). Set $\alpha := d(x, t) = d(x, z)$, and $\beta := d(t, y) = d(z, y)$.

Let $\kappa(G) = 2k$ be even (thus k is an integer). Clearly $\alpha \geq k$ and $\beta \geq k$, otherwise $d(z, t) < 2k$. By Lemma 10, there is a vertex x' such that $d(x, x') = \alpha - k$, $d(z, x') = k$, and $d(t, x') = k$, and there is a vertex y' such that $d(y, y') = \beta - k$, $d(z, y') = k$, and $d(z, y') = k$. By the triangle inequality, $d(x', y') \leq d(x', z) + d(z, y') = 2k$ and $\alpha + \beta = d(x, y) \leq \alpha - k + d(x', y') + \beta - k \leq \alpha + \beta$. Therefore, $d(x', y') = 2k$ must hold. Then, by Lemma 13, G contains an isometric subgraph H_1^k with $\{x', z, y', t\}$ as corner points.

Let $\kappa(G) = 2k$ be odd (thus k is a half-integer). Let $k = p + \frac{1}{2}$ for an integer p. Then $d(z, t) = 2p + 1$. Clearly $\alpha > p$ and $\beta > p$, otherwise $d(z, t) < 2p + 1$. By Lemma 10, there is a triangle $\triangle(x', x_z, x_t)$ such that $d(x, x') = \alpha - p - 1$, $d(x_z, z) = p$, and $d(x_t, t) = p$, and there is a triangle $\triangle(y', y_z, y_t)$ such that $d(y, y') = \beta - p - 1$, $d(y_z, z) = p$, and $d(y_t, t) = p$. By the triangle inequality, $d(x', y') \leq d(x', x_z) + d(x_z, z) + d(z, y_z,) + d(y_z, y') = 2p + 2$ and $\alpha + \beta = d(x, y) \leq \alpha - p - 1 + d(x', y') + \beta - p - 1 = \alpha + \beta$. Therefore, $d(x', y') = 2p + 2$. Since $p = k - \frac{1}{2}$, by Lemma 14, G contains an isometric subgraph $H_2^{k-\frac{1}{2}}$ with $\{x', z, y', t\}$ as corner points. $\square$

Using the previous lemma, we can now characterize Helly graphs G with $\delta(G) \leq k$ based on three forbidden isometric subgraphs. Whether k is an integer or a half-integer determines which of the H_1^ℓ, H_2^ℓ, H_3^ℓ graphs are forbidden and the value of ℓ.

Theorem 3. Let G be a Helly graph and k be a non-negative integer.

- $\delta(G) \leq k$ if and only if G contains no H_2^k as an isometric subgraph.

- $\delta(G) \leq k + \frac{1}{2}$ if and only if G contains neither H_1^{k+1} nor H_3^k as an isometric subgraph.

Proof. Assume $\delta(G) \leq k$ and that G has H_2^k as an isometric subgraph. It is easy to check that $\delta(H_2^k) = k + \frac{1}{2} > k$ (the hyperbolicity realizes on four extreme vertices). As the hyperbolicity of a graph is at least the hyperbolicity of its isometric subgraph, $\delta(G) > k$, giving a contradiction.

Assume $\delta(G) \leq k + \frac{1}{2}$, and that G has H_1^{k+1} or H_3^k as an isometric subgraph. It is easy to check that $\delta(H_1^{k+1}) = k + 1 > k + \frac{1}{2}$ and $\delta(H_3^k) = k + 1 > k + \frac{1}{2}$ (the hyperbolicity of each realizes on four extreme vertices). As the hyperbolicity of a graph is at least the hyperbolicity of its isometric subgraph, $\delta(G) > k + \frac{1}{2}$, giving a contradiction.

For the other direction, let $\delta(G) =: \delta$. Then, by Lemma 16, G has one of H_1^δ, $H_2^{\delta - \frac{1}{2}}$, $H_3^{\delta-1}$ as an isometric subgraph. Note that, for any integer m, H_1^m is an isometric subgraph of H_2^m and H_1^{m+1}, H_2^m is an isometric subgraph of H_3^m, H_2^{m+1} and H_1^{m+1}, and H_3^m is an isometric subgraph of H_3^{m+1}. If δ is an integer, G contains H_1^δ or $H_3^{\delta-1}$, and hence H_1^{k+1} or H_3^k when $\delta > k + \frac{1}{2}$, as an isometric subgraph. If δ is a half-integer, G contains $H_2^{\delta-\frac{1}{2}}$, and hence H_2^k when $\delta \geq k + \frac{1}{2}$, as an isometric subgraph. $\square$

Theorem 3 can easily be applied to determine the forbidden subgraphs characterizing any δ-hyperbolic Helly graph. The corollaries that follow exemplify this for δ-hyperbolic graphs, where $\delta \in \{0, 1/2, 1, 3/2, 2\}$.

Corollary 3. A Helly graph is 0-hyperbolic if and only if it contains no H_2^0 as an isometric subgraph (as shown in Figure 3.12).

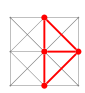


Figure 3.12: H_2^0 is a forbidden isometric subgraph for 0-hyperbolic Helly graphs.

Note that forbidding the graph shown in Figure 3.12 results in exactly block graphs; it has been shown that block graphs are 0-hyperbolic [13].

Corollary 4. A Helly graph is $\frac{1}{2}$-hyperbolic if and only if it contains neither H_1^1 nor H_3^0 as an isometric subgraph (see Figure 3.13).

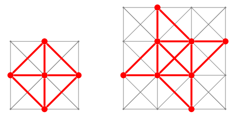


Figure 3.13: H_1^1 (left) and H_3^0 (right) are forbidden isometric subgraphs for $\frac{1}{2}$-hyperbolic Helly graphs.

Corollary 5. A Helly graph is 1-hyperbolic if and only if it contains no H_2^1 isometric subgraph (see Figure 3.14).

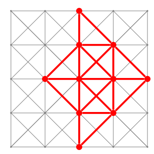


Figure 3.14: H_2^1 is a forbidden isometric subgraph for 1-hyperbolic Helly graphs.

Corollary 6. A Helly graph is $\frac{3}{2}$-hyperbolic if and only if it contains neither H_1^2 nor H_3^1 as an isometric subgraph (see Figure 3.15).

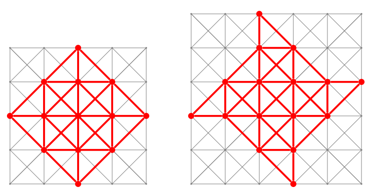


Figure 3.15: H_1^2 (left) and H_3^1 (right) are forbidden isometric subgraphs for $\frac{3}{2}$-hyperbolic Helly graphs.

Corollary 7. A Helly graph is 2-hyperbolic if and only if it contains no H_2^2 as an isometric subgraph (see Figure 3.16).

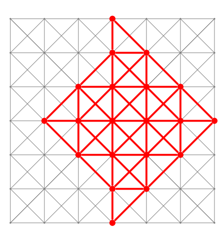


Figure 3.16: H_2^2 is a forbidden isometric subgraph for 2-hyperbolic Helly graphs.

To give a few equivalent characterizations of $\frac{1}{2}$-hyperbolic Helly graphs, we will need one more lemma. Let C_4 denote an induced cycle on four vertices. We say that a graph G is C_4-*free* if it does not contain C_4 as an induced subgraph. The graph H_3^0 is also known in the literature as the 4-sun S_4.

Lemma 17 ([51])**.** For any C_4-free Helly graph G, every C_4 in G^2 forms in G an isometric subgraph S_4.

Coudert and Ducoffe prove in [41] that a graph is $\frac{1}{2}$-hyperbolic if and only if every graph power G^i, $i \geq 1$, is C_4-free and one additional graph is C_4-free. By combining Theorem 2, Theorem 3 and Lemma 17, we have:

Corollary 8. The following statements are equivalent for any Helly graph G:

i) G is $\frac{1}{2}$-hyperbolic;

ii) G has neither C_4 nor S_4 as an isometric subgraph;

iii) Neither G nor G^2 has an induced C_4;

iv) $\kappa(G) \leq 1$ and G has no S_4 as an isometric subgraph.

The following lemmas describe the three forbidden isometric subgraphs in terms of graph powers.

Lemma 18. Let G be a Helly graph and k be a non-negative integer. Then G has H_1^{k+1} as an isometric subgraph if and only if there exist four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+1, 2k+1]$.

Proof. Suppose G has H_1^{k+1} as an isometric subgraph. Then, for four extreme vertices x, y, z, t of H_1^{k+1}, we have $d(x,y) = d(y,z) = d(z,t) = d(t,x) = k+1$ and $d(x,z) = d(y,t) = 2k+2$. Thus, x, y, z, t, form C_4 in G^ℓ for all $\ell \in [k+1, 2k+1]$.

Now, let x, y, z, t be four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+1, 2k+1]$. Then, each of $d(x,y)$, $d(y,z)$, $d(z,t)$, $d(t,x)$ is less than or equal to $k+1$, and $d(x,z)$, $d(y,t)$ are greater than or equal to $2k+2$. From these distance requirements, necessarily, $d(x,y) = d(y,z) = d(z,t) = d(t,x) = k+1$ and $d(x,z) = d(y,t) = 2k+2$. By Lemma 13, G has isometric H_1^{k+1}. □

A cycle on 4 vertices with one diagonal is called a *diamond*.

Lemma 19. Let G be a Helly graph and k be a non-negative integer. Then G has H_2^k as an isometric subgraph if and only if there exist four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+1, 2k]$ and form a diamond in G^{2k+1}.

Proof. Suppose G has H_2^k as an isometric subgraph. Then, for four extreme vertices x, y, z, t of H_2^k, we have $d(x,y) = d(y,z) = d(z,t) = d(t,x) = k+1$ and $d(x,z) = 2k+2$ and $d(y,t) = 2k+1$. Thus, x, y, z, t, form C_4 in G^ℓ for all $\ell \in [k+1, 2k]$ and form a diamond in G^{2k+1}.

Next, let x, y, z, t be four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+1, 2k]$ and form a diamond in G^{2k+1}. Then, each of $d(x,y)$, $d(y,z)$, $d(z,t)$, $d(t,x)$ is less than or equal to $k+1$. Without loss of generality, let yt be the chord of a diamond in G^{2k+1} formed by x, y, z, t. Thus, $d(y,t) \geq 2k+1$ and $d(x,z) \geq 2k+2$. From these distance requirements, necessarily, $d(x,y) = d(y,z) = d(z,t) = d(t,x) = k+1$, $2k+1 \leq d(y,t) \leq 2k+2$ and $d(x,z) = 2k+2$. If $d(y,t) = 2k+2$, then by Lemma 13, G has an isometric H_1^{k+1}, and hence an isometric H_2^k (note that H_1^{k+1} contains an isometric H_2^k). Let now $d(y,t) = 2k+1$. By Lemma 10, there exist shortest paths $P(x,y)$ and $P(x,t)$ such that the neighbors of x on those paths are adjacent. Similarly, there exist shortest paths $P(z,y)$ and $P(z,t)$ such that the neighbors of z on those paths are adjacent. Thus, x, y, z, t form $\{x,y,z,t\}$-distance preserving subgraph depicted on Figure 3.7(b). By Lemma 14, G has an isometric H_2^k. □

The following result generalizes Lemma 17.

Lemma 20. Let G be a Helly graph and k be a non-negative integer. Then G has H_1^{k+1} or H_3^k as an isometric subgraph if and only if there exist four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+1, 2k+1]$ or there exist four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+2, 2k+2]$.

Proof. By Lemma 18, G has H_1^{k+1} as an isometric subgraph if and only if there exist four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+1, 2k+1]$. Suppose G has H_3^k as an isometric subgraph. Then, for four extreme vertices x, y, z, t, we have $d(x,y) = d(y,z) = d(z,t) = d(t,x) = k+2$ and $d(x,z) = d(y,t) = 2k+3$. Thus, x, y, z, t form C_4 in G^ℓ for all $\ell \in [k+2, 2k+2]$.

Next, let x, y, z, t be four vertices in G that form C_4 in G^ℓ for all $\ell \in [k+2, 2k+2]$. Then, each of $d(x,y)$, $d(y,z)$, $d(z,t)$, and $d(t,x)$ is less than or equal to $k+2$, and each of $d(x,z)$ and $d(y,t)$ is greater than or equal to $2k+3$. Additionally, $d(x,y)$, $d(y,z)$, $d(z,t)$, and $d(t,x)$ must be greater than or equal to $k+1$, since otherwise $d(x,z) < 2k+3$ and $d(y,t) < 2k+3$. Thus, $2k+3 \leq d(x,z) \leq 2k+4$ and $2k+3 \leq d(y,t) \leq 2k+4$. We consider three cases.

In case 1, let $d(x,z) = d(y,t) = 2k+4$. Then, necessarily, $d(x,y) = d(y,z) = d(t,z) = d(t,x) = k+2$. By Lemma 13, G has an isometric H_1^{k+2}. Since H_1^{k+1} is an isometric subgraph of H_1^{k+2}, G

has an isometric H_1^{k+1}.

In case 2, let $d(x,z) = 2k+3$ and $d(y,t) = 2k+4$. Then $d(x,y) = d(y,z) = d(t,z) = d(t,x) = k+2$ (otherwise, $d(y,t) < 2k+4$). As in the proof of Lemma 19, we conclude that G has an isometric H_2^{k+1}. Thus, G has an isometric H_1^{k+1} (recall that H_2^{k+1} contains an isometric H_1^{k+1}).

In case 3, let $d(x,z) = d(y,t) = 2k+3$. First assume, without loss of generality, that $d(x,y) = k+1$. Then, necessarily, $d(x,t) = d(y,z) = k+2$. If also $d(t,z) = k+1$ then, by Lemma 13, G has an isometric $H_1^{k+1,k+2}$. Since H_1^{k+1} is an isometric subgraph of $H_1^{k+1,k+2}$, G has an isometric H_1^{k+1}. If now $d(t,z) = k+2$ then, by Lemma 10 applied to y,z,t, there exist shortest paths $P(z,y)$ and $P(z,t)$ such that the neighbors of z on those paths are adjacent. Let z' be the neighbor of z on $P(z,y)$. We have $d(x,t) = d(t,z') = k+2$, $d(x,y) = d(y,z') = k+1$ and hence $d(x,z') = 2k+2$ as $d(x,z) = 2k+3$. By Lemma 10 applied to x,z',t, there exists a vertex t' adjacent to t such that $d(t',x) = k+1$ and $d(t',z') = k+1$. Since $d(x,y) = d(x,t') = k+1$ and $d(y,t) = 2k+3$, necessarily, $d(y,t') = 2k+2$. By Lemma 13, G has an isometric H_1^{k+1} with x,y,z',t' as corner points.

To finish case 3, it remains to analyze the situation when $d(x,z) = d(y,t) = 2k+3$ and $d(x,y) = d(y,z) = d(z,t) = d(t,x) = k+2$. By Lemma 10 applied to x,z,t, there is a triangle $\triangle(t_x,t,t_z)$ such that t_x is the neighbor of t on a shortest (x,t)-path, and t_z is the neighbor of t on a shortest (z,t)-path. Similarly, by Lemma 10 applied to x,z,y, there is a triangle $\triangle(y_x,y,y_z)$ such that y_x is the neighbor of y on a shortest (x,y)-path, and y_z is the neighbor of y on a shortest (z,y) path. From the distance requirements, $2k+1 \le d(t_x,y_x) \le 2k+2$ and $2k+1 \le d(t_z,y_z) \le 2k+2$ (recall that $d(x,z) = d(y,t) = 2k+3$).

If $d(t_x,y_x) = 2k+2$ then, by Lemma 10 applied to y_x,z,t_x, there exists a vertex z' adjacent to z such that $d(z',y_x) = d(z',t_x) = k+1$. Necessarily, $d(x,z') = 2k+2$ as $d(x,z) = 2k+3$. Now, $d(x,y_x) = d(x,t_x) = d(z',y_x) = d(z',t_x) = k+1$ and $d(y_x,t_x) = d(x,z') = 2k+2$, and we can apply Lemma 13 and get in G an isometric H_1^{k+1} with x,y_x,z',t_x as corner points. Thus, we may assume that $d(t_x,y_x) = 2k+1$. Similarly, we may assume that $d(t_z,y_z) = 2k+1$.

By Lemma 10 applied to t_x,x,y_x, there exist shortest paths $P(y_x,x)$ and $P(t_x,x)$ such that the neighbors of x on those paths are adjacent. By Lemma 10 applied to y_z,z,t_z, there exist shortest paths $P(y_z,z)$ and $P(t_z,z)$ such that the neighbors of z on those paths are adjacent. Thus, we have constructed an $\{x,y,z,t\}$-distance preserving subgraph depicted on Figure 3.7(c). Hence, by Lemma 15, G has an isometric subgraph H_3^k. □

The following result reformulates Theorem 3 in terms of graph powers. It follows directly from Theorem 3, Lemma 19, and Lemma 20. It generalizes a result of Coudert and Ducoffe [41] from $\frac{1}{2}$-hyperbolic graphs to all δ-hyperbolic Helly graphs for all values of δ.

Theorem 4. Let G be a Helly graph and k be a non-negative integer.
- $\delta(G) \leq k$ if and only if there are no four vertices that form C_4 in G^ℓ for all $\ell \in [k+1, 2k]$ and form a diamond in G^{2k+1}.
- $\delta(G) \leq k + \frac{1}{2}$ if and only if there are no four vertices that form C_4 in G^ℓ for all $\ell \in [k+1, 2k+1]$, and there are no four vertices that form C_4 in G^ℓ for all $\ell \in [k+2, 2k+2]$.

Chapter 4

Eccentricity function in distance-hereditary graphs*

Recall that a graph is distance hereditary if every induced path is a shortest path [82]. In this chapter, we show that the eccentricity function in any distance-hereditary graph G is almost unimodal, that is, every vertex v with $e(v) > rad(G) + 1$ has a neighbor with smaller eccentricity. Moreover, we use this result to fully characterize the centers of distance-hereditary graphs. Several bounds on the eccentricity of a vertex with respect to its distance to the center of G or to the ends of a diametral path are explored. Finally, we introduce a new linear time algorithm to compute all eccentricities of a distance-hereditary graph.

The diameter and radius has been extensively studied in distance-hereditary graphs. A close relationship between the two was discovered in [52, 116], where it was shown that $diam(G) \geq 2rad(G) - 2$. It was shown in [52] that with two sweeps of a Breadth-First Search (BFS) one can obtain a value that is very close to the diameter. In fact, any vertex v that is furthest from an arbitrary vertex u has eccentricity $e(v) \geq diam(G) - 2$. Later, Feodor Dragan and Falk Nicolai [64] showed that by using instead LexBFS (Lexicographic Breadth-First Search) one can get a vertex v (last visited by a LexBFS starting at any vertex u) with $e(v) \geq diam(G) - 1$, and additionally if $e(v)$ is even, then $e(v)$ exactly realizes the diameter of G. This yielded a linear time algorithm to compute the diameter as well as a diametral pair of vertices [52, 64]. There is also a linear time algorithm to find a central vertex and calculate the radius [52]. These results were very recently generalized in [42]; it follows from [42] that all vertex eccentricities of a distance-hereditary graph G can be computed in total linear time via a split decomposition of G.

*Results from this chapter have been published in *Theoretical Computer Science* [59]

Here, we establish further properties of the eccentricity function in distance-hereditary graphs. The unimodality of the eccentricity function has been studied in a variety of graph classes; for example, it is exactly unimodal in Helly graphs [49] and almost unimodal in (α_1, Δ)-metric graphs [63] (that includes all chordal graphs) and in hyperbolic graphs [8]. In particular, it was shown [63] that a vertex v of a chordal graph G can have $loc(v) > 1$ only under very specific conditions: that $diam(G) = 2rad(G)$, that $e(v) = rad(G) + 1$, and that v is at distance 2 from a central vertex. We show in the main theorem of Section 4.1 that the same conditions hold for vertices of distance-hereditary graphs with locality larger than 1. This result, which is of independent interest, is a crucial intermediate step to establish the remaining results of this chapter.

The center $C(G)$ of a distance-hereditary graph G is also of interest. Many graph classes have a well defined center. The center of a tree is either K_1 or K_2 [86], the center of a maximal outerplanar graph is one of seven special graphs [104], and more generally all possible centers of 2-trees are known [103]. Graph centers have also been characterized fully for chordal graphs [30]. In distance-hereditary graphs it is known [116] that the diameter of the center is no more than 3. This was later improved by Hong-Gwa Yeh and Gerard Chang [115] that either $diam(C(G)) = 3$ and $C(G)$ is connected or $C(G)$ is a cograph (which may not be connected), i.e., a P_4-free graph. Furthermore, any cograph is the center of some distance-hereditary graph. We complete the characterization of centers of distance-herediatary graphs by investigating the instance that $C(G)$ is not a cograph (i.e., $diam(C(G)) = 3$), in which case $C(G)$ takes the form of a graph H which is further described in Section 4.4, and moreover each such H is the center of some distance-hereditary graph. Finally, we obtain several bounds on the eccentricity of an arbitrary vertex v with respect to its distance to a mutually distant pair and also its distance to the center $C(G)$. A simple dynamic programming algorithm is also presented which computes all eccentricities of a distance-hereditary graph in total linear time by utilizing a pruning sequence which is a characteristic property of distance-hereditary graphs.

Unless otherwise stated, the graph G is assumed to be distance-hereditary in all subsequent results of this chapter.

4.1 Unimodality of the eccentricity function

Recall that the eccentricity function is unimodal in G if every non-central vertex v of G has a neighbor u such that $e(u) < e(v)$. Our main result of this section is that in distance-hereditary graphs any vertex with sufficiently large eccentricity does have a neighbor with strictly smaller eccentricity. Unimodality can break only at vertices with eccentricity equal to $rad(G)+1$, but those vertices are close (within 2) to the center $C(G)$. Moreover, this can only occur when $diam(G) = 2rad(G)$. This property of the eccentricity function of distance-hereditary graphs aligns with known results for other graph classes, such as chordal graphs and the underlying graphs of 7-systolic complexes [63].

Our proof will be based on the following two lemmas.

Lemma 21. If a vertex x of G has $e(x) = rad(G) + 1$, then $d(x, C(G)) \leq 2$.

Proof. Let c be a central vertex closest to x. Consider any vertex $v \in S_1(c, x)$ and vertex $u \in F(v)$ furthest from v. As v is not central, $d(v, u) = rad(G) + 1$ and, by distance requirements, $d(c, u) = e(c) = rad(G)$. Hence, $u \in F(c) \cap F(v)$.

First we claim that $d(c, x) \leq 3$. Since $c \in I(v, u)$ and $v \in I(c, x)$, we have $d(u, v) + d(c, x) = d(u, c) + d(v, x) + 2$. Consider the 4-point condition on vertices u, x, v, c. As two distance sums must be equal, then either $d(u, x) + d(c, v) = d(u, v) + d(c, x)$ or $d(u, x) + d(c, v) = d(u, c) + d(v, x)$. We have $d(u, x) + d(c, v) \leq e(x) + 1 = rad(G) + 2$. In the first case we get $d(u, x) + d(c, v) = d(u, v) + d(c, x) = d(u, c) + d(v, x) + 2 = rad(G) + d(v, x) + 2$. Hence, $d(v, x) \leq 0$ and, by the triangle inequality, $d(x, C(G)) \leq 1$. In the second case we get $d(u, x) + d(c, v) = d(u, c) + d(v, x) = rad(G) + d(v, x)$. Hence, $d(v, x) \leq 2$ and, by the triangle inequality, $d(x, C(G)) \leq 3$, establishing the claim.

Assume now that $d(c, x) = 3$ and consider $y \in S_1(v, x)$. We next claim that $e(y) = rad(G) + 1$. By the choice of c, vertex y is non-central and so $e(y) \geq rad(G) + 1$. Since $y \in N(v) \cap N(x)$ with $e(v) = e(x) = rad(G) + 1$, by distance requirements, $e(y) \leq rad(G) + 2$. By way of contradiction assume that $e(y) = rad(G) + 2$. Consider a furthest vertex $y^* \in F(y)$. By distance requirements, $d(v, y^*) = d(x, y^*) = rad(G) + 1$ and $d(c, y^*) = rad(G)$. Since there is a (v, x)-path via vertex y in $\langle V \setminus N^{rad(G)}(y^*) \rangle$, by Proposition 2(iii), the neighbors of v and x in $N^{rad(G)}(y^*)$ are shared. Therefore, $cx \in E$, contradicting with $d(c, x) = 3$. Thus, $e(y) = rad(G) + 1$ must hold.

We now obtain a general contradiction in two steps. Recall that $e(y) = e(v) = e(x) = rad(G)+1$ and $e(c) = rad(G)$. First, consider the 4-point condition on vertices y, y^*, c, v. Consider three sums: $d(y, y^*) + d(c, v) = rad(G) + 2$, $d(y^*, c) + d(y, v) \le rad(G) + 1$, and $d(y^*, v) + d(c, y) \le rad(G) + 3$. Clearly, the first and the second sums are not equal. If the second and the third sums are equal, then $d(y^*, v) = d(y^*, c) + d(y, v) - d(c, y) \le rad(G) - 1$, contradicting with $d(y^*, y) = rad(G) + 1$. Therefore, the first and the third sums are equal. Then $d(y^*, v) = d(y, y^*) + d(c, v) - d(c, y) = rad(G)$. Let $P(y^*, v)$ be any shortest path between y^* and v. Its length is $rad(G)$. Consider also the path $Q = P(y^*, v), y, x$ (extension of $P(y^*, v)$ that includes also y and x). In distance-hereditary graphs, every induced path is a shortest path. As $d(x, y^*) \le e(x) = rad(G) + 1$, Q cannot be induced. As $d(y, y^*) = rad(G)+1$, vertex x must be adjacent to some vertex on $P(y^*, v)$. To avoid large induced cycles C_k of length $k \ge 5$, x must be adjacent to a vertex $z \in P(y^*, v)$ which is a neighbor of v. Thus, $d(y^*, x) = rad(G)$. Necessarily, $zc \notin E$ since $d(x, c) = 3$. We also have that $d(y^*, c) \le e(c) = rad(G)$.

Next, consider the 4-point condition on vertices y^*, c, x, v. We have $d(y^*, c)+d(x, v) \le rad(G)+2$, $d(y^*, v) + d(c, x) = rad(G) + 3$, and $d(y^*, x) + d(c, v) = rad(G) + 1$. Since at least two sums must be equal, necessarily $d(y^*, c) = d(y^*, x) + d(c, v) - d(x, v) = rad(G) - 1$. Now all distances from y^* are known. We have $d(c, y^*) = d(z, y^*) = rad(G) - 1$, $d(v, y^*) = d(x, y^*) = rad(G)$, and $d(y, y^*) = rad(G) + 1$. Since there is a (v, x)-path in $\langle V \setminus N^{rad(G)-1}(y^*)\rangle$, by Proposition 2(iii), the neighbors of v and x in $N^{rad(G)-1}(y^*)$ are shared. Therefore, vertices x and c must be adjacent, contradicting with $d(x, c) = 3$.

Obtained contradiction proves the lemma. □

Lemma 22. If there is a non-central vertex v of G such that each vertex $w \in N(v)$ has $e(w) \ge e(v)$, then $e(v) = rad(G) + 1$ and $diam(G) = 2rad(G)$.

Proof. Consider a vertex $u \in F(v)$, a vertex $x \in S_1(v, u)$ with minimal $|F(x)|$, and let $y \in F(x)$. By assumption, $e(x) \ge e(v)$ and v is non-central. If $d(v, y) = d(x, y) + 1$, then $e(v) \ge d(v, y) = d(x, y) + 1 = e(x) + 1 \ge e(v) + 1$, a contradiction. Thus, $d(x, y) - 1 \le d(v, y) \le d(x, y)$.

Consider the 4-point condition on vertices u, v, x, y. We have $d(u, v) + d(x, y) = e(v) + e(x) \ge 2rad(G)+2$, $d(u, y)+d(x, v) \le 2rad(G)+1$, and $d(u, x)+d(v, y) = e(v)-1+d(v, y) \le e(v)+e(x)-1$. Since two sums must be equal, necessarily $d(u, y)+d(x, v) = d(u, x)+d(v, y)$. Hence $2rad(G)+1 \ge$

$d(u, y) + d(x, v) = d(u, x) + d(v, y) \geq (e(v) - 1) + (e(x) - 1) \geq 2e(v) - 2 \geq 2rad(G)$. Therefore, $2rad(G) \geq d(u, y) \geq 2rad(G) - 1$ and $e(v) \leq rad(G) + 3/2$. Since eccentricity is an integer and v is non-central, $e(v) = rad(G) + 1$ must hold.

It remains only to show that $diam(G) = 2rad(G)$. If $d(u, y) = 2rad(G)$, we are done. So, assume that $d(u, y) = 2rad(G) - 1$. We get $d(y, v) = d(u, y) + d(x, v) - d(u, x) = rad(G)$, and so $e(x) = rad(G) + 1$. Furthermore, $v \in I(x, y)$. The length of path $Q = P(u, v) \cup P(v, y)$ (the concatenation of $P(u, v)$ with $P(v, y)$) is $2rad(G)+1$. As $d(u, y) = 2rad(G)-1$, Q is not an induced path. Hence, there are vertices $s \in P(v, u)$ and $w \in P(v, y)$ such that $sw \in E$. To avoid large induced cycles C_k of length $k \geq 5$, necessarily $s \in S_1(x, u)$ and $w \in S_1(v, y)$ must hold. Then, w belongs to $S_1(v, u)$ as well as x. Since $y \in F(x) \setminus F(w)$ (note that $e(w) \geq e(v) = rad(G) + 1$ by assumption), by minimality of $|F(x)|$, there is a vertex $t \in F(w) \setminus F(x)$. Hence, $d(t, x) < rad(G)+1$ and $d(t, w) = e(w) \geq rad(G) + 1$.

Now consider the 4-point condition on vertices x, y, w, t. We have $d(x, y) + d(w, t) = e(x) + e(w) \geq 2rad(G)+2$, whereas $d(t, y)+d(w, x) \leq 2rad(G)+2$ and $d(x, t)+d(w, y) \leq 2rad(G)-1$. As $d(x, y)+d(w, t)-d(x, t)-d(w, y) \geq 3$, then only the two largest sums are equal: $d(x, y)+d(w, t) = d(t, y) + d(w, x)$. Hence, $diam(G) \geq d(t, y) = d(x, y) + d(w, t) - d(w, x) \geq 2rad(G)$. That is, $diam(G) = d(t, y) = 2rad(G)$. □

We are ready to prove the main result of this section.

Theorem 5. Every vertex $v \notin C(G)$ either has an adjacent vertex w with $e(w) < e(v)$ or $e(v) = rad(G) + 1$, $diam(G) = 2rad(G)$, and $d(v, C(G)) = 2$.

Proof. If a non-central vertex v has no neighbors with smaller eccentricity then $d(v, C(G)) > 1$ and, by Lemma 22, $e(v) = rad(G)+1$ and $diam(G) = 2rad(G)$. Thus, by Lemma 21, $d(v, C(G)) = 2$. □

4.2 Certificates for eccentricities

We obtain as a consequence of Theorem 5 several new results for distance-hereditary graphs on lower and upper certificates for eccentricities, which were introduced in [61] as a way to compute exactly or approximately eccentricities in a graph by maintaining upper and lower bounds. A set L (set U) of vertices is a *lower certificate* (respectively, an *upper certificate*) for eccentricities of G

if it is used to obtain lower bounds (respectively, upper bounds) of eccentricities in G. Given all distances from a vertex v to all vertices in $L \cup U$ as well as the eccentricities of vertices in U, we have the following lower and upper bounds for the eccentricity of any vertex v [61]:

$$e_L(v) \leq e(v) \leq e^U(v), \text{ where } \begin{cases} e^U(v) = \min_{x \in U} d(v,x) + e(x), \\ e_L(v) = \max_{x \in L} d(v,x). \end{cases}$$

A lower certificate L (an upper certificate U) is said to be *tight* if $e_L(v) = e(v)$ ($e^U(v) = e(v)$, respectively) for all $v \in V$. A *diameter certificate* is a set U such that $e^U(v) \leq diam(G)$ for all $v \in V$, and therefore the diameter is realized by $\max_{v \in V} e^U(v)$. A *radius certificate* is a set L such that $e_L(v) \geq rad(G)$ for all $v \in V$, and therefore the radius is realized by $\min_{v \in V} e_L(v)$. Recall that the set of all diametral vertices of G is $D(G) = \{v \in V : e(v) = diam(G)\}$.

In this section we show that all eccentricities can exactly be determined in distance-hereditary graphs by computing distances from vertices of $C^1(G)$ to all vertices, since $C^1(G)$ forms a tight upper certificate. We also show that in distance-hereditary graphs the set $C(G)$ is a diameter certificate and the set $D(G)$ is a radius certificate (a kind of duality between $C(G)$ and $D(G)$). This agrees with radius and diameter certificates in chordal graphs [61] but, as we show later, this does not hold for arbitrary graphs.

We use the following corollary to Theorem 5.

Corollary 9. Let G be a distance-hereditary graph.

(i) If $diam(G) < 2rad(G)$ then, for every pair of vertices $v \in V$ and $u \in F(v)$, there is a vertex $w \in I(v,u) \cap C(G)$ such that $u \in F(w)$.

(ii) For every pair of vertices $v \in V \setminus C(G)$ and $u \in F(v)$, there is a vertex $w \in I(v,u) \cap C^1(G)$ such that $u \in F(w)$.

Proof. Consider any vertex $v \in V$ and $u \in F(v)$ and proceed by induction on $k := e(v)$. If $k = rad(G)$, then $w = v$ and we are done. If $k = rad(G) + 1$ and $diam(G) = 2rad(G)$ then again $w = v$ and we are done. If $k > rad(G) + 1$ or $k = rad(G) + 1$ and $diam(G) < 2rad(G)$ then, by Theorem 5, a neighbor z of v with $e(z) = k - 1$ satisfies $u \in F(z)$, and we can apply the induction hypothesis. □

Lemma 23. The set $C^1(G)$ is a tight upper certificate for all eccentricities of G.

Proof. The statement follows from Corollary 9 and the definition of a tight upper certificate. $\square$

Lemma 24. The center $C(G)$ is a diameter certificate of G.

Proof. This is clear by Corollary 9 if $diam(G) < 2rad(G)$. Additionally, in any graph G with $diam(G) = 2rad(G)$ all central vertices $c \in C(G)$ and every diametral pair of vertices x, y satisfy $d(x,y) = d(x,c) + d(c,y) = d(x,c) + rad(G) = rad(G) + d(y,c) = 2rad(G)$. $\square$

Lemma 25. The set $D(G)$ is a radius certificate of G.

Proof. We first show that $D(G)$ is a radius certificate for any graph G if $diam(G) \geq 2rad(G)-1$. If $D(G)$ is not a radius certificate, then there is a vertex $u \in V$ such that $max_{v \in D(G)} d(u,v) < rad(G)$. Thus, for any diametral pair x, y, $d(x,y) \leq d(x,u) + d(u,y) \leq (rad(G)-1) + (rad(G)-1) = 2rad(G)-2$, a contradiction with $d(x,y) = diam(G) \geq 2rad(G)-1$.

As in a distance-hereditary G, $diam(G) \geq 2rad(G)-2$ holds [52,116], it remains to consider only the case when $diam(G) = 2rad(G)-2$. Let S be the set of vertices u such that $d(u,t) \leq rad(G)-1$ for all $t \in D(G)$. By contradiction assume $D(G)$ is not a radius certificate and therefore S is not empty. Let $u \in S$ be a vertex which minimizes $|F(u)|$. Consider any diametral pair x, y and furthest from u vertex $v \in F(u)$. Necessarily $v \notin D(G)$ by the choice of u. Since $d(x,y) = 2rad(G)-2$, $d(u,x) \leq rad(G)-1$ and $d(u,y) \leq rad(G)-1$, clearly $d(u,x) = d(u,y) = rad(G)-1$.

Consider the 4-point condition on vertices v, u, x, y. We have that the largest distance sum is $d(v,u) + d(x,y) = d(v,u) + 2rad(G) - 2 \geq 3rad(G) - 2$, given that $d(v,x) + d(u,y) \leq d(x,y) - 1 + rad(G) - 1 = 3rad(G) - 4$ and that $d(v,y) + d(u,x) \leq d(x,y) - 1 + rad(G) - 1 = 3rad(G) - 4$. Therefore, the smaller sums are equal, establishing $d(v,x) = d(v,y)$. Moreover, since the difference between the largest sum and the other sums is at most 2, we get $d(v,u) = rad(G)$ and $d(v,x) = d(v,y) = 2rad(G) - 3$. So, $u \in C(G)$.

We claim that there is a vertex w such that $d(w,x) = rad(G)-1$, $d(w,y) = rad(G)-1$ and $v \notin F(w)$. Fix arbitrary shortest path $P(x,u)$, $P(y,u)$ and $P(u,v)$. Since $d(v,x) < d(v,u) + d(u,x) = diam(G) + 1$, path $Q = P(x,u) \cup P(u,v)$ is not induced. Hence, there must exist a chord between shortest path $P(x,u)$ and shortest path $P(u,v)$. Define vertices $t, w \in P(u,v)$, $s, z \in P(x,u)$, $q, p \in P(y,u)$, as shown in Figure 4.1. Since $d(v,x) = 2rad(G) - 3$, we must have the chord $zt \in E$ or the chord $sw \in E$. By the same argument, there must exist a chord between shortest path $P(y,u)$ and shortest path $P(u,v)$ which is realized by chord $pt \in E$ or $qw \in E$. We note that if $zt \in E$ then $pt \notin E$ since $d(x,y) = 2rad(G) - 2$. Up to symmetry, we have two cases as shown in Figure 4.1. In case (a) we have $zt, qw \in E$, and in case (b) we have $sw, qw \in E$. In either case vertex w satisfies the desired properties, establishing the claim.

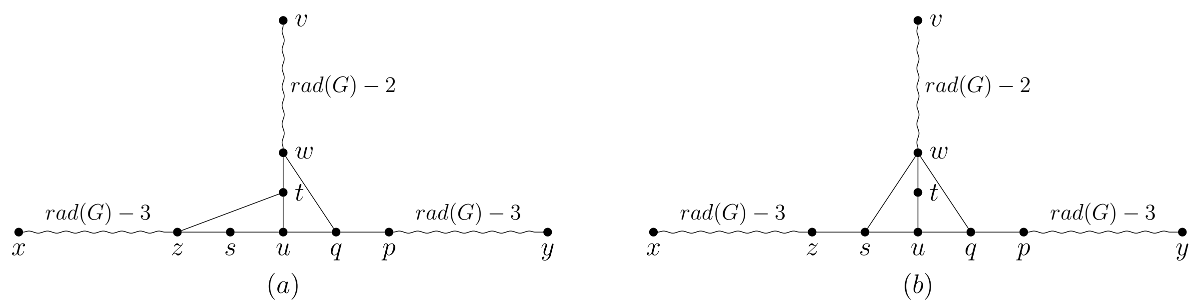


Figure 4.1: Illustration to the proof of Lemma 25.

We next claim that there is a vertex $\overline{w}$ such that $d(w,\overline{w}) \geq rad(G)$ and $d(u,\overline{w}) \leq rad(G) - 1$. On one hand, if $w \notin S$ then, by definition of S, there exists a vertex $\overline{w} \in D(G)$ such that $d(w,\overline{w}) \geq rad(G)$ and, by the choice of $u \in S$, we have $d(u,\overline{w}) \leq rad(G) - 1$. On the other hand, if $w \in S$ then, by minimality of $|F(u)|$ and since $v \notin F(w)$, there exists a vertex $\overline{w} \in F(w) \setminus F(u)$. As $\overline{w} \notin F(u)$, $d(u,\overline{w}) \leq rad(G) - 1$ and, as $\overline{w} \in F(w)$, $d(w,\overline{w}) \geq rad(G)$, establishing the claim.

Consider now the 4-point condition on vertices $v, u, w, \overline{w}$. Since $v \notin D(G)$ we have $d(v,\overline{w}) + d(w,u) \leq 2rad(G) - 3 + 2 = 2rad(G) - 1$. We also have $d(v,w) + d(\overline{w},u) \leq rad(G) - 2 + rad(G) - 1 = 2rad(G) - 3$ and $d(v,u) + d(w,\overline{w}) \geq rad(G) + rad(G) = 2rad(G)$. Given that $d(v,u) + d(w,\overline{w})$ is strictly larger than the other sums, it must differ from them by at most 2. However, it differs by at least 3, giving a contradiction. $\square$

As a consequence of these results, if the set $C^1(G)$ of a distance-hereditary graph G is known, then all vertex eccentricities in G can straightforwardly be found by performing a BFS from each vertex of $C^1(G)$. Similarly, if the set $C(G)$ ($D(G)$) is known, then the entire set $D(G)$ ($C(G)$, respectively) of G can be found. However, as we will discuss in Section 4.5, there is a more efficient approach to compute all eccentricities of a distance-hereditary graph.

