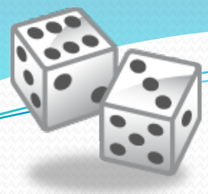
An Introduction to Discrete Probability

Section 7.1

Section Summary

- Finite Probability
- Probabilities of Complements and Unions of Events
- Probabilistic Reasoning



Probability of an Event

Pierre-Simon Laplace (1749-1827)

- An *experiment* is a procedure that yields one of a given set of possible outcomes.
- The *sample space* (S) of the experiment is the set of possible outcomes.
- An *event* (E) is a subset of the sample space.

Laplace's Definition: If S is a finite sample space of equally likely outcomes, and E is an event (a subset of S), then the *probability* of E is

$$p(E) = \frac{|E|}{|S|}$$

- For every event E, $0 \leq p(E) \leq 1$.

Applying Laplace's Definition

Ex: An urn contains four blue balls and five red balls. What is the probability that a ball chosen from the urn is blue?

Solution: 4/9 (9 possible outcomes, and 4 of these produce a blue ball)

Ex: What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

Solution: $6/36 = 1/6$ (by the product rule there are $6^2 = 36$ possible outcomes, and 6 of these outcomes have a sum of 7)

Applying Laplace's Definition

Ex: In a lottery, a player wins a large prize when they pick four digits that