We note that Lemma 24 and Lemma 25 do not hold for general graphs, as illustrated in Figure 4.2 by a graph G with $diam(G) = 6$ and $rad(G) = 4$. Here $D(G) = \{x, y\}$ and

$C(G) = \{u\}$, and all other vertices have eccentricity 5. However, $D(G)$ is not a radius certificate since $e_{D(G)}(u) = 3 < rad(G)$. Moreover, $C(G)$ is not a diameter certificate since $e^{C(G)}(v) = d(v,u) + e(u) = 8 > diam(G)$.

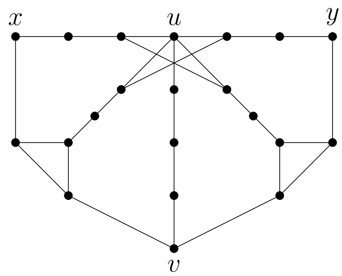


Figure 4.2: A (non-distance-hereditary) graph G where $D(G)$ is not a radius certificate and $C(G)$ is not a diameter certificate.

One aims also to minimize the size of a certificate. In trees, for example, a single diametral pair is a sufficient radius certificate rather than the full set of diametral vertices. Unfortunately, this is not true for distance-hereditary graphs. The graph G in Figure 4.3 illustrates that every diametral vertex is necessary to establish a radius certificate. Graph G consists of a clique of vertices $\{u_1, ..., u_\ell\}$ and a clique of vertices $\{v_1, ..., v_\ell\}$, where each u_i is adjacent to all vertices $v_{j \neq i}$, and each u_i and v_i has a pendant vertex x_i and y_i, respectively. G is distance-hereditary as it can be dismantled via a sequence of pendant and twin vertex eliminations. All vertices x_i and y_i are pendant, each u_i vertex is a false twin to v_i, and the remaining v_i vertices are true twins (as they form a clique in the remaining graph). Here $D(G)$ consists of all x_i and y_i vertices and $C(G)$ consists of all u_i and v_i vertices, where $diam(G) = d(x_i, y_i) = 4$ and $rad(G) = d(v_i, x_i) = d(u_i, y_i) = 3$. However, any $x_i \in D(G)$ has a vertex v_i such that $d(v_i, t) < rad(G)$ for all $t \in D(G) \setminus \{x_i\}$. By symmetry, the same is true for y_i and its counterpart u_i. Hence, all vertices of $D(G)$ are necessary to form a radius certificate. One can also show that all vertices of $C(G)$ are necessary to form a diameter certificate.

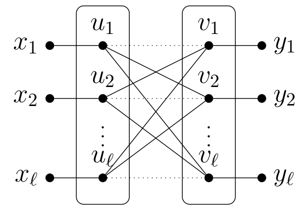


Figure 4.3: A distance-hereditary graph G for which $D(G) \setminus \{t\}$ is not a radius certificate for any $t \in D(G)$ and for which $C(G) \setminus \{c\}$ is not a diameter certificate for any $c \in C(G)$.

4.3 Eccentricities, mutually distant pairs and distances to the center

In this section, we show that the eccentricity of any vertex of a distance-hereditary graph is bounded by its distances to just two mutually distant vertices. Furthermore, the distance between any two mutually distant vertices is bounded by their distances to an arbitrary peripheral vertex (a vertex which is furthest for some other vertex). The unimodality behavior of the eccentricity function described in Theorem 5 gives also a relation between the eccentricity of a vertex and its distance to $C^1(G)$.

Lemma 26. Let x, y be a mutually distant pair of G, and let $u \in V$ and $v \in F(u)$ be a furthest vertex from u. Let also $\alpha := d(u,x)$ and $\beta := d(u,y)$. Then,

$$\max\{\alpha,\beta\} \leq e(u) \leq \max\{\max\{\alpha,\beta\},\ \min\{\alpha,\beta\}+2\} \leq \max\{\alpha,\beta\}+2.$$

Moreover, if $e(u) = \max\{\alpha,\beta\}+2$, then $\alpha = \beta = e(u)-2$ and $d(v,x) = d(v,y) = d(x,y)$.

Proof. Let $v \in F(u)$. By the choice of v, we have $e(u) = d(u,v) \geq \max\{d(u,x), d(u,y)\}$. Consider the 4-point condition on vertices u, v, x, y. As x,y is a mutually distant pair, we have for the three distance sums that $d(u,v)+d(x,y) = e(u)+d(x,y)$, $d(u,x)+d(v,y) \leq d(u,x)+d(x,y) \leq e(u)+d(x,y)$, and $d(u,y)+d(v,x) \leq d(u,y)+d(x,y) \leq e(u)+d(x,y)$. Clearly the first sum is largest.

We first consider the case when the first sum equals one of the latter. Suppose that $d(u,v)+d(x,y) = d(u,x)+d(v,y)$. Then, $e(u)+d(x,y) = d(u,x)+d(v,y) \leq e(u)+d(x,y)$. Hence,

$e(u) = d(u,x) = \max\{d(u,x), d(u,y)\}$. Suppose now that $d(u,v)+d(x,y) = d(u,y)+d(v,x)$. Then, $e(u)+d(x,y) = d(u,y)+d(v,x) \leq e(u)+d(x,y)$. Hence, $e(u) = d(u,y) = \max\{d(u,x), d(u,y)\}$. In either case, $e(u) = \max\{d(u,x), d(u,y)\}$.

We next consider the case when the two smaller sums are equal and differ from the largest one by at most 2. We have $e(u) = d(u,v) \leq d(v,y)+d(u,x)-d(x,y)+2 = d(v,x)+d(u,y)-d(x,y)+2$. Since $d(x,y)$ is not smaller than $d(v,y)$ and $d(v,x)$, we obtain $e(u) \leq d(u,x)+2$ and $e(u) \leq d(u,y)+2$, i.e., $e(u) \leq \min\{d(u,x), d(u,y)\}+2$. Moreover, if $e(u) = \max\{d(u,x), d(u,y)\}+2$, we must be in the latter case when the two smaller sums are equal (otherwise, $e(u) = \max\{d(u,x), d(u,y)\}$ as shown previously), and so $d(u,v) = e(u) = \min\{d(u,x), d(u,y)\}+2$. Hence, $d(u,x) = d(u,y) = d(u,v)-2$ and, since $d(u,x)+d(v,y) = d(u,y)+d(x,v)$, $d(v,y) = d(x,v)$ holds too. Combining this with the fact that $d(u,v)+d(x,y)-d(u,x)-d(y,v) \leq 2$, we obtain $d(x,y) \leq d(v,y)$ and, since x,y are mutually distant, necessarily $d(x,y) = d(v,y) = d(x,v)$, completing the proof. □

Lemma 27. Let x, y be a mutually distant pair of G, and let $u \in V$ and $v \in F(u)$ be a furthest vertex from u. Let also $\alpha \coloneqq d(v,x)$ and $\beta \coloneqq d(v,y)$. Then,

$$\max\{\alpha,\beta\} \leq d(x,y) \leq \max\{\max\{\alpha,\beta\},\ \min\{\alpha,\beta\}+2\} \leq \max\{\alpha,\beta\}+2.$$

Moreover, if $d(x,y) = \max\{\alpha,\beta\}+2$ then $\alpha = \beta = d(x,y)-2$ and $d(u,x) = d(u,y) = d(u,v)$.

Proof. The proof is analogous to that of Lemma 26 and is omitted. The only difference is that now we argue from the perspective of $d(x,y)$ and not of $e(u)$. □

Figure 4.4(a) illustrates that the upper bounds of Lemma 26 and Lemma 27 are sharp using vertices z and w for two opposing purposes. First, $e(z) = d(z,w) = \max\{d(x,z), d(y,z)\}+2$, whereas $w \in F(z)$ has $d(x,y) = \max\{d(x,w), d(y,w)\}$. Secondly, $e(w) = d(z,w) = \max\{d(x,w), d(y,w)\}$, whereas $z \in F(w)$ has $d(x,y) = \max\{d(x,z), d(y,z)\}+2$. Recall the implications of Lemma 26 and Lemma 27 which state that for a mutually distant pair x,y and fixed vertices $u \in V$ and $v \in F(u)$, if either $e(u)$ or $d(x,y)$ is realized by its upper bound as given in the above inequalities, then the other value is realized by its lower bound. So, it is not possible to obtain for the same $u \in V$ and $v \in F(u)$ that both $e(u) = \max\{d(x,u), d(y,u)\}+2$ and $d(x,y) = \max\{d(x,v), d(y,v)\}+2$ are true (Figure 4.4(a) uses a different starting vertex u to illustrate both upper bounds - with $u := z$

and then with $u := w$). However, we show in Figure 4.4(b) an example when for fixed vertices u and $v \in F(u)$, both $e(u) = \max\{d(x,u), d(y,u)\} + 1$ and $d(x,y) = \max\{d(x,v), d(y,v)\} + 1$ are true.

Figure 4.4: Illustration to the sharpness of Lemma 26 and Lemma 27.

In the case when x and y form a diametral pair, Lemma 27 yields a result known from [115]. We have $diam(G) = d(x,y) \leq \max\{d(v,x), d(v,y)\} + 2 \leq e(v) + 2$.

Corollary 10. [115] If vertex v of G is a furthest vertex from any $u \in V$, then $e(v) \geq diam(G) - 2$.

Corollary 10 can be used to find a pair of mutually distant vertices of a distance-hereditary graph in linear time. Pick an arbitrary start vertex v_0. With at most five BFSs find vertices $v_1, \ldots, v_k$ ($2 \leq k \leq 5$) such that $v_i \in F(v_{i-1})$ and $d(v_k, v_{k-1}) = d(v_{k-1}, v_{k-2})$. Since, by Corollary 10, $e(v_1) \geq diam(G) - 2$, there are at most two improvements on $e(v_1)$ to get a required mutually distant pair v_{k-1}, v_{k-2}.

Now, by Lemma 26, the obtained distances from v_{k-1} and v_{k-2} to all vertices $u \in V$ already yield good lower bounds for vertex eccentricities in G (they are within 2 from exact eccentricities). We next show that, in some cases, they are even closer.

Corollary 11. Let x, y be a mutually distant pair of G and let $u \in V$.

(i) If $|d(x,u) - d(y,u)| \geq 2$, then $e(u) = \max\{d(x,u), d(y,u)\}$.

(ii) If $|d(x,u) - d(y,u)| = 1$ or $u \in C(G)$, then $\max\{d(x,u), d(y,u)\} \leq e(u) \leq \max\{d(x,u), d(y,u)\} + 1$.

Proof. By Lemma 26, $\max\{d(x,u), d(y,u)\} \leq e(u) \leq \max\{\max\{d(x,u), d(y,u)\}, \min\{d(x,u), d(y,u)\} + 2\}$. If $|d(x,u) - d(y,u)| \geq 2$, then $\max\{d(x,u), d(y,u)\} \geq \min\{d(x,u), d(y,u)\} + 2$ and therefore $e(u) = \max\{d(x,u), d(y,u)\}$. If $|d(x,u) - d(y,u)| = 1$, then $e(u) \leq \max\{d(x,u), d(y,u)\} + 1$. By contradiction assume now that $u \in C(G)$, $d(x,u) = d(y,u)$ and $e(u) = d(x,u) + 2$.

By Proposition 4, we have $d(x,y) = e(x) \geq 2rad(G) - 3$. By the triangle inequality, $d(x,y) \leq d(x,u) + d(u,y) = 2(rad(G) - 2) = 2rad(G) - 4$, a contradiction. $\square$

In what follows, we analyze deeper the case when $d(u,x) = d(u,y)$. As the graph on Figure 4.4 showed, in this case, $e(u) = \max\{d(x,u), d(y,u)\} + 2$ may happen. However, we demonstrate that it happens not very often. First we show that if $d(x,y)$ is odd, then still $e(u) \leq \max\{d(x,u), d(y,u)\} + 1$. For this we will need one auxiliary lemma.

Lemma 28. Let u, x, y be vertices of G. If $d(u,x) = d(u,y)$ and $d(x,y) = 2k+1$ for some integer k, then all vertices $s \in S_k(x,y)$ satisfy $d(u,x) = d(u,s) + k$.

Proof. Let $s \in S_k(x,y)$ and $d(u,x) = d(u,y)$. Necessarily, $d(s,y) = k+1$. Consider the 4-point condition on vertices x, y, u, s. We have $d(u,s) + d(x,y) = d(u,s) + 2k + 1$, $d(u,x) + d(s,y) = d(u,x) + k + 1$, and $d(u,y) + d(x,s) = d(u,x) + k$. Since at least two sums must be equal and the latter two sums are not, we consider the two remaining cases. If $d(u,s) + d(x,y) = d(u,x) + d(s,y)$, then $d(u,x) = d(u,s) + (2k+1) - (k+1) = d(u,s) + k$, and we are done. If $d(u,s) + d(x,y) = d(u,y) + d(x,s)$, then $d(u,x) = d(u,y) = d(u,s) + (2k+1) - k = d(u,s) + k + 1$. However, by the triangle inequality, $d(u,x) \leq d(u,s) + d(s,x) = d(u,s) + k$, a contradiction. $\square$

Next lemma handles the case when $d(x,y)$ is odd.

Lemma 29. Let x, y be a mutually distant pair of G. If $d(x,y)$ is odd, then any vertex $u \in V$ has $e(u) \leq max\{d(x,u), d(y,u)\} + 1$.

Proof. Let $d(x,y) = 2k+1$ for some integer k. By contradiction assume that $e(u) = \max\{d(x,u), d(y,u)\} + 2$. Consider a vertex $v \in F(u)$. By Lemma 26, when $e(u) = \max\{d(x,u), d(y,u)\} + 2$, we have $d(u,x) = d(u,y) = e(u) - 2$, and $d(x,y) = d(x,v) = d(y,v)$. Let $s \in S_k(x,y)$. By Lemma 28 applied to vertex u and to vertex v, we have $d(u,x) = d(u,s) + k$ and $d(v,x) = d(v,s) + k$. By the triangle inequality, $e(u) = d(u,v) \leq d(u,s) + d(s,v) = (d(u,x) - k) + (d(v,x) - k) = d(u,x) + d(v,x) - 2k = d(u,x) + d(x,y) - 2k = d(u,x) + 1$, contradicting with $e(u) = d(u,x) + 2$. $\square$

The following theorem summarizes the obtained bounds on the eccentricity of any vertex $u \in V$. The distances from u to a selected pair of vertices is very close to the eccentricity of u, and in some cases, measures it exactly.

Theorem 6. Let x, y be mutually distant vertices of G. For every vertex $u \in V$, the following holds:

$$\max\{d(x,u), d(y,u)\} \leq e(u) \leq \max\{d(x,u), d(y,u)\} + \begin{cases} 0, & \text{if } |d(x,u) - d(y,u)| \geq 2, \\ 1, & \text{if } |d(x,u) - d(y,u)| = 1 \text{ or } d(x,y) \text{ is odd}, \\ 2, & \text{otherwise.} \end{cases}$$

As a consequence of Theorem 6, in distance-hereditary graphs, all eccentricities with an additive one-sided error of at most 2 can be computed in linear time. We remark that a 2-approximation of eccentricities is known for distance-hereditary graphs. One common approach to approximating eccentricities in a graph G is via an eccentricity k-approximating spanning tree T [36, 53, 63, 102], i.e., a spanning tree T of G such that $e_T(v) - e_G(v) \leq k$ holds for each vertex v of G. Note that every additive tree k-spanner (a spanning tree T of G such that $d_T(x,y) \leq d_G(x,y) + k$ holds for every vertex pair x, y) is eccentricity k-approximating. However, there are graph families which do not admit any additive tree k-spanners and yet they have very good eccentricity approximating spanning trees. The introduction of eccentricity approximating spanning trees is an attempt to weaken the restriction of additive tree spanners and instead closely approximate only distances to most distant vertices, the eccentricities. This is fruitful especially for those graphs for which additive tree k-spanners with small k do not exist. For example, for every k there is a chordal graph without an additive tree k-spanner, though every chordal graph has an eccentricity 2-approximating spanning tree [63, 102] computable in linear time [53]. More generally, all δ-hyperbolic graphs (note that chordal graphs are 1-hyperbolic) have an eccentricity $(4\delta + 1)$-approximating spanning tree [36, 60]. As distance-hereditary graphs are also 1-hyperbolic, that general result for δ-hyperbolic graphs implies that all distance-hereditary graphs have an eccentricity 5-approximating spanning tree. In fact, the situation with distance-hereditary graphs is even simpler. They have additive tree 2-spanners [101] (computable in linear time) and therefore eccentricity 2-approximating spanning trees. Furthermore, in general, the additive error 2 in an eccentricity approximating spanning tree cannot be improved. A distance-hereditary graph in Figure 4.5 has no eccentricity 1-approximating spanning tree. Consider any edge uv of the inner C_4 which is not present in T; either $e_T(u) = e_G(u) + 2$ or $e_T(v) = e_G(v) + 2$. So, another approach is needed to efficiently compute all eccentricities in a distance-hereditary graph. In Section 4.5, we present a new linear time algorithm for that which utilizes a characteristic pruning sequence.

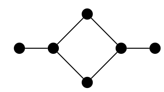


Figure 4.5: A distance-hereditary graph in which every spanning tree is eccentricity 2-approximating.

We turn now to a relation between the eccentricity of a vertex and its distance to $C(G)$ or $C^1(G)$.

Lemma 30. Let $v \in V$ be any vertex of an arbitrary graph G and let k be an integer. If $e(v) = d(v, C^k(G)) + rad(G) + k$, then $e(v) = d(v, C^{k+1}(G)) + rad(G) + k + 1$.

Proof. Suppose $e(v) = d(v, C^k(G)) + rad(G) + k$ and let $u \in F(v)$. Let $c \in C^k(G)$ be a closest vertex to v in $C^k(G)$. Consider an adjacent vertex $z \in S_1(c, v)$. By the choice of c, $e(z) = rad(G) + k + 1$ and therefore $z \in C^{k+1}(G)$. Then, $e(v) = d(v, c) + rad(G) + k = d(v, z) + d(z, c) + rad(G) + k \geq d(v, C^{k+1}(G)) + rad(G) + k + 1$. By the triangle inequality, also $e(c) \leq d(v, C^{k+1}(G)) + rad(G) + k + 1$. □

Lemma 31. Let G be a distance-hereditary graph.

(i) If $diam(G) < 2rad(G)$, then all vertices $v \in V$ satisfy $e(v) = d(v, C(G)) + rad(G)$.

(ii) All vertices $v \in V \setminus C(G)$ satisfy $e(v) = d(v, C^1(G)) + rad(G) + 1$.

Proof. The statements follow from Theorem 5 and Lemma 30 (see also Corollary 9). □

Lemma 32. Let v be an arbitrary vertex of G and v' be an arbitrary vertex of $C(G)$ closest to v. Then, $d(v, C(G)) + rad(G) - 1 \leq e(v) \leq d(v, C(G)) + rad(G)$.

Furthermore, all shortest paths $P := (v' = x_0, x_1, ..., x_\ell = v)$, connecting v with v', satisfy the following:

(a) if $e(v) = d(v, C(G)) + rad(G)$, then $e(x_i) = d(x_i, C(G)) + rad(G) = i + rad(G)$ for each $i \in \{0, ..., \ell\}$;

(b) if $e(v) = d(v, C(G)) + rad(G) - 1$, then $e(x_i) = d(x_i, C(G)) - 1 + rad(G) = i - 1 + rad(G)$ for each $i \in \{3, ..., \ell\}$ and $e(x_1) = e(x_2) = rad(G) + 1$.

Proof. Let $r := rad(G)$. By the triangle inequality, $e(v) \leq d(v, C(G)) + r$. Let $e(v) \geq r + 1$ and w be a vertex of $C^1(G)$ closest to v. By the triangle inequality and Lemma 21, $d(v, v') = d(v, C(G)) \leq d(v, w) + d(w, C(G)) \leq d(v, C^1(G)) + 2$. Combining with Lemma 31, one obtains $e(v) = d(v, C^1(G)) + r + 1 \geq d(v, C(G)) + r - 1$.

Consider an arbitrary shortest path $P := (v' = x_0, x_1, ..., x_\ell = v)$ connecting v with v'. As adjacent vertices can have eccentricities which differ by at most one, $e(v) = d(v, C(G)) + r$ implies $e(x_i) = d(x_i, C(G)) + r = i + r$ for each $i \in \{0, ..., \ell\}$. If $e(v) = d(v, C(G)) + r - 1$, there must exist an index $i \in \{1, ..., \ell - 1\}$ such that $e(x_i) = e(x_{i+1}) = r + i$ and $e(x_j) = r + j$ for $j \leq i$. We claim that $i = 1$ must hold.

We first establish that $i \leq 2$. By Theorem 5, there is a shortest path $Q^* := (w = u_2, ..., u_\rho = v)$ connecting v to a vertex $w \in C^1(G)$ closest to v, where $d(w, C(G)) \leq 2$ and $e(u_k) = r + k - 1$ for all $k \in 2, ..., \rho$. Let c be a vertex of $C(G)$ closest to w. We have that $d(v, C(G)) \leq d(v, c) \leq d(v, w) + d(w, c) \leq e(v) - e(w) + 2 = (d(v, C(G)) + r - 1) - (r + 1) + 2 = d(v, C(G))$. Therefore, vertex c is also a closest to v central vertex, $d(w, C(G)) = 2$, and $\rho = \ell$. Let now $Q := (c = u_0, u_1, u_2, ..., u_\ell = v)$ be a shortest path which joins shortest paths Q^* and any shortest path connecting u_2 to c. As c is a closest to v central vertex, necessarily $e(u_1) = e(u_2) = r + 1$. We claim that central vertices u_0 and x_0 are connected in $\langle V \setminus N^{d(v,C(G))-1}(v)\rangle$ by respective paths to a vertex $t \in F(v)$. Let $u^* \in I(u_0, t)$, $u^* \neq u_0$. We have $d(u_0, t) \geq d(u^*, t) + 1$. Assume that $d(u^*, v) \leq d(v, C(G)) - 1$. Then, $e(v) = d(v, t) \leq d(v, u^*) + d(u^*, t) \leq (d(v, C(G)) - 1) + (d(u_0, t) - 1) \leq d(v, C(G)) + r - 2$, a contradiction with $e(v) = d(v, C(G)) + r - 1$. Therefore, any vertex u^* on a shortest (u_0, t)-path satisfies $d(v, u^*) \geq d(v, C(G))$ and, by symmetry, any vertex x^* on a shortest (x_0, t)-path also satisfies $d(v, x^*) \geq d(v, C(G))$. The combined paths connect vertices u_0 and x_0 in $< V \setminus N^{d(v,C(G))-1}(v) >$. As a result, for each $k \in \{0, ..., \ell - 1\}$, vertices u_k and x_k are connected in $\langle V \setminus N^{d(v,C(G))-k-1}(v)\rangle$. Then, by Proposition 2(iii), $u_k x_{k+1} \in E$ and $u_{k+1} x_k \in E$. In particular, $u_2 x_3 \in E$, $u_2 x_1 \in E$, $u_1 x_2 \in E$, and $x_0 u_1 \in E$. Recall that $e(u_2) = r + 1$. If $i \geq 3$, then $e(x_3) = r + 3$; however, $e(x_3) \leq 1 + e(u_2) = r + 2$, a contradiction.

Assume now that $i = 2$. Hence, $e(x_2) = e(x_3) = r + 2$. Recall that $e(u_2) = e(u_1) = e(x_1) = r + 1$ and $e(x_0) = r$. Let $z \in F(x_2)$; then z is also furthest from vertices u_1, x_1, and x_0. Denote by Z a shortest path connecting x_0 and z, and let $z_0 \in Z$ be the vertex adjacent to x_0. By distance requirements from each vertex x_2, x_1, and u_1 to furthest vertex z, necessarily $z_0 x_2 \notin E$, $z_0 x_1 \notin E$, and $u_1z_0 \notin E$. Since $d(x_3, x_0) = 3$, $z_0x_3 \notin E$. As $d(u_2, z) \leq e(u_2) = r + 1$, the path obtained by joining shortest paths Z and (x_0, u_1, u_2) by their common end-vertex x_0 is not induced. The only possible chord is u_2z_0. As u_2 and x_2 are connected in $\langle V \setminus N(x_0)\rangle$, by Proposition 2(iii), $z_0x_2 \in E$, a contradiction. □

Lemma 31 and Lemma 32 establish a close relationship between the eccentricity of a vertex and its distance to a closest vertex of $C(G)$ or of $C^1(G)$. We now use the following section to fully describe the structure of centers of distance-hereditary graphs.

4.4 Centers of distance-hereditary graphs

In this section, we investigate the structure of centers of distance-hereditary graphs and provide their full characterization. A subset $S \subseteq V$ is called *m^3-convex* if and only if S contains every induced path of length at least three between vertices of S. It is known from [65] that the centers of HHD-free graphs are m^3-convex. As every distance-hereditary graph is HHD-free, the centers of distance-hereditary graphs are m^3-convex, too. It is also known from [115] that $C(G)$ is either a cograph or a connected graph with $diam(C(G)) = 3$. As every connected subgraph of a distance-hereditary graph is isometric, when $diam(C(G)) = 3$, $C(G)$ is isometric and a distance-hereditary graph. We remark that if the diameter of a set S in a distance-hereditary graph is no more than 2 then, by definition, it induces a cograph (or, equivalently, a distance-hereditary graph of diameter at most 2).

To prove our main result of this section, we will need the following auxiliary lemmas.

Lemma 33. Let x, y be a diametral pair of G, $diam(G) = 2rad(G)$, and $S \coloneqq S_{rad(G)}(x, y)$. Then, S and $C(G)$ are cographs with $C(G) \subseteq S$, any vertex of $S_{rad(G)+1}(x, y) \cup S_{rad(G)-1}(x, y)$ is universal to S, and $C^1(G) \subseteq D(S, 1)$.

Proof. Any central vertex $c \in C(G)$ has $d(x, c) \leq rad(G)$ and $d(y, c) \leq rad(G)$, therefore, by distance requirements, $C(G) \subseteq S$. By Proposition 5, any vertex $w \in S_{rad(G)+1}(x, y) \cup S_{rad(G)-1}(x, y)$ is universal to slice S and therefore universal to $C(G)$. Moreover, since the diameters of S and $C(G)$ are no more than 2, both are cographs.

Let now $c \in C^1(G)$ and by contradiction assume $c \notin D(S, 1)$. Necessarily $d(x, c) \leq rad(G) + 1$ and $d(y, c) \leq rad(G) + 1$. If $d(x, c) < rad(G)$ then, by distance requirements, $d(y, c) = rad(G) + 1$ and $c \in S_{rad(G)-1}(x, y)$. By Proposition 5, $c \in D(S, 1)$, a contradiction. If $d(x, c) = d(y, c) = rad(G)$ then $c \in S$, a contradiction. Hence, we can assume, without loss of generality, that $d(x, c) = rad(G)+1$. Consider vertex $u \neq c$ on a shortest path $P(c, y)$ closest to x, and let $b \in S_{rad(G)+1}(x, y)$. Then, $d(y, u) \leq rad(G)$ as $d(y, c) \leq rad(G) + 1$. If $d(x, u) \geq rad(G) + 1$, then vertices b and c are connected in $\langle V \setminus N^{rad(G)}(x)\rangle$ and therefore, by Proposition 2 (iii), share common neighbors in S. So, $c \in D(S, 1)$, a contradiction. If $d(x, u) \leq rad(G)$, then $d(x, u) = d(y, u) = d(y, c) - 1 = rad(G)$ as $d(x, y) = 2rad(G)$. Hence, $u \in S \cap N(c)$, a contradiction. $\square$

Lemma 34. Let x, y be a diametral pair of G, $diam(G) = 2rad(G)-1$, and let $A \coloneqq S_{rad(G)-1}(x, y)$ and $B \coloneqq S_{rad(G)-1}(y, x)$. Then, $C(G)$ is a cograph and any edge $ab \in E$, where $a \in A$ and $b \in B$, satisfies $C(G) \subseteq D(\{a, b\}, 1)$. Moreover, there is a vertex $a \in A \cap C(G)$ and a vertex $b \in B \cap C(G)$.

Proof. By Proposition 5, slices A and B are joined. Consider any $c \in C(G)$ and edge $ab \in E$ for any vertex pair $a \in A$ and $b \in B$. As $d(x, y) = 2rad(G) - 1$ and $e(c) = rad(G)$, $d(x, c) < rad(G)$ implies $d(x, c) = rad(G) - 1$ and $d(y, c) = rad(G)$. Hence, $c \in A$. By symmetry, if $d(y, c) < rad(G)$ then $c \in B$. Assume now that $d(x, c) = d(y, c) = rad(G)$. Then, $b, c \in N^{rad(G)}(x)$. Consider vertex $u \neq c$ on a shortest path $P(c, y)$ closest to x. We have $d(y, u) \leq rad(G) - 1$ by the choice of u. If $d(x, u) \leq rad(G) - 1$, then $d(x, y) \leq d(x, u) + d(u, y) \leq 2rad(G) - 2$, a contradiction. Thus, $d(x, u) \geq rad(G)$. Vertices b and c are connected in $\langle V \setminus N^{rad(G)-1}(x)\rangle$ by shortest paths to y. By Proposition 2 (iii), b and c share neighbors in A. Therefore, $a \in N(c)$ and, by symmetry, $b \in N(c)$. Hence, any central vertex either belongs to $A \cup B$ or is universal to $A \cup B$. Thus, $C(G) \subseteq D(\{a, b\}, 1)$. Additionally, since any pair of vertices in $C(G)$ is at most distance 2 apart, $C(G)$ is a cograph.

We now show the existence of vertices $a \in A$ and $b \in B$ such that $a, b \in C(G)$. Consider the family of disks $D(v, r(v))$ centered at each vertex v, where $r(v) = 1$ for all central vertices $v \in C(G)$ and $r(v) = rad(G) - 1$ for all others. Any two non-central vertices $u, v \in V \setminus C(G)$ have distance no more than the diameter, therefore $d(u, v) \leq 2rad(G) - 1 = r(u) + r(v) + 1$. Any two central vertices $u, v \in C(G)$ have distance no more than the diameter of the center, therefore $d(u, v) \leq 2 = r(u) + r(v)$. By definition, any central vertex $u \in C(G)$ sees any vertex $v \in V$ within $rad(G)$, and therefore $d(u, v) \leq rad(G) = r(u) + r(v)$. Hence, by Proposition 3, there is an r-dominating clique K. As any non-central vertex has distance $rad(G) - 1$ to a vertex of K, we have $K \subseteq C(G)$. Let $a \in K$ be closest to x and let $b \in K$ be closest to y. By distance requirements, ab must be an edge with $d(x, a) = rad(G) - 1$ and $d(b, y) = rad(G) - 1$. Therefore, $a \in A$ and $b \in B$. $\square$

Corollary 12. If $diam(G) \geq 2rad(G) - 1$ then $C(G)$ is a cograph.

It remains now to investigate the case when $diam(G) = 2rad(G) - 2$.

Lemma 35. Let $diam(G) = 2rad(G) - 2$, and let $M \subseteq C(G)$. If all $u, v \in M$ satisfy $d(u, v) = 2$, then there is a vertex $c \in C(G)$ that is universal to M.

Proof. Consider a disk of radius 1 centered at each $s \in M$ and a disk of radius $rad(G)-1$ centered at each $v \in V \backslash M$. Any two vertices $u, v \in V \backslash M$ satisfy $d(u, v) \leq diam(G) = 2rad(G)-2 = r(u)+r(v)$. Since $M \subseteq C(G)$, any $s \in M$ and $v \in V$ satisfy $d(s, v) \leq rad(G) = r(s) + r(v)$. By assumption, any two $s, t \in M$ satisfy $d(s, t) = 2 = r(s) + r(t)$. By Proposition 3, there is a single vertex or a pair of adjacent vertices r-dominating G. In the former case, we are done. Thus, consider the case when there is an r-dominating edge $ab \in E$. We have $a, b \in C(G)$ since all vertices $v \in V$ see some end-vertex of edge ab within $rad(G) - 1$. We claim that at least one end-vertex of edge ab is universal to M. By contradiction assume there exist vertices $u, v \in M$ which are adjacent to opposite ends of edge ab. Without loss of generality, let $u \in N(a) \setminus N(b)$ and $v \in N(b) \setminus N(a)$. Since $d(u, v) = 2$, we get in G either an induced C_5, or an induced house, or an induced gem. A contradiction obtained proves the lemma. $\square$

Lemma 36. Let x, y be a diametral pair of G, $diam(G) = 2rad(G)-2$, and let $A := S_{rad(G)-2}(x, y)$, $S := S_{rad(G)-1}(x, y)$, and $B := S_{rad(G)-2}(y, x)$. Then $A \cup B \cup (S \cap C(G)) \subseteq C(G)$ and there is a vertex $c \in S \cap C(G)$. Moreover, $C(G) \subseteq D(S \cap C(G), 1)$.

Proof. Consider any $s \in A$. By Lemma 26, $e(s) \leq \max\{\max\{d(s, x), d(s, y)\}, \min\{d(s, x), d(s, y)\}+ 2\} = d(s, y) = rad(G)$. Hence, $s \in C(G)$ and, by symmetry, $A \cup B \subseteq C(G)$.

By contradiction assume there is no central vertex in S. Let w be a vertex from S minimizing $|F(w)|$, and let $v \in F(w)$. Since $w \notin C(G)$, $d(w, v) \geq rad(G)+1$. Denote by $s_1 \in S_1(w, x)$ and $s_2 \in S_1(w, y)$ two adjacent to w vertices on a shortest path from x to y. As previously established, both s_1 and s_2 are central since they belong to A and B, respectively. Thus, $d(s_1, v) \leq rad(G)$ and $d(s_2, v) \leq$

$rad(G)$. Therefore, $d(s_1, v) = d(s_2, v) = rad(G)$ and $d(w, v) = rad(G)+1$. Since $s_1, s_2 \in N^{rad(G)}(v)$ and are connected via w in the graph $\langle V \setminus N^{rad(G)-1}(v)\rangle$, by Proposition 2(iii), there is a vertex $t \in N^{rad(G)-1}(v)$ adjacent to s_1 and s_2. As $t \in S_{rad(G)-1}(y, x)$ and $v \in F(w)\setminus F(t)$, by minimality of $|F(w)|$, there is a vertex $u \in F(t) \setminus F(w)$. By our assumption, $u \notin C(G)$, i.e., $d(u, t) \geq rad(G) + 1$. Consider the 4-point condition on vertices t, w, v, u. We have $d(v, u) + d(w, t) = d(v, u) + 2$, $d(v, t)+d(u, w) \leq rad(G)-1+rad(G)$, and $d(v, w)+d(u, t) \geq rad(G)+1+rad(G)+1 = 2rad(G)+2$. Since the latter two sums differ by more than 2, necessarily, $d(v, u) + d(w, t) = d(v, w) + d(u, t)$. Hence, $d(v, u) = d(v, w)+d(u, t)-d(w, t) \geq 2rad(G)$, a contradiction with $diam(G) = 2rad(G)-2$. Thus, there is a vertex $c \in S \cap C(G)$.

Next, we establish an intermediate claim that $C(G) \subseteq D(M, 1)$, where $M := A\cup B\cup(S\cap C(G))$. By contradiction suppose there is a vertex $w \in C(G)$ with $w \notin D(M, 1)$. Consider arbitrary vertices $a \in A$ and $b \in B$. Thus, $d(a, b) = 2$, and $a, b \in M$ and, by the choice of w, necessarily $d(w, a) \geq 2$ and $d(w, b) \geq 2$. If $d(w, a) = d(w, b) = 2$ then, by Lemma 35 applied to the set $\{w, a, b\}$, there is a central vertex u adjacent to w, a, b. In this case $u \in S \cap C(G)$ and therefore $u \in M$, contradicting with $w \notin D(M, 1)$. Assume now, without loss of generality, that $d(w, a) \geq 3$. Consider the 4-point condition on vertices w, x, y, a. We have $d(x, y) + d(w, a) \geq 2rad(G) + 1$ is the largest sum since $d(x, w)+d(y, a) \leq 2rad(G)$ and $d(x, a)+d(w, y) \leq 2rad(G)-2$. Since the smaller two sums must be equal and differ from the larger one by at most two, inequality $d(x, y)+d(w, a) \geq d(x, a)+d(y, w)+3$ gives a contradiction which establishes the claim that $C(G) \subseteq D(M, 1)$.

Finally, we establish that $C(G) \subseteq D(S \cap C(G), 1)$. By contradiction assume there is a central vertex $w \in C(G) \setminus S$ which is not adjacent to any vertex of $S \cap C(G)$. By the previous claim, w is adjacent to some vertex from A or B. Without loss of generality, let $wa \in E$ for some vertex $a \in A$. Since $d(a, y) = rad(G)$, necessarily, $d(w, y) \geq rad(G) - 1$. If $d(w, y) = rad(G) - 1$ then $w \in S$, a contradiction. So, $d(w, y) = rad(G)$ must hold. Now, vertices w and a are connected in $\langle V\setminus N^{rad(G)-1}(y)\rangle$. By Proposition 2 (iii), $N(w)\cap N^{rad(G)-1}(y) = N(a)\cap N^{rad(G)-1}(y)$. By Proposition 5, also $S \cap C(G) \subseteq N(a) \cap N^{rad(G)-1}(y)$. Thus, w is universal to $S \cap C(G)$, a contradiction. □

We are ready to prove the main result of this section.

Theorem 7. Let H be a subgraph of a distance-hereditary graph G induced by $C(G)$. Either

(i) H is a cograph, or

(ii) H is a connected distance-hereditary graph with $diam(H) = 3$ and $C(H)$ is a connected cograph with $rad(C(H)) = 2$.

Furthermore, any such graph H is the center of some distance-hereditary graph.

Proof. If $diam(G) \geq 2rad(G) - 1$ then, by Lemma 33 and Lemma 34, H is a cograph. Assume now that $diam(G) = 2rad(G) - 2$ and $diam(H) = 3$ (if $diam(H) \leq 2$ then, by definition, H is a cograph). Then, H is a connected distance-hereditary graph [115], and so $2rad(H) - 2 \leq diam(H) \leq 2rad(H)$. On one hand, $rad(H) \geq \lceil (diam(H)/2) \rceil = 2$. On the other hand, $rad(H) \leq \lfloor (diam(H)+2)/2 \rfloor = 2$. Hence, $rad(H) = 2$.

So, H is a connected distance-hereditary graph with $diam(H) = 3$ and $rad(H) = 2$. Consider the center $C(H)$ of H. First we show that $C(H)$ is connected. Let $r(u) = 1$ for each vertex $u \in H$. Then, any pair $u, v \in H$ satisfies $d_H(u, v) \leq diam(H) = r(u) + r(v) + 1$. By Proposition 3, there is a clique K in H dominating H. Since each vertex of K is at most distance $2 = rad(H)$ from every vertex of H, $K \subseteq C(H)$ holds. Moreover, every two vertices of $C(H)$ are connected through vertices of $K \subseteq C(H)$, implying that $C(H)$ is connected in H. In distance-hereditary graphs every connected subgraph is isometric. Hence, $C(H)$ is an isometric subgraph of H. As $rad(H) = 2$, every two vertices of $C(H)$ are at distance at most 2 from each other, implying $diam(C(H)) = 2$. Thus, $C(H)$ is a connected cograph with $rad(C(H)) \leq 2$.

We will show next that $rad(C(H)) = 2$, i.e., for any $c \in C(H)$ there is a vertex $z \in C(H)$ such that $cz \notin E$. Consider a vertex $t \in F(c)$ furthest from c in G. We have $d_G(c, t) = rad(G)$. Let $z \in H$ be a closest vertex to t which is central in G. Since $diam(G) = 2rad(G) - 2$, by Lemma 31, we have $d_G(t, z) = d_G(t, C(G)) = e_G(t) - rad(G) \leq diam(G) - rad(G) = rad(G) - 2$. Moreover, vertices z and c are not adjacent since $d_G(c, t) = rad(G)$ and $d_G(t, z) \leq rad(G) - 2$. But, since $c \in C(H)$, $d_G(c, z) = d_H(c, z) \leq 2$. Therefore, $d_H(c, z) = 2$ and $d_G(t, z) = rad(G) - 2$. We next establish that z belongs to $C(H)$. By contradiction, assume that there is a vertex $u \in H$ such that $d_H(z, u) > rad(H) = 2$. Then, $d_H(z, u) = diam(H) = 3$ and, by the choice of c ($c \in C(H)$), necessarily $d_H(c, u) \leq 2$. Consider the 4-point condition on vertices c, u, z, t. We have that $d(c, t) + d(u, z) = rad(G) + 3$ is the largest sum since $d(c, z) + d(u, t) \leq rad(G) + 2$ and $d(c, u) + d(z, t) \leq rad(G)$. However, $d(c, t) + d(u, z) \geq d(c, u) + d(z, t) + 3$, giving a contradiction since the smaller two sums must be equal and differ from the larger one by at most two. Hence, z belongs to $C(H)$ showing that every $c \in C(H)$ has a non-adjacent vertex $z \in C(H)$.

Finally, we show that any such graph H is the center of some distance-hereditary graph G. In what follows, we refer to Figure 4.6 for an illustration. If H is a cograph, then one can construct a graph G by simply adding to H four new vertices x, x^*, y, y^*. Vertices x and y are universal to H, and vertices x^*, y^* are pendant to x and y, respectively. Now graph H is the center of G as any vertex u of the cograph H is at most distance 2 to any vertex of G, whereas $d_G(x, y^*) = 3$ and $d_G(y, x^*) = 3$. Suppose now that H is a connected distance-hereditary graph with $diam(H) = 3$ and $C(H)$ is a connected cograph with $rad(C(H)) = 2$. One can construct a graph G by adding to H (with $C(H) = \{c_1, c_2, ..., c_\ell\}$) ℓ new vertices $x_1, x_2, ..., x_\ell$ such that each x_i is pendant to $c_i \in C(H)$. Each $c_i \in C(H)$ has $d_G(c_i, u) \leq 2$ for all $u \in H$. Since $rad(C(H)) = 2$, each c_i has a non-adjacent vertex $c_k \in C(H)$, and therefore $d_G(c_i, x_k) = 3$. Any vertex $u \in H \backslash C(H)$ has a vertex $v \in H \backslash C(H)$, for which $d_G(u, v) = 3$. Furthermore, any such u satisfies $d_G(u, x_i) = d_G(u, c_i) + 1 \leq 3$ for each x_i. Since $rad(C(H)) = 2$, for any pendant x_i, vertex c_i has a non-adjacent vertex $c_k \in C(H)$ and therefore $d_G(x_i, x_k) = 4$. Hence, H is the center of G. □

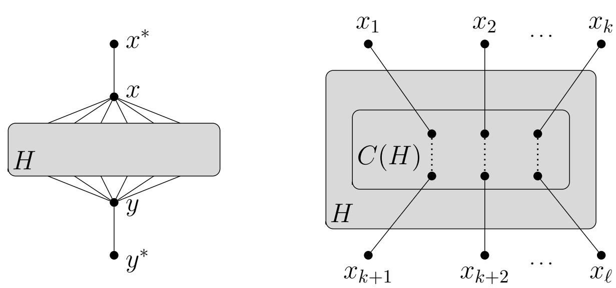


Figure 4.6: Any cograph H (left), and any connected distance-hereditary graph H with diameter 3 where $C(H)$ is a connected cograph with radius 2 (right), is the center of some distance hereditary graph.

4.5 Computing all eccentricities

We provide in this section a simple linear time algorithm to compute all eccentricities of a distance-hereditary graph G using a weight function and a special pruning sequence produced by processing layers of a breadth-first search tree of G (see [44]).

For a given graph $G = (V(G), E(G))$ and an n-tuple $(p(v_1), p(v_2), ..., p(v_n))$ of non-negative vertex weights, we define the p-weighted eccentricity of each vertex $v \in V(G)$ as $e_{G,p}(v) = \max_{u\in V(G)}\{d_G(v, u) + p(u)\}$. We refer to the set of furthest vertices from a vertex v under weight function p in G as $F_{G,p}(v) = \{u \in V(G) : e_{G,p}(v) = d_G(v, u) + p(u)\}$. Clearly, when $p(v) = 0$ for all $v \in V(G)$, we have $e_{G,p}(v) = e_G(v)$ and $F_{G,p}(v) = F_G(v)$ agreeing with earlier definitions.

Lemma 37. Let $x, y \in V(G)$ be twins with $p(x) \leq p(y)$. Set $G' = G - \{x\}$ and $p'(v) = p(v)$ for all $v \in V(G')$. Then, $e_{G,p}(v) = e_{G',p'}(v)$ for all $v \in V(G') \setminus \{y\}$, $e_{G,p}(y) = \max\{p(x) + d_G(x, y), e_{G',p'}(y)\}$, and $e_{G,p}(x) = \max\{p(y) + d_G(x, y), e_{G,p}(y)\}$. Moreover, if $F_{G,p}(y) \setminus \{x\} \neq \emptyset$, then $e_{G,p}(y) = e_{G',p'}(y)$.

Proof. Let $v \in V(G') \setminus \{y\}$. As x and y are twins, $d_G(v, y) = d_G(v, x)$. Since G' is isometric in G, $d_G(v, u) + p(u) = d_{G'}(v, u) + p'(u)$ for any $u \in V(G')$. Then, by definition of eccentricity, $e_{G',p'}(v) = \max_{u\in V(G')}\{d_{G'}(v, u) + p'(u)\} = \max_{u\in V(G)\setminus\{x\}}\{d_G(v, u) + p(u)\}$. As $p(x) \leq p(y)$, $d_G(v, y) + p(y) \geq d_G(v, x) + p(x)$. Thus, $e_{G',p'}(v) = \max\{d_G(v, x) + p(x), \max_{u\in V(G)\setminus\{x\}}\{d_G(v, u) + p(u)\}\} = \max_{u\in V(G)}\{d_G(v, u) + p(u)\} = e_{G,p}(v)$. Similarly, $e_{G,p}(y) = \max\{d_G(x, y) + p(x), \max_{u\in V(G')}\{d_G(y, u) + p(u)\}\} = \max\{d_G(x, y) + p(x), e_{G',p'}(y)\}$. Moreover, if $F_{G,p}(y) \setminus \{x\} \neq \emptyset$, then $e_{G,p}(y) = e_{G',p'}(y)$ as realized by the weighted distance in G' from y to any $v \in F_{G,p}(y) \setminus \{x\}$.

As $d(x, y) \geq 1$ and $p(x) \leq p(y)$, $d_G(x, y) + p(y) \geq \max\{d_G(x, y) + p(x), p(y)\}$. Again, by definition of eccentricity,

$$
\begin{aligned}
e_{G,p}(x) &= \max\{p(x), d_G(x, y) + p(y), \max_{u\in V(G)\setminus\{x,y\}}\{d_G(x, u) + p(u)\}\} \\
&= \max\{d_G(x, y) + p(y), \max_{u\in V(G)\setminus\{x,y\}}\{d_G(y, u) + p(u)\}, d_G(x, y) + p(x), p(y)\} \\
&= \max\{d_G(x, y) + p(y), e_{G,p}(y)\}.
\end{aligned}
$$

□

Lemma 38. Let x be a vertex pendant to y. Set $G' = G - \{x\}$, $p'(y) = \max\{1 + p(x), p(y)\}$, and $p'(v) = p(v)$ for all $v \in V(G')\setminus\{y\}$. Then, $e_{G,p}(v) = e_{G',p'}(v)$ for all $v \in V(G')$. If $F_{G,p}(y)\setminus\{x\} \neq \emptyset$, then $e_{G,p}(x) = \max\{p(x), e_{G',p'}(y) + 1\}$.

Proof. Let $v \in V(G')$. As x is pendant to y, $d_G(v, x) = d_G(v, y) + 1$. Since G' is isometric in G, $d_{G'}(v, y) + p'(y) = \max\{d_G(v, y) + 1 + p(x), d_G(v, y) + p(y)\} = \max\{d_G(v, x) + p(x), d_G(v, y) + p(y)\}$.

Then, by definition of eccentricity,

$$
\begin{aligned}
e_{G',p'}(v) &= \max\{d_{G'}(v,y) + p'(y), \max_{u \in V(G') \setminus \{y\}}\{d_{G'}(v,u) + p'(u)\}\} \\
&= \max\{d_G(v,x) + p(x), d_G(v,y) + p(y), \max_{u \in V(G') \setminus \{y\}}\{d_G(v,u) + p(u)\}\} \\
&= \max_{u \in V(G)}\{d_G(v,u) + p(u)\} = e_{G,p}(v).
\end{aligned}
$$

Assume now that $F_{G,p}(y) \setminus \{x\} \neq \emptyset$. Hence, $e_{G',p'}(y) = e_{G,p}(y) = \max_{u \in V(G) \setminus \{x\}}\{d_G(y,u) + p(u)\}$. Thus, by definition of eccentricity,

$$
\begin{aligned}
e_{G,p}(x) &= \max\{p(x), p(y) + 1, \max_{u \in V(G) \setminus \{x,y\}}\{d_G(x,u) + p(u)\}\} \\
&= \max\{p(x), p(y) + 1, \max_{u \in V(G) \setminus \{x,y\}}\{d_G(y,u) + p(u)\} + 1\} \\
&= \max\{p(x), \max_{u \in V(G) \setminus \{x\}}\{d_G(y,u) + p(u)\} + 1\} = \max\{p(x), e_{G',p'}(y) + 1\}. \qquad \square
\end{aligned}
$$

We use the pruning sequence (the vertex elimination ordering) $\sigma = (v_1, ..., v_n)$ that can be constructed in linear time via ρ iterations of a systematic removal of pendants/twins from each layer $\mathcal{L}_\rho, ..., \mathcal{L}_1$ of a breadth-first search tree rooted at v_n (see [44]). Iteration k, where $k = \rho, ..., 1$, consists of four consecutive steps:

(a) remove any $x \in \mathcal{L}_k$ twin to some $y \in \mathcal{L}_k$ of the same connected component in $\mathcal{L}_k$ (i.e., x and y belong to the same connected component of the subgraph of G induced by vertices of $\mathcal{L}_k$),

(b) remove any $x \in \mathcal{L}_k$ pendant to some $y \in \mathcal{L}_{k-1}$,

(c) remove any $x \in \mathcal{L}_{k-1}$ twin to some $y \in \mathcal{L}_{k-1}$ belonging to the same neighborhood $N(z) \cap \mathcal{L}_{k-1}$ of some $z \in \mathcal{L}_k$, and

(d) remove any $x \in \mathcal{L}_k$ pendant to some $y \in \mathcal{L}_{k-1}$.

Note that we move to the next step only when no vertex remains satisfying the condition of the previous step. At the end of iteration k, all vertices of $\mathcal{L}_k$ have been removed (see [44]). By this ordering, any $u \in \mathcal{L}_k$ satisfies that if u is a pendant to v, then $v \in \mathcal{L}_{k-1}$, and if u is a twin to v, then $v \in \mathcal{L}_k$. Let G_i denote the graph induced by $\{v_i, ..., v_n\}$ for each $i = 1, ..., n$.

Theorem 8. There is a linear time algorithm to compute all eccentricities in a distance-hereditary graph.

Proof. Let $\sigma = (v_1, ..., v_n)$ be the pruning sequence constructed as described above by each iteration $k = rad(G), ..., 1$ of removing vertices from layer $\mathcal{L}_k$ of a BFS tree rooted at a central vertex v_n. Denote by $v_y \in \mathcal{L}_2$ the first pendant vertex of σ encountered in step (d) of iteration 2 (or in step (b) if $\mathcal{L}_2$ becomes empty after steps (a) and (b)). Denote by v_z the first vertex of σ encountered in iteration 1. Thus, the graph G_z consists of v_n and some twins/pendants in $\mathcal{L}_1$ adjacent to v_n. The algorithm is summarized as follows. We process vertices v_i, $i < z$, from v_1 to v_z (from left to right along σ). We denote by p_i the weight function of each vertex immediately before vertex v_i is processed. For each vertex $v_j \in \sigma$, set $p_1(v_j) = 0$. As each v_i is processed, p_{i+1} is invariant (that is, $p_{i+1}(v_j) = p_i(v_j)$ for every v_j) with the exception of one case: if v_i is a pendant to v_j in G_i, then let $p_{i+1}(v_j) = \max\{p_i(v_i) + 1, p_i(v_j)\}$. We can assume that if v_i is a twin to vertex v_j in G_i, then $p_i(v_i) \leq p_i(v_j)$, since otherwise, their positions as twins can be swapped in σ. Observe that the weight function of a vertex $v_j \in \mathcal{L}_k$ can only increase when a vertex $v_i \in \mathcal{L}_{k+1}$ is pendant to v_j, where $i < j$. Hence, if any vertex v_i belongs to layer $\mathcal{L}_k$, then for any integer ℓ, $p_\ell(v_i) \leq rad(G) - k$. Additionally, every vertex v_i and integers $\ell < \kappa$ satisfy $p_\ell(v_i) \leq p_\kappa(v_i)$. After all vertices v_i, $i < z$, are processed (along σ from left to right), we next compute all p_z-weighted eccentricities in G_z. Then, we compute all p_y-weighted eccentricities in G_y. Finally, we process each vertex v_i, $i < y$, along the reverse direction of σ. The p_{i+1}-weighted eccentricities in G_{i+1} are used to obtain the p_i-weighted eccentricities in G_i.

Backward phase 1: Compute all p_z-weighted eccentricities in G_z. Denote by V^* and N_i^* the vertex lists ordered by decreasing weight p_z from the respective vertex sets $V(G_z)$ and $N(v_i) \cap V(G_z)$ for each $v_i \in V(G_z)$. The lists are ordered in total linear time with a bucket sort. The first vertex w of N_i^* has maximum $p_z(w)$ among neighbors of v_i in G_z, and the first vertex u of $V^* \setminus N_i^*$ has maximum $p_z(u)$ among non-neighbors of v_i. By definition, the weighted eccentricity of each v_i is $e_{G_z,p_z}(v_i) = \max\{p_z(v_i), 1 + p_z(w), 2 + p_z(u)\}$ if there exists a non-neighbor $u \in V^* \setminus N_i^*$, and $e_{G_z,p_z}(v_i) = \max\{p_z(v_i), 1 + p_z(w)\}$ otherwise.

Backward phase 2: Compute all p_y-weighted eccentricities in G_y. By choice of y, each vertex v_i for $y \leq i < z$ satisfies $v_i \in \mathcal{L}_2$ and v_i is pendant to a vertex $v_j \in \mathcal{L}_1$ in G_y. Hence, for $M = \{v_z, ..., v_n\}$ and $S = \{v_y, ..., v_{z-1}\}$, $\max_{m \in M} p_z(m) = \max\{\max_{m \in M} p_y(m), \max_{u \in S} p_y(u) + 1\}$ holds. Therefore, we may again use ideas from the previous phase. For each vertex v_i, $y \leq i < z$, pendant to v_j, we have a vertex $w \in N(v_j)$ in G_z with maximum $p_z(w)$ among neighbors of v_j in G_z

and a vertex $u \notin N(v_j)$ in G_z with maximum $p_z(u)$ among non-neighbors of v_j in G_z. By definition, the weighted eccentricity of each v_i is $e_{G_y,p_y}(v_i) = \max\{p_i(v_i), p_i(v_j)+1, p_z(w)+2, p_z(u)+3\}$ if there exists in G_z a non-neighbor $u \notin N(v_j)$, and $e_{G_y,p_y}(v_i) = \max\{p_i(v_i), p_z(v_j)+1, p_z(w)+2\}$ otherwise. Additionally, by Lemma 38, each vertex v_ℓ for $\ell \geq z$ has equal p_z and p_y weighted eccentricities, that is, $e_{G_y,p_y}(v_\ell) = e_{G_z,p_z}(v_\ell)$.

Backward phase 3: Compute all p_i-weighted eccentricities in G_i for $i < y$ along a reverse direction of σ. If v_i is a twin to v_j in G_i, by Lemma 37, $e_{G_i,p_i}(v_j) = \max\{p_i(v_i) + d_G(v_i,v_j), e_{G_{i+1},p_{i+1}}(v_j)\}$, $e_{G_i,p_i}(v_i) = \max\{p_i(v_j) + d_G(v_i,v_j), e_{G_i,p_i}(v_j)\}$, and for all $u \in V(G_i) \setminus \{v_i,v_j\}$ $e_{G_i,p_i}(u) = e_{G_{i+1},p_{i+1}}(u)$. We consider now the case that v_i is a pendant to v_j. By Lemma 38, each $u \in V(G_i) \setminus \{v_i\}$ satisfy $e_{G_i,p_i}(u) = e_{G_{i+1},p_{i+1}}(u)$. We claim that $e_{G_i,p_i}(v_i) = \max\{p_i(v_i), e_{G_{i+1},p_{i+1}}(v_j)+1\}$. It remains only to show that any pendant v_i, where $i < y$, satisfies $F_{G_i,p_i}(v_j) \setminus \{v_i\} \neq \emptyset$; applying Lemma 38 then proves the claim.

By contradiction, let $v_i \in \mathcal{L}_\gamma$ pendant to $v_j \in \mathcal{L}_{\gamma-1}$ be the earliest vertex in σ with $i < y$ and $F_{G_i,p_i}(v_j) = \{v_i\}$. Hence, $e_{G_i,p_i}(v_j) = d_{G_i}(v_j,v_i) + p_i(v_i) > \max_{i+1 \leq t \leq n}\{d_{G_i}(v_j,v_t) + p_i(v_t)\}$. As $i < y$ and $v_y \in \mathcal{L}_2$, then $\gamma \geq 2$. Thus, $p_i(v_i) \leq rad(G) - 2$ and $e_{G_i,p_i}(v_j) = d_G(v_j,v_i) + p_i(v_i) \leq rad(G) - 1$. If $e_{G_i,p_i}(v_j) = e_G(v_j)$, we obtain a contradiction with $e_G(v_j) \leq rad(G) - 1$. Therefore, $e_{G_i,p_i}(v_j) < e_G(v_j)$. Let $v_\ell \in \sigma$ be the earliest vertex such that $e_{G_\ell,p_\ell}(v_j) > e_{G_{\ell+1},p_{\ell+1}}(v_j)$, where $\ell < i < j$. By Lemma 37 and Lemma 38, v_ℓ is a twin to v_j in G_ℓ such that $F_{G_\ell,p_\ell}(v_j) = \{v_\ell\}$ and $p_\ell(v_\ell) \leq p_\ell(v_j)$. Then, $rad(G) \leq e_G(v_j) = e_{G_\ell,p_\ell}(v_j) = d_G(v_\ell,v_j) + p_\ell(v_\ell)$. Hence, $p_i(v_j) \geq p_\ell(v_j) \geq p_\ell(v_\ell) \geq rad(G) - d_G(v_\ell,v_j)$. If $p_i(v_j) = rad(G) - 1$ or if v_ℓ is a true twin to v_j, then $p_i(v_j) \geq rad(G) - 1$, and a contradiction arises with $p_i(v_j) < d_G(v_j,v_i) + p_i(v_i) \leq rad(G) - 1$ (recall that $F_{G_i,p_i}(v_j) = \{v_i\}$). Necessarily, v_ℓ is a false twin to v_j and $p_i(v_j) = rad(G) - 2$. As $rad(G) - 2 = p_i(v_j) < e_{G_i,p_i}(v_j) = d_G(v_i,v_j) + p_i(v_i) \leq 1 + rad(G) - \gamma$, we obtain $\gamma < 3$. Hence, $v_i \in \mathcal{L}_2$ and $v_\ell \in \mathcal{L}_1$. As $\ell < i$, necessarily v_ℓ is removed in iteration 2 step (c) of σ construction. However, this implies that v_i is removed in iteration 2 step (d); therefore, $i \geq y$, a contradiction that proves the claim. □

Chapter 5

Eccentricity terrain of hyperbolic graphs*

This chapter further investigates the eccentricity function in δ-hyperbolic graphs from the eccentricity terrain prospective. Recall that Gromov [79] defines δ-hyperbolic graphs via a simple 4-point condition: for any four vertices u, v, w, x, the two larger of the three distance sums $d(u,v)+d(w,x)$, $d(u,w)+d(v,x)$, and $d(u,x)+d(v,w)$ differ by at most $2\delta \geq 0$. Such graphs have become of recent interest due to the empirically established presence of a small hyperbolicity in many real-world networks, such as biological networks, social networks, Internet application networks, and collaboration networks, to name a few (see, e.g., [2, 3, 19, 89, 99, 111]). Under plausible assumptions, even distinguishing the radius [1] or the diameter [108] between exact values 2 or 3 cannot be accomplished in subquadratic time for sparse graphs. Since the graphs constructed in the reductions [1, 108] are 1-hyperbolic, the same result holds for 1-hyperbolic graphs. Therefore, we are interested in fast approximation algorithms with additive errors depending linearly only on the hyperbolicity.

We also define a β-pseudoconvexity which implies the quasiconvexity found by Gromov in hyperbolic graphs, but additionally, is closed under intersection. Interestingly, all disks and all sets $C^k(G)$, for any integer $k \geq 0$, are $(2\delta-1)$-pseudoconvex in δ-hyperbolic graphs. We also identify several important features of terrain shapes on a shortest path to a central vertex, which are insightful to the eccentricity bounds for a vertex. Such results also give lower bounds on $diam(G)$ and upper bounds on $rad(G)$ consistent with those found in literature [33, 61]. More importantly, they are very useful in approximating all eccentricities in G. We present three approximation algorithms for all eccentricities:

*Results from this chapter have been published in *Journal of Computer and System Sciences* [60]

- a $O(\delta m)$ time eccentricity approximation $\hat{e}(v)$ based on the distances from any vertex to two mutually distant vertices which satisfies $e_G(v) - 2\delta \leq \hat{e}(v) \leq e_G(v)$
- a spanning tree constructible in $O(\delta m)$ time which satisfies $e_G(v) \leq e_T(v) \leq e_G(v) + 4\delta + 1$, and
- a spanning tree constructible in $O(m)$ time which satisfies $e_G(v) \leq e_T(v) \leq 6\delta$.

Thus, the eccentricity terrain of a tree gives a good approximation (up-to an additive error $O(\delta)$) of the eccentricity terrain of a δ-hyperbolic graph. Furthermore, we obtain an approximation for the distance from an arbitrary vertex v to $C(G)$ or $C^{2\delta}(G)$ based on the eccentricity of v.

We will often use the following lemma.

Lemma 39. Let G be a δ-hyperbolic graph. For any $x, y, v \in V$ and any vertex $c \in I(x, y)$ the following holds.

(*i*) If $d(x, c) \leq (v|y)_x$, then $d(c, v) \leq d(x, v) - d(x, c) + 2\delta$ and $d(c, v) \leq d(x, v) + \delta$. Moreover, $e(c) \leq e(x) - d(x, c) + 2\delta$ and $e(c) \leq e(x) + \delta$, when $v \in F(c)$.

(*ii*) If $d(y, c) \leq (v|x)_y$, i.e., $d(x, c) \geq (v|y)_x$, then $d(c, v) \leq d(y, v) - d(y, c) + 2\delta$ and $d(c, v) \leq d(y, v) + \delta$. Moreover, $e(c) \leq e(y) - d(y, c) + 2\delta$ and $e(c) \leq e(y) + \delta$, when $v \in F(c)$.

Proof. Consider the three distance sums $d(x, y) + d(c, v)$, $d(x, v) + d(c, y)$, and $d(x, c) + d(v, y)$. Since $c \in I(x, y)$, $d(x, y) = d(x, c) + d(c, y)$. By the triangle inequality, $d(x, v) + d(c, y) \leq d(x, c) + d(c, v) + d(c, y)$ and $d(x, c) + d(v, y) \leq d(x, c) + d(v, c) + d(c, y)$. Thus, $d(x, y) + d(c, v) \geq \max\{d(x, v) + d(c, y),\ d(x, c) + d(v, y)\}$.

Suppose $d(x, c) \leq (v|y)_x = \frac{1}{2}(d(v, x) + d(y, x) - d(v, y))$. We have $2d(x, c) + 2d(v, y) \leq (d(v, x) + d(y, x) - d(v, y)) + 2d(v, y) = d(v, x) + d(y, x) + d(v, y) = d(v, x) + d(x, c) + d(c, y) + d(v, y)$. Subtracting $d(x, c) + d(v, y)$ from this inequality, one obtains $d(x, c) + d(v, y) \leq d(v, x) + d(c, y)$. Since G is δ-hyperbolic, $2\delta \geq (d(x, y) + d(c, v)) - (d(v, x) + d(c, y)) = d(x, c) + d(c, v) - d(v, x)$. Therefore, $d(c, v) \leq d(v, x) - d(x, c) + 2\delta$. By adding the triangle inequality $d(c, v) \leq d(v, x) + d(x, c)$ to this, we obtain $d(c, v) \leq d(v, x) + \delta$. Applying both inequalities to the case in which v is furthest from c, we get $e(c) = d(c, v) \leq d(v, x) - d(x, c) + 2\delta \leq e(x) - d(x, c) + 2\delta$ and $e(c) = d(c, v) \leq d(v, x) + \delta \leq e(x) + \delta$. Thus, (*i*) is true.

Suppose now that $d(x, c) \geq (v|y)_x = \frac{1}{2}(d(v, x) + d(y, x) - d(v, y))$. First, we claim that this is equivalent to $d(c, y) \leq (v|x)_y$. By assumption, $d(x, y) = d(x, c) + d(c, y) \geq \frac{1}{2}(d(v, x) + d(y, x) -$

$d(v,y)) + d(c,y)$. Therefore, $d(c,y) \le d(x,y) - \frac{1}{2}(d(v,x) + d(y,x) - d(v,y)) = \frac{1}{2}(d(x,y) - d(v,x) + d(v,y)) = (v|x)_y$, establishing the claim. Now, (ii) is true by symmetry with (i). $\square$

Lemma 39 has a few important corollaries.

Corollary 13. Let G be a δ-hyperbolic graph. Any $x, y, v \in V$ and $c \in I(x,y)$ satisfies $d(c,v) \le \max\{d(x,v), d(y,v)\} - \min\{d(x,c), d(y,c)\} + 2\delta$.

We next combine both cases of Lemma 39 to form an upper bound on all distances from vertex c on a shortest (x,y)-path, including $e(c)$, as well as improvements to this bound when c is sufficiently far from the endpoints x and y. By these we generalize greatly some known results from [8].

Corollary 14. Let G be a δ-hyperbolic graph. Any vertices $x, y, v \in V$ and $c \in I(x,y)$ satisfy $d(c,v) \le \max\{d(x,v),\ d(y,v)\} + \delta$. Furthermore, if $d(x,y) \ge 4\delta$, then any vertex $c^* \in I(x,y)$ with $d(x,c^*) \ge 2\delta$ and $d(y,c^*) \ge 2\delta$ satisfies $d(c^*,v) \le \max\{d(x,v),\ d(y,v)\}$. If $d(x,y) > 4\delta+1$ then any vertex $c^* \in I(x,y)$ with $d(x,c^*) > 2\delta$ and $d(y,c^*) > 2\delta$ satisfies $d(c^*,v) < \max\{d(x,v),\ d(y,v)\}$ [8].

Proof. By Lemma 39, $d(c,v) \le d(x,v) + \delta$ or $d(c,v) \le d(y,v) + \delta$. Therefore, $d(c,v) \le \max\{d(x,v),\ d(y,v)\} + \delta$. If $d(x,y) \ge 4\delta$ then, by Corollary 13, any vertex $c^* \in I(x,y)$ with $d(x,c^*) \ge 2\delta$ and $d(c^*,y) \ge 2\delta$ satisfies $e(c^*) \le \max\{d(x,v), d(y,v)\} - \min\{d(x,c^*), d(y,c^*)\} + 2\delta \le \max\{d(x,v), d(y,v)\}$. If $d(x,y) > 4\delta + 1$, i.e., $d(x,y) \ge 4\delta + 2$ then, by Corollary 13, any vertex $c^* \in I(x,y)$ with $d(x,c^*) > 2\delta$ and $d(c^*,y) > 2\delta$ satisfies $e(c^*) \le \max\{d(x,v), d(y,v)\} - \min\{d(x,c^*), d(y,c^*)\} + 2\delta < \max\{d(x,v), d(y,v)\}$. $\square$

Corollary 15. Let G be a δ-hyperbolic graph. Any vertices $x, y \in V$ and $c \in I(x,y)$ satisfy $e(c) \le \max\{e(x),\ e(y)\} + \delta$. Furthermore, if $d(x,y) \ge 4\delta$, then any vertex $c^* \in I(x,y)$ with $d(x,c^*) \ge 2\delta$ and $d(y,c^*) \ge 2\delta$ satisfies $e(c^*) \le \max\{e(x),\ e(y)\}$. If $d(x,y) > 4\delta + 1$ then any vertex $c^* \in I(x,y)$ with $d(x,c^*) > 2\delta$ and $d(y,c^*) > 2\delta$ satisfies $e(c^*) < \max\{e(x),\ e(y)\}$ [8].

Proof. By Lemma 39, $e(c) \le e(x) + \delta$ or $e(c) \le e(y) + \delta$. Therefore, $e(c) \le \max\{e(x),\ e(y)\} + \delta$. Suppose that $d(x,y) \ge 4\delta$ and consider any vertex $c^* \in I(x,y)$ satisfying $d(x,c^*) \ge 2\delta$ and $d(c^*,y) \ge 2\delta$. By Lemma 39, $e(c^*) \le e(x) - d(x,c^*) + 2\delta \le e(x)$ or $e(c^*) \le e(y) - d(y,c^*) + 2\delta \le e(y)$. Hence, $e(c^*) \le \max\{e(x),\ e(y)\}$.

Suppose now that $d(x, y) > 4\delta + 1$, i.e., $d(x, y) \geq 4\delta + 2$. Consider any vertex $c^* \in I(x, y)$ satisfying $d(x, c^*) > 2\delta$ and $d(c^*, y) > 2\delta$. By Lemma 39, $e(c^*) \leq e(x) - d(x, c^*) + 2\delta < e(x)$ or $e(c^*) \leq e(y) - d(y, c^*) + 2\delta < e(y)$. Hence, $e(c^*) < \max\{e(x),\ e(y)\}$. □

Corollary 16. Let G be a δ-hyperbolic graph where $x, y \in V$, $d(x, y) \geq 2\delta+1$, and $c \in S_{2\delta+1}(x, y)$. If $e(c) \geq \max\{e(x),\ e(y)\}$ then $d(x, y) \leq 4\delta + 1$.

Proof. By contradiction assume that $e(c) \geq \max\{e(x),\ e(y)\}$ and $d(x, y) > 4\delta + 1$, i.e., $d(x, y) \geq 4\delta + 2$. By Corollary 15, $e(c) < \max\{e(x),\ e(y)\}$ must hold, giving a contradiction. □

5.1 Pseudoconvexity of the sets $C^k(G)$ and their diameters

A subset S of a geodesic metric space or a graph is *convex* if for all $x, y \in S$ the metric interval $I(x, y)$ is contained in S. This notion was extended by Gromov [79] as follows: for $\epsilon \geq 0$, a subset S of a geodesic metric space or a graph is called ϵ-*quasiconvex* if for all $x, y \in S$ the metric interval $I(x, y)$ is contained in the disk $D(S, \epsilon)$. S is said to be *quasiconvex* if there is a constant $\epsilon \geq 0$ such that S is ϵ-quasiconvex. Quasiconvexity plays an important role in the study of hyperbolic and cubical groups, and hyperbolic graphs contain an abundance of quasiconvex sets [37]. Unfortunately, ϵ-quasiconvexity is not closed under intersection. Consider a path $P = (v_0, \ldots, v_{2k})$ of length $2k$. Let $S_1 = \{v_0, v_{2k}\} \cup \{v_i : i \text{ is odd}\}$ and $S_2 = \{v_0, v_{2k}\} \cup \{v_i : i \text{ is even}\}$. Both S_1 and S_2 are 1-quasiconvex, however, their intersection is only k-quasiconvex.

In this section, we introduce β-pseudoconvexity which satisfies this important intersection axiom of convexity and we illustrate the presence of pseudoconvex sets in hyperbolic graphs. For $\beta \geq 0$, we define a set $S \subseteq V$ to be β-*pseudoconvex* if, for any vertices $x, y \in S$, any vertex $z \in I(x, y) \setminus S$ satisfies $\min\{d(z, x), d(z, y)\} \leq \beta$. Note that when $\beta = 0$ the definitions of convex sets and β-pseudoconvex sets coincide. Moreover, β-pseudoconvexity implies β-quasiconvexity. Consider a β-pseudoconvex set S and its arbitrary two vertices x and y. As any vertex $z \in I(x, y) \setminus S$ satisfies $\min\{d(z, x), d(z, y)\} \leq \beta$, necessarily, z belongs to disk $D(S, \beta)$. Since the empty set and V are β-pseudoconvex, the following lemma establishes that β-pseudoconvex sets form a convexity.

Lemma 40. If sets $S_1 \subseteq V$ and $S_2 \subseteq V$ are β-pseudoconvex, then $S_1 \cap S_2$ is β-pseudoconvex.

Proof. Consider any two vertices $x, y \in S_1 \cap S_2$. If there is a vertex $z \in I(x, y)$ which does not belong to $S_1 \cap S_2$, then, $z \notin S_1$ or $z \notin S_2$. Without loss of generality, assume $z \notin S_1$. Then, $\min\{d(z, x), d(z, y)\} \leq \beta$ because S_1 is β-pseudoconvex. □

It is easy to see that in 0-hyperbolic graphs (which are block graphs, i.e., graphs in which every 2-connected component is a complete graph) all disks are convex. We next show that all disks are $(2\delta - 1)$-pseudoconvex in δ-hyperbolic graphs with $\delta > 0$.

Lemma 41. Let G be a δ-hyperbolic graph. Any disk of G is $(2\delta - 1)$-pseudoconvex, when $\delta > 0$, and is convex, when $0 \leq \delta \leq 1/2$.

Proof. Consider a disk $D(v, r)$ centered at a vertex $v \in V$ and with radius r. Let $x, y \in D(v, r)$ and let $z \in I(x, y)$ be a vertex which is not contained in $D(v, r)$. By contradiction, assume that $d(z, x) \geq 2\delta$ and $d(z, y) \geq 2\delta$. Since $z \notin D(v, r)$, $d(v, z) > \max\{d(v, y), d(v, x)\}$. By Corollary 14 applied to vertices x, y, v and vertex $z \in I(x, y)$, necessarily, $d(z, v) \leq \max\{d(x, v), d(y, v)\}$, a contradiction. □

It is known that in chordal graphs (including 0-hyperbolic graphs) all sets $C^k(G)$, $k \in \mathbb{N}$, are convex (see, e.g., [30,63]). We next show that all such sets are $(2\delta-1)$-pseudoconvex in δ-hyperbolic graphs with $\delta > 0$.

Lemma 42. Let G be a δ-hyperbolic graph and $k \geq 0$ be an arbitrary integer. Any set $C^k(G)$ of G is $(2\delta - 1)$-pseudoconvex, when $\delta > 0$, and is convex, when $0 \leq \delta \leq 1/2$.

Proof. Let S be the intersection of disks $D(v, rad(G) + k)$ centered at each vertex $v \in V$. By Lemma 41, each disk is $(2\delta - 1)$-pseudoconvex. By Lemma 40, S is also $(2\delta - 1)$-pseudoconvex. It remains only to show that $S = C^k(G)$. Recall that $C^k(G) = \{v \in V : e(v) \leq rad(G) + k\}$. If $x \in S$, then $d(x, v) \leq rad(G) + k$ for all $v \in V$. Therefore, $e(x) \leq rad(G) + k$ and so $x \in C^k(G)$. On the other hand, if $x \notin S$, then $d(x, v) > rad(G) + k$ for some $v \in V$. Therefore, $e(x) > rad(G) + k$ and so $x \notin C^k(G)$. Hence, $S = C^k(G)$. □

As a consequence of Lemma 42, we obtain several interesting features of any shortest path between vertices of $C^k(G)$.

Corollary 17. Let G be a δ-hyperbolic graph, and let $x, y \in C^k(G)$ for an integer $k \geq 0$. If there is a vertex $c \in I(x,y)$ where $c \notin C^k(G)$, then $d(y,c) < 2\delta$ or $d(x,c) < 2\delta$.

Corollary 18. Let G be a δ-hyperbolic graph with $\delta > 0$, and let $x, y \in C^k(G)$ for an integer $k \geq 0$. If there is a shortest path $P(x,y)$ where $P(x,y) \cap C^k(G) = \{x, y\}$, then $d(x,y) \leq 4\delta - 1$.

Proof. Assume $d(x,y) \geq 4\delta$ for some $x, y \in C^k(G)$ and let $P(x,y)$ be a shortest path such that $P(x,y) \cap C^k(G) = \{x,y\}$. Consider vertex $c \in P(x,y)$ with $d(x,c) = 2\delta$. Since $d(x,c) > 2\delta - 1$, by Lemma 42, necessarily $d(y,c) \leq 2\delta - 1$. Thus, $d(x,y) = d(x,c) + d(c,y) \leq 2\delta + 2\delta - 1 = 4\delta - 1$, a contradiction. □

Note that for 0-hyperbolic graphs any shortest path $P(x,y)$ with $x, y \in C^k(G)$ is contained in $C^k(G)$ due to convexity of $C^k(G)$.

We next obtain a bound on the diameter of set $C^k(G)$. It is known [36] that if G is τ-thin, then $diam(C^k(G)) \leq 2k + 2\tau + 1$. Applying the inequality $\tau \leq 4\delta$ from Proposition 1 yields $diam(C^k(G)) \leq 2k + 8\delta + 1$, which can be improved working directly with δ, hereby generalizing also a result from [33, 61].

Lemma 43. Any δ-hyperbolic graph G has $diam(C^k(G)) \leq 2k + 4\delta + 1$ for every $k \in \mathbb{N}$. In particular, $diam(C^{2\delta}(G)) \leq 8\delta + 1$ [61], $diam(C(G)) \leq 4\delta + 1$ and $diam(G) \geq 2rad(G) - 4\delta - 1$ [33, 61].

Proof. Let $x, y \in C^k(G)$ realize the diameter of $C^k(G)$. We have $e(x) \leq rad(G) + k$ and $e(y) \leq rad(G) + k$. Consider a (middle) vertex $c \in I(x,y)$ so that $\min\{d(x,c), d(y,c)\} = \lfloor d(x,y)/2 \rfloor$. Let $v \in F(c)$ be a vertex furthest from c. Hence, $d(c,v) \geq rad(G)$. By Corollary 13, $\lfloor d(x,y)/2 \rfloor = \min\{d(x,c),\ d(y,c)\} \leq \max\{d(v,x),\ d(v,y)\} - d(c,v) + 2\delta \leq rad(G) + k - rad(G) + 2\delta = k + 2\delta$. Thus, $diam(C^k(G)) = d(x,y) = d(x,c) + d(c,y) \leq 2\lfloor d(x,y)/2 \rfloor + 1 \leq 2k + 4\delta + 1$. In particular, when $k = diam(G) - rad(G)$, we get $diam(G) \geq 2rad(G) - 4\delta - 1$ as $C^{diam(G)-rad(G)}(G) = V$. □

Thus, combining this with the result from [36], we get $diam(C^k(G)) \leq 2k + 2\min\{\tau(G), 2\delta(G)\} + 1$ for any graph G and any $k \in \mathbb{N}$.

Summarizing the results of this section, we have.

Theorem 9. Every disk and every set $C^k(G)$, $k \geq 0$, of a δ-hyperbolic graph G is $(2\delta - 1)$-pseudoconvex, when $\delta > 0$, and is convex, when $0 \leq \delta \leq 1/2$. Furthermore, $diam(C^k(G)) \leq 2k + 4\delta + 1$.

For a δ-hyperbolic graph G, although its center $C(G)$ has a bounded diameter in G, the graph $\langle C(G) \rangle$ induced by $C(G)$ may not be connected. This is the case even for distance-hereditary graphs (as shown in Chapter 4) which are 1-hyperbolic. The following simple construction shows that even if the center of G induces a connected subgraph, it may induce an arbitrary connected graph. Thus, even if G has a bounded hyperbolicity, its center graph $\langle C(G) \rangle$ may have an arbitrarily large hyperbolicity. Consider any connected graph H with sufficiently large $\delta(H)$, and construct a new graph G from H by adding four new vertices x, y, x^*, y^* to H, making x and y adjacent to each vertex of G, and making x^* and y^* adjacent only to x and y, respectively. It is easy to see that G is 1-hyperbolic and $\langle C(G) \rangle$ is isomorphic to H. However, H has a large hyperbolicity.

5.2 Terrain shapes

In this section, we find a limit on the number of up-edges and horizontal-edges which can occur on a shortest path $P(y, x)$ from a vertex y to a vertex x. Moreover, we discover that the length of any up-hill or the width of any plain of $P(y, x)$ is small and depends only on the hyperbolicity of G. As a consequence, we get that on any given shortest path P from an arbitrary vertex to a closest central vertex, the number of vertices with locality more than 1 does not exceed $\max\{0, 4\delta - 1\}$. Furthermore, only at most 2δ of them are located outside $C^{\delta}(G)$ and only at most $2\delta + 1$ of them are at distance $> 2\delta$ from $C(G)$.

First, we consider any shortest path between two arbitrary vertices and show that any plain on it has width at most $4\delta + 1$. If a plain is elevated and/or is far enough from the end-vertices of the shortest path, then its width is even smaller.

Theorem 10. Let G be a δ-hyperbolic graph and let $P(y, x)$ be a shortest path between any vertex $y \in V$ and any vertex $x \in V$.

(*i*) Any plain $(u, ..., v)$ of $P(y, x)$ has width $d(u, v) \leq 4\delta + 1$. Terraces are absent, if $\delta = 0$, and have width at most $4\delta - 1$, otherwise. Plateaus are absent, if $\delta \leq \frac{1}{2}$, and have width at most $4\delta - 3$, otherwise.

(ii) Any plain $(u, ..., v)$ of $P(y, x)$ with $e(u) > \min\{e(x), e(y)\} + \delta$ has width $d(u, v) \leq 2\delta$, and, if $(u, ..., v)$ is a plateau, it has width at most $2\delta - 2$.

In particular, if $\delta = 1$, then plateaus in any shortest path $P(y, x)$ are absent in eccentricity layers $C^{=k}(G)$ for all $k > \min\{e(x), e(y)\} + 1 - rad(G)$. Moreover, plateaus are completely absent if $\delta = 1$ and every vertex c in $P(y, x)$, $c \neq x$, has $e(c) > e(x)$.

(iii) If there are two vertices $u, v \in P(y, x)$ with $e(u) = e(v) \geq \min\{e(x), e(y)\}$, $d(x, \{u, v\}) > 2\delta$ and $d(y, \{u, v\}) > 2\delta$, then $d(u, v) \leq 2\delta$.

Proof. *(i)* By definition of a plain, terrace, and plateau $(u, ..., v)$, the eccentricity of each vertex $z \in (u, ..., v)$ satisfies $e(z) = e(u)$. In the case of terraces and plateaus, denote by y' (x') a vertex of $P(y, x)$ adjacent to u (to v, respectively) but not in $(u, ..., v)$. Suppose, by contradiction, that $d(u, v) > 4\delta + 1$. By Corollary 15 applied to u and v, any vertex $c \in I(u, v)$ with $d(u, c) > 2\delta$ and $d(c, v) > 2\delta$ has eccentricity $e(c) < \max\{e(u), e(v)\} = e(u)$. The latter contradicts with $e(z) = e(u)$.

Let $(u, ..., v)$ be a terrace of $P(y, x)$ and, without loss of generality, let $e(x') = e(u) - 1$. Suppose, by contradiction, that $d(u, v) > 4\delta$, i.e., $d(u, x') > 4\delta + 1$. By Corollary 15 applied to u and x', we again obtain a contradiction with $e(z) = e(u)$. Hence, terraces have width at most 4δ. If additionally $\delta > 0$, by contradiction, suppose that $d(u, v) = 4\delta$. Let $c \in (u, ..., v)$ be a vertex at distance $2\delta - 1$ from v, i.e., $d(c, x') = 2\delta$ and $d(u, c) = 2\delta + 1$. We apply Lemma 39 to a (u, x')-subpath of $P(y, x)$ and to a vertex $w \in F(c)$. If $d(x', c) \leq (w|u)_{x'}$, then $e(c) \leq e(x') - d(x', c) + 2\delta = e(c) - 1$, a contradiction. If $d(x', c) \geq (w|u)_{x'}$, then $e(c) \leq e(u) - d(u, c) + 2\delta = e(c) - 1$, a contradiction.

Let now $(u, ..., v)$ be a plateau of $P(y, x)$. By maximality of m-segment $(u, ..., v)$ and the definition of a plateau, $x', y' \in C^{=e(u)-1}(G)$. As $(y', u, ..., v, x') \cap C^{e(u)-1}(G) = \{x', y'\}$, by Corollary 18, $d(x', y') \leq 4\delta - 1$, implying $d(u, v) \leq 4\delta - 3$. As $C^{e(u)-1}(G)$ is convex when $\delta \leq \frac{1}{2}$, plateaus can occur only if $\delta \geq 1$.

(ii) Without loss of generality, assume that $e(x) \leq e(y)$, i.e., $e(u) > e(x) + \delta$. By definition of a plain, the eccentricity of each vertex $z \in (u, ..., v)$ satisfies $e(z) = e(u)$. By contradiction, assume that $d(u, v) > 2\delta$. Let $c \in (u, ..., v)$ be a vertex at distance $2\delta + 1$ from u. We apply Lemma 39 to a (u, x)-subpath of $P(y, x)$ containing $(u, ..., v)$ and to a vertex $w \in F(c)$. If $d(x, c) \leq (w|u)_x$, then $e(u) = e(c) \leq e(x) + \delta$, a contradiction. On the other hand, if $d(x, c) \geq (w|u)_x$, then

$e(u) = e(c) \leq e(u) - d(u, c) + 2\delta = e(u) - 1$, a contradiction.

Let now $(u, ..., v)$ be a plateau of $P(y, x)$. By (i), $\delta \geq 1$. By contradiction, assume that $d(u, v) > 2\delta - 2$. Apply Lemma 39 to a (y', x)-subpath of $P(y, x)$ containing $(u, ..., v)$ and to a vertex $w \in F(v)$. If $d(x, v) \leq (w|y')_x$, then $e(u) = e(v) \leq e(x) + \delta$, a contradiction. On the other hand, if $d(x, v) \geq (w|y')_x$, then $e(v) \leq e(y') - d(y', v) + 2\delta = e(v) - 1 - d(u, v) - 1 + 2\delta < e(v)$, a contradiction. Consider now the case when $\delta = 1$. By the previous claim, any plateau can occur only at eccentricity layer $C^{=k}(G)$ for $k \leq \min\{e(x), e(y)\} + \delta - rad(G) = e(x) + 1 - rad(G)$. Finally, suppose that every vertex c in $P(y, x)$, $c \neq x$, has $e(c) > e(x)$ and there is a plateau $(u, ..., v)$ at eccentricity layer $C^{=k}(G)$. By maximality of m-segment $(u, ..., v)$ and the definition of a plateau, $e(y') = k - 1 + rad(G) \leq e(x)$, a contradiction with $e(x) < e(z)$ for all $z \in P(y, x)$.

(iii) Assume $d(x, \{u, v\}) > 2\delta$ and $d(y, \{u, v\}) > 2\delta$ for vertices $u, v \in P(y, x)$ with $e(u) = e(v)$. Without loss of generality, let $d(x, v) \leq d(x, u)$ and $e(x) \leq e(y)$. Consider an arbitrary $z \in F(v)$. If $d(x, v) \leq (z|y)_x$ then, by Lemma 39, $e(v) \leq e(x) - d(x, v) + 2\delta < e(x)$, and a contradiction with $e(x) \leq e(v)$ arises. Thus, $d(x, v) > (z|y)_x$ and, by Lemma 39, $d(y, u) \leq d(y, v) \leq (z|x)_y$ and $e(v) \leq e(y) - d(y, v) + 2\delta = e(y) - (d(y, u) + d(u, v)) + 2\delta$. By the triangle inequality, $e(v) = e(u) \geq e(y) - d(y, u)$. Combining the previous two inequalities, we have $e(y) - d(y, u) \leq e(v) \leq e(y) - d(y, u) - d(u, v) + 2\delta$. Therefore, $d(u, v) \leq 2\delta$. □

We define a shortest path $P(y, x)$ from a vertex $y \in V$ to a vertex $x \in V$ to be *end-minimal* if $e(x)$ is minimal among all vertices of $P(y, x)$, that is, all $v \in P(y, x)$ satisfy $e(x) \leq e(v)$. $P(y, x)$ is referred to as *strict end-minimal* if all $v \in P(y, x)$ with $v \neq x$ satisfy $e(x) < e(v)$. Notice that any shortest path from an arbitrary vertex to a closest central vertex is strict end-minimal. We turn our focus now to end-minimal shortest paths because any shortest path can be decomposed into two end-minimal subpaths. Let $v \in P(y, x)$ be a vertex closest to x of minimal eccentricity on shortest path $P(y, x)$. Then $P(y, x)$ is represented by end-minimal shortest path $P(y, v)$ joined with (strict) end-minimal shortest path $P(x, v)$.

The following theorem shows that, in an end-minimal shortest path $P(y, x)$ from y to x, all up-hills have bounded height, each vertex that is far from x cannot have eccentricity higher than $e(y) + \delta$, and for each vertex c that is far from the extremities of $P(y, x)$, all vertices z of $P(y, x)$ which are between y and c and with $d(c, z) \geq 2\delta + 1$ have eccentricity larger than $e(c)$.

Theorem 11. Let G be a δ-hyperbolic graph and let $P(y,x) = (y = v_0, v_1, ..., v_p = x)$ be an end-minimal shortest path from y to x.

(i) Any up-hill $(u, ..., v)$ of $P(y,x)$ has height $d(u,v) \leq \delta$.

(ii) Any $c \in P(y,x)$ with $d(c,x) > 2\delta$ satisfies $e(c) \leq e(y) + \delta$ and $e(c) \leq e(y) - d(c,y) + 2\delta$.

(iii) If $d(y,x) > 4\delta + 1$, then all vertices $v_i \in P(y,x)$ with $i \in [2\delta+1, p-2\delta-1]$ have $e(v_i) < \min\{e(v_k) : k \in [0, i-2\delta-1]\}$ *(a kind of pseudodescending).*

Proof. (i) Let $(u, ..., v)$ be an m-segment on $P(y,x)$ which forms an up-hill. By definition of an up-hill, eccentricity increases by one along each edge. Therefore, $e(v) = e(u) + d(u,v)$. By Corollary 15 applied to a (u,x)-subpath of $P(y,x)$, and because $e(x)$ is minimal on $P(y,x)$, we have $e(u) + d(u,v) = e(v) \leq \max\{e(u), e(x)\} + \delta = e(u) + \delta$. Thus, $d(u,v) \leq \delta$.

(ii) Let v be an arbitrary vertex from $F(c)$. Assume $d(x,c) \leq (v|y)_x$. By Lemma 39, $e(c) \leq e(x) - d(x,c) + 2\delta < e(x)$, a contradiction with $e(c) \geq e(x)$. Let now $d(x,c) > (v|y)_x$. Then, by Lemma 39, $e(c) \leq e(y) + \delta$ and $e(c) \leq e(y) - d(c,y) + 2\delta$.

(iii) By contradiction assume that a vertex $v_i \in P(y,x)$ with $i \in [2\delta+1, p-2\delta-1]$ has $e(v_i) \geq e(v_k)$ for some $k \in [0, i-2\delta-1]$. Then, $d(v_k, v_i) \geq 2\delta+1$ and $d(v_i, x) \geq 2\delta+1$, and therefore $d(v_k, x) \geq 4\delta+2$. By Corollary 15 applied to a subpath $P(v_k, x)$ of $P(y,x)$, vertex $v_i \in P(v_k, x)$ with $d(v_k, v_i) > 2\delta$ and $d(v_i, x) > 2\delta$ satisfies $e(v_i) < \max\{e(v_k),\ e(x)\}$. As $e(x)$ is minimal on $P(v_k, x)$, $e(v_i) < e(v_k)$, a contradiction. □

Theorem 11 in part (iii) greatly generalizes a result from [8] where it was shown that any vertex v has $loc(v) \leq 2\delta+1$ or $C(G) \subseteq D(v, 4\delta+1)$. In particular, we have the following corollary.

Corollary 19. Let G be a δ-hyperbolic graph and $P(v,c)$ be a shortest path from an arbitrary vertex v to an arbitrary central vertex $c \in C(G)$. Then, either the length of $P(v,c)$ is at most $4\delta+1$ or the vertex u of $P(v,c)$ at distance $2\delta+1$ from v satisfies $e(u) < e(v)$.

An illustration of several results for an end-minimal shortest path $P(y,x)$ is shown in Figure 5.1.

We next show that any end-minimal shortest path $P(y,x)$ from y to x has no more than $4\delta+1$ up-edges and horizontal-edges combined. Moreover, our result implies that, on any strict end-minimal shortest path $P(y,x)$ from an arbitrary vertex y to a vertex x, the number of vertices

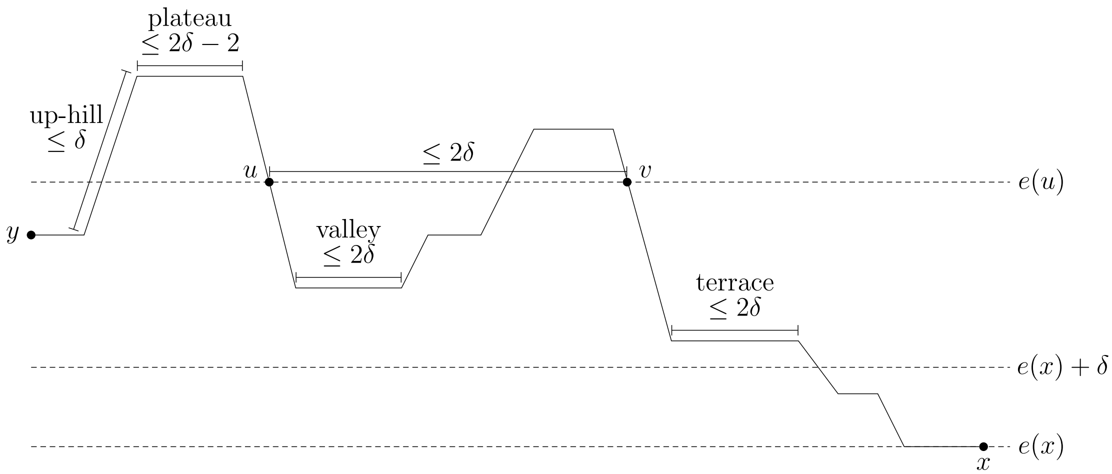


Figure 5.1: End-minimal shortest path $P(y, x)$ is depicted as it travels through the eccentricity layers of G. By Theorem 10 and Theorem 11, any up-hill on $P(y, x)$ has height at most δ and any plain (including plateau, valley, or terrace) that is above the layer $C^{=e(x)+\delta}(G)$ has width at most 2δ. Moreover, by Theorem 10, two vertices u, v with the same eccentricity have distance at most 2δ provided they are far (at least $2\delta + 1$) from both end-vertices y and x.

with locality more than 1 does not exceed 4δ. We give also two simple conditions which limit this number to 2δ.

Lemma 44. Let G be a δ-hyperbolic graph and $P(y, x)$ be a shortest path from y to x. Then, the following holds:

$$d(x,y) \leq e(y) - e(x) + \begin{cases} \max\{0, 4\delta - 1\}, & \text{if } P(y,x) \text{ is strict end-minimal,} \\ 4\delta + 1, & \text{if } P(y,x) \text{ is end-minimal.} \end{cases}$$

Moreover, for a vertex x' of $P(y, x)$, $d(y, x') \leq e(y) - e(x') + 2\delta$ if $e(x') > e(x) + \delta$ or $P(y, x)$ is end-minimal and $d(x, x') > 2\delta$.

Proof. Let $e(x) \leq e(c)$ for all $c \in P(y, x)$. By contradiction, assume $d(x, y) \geq e(y) - e(x) + 4\delta + 2$. As $e(y) - e(x) \geq 0$, $d(x, y) \geq 4\delta + 2$. Pick a vertex $c \in P(x, y)$ at distance $2\delta + 1$ from x. Let v be an arbitrary vertex from $F(c)$. If $d(x, c) \leq (v|y)_x$ then, by Lemma 39, $e(c) \leq e(x) - d(x, c) + 2\delta = e(x) - 1$, a contradiction with $e(c) \geq e(x)$. Hence, $d(x, c) > (v|y)_x$ and, by Lemma 39, $e(c) \leq e(y) - d(y, c) + 2\delta$. Therefore, $d(y, c) \leq e(y) - e(c) + 2\delta \leq e(y) - e(x) + 2\delta$. Since $c \in I(x, y)$, we obtain $d(x, y) = d(x, c) + d(c, y) \leq 2\delta + 1 + e(y) - e(x) + 2\delta = e(y) - e(x) + 4\delta + 1$, a contradiction.

In the remaining case, when $P(y,x)$ is strict end-minimal, we apply similar arguments as above but with vertex c at distance ℓ from x, where $\ell = 1$ when $\delta = 0$ and $\ell = 2\delta$ when $\delta \geq 1/2$. We have $e(y) > e(x)$ and $e(c) > e(x)$. By contradiction, assume $d(x,y) \geq e(y) - e(x) + 4\delta$. That is, $d(x,y) \geq 4\delta + 1$. If $d(x,c) \leq (v|y)_x$ then, by Lemma 39, $e(c) \leq e(x) - d(x,c) + 2\delta \leq e(x)$, a contradiction with $e(c) > e(x)$. If $d(x,c) > (v|y)_x$ then, by Lemma 39, $e(c) \leq e(y) - d(y,c) + 2\delta$. Therefore, $d(y,c) \leq e(y) - e(c) + 2\delta \leq e(y) - e(x) - 1 + 2\delta$. Hence, $d(x,y) = d(x,c) + d(c,y) \leq d(x,c) + e(y) - e(x) + 2\delta - 1$. That is, $d(x,y) \leq e(y) - e(x) + 4\delta - 1$ when $\delta > 0$ (contradicting our assumption), and $d(x,y) \leq e(y) - e(x) = e(y) - e(x) + 4\delta$ when $\delta = 0$.

Let now x' be a vertex of $P(y,x)$ with $e(x') > e(x) + \delta$ or $d(x,x') > 2\delta$ and $P(y,x)$ is end-minimal. By Lemma 39, either $e(x') \leq e(x) + \delta$ and $e(x') \leq e(x) - d(x,x') + 2\delta$ holds or $e(x') \leq e(y) - d(y,x') + 2\delta$ holds. If the former case is true, necessarily, $P(y,x)$ is end-minimal, $d(x,x') > 2\delta$ and $e(x') \leq e(x) - d(x,x') + 2\delta < e(x)$, contradicting with $P(y,x)$ being end-minimal. In the latter case, $d(y,x') \leq e(y) - e(x') + 2\delta$. □

Theorem 12. If G is δ-hyperbolic, then for any shortest path $P(y,x)$ from y to x the following holds:

$$2\mathbb{U}(P(y,x)) + \mathbb{H}(P(y,x)) \leq \begin{cases} \max\{0, 4\delta - 1\}, & \text{if } P(y,x) \text{ is strict end-minimal,} \\ 4\delta + 1, & \text{if } P(y,x) \text{ is end-minimal.} \end{cases}$$

Moreover, for a vertex x' of $P(y,x)$, $2\mathbb{U}(P(y,x')) + \mathbb{H}(P(y,x')) \leq 2\delta$ if $e(x') > e(x) + \delta$ or $P(y,x)$ is end-minimal and $d(x,x') > 2\delta$.

Proof. The proof follows directly from Lemma 1 and Lemma 44. □

Corollary 20. Let G be a δ-hyperbolic graph. Then, on any strict end-minimal shortest path $P(y,x)$ from a vertex y to a vertex x, the number of vertices with locality more than 1 does not exceed 4δ. If additionally $x \in C(G)$, then the number of vertices with locality more than 1 does not exceed $\max\{0, 4\delta - 1\}$.

Proof. As we go from y to x along $P(y,x)$, every vertex u of $P(y,x)$, except x, is the beginning of an edge (u,v) on $P(y,x)$. If an ordered pair (u,v) forms a down-edge, then the vertex u has locality 1 in G. Only when an ordered pair (u,v) forms an up-edge or a horizontal-edge on $P(y,x)$, then the vertex u may have locality more than 1 in G. If $\delta > 0$, then any strict end-minimal shortest path $P(y, x)$ has no more than $4\delta - 1$ up-edges and horizontal-edges combined. Hence, together with x, there are at most 4δ vertices on $P(y, x)$ with locality more than 1. If $\delta = 0$, then G is a block (and hence, a Helly) graph and each of its non-central vertices has locality 1 [49]. Recall that, by definition, the locality of a central vertex is 0. □

Corollary 21. Let G be a δ-hyperbolic graph. Then, on any shortest path $P(y, x)$ between a vertex y and a vertex x, the number of vertices with locality more than 1 does not exceed $8\delta + 1$. If $P(y, x)$ is end-minimal, then the number of vertices with locality more than 1 does not exceed $4\delta + 2$.

Proof. Let $P(y, x)$ be an end-minimal shortest path from y to x. Using Theorem 12 and same arguments as in the proof of Corollary 20, we get that at most $4\delta + 1$ vertices of $P(y, x) \setminus \{x\}$ have locality more than 1. Hence, together with x, there are at most $4\delta + 2$ vertices on $P(y, x)$ with locality more than 1.

Let now $P(y, x)$ be an arbitrary shortest path between y and x. Let also v be a vertex from $P(y, x)$ with minimal eccentricity closest to x. Then, subpath $P(x, v)$ of $P(x, y)$ is strict end-minimal and subpath $P(y, v)$ of $P(y, x)$ is end-minimal. There are at most 4δ vertices with locality more than 1 in $P(x, v)$ and there are at most $4\delta+1$ vertices with locality more than 1 in $P(y, v)\setminus\{v\}$. Thus, $P(y, x)$ has at most $8\delta + 1$ such vertices. □

These corollaries can be refined in the following way.

Corollary 22. Let $P(y, x)$ be a shortest path from any vertex y to any vertex x. Then, a (prefix) subpath $P(y, x')$ of $P(y, x)$ has at most k vertices with locality more than 1, where

$$k = \begin{cases} 2\delta, & \text{if } e(x') > e(x) + \delta, \\ 2\delta + 1, & \text{if } P(y, x) \text{ is end-minimal and } d(x, x') > 2\delta. \end{cases}$$

Proof. As, by Theorem 12, $2\mathbb{U}(P(y, x')) + \mathbb{H}(P(y, x')) \leq 2\delta$, we can use same arguments as in the proof of Corollary 20 to show that in $P(y, x') \setminus \{x'\}$ there are at most 2δ vertices with locality more than 1. It remains only to show that the entire $P(y, x')$ has at most 2δ vertices with locality more than 1 when $e(x') > e(x) + \delta$. Without loss of generality, we can pick a vertex $x' \in P(y, x)$, with

$e(x') > e(x) + \delta$, that is furthest from y. By the choice of x', the neighbor x'' of x' on $P(y,x)$ that is closer to x satisfies $e(x'') = e(x) + \delta = e(x') - 1$. Hence, $loc(x') = 1$. □

Thus, on any shortest path from an arbitrary vertex to a closest central vertex, there are at most $\max\{0, 4\delta - 1\}$ vertices with locality more than 1, and only at most 2δ of them are located outside $C^{\delta}(G)$ and only at most $2\delta + 1$ of them are at distance $> 2\delta$ from $C(G)$.

In certain graph classes up-hills and plains are restricted in their location along a shortest path $P(y,x)$ connecting a vertex y to a closest central vertex $x \in C(G)$. Helly graphs contain no such non-descending shapes [49], whereas chordal graphs [63] and distance-hereditary graphs (cf. Chapter 4) have no up-hills but plains of width at most 1 may occur. Furthermore, for every vertex y, there is a shortest path $P(y,x)$ from y to any closest central vertex x such that it has at most one plain of width 1, and if a plain exists then it is located in layer $C^{=1}(G)$. We observe that in hyperbolic graphs even up-hills can occur anywhere on any shortest path - it can be close or far from a central vertex or endpoints of the path.

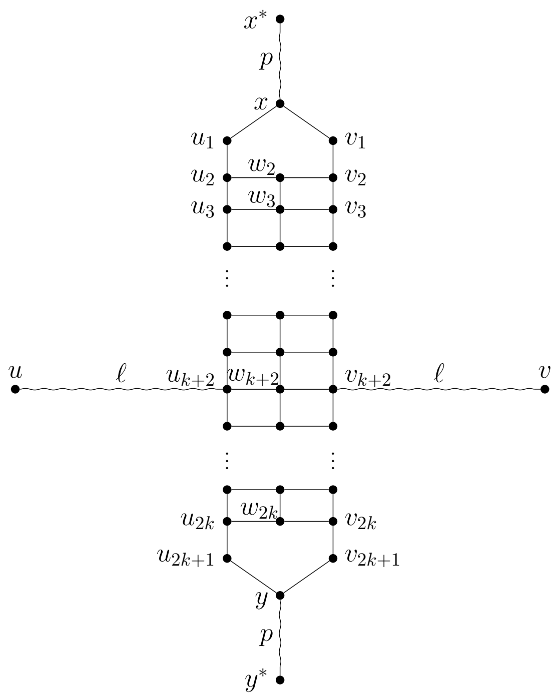


Figure 5.2: An illustration that in a 2-hyperbolic graph up-hills on each shortest path to the center or to a furthest vertex can occur very far from the center and from the endpoints of the path.

Consider a 2-hyperbolic graph $G = (V,E)$ depicted in Figure 5.2. It has two paths

$(x, u_1, u_2, ..., u_{2k+1}, y)$ and $(x, v_1, v_2, ..., v_{2k+1}, y)$ of length $2k+2$, a path $(w_2, w_3, ..., w_{2k})$ of length $2k-2$, and edges $u_i w_i \in E$ and $w_i v_i \in E$ for each $i \in [2, 2k]$. It has also two paths each of length ℓ connecting vertex u_{k+2} to vertex u as well as vertex v_{k+2} to vertex v, and two paths each of length $p > 0$ connecting x to x^* as well as y to y^*. If $\ell = k+p$, then $diam(G) = 2\ell+2 = d(u, v) = d(x^*, y^*)$, $rad(G) = \ell + 2$, and $C(G) = \{u_{k+2}, w_{k+1}, v_{k+2}\}$. Observe that $e(x) = d(x, u) = \ell + k + 2$ whereas $e(u_1) = d(u_1, v) = \ell + k + 3$ and $e(v_1) = d(v_1, u) = \ell + k + 3$. Any shortest (x^*, z)-path where $z \in C(G)$ or $z \in F(x^*)$ contains either the up-hill (x, u_1) or the up-hill (x, v_1). However, both up-hills are arbitrarily far from the center $C(G)$ and far from any furthest vertex in $F(x^*)$. Both up-hills also occur arbitrarily far from the starting vertex x^* of the path. Up-hills also occur on all shortest paths between the diametral pair (x^*, y^*).

5.3 Bounds on the eccentricity of a vertex

In this section, we show that the auxiliary lemmas stated earlier yield several known from [33, 61] results on finding a vertex with small or large eccentricity, as well as intermediate results regarding the relationship between diameter and radius. We obtain also new efficient algorithms for approximating all vertex eccentricities in δ-hyperbolic graphs and compare them with known results on graphs with τ-thin triangles [36]. We present the following algorithms for approximating all eccentricities: a $O(\delta|E|)$ time left-sided additive 2δ-approximation, a $O(\delta|E|)$ time right-sided additive $(4\delta + 1)$-approximation, and a $O(|E|)$ time right-sided additive 6δ-approximation.

But first, we establish some lower and upper bounds on the eccentricity of any vertex based on its distance to either $C(G)$ or $C^{2\delta}(G)$, and vice versa.

Relationship between eccentricity of a vertex and its distance to $C(G)$ or $C^{2\delta}(G)$

In this subsection, we show that the eccentricity of a vertex is closely related to its distance to both $C(G)$ and $C^{2\delta}(G)$, analogous up to $O(\delta)$ to that of Helly graphs.

We will need the following lemma which is a consequence of Lemma 39.

Lemma 45. Let G be a δ-hyperbolic graph and let $x \in V$, $y \in F(x)$. Any vertex $c \in S_{rad(G)+k}(y, x)$, $0 \leq k \leq d(x, y) - rad(G)$, has $e(c) \leq rad(G) + 2\delta + k$. In particular, $c \in S_{rad(G)}(y, x)$ has $e(c) \leq rad(G) + 2\delta$ [33, 61].

Proof. Let v be any vertex from $F(c)$. If $d(x,c) \leq (v|y)_x$ then, by Lemma 39 and the fact that $d(v,x) \leq d(x,y)$, we have $e(c) = d(c,v) \leq d(x,v) - d(x,c) + 2\delta \leq d(x,y) - (d(x,y) - d(y,c)) + 2\delta = rad(G) + 2\delta + k$. On the other hand, if $d(x,c) \geq (v|y)_x$ then, by Lemma 39 and the fact that $2rad(G) \geq d(v,y)$, we have $e(c) = d(c,v) \leq d(y,v) - d(y,c) + 2\delta \leq 2rad(G) - rad(G) - k + 2\delta = rad(G) + 2\delta - k$. □

Theorem 13. Let G be a δ-hyperbolic graph. Any vertex x of G satisfies the following inequalities:

$$d(x, C^{2\delta}(G)) + rad(G) + 2\delta \geq e(x) \geq d(x, C^{2\delta}(G)) + rad(G).$$

$$d(x, C(G)) + rad(G) - 4\delta \leq e(x) \leq d(x, C(G)) + rad(G).$$

Proof. Let $y \in F(x)$ be a furthest vertex from x, c (c') be a vertex closest to x in $C(G)$ ($C^{2\delta}(G)$, respectively). On one hand, by the triangle inequality, $e(x) = d(x,y) \leq d(x,c) + d(c,y) \leq d(x,c) + e(c) = d(x, C(G)) + rad(G)$ and $e(x) = d(x,y) \leq d(x,c') + d(c',y) \leq d(x,c') + e(c') = d(v, C^{2\delta}(G)) + rad(G) + 2\delta$. On the other hand, by Lemma 44, $d(x,c) \leq e(x) - e(c) + 4\delta$. Thus, $e(x) \geq d(x,c) + e(c) - 4\delta = d(x, C(G)) + rad(G) - 4\delta$. Furthermore, by Lemma 45, any vertex $c^* \in S_{rad(G)}(y,x)$ satisfies $e(c^*) \leq rad(G) + 2\delta$. Hence, $e(x) = d(x,c^*) + d(c^*,y) \geq d(x, C^{2\delta}(G)) + rad(G)$. □

It is known [36] that if G is a τ-thin graph, then for any vertex $x \in V$, $d(x, C(G)) + rad(G) - 4\tau - 2 \leq e(x)$. Applying the inequality $\tau \leq 4\delta$ from Proposition 1 yields $d(x, C(G)) + rad(G) - 16\delta - 2 \leq e(x)$. Working directly with δ, in Theorem 13, we obtained a significantly better bound with δ, which, as $\delta \leq \tau$, also improves the bound known with τ.

Corollary 23. Let G be a τ-thin graph. Any vertex x satisfies the following inequality:

$$d(x, C(G)) + rad(G) - 4\tau \leq e(x) \leq d(x, C(G)) + rad(G).$$

Let x be an arbitrary vertex with eccentricity $e(x) = rad(G) + k$ for some integer $k \geq 0$. By

Theorem 13, we have:

$$k \geq d(x, C^{2\delta}(G)) \geq k - 2\delta \tag{5.1}$$

$$k \leq d(x, C(G)) \leq k + 4\delta \tag{5.2}$$

Hence, one obtains a relationship also between the distance from x to $C(G)$ and to $C^{2\delta}(G)$.

Corollary 24. Let G be a δ-hyperbolic graph and let $x \in V$ with $e(x) = rad(G) + k$. Then, $d(x, C^{2\delta}(G)) \leq k \leq d(x, C(G))$. Moreover, $d(x, C^{2\delta}(G)) = \ell$ implies $d(x, C(G)) \leq \ell + 6\delta$.

Proof. Combining equations (5.1) and (5.2) yields $d(x, C^{2\delta}(G)) \leq k \leq d(x, C(G))$. Assume now that $d(x, C^{2\delta}(G)) = \ell$. By equation (5.1), $\ell \geq k - 2\delta$. By equation (5.2), $d(x, C(G)) \leq k + 4\delta \leq \ell + 6\delta$. □

Now, we turn our focus from end-minimal shortest paths to shortest (x, y)-paths wherein $y \in F(x)$ and $x \in F(z)$ for some vertex $z \in V$ or when $\{x, y\}$ is a mutually distant pair.

Finding a vertex with small or large eccentricity and left-sided additive approximation of all vertex eccentricities

Let $\{x, y\}$ be a pair of vertices such that $y \in F(x)$ and $x \in F(z)$ for some vertex $z \in V$. In Lemma 45, we established that the eccentricity of vertex c_r on any shortest (x, y)-path at distance $rad(G)$ from y has small eccentricity (within 2δ of radius). Of more algorithmic convenience, we show here that even a middle vertex c_m of any shortest (x, y)-path has small eccentricity (within 3δ of radius), and its eccentricity is even smaller (within 2δ of radius) if $x \in F(y)$ as well, i.e., when $\{x, y\}$ is a mutually distant pair.

We will need the following lemma from [61].

Lemma 46. [61] Let G be a δ-hyperbolic graph. For every quadruple $c, v, x, y \in V$, $d(x, v) - d(x, y) \geq d(c, v) - d(y, c) - 2\delta$ or $d(y, v) - d(x, y) \geq d(c, v) - d(x, c) - 2\delta$ holds.

Interestingly, the distances from any vertex c to two mutually distant vertices give a very good estimation on the eccentricity of c.

Theorem 14. Let G be a δ-hyperbolic graph, and let $\{x, y\}$ be a mutually distant pair of vertices. Any vertex $c \in V$ has $\max\{d(x,c), d(y,c)\} \leq e(c) \leq \max\{d(x,c), d(y,c)\} + 2\delta$. Moreover, any vertex $c^* \in S_{\lfloor d(x,y)/2 \rfloor}(x, y) \cup S_{\lfloor d(x,y)/2 \rfloor}(y, x)$ has $e(c^*) \leq \lceil d(x,y)/2 \rceil + 2\delta \leq rad(G) + 2\delta$ [61]. In particular, $diam(G) \geq d(x,y) \geq 2rad(G) - 4\delta - 1$ [33].

Proof. The inequality $e(c) \geq \max\{d(x,c), d(y,c)\}$ holds for any three vertices by definition of eccentricity. To prove the upper bound on $e(c)$ for any $c \in V$, consider a furthest vertex $v \in F(c)$. Note that, as x and y are mutually distant, $d(x,y) \geq \max\{d(x,v), d(y,v)\}$. By Lemma 46, for every $x, y, v, c \in V$ either $d(x,v) - d(x,y) \geq d(c,v) - d(y,c) - 2\delta$ or $d(y,v) - d(x,y) \geq d(c,v) - d(x,c) - 2\delta$ holds. If the former is true, then $d(c,v) \leq d(x,v) - d(x,y) + d(y,c) + 2\delta \leq d(y,c) + 2\delta$. If the latter is true, then $d(c,v) \leq d(y,v) - d(x,y) + d(x,c) + 2\delta \leq d(x,c) + 2\delta$. Thus, $e(c) \leq \max\{d(x,c), d(y,c)\} + 2\delta$.

Moreover, if c^* is a middle vertex of $I(x,y)$, i.e., $c^* \in S_{\lfloor d(x,y)/2 \rfloor}(x,y) \cup S_{\lfloor d(x,y)/2 \rfloor}(y,x)$, then $e(c^*) \leq \max\{d(x,c^*), d(y,c^*)\} + 2\delta = \lceil d(x,y)/2 \rceil + 2\delta \leq \lceil 2rad(G)/2 \rceil + 2\delta = rad(G) + 2\delta$. In particular, since $\lceil d(x,y)/2 \rceil \geq e(c^*) - 2\delta \geq rad(G) - 2\delta$, $diam(G) \geq d(x,y) \geq 2rad(G) - 4\delta - 1$. □

Furthermore, the eccentricity of a vertex, that is most distant from some other vertex, is close to the distance between any two mutually distant vertices.

Lemma 47. Let G be a δ-hyperbolic graph. For any $x, y, c \in V$, and any furthest vertex $v \in F(c)$, $d(x,y) \leq e(v) + 2\delta$. In particular, $e(v) \geq diam(G) - 2\delta$ [33, 61].

Proof. From the choice of v, necessarily $d(c,v) \geq \max\{d(y,c), d(x,c\}$. By Lemma 46, either $d(x,v) - d(x,y) \geq d(c,v) - d(y,c) - 2\delta$ or $d(y,v) - d(x,y) \geq d(c,v) - d(x,c) - 2\delta$ holds. If the former is true, then $d(x,y) \leq d(x,v) + d(y,c) - d(c,v) + 2\delta \leq d(x,v) + 2\delta$. If the latter is true, then $d(x,y) \leq d(y,v) + d(x,c) - d(c,v) + 2\delta \leq d(y,v) + 2\delta$. In either case $d(x,y) \leq \max\{d(x,v), d(y,v)\} + 2\delta \leq e(v) + 2\delta$, establishing the result. In particular, when $d(x,y) = diam(G)$, then $e(v) \geq diam(G) - 2\delta$. □

With Theorem 14 and Lemma 47, one obtains the following corollary known from [33, 61].

Corollary 25. [33,61] Let G be a δ-hyperbolic graph. For any vertex $x \in V$, every vertex $y \in F(x)$ has $e(y) \geq diam(G) - 2\delta \geq 2rad(G) - 6\delta - 1$.

Of a great algorithmic interest is also the following result.

Corollary 26. Let G be a δ-hyperbolic graph, where $z \in V$, $x \in F(z)$, and $y \in F(x)$. Any vertex $c \in S_{\lfloor d(x,y)/2 \rfloor}(x, y)$ has $e(c) \leq rad(G) + 3\delta$ [33, 61]. Moreover, $e(c) \leq \lceil d(x, y)/2 \rceil + 4\delta$.

Proof. Let $v \in F(c)$ be a furthest vertex from c. Since $y \in F(x)$, $d(x, v) \leq d(x, y)$. We also have $d(x, y) \leq 2rad(G)$. If $d(x, c) \leq (v|y)_x$ then, by Lemma 39, $e(c) = d(c, v) \leq d(x, v) - d(x, c) + 2\delta \leq d(x, y) - d(x, c) + 2\delta = \lceil d(x, y)/2 \rceil + 2\delta \leq rad(G) + 2\delta$.

On the other hand, if $d(x, c) \geq (v|y)_x$ then, by Lemma 39, $e(c) = d(c, v) \leq d(y, v) - d(y, c) + 2\delta$. By Corollary 25, $d(y, c) = \lceil d(x, y)/2 \rceil = \lceil e(x)/2 \rceil \geq \lceil (diam(G) - 2\delta)/2 \rceil = \lceil diam(G)/2 \rceil - \delta$. Therefore, $e(c) \leq d(y, v) - d(y, c) + 2\delta \leq diam(G) - \lceil diam(G)/2 \rceil + 3\delta \leq rad(G) + 3\delta$. Thus, $e(c) \leq rad(G) + 3\delta$. Moreover, by Corollary 25, $diam(G) \leq d(x, y) + 2\delta$. Hence, $e(c) \leq d(y, v) - d(y, c) + 2\delta \leq diam(G) - d(y, c) + 2\delta \leq d(x, y) - d(y, c) + 4\delta \leq \lfloor d(x, y)/2 \rfloor + 4\delta \leq \lceil d(x, y)/2 \rceil + 4\delta$. $\square$

In Section 5.1, we saw that the diameter in G of any set $C^k(G)$, $k \in \mathbb{N}$, is bounded by $2k + 4\delta + 1$. In particular, $diam(C(G)) \leq 4\delta + 1$ holds [33]. Note that, in [33], it was additionally shown that all central vertices are close to a middle vertex c of a shortest (x, y)-path, provided that x is furthest from some vertex and that y is furthest from x. Namely, $D(c, 5\delta + 1) \supseteq C(G)$ holds. Here, we provide such a result with respect to $C^k(G)$ for all $k \in \mathbb{N}$.

Lemma 48. Let G be a δ-hyperbolic graph, and let $z \in V$, $x \in F(z)$, and $y \in F(x)$. Any middle vertex $c \in S_{\lceil d(x,y)/2 \rceil}(x, y) \cup S_{\lceil d(x,y)/2 \rceil}(y, x)$ satisfies $D(c, 5\delta + 1 + k) \supseteq C^k(G)$. In particular, $D(c, 5\delta + 1) \supseteq C(G)$ [33].

Proof. Consider an arbitrary vertex $u \in C_{\leq k}(G)$. By Corollary 13, $d(c, u) \leq \max\{d(x, u), d(y, u)\} - \min\{d(x, c), d(y, c)\} + 2\delta$. As $e(u) \leq rad(G) + k$, $\max\{d(x, u), d(y, u)\} \leq rad(G) + k$ holds. As c is a middle vertex of a shortest (x, y)-path, $\min\{d(x, c), d(y, c)\} = \lfloor d(x, y)/2 \rfloor$. Since $x \in F(z)$, by Corollary 25, $e(x) = d(x, y) \geq 2rad(G) - 6\delta - 1$. Hence, $d(c, u) \leq rad(G) + k - \lfloor (2rad(G) - 6\delta - 1)/2 \rfloor + 2\delta \leq 5\delta + 1 + k$. $\square$

In [36], it was proven that if G is a τ-thin graph, then any middle vertex c of any shortest path $P(x, y)$ between two mutually distant vertices x and y satisfies $D(c, k + 2\tau + 1) \supseteq C^k(G)$. Applying the inequality $\tau \leq 4\delta$ from Proposition 1 to this result, one obtains $D(c, k + 8\delta + 1) \supseteq C^k(G)$. The latter result can be improved working directly with δ.

Lemma 49. Let G be a δ-hyperbolic graph, and let $\{x, y\}$ be a mutually distant pair. Any middle vertex $c \in S_{\lceil d(x,y)/2 \rceil}(x, y) \cup S_{\lceil d(x,y)/2 \rceil}(y, x)$ satisfies $D(c, 4\delta + 1 + k) \supseteq C^k(G)$. In particular, $D(c, 4\delta + 1) \supseteq C(G)$.

Proof. The proof is analogous to that of Lemma 48. However, since x, y are mutually distant, by Theorem 14, $d(x, y) \geq 2rad(G) - 4\delta - 1$. Hence, for any $u \in C_{\leq k}(G)$, $d(c, u) \leq rad(G) + k - \lfloor (2rad(G) - 4\delta - 1)/2 \rfloor + 2\delta \leq 4\delta + 1 + k$. □

Thus, combining this with the result from [36], we get $D(c, \min\{4\delta(G), 2\tau(G)\} + 1 + k) \supseteq C^k(G)$ for any graph G.

There are several algorithmic implications of the results of this subsection. For an arbitrary connected graph $G = (V, E)$ and a given vertex $z \in V$, a most distant from z vertex $x \in F(z)$ can be found in linear ($O(|E|)$) time by a *breadth-first-search* $BFS(z)$ started at z. A pair of mutually distant vertices of a δ-hyperbolic graph $G = (V, E)$ can be computed in $O(\delta|E|)$ total time as follows. By Lemma 47, if x is a most distant vertex from an arbitrary vertex z and y is a most distant vertex from x, then $d(x, y) \geq diam(G) - 2\delta$. Hence, using at most $O(\delta)$ *breadth-first-searches*, one can generate a sequence of vertices $x := v_1, y := v_2, v_3, \ldots v_k$ with $k \leq 2\delta + 2$ such that each v_i is most distant from v_{i-1} (with $v_0 = z$) and v_k, v_{k-1} are mutually distant vertices (the initial value $d(x, y) \geq diam(G) - 2\delta$ can be improved at most 2δ times).

Thus, by Theorem 14, Lemma 47, Corollary 26, Lemma 48, and Lemma 49, we get the following additive approximations for the radius and the diameter of a δ-hyperbolic graph G.

Corollary 27. Let $G = (V, E)$ be a δ-hyperbolic graph.

(*i*) [33,61] There is a linear ($O(|E|)$) time algorithm which finds in G a vertex c with eccentricity at most $rad(G) + 3\delta$ and a vertex v with eccentricity at least $diam(G) - 2\delta$. Furthermore, $C(G) \subseteq D(c, 5\delta + 1)$ holds.

(*ii*) There is an almost linear ($O(\delta|E|)$) time algorithm which finds in G a vertex c with eccentricity at most $rad(G) + 2\delta$ [61]. Furthermore, $C(G) \subseteq D(c, 4\delta + 1)$ holds.

In a graph G where vertex degrees and the hyperbolicity $\delta(G)$ are bounded by constants, the entire center $C(G)$ can be found in linear time by running a breadth-first-search from each vertex of $D(c, 5\delta + 1)$ and selecting among them the vertices with smallest eccentricity. In this case, $D(c, 5\delta + 1)$ contains only a constant number of vertices.

Corollary 28. Let G be a δ-hyperbolic graph with maximum vertex degree $\Delta(G)$. A vertex c with $C(G) \subseteq D(c, 5\delta + 1)$ can be found in $O(|E|)$ time and the center $C(G)$ can be computed in $O(\Delta(G)^{5\delta+1}|E|)$ time [33]. A vertex c with $C(G) \subseteq D(c, 4\delta + 1)$ can be found in $O(\delta|E|)$ time and the center $C(G)$ can be computed in $O(\Delta(G)^{4\delta+1}|E|)$ time. If the degrees of vertices and the hyperbolicity of G are uniformly bounded by a constant, then $C(G)$ can be found in total linear time [33, 61].

By Theorem 14, we get also the following left-sided additive approximations of all vertex eccentricities. Let $\{x, y\}$ be a mutually distant pair of vertices of G. For every vertex $v \in V$, set $\hat{e}(v) := \max\{d(x, v), d(y, v)\}$.

Corollary 29. Let $G = (V, E)$ be a δ-hyperbolic graph. There is an algorithm which in total almost linear ($O(\delta|E|)$) time outputs for every vertex $v \in V$ an estimate $\hat{e}(v)$ of its eccentricity $e_G(v)$ such that $e_G(v) - 2\delta \leq \hat{e}(v) \leq e_G(v)$.

If the hyperbolicity $\delta(G)$ of G is known in advance, we can transform $\hat{e}$ into a right-sided additive 2δ-approximation by setting $\hat{e}(v) := \max\{d(x, v), d(y, v)\} + 2\delta$. This approach generalizes the eccentricity approximation for distance-hereditary graphs described in Chapter 4 (the hyperbolicity of any distance-hereditary graph is at most 1). Unfortunately, if $\delta(G)$ is not known in advance, the best to date algorithm for computing $\delta(G)$ has complexity $O(n^{3.69})$ and relies on some (rather impractical) matrix multiplication results [76] (see also [24] for some recent approximation algorithms). In the next subsection, we will give some right-sided additive approximations of all vertex eccentricities which do not assume any knowledge of $\delta(G)$.

Right-sided additive approximations of all vertex eccentricities

In what follows, we illustrate two right-sided additive eccentricity approximations for all vertices using a notion of *eccentricity approximating spanning tree* introduced in [102] and investigated in [36, 53, 63, 67]. We get a $O(|E|)$ time right-sided additive (6δ)-approximations and a $O(\delta|E|)$ time right-sided additive $(4\delta + 1)$-approximations.

A spanning tree T of a graph G is called an *eccentricity k-approximating spanning tree* if for every vertex v of G $e_T(v) \leq e_G(v) + k$ holds [102]. All $(\alpha_1, \triangle)$-metric graphs (including chordal graphs and the underlying graphs of 7-systolic complexes) admit eccentricity 2-approximating spanning trees [63]. An eccentricity 2-approximating spanning tree of a chordal graph can be computed in linear time [53]. An eccentricity k-approximating spanning tree with minimum k can be found in $O(|V||E|)$ time for any graph G [67]. It is also known [36] that if G is a τ-thin graph, then G admits an eccentricity (2τ)-approximating spanning tree constructible in $O(\tau|E|)$ time and an eccentricity $(6\tau+1)$-approximating spanning tree constructible in $O(|E|)$ time. Applying the inequality $\tau \leq 4\delta$ from Proposition 1, we get that every δ-hyperbolic graph admits an eccentricity 8δ-approximating spanning tree constructible in $O(\delta|E|)$ time and an eccentricity $(24\delta+1)$-approximating spanning tree constructible in $O(|E|)$ time. Both these results can be significantly improved working directly with δ.

Theorem 15. Let G be a δ-hyperbolic graph. If c is a middle vertex of any shortest (x, y)-path between a pair $\{x, y\}$ of mutually distant vertices of G and T is a $BFS(c)$-tree of G, then, for every vertex v of G, $e_G(v) \leq e_T(v) \leq e_G(v) + 4\delta + 1$. That is, G admits an eccentricity $(4\delta + 1)$-approximating spanning tree constructible in $O(\delta|E|)$ time.

Proof. The eccentricity in T of any vertex v can only increase compared to its eccentricity in G. Hence, $e_G(v) \leq e_T(v)$. By the triangle inequality and the fact that all distances from vertex c are preserved in T, $e_T(v) \leq d_T(v, c) + e_T(c) = d_G(v, c) + e_G(c)$. We know that $e_G(v) \geq \max\{d_G(y, v), d_G(x, v)\}$. By Corollary 13, also $d_G(v, c) - \max\{d_G(y, v),\ d_G(x, v)\} \leq 2\delta - \min\{d_G(y, c),\ d_G(x, c)\}$ holds. Since c is a middle vertex of a shortest (x, y)-path, necessarily $\min\{d_G(y, c),\ d_G(x, c)\} \geq \lfloor d_G(x, y)/2 \rfloor$ and, by Theorem 14, $e_G(c) \leq \lceil d_G(x, y)/2 \rceil + 2\delta$. Combining all these, we get

$$\begin{aligned}
e_T(v) - e_G(v) &\le d_G(v,c) + e_G(c) - e_G(v) \\
&\le d_G(v,c) - \max\{d_G(y,v), d_G(x,v)\} + e_G(c) \\
&\le 2\delta - \min\{d_G(y,c),\ d_G(x,c)\} + e_G(c) \\
&\le 2\delta - \lfloor d_G(x,y)/2 \rfloor + e_G(c) \\
&\le 2\delta - \lfloor d_G(x,y)/2 \rfloor + \lceil d_G(x,y)/2 \rceil + 2\delta \\
&\le 4\delta + 1. \qquad \square
\end{aligned}$$

We next give a $O(|E|)$ time right-sided additive eccentricity $(6\delta + 1 - k)$-approximation for any constant integer k, $0 \le k \le 2\delta$.

Theorem 16. Let G be a δ-hyperbolic graph and k be an integer from $[0, 2\delta]$. Let $u_0, u_1, \ldots, u_{k+2}$ be a sequence of vertices of G such that u_0 is an arbitrary start vertex and each u_{i+1} is a vertex furthest from u_i $(0 \le i \le k+1)$. If c is a middle vertex of any shortest (u_{k+1}, u_{k+2})-path and T is a $BFS(c)$-tree of G, then, for every vertex v of G, $e_G(v) \le e_T(v) \le e_G(v) + 6\delta + 1 - k$. That is, G admits an eccentricity $(6\delta + 1 - k)$-approximating spanning tree constructible in $O(k|E|)$ time.

Proof. Recall that if $\{u_{k+1}, u_{k+2}\}$ or an earlier pair $\{u_{i+1}, u_{i+2}\}$ $(i < k)$ is a mutually distant pair then, by Theorem 15, T is an eccentricity $(4\delta+1)$-approximating spanning tree. Therefore, in what remains, we assume that $d_G(u_{i+1}, u_{i+2}) \ge d_G(u_i, u_{i+1}) + 1$ for all $i \le k$. Hence, by Corollary 25, $d_G(u_{k+1}, u_{k+2}) \ge d_G(u_1, u_2) + k \ge diam(G) - 2\delta + k$.

We first claim that a middle vertex c of any shortest (u_{k+1}, u_{k+2})-path satisfies $e_G(c) \le \lceil d_G(u_{k+1}, u_{k+2})/2 \rceil + 4\delta - k$. Let $t \in F(c)$, i.e., $e_G(c) = d_G(c,t)$. By Lemma 39, either $d_G(c,t) \le d_G(u_{k+1}, t) - d_G(u_{k+1}, c) + 2\delta$ or $d_G(c,t) \le d_G(u_{k+2}, t) - d_G(u_{k+2}, c) + 2\delta$. If the former is true then, since $d_G(u_{k+1}, t) \le d_G(u_{k+1}, u_{k+2})$, we have $e_G(c) \le d_G(u_{k+1}, t) - d_G(u_{k+1}, c) + 2\delta \le d_G(u_{k+1}, u_{k+2}) - d(u_{k+1}, c) + 2\delta \le \lceil d_G(u_{k+1}, u_{k+2})/2 \rceil + 2\delta$. If the latter is true then, since $d_G(u_{k+1}, u_{k+2}) \ge diam(G) - 2\delta + k \ge d_G(u_{k+2}, t) - 2\delta + k$, we get $e_G(c) \le d_G(u_{k+2}, t) - d_G(u_{k+2}, c) + 2\delta \le d_G(u_{k+1}, u_{k+2}) + 2\delta - k - d_G(u_{k+2}, c) + 2\delta \le \lceil d_G(u_{k+1}, u_{k+2})/2 \rceil + 4\delta - k$. As $k \le 2\delta$, in either case, $e_G(c) \le \lceil d_G(u_{k+1}, u_{k+2})/2 \rceil + 4\delta - k$, establishing the claim.

Set now $x := u_{k+1}$ and $y := u_{k+2}$. The remainder of the proof follows the proof of Theorem 15 with one adjustment: replace the application of Theorem 14 which yields $e_G(c) \leq \lceil d_G(x,y)/2 \rceil + 2\delta$ with our claim which yields $e_G(c) \leq \lceil d_G(x,y)/2 \rceil + 4\delta - k$. Hence, $e_T(v) - e_G(v) \leq 2\delta - \lfloor d_G(x,y)/2 \rfloor + \lceil d_G(x,y)/2 \rceil + 4\delta - k \leq 6\delta + 1 - k$. □

Theorem 16 generalizes Theorem 15 (when $k = 2\delta$, we obtain Theorem 15). Also, when $k = 1$, we get an eccentricity (6δ)-approximating spanning tree constructible in $O(|E|)$ time. Note that the eccentricities of all vertices in any tree $T = (V, U)$ can be computed in $O(|V|)$ total time. It is a folklore by now that for trees the following facts are true: (1) the center $C(T)$ of any tree T consists of one vertex or two adjacent vertices; (2) the center $C(T)$ and the radius $rad(T)$ of any tree T can be found in linear time; (3) for every vertex $v \in V$, $e_T(v) = d_T(v, C(T)) + rad(T)$. Hence, using $BFS(C(T))$ on T one can compute $d_T(v, C(T))$ for all $v \in V$ in total $O(|V|)$ time. Adding now $rad(T)$ to $d_T(v, C(T))$, one gets $e_T(v)$ for all $v \in V$. Consequently, by Theorem 15 and Theorem 16, we get the following additive approximations for the vertex eccentricities in δ-hyperbolic graphs.

Corollary 30. Let $G = (V, E)$ be a δ-hyperbolic graph.

- (i) There is an algorithm which in total linear $(O(|E|))$ time outputs for every vertex $v \in V$ an estimate $\hat{e}(v)$ of its eccentricity $e_G(v)$ such that $e_G(v) \leq \hat{e}(v) \leq e_G(v) + 6\delta$.
- (ii) There is an algorithm which in total almost linear $(O(\delta|E|))$ time outputs for every vertex $v \in V$ an estimate $\hat{e}(v)$ of its eccentricity $e_G(v)$ such that $e_G(v) \leq \hat{e}(v) \leq e_G(v) + 4\delta + 1$.

As $\delta(G) \leq \tau(G)$ for any graph G, Corollary 30(i) improves the corresponding result from [36]. Also, combining Corollary 30(ii) with the corresponding result from [36], we get that, for any graph G, there is an algorithm which in total almost linear $(O(\delta|E|))$ time outputs for every vertex $v \in V$ an estimate $\hat{e}(v)$ of its eccentricity $e_G(v)$ such that $e_G(v) \leq \hat{e}(v) \leq e_G(v) + \min\{4\delta(G) + 1, 2\tau(G)\}$.

Chapter 6

Helly-gap of a graph and vertex eccentricities

A new metric parameter for a graph, Helly-gap, is introduced. A graph G is called α-weakly-Helly if any system of pairwise intersecting disks in G has a nonempty common intersection when the radius of each disk is increased by an additive value α. The minimum α for which a graph G is α-weakly-Helly is called the Helly-gap of G and denoted by $\alpha(G)$. The Helly-gap of a graph G is characterized by distances in the injective hull $\mathcal{H}(G)$, which is a (unique) minimal Helly graph which contains G as an isometric subgraph. This characterization is used as a tool to generalize many eccentricity related results known for Helly graphs ($\alpha(G) = 0$), as well as for chordal graphs ($\alpha(G) \leq 1$), distance-hereditary graphs ($\alpha(G) \leq 1$) and δ-hyperbolic graphs ($\alpha(G) \leq 2\delta$), to all graphs, parameterized by their Helly-gap $\alpha(G)$. Several additional graph classes are shown to have a bounded Helly-gap, including AT-free graphs and graphs with bounded tree-length, bounded chordality or bounded α_i-metric. First, recall the following definition of a Helly graph.

Definition 7. A graph G is Helly if, for any system of disks $\mathcal{F} = \{D_G(v, r(v)) : v \in S \subseteq V(G)\}$, the following Helly property holds: if $X \cap Y \neq \emptyset$ for every $X, Y \in \mathcal{F}$, then $\bigcap_{v \in S} D_G(v, r(v)) \neq \emptyset$.

Helly graphs are well investigated. They have several characterizations and important features as established in [15, 16, 49, 50, 100, 105]. They are exactly the so-called *absolute retracts of reflexive graphs* and possess a certain elimination scheme [15, 16, 49, 50, 100]. They can be recognized in $O(n^2m)$ time [49] where n is the number of vertices and m is the number of edges. The Helly property works as a compactness criterion on graphs [105]. More importantly, every graph is isometrically embeddable into a Helly graph [66, 83, 95] (see Section 6.1 for more details).

Many nice properties of Helly graphs are based on the eccentricity $e_G(v)$ of a vertex v. Graph's diameter is tightly bounded in Helly graphs as $2rad(G) \geq diam(G) \geq 2rad(G) - 1$ [49, 50]. Moreover, the eccentricity function in Helly graphs is unimodal [50], that is, any local minimum coincides with the global minimum; this is equivalent to the condition that, for any vertex $v \in V(G)$, $e_G(v) = d_G(v, C(G)) + rad(G)$ holds [49]. The unimodality of the eccentricity function was recently used in [55, 68] to compute the radius, diameter and a central vertex of a Helly graph in subquadratic time. For a vertex $v \in V(G)$, let $F_G(v)$ denote the set of all vertices farthest from v, that is, $F_G(v) = \{u \in V(G) : e_G(v) = d_G(v, u)\}$. For every vertex v of a Helly graph G, each vertex $u \in F_G(v)$ satisfies $e_G(u) \geq 2rad(G) - diam(C(G))$ [49]. Although the center $C(G)$ of a Helly graph G may have an arbitrarily large diameter (as any Helly graph is the center of some Helly graph), $C(G)$ induces a Helly graph and is isometric in G [49]. Additionally, any power of a Helly graph is a Helly graph as well [49].

In this chapter, we introduce a far reaching generalization of Helly graphs, α-weakly-Helly graphs. We define *α-weakly-Helly graphs* as those graphs which are "weakly" Helly with the following simple generalization: for any system of disks, if the disks pairwise intersect, then by expanding the radius of each disk by some integer α there forms a common intersection. Thus, Helly graphs are exactly 0-weakly-Helly graphs.

Definition 8 (α-weakly-Helly graph)**.** A graph G is α-weakly-Helly if, for any system of disks $\mathcal{F} = \{D_G(v, r(v)) : v \in S \subseteq V(G)\}$, the following α-weakly-Helly property holds: if $X \cap Y \neq \emptyset$ for every $X, Y \in \mathcal{F}$, then $\bigcap_{v \in S} D_G(v, r(v) + \alpha) \neq \emptyset$.
Clearly, every graph is α-weakly-Helly for some α. We call the minimum α for which a graph G is α-weakly-Helly the Helly-gap of G, denoted by $\alpha(G)$.

Interestingly, there are a few results in the literature which demonstrate that such a α-weakly-Helly property with bounded α is present in some graphs and in some metric spaces. In [38], Chepoi and Estellon showed that for every δ-hyperbolic geodesic space (X, d) (and for every δ-hyperbolic graph G), if disks of the family $\mathcal{F} = \{D(x, r(x)) : v \in S \subseteq X, S \text{ is compact}\}$ pairwise intersect, then the disks $\{D(x, r(x) + 2\delta) : x \in S\}$ have a nonempty common intersection (see also [33]). That is, the disks in δ-hyperbolic geodesic spaces and in δ-hyperbolic graphs satisfy (2δ)-weakly-Helly property. Even earlier, Lenhart, Pollack et al. [96, Lemma 9] established that the disks of simple polygons endowed with the link distance satisfy a Helly-type property which implies 1-weakly-Helly property. For chordal graphs [54] and for distance-hereditary graphs [52] (as well as for a more general class of graphs [31]) the following similar result is known: if for a set $M \subseteq V(G)$ and radius function $r : M \to \mathbb{N}$, every two vertices $x, y \in M$ satisfy $d_G(x, y) \leq r(x) + r(y) + 1$, then there is a clique K in G such that $d_G(v, K) \leq r(v)$ for all $v \in M$. Hence, clearly, every chordal graph and every distance-hereditary graph is 1-weakly-Helly. Note that two disks $D_G(v, p)$ and $D_G(u, q)$ intersect if and only if $d_G(v, u) \leq p + q$.

There are numerous other approaches to generalize in some form the Helly graphs. One may loosen the restriction on the type of sets which must satisfy the Helly property. If one considers neighborhoods or cliques, rather than arbitrary disks, then one gets the neighborhood-Helly graphs and the clique-Helly graphs as superclasses of Helly graphs (see [21,80,97] and papers cited therein). One may also generalize the Helly-property that a family of sets satisfies. This can be accomplished by specifying a minimum number or size of sets that pairwise intersect, a minimum number of subfamilies that have a common intersection, or the size of the intersection (see [69] and papers cited therein). Far reaching examples include the (p, q)-Helly property (see [5, 6] and papers cited therein) and the fractional Helly property (see [17, 87] and papers cited therein). There is also a related to ours notion of λ-hyperconvexity in metric spaces (X, d) [71, 90], where λ is the smallest multiplicative constant such that for any system of disks $\mathcal{F} = \{D(x, r(x)) : x \in S \subseteq X\}$, the following property holds: if $X \cap Y \neq \emptyset$ for every $X, Y \in \mathcal{F}$, then $\bigcap_{x \in S} D(x, \lambda \cdot r(x)) \neq \emptyset$. Several classical metric spaces are λ-hyperconvex for a bounded λ (see [71] and papers cited therein). These include reflexive Banach spaces and dual Banach spaces ($\lambda \leq 2$) and Hilbert spaces ($\lambda \leq \sqrt{2}$). The classical Jung Theorem asserting that each subset S of the Euclidean space $\mathbb{E}^m$ with finite diameter D is contained in a disk of radius at most $\sqrt{\frac{m}{2(m+1)}}D$ belongs to this kind of results, too.

In this chapter, we are interested in α-weakly-Helly graphs, where α describes an additive constant by which radius of each disk in the family of pairwise intersecting disks can be increased in order to obtain a nonempty common intersection of all expanded disks. By definition, any α-weakly-Helly graph is $(\alpha + 1)$-hyperconvex. We find that there are also an abundance of graph classes with a bounded Helly-gap $\alpha(G)$. Here, we generalize many eccentricity related results known for Helly graphs (as well as for chordal graphs, distance-hereditary graphs and δ-hyperbolic graphs) to all graphs, parameterized by their Helly-gap $\alpha(G)$. We provide a characterization of weakly-Helly graphs with respect to their injective hulls and use this as a tool to prove those generalizations.

Several additional well-known graph classes that are α-weakly-Helly for some constant α are also identified.

The main results of this chapter can be summarized as follows. In Section 6.1 we give a characterization of α-weakly-Helly graphs through distances in injective hulls. Recall that the *injective hull* of a graph G, denoted by $\mathcal{H}(G)$, is a (unique) minimal Helly graph which contains G as an isometric subgraph [66, 83]. We show that G is an α-weakly-Helly graph if and only if for every vertex $h \in V(\mathcal{H}(G))$ there is a vertex $v \in V(G)$ such that $d_{\mathcal{H}(G)}(h, v) \leq \alpha$ (Theorem 17). In Section 6.2, we relate the diameter, radius, and all eccentricities in G to their counterparts in $H := \mathcal{H}(G)$. In particular, we show that $e_G(v) = e_H(v)$ for all $v \in V(G)$, $diam(G) = diam(H)$, and $rad(G) - \alpha(G) \leq rad(H) \leq rad(G)$. Additionally, we show $diam_G(C(G)) - 2\alpha(G) \leq diam_H(C(H)) \leq diam_G(C^{\alpha(G)}(G)) + 2\alpha(G)$. We also provide several bounds on $\alpha(G)$ including its relation to diameter and radius as well as investigate the Helly-gap of powers of weakly-Helly graphs (Theorem 19). In particular, $\lfloor (2rad(G) - diam(G))/2 \rfloor \leq \alpha(G) \leq \lfloor diam(G)/2 \rfloor$ holds. The eccentricity function in α-weakly-Helly graphs is shown to exhibit the property that any vertex $v \notin C^{\alpha}(G)$ has a nearby vertex, within distance $2\alpha + 1$ from v, with strictly smaller eccentricity (Theorem 20). In Section 6.3, we give upper and lower bounds on the eccentricity $e_G(v)$ of a vertex v. We consider bounds based on the distance from v to a closest vertex in $C^{\alpha(G)}(G)$, whether v is farthest from some other vertex and if $diam_G(C^{\alpha(G)}(G))$ is bounded. In particular, we show that $|e_G(v) - d_G(v, C^{\alpha(G)}(G)) - rad(G)| \leq \alpha(G)$ holds for any vertex $v \in V(G)$ (Theorem 21). We also prove the existence of a spanning tree T of G which gives an approximation of all vertex eccentricities in G with an additive error depending only on $\alpha(G)$ and $diam_G(C^{\alpha(G)}(G))$ (Theorem 23). We find that in any shortest path to $C^{\alpha(G)}(G)$, the number of vertices with locality more than 1 does not exceed $2\alpha(G)$ (Theorem 24). All these results greatly generalize some known facts about distance-hereditary graphs, chordal graphs, and δ-hyperbolic graphs. Those graphs have bounded Helly gap. In Section 6.4, we identify several more (well-known) graph classes with a bounded Helly-gap, including k-chordal graphs, AT-free graphs, rectilinear grids, graphs with a bounded α_i-metric, and graphs with bounded tree-length or tree-breadth.

6.1 Distances in injective hulls characterize weakly-Helly graphs

Recall that the *injective hull* of G, denoted by $\mathcal{H}(G)$, is a minimal Helly graph which contains G as an isometric subgraph [66, 83]. It turns out that $\mathcal{H}(G)$ is unique for every G [83]. When G is known by context, we often let $H := \mathcal{H}(G)$.

We will show in Theorem 17 that a graph G is α-weakly-Helly if and only if the distance from any Helly vertex in $\mathcal{H}(G)$ to a closest real vertex in $V(G)$ is no more than α. We recently learned that this result was independently discovered by Chalopin et al. [25]. They use the name *coarse Helly property* for α-weakly-Helly property used here.

The following properties will be useful to the main result of this section and to later sections. A vertex x is called a *peripheral* vertex if $I(y, x) \not\subset I(y, z)$ for some vertex y and all vertices $z \neq x$. We show next that the peripheral vertices of $\mathcal{H}(G)$ are real vertices. This adheres to the intuitive notion that an injective hull contains all of the Helly vertices "between" the vertices of G, so that the outermost vertices of $\mathcal{H}(G)$ are real.

Proposition 8. Peripheral vertices of $\mathcal{H}(G)$ are real.

Proof. By contradiction, suppose there is a peripheral Helly vertex $u \in V(\mathcal{H}(G)) \setminus V(G)$. By definition, there is a vertex $s \in V(\mathcal{H}(G))$ such that, for all $x \in V(\mathcal{H}(G))$, $I(s, u) \not\subset I(s, x)$. Let $k := d(u, s)$. Consider pairwise intersecting disks $D(s, k-1)$ and $D(x, 1)$ for each $x \in D(u, 1)$. By the Helly property, there exists vertex $w \in V(\mathcal{H}(G))$ with $d(w, s) = k - 1$ and $D(w, 1) \supseteq D(u, 1)$. By Lemma 4, $\mathcal{H}(G) - \{u\}$ is Helly and is an isometric subgraph of $\mathcal{H}(G)$. Since u is a Helly vertex, G is an isometric subgraph of $\mathcal{H}(G) - \{u\}$, a contradiction with the minimality of $\mathcal{H}(G)$. □

Moreover, we show that any shortest path of $\mathcal{H}(G)$ is a subpath of a shortest path between real vertices, which will later prove a useful property of injective hulls.

Proposition 9. Let H be the injective hull of G. For any shortest path $P(x, y)$, where $x, y \in V(H)$, there is a shortest path $P(x', y')$, where $x', y' \in V(G)$ such that $P(x', y') \supseteq P(x, y)$.

Proof. If x and y are both real vertices, then the proposition is trivially true. Without loss of generality, let y be a Helly vertex. Consider a breadth-first search layering where y belongs to layer L_i of BFS(H, x). Let $y' \in L_k$ be a vertex with $y \in I(x, y')$ that maximizes $k = d_H(x, y')$. Then, for any vertex $z \in V(\mathcal{H}(G))$, $I(x, y') \not\subset I(x, z)$. By Proposition 8, $y' \in V(G)$. If $x \notin V(G)$, then applying the previous step using BFS(H, y') yields vertex $x' \in V(G)$. $\square$

We are now ready to prove the main result of this section.

Theorem 17. For any vertex $h \in V(\mathcal{H}(G))$ there is a real vertex $v \in V(G)$ such that $d_{\mathcal{H}(G)}(h, v) \leq \alpha$ if and only if G is an α-weakly-Helly graph.

Proof. Suppose any Helly vertex in $H := \mathcal{H}(G)$ is within distance at most α from a vertex of G. Consider in G a family of pairwise intersecting disks $\mathcal{F}_G = \{D_G(v, r(v)) : v \in S \subseteq V(G)\}$. As H contains G as an isometric subgraph, the disks $\mathcal{F}_H = \{D_H(v, r(v)) : v \in S\}$ are also pairwise intersecting in H. By the Helly property, in H there is a vertex $x \in \bigcap_{v \in S} D_H(v, r(v))$. By assumption, there is a vertex $u \in V(G)$ such that $d_H(u, x) \leq \alpha$. Thus $u \in \bigcap_{v \in S} D_G(v, r(v) + \alpha)$ and so G is α-weakly-Helly.

Assume now that G is α-weakly-Helly. Let $h \in V(H)$ be an arbitrary vertex represented as a vector with nonnegative integer values $h(x)$ for each $x \in V(G)$ satisfying conditions (2.1) and (2.2) from the definition of an injective hull. Then, $\{D_G(x, h(x)) : x \in V(G)\}$ is a family of pairwise intersecting disks in G. By the α-weakly-Helly property, there is a real vertex $z \in V(G)$ belonging to $\bigcap_{x \in V(G)} D_G(x, h(x) + \alpha)$. To establish that $d_H(z, h) \leq \alpha$ and complete the proof, we will show that $\max_{t \in V(G)} |z(t) - h(t)| \leq \alpha$.

First, we show that $z(t) - h(t) \leq \alpha$ for all $t \in V(G)$. As z is a real vertex, by definition, $z(t) = d_G(z, t)$ and, by the α-weakly-Helly property (recall that $z \in \bigcap_{x \in V(G)} D_G(x, h(x) + \alpha)$), $z(t) \leq h(t) + \alpha$. Next, we show that $h(t) - z(t) \leq \alpha$ for all $t \in V(G)$. Suppose that $h(x) - z(x) > \alpha$ for some $x \in V(G)$. Then, for all vertices $y \neq x$, $h(x) + h(y) > z(x) + \alpha + h(y) \geq z(x) + z(y) \geq d(x, y)$, a contradiction with condition (2.2). $\square$

The injective hull is also useful to prove that α-weakly-Helly graphs are closed under pendant vertex addition.

Lemma 50. Let $G + \{x\}$ be the graph obtained from G by adding a vertex x pendant to any fixed vertex $v \in V(G)$. Then, $\mathcal{H}(G + \{x\}) = \mathcal{H}(G) + \{x\}$.

Proof. As x is pendant to v, then all $u \in V(G)$ have $d_{G+\{x\}}(u, x) = d_G(u, v) + 1$. Let $H_1 := \mathcal{H}(G+\{x\})$ and $H_2 := \mathcal{H}(G)+\{x\}$. Note that H_2 is a Helly graph containing $G+\{x\}$ as an isometric subgraph. We first show that any $h \in V(H_1)$ also belongs to $V(H_2)$. The statement clearly holds if h is a real vertex, so assume that h is a Helly vertex of H_1 represented as a vector with nonnegative integer values for each $u \in V(G+\{x\})$ satisfying conditions (2.1) and (2.2) from the definition of an injective hull. We will show h also satisfies the conditions under G. By condition (2.1) under $G+\{x\}$, and since G is isometric in $G+\{x\}$, for all $u, y \in V(G)$, $h(u)+h(y) \geq d_{G+\{x\}}(u,y) = d_G(u,y)$. Thus, h satisfies condition (2.1) in G. By condition (2.2) under $G+\{x\}$, for every $u \in V(G)$ there is a vertex $y \in V(G+\{x\})$ with $h(u)+h(y) = d_{G+\{x\}}(u,y)$. We claim that if $y = x$, then $h(x) = h(v)+1$ and so vertex v also satisfies $h(u) + h(v) = h(u) + h(y) - 1 = d_{G+\{x\}}(u,y) - 1 = d_G(u,v)$. On one hand, $h(x) \leq h(v) + 1$ since $h(u) + h(x) = d_{G+\{x\}}(u,x) = d_G(u,v) + 1 \leq h(u) + h(v) + 1$. On the other hand, let $z \in V(G) \cup \{x\}$ be a vertex such that $h(z) + h(v) = d_{G+\{x\}}(z,v)$. Note that $z \neq x$ as h is not a real vertex and therefore $h(v) = d_{H_1}(h,v) > 0$ and $h(z) = d_{H_1}(h,z) > 0$. Then, $h(x) \geq d_{G+\{x\}}(z,x) - h(z) = d_{G+\{x\}}(z,v) + 1 - h(z) = h(v) + 1$. With the claim established, h satisfies condition (2.2) in G. Thus, $h \in V(H_2)$ and $V(H_1) \subseteq V(H_2)$. By minimality of $\mathcal{H}(G)$, $V(H_1) = V(H_2)$. □

Corollary 31. Let $G + \{x\}$ be the graph obtained from G by adding a pendant vertex x adjacent to any fixed vertex $v \in V(G)$. Then, $\alpha(G) = \alpha(G + \{x\})$.

6.2 Diameter, radius, and all eccentricities

We establish several bounds on all eccentricities, and the diameter and radius in particular, for α-weakly-Helly graphs. As an intermediate step, we relate the diameter, radius, and all eccentricities in G to their counterparts in $\mathcal{H}(G)$.

6.2.1 Eccentricities and centers

First, we provide a few immediate consequences of Proposition 8 which establishes that farthest vertices in $\mathcal{H}(G)$ are real vertices. That is, for any $v \in V(\mathcal{H}(G))$, $F_{\mathcal{H}(G)}(v) \subseteq V(G)$. It follows that all eccentricities in G and the diameter of G are preserved in $\mathcal{H}(G)$, including with respect to any subset $M \subseteq V(G)$ of real vertices.

Proposition 10. Let H be the injective hull of G. For any $M \subseteq V(G)$ and $v \in V(G)$, $e_G^M(v) = e_H^M(v)$. Moreover, $e_G(v) = e_H(v)$.

Proposition 11. Let H be the injective hull of G. For any $M \subseteq V(G)$, $diam_G(M) = diam_H(M)$. Moreover, $diam(G) = diam(H)$.

Proposition 12. Let H be the injective hull of an α-weakly-Helly graph G. For any $M \subseteq V(G)$, $rad_G(M) - \alpha \leq rad_H(M) \leq rad_G(M)$. In particular, $rad(G) - \alpha \leq rad(H) \leq rad(G)$.

Proof. By Proposition 10, the eccentricity of any vertex of G is preserved in H. Hence, $rad_H(M) \leq rad_G(M)$. Consider any vertex $h \in C_H(M)$. We have $d_H(h,x) \leq rad_H(M)$ for all $x \in M$. By Theorem 17, there is a real vertex $v \in V(G)$ such that $d_H(v,h) \leq \alpha$. By the triangle inequality, $rad_G(M) \leq e_G^M(v) \leq max_{x \in M}\{d_H(v,h) + d_H(h,x)\} \leq rad_H(M) + \alpha$. □

Though the diameter is preserved, the radius in $\mathcal{H}(G)$ may be smaller than the radius in G by at most $\alpha(G)$. In this case, the radius in $\mathcal{H}(G)$ is realized by Helly vertices which are not present in G, resulting in different centers in $\mathcal{H}(G)$ and G. In what follows, we establish that, for any $M \subseteq V(G)$, any vertex of $C_G(M)$ is close to a vertex of $C_H(M)$. Moreover, the diameter of the center of M in G is at most the diameter of the center of M in the injective hull $\mathcal{H}(G)$ plus $2\alpha(G)$.

Lemma 51. Let H be the injective hull of an α-weakly-Helly graph G. For any $M \subseteq V(G)$ and integer $\ell \geq 0$, $C_G^\ell(M) \subseteq D_H(C_H(M), \alpha + \ell) = D_H(C_H^\ell(M), \alpha)$.

Proof. Consider any vertex $x \in C_G^\ell(M)$. By Proposition 12, $e_G^M(x) \leq rad_G(M) + \ell \leq rad_H(M) + \alpha + \ell$. By Proposition 10 and Lemma 8 (since H is Helly), $e_G^M(x) = e_H^M(x) = rad_H(M) + d_H(x, C_H(M))$. Therefore, $d_H(x, C_H(M)) \leq \alpha + \ell$. By Corollary 1, $D_H(C_H(M), \alpha + \ell) = D_H(C_H^\ell(M), \alpha)$. □

Theorem 18. Let H be the injective hull of an α-weakly-Helly graph G. For any integer $\ell \geq 0$ and any $M \subseteq V(G)$, $diam_G(C_G^\ell(M)) - 2\alpha \leq diam_H(C_H^\ell(M)) \leq diam_G(C_G^{\alpha+\ell}(M)) + 2\alpha$.

Proof. Let $d_H(u,v) = diam_H(C_H^\ell(M))$ for some $u, v \in C_H^\ell(M)$. By Theorem 17, there is a real vertex $u^* \in V(G)$ at distance at most α from u. By Proposition 10 and Proposition 12, $e_G^M(u^*) = e_H^M(u^*) \leq e_H^M(u) + \alpha \leq rad_H(M) + \ell + \alpha \leq rad_G(M) + \ell + \alpha$. Similarly, there is a real vertex $v^* \in V(G)$ at distance at most α from v with $e_G^M(v^*) \leq rad_G(M) + \ell + \alpha$. Both vertices v^*, u^*

belong to $C_G^{\alpha+\ell}(M)$. By the triangle inequality, $diam_H(C_H^\ell(M)) = d_H(u,v) \leq 2\alpha + d_H(u^*,v^*) \leq 2\alpha + diam_G(C_G^{\alpha+\ell}(M))$.

On the other hand, by Lemma 51, any vertex of $C_G^\ell(M)$ has distance at most α to $C_H^\ell(M)$. Let $x,y \in C_G^\ell(M)$ such that $d_G(x,y) = diam_G(C_G^\ell(M))$. By the triangle inequality, $d_G(x,y) = d_H(x,y) \leq diam_H(C_H^\ell(M)) + 2\alpha$. A small graph depicted on Figure 6.1 demonstrates that this inequality is tight. $\square$

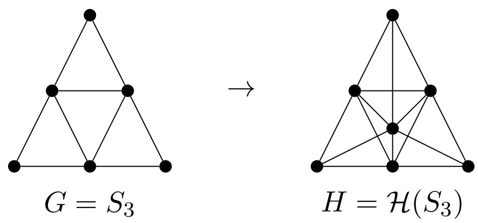


Figure 6.1: A graph G (left) and its injective hull H (right) which show that inequalities in Proposition 12 and Theorem 18 are tight: $\alpha(G) = 1$, $rad(G) = 2$, $rad(H) = 1$, $diam(C(H)) = 0$, $diam(C(G)) = 2$.

Corollary 32. For each α-weakly-Helly graph G with the injective hull $H = \mathcal{H}(G)$ and every $\ell \geq 0$, $C^\ell(H) \cap V(G) \subseteq C^\ell(G) \subseteq C^{\ell+\alpha}(H) \cap V(G)$.

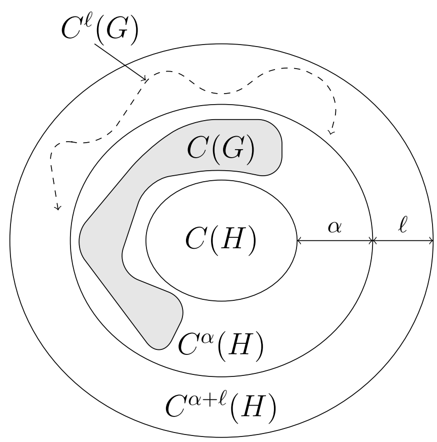


Figure 6.2: Inclusions for sets $C(G)$, $C(H)$, $C^\alpha(H)$, $C^\ell(G)$, and $C^{\alpha+\ell}(H)$ in $H = \mathcal{H}(G)$.

Proof. Let $v \in C^\ell(H) \cap V(G)$. By Proposition 12 and Proposition 10, $e_G(v) = e_H(v) \leq rad(H) + \ell \leq rad(G) + \ell$. Hence, $v \in C^\ell(G)$. Consider now a vertex $v \in C^\ell(G)$. It follows from Lemma 51 that $d_H(v, C(H)) \leq \ell + \alpha$. By Lemma 8, $e_H(v) = rad(H) + d_H(v, C(H)) \leq rad(H) + \ell + \alpha$. Thus, $v \in C^{\ell+\alpha}(H)$. $\square$

The results from this section on inclusions for central sets are illustrated in Figure 6.2.

6.2.2 Relation between Helly-gap and diameter, radius and graph powers

We obtain a few lower and upper bounds on the Helly-gap $\alpha(G)$. The first follows directly from Proposition 12.

Corollary 33. Let $H := \mathcal{H}(G)$. For any $M \subseteq V(G)$, $\alpha(G) \geq rad_G(M) - rad_H(M)$.

If $\mathcal{H}(G)$ were given, one could compute $\alpha(G)$ by computing the maximum distance from a Helly vertex to a closest real vertex. However, we provide in Corollary 34 and Lemma 52 upper and lower bounds which do not necessitate computing the injective hull.

Corollary 34. Any graph G is α-weakly-Helly for $\alpha \leq \lfloor diam(G)/2 \rfloor$.

Proof. By contradiction, suppose $\alpha(G) > \lfloor diam(G)/2 \rfloor$. Let $H := \mathcal{H}(G)$. By Theorem 17, in H there exists a Helly vertex u with $d_H(u,v) > \lfloor diam(G)/2 \rfloor$ for all $v \in V(G)$. By Proposition 9, u belongs to a shortest (x,y)-path for two real vertices $x, y \in V(G)$. As G is isometric in H, $d_G(x,y) = d_H(x,y) = d_H(x,u) + d_H(u,y) \geq 2(\lfloor diam(G)/2 \rfloor + 1) > diam(G)$, a contradiction with $d_G(x,y) \leq diam(G)$. □

Lemma 52. Let H be the injective hull of G, M be any subset of $V(G)$ and $k \geq 0$ be an integer. If $diam_G(M) = 2rad_G(M) - k$, then $rad_G(M) = rad_H(M) + \lfloor k/2 \rfloor$. Moreover, $\alpha(G) \geq \lfloor (2rad_G(M) - diam_G(M))/2 \rfloor$, that is, $2rad_G(M) \geq diam_G(M) \geq 2rad_G(M) - 2\alpha(G) - 1$.

Proof. By Proposition 11, $2rad_G(M) - k = diam_G(M) = diam_H(M)$. Since H is Helly, by Lemma 7, $rad_H(M) = \lceil (2rad_G(M) - k)/2 \rceil = rad_G(M) - \lfloor k/2 \rfloor$. By Corollary 33, $\alpha(G) \geq \lfloor k/2 \rfloor$. □

***Remark* 1.** Observe that the Helly-gap of a graph G can be much larger than $rad(G) - rad(H)$ and $\lfloor (2rad(G) - diam(G))/2 \rfloor$. Consider a graph G formed by a cycle C_{4k} of size $4k$ and two paths of length k each connected to opposite ends of the cycle, as illustrated in Figure 6.3. By Lemma 52, $\alpha(C_{4k}) \geq k$. By Corollary 31, $\alpha(G) = \alpha(C_{4k}) \geq k$. However, $diam(G) = 4k$ and $rad(G) = 2k$, and by Lemma 52, $rad(G) = rad(H)$. In this case, $rad(G) - rad(H) = \lfloor (2rad(G) - diam(G))/2 \rfloor = 0$.

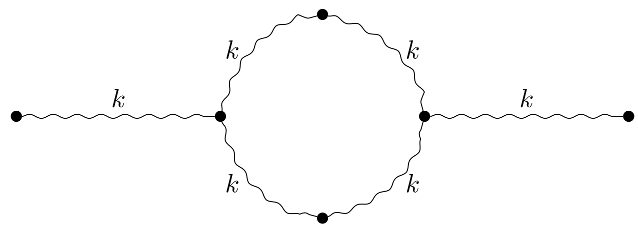


Figure 6.3: A graph G with Helly-gap k and $rad(G) = rad(H) = 2k$.

Corollary 35. Let H be the injective hull of G and $M \subseteq V(G)$. Then, $diam_G(M) \geq 2rad_G(M)-1$ if and only if $C_G(M) \subseteq C_H(M)$.

Proof. If $diam_G(M) \geq 2rad_G(M) - 1$, then by Lemma 52, $rad_G(M) = rad_H(M)$. Thus, any $v \in C_G(M)$ has $e_H^M(v) \leq rad_H(M)$. On the other hand, if $C_G(M) \subseteq C_H(M)$, since eccentricities are preserved in H by Proposition 10, then also $rad_G(M) = rad_H(M)$. Since H is Helly, by Lemma 7 and Proposition 11, $diam_G(M) = diam_H(M) \geq 2rad_H(M) - 1 = 2rad_G(M) - 1$ holds. □

Interestingly, the Helly-gap of a graph G decreases in powers of G.

Lemma 53. Let G be an α-weakly-Helly graph. For every integer $k \geq 1$, G^k is $\lceil \alpha/k \rceil$-weakly-Helly.

Proof. Let $\mathcal{F} = \{D_{G^k}(v, r(v)) : v \in S\}$ be a system of disks which pairwise intersect in G^k so that any two vertices $u, v \in S$ satisfy $d_{G^k}(u, v) \leq r(u) + r(v)$. Then, their distances in G satisfy $d_G(u, v) \leq k(r(u) + r(v))$. Consider now a corresponding system of disks in G centered at same vertices, defined as $\mathcal{M} = \{D_G(v, kr(v)) : v \in S\}$. $\mathcal{M}$ is a family of pairwise intersecting disks of G as any two vertices $u, v \in S$ satisfy $d_G(u, v) \leq kr(u) + kr(v)$. As G is α-weakly-Helly, there exists a common vertex $z \in \cap\{D_G(v, kr(v) + \alpha) : v \in S\}$. Since, for any $v \in S$, $d_G(z, v) \leq kr(v) + \alpha$, necessarily, $d_{G^k}(z, v) \leq r(v) + \lceil \alpha/k \rceil$. Hence, vertex z intersects all disks of $\mathcal{F}$ when the radii of each disk is extended by $\lceil \alpha/k \rceil$. Therefore, G^k is $\lceil \alpha/k \rceil$-weakly-Helly. □

The results of this subsection are summarized in Theorem 19.

Theorem 19. Let G be an arbitrary graph. Then, the following holds:

i) $\lfloor (2rad(G) - diam(G))/2 \rfloor \leq \alpha(G) \leq \lfloor diam(G)/2 \rfloor$, and

ii) $\alpha(G^k) \leq \lceil \alpha(G)/k \rceil$.

This Theorem 19 generalizes some known results for Helly graphs. Recall that, in Helly graphs, $\lfloor(2rad(G) - diam(G))/2\rfloor = 0$ holds and that every power of a Helly graph is a Helly graph as well [49]. We know also (see also Section 6.4) that the Helly-gap of a chordal graph or a distance-hereditary graph is at most 1 and the Helly-gap of a δ-hyperbolic graph is at most 2δ. Hence, Theorem 19 generalizes some known results on those graphs, too. For every chordal graph G as well as for every distance-hereditary graph G, $\lfloor(2rad(G) - diam(G))/2\rfloor \leq 1$ holds [30,52,116]. For every δ-hyperbolic graph G, $\lfloor(2rad(G) - diam(G))/2\rfloor \leq 2\delta$ holds [33,61].

6.2.3 The eccentricity function is almost unimodal in α-weakly-Helly graphs

Recall that the *locality* $loc(v)$ of a vertex v is defined as the minimum distance from v to a vertex of strictly smaller eccentricity. The eccentricity function $e_G(\cdot)$ is *unimodal* in G if every vertex $v \in V(G) \setminus C(G)$ has a neighbor u such that $e_G(u) < e_G(v)$, i.e., $loc(v) = 1$. In a graph G with a unimodal eccentricity function, any local minimum of the eccentricity function (i.e., a vertex whose eccentricity is not larger than the eccentricity of any of its neighbors) is the global minimum of the eccentricity function in G (i.e., it is a central vertex). Helly graphs are characterized by the property that every eccentricity function e_G^M is unimodal for any $M \subseteq V(G)$; therefore, $e_G^M(v) = d(v, C_G(M)) + rad_G(M)$ holds (see Lemma 8). A natural question for α-weakly-Helly graphs is whether similar results on the unimodality of the eccentricity function hold up to a function of α, that is, if any vertex $v \in V(G) \setminus C^\alpha(G)$ has $loc(v) \leq f(\alpha)$. The following lemmas answer in the positive.

Lemma 54. Let G be an α-weakly-Helly graph and let $M \subseteq V(G)$. If there is a vertex $v \in V(G)$ such that $e_G^M(v) > rad_G(M) + \alpha$, then there is a vertex $u \in D_G(v, 2\alpha + 1)$ with $e_G^M(u) < e_G^M(v)$.

Proof. Let $S = M \cup \{v\}$. Consider in G a system of disks $\mathcal{F} = \{D(u, \rho_u) : u \in S\}$, where the radii are defined as $\rho_w = e_G^M(v) - 1 - \alpha$ for any $w \in M$, and $\rho_v = \alpha + 1$. We assert that all disks of $\mathcal{F}$ are pairwise intersecting. Clearly for any $w \in M$, disks $D(v, \rho_v)$ and $D(w, \rho_w)$ intersect as $d(w, v) \leq e_G^M(v)$. We now show that for any $w, w' \in M$ the disks $D(w, \rho_w)$ and $D(w', \rho_{w'})$ intersect.

Consider a vertex $c \in C_G(M)$. By choice of v, $e_G^M(v) \geq rad_G(M) + \alpha + 1 = e_G^M(c) + \alpha + 1$. By the triangle inequality, for any two vertices $w, w' \in M$ we have $d(w, w') \leq d(w, c) + d(w', c) \leq e_G^M(c) + e_G^M(c) \leq (e_G^M(v) - 1 - \alpha) + (e_G^M(v) - 1 - \alpha) = \rho_w + \rho_{w'}$. Then, by the α-weakly-Helly property, the system $\mathcal{F}$ of pairwise intersecting disks has a common intersection when radii of all disks are extended by α. Therefore, there is a vertex u such that $d(u,v) \leq 2\alpha+1$ and $d(u,w) \leq e_G^M(v) - 1$ for all $w \in M$. $\square$

For $\alpha = 0$ we obtain a result known for Helly graphs. As δ-hyperbolic graphs are (2δ)-weakly-Helly (this follows from a result in [38], see also Section 6.4), Lemma 54 also greatly generalizes a result known for δ-hyperbolic graphs (cf. Chapter 5).

Lemma 55. Let H be the injective hull of an α-weakly-Helly graph G, and let $M \subseteq V(G)$. If there is a vertex $v \in V(G)$ such that $d_H(v, C_H(M)) > \alpha$, then there is a real vertex $u \in D_G(v, 2\alpha+1)$ such that $e_G^M(u) < e_G^M(v)$.

Proof. Let $c \in C_H(M)$ be a vertex closest in H to v, and assume that $d_H(v,c) \geq \alpha+1$. Let $w \in V(H)$ be a vertex on a shortest (v,c)-path of H such that $d_H(v,w) = \alpha+1$. Since H is Helly, by Proposition 10 and Lemma 8, $e_G^M(v) = e_H^M(v) = d_H(v, C_H(M)) + rad_H(M) = d_H(v,w) + d_H(w,c) + rad_H(M) = \alpha+1+e_H^M(w)$. Therefore, $e_H^M(w) = e_G^M(v) - \alpha - 1$. Since G is α-weakly-Helly, by Theorem 17, there is a vertex $u \in V(G)$ such that $d_H(u,w) \leq \alpha$. Hence, $e_G^M(u) = e_H^M(u) \leq e_H^M(w) + \alpha \leq e_G^M(v) - 1$. See Figure 6.4 for an illustration. $\square$

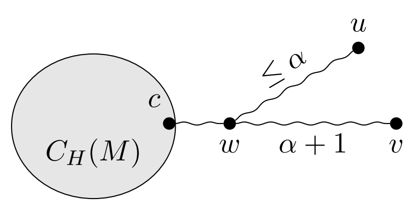


Figure 6.4: Illustration to the proof of Lemma 55.

Theorem 20 summarizes the results of Lemma 54 and Lemma 55.

Theorem 20. Let H be the injective hull of an α-weakly-Helly graph G, and let $M \subseteq V(G)$. If there is a vertex $v \in V(G)$ such that $e_G^M(v) > rad_G(M) + \alpha$ or $d_H(v, C_H(M)) > \alpha$, then there is a real vertex $u \in D_G(v, 2\alpha+1)$ such that $e_G^M(u) < e_G^M(v)$.

6.3 Estimating all eccentricities

This section provides upper and lower bounds on the eccentricity of a vertex v based on a variety of conditions: the distance from v to a closest almost central vertex (i.e., a closest vertex with eccentricity at most $rad_G(M) + \alpha$) and whether v is a farthest vertex from some other vertex and if $diam(C_G^{2\alpha}(M))$ is bounded. We also prove that any α-weakly-Helly graph has an eccentricity approximating spanning tree where the additive approximation error depends on the diameter of the set $C^\alpha(G)$. Finally, we describe the eccentricity terrain in α-weakly-Helly graphs, that is, how vertex eccentricities change along vertices of a shortest path to $C_G^\alpha(M)$.

6.3.1 Using distances to $C_G^\alpha(M)$

Lemma 56. Let G be a graph, $M \subseteq V(G)$, and $k \geq 0$. For every vertex $x \in V(G)$, $e_G^M(x) \leq d(x, C_G^k(M)) + rad_G(M) + k$ holds.

Proof. Let $e_G^M(x) = d(x, y)$ where $y \in F_G^M(x)$, and let $x_p \in C_G^k(M)$ be closest to x. By the triangle inequality, $e_G^M(x) \leq d(x, x_p) + d(x_p, y) \leq d(x, C_G^k(M)) + rad_G(M) + k$. □

Lemma 57. Let G be an α-weakly-Helly graph and $M \subseteq V(G)$. For every vertex $x \in V(G)$, $e_G^M(x) \geq d(x, C_G^\alpha(M)) + rad_G(M) - \alpha$ holds.

Proof. Let $H := \mathcal{H}(G)$ and $h \in C_H(M)$ be a closest vertex to x in H. By Lemma 8 and Proposition 10, $d_H(x, h) = d_H(x, C_H(M)) = e_H^M(x) - rad_H(M) = e_G^M(x) - rad_H(M) \geq rad_G(M) - rad_H(M)$. Then, let y be a vertex on a shortest (x, h)-path with $d_H(h, y) = rad_G(M) - rad_H(M)$. By Lemma 8, $e_H^M(y) = d_H(y, C_H(M)) + rad_H(M) = d_H(y, h) + rad_H(M) = rad_G(M)$. By Theorem 17, there is a real vertex $y^* \in V(G)$ such that $d_H(y, y^*) \leq \alpha$. Applying Proposition 10 one obtains $e_G^M(y^*) = e_H^M(y^*) \leq rad_G(M) + \alpha$. By the triangle inequality, $d_H(x, y) \geq d_H(x, y^*) - d_H(y, y^*) \geq d(x, C_G^\alpha(M)) - \alpha$. Thus, $e_G^M(x) = d_H(x, h) + rad_H(M) = d_H(x, y) + rad_G(M) \geq d(x, C_G^\alpha(M)) + rad_G(M) - \alpha$. □

Applying Lemma 56 with $k = \alpha$ and Lemma 57, we obtain the following approximation of eccentricities in α-weakly-Helly graphs.

Theorem 21. Let G be an α-weakly-Helly graph. For every $M \subseteq V(G)$, every $x \in V(G)$ satisfies

$$d(x, C_G^\alpha(M)) + rad_G(M) - \alpha \leq e_G^M(x) \leq d(x, C_G^\alpha(M)) + rad_G(M) + \alpha.$$

If $C_G^\alpha(M)$ is known in advance, then one obtains a linear time additive 2α-approximation of all eccentricities by using a breadth-first search from $C_G^\alpha(M)$ $(BFS(C_G^\alpha(M)))$, to obtain all distances to $C^{\alpha}_G(M)$, and a $BFS(c)$ from any fixed $c \in C^{\alpha}_G(M)$ that has a neighbor not in $C^{\alpha}_G(M)$, to compute $rad_G(M) + \alpha$. Then, for each vertex v, set an approximate eccentricity $\hat{e}^M(v) = d_G(v, C^{\alpha}_G(M)) + rad_G(M) + \alpha$. By Theorem 21, $e^M_G(v) \leq \hat{e}^M(v) \leq e^M_G(v) + 2\alpha$.

By Corollary 1, any Helly graph G satisfies $C^{\ell}_G(M) = D(C_G(M), \ell)$. This also extends to α-weakly-Helly graphs in the following way.

Corollary 36. Let G be an α-weakly-Helly graph. For any $M \subseteq V(G)$ and any integer $\ell \geq 0$, $C^{\ell}_G(M) \subseteq D_G(C^{\alpha}_G(M), \alpha + \ell)$.

Proof. If $x \in C^{\ell}_G(M)$ then, by Theorem 21, $d(x, C^{\alpha}_G(M)) + rad_G(M) - \alpha \leq e^M_G(x) \leq \ell + rad_G(M)$. Hence, $d(x, C^{\alpha}_G(M)) \leq \alpha + \ell$. □

We can restate the lower bound on $e^M_G(x)$ in Theorem 21 by using thinness $\kappa(\mathcal{H}(G))$ of metric intervals of a graph's injective hull.

Lemma 58. Let H be the injective hull of a graph G. For any $M \subseteq V(G)$, every $x \in V(G)$ satisfies $e^M_G(x) \geq d_G(x, C^{\kappa(H)}_G(M)) + rad_G(M)$.

Proof. We apply the same idea as in the proof of Lemma 57 with vertex x, where $e^M_G(x) = e^M_H(x) = d_H(x,t)$ for some vertex $t \in M$. Let $h \in C_H(M)$ be a closest vertex to x, and let y be a vertex on a shortest (x,h)-path with $d_H(h,y) = rad_G(M) - rad_H(M)$ and $e^M_H(y) = rad_G(M)$. Then, since G is isometric in H, there is a shortest (x,t)-path P in H such that each vertex $v \in P$ belongs to $V(G)$. Note also that, by Lemma 8, h is on a shortest path from x to t in H. Let $y^* \in P$ such that $d_G(y^*, t) = rad_G(M)$ so that y and y^* belong to the same interval slice $S_{rad_G(M)}(t,x)$ in H. Therefore, $d_H(y, y^*) \leq \kappa(H)$ and, by Proposition 10, $e^M_G(y^*) = e^M_H(y^*) \leq e^M_H(y) + \kappa(H) = rad_G(M) + \kappa(H)$. Hence, $e^M_G(x) = d_H(x,y) + rad_G(M) = d_H(x,y^*) + rad_G(M) \geq d_H(x, C^{\kappa(H)}_G(M)) + rad_G(M)$ as $y^* \in C^{\kappa(H)}_G(M)$. □

As in a δ-hyperbolic graph G, $\kappa(\mathcal{H}(G)) \leq 2\delta$ (see Subsection 6.4.2), Lemma 58 generalizes a result for δ-hyperbolic graphs from Chapter 5): $e_G(x) \geq d_G(x, C^{2\delta}(G)) + rad(G)$ for all $x \in V(G)$. Note that similar results are known for chordal graphs: $e_G(x) \geq d_G(x, C(G)) + rad(G) - 1$ for every $x \in V(G)$ [63], and for distance-hereditary graphs (cf. Chapter 4: $e_G(x) = d_G(x, C^1(G)) + rad(G) + 1$ for every $x \in V(G) \setminus C(G)$.

6.3.2 A vertex furthest from some other vertex

Recall that in a Helly graph G, for every vertex $x \in V$ and every farthest vertex $y \in F(x)$, $e(y) \geq 2rad(G) - diam(C(G))$ holds [49].

Theorem 22. Let G be an α-weakly-Helly graph. Then, for every $M \subseteq V(G)$ and every $x \in V(G)$, each vertex $y \in F_G^M(x)$ satisfies

$$e_G^M(y) \geq 2rad_G(M) - diam_G(C_G^{2\alpha}(M)) - 2\alpha,$$

$$e_G^M(y) \geq 2rad_G(M) - diam_G(C_G^{\alpha}(M)) - 4\alpha.$$

Proof. Let $H := \mathcal{H}(G)$ and let $e_G^M(y) = d_G(y, w)$ for some $w \in M$. As $e_H^M(\cdot)$ is unimodal, in H there is a closest to x vertex $b_h \in C_H(M)$ such that $e_G^M(x) = d_H(x, b_h) + rad_H(M)$. Similarly, in H there is a closest to y vertex $c_h \in C_H(M)$ such that $e_G^M(y) = d_H(y, c_h) + rad_H(M)$. Then, $d_H(x, C_H(M)) = d_H(x, b_h) = e_H^M(x) - rad_H(M) \geq rad_G(M) - rad_H(M)$ and, by symmetry, $d_H(y, c_h) \geq rad_G(M) - rad_H(M)$. Now let $c \in I(c_h, y)$ be a vertex in H such that $d_H(c_h, c) = rad_G(M) - rad_H(M)$, and also $b \in I(b_h, x)$ be a vertex in H such that $d_H(b_h, b) = rad_G(M) - rad_H(M)$, as illustrated in Figure 6.5. By Proposition 12, $e_H^M(b) \leq rad_G(M) - rad_H(M) + e_H^M(b_h) = rad_G(M) \leq rad_H(M) + \alpha$. By symmetry, $e_H^M(c) \leq rad_H(M) + \alpha$, and so both b and c belong to $C_H^{\alpha}(M)$. By the triangle inequality, $rad_G(M) = d_H(b, y) \leq d_H(b, c) + d_H(c, y) = d_H(b, c) + d_H(w, y) - rad_G(M)$. That is, $e_G^M(y) = d_H(w, y) \geq 2rad_G(M) - d_H(b, c) \geq 2rad_G(M) - diam_H(C_H^{\alpha}(M))$. By Corollary 1, as H is Helly, $e_G^M(y) \geq 2rad_G(M) - diam_H(C_H^{\alpha}(M)) = 2rad_G(M) - diam_H(C_H(M)) - 2\alpha$. Applying now Theorem 18 with $\ell = \alpha$, we get $diam_H(C_H^{\alpha}(M)) \leq 2\alpha + diam_G(C_G^{2\alpha}(M))$ and hence $e_G^M(y) \geq 2rad_G(M) - diam_G(C_G^{2\alpha}(M)) - 2\alpha$. Applying Theorem 18 with $\ell = 0$, we get $diam_H(C_H(M)) \leq 2\alpha + diam_G(C_G^{\alpha}(M))$ and hence $e_G^M(y) \geq 2rad_G(M) - diam_G(C_G^{\alpha}(M)) - 4\alpha$. $\square$

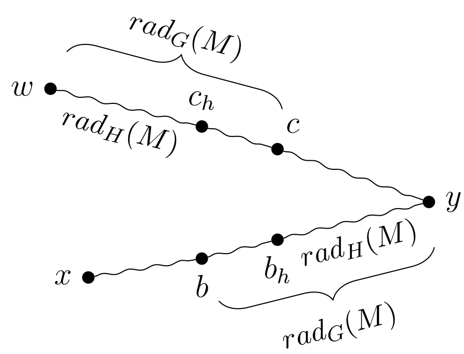


Figure 6.5: Illustration to the proof of Theorem 22.

It is known that for every vertex x, any vertex $y \in F_G(x)$ satisfies $e_G(y) \geq 2rad(G) - 3$ if G is a chordal graph or a distance-hereditary graph [34,52] and $e_G(y) \geq diam(G) - 2\delta \geq 2rad(G) - 6\delta - 1$ if G is a δ-hyperbolic graph [33,61]. Furthermore, for every chordal or distance-hereditary graph G, $diam_G(C(G)) \leq 3$ holds [30,115,116] and for every δ-hyperbolic graph G, $diam_G(C^{2\delta}(G)) \leq 8\delta + 1$ and $diam_G(C(G)) \leq 4\delta + 1$ hold [33,61]. Recall that the diameter of the center of a Helly graph cannot be bounded. Although $C(G)$ itself is Helly and is isometric in G for a Helly graph G [49], $C(G)$ can have an arbitrarily large diameter as any Helly graph is the center of some Helly graph [49]. Thus, $diam_G(C^{\alpha(G)}(G))$ cannot be bounded by a function of $\alpha(G)$ (even for $\alpha(G) = 0$, i.e., for Helly graphs).

6.3.3 Eccentricity approximating spanning tree

An *eccentricity t-approximating spanning tree* is a spanning tree T such that $e_T(v) - e_G(v) \leq t$ holds for all $v \in V(G)$. Such a tree tries to approximately preserve only distances from v to its farthest vertices, allowing for a larger increase in distances to nearby vertices. Many classes of graphs have an eccentricity t-approximating spanning tree, where t is a small constant (e.g., see [36, 63, 98, 102] and Chapter 5 and papers cited therein). We will show that α-weakly-Helly graphs have an eccentricity t-approximating spanning tree where t depends linearly on α and on the diameter of $C^{\alpha}(G)$.

Lemma 59. Let H be the injective hull of an α-weakly-Helly graph G. For every $M \subseteq V(G)$, there is a real vertex $c^* \in V(G)$ such that, for every vertex $x \in V(G)$, $d_G(x, c^*) + e_G^M(c^*) - e_G^M(x) \leq \lceil \frac{diam_H(C_H(M))}{2} \rceil + 2\alpha$ holds.

Proof. Let $c \in V(H)$ be a vertex belonging to $C_H(C_H(M))$. By Theorem 17, there is a real vertex $c^* \in V(G)$ such that $d_H(c, c^*) \leq \alpha$.

We first establish a few useful inequalities. Since H is Helly, by Lemma 3, $C_H(M)$ and $C_H(C_H(M))$ are also Helly and isometric in H. Thus, by Lemma 7, $rad_H(M) = \lceil diam_H(M)/2 \rceil$ and $rad_H(C_H(M)) = \lceil diam_H(C_H(M))/2 \rceil$. Let $c_x \in C_H(M)$ be a closest vertex to x. By Lemma 8, $e_G^M(x) = e_H^M(x) = d_H(x, C_H(M)) + rad_H(M) = d_H(x, c_x) + \lceil diam_H(M)/2 \rceil$. By the triangle inequality, $e_G^M(c^*) = e_H^M(c^*) \leq e_H^M(c) + \alpha = \lceil diam_H(M)/2 \rceil + \alpha$. Since G is isometric in H, $d_G(x, c^*) = d_H(x, c^*) \leq d_H(x, c_x) + d_H(c_x, c) + d_H(c, c^*) \leq d_H(x, c_x) + rad_H(C_H(M)) + \alpha = d_H(x, c_x) + \lceil diam_H(C_H(M))/2 \rceil + \alpha$. We are now ready to prove the claim.

$$\begin{aligned} e_G^M(x) = e_H^M(x) = d_H(x, c_x) + \lceil diam_H(M)/2 \rceil \\ &\geq (d_G(x, c^*) - \lceil diam_H(C_H(M))/2 \rceil - \alpha) + \lceil diam_H(M)/2 \rceil \\ &\geq d_G(x, c^*) - \lceil diam_H(C_H(M))/2 \rceil + e_G^M(c^*) - 2\alpha. \end{aligned}$$

Therefore, $d_G(x, c^*) + e_G^M(c^*) - e_G^M(x) \leq diam_H(C_H(M))/2 \rceil + 2\alpha$. □

Theorem 23. Let G be an α-weakly-Helly graph. For every $M \subseteq V(G)$, G has a spanning tree T such that, for all $x \in V(G)$, $e_T^M(x) - e_G^M(x) \leq \lceil diam_G(C_G^\alpha(M))/2 \rceil + 3\alpha$.

Proof. By Lemma 59, there exists a vertex $c^* \in V(G)$ such that any vertex $x \in V(G)$ satisfies $d_G(x, c^*) + e_G^M(c^*) - e_G^M(x) \leq \lceil diam_H(C_H(M))/2 \rceil + 2\alpha$. Let T be a BFS tree of G rooted at c^*. Then, any vertex $x \in V(G)$ has $e_T^M(x) - e_G^M(x) \leq d_T(x, c^*) + e_T^M(c^*) - e_G^M(x)$. As distances to c^* are preserved in T, one obtains $e_T^M(x) - e_G^M(x) \leq \lceil diam_H(C_H(M))/2 \rceil + 2\alpha$. Applying Theorem 18 with $\ell = 0$, $diam_H(C_H(M)) \leq diam_G(C_G^\alpha(M)) + 2\alpha$. Hence, $e_T^M(x) - e_G^M(x) \leq \lceil (diam_G(C_G^\alpha(M)) + 2\alpha)/2) \rceil + 2\alpha$, establishing the desired result. □

It is known that every chordal graph and every distance-hereditary graph has an eccentricity 2-approximating spanning tree [53, 63, 93, 102] and every δ-hyperbolic graph has an eccentricity $(4\delta+1)$-approximating spanning tree (as shown in Chapter 5). These graph classes have additional nice properties that allowed to get for them sharper results than the one of Theorem 23 which holds for general α-weakly-Helly graphs.

6.3.4 Eccentricity terrain

We can describe the behavior of the eccentricity function in a graph in terms of graph's *eccentricity terrain*, that is, how vertex eccentricities change along vertices of a shortest path to $C^{\alpha}_{G}(M)$.

Theorem 24. Let G be an α-weakly-Helly graph and $M \subseteq V(G)$. For any shortest path $P(y, x)$ of G from a vertex $y \notin C^{\alpha}_{G}(M)$ to a closest vertex $x \in C^{\alpha}_{G}(M)$, $2\mathbb{U}(P(y,x)) + \mathbb{H}(P(y,x)) \leq 2\alpha$ holds.

Proof. Let $x \in C^{\alpha}_{G}(M)$ be closest to y. Hence, $e^{M}_{G}(x) = rad_G(M) + \alpha$. Applying Theorem 21, $e^{M}_{G}(y) \geq d_G(y, C^{\alpha}_{G}(M)) + rad_G(M) - \alpha$. Hence, $d_G(y, x) = d_G(y, C^{\alpha}_{G}(M)) \leq e^{M}_{G}(y) - rad_G(M) + \alpha = e^{M}_{G}(y) - e^{M}_{G}(x) + 2\alpha$. Applying Lemma 1, we obtain the desired result. □

As a consequence of Theorem 24, at most $2\alpha(G)$ non-descending edges can occur along every shortest path from any vertex $y \notin C^{\alpha}(G)$ to $C^{\alpha}(G)$. Hence, in any shortest path to $C^{\alpha}(G)$, the number of vertices with locality more than 1 does not exceed $2\alpha(G)$. These kind of results were only known for chordal graphs [63] and distance-hereditary graphs [Chapter 4] (at most one non-descending edge, i.e., horizontal-edge and for δ-hyperbolic graphs (at most 4δ non-descending edges [Chapter 5]).

6.4 Classes of graphs having small Helly-gap

Let G be a graph and let $\alpha := \alpha(G)$ be its Helly-gap. We now focus on the upper bound on α for some special graph classes and identify those classes which exhibit a topological Helly-likeness, that is, classes which have small α. Recall that, by definition, Helly graphs are 0-weakly-Helly. It was also recently shown that each Helly vertex in the injective hull of a distance-hereditary graph is adjacent to a real vertex [81]; thus distance-hereditary graphs are 1-weakly-Helly (this result follows also from an earlier result on existence of r-dominating cliques in distance-hereditary graphs [52]). *Distance-hereditary graphs* can be defined as the graphs where each induced path is a shortest path.

We consider several other graph classes and summarize our findings in Table 6.1.

6.4.1 Rectilinear grids

Let G be an $n \times n$ rectilinear grid (the cartesian product of two paths of length n). G can be isometrically embedded into a $2n \times 2n$ king grid H (the strong product of two paths of length $2n$,

Graph class	**Helly-gap**
Helly	$\alpha(G) = 0$
Distance-hereditary	$\alpha(G) \leq 1$
k-Chordal	$\alpha(G) \leq \lfloor k/2 \rfloor$
Chordal	$\alpha(G) \leq 1$
AT-free	$\alpha(G) \leq 2$
$n \times n$ Rectilinear grid	$\alpha(G) = 1$
δ-Hyperbolic	$\alpha(G) \leq 2\delta$
Cycle C_n	$\alpha(G) = \lfloor n/4 \rfloor$
α_i-Metric	$\alpha(G) \leq \lceil i/2 \rceil$
Tree-breadth $tb(G)$	$\alpha(G) \leq tb(G)$
Tree-length $tl(G)$	$\alpha(G) \leq tl(G)$
Tree-width $tw(G)$	no relation
Bridged	unbounded

Table 6.1: The bound on the Helly-gap $\alpha(G)$ for various graph classes.

and a natural subclass of Helly graphs) wherein the four extreme vertices of G are the midpoints of each side of H (see Figure 6.6). One obtains $\mathcal{H}(G)$ by removing from H any peripheral vertices that are not present in G. Therefore, each Helly vertex h of $\mathcal{H}(G)$ corresponds to one of the n^2 squares of the grid G, where h is adjacent to the four corners of the square and to each of the Helly vertices corresponding to the 2-4 adjacent squares. Hence, the Helly-gap of G is 1.

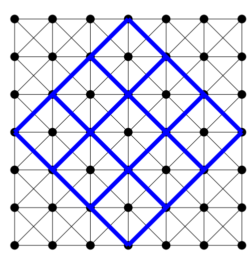


Figure 6.6: An $n \times n$ rectilinear grid (shown in bold blue lines) is isometrically embedded into a $2n \times 2n$ king grid (shown in thin black lines).

6.4.2 Graphs of bounded hyperbolicity

Recall that $\kappa(G)$ denotes the smallest value κ for which all intervals of G are κ-thin. We show that the Helly-gap of any graph G is at most $\kappa(\mathcal{H}(G))$.

Lemma 60. For any vertex $h \in V(\mathcal{H}(G))$ there is a real vertex $v \in V(G)$ such that $d(h, v) \leq \kappa(\mathcal{H}(G))$.

Proof. Let $H := \mathcal{H}(G)$ and consider a vertex $h \in V(H)$. By Proposition 9, h belongs to a shortest path $P_H(x, y)$ where $x, y \in V(G)$. Let $d_H(x, h) = \ell$. Because G is isometric in H, there is a shortest path $P_G(x, y) := (v_0, ..., v_k)$, where $x = v_0$, $y = v_k$ and all v_i in P_G are real vertices. By assumption, $d_H(h, v_\ell) \leq \kappa(H)$. □

Corollary 37. Any graph G is α-weakly-Helly for $\alpha \leq \kappa(\mathcal{H}(G))$.

Proof. The result follows from Theorem 17 and Lemma 60. □

Note that an $n \times n$ rectilinear grid gives an example of a graph where $\alpha = 1$ and $\kappa(\mathcal{H}(G)) = 2n$.

Urs Lang [95] has established that the δ-hyperbolicity of a graph G is preserved in $H = \mathcal{H}(G)$. As hyperbolicity is preserved in $\mathcal{H}(G)$, by Lemma 60 and Lemma 9, for any Helly vertex $h \in V(H)$ there is a real vertex $v \in V(G)$ such that $d_H(h, v) \leq \kappa(\mathcal{H}(G)) \leq 2\delta(\mathcal{H}(G)) = 2\delta(G)$. Hence, δ-hyperbolic graphs are 2δ-weakly-Helly. The latter result follows also from [38].

***Remark* 2.** Several graph classes are δ-hyperbolic for a bounded value δ, including k-chordal graphs and graphs without an asteroidal triple (AT). Three pairwise non-adjacent vertices of a graph form an AT if every two of them are connected by a path that avoids the neighborhood of the third. A graph is k-chordal provided it has no induced cycle of length greater than k. It is known [114] that k-chordal graphs have hyperbolicity at most $\frac{\lfloor k/2 \rfloor}{2}$. Therefore, k-chordal graphs are $\lfloor k/2 \rfloor$-weakly-Helly. Chordal graphs, which are exactly the 3-chordal graphs, are 1-weakly-Helly (this result follows also from an earlier result on existence of r-dominating cliques in chordal graphs [54]). The 2-weakly-Helly graphs also include all 5-chordal graphs such as AT-free graphs, all 4-chordal graphs such as weakly-chordal and cocomparability graphs, as well as all 1-hyperbolic graphs including interval graphs and permutation graphs.

***Remark* 3.** Observe that $\alpha(G)$ can be arbitrarily smaller than the δ-hyperbolicity of a graph. Consider a $(2r \times 2r)$ king grid G, i.e., the strong product of two paths each of even length $2r$. King grids form a natural subclass of Helly graphs, and therefore $\mathcal{H}(G) = G$. Thus, G is 0-weakly-Helly, whereas $\delta(G) = r$.

6.4.3 Cycles

Recall that C_n denotes a cycle of length n.

Lemma 61. The Helly-gap of a cycle C_n is $\lfloor n/4 \rfloor$.

Proof. Let G be a cycle of length n. Clearly, $diam(G) = \lfloor n/2 \rfloor = rad(G)$. By Proposition 34, G is α-weakly-Helly for $\alpha \leq \lfloor diam(G)/2 \rfloor = \lfloor n/4 \rfloor$. By Lemma 52, $\alpha \geq \lfloor (2rad(G) - diam(G))/2 \rfloor = \lfloor n/4 \rfloor$. $\square$

It should also be noted that for a self-centered graph (i.e., a graph where all vertex eccentricities are equal), from $\lfloor (2rad(G) - diam(G))/2 \rfloor \leq \alpha(G) \leq \lfloor diam(G)/2 \rfloor$ and since for a self-centered graph G, $rad(G) = diam(G)$, we get $\alpha(G) = \lfloor diam(G)/2 \rfloor = \lfloor rad(G)/2 \rfloor$. Cycles are self-centered graphs.

6.4.4 Graphs with an α_i-metric

Introduced by Chepoi and Yushmanov [30,116], a graph G is said to have an α_i-metric if it satisfies the following: for any $x, y, z, v \in V(G)$ such that $z \in I(x, y)$ and $y \in I(z, v)$, $d_G(x, v) \geq d_G(x, y) + d_G(y, v) - i$ holds. For every graph G with an α_i-metric, $diam(G) \geq 2rad(G) - i - 1$ holds [116]. Several graph classes have an α_i-metric [116]. Ptolemaic graphs are exactly the graphs with α_0-metric [116]. Chordal graphs are a subclass of graphs with α_1-metric [30]. All graphs with α_1-metric are characterized in [116]. In a private communication, discussing the results of this chapter, Victor Chepoi asked if graphs with an α_i-metric are α-weakly-Helly for an α depending only on i. The following lemma answers this question in the affirmative.

Lemma 62. Any graph G with an α_i-metric is α-weakly-Helly for $\alpha \leq \lceil i/2 \rceil$.

Proof. We prove by induction on the number k of disks in a family of pairwise intersecting disks $\mathcal{F} = \{D(v, r(v)) : v \in M \subseteq V(G)\}$. Let $y \in M$ and pick a vertex c which belongs to $\bigcap_{v \in M \setminus \{y\}} D(v, r(v) + \alpha)$ such that c is closest to y. If $d(c, y) \leq r(y) + \alpha$, then c is common to all disks of $\mathcal{F}$ inflated by α and we are done. Assume now that $d(c, y) > r(y) + \alpha$. Let $c' \in S_1(c, y)$. By choice of c, there is a disk $D(x, r(x)) \in \mathcal{F}$ such that $d(c, x) = r(x) + \alpha = d(c', x) - 1$. Hence, $c \in I(x, c')$ and $c' \in I(c, y)$. Applying α_i-metric to x, c, c', y, one obtains $d(x, y) \geq d(x, c) + d(c, y) - i > (r(x) + \alpha) + (r(y) + \alpha) - i = r(x) + r(y) + 2\alpha - i$. If now $i \leq 2\alpha$, we get $d(x, y) > r(x) + r(y)$, contradicting the fact that disks $D(x, r(x))$ and $D(y, r(y))$ intersect. Thus, for every i (even or odd) such that $i \leq 2\alpha$, $d(c, y) \leq r(y) + \alpha$ must hold. That is, G is α-weakly-Helly for $\alpha = \lceil i/2 \rceil$ (pick $\alpha = i/2$, when i is even, and $\alpha = (i + 1)/2$, when i is odd). $\square$

***Remark* 4.** Observe that a graph G with an α_i-metric can have Helly-gap $\alpha(G)$ that is arbitrarily smaller than $\lceil i/2 \rceil$. Consider a $(1 \times \ell)$ rectilinear grid G. Let (x, y) and (z, v) be edges on extreme ends of G so that $d(x, z) = d(y, v) = \ell$ and $d(y, z) = d(x, v) = \ell + 1$. Then, $z \in I(x, v)$, $v \in I(z, y)$, and $1 = d(x, y) \geq d(x, v) + d(v, y) - i = 2\ell + 1 - i$. Thus, G has an α_i-metric for $i \geq 2\ell$ whereas $\alpha(G) = 1$.

6.4.5 Graphs of bounded tree-breadth, tree-length, or tree-width

Recall that a tree-decomposition $(\mathcal{T}, T)$ [107] for a graph $G = (V, E)$ is a family $\mathcal{T} = \{B_1, B_2, ...\}$ of subsets of V, called *bags*, such that $\mathcal{T}$ forms a tree T with the bags in $\mathcal{T}$ as nodes which satisfy the following conditions: (i) each vertex is contained in a bag, (ii) for each edge $(u, v) \in E$, $\mathcal{T}$ has a bag B with $u, v \in B$, and (iii) for each vertex $v \in V$, the bags containing v induce a subtree of T. The width of a tree decomposition is the size of its largest bag minus one. A tree decomposition has breadth ρ if, for each bag B, there is a vertex v in G such that $B \subseteq D_G(v, \rho)$. A tree decomposition has length λ if the diameter in G of each bag B is at most λ. The *tree-width* $tw(G)$ [107], *tree-breadth* $tb(G)$ [62] and *tree-length* $tl(G)$ [48] are the minimum width, breadth, and length, respectively, among all possible tree decompositions of G. By definition, $tb(G) \leq tl(G) \leq 2tb(G)$, as for any graph G and any set $M \subseteq V(G)$, $rad_G(M) \leq diam_G(M) \leq 2rad_G(M)$ holds.

Lemma 63. Any graph G is α-weakly-Helly for $\alpha \leq tb(G)$ and $\alpha \leq tl(G)$.

Proof. Let G have tree-breadth $tb(G) \leq \rho$. Consider a corresponding tree-decomposition $(\mathcal{T}, T)$ of G of breadth ρ, and let $\mathcal{F} = \{D_G(v, r(v)) : v \in S \subseteq V(G)\}$ be a family of disks of G that pairwise intersect. Observe that bags containing vertices of a disk induce a subtree of T and the subtrees of T corresponding to disks of $\mathcal{F}$ pairwise intersect in T. If the subtrees of a tree pairwise intersect, then they have a common node in T. Therefore, there is a bag $B \in \mathcal{T}$ such that each disk in $\mathcal{F}$ intersects B in G. Let vertex w be the center of bag B, i.e., $B \subseteq D_G(w, \rho)$. So, each $v \in S$ satisfies $w \in D_G(v, r(v) + \rho)$. Hence, G is α-weakly-Helly where $\alpha \leq tb(G) \leq tl(G)$. $\square$

***Remark* 5.** While $\alpha := \alpha(G)$ is upper bounded by tree-breadth and tree-length, α can be arbitrarily far from tree-width and arbitrarily smaller than tree-length. Consider the $(r \times r)$ rectilinear grid G which is 1-weakly-Helly but $tw(G) = r$ (cf. [106]) and $tl(G) = 2r$ (cf. [4, 48]). On the other hand, let G be a cycle of size $4k$; the Helly-gap of G is k, yet $tw(G) = 2$.

6.4.6 Bridged graphs

A graph is *bridged* [72] if it contains no isometric cycles of length greater than 3. As any isometric subgraph is an induced subgraph, bridged graphs form a superclass of chordal graphs. Interestingly, we find that although chordal graphs are 1-weakly-Helly, the Helly-gap of a bridged graph can be arbitrarily large. Consider the bridged graph G in Figure 6.7 with each side of length $s = 4k$ for some integer k. Clearly the disks centered at x, y, z with radius $2k$ pairwise intersect and have no common intersection. However, only extending all radii by k yields middle vertex u as a common intersection. Thus, the Helly-gap of G is at least k, where k can be arbitrarily large.

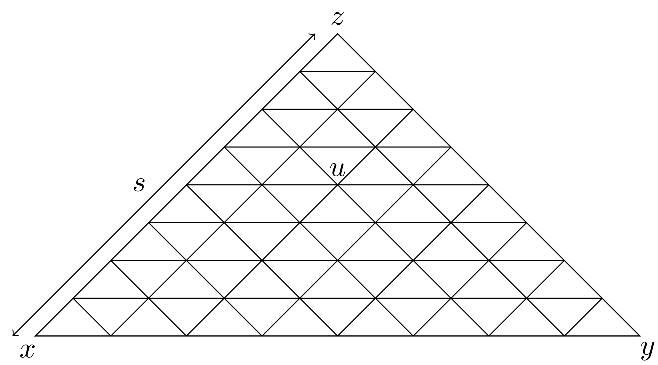


Figure 6.7: Example that bridged graphs have arbitrarily large Helly-gap.

Chapter 7

Injective hulls of various graph classes

In this chapter, we investigate the structural properties of injective hulls of various graph classes. We say that a class of graphs $\mathcal{C}$ is closed under Hellification if $G \in \mathcal{C}$ implies $\mathcal{H}(G) \in \mathcal{C}$. We identify several graph classes that are closed under Hellification. We show that permutation graphs are not closed under Hellification, but chordal graphs, square-chordal graphs, and distance-hereditary graphs are. Graphs that have an efficiently computable injective hull are of particular interest. A polynomial time algorithm to construct the injective hull of any distance-hereditary graph is provided and we show that the injective hull of several graphs from some other well-known classes of graphs are impossible to compute in subexponential time. In particular, there are split graphs, cocomparability graphs, chordal bipartite graphs G such that $\mathcal{H}(G)$ contains $\Omega(a^n)$ vertices, where $n = |V(G)|$ and $a > 1$.

The rich theory behind Helly graphs entices the use of injective hulls as an underlying structure to solve (approximately) problems on G. Problems such as finding the diameter or computing vertex eccentricities in G are translatable to finding the diameter of $\mathcal{H}(G)$ and computing eccentricities of vertices of the Helly graph $\mathcal{H}(G)$. For example, there is a subquadratic time approximation for radius $rad(G)$ of a graph with an additive error depending on $\alpha(G)$ [68]. Moreover, the existence of the injective hull of a graph G is useful to prove properties that appear in G. For example, the existence of injective hulls has been used to prove the existence of a core which intersects shortest paths for a majority of pairs of vertices, establishing that traffic congestion is inherent in graphs with global negative curvature [37]. In Chapter 6, injective hulls were also used to prove the existence of an eccentricity approximating spanning tree T of G which gives an approximation of all vertex eccentricities with additive error depending essentially on $\alpha(G)$.

The importance of $\mathcal{H}(G)$ as an underlying structure drives our interest in the injective hulls of various graph classes. Our main contributions are summarized in Table 7.1 and organized as follows. We identify in Section 7.1 several universal properties of the injective hull of any graph. Next, we focus on a graph G that belongs to a particular graph class $\mathcal{C}$. In particular, we are interested in whether $\mathcal{C}$ is closed under Hellification, i.e., whether $G \in \mathcal{C}$ implies $\mathcal{H}(G) \in \mathcal{C}$. In Section 7.2, we give a graph theoretic proof that hyperbolic graphs are closed under Hellification. Moreover, we prove that satisfying the Helly property in disks of radii at most $\delta + 1$ is sufficient to satisfy the Helly property in all disks of a δ-hyperbolic graph. In Section 7.3, we show that permutation graphs are not closed under Hellification and provide conditions in which AT-free graphs are. In Section 7.4, we prove that chordal graphs and square-chordal graphs are closed under Hellification. In Section 7.5, we add distance-hereditary graphs to the growing list of graph classes closed under Hellification and provide a polynomial time algorithm to compute $\mathcal{H}(G)$ of a distance-hereditary graph G. We demonstrate in Section 7.6 that the injective hull of several graphs from some other well-known classes of graphs are impossible to compute efficiently. Specifically, there is a graph G where the number of vertices in $\mathcal{H}(G)$ is $\Omega(a^n)$, where $a > 1$, $n = |V(G)|$. We construct three such graphs: a split graph, a cocomparability graph, and a chordal bipartite graph.

Graph Class $\mathcal{C}$	Closed under Hellification	Hardness to compute $\mathcal{H}(G)$ for any $G \in \mathcal{C}$
δ-Hyperbolic	Yes	$\Omega(a^n)$
Chordal	Yes	$\Omega(a^n)$
Square-Chordal	Yes	?
Distance-Hereditary	Yes	$O(n^3)$
Permutation	No	?
Cocomparability	?	$\Omega(a^n)$
AT-free	?	$\Omega(a^n)$
Chordal Bipartite (or any triangle-free)	No	$\Omega(a^n)$

Table 7.1: A summary of our results on injective hulls of various graph classes, where $a > 1$ and $n = |V(G)|$. "?" means that this question is still open.

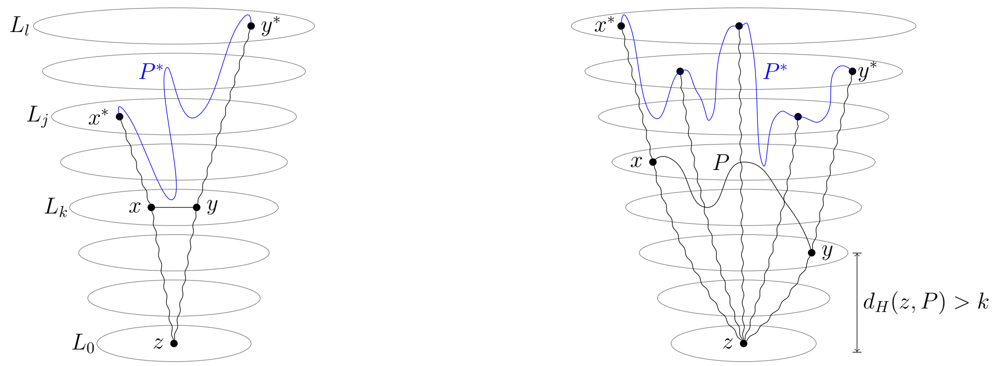


Figure 7.1: Illustration to the proofs of (a) Lemma 64 and (b) Theorem 25, where real paths are shown in blue.

7.1 Properties of injective hulls

Let the distance $d(z, P)$ from a vertex z to an (x, y)-path P be the minimum distance from z to any vertex $u \in P$. We will next show that for any vertex $z \in V(\mathcal{H}(G))$ and any (x, y)-path P in $\mathcal{H}(G)$, there is a real (x^*, y^*)-path P^* in G which behaves similarly to P with respect to some distance properties. In particular, we show that if $x, y, z \in V(G)$ then for every (x, y)-path P in $\mathcal{H}(G)$, there is a real (x, y)-path P^* in G such that $d(z, P^*) \geq d(z, P)$.

We have the following lemma.

Lemma 64. Let H be the injective hull of G. For any vertex $z \in V(H)$ and edge $xy \in E(H)$, there is a real (x^*, y^*)-path P^* in G such that $d_H(z, P^*) \geq d_H(z, \{x, y\})$, $I(z, x) \subseteq I(z, x^*)$, and $I(z, y) \subseteq I(z, y^*)$.

Proof. Let $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{e(z)}$ be layers of H produced by a breadth-first search rooted at vertex z. Without loss of generality, let $d_H(z, \{x, y\}) = d_H(z, x) = k$. Hence, $x \in \mathcal{L}_k$ and $y \in \mathcal{L}_p$ where $p = k$ or $p = k + 1$. By Proposition 9, there is a vertex $x^* \in V(G)$ such that $x \in I(z, x^*)$ and there is a vertex $y^* \in V(G)$ such that $y \in I(z, y^*)$. Then, $x^* \in \mathcal{L}_j$ for some $j \geq k$ and $y^* \in \mathcal{L}_\ell$ for some $\ell \geq p$. Since G is isometric in H, there is a shortest (x^*, y^*)-path P^* in G of length $d_H(x^*, y^*)$ consisting of all real vertices, as illustrated in Figure 7.1(a). By the triangle inequality, $d_G(x^*, y^*) = d_H(x^*, y^*) \leq d_H(x^*, x) + 1 + d_H(y, y^*) \leq (j - k) + 1 + (\ell - p)$. By contradiction, assume there is a vertex $w^* \in P^*$ such that $d_H(z, w^*) < k$. Then, $d_H(x^*, y^*) = d_H(x^*, w^*) + d_H(w^*, y^*) \geq (j - k + 1) + (\ell - k + 1)$, a contradiction. $\square$

Theorem 25. Let H be the injective hull of any graph G. For any $x, y, z \in V(G)$, the disk $D_G(z, k)$ separates vertices x, y in G if and only if disk $D_H(z, k)$ separates vertices x, y in H.

Proof. ($\leftarrow$) It suffices to remark that if $D_G(z, k)$ does not separate x, y in G due to a path P^* connecting them, then the same path establishes that $D_H(z, k)$ does not separate x, y in $\mathcal{H}(G)$.

($\rightarrow$) Suppose the disk $D_H(z, k)$ does not separate vertices x, y in H and assume, without loss of generality, that $D_H(z, k) \cap \{x, y\} = \emptyset$. Then, there is an (x, y)-path P in H such that $d_H(z, P) > k$. Let $P = v_0, v_1, v_2, \ldots, v_j$, where $v_0 := x$ and $v_j := y$. By Lemma 64, for each edge $v_i v_{i+1}$ on P, there is a real $({v_i}^*, {v_{i+1}}^*)$-path ${P_i}^*$ in G such that $d_H(z, {P_i}^*) \geq d_H(z, \{v_i, v_{i+1}\}) > k$, as shown in Figure 7.1(b). Let P^* be the real path obtained by joining, for $i \in [0, j-1]$, each real path P_i^* by their end vertices. Then, $d_H(z, P^*) \geq d_H(z, P) > k$. As a result, the disk $D_G(z, k)$ does not separate vertices x, y in G. $\square$

Corollary 38. Let H be the injective hull of a graph G. For any $x, y, z \in V(G)$ and every (x, y)-path P in H, there is a real (x, y)-path P^* in G such that $d(z, P^*) \geq d(z, P)$.

7.2 δ-Hyperbolic graphs

Recall that a graph G is δ-hyperbolic if it satisfies Gromov's 4-point condition: for any four vertices u, v, w, x the two larger of the three distance sums $d(u, v) + d(w, x)$, $d(u, x) + d(v, w)$, and $d(u, w) + d(v, x)$ differ by at most $2\delta \geq 0$. For a quadruple of vertices $u, v, w, x \in V(G)$, it will be convenient to denote by $hb(u, v, w, x)$ half the difference of the largest two distance sums among $d(u, v) + d(w, x)$, $d(u, x) + d(v, w)$, and $d(u, w) + d(v, x)$.

It is known [83,95] that the hyperbolicity of any metric space is preserved in its injective hull. For completeness, we provide a graph-theoretic proof of this result and show that in fact hyperbolicity is preserved in any host H as long as distances in G are preserved in H and that peripheral vertices of H are real.

Proposition 13. If H is a host graph such that G embeds isometrically into H and all peripheral vertices in H are from G, then $\delta(G) = \delta(H)$.

Proof. As G embeds isometrically into H, $\delta(G) \leq \delta(H)$. By contradiction, assume $\delta(H) > \delta(G)$. Let $x, y, z, t \in V(H)$ with $hb(x, y, z, t) > \delta(G)$ such that $|V(G) \cap \{x, y, z, t\}|$ is maximized. Without loss of generality, let $d_H(x,t)+d_H(z,y) \geq d_H(x,z)+d_H(t,y) \geq d_H(x,y)+d_H(z,t)$. If $\{x,y,z,t\} \subseteq V(G)$, then $hb(x,y,z,t) \leq \delta(G)$, a contradiction. Thus, without loss of generality, suppose $x \notin V(G)$. By Proposition 9, there is a peripheral vertex $x^* \in V(G)$ such that $I(t,x) \subset I(t,x^*)$ for vertex $t \in V(H)$. Let $d_H(t,x^*) = d_H(t,x) + \gamma$. Clearly, $d_H(x^*,t) + d_H(z,y) \geq \max\{d_H(x^*,y) + d_H(z,t), d_H(x^*,z) + d_H(t,y)\}$.

Suppose that $d_H(x^*,y)+d_H(z,t) \geq d_H(x^*,z)+d_H(t,y)$. By the triangle inequality and definition of hyperbolicity, we have

$$\begin{aligned}
2hb(x^*,y,z,t) &= d_H(x^*,t) + d_H(z,y) - d_H(x^*,y) - d_H(z,t) \\
&\geq d_H(x,t) + d_H(z,y) + \gamma - d_H(x,y) - d_H(z,t) - \gamma \\
&= d_H(x,t) + d_H(z,y) - d_H(x,y) - d_H(z,t) \\
&\geq d_H(x,t) + d_H(z,y) - d_H(x,z) - d_H(t,y) \\
&= 2hb(x,y,z,t).
\end{aligned}$$

Thus, $hb(x^*,y,z,t) \geq hb(x,y,z,t)$, a contradiction with the maximality of the number of real vertices in the quadruple.

Suppose now that $d_H(x^*,z) + d_H(t,y) \geq d_H(x^*,y) + d_H(z,t)$. By the triangle inequality and definition of hyperbolicity, we have

$$\begin{aligned}
2hb(x^*,y,z,t) &= d_H(x^*,t) + d_H(z,y) - d_H(x^*,z) - d_H(t,y) \\
&\geq d_H(x,t) + d_H(z,y) + \gamma - d_H(x,z) - d_H(t,y) - \gamma \\
&= d_H(x,t) + d_H(z,y) - d_H(x,z) - d_H(t,y) \\
&= 2hb(x,y,z,t).
\end{aligned}$$

Thus, $hb(x^*,y,z,t) \geq hb(x,y,z,t)$, again a contradiction with the maximality of the number of real vertices in the quadruple. □

Theorem 26. For any graph G, $\delta(G) = \delta(\mathcal{H}(G))$. That is, the δ-hyperbolic graphs are closed under Hellification.

We next show that a δ-hyperbolic graph G is Helly if its disks up to radii $\delta+1$ satisfy the Helly property. In this sense, a localized Helly property implies a global Helly property, akin to what is known for pseudo-modular graphs wherein all disks of radii at most 1 satisfy the Helly property implies all disks (of all radii) satisfy the Helly property.

Lemma 65. If G is δ-hyperbolic and all disks with up to $\delta + 1$ radii satisfy the Helly property, then G is a Helly graph.

Proof. Assume all disks with radii at most $\delta + 1$ satisfy the Helly property. Clearly G is neighborhood-Helly. By Proposition 7, it remains only to prove that G is pseudo-modular. We apply Proposition 6(iii). Consider three vertices u, v, w such that $d(u, v) = d(u, w) = k \geq 2$, and either v and w are adjacent or have a common neighbor z. We claim that $d(u, v) = d(u, w) = k$ implies there is a vertex t adjacent to v and w and at distance $k - 1$ from u. We use an induction on $d(u, v)$. By assumption, it is true for $k \leq \delta + 2$ as the pairwise-intersecting disks $D(u, k - 1)$, $D(v, 1)$, $D(w, 1)$ have a common vertex t by the Helly property.

Consider the case when $d(u, v) = d(u, w) = k > \delta + 2$. Let $x \in I(v, u)$ and $y \in I(w, u)$ be vertices such that $d(x, u) = d(y, u) = \delta + 2$. We claim the disks $D(x, \delta + 1)$, $D(y, \delta + 1)$, and $D(u, 1)$ pairwise intersect; then, vertex u^* exists by the Helly property and applying the inductive hypothesis to vertex u^* equidistant to v, w yields the desired vertex t. Clearly, $D(u, 1)$ intersects both $D(x, \delta + 1)$ and $D(y, \delta + 1)$. It remains to show that $d(x, y) \leq 2\delta + 2$.

Consider vertices u, x, y, w and three distance sums: $A := d(u, w) + d(x, y)$, $B := d(u, y) + d(x, w)$ and $C := d(u, x) + d(y, w)$. We have $A = k + d(x, y)$ and $C = k$. Moreover, $k \leq B \leq k + 2$ as $k = d(u, w) \leq d(u, x) + d(x, v) + d(v, w) \leq d(u, v) + 2 = k + 2$. Hence, C is a smallest sum. If $B \geq A$ then $k + 2 \geq B \geq A = k + d(x, y)$ implies $d(x, y) \leq 2 \leq 2\delta + 2$. If $A \geq B$ then, by 4-point condition, $2\delta \geq A - B \geq k + d(x, y) - k - 2$, i.e., $d(x, y) \leq 2\delta + 2$. □

7.3 Permutation graphs and relatives

Permutation graphs can be defined as follows. Consider two parallel lines (upper and lower) in the plane. Assume that each line contains n points, labeled 1 to n, and each two points with the same label define a segment with that label. The intersection graph of such a set of segments between two parallel lines is called a *permutation graph* [21]. An *asteroidal triple* is an independent set of three vertices such that each pair is joined by a path that avoids the closed neighborhood of the third. A far reaching superclass of permutation graphs are the AT-free graphs, i.e., the graphs that do not contain any asteroidal triples [40].

We show that permutation graphs are not closed under Hellification. Moreover, if the Helly-gap of some AT-free graph is 2, then AT-free graphs are also not closed under Hellification.

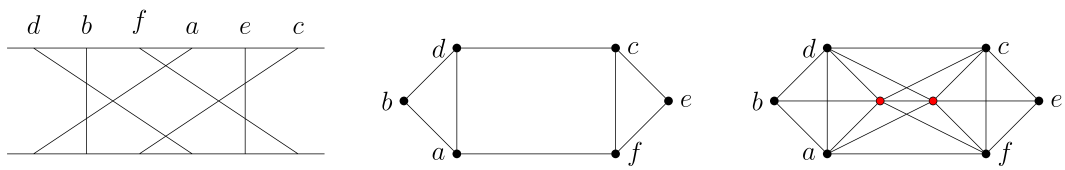


Figure 7.2: A permutation model (left) corresponding to permutation graph G (center) and its injective hull $\mathcal{H}(G)$ (right), where $\mathcal{H}(G)$ is not a permutation graph.

Lemma 66. Permutation graphs are not closed under Hellification.

Proof. The graph G illustrated in Figure 7.2 is an example of a permutation graph G for which $\mathcal{H}(G)$ is not a permutation graph (although, $\mathcal{H}(G)$ is AT-free). Note that only two Helly vertices h_1 and h_2 are added to produce $\mathcal{H}(G)$, where h_1 is adjacent to real vertices b, d, a, c, f and $h_1 e \notin E(\mathcal{H}(G))$. The resulting graph $\mathcal{H}(G)$ is not a permutation graph since such a vertex/segment h_1 cannot be added to the essentially unique permutation model of G depicted in Figure 7.2; any segment h_1 intersecting the segments b, d, a, c, f needs to intersect also the segment e. $\square$

For an AT-free graph G, the Helly-gap $\alpha(G)$ is impacted by whether $\mathcal{H}(G)$ is AT-free. Recall that the Helly gap $\alpha(G)$ is the minimum integer α such that the distance from any Helly vertex $h \in V(H)$ to a closest real vertex $x \in V(G)$ is at most α. It is known [57] that any AT-free graph G has $\alpha(G) \leq 2$.

Lemma 67. For any graph G, $\alpha(G) \leq 1$ if $\mathcal{H}(G)$ is AT-free.

Proof. By contradiction, suppose $\alpha(G) \geq 2$ for some graph G and $H := \mathcal{H}(G)$ is AT-free. Then, there is a vertex $h \in V(H)$ such that $d_H(h, v) \geq \alpha(G)$ for all $v \in V(G)$. Let $x \in V(G)$ be closest to h; then, $d_H(h, x) = \alpha(G) \geq 2$. By Proposition 9, there is a real vertex $y \in V(G)$ such that $h \in I(x, y)$. Moreover, $d_H(h, y) \geq d_H(h, x) \geq 2$. Let P be a shortest (x, y)-path of H with $h \in P$. As G is isometric in H, there is a (real) shortest (x, y)-path P^* in G. By $d_H(x, y)$ distance requirements, vertex x avoids $I(h, y)$ and vertex y avoids $I(h, x)$. As $P^* \subseteq V(G)$ and $\alpha(G) \geq 2$, vertex h also avoids all vertices on P^*. Therefore, $\{x, y, h\}$ forms an asteroidal triple in H, a contradiction. □

Corollary 39. If there is an AT-free graph G with $\alpha(G) = 2$, then AT-free graphs are not closed under Hellification.

Currently, we do not know whether there is an AT-free graph G with $\alpha(G) = 2$.

7.4 Chordal Graphs and Square-Chordal Graphs

Recall that a graph is *chordal* if it contains no induced cycle C_k of length $k \geq 4$. A graph G is *square-chordal* if G^2 is chordal. In this section, we will show that for a chordal (square-chordal) graph G, its injective hull $\mathcal{H}(G)$ is also chordal (square-chordal). That is, chordal graphs and square-chordal graphs are closed under Hellification.

The following fact is a folklore.

Proposition 14. Let G be a chordal graph, and let C be a cycle of G. For any vertex $x \in C$, if x is not adjacent to any third vertex of C, then the neighbors in C of x are adjacent.

We will need a few auxiliary lemmas. The following characterizations of chordal graphs within the class of the α_1-metric graphs will be useful. A graph is said to be an α_1-metric graph if it satisfies the following: for any $x, y, z, v \in V(G)$ such that $zy \in E(G)$, $z \in I(x, y)$ and $y \in I(z, v)$, $d_G(x, v) \geq d_G(x, y) + d_G(y, v) - 1$ holds [30, 116].

Lemma 68. [116] G is a chordal graph if and only if it is an α_1-metric graph not containing any induced subgraphs isomorphic to cycle C_5 and wheel W_k, $k \geq 5$.

A graph is *bridged* [72] if it contains no isometric cycle C_k of length $k \geq 4$. Bridged graphs are a natural generalization of chordal graphs. Directly combining two results from [63, 116], we obtain the following lemma.

Lemma 69. [63, 116] G is an α_1-metric graph not containing an induced C_5 if and only if G is a bridged graph not containing W_6^{++} as an isometric subgraph (see Figure 7.3).

Next lemma establishes conditions in which a Helly graph is chordal.

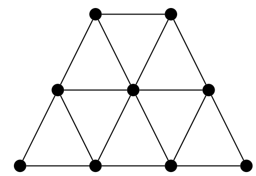


Figure 7.3: Forbidden isometric subgraph W_6^{++}

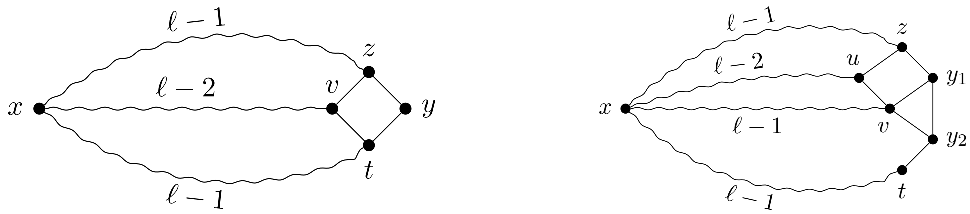
Lemma 70. If G is a Helly graph with no induced wheels W_k, $k \geq 4$, then G is chordal.

Proof. We first claim that G has no induced C_4 nor C_5. By contradiction, assume C_4 or C_5 is induced in G. Consider the system of pairwise intersecting unit disks centered at each vertex of the cycle. By the Helly property, there is a vertex universal to the cycle. Thus, G contains W_4 or W_5, a contradiction establishing the claim that G has no induced C_4 nor C_5.

We next claim that G is a bridged graph. Suppose G has an isometric cycle $C_{2\ell}$ for some integer $\ell \geq 3$ (when $\ell = 2$, G has induced C_4). Let $x, y \in C_{2\ell}$ be opposite vertices such that $d_G(x, y) = \ell$. Let $z, t \in C_{2\ell}$ be the distinct neighbors of y, as illustrated in Figure 70(a). As the disks $D(x, \ell - 2)$, $D(z, 1)$, and $D(t, 1)$ pairwise intersect, then by the Helly property, there is a vertex $v \in I(x, t) \cap I(x, z) \cap I(t, z)$. Since $C_{2\ell}$ is isometric, necessarily $vy \notin E(G)$ and $zt \notin E(G)$. A contradiction arises with the C_4 induced by v, z, y, t.

Suppose now that G has an isometric cycle $C_{2\ell+1}$ for some integer $\ell \geq 3$ (when $\ell = 2$, G has induced C_5). Let $x, y_1, y_2 \in C_{2\ell+1}$ be vertices such that $y_1 y_2 \in E(G)$ and $d_G(x, y_1) = d_G(x, y_2) = \ell$. As the disks $D(x, \ell - 1)$, $D(y_1, 1)$, and $D(y_2, 1)$ pairwise intersect, by the Helly property, there is a vertex v adjacent to y_1 and y_2 such that $d_G(x, v) = \ell - 1$. Let $z, t \in C_{2\ell+1}$ be vertices such that $z \in N(y_1) \cap I(y_1, x)$ and $t \in N(y_2) \cap I(y_2, x)$, as illustrated in Figure 70(b). Since $\ell \geq 3$ and by choice of the vertices z, t on isometric cycle $C_{2\ell+1}$, necessarily $d_G(z, t) = 3$. Therefore, $vz \notin E(G)$ or $vt \notin E(G)$; without loss of generality, let $vz \notin E(G)$. As the disks $D(x, \ell - 2)$, $D(v, 1)$, and $D(z, 1)$ pairwise intersect, by the Helly property, there is a vertex $u \in I(x, v) \cap I(x, z) \cap I(v, z)$. Necessarily $uy_1 \notin E(G)$, otherwise $d_G(x, y_1) < \ell$. A contradiction arises with the C_4 induced by u, z, y_1, v.

Hence, G is a bridged graph. Since G has no induced W_k for $k \geq 4$, G does not contain W_6^{++} as an isometric subgraph (observe that W_6 is an isometric subgraph of W_6^{++}). By Lemma 69, G is an α_1-metric graph not containing an induced C_5. Since G also has no induced W_k for $k \geq 4$, by Lemma 68, G is chordal. $\square$



(a) Case of isometric $C_{2\ell}$ (b) Case of isometric $C_{2\ell+1}$

Figure 7.4: Illustration to the proof of Lemma 70.

We will also use the fact that chordal graphs and square-chordal graphs can be characterized by the chordality of their so-called visibility graph and intersection graph, respectively. Let $M = \{S_1, \dots, S_\ell\}$ be a family of subsets of $V(G)$, i.e., each $S_i \subseteq V(G)$. An *intersection graph* $\mathcal{L}(M)$ and a *visibility graph* $\Gamma(M)$ are both a generalization of graph powers and are defined by Brandstädt et al. [20] as follows. The sets from M are the vertices of $\mathcal{L}(M)$ and $\Gamma(M)$. Two vertices of $\mathcal{L}(M)$ are joined by an edge if and only if their corresponding sets intersect. Two vertices of $\Gamma(M)$ are joined by an edge if and only if their corresponding sets are visible to each other; two sets S_i and S_j are visible to each other if $S_i \cap S_j \neq \emptyset$ or there is an edge of G with one end in S_i and the other end in S_j. Denote by $\mathcal{D}(G) = \{D(v, r) : v \in V(G), r \text{ a non-negative integer}\}$ the family of all disks of G.

Lemma 71. [20] For a graph G, $\Gamma(\mathcal{D}(G))$ is chordal if and only if G is chordal.

Lemma 72. [20] For a graph G, $\mathcal{L}(\mathcal{D}(G))$ is chordal if and only if G^2 is chordal.

We are now ready to prove the main results of this section.

Theorem 27. Let G be a chordal graph. Then $\mathcal{H}(G)$ is also chordal.

Proof. By contradiction, assume G is chordal and $H := \mathcal{H}(G)$ is not. By Lemma 70, there is an induced wheel W_k in H for some $k \geq 4$. Let $S = \{v_1, \dots, v_k\}$ be the set of vertices of W_k that induce a cycle C_k suspended by universal vertex c.

We first claim that there is a real vertex u_2 such that $d_H(u_2, v_i) = d_H(u_2, v_2) + d_H(v_2, v_i)$ for each $v_i \in S$. Consider the layering $\mathcal{L}_0, \dots, \mathcal{L}_\lambda$ produced by a multi-source breadth-first search rooted at the vertex set $\{v_4, v_5, \dots, v_k\}$; as the vertices of S have pairwise distance at most 2, this can be simulated with a BFS rooted at an artificial vertex s adjacent to only $\{v_4, v_5, \dots, v_k\}$. Then, $\mathcal{L}_0 = \{s\}$, $\mathcal{L}_1 = \{v_4, \dots, v_k\}$, $\{v_1, v_3, c\} \subseteq \mathcal{L}_2$, and $v_2 \in \mathcal{L}_3$. Let vertex $u_2 \in \mathcal{L}_\rho$ be an ancestor of v_2

with maximal ρ; then, $d_H(u_2, v_i) = d_H(u_2, v_2) + d_H(v_2, v_i)$ holds for each $v_i \in S$. By maximality of ρ, there is no vertex $z \in V(H)$ with $I(v_2, u_2) \subset I(v_2, z)$. Therefore, u_2 is a peripheral vertex and, by Proposition 8, is real (see Figure 7.5).

For each remaining vertex $v_i \in S$, we define a corresponding real vertex u_i in the following way. By Proposition 9, there is a shortest path between real vertices $u_1, u_3 \in V(G)$ which contains shortest path $P(v_1, v_3) = v_1 c v_3$ as a subpath. Thus, $d_H(u_1, u_3) = d_H(u_1, v_1) + 2 + d_H(v_3, u_3)$. Now let $j \in [4, k]$ be an integer. By choice of u_2, vertices c and v_2 belong to $I(u_2, v_j)$. Denote by $P(u_2, v_j)$ a shortest path containing c, v_2. By Proposition 9, there is a (not necessarily distinct) real vertex u_j such that shortest path $P(u_2, u_j)$ contains $P(u_2, v_j)$. Thus, $d_H(u_2, u_j) = d_H(u_2, v_2) + 2 + d_H(v_j, u_j)$.

With all distances established, we consider in G the family of disks $\{D(u_i, r(u_i))\}$, where $r(u_i) = d_H(u_i, v_i)$, for each $v_i \in S$. The disks centered at each vertex $u_i \in V(G)$ are visible to each other if their corresponding vertices $v_i \in V(H)$ are adjacent, i.e., $d_G(u_i, u_j) \leq d_H(u_i, v_i) + 1 + d_H(v_j, u_j) = r(u_i) + r(v_i) + 1$ if $v_i v_j \in E(H)$. As $d_H(u_1, u_3) = r(u_1) + r(u_3) + 2$, the disk $D(u_1, r(u_1))$ and disk $D(u_3, r(u_3))$ are not visible to each other. As $d_H(u_2, u_j) = r(u_2) + r(u_j) + 2$, for each integer $j \in [4, k]$, the disk $D(u_2, r(u_2))$ is not visible to the disk $D(u_j, r(u_j))$. Consider the visibility graph $\Gamma(\mathcal{D}(G))$. The vertices $D(u_i, r(u_i)) \in V(\Gamma(\mathcal{D}(G)))$, $i \in \{1, \ldots, k\}$, form a cycle in $\Gamma(\mathcal{D}(G))$. As vertex $D(u_2, r(u_2))$ is not adjacent to any vertex $D(u_j, r(u_j))$, where $j \in \{4, \ldots, k\}$, and its neighbors $D(u_1, r(u_1))$ and $D(u_3, r(u_3))$ on the cycle are not adjacent, by Proposition 14, $\Gamma(\mathcal{D}(G))$ is not chordal. By Lemma 71, G is also not chordal, a contradiction. $\square$

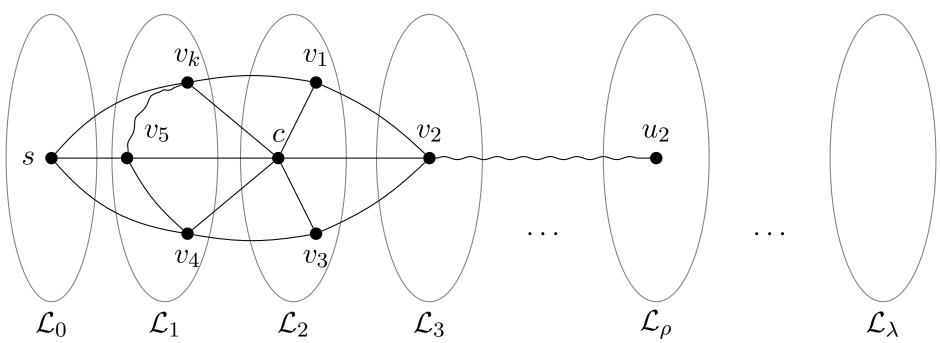


Figure 7.5: Illustration to the proof of Theorem 27.

A similar proof shows that the injective hull of a square-chordal graph G is also square-chordal.

Theorem 28. If G is square-chordal, then $\mathcal{H}(G)$ is square-chordal.

Proof. Let $H := \mathcal{H}(G)$. By Lemma 5, H^2 is Helly. Assume, by contradiction, that G^2 is chordal but H^2 is not. By Lemma 70, there is an induced wheel W_k in H^2 for some $k \geq 4$. Let $S = \{v_1, \ldots, v_k\}$ be the set of vertices of W_k that induce a cycle C_k suspended by universal vertex c. As $cv_i \in E(H^2)$ for each $v_i \in S$, then $d_H(c, v_i) \leq 2$. We denote by v_z a particular vertex of S defined as follows. If there is a vertex $v_i \in S$ such that $cv_i \in E(H)$, then set $v_z := v_i$. In this case, observe that all vertices $v_j \in S \setminus D_{H^2}(v_i, 1)$ satisfy $d_H(v_j, c) = 2$, else S would not induce an induced cycle in H^2. As $k \geq 4$, there is at least one such vertex v_j. On the other hand, if $d_H(c, v_i) = 2$ for each $v_i \in S$, then let v_z be any vertex of S. Without loss of generality, in what follows, we can assume that v_z is v_2.

In the next few steps, we define for each vertex $v_i \in S$ a real vertex u_i satisfying particular distance requirements. By Proposition 9, there are real vertices $u_1, u_3 \in V(G)$ such that a shortest (u_1, u_3)-path in H contains a shortest (v_1, v_3)-path in H. As v_1 and v_3 are non-adjacent in H^2, $d_H(v_1, v_3) \geq 3$ and, therefore, $d_H(u_1, u_3) \geq d_H(u_1, v_1) + d_H(v_3, u_3) + 3$. Consider now a multi-source breadth-first search in H rooted at $M = S \setminus \{v_1, v_2, v_3\}$; this can be simulated with a BFS rooted at an artificial vertex s adjacent to only the vertices of M. Then, $\mathcal{L}_0 = \{s\}$, $\mathcal{L}_1 = M$, $v_1, v_3 \in \mathcal{L}_2 \cup \mathcal{L}_3$, $c \in \mathcal{L}_3$, and finally, $v_2 \in \mathcal{L}_4 \cup \mathcal{L}_5$. Let vertex $u_2 \in \mathcal{L}_\rho$ be an ancestor of v_2 with maximal ρ. By maximality of ρ, there is no vertex $f \in V(H)$ with $I(v_2, u_2) \subset I(v_2, f)$. Therefore, u_2 is a peripheral vertex and, by Proposition 8, is real.

For each remaining vertex $v_i \in M$, we define a corresponding real vertex u_i in the following way. Note that, by choice of v_i, v_2 and v_i are non-neighbors in H^2; thus, $d_H(v_i, v_2) \geq 3$. On one hand, if $v_2 \in I(v_i, u_2)$ then, by Proposition 9, there is a real u_i vertex such that a shortest (u_i, u_2)-path in H contains a shortest (v_i, v_2)-path in H. Hence, $d_H(u_i, u_2) \geq d_H(u_i, v_i) + d_H(v_2, u_2) + 3$. On the other hand, if $v_2 \notin I(v_i, u_2)$, then necessarily $v_2 \in \mathcal{L}_4$ and $d_H(v_i, v_2) = 4$. Then, there exists a vertex $z \in I(v_i, u_2) \cap \mathcal{L}_4$ with $d_H(v_i, z) = 3$ and $d_H(z, u_2) = d_H(v_2, u_2)$. By Proposition 9, there is a real vertex u_i such that a shortest (u_i, u_2)-path in H contains a shortest (v_i, z)-path in H. Hence, $d_H(u_i, u_2) = d_H(u_i, v_i) + d_H(v_i, z) + d_H(z, u_2) = d_H(u_i, v_i) + 3 + d_H(v_i, u_2)$.

With all distances established, we consider in G the family of disks $\{D(u_i, r(u_i))\}$, where $r(u_i) = d_H(u_i, v_i) + 1$, for each $v_i \in S$. The disks centered at each vertex $u_i \in V(G)$ intersect if their corresponding vertices $v_i \in V(H^2)$ are adjacent in H^2, i.e., $d_G(u_i, u_j) \leq d_H(u_i, v_i) + 2 + d_H(v_j, u_j) = r(u_i) + r(u_j)$ if $v_i v_j \in E(H^2)$. As $d_H(u_1, u_3) \geq r(u_1) + r(u_3) + 1$, the disk $D(u_1, r(u_1))$ and disk

$D(u_3, r(u_3))$ do not intersect. As $d_H(u_2, u_j) \geq r(u_2) + r(u_j) + 1$, for each $j \in \{4, \ldots, k\}$, the disks $D(u_2, r(u_2))$ and $D(u_j, r(u_j))$ do not intersect. Consider the intersection graph $\mathcal{L}(\mathcal{D}(G))$. The vertices $D(u_i, r(u_i)) \in V(\mathcal{L}(\mathcal{D}(G)))$, $i \in \{1, \ldots, k\}$, form a cycle in $\mathcal{L}(\mathcal{D}(G))$. As vertex $D(u_2, r(u_2))$ is not adjacent to any vertex $D(u_j, r(u_j))$, where $j \in \{4, \ldots, k\}$, and its neighbors $D(u_1, r(u_1))$ and $D(u_3, r(u_3))$ on the cycle are not adjacent, by Proposition 14, $\mathcal{L}(\mathcal{D}(G))$ is not chordal. By Lemma 72, G^2 is also not chordal, a contradiction. □

A graph G is *dually chordal* if it has a so-called *maximum neighborhood ordering* (see [73] for definitions and various characterizations of this class of graphs). A maximum neighborhood ordering can be constructed in total linear time [73]. For us here, the following characterization is relevant: a graph G is dually chordal if and only if G is neighborhood-Helly and G^2 is chordal [73]. So, we can state the following corollary.

Corollary 40. If G is a square-chordal graph, then $\mathcal{H}(G)$ is dually chordal.

7.5 Distance-hereditary graphs

Recall that a graph is *distance-hereditary* if and only if each of its connected induced subgraphs is isometric [82], that is, the length of any induced path between two vertices equals their distance in G. In this section, we show that distance-hereditary graphs are closed under Hellification. We give a characterization of the distance-hereditary Helly graphs and provide an algorithm to construct the injective hull of a distance-hereditary graph in linear time. Moreover, if G is distance-hereditary, then $|V(\mathcal{H}(G))| \in O(n)$, where $n = |V(G)|$.

Let w, x, y, z be four vertices that induce a C_4. We denote by $S(w, x, y, z)$ an extended square that includes the vertices w, x, y, z which induce a C_4 and any vertex adjacent to at least three of them, i.e., $S(w, x, y, z) = \{v \in D(\{w, x, y, z\}, 1) : |N[v] \cap \{w, x, y, z\}| \geq 3\}$. We show that a distance-hereditary graph is Helly if and only if all extended squares are suspended. The result is analogous to a characterization of chordal Helly graphs [49]: a chordal graph is Helly if and only if all extended triangles are suspended, where an extended triangle $S(x, y, z)$ is defined as the set of vertices that see at least two vertices of the triangle $\Delta(x, y, z)$.

Lemma 73. A distance-hereditary graph G is Helly if and only if, for every C_4 induced by $w, x, y, z \in V(G)$, the extended square $S(w, x, y, z)$ is suspended.

Proof. It is known that distance-hereditary graphs are pseudo-modular [14]. Hence, by Proposition 7, G is Helly if and only if it is neighborhood-Helly, i.e., all 2-sets are suspended. If G is neighborhood-Helly, then all extended squares are suspended since all vertices of an extended square are pairwise at distance at most 2. We assert that if every extended square is suspended, then all 2-sets are suspended.

We use an induction on the cardinality of a 2-set. Assume any 2-set $M \subseteq V(G)$ with $|M| \leq k$ is suspended. Clearly, it is true for $k \leq 2$. By contradiction, assume every extended square is suspended but there is an unsuspended 2-set M with $|M| = k + 1 \geq 3$. Let $v_1, v_2, v_3 \in M$. By the inductive hypothesis, there is a vertex c_1 universal to $M \setminus \{v_1\}$, a vertex c_2 universal to $M \setminus \{v_2\}$, and a vertex c_3 universal to $M \setminus \{v_3\}$. Since M is not suspended, necessarily each $c_i \in \{c_1, c_2, c_3\}$ has $c_i v_i \notin E(G)$, $c_i \neq v_i$, and c_i is distinct from the other two vertices of $\{c_1, c_2, c_3\}$. We consider three cases based on how many of $\{c_1, c_2, c_3\}$ are distinct from the three vertices $\{v_1, v_2, v_3\}$. We will obtain a forbidden induced subgraph which contradicts Proposition 2(ii) or will show that M is a subset of some extended square which is suspended, giving a contradiction with M being unsuspended.

Case 1. $c_3 \notin \{v_1, v_2, v_3\}$ and $c_1, c_2 \in \{v_1, v_2, v_3\}$.

Without loss of generality, let $c_2 = v_1$. If $c_1 = v_3$, then $v_3 c_2 \in E(G)$, i.e., $c_1 v_1 \in E(G)$, a contradiction. Hence, $c_1 = v_2$ and therefore, c_1, c_3, c_2, v_3 induce C_4. Any $x \in M$ belongs to $S(c_1, c_3, c_2, v_3)$ since x is either one of c_1, c_2, v_3 or adjacent to all of c_1, c_2, c_3. Thus, $M \subseteq S(c_1, c_3, c_2, v_3)$.

Case 2. $c_2, c_3 \notin \{v_1, v_2, v_3\}$ and $c_1 \in \{v_1, v_2, v_3\}$.

Without loss of generality, let $c_1 = v_2$. Then, c_1 is adjacent to v_3, but $c_1 v_1 \notin E(G)$. Since $c_3 v_2 \in E(G)$ and $c_2 v_2 \notin E(G)$, by equality $c_3 c_1 \in E(G)$ and $c_2 c_1 \notin E(G)$. By assumption, c_2 is adjacent to v_3 and v_1, c_3 is adjacent to v_1, and $c_3 v_3 \notin E(G)$. It only remains whether $c_2 c_3 \in E(G)$ and/or $v_1 v_3 \in E(G)$. If at most one of those edges occurs, we obtain C_5 or a house induced by $\{c_1, c_3, v_1, c_2, v_3\}$. Therefore, both edges $c_2 c_3$ and $v_1 v_3$ must be present. Now, any $x \in M \setminus \{c_2, v_3, v_1\}$ is adjacent to both c_1, c_3. Furthermore, if $x v_1 \notin E(G)$ and $x v_3 \notin E(G)$, then x, c_3, c_1, v_3, v_1 induce a house. Thus, x is also adjacent to at least one of v_1, v_3. Since any $x \in M$

is either one of v_1, v_3 or is adjacent to at least three of $\{c_1, c_3, v_1, v_3\}$, we get $M \subseteq S(c_1, c_3, v_1, v_3)$.

Case 3. $c_1, c_2, c_3 \notin \{v_1, v_2, v_3\}$.

By assumption, c_1 is adjacent to v_2 and v_3, c_2 is adjacent to v_1 and v_3, c_3 is adjacent to v_1 and v_2, and $c_1v_1, c_2v_2, c_3v_3 \notin E(G)$. If each $i, j \in \{1, 2, 3\}$ with $i \neq j$ satisfies $c_ic_j \notin E(G)$ and $v_iv_j \notin E(G)$, then $v_1, c_3, v_2, c_1, v_3, c_2$ induce C_6. Thus, there is some chord $c_ic_j \in E(G)$ or $v_iv_j \in E(G)$. We consider two subcases without loss of generality.

Case 3(a). There is a chord $c_2c_3 \in E(G)$.

If there are no other edges between vertices $\{c_2, c_3, v_2, c_1, v_3\}$, then those vertices induce a C_5. Thus, there is at least one of the following chords: c_1c_2, v_3v_2, or c_1c_3. If $v_2v_3 \notin E(G)$, we get in G a house or gem induced by v_3, c_2, c_3, v_2, c_1. Hence, $v_2v_3 \in E(G)$. Consider now C_4 induced by c_2, c_3, v_2, v_3. Any vertex $x \in M \setminus \{c_2, c_3, v_2, v_3\}$ is adjacent to both c_2, c_3. Furthermore, if $xv_2 \notin E(G)$ and $xv_3 \notin E(G)$, then x, c_3, c_2, v_3, v_2 induce a house. Thus, x is also adjacent to at least one of v_2, v_3. Since any $x \in M$ is either one of c_2, c_3, v_2, v_3 or is adjacent to at least three of $\{c_2, c_3, v_2, v_3\}$, we get $M \subseteq S(c_2, c_3, v_2, v_3)$.

Case 3(b). There is a chord $v_2v_3 \in E(G)$ and $c_ic_j \notin E(G)$ for distinct $i, j \in \{1, 2, 3\}$.

If there are no other edges between vertices $\{v_3, c_2, v_1, c_3, v_2\}$, then those vertices induce C_5. Thus, there is at least one of the following chords: v_1v_3 or v_1v_2. But then, vertices v_2, v_3, c_2, v_1, c_3 induce a house or a gem. Obtained contradictions prove the lemma. □

We next show that one can efficiently compute the injective hull of a distance-hereditary graph G. Moreover, as a byproduct, we get that $\mathcal{H}(G)$ is distance-hereditary and $|V(\mathcal{H}(G))| \in O(|V(G)|)$. One attempt to compute $\mathcal{H}(G)$ is to add Helly vertices suspending all maximal 2-sets of G. We observe that G has $O(n)$ maximal 2-sets. Indeed, since G^2 is chordal [12], there are $O(n)$ maximal cliques in G^2 obtainable via a perfect elimination ordering of G^2, and there is one-to-one correspondence between maximal cliques of G^2 and maximal 2-set of G. Since adding a Helly vertex h to suspend a single 2-set in G may create another unsuspended 2-set in $G + \{h\}$, this information alone is insufficient to conclude but gave a promising indication that possibly $|V(\mathcal{H}(G)| \in O(|V(G)|)$. Second attempt based on Lemma 73 is to suspend all extended squares, which incurs a similar problem that $G + \{h\}$ may have a new unsuspended extended square. Additionally, there can be more extended squares than there are maximal 2-sets.

We found it advantageous to use a characteristic pruning sequence of G (see Proposition 2(iii)). A *pruning sequence* $\sigma_G : V(G) \to \{1, \ldots, n\}$ of G is a total ordering of its vertex set $V(G) = \{v_1, \ldots, v_n\}$ such that each vertex v_i satisfies one of the following conditions in the induced subgraph $G_i := \langle v_1, \ldots, v_i \rangle$:

(i) v_i is a pendant vertex to some vertex v_j with $\sigma_G(v_j) < \sigma_G(v_i)$,

(ii) v_i is a true twin of some vertex v_j with $\sigma_G(v_j) < \sigma_G(v_i)$, or

(iii) v_i is a false twin of some vertex v_j with $\sigma_G(v_j) < \sigma_G(v_i)$.

Next lemmas give conditions under which adding a vertex to a Helly graph keeps it Helly. Consider a graph H obtained by adding to a Helly graph G a vertex u as a pendant or twin to some vertex in G. We show that any family $\mathcal{F} = \big\{D_H(w, r(w)) : w \in M \subseteq V(H)\big\}$ of pairwise intersecting disks in H has a common intersection. Note that this is trivially true if any vertex $w \in M$ has $r(w) = 0$ (since w is common to all disks of $\mathcal{F}$) or if $u \notin M$ (since G is isometric in H and the family of pairwise intersecting disks $\big\{D_G(w, r(w)) : w \in M \subseteq V(G)\big\}$ have a common intersection in G).

Lemma 74. Let $G + \{u\}$ be a graph obtained by adding a vertex u pendant to $v \in V(G)$. If G is Helly, then $G + \{u\}$ is Helly.

Proof. Let $H := G+\{u\}$, $\mathcal{F} = \big\{D_H(w, r(w)) : w \in M \subseteq V(H)\big\}$ be a family of pairwise intersecting disks in H, and $u \in M$. If $r(u) \geq 2$, one may substitute in $\mathcal{F}$ the disk $D_H(u, r(u))$ with the equivalent disk $D_H(v, r(u) - 1)$. Since G is isometric in H and is Helly, the corresponding disks in G have a common intersection. Assume now that $r(u) = 1$. Then, $v \in D_H(u, r(u))$. As the disks of $\mathcal{F}$ pairwise intersect, every $w \in M \setminus \{u\}$ satisfies $r(w) + r(u) \geq d_H(w, u) = d_H(w, v) + 1 = d_H(w, v) + r(u)$. Hence, $d_H(w, v) \leq r(w)$ and vertex v is common to all disks. □

Lemma 75. Let $G + \{u\}$ be a graph obtained by adding a vertex u as a true twin to $v \in V(G)$. If G is Helly, then $G + \{u\}$ is Helly.

Proof. Let $H := G+\{u\}$, $\mathcal{F} = \big\{D_H(w, r(w)) : w \in M \subseteq V(H)\big\}$ be a family of pairwise intersecting disks in H, and $u \in M$. Because u is a true twin of v, $D_H(u, r) = D_H(v, r)$ for any radius $r \geq 1$. Hence, we can assume that each disk of $\mathcal{F}$ is centered at a vertex of G, therefore there is a common intersection of all disks of $\mathcal{F}$. □

Lemma 76. Let $G + \{u\}$ be the graph obtained by adding a vertex u as a false twin to $v \in V(G)$. If G is Helly and v is a true twin to some $y \in V(G)$, then $G + \{u\}$ is Helly.

Proof. Let $H := G+\{u\}$, $\mathcal{F} = \{D_H(w, r(w)) : w \in M \subseteq V(H)\}$ be a family of pairwise intersecting disks in H, and $u \in M$. If $r(u) > 1$, then $D_H(v, r(u)) = D_H(u, r(u))$ and so we can assume that each disk of $\mathcal{F}$ is centered at a vertex of G, implying a non-empty common intersection of all disks of $\mathcal{F}$. Assume now that $r(u) = 1$. As each disk of $\mathcal{F}' = (\mathcal{F} \setminus N[u]) \cup N[v]$ is centered at a vertex of G, there is a common intersection R of all disks of $\mathcal{F}'$. Since $N[v] \in \mathcal{F}'$, $R \subseteq N[v]$. Recall that v has true twin y in G. Therefore, $v \in R$ implies $y \in R$. Thus, there exists vertex $s \in R \cap N(v)$. By definition of u, $N(u) = N(v)$. Thus, $s \in N(u)$ and hence s is contained in each disk in $\mathcal{F}$. $\square$

Using Corollary 40, Lemma 74, Lemma 75, Lemma 76 and a pruning sequence of a distance-hereditary graph G, we can efficiently compute a Helly graph which contains G as an isometric subgraph.

Theorem 29. If G is a distance-hereditary graph, then $\mathcal{H}(G)$ can be computed in $O(nm)$ time, where $n = |V(G)|$ and $m = |E(G)|$. Moreover, $|V(\mathcal{H}(G))| \in O(n)$.

Proof. We use a pruning sequence $\sigma_G = (v_1, \dots, v_n)$ of G that can be constructed in linear time [13, 44, 46]. Let G_i denote the graph induced by $\{v_1, \dots, v_i\}$ and let H be a graph that initially contains only v_1; clearly H is Helly, distance-hereditary, and contains G_1 as an isometric subgraph. We iterate over the remaining vertices $v_i \in \sigma_G$, $i \geq 2$, to carefully attach new vertices to H as pendants/twins to old vertices in H, thereby constructing for H a pruning sequence σ_H. Then, by Proposition 2(v), H is distance-hereditary. For clarity, denote by H_k the graph induced by $\{u \in V(H) : \sigma_H(u) \leq k\}$. We claim that the resulting graph H_k, where $k = |V(H)|$, is Helly and contains G as an isometric subgraph. There are two cases.

Case 1. Next vertex $v_i \in \sigma_G$ is a pendant or true twin to some vertex v_j in G_i. Set $H = H+\{v_i\}$ (the graph obtained by adding v_i as a pendant or true twin to v_j in H). By Lemma 74, and Lemma 75 H remains Helly. As H is distance-hereditary and contains G_i as an induced subgraph, G_i is isometric in H.

Case 2. Next vertex $v_i \in \sigma_G$ is a false twin to some vertex v_j in G_i. H as it is constructed so far is distance-hereditary and Helly, therefore, it is dually chordal. Clearly $H+\{v_i\}$ (the graph obtained by adding v_i as a false twin to v_j in H) is distance-hereditary and, thus, square-chordal. If $H+\{v_i\}$ is dually chordal, then it is Helly. In this case, set $H = H+\{v_i\}$. Otherwise, $H+\{v_i\}$ is not dually chordal, therefore it is not Helly. Hence, there is a family $\mathcal{F} = \{D(u, r(u)) : u \in M\}$ of pairwise intersecting disks which have no common intersection. Necessarily, $v_i \in M$. If $r(v_i) \geq 2$, then there is a common intersection in the Helly graph H as the family of disks $(\mathcal{F} \setminus D(v_i, r(v_i))) \cup D(v_j, r(v_j))$ pairwise intersect. Hence, $r(v_i) = 1$. Thus, the 2-set $N[v_j] \cup \{v_i\}$ is unsuspended. Set $H = H+\{y\}$, where y is a new true twin to v_j in H. Next, set $H = H+\{v_i\}$ (the graph obtained by adding v_i as a false twin to v_j in H). By Lemma 75, and Lemma 76, H remains Helly. As H is distance-hereditary and contains G_i as an induced subgraph, G_i is isometric in H.

Clearly, H contains at most $2n$ vertices. Moreover, as we can recognize dually chordal graphs $O(n+m)$ time [73], then H is obtainable in total $O(nm)$ time. We finally claim that H is a *minimal* Helly graph that contains G as an isometric subgraph, i.e., $H = \mathcal{H}(G)$. By contradiction, assume there is vertex $y \in V(H) \setminus V(G)$ with minimal $\sigma_H(y)$ such that $H \setminus \{y\}$ is Helly. Since $y \notin V(G)$, by algorithm construction, there is a vertex $v_i \in V(G_i)$ which is a false twin to v_j in G_i, but no vertex in $V(H_{\sigma_H(v_i)}) \setminus \{y\}$ suspends the 2-set $M = D_{G_i}(v_j, 1) \cup \{v_i\}$ in $H_{\sigma_H(v_i)} \setminus \{y\}$. Since $H \setminus \{y\}$ is Helly, there is a vertex u with minimal $\sigma_H(u)$ such that u suspends M in $H \setminus \{y\}$, where $\sigma_H(u) > \sigma_H(y)$. Each $v \in M$ has $\sigma_H(v) < \sigma_H(u)$. Let u be a pendant/twin to vertex z in the graph $H_{\sigma_H(u)}$, where $\sigma_H(z) < \sigma_H(u)$. Since $M \subseteq D_{H_{\sigma_H(u)}}(u, 1)$ and $|M| \geq 4$, clearly, u is not pendant to z. Hence, u is a twin to z and therefore, $M \subseteq D_{H_{\sigma_H(u)}}(z, 1)$. Thus, z suspends M, a contradiction with the minimality of $\sigma_H(u)$. $\square$

We remark that one of the motivations to find the injective hull of a graph is to obtain, using the least number of vertex additions, the many nice properties of Helly graphs while preserving distances of vertices in the original graph. By relaxing the requirement that we use the minimal number of vertices to obtain a Helly graph into which a distance-hereditary graph G isometrically embeds, we may obtain in linear time a Helly and distance-hereditary graph H with the desired distance-preserving properties which contains only $O(|V(G)|)$ vertices.

Corollary 41. Let G be a distance-hereditary graph, $n = |V(G)|$, and $m = |E(G)|$. A (not necessarily smallest) Helly graph H such that H is distance-hereditary, contains G as an isometric subgraph, and $|V(H)| \in O(|V(G)|)$ can be computed in $O(n+m)$ time.

Proof. The proof is analogous to the algorithm and analysis given in the proof of Theorem 29, but does not necessitate minimality of H. Compute a pruning sequence σ_G for G in $O(n+m)$ time and create a Helly distance-hereditary graph H by appending to H each vertex $v_i \in \sigma_G$ as follows. If v_i is a pendant or true twin to v_j in G, add it as a pendant or true twin to v_j in H. If v_i is a false twin to v_j in G, create a new true twin to v_j in H, then add v_i as a false twin to v_j in H. By Proposition 2(v), H is distance-hereditary. As G is an induced subgraph of H, then G is isometric in H. By Lemma 74, Lemma 75, and Lemma 76, H is Helly. Moreover, at most one new vertex $v \in V(H) \setminus V(G)$ is added to H for each vertex of $V(G)$. $\square$

7.6 Graphs with exponentially large injective hulls

We show that several restrictive graph classes, including split graphs, cocomparability graphs, chordal bipartite graphs, and consequently graphs of bounded hyperbolicity, graphs of bounded chordality, graphs of bounded tree-length or tree-breadth, and graphs of bounded diameter can have injective hulls that are exponential in size. In particular, there is a graph G of that class such that $|V(\mathcal{H}(G))| \in \Omega(a^n)$ for some constant $a > 1$ and $n = |V(G)|$.

We will use the following lemma to obtain a lower bound on the number of vertices in the injective hull of a particular graph. Recall that a set $S \subseteq V(G)$ is said to be a *2-set* if all vertices of S have pairwise distance at most 2.

Lemma 77. If G has at least k unsuspended maximal 2-sets, then $|V(\mathcal{H}(G)) \setminus V(G)| \geq k$.

Proof. Each unsuspended maximal 2-set $S = \{v_1, \dots, v_\ell\}$ corresponds to a unique family of pairwise intersecting disks $\{D(v_i, 1) : v_i \in S\}$ that have no common intersection in G. As $\mathcal{H}(G)$ is Helly, then for each S there is a unique Helly vertex $h \in V(\mathcal{H}(G))$ universal to S in $\mathcal{H}(G)$, i.e., $h(v_i) = 1$ for each $v_i \in S$ and $h(x) = d_G(x, S) + 1$ for each $x \in V(G) \setminus S$ (see Section 7.1). $\square$

7.6.1 Split graphs

A graph is a *split graph* if there is a partition of its vertices into a clique and an independent set [74, 112]. We construct a special split graph G as follows. Let $X = (x_1, x_2, \dots, x_k)$ be an independent set and let $Y = (y_1, y_2, \dots, y_k)$ be an independent set. Let also $M =$

$(u_1, v_1, w_1, z_1, u_2, v_2, w_2, z_2, \ldots, u_k, v_k, w_k, z_k)$ be a clique partitioned into k complete graphs K_4. For each integer $i \in [1, k]$, let x_i be adjacent to u_i and v_i, and let y_i be adjacent to w_i and z_i. Additionally, for all distinct integers $i, j \in [1, k]$, let x_i be adjacent to u_j and z_j, and let y_i be adjacent to w_j and v_j. See Figure 7.6 for an illustration. By construction, each vertex $x_i \in X$ is within distance 2 of every vertex in the graph except y_i. Every shortest (x_i, y_i)-path goes through M, but y_i and x_i share no common vertex of M. However, each x_i and y_j share a common vertex v_i. Observe that the resulting graph G has the following distance properties:

- $\forall x_i \in X,\ \forall m \in M,\ d_G(x_i, m) \leq 2$ via common neighbor u_i;
- $\forall y_i \in Y,\ \forall m \in M,\ d_G(y_i, m) \leq 2$ via common neighbor w_i;
- $\forall i, j \in [1, k],\ i \neq j,\ d_G(x_i, y_j) \leq 2$ via common neighbor z_j;
- $\forall i, j \in [1, k],\ i \neq j,\ d_G(x_i, x_j) = 2$ via common neighbor u_j;
- $\forall i, j \in [1, k],\ i \neq j,\ d_G(y_i, y_j) = 2$ via common neighbor w_j;
- $\forall i \in [1, k],\ d_G(x_i, y_i) = 3$ because y_i and x_i share no common vertex of M.

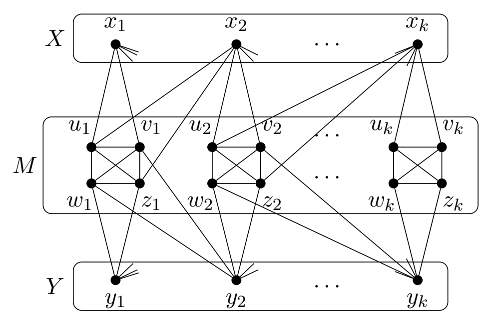


Figure 7.6: G is a split graph that requires exponentially many new Helly vertices. For readability, some edges are not shown. X and Y are independent sets and M is a clique of k complete graphs K_4.

Theorem 30. There is a split graph G such that $|V(\mathcal{H}(G))| \geq 2^{n/6} + 2n/3 - 2$, where $n = |V(G)|$.

Proof. Clearly, G is a split graph with independent set $X \cup Y$ and clique M.

We first claim that G described above has 2^k maximal 2-sets, where $k = n/6$. Let S be a maximal 2-set in G. Since all vertices are within distance at most 2 from M, then $M \subset S$. It remains only to observe that for each $i \in [1, k]$, either $x_i \in S$ or $y_i \in S$, but not both since $d_G(x_i, y_i) = 3$.

We next claim that any maximal 2-set S that contains at least two vertices from X and at least two vertices from Y is unsuspended. By contradiction, suppose a vertex $m \in V(G)$ suspends S.

As X and Y are independent sets, necessarily $m \in M$. Thus, $m \in \{u_i, w_i, v_i, z_i\}$ for some $i \in [1, k]$. However, for all $j \in [1, k]$, $d_G(u_i, y_j) = 2$ and $d_G(w_i, x_j) = 2$ holds. Hence, $m \neq u_i$ and $m \neq w_i$. As there are at least two vertices of X in S, there is an $x_j \in S$ such that $d_G(v_i, x_j) = 2$. As there are at least two vertices of Y in S, there is a $y_j \in S$ such that $d_G(z_i, y_j) = 2$. Thus, $m \neq v_i$ and $m \neq z_i$, a contradiction with the choice of m.

Moreover, there are at least $2^k - 2k - 2$ unsuspended maximal 2-sets in G. Observe that only 2 maximal 2-sets S have no $x_i \in S$ or have no $y_i \in S$. There are k maximal 2-sets which have only one $x_i \in S$ (one $i \in [1, k]$ is reserved for $x_i \in S$ and all other $j \in [1, k]$, $j \neq i$, have $y_i \in S$). Similarly, there are k maximal 2-sets which have only one $y_i \in S$. By Lemma 77, $|V(\mathcal{H}(G)) \setminus V(G)| \geq 2^k - 2k - 2$. Including the $6k$ vertices of $V(G)$, one obtains $|V(\mathcal{H}(G))| \geq 2^k + 4k - 2$. □

We remark that split graphs are chordal graphs. Additionally, chordal graphs are 1-hyperbolic [114]. G also has tree-length $tl(G) \leq 1$ [48] and tree-breadth $tb(G) \leq 1$ [62].

Corollary 42. Split graphs, chordal graphs, α_1-metric graphs, 1-hyperbolic graphs, graphs with $tl(G) \leq 1$, $tb(G) \leq 1$, and graphs with $diam(G) \leq 3$ can have exponentially large injective hulls. Specifically, there is a graph G of that class with $|V(\mathcal{H}(G))| \in \Omega(a^n)$, where $a > 1$ and $n = |V(G)|$.

7.6.2 Cocomparability graphs

Cocomparability graphs are exactly the graphs which admit a *cocomparability ordering* [43], i.e., an ordering $\sigma = [v_1, v_2, \ldots, v_n]$ of its vertices such that if $\sigma(x) < \sigma(y) < \sigma(z)$ and $xz \in E(G)$, then $xy \in E(G)$ or $yz \in E(G)$ must hold. Cocomparability graphs form a subclass of AT-free graphs.

A special cocomparability graph G is constructed as follows. Let $X = (x_1, x_2, \ldots, x_k)$ be a clique and $Y = (y_1, y_2, \ldots, y_k)$ be a clique. Let also $M = (u_1, v_1, w_1, z_1, u_2, v_2, w_2, z_2, \ldots, u_k, v_k, w_k, z_k)$ be a clique partitioned into k complete graphs K_4. For each integer $i \in [1, k]$, let x_i be adjacent to u_i and v_i, and let y_i be adjacent to w_i and z_i. Additionally, for all distinct integers $i, j \in [1, k]$, let x_i be adjacent to u_j and z_j, and let y_i be adjacent to w_j and v_j. See Figure 7.7 for an illustration. We emphasize that the key difference between graph G described above and the chordal graph construction in Figure 7.6 is that, here, X and Y are cliques.

By construction, each vertex $x_i \in X$ is within distance 2 of every vertex in the graph except y_i. Every shortest (x_i, y_i)-path goes through M, but y_i and x_i share no common vertex of M.

However, each x_i and y_j share a common vertex v_i. Observe that the resulting graph G has the following distance properties:

- $\forall i, j \in [1, k],\ i \neq j,\ d_G(x_i, x_j) = d_G(y_i, y_j) = 1$;
- $\forall x_i \in X,\ \forall m \in M,\ d_G(x_i, m) \leq 2$ via common neighbor u_i;
- $\forall y_i \in Y,\ \forall m \in M,\ d_G(y_i, m) \leq 2$ via common neighbor w_i;
- $\forall i, j \in [1, k],\ i \neq j,\ d_G(x_i, y_j) \leq 2$ via common neighbor z_j;
- $\forall i \in [1, k],\ d_G(x_i, y_i) = 3$ because y_i and x_i share no common vertex.

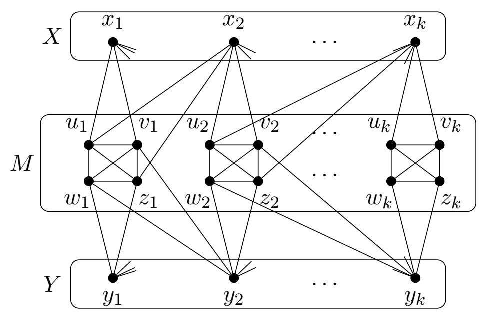


Figure 7.7: G is a cocomparability graph that requires exponentially many new Helly vertices. For readability, some edges are not shown. X and Y are each cliques and M is a clique of k complete graphs K_4.

Theorem 31. There is a cocomparability graph G such that $|V(\mathcal{H}(G))| \geq 2^{n/6} + 2n/3 - 2$, where $n = |V(G)|$.

Proof. The proof that G has an exponential number of maximal unsuspended 2-sets is the same as in the proof of Theorem 30, establishing $|V(\mathcal{H}(G))| \geq 2^{n/6} + 2n/3 - 2$.

It remains only to show that G is a cocomparability graph. Let m_1, m_2 be two vertices of M. Let σ be a vertex ordering of G such that $\sigma(x) < \sigma(m_1)$ for all $x \in X$, $\sigma(m_1) \leq \sigma(m) \leq \sigma(m_2)$ for all $m \in M$, and $\sigma(m_2) < \sigma(y)$ for all $y \in Y$. That is, σ is an ordering which consists of all vertices of X, followed by all vertices of M, followed by all vertices of Y. We claim that σ is a cocomparability ordering. Since X is a clique, for any $x_i x_j \in E(G)$ and any $x \in X$ such that $\sigma(x_i) < \sigma(x) < \sigma(x_j)$ has an edge to x_i and x_j. We apply the same argument to vertices of cliques M and Y. Thus, any ordering of the vertices of X alone is a cocomparability ordering, any ordering of the vertices of M alone is a cocomparability ordering, and any ordering of the vertices of Y alone is a cocomparability ordering.

We next show that any other possible edges between the sets X, M, Y satisfy the constraints of a cocomparability ordering. Consider any vertices $x \in X$, $m \in M$, and $v \in V(G)$ with $\sigma(x) < \sigma(v) < \sigma(m)$ and $xm \in E(G)$. Then, either $v \in M$ and therefore $vm \in E(G)$, or $v \in X$ and therefore $vx \in E(G)$. By symmetry, any $m \in M$, $v \in V(G)$, and $y \in Y$ with $\sigma(m) < \sigma(v) < \sigma(y)$ and $xm \in E(G)$ satisfies that either $vm \in E(G)$ or $vy \in E(G)$. By construction, all vertices $x \in X$ and all $y \in Y$ satisfy $xy \notin E(G)$. Therefore, σ is a cocomparability ordering. □

Corollary 43. Cocomparability graphs and AT-free graphs can have exponentially large injective hulls. Specifically, there is a graph G of that class with $|V(\mathcal{H}(G))| \in \Omega(a^n)$, where $a > 1$ and $n = |V(G)|$.

Currently, we do not know whether there is a permutation graph G with exponentially large injective hull.

7.6.3 Chordal bipartite graphs

A graph is *chordal bipartite* if it is bipartite and every its induced cycle is of length 4. We construct a special chordal bipartite graph G with $2k$ $(k \geq 3)$ vertices as follows. Let $X = \{x_1, x_2, \ldots, x_k\}$ be an independent set and let $Y = \{y_1, y_2, \ldots, y_k\}$ be an independent set. For each $i, j \in [1, k]$ and $i \neq j$, let $x_i y_j \in E(G)$. See Figure 7.8 for an illustration. Clearly, no two vertices in X are adjacent and no two vertices in Y are adjacent. By construction, G has the following distance properties:

- $\forall i, j \in [1, k],\ i \neq j,\ d_G(x_i, y_j) = 1$;
- $\forall i, j \in [1, k],\ i \neq j,\ d_G(x_i, x_j) = 2$ via common neighbor y_p, $p \neq i, j$;
- $\forall i, j \in [1, k],\ i \neq j,\ d_G(y_i, y_j) = 2$ via common neighbor x_p, $p \neq i, j$;
- $\forall i \in [1, k],\ d_G(x_i, y_i) = 3$ as any x_i is adjacent to only vertices $y_j \in Y$, $j \neq i$.

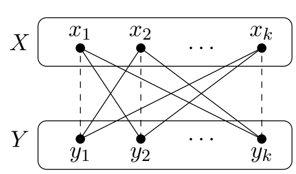


Figure 7.8: G is a chordal bipartite graph that requires exponentially many new Helly vertices. Non-edges are drawn in dashed lines. X and Y are independent sets.

Theorem 32. There is a chordal bipartite graph G such that $|V(\mathcal{H}(G))| \geq 2^{n/2} - 2$, where $n = |V(G)|$.

Proof. Clearly, G as constructed above is chordal bipartite. Next, we show that G has exponentially many unsuspended maximal 2-sets. Observe that there are 2^k maximal 2-sets in G that are suspended or unsuspended; for each $j \in [1, k]$, either $x_j \in S$ or $y_j \in S$, but not both since $d_G(x_j, y_j) = 3$. We claim that any maximal 2-set S that contains at least two vertices from X and at least two vertices from Y is unsuspended. Let $x_i, x_j, y_k, y_\ell \in S$, where i, j, k, ℓ are pairwise distinct integers, $x_i, x_j \in X$, and $y_k, y_\ell \in Y$. By construction, x_i, x_j, y_k, y_ℓ induce a C_4. As X and Y are independent sets, there is no vertex of G that suspends this C_4 and hence S. As there are $2k + 2$ maximal 2-sets which do not contain at least two vertices from X and at least two vertices from Y, by Lemma 77, $|V(\mathcal{H}(G)) \setminus V(G)| \geq 2^k - 2k - 2$. Including the $2k$ vertices of $V(G)$, one obtains $|V(\mathcal{H}(G))| \geq 2^k - 2$, where $k = n/2$. $\square$

Corollary 44. Chordal bipartite graphs can have exponentially large injective hulls. Specifically, there is a graph G of that class with $|V(\mathcal{H}(G))| \in \Omega(a^n)$, where $a > 1$ and $n = |V(G)|$.

Chapter 8

Conclusion

We investigated the δ-hyperbolicity of Helly graphs, motivated by the fact that every graph G can be isometrically embedded into a unique smallest hyperbolicity-preserving Helly graph $\mathcal{H}(G)$, called its injective hull. In Helly graphs, we characterized the δ-hyperbolicity via three forbidden isometric subgraphs of the kings grid and found a tight bound on hyperbolicity with respect to interval thinness. In later chapters, we leveraged the eccentricity terrain to efficiently approximate the diameter, radius, and all vertex eccentricities.

First, we showed that the eccentricity function in distance-hereditary graphs, a subclass of 1-hyperbolic graphs, is almost unimodal. We used this result to fully characterize centers of distance-hereditary graphs and to provide several bounds on the eccentricity of a vertex. We presented a new linear time algorithm to calculate all eccentricities in distance-hereditary graphs.

Second, our results which capture the almost unimodality of the eccentricity function are extended to δ-hyperbolic graphs. We defined β-pseudoconvexity, which implies Gromov's ϵ-quasiconvexity, but additionally satisfies an important axiom that pseudoconvexity is closed under intersection. Additional bounds on the eccentricity of a vertex yielded a linear ($O(|E|)$) time algorithm to approximate all eccentricities within $O(\delta)$ additive error and almost linear ($O(\delta|E|)$) time algorithms that more closely approximate all eccentricities in hyperbolic graphs.

Next, we found that the eccentricity function in a graph G is tightly tied to the eccentricity function in its injective hull. We related the diameter, radius, center, and all eccentricities in G to their counterparts in $\mathcal{H}(G)$. We introduced a new metric parameter for a graph, the Helly-gap $\alpha(G)$. We showed that the Helly-gap can also be characterized by the minimum distance from a Helly vertex to a real vertex. We found many graph classes have a small Helly-gap, including distance-hereditary graphs ($\alpha(G) \leq 1$), chordal graphs ($\alpha(G) \leq 1$), δ-hyperbolic graphs ($\alpha(G) \leq 2\delta$), as well as graphs of bounded tree-length ($\alpha(G) \leq tl(G)$) and graphs with an α_i metric ($\alpha(G) \leq \lceil i/2 \rceil$). Moreover, we use the Helly-gap as a tool to generalize many known eccentricity related results to a much larger family of graphs, α-weakly-Helly graphs.

Finally, we proved that chordal graphs, square chordal graphs, and distance-hereditary graphs are closed under Hellification; permutation graphs are not. We provided a polynomial time algorithm to compute $\mathcal{H}(G)$ when G is distance-hereditary. Additional graph classes are identified for which $\mathcal{H}(G)$ is impossible to compute in subexponential time, including split graphs, cocomparability graphs, AT-free graphs, chordal bipartite graphs, and graphs with a constant bound on any of the following parameters: diameter, hyperbolicity, tree-length, tree-breadth, or chordality.

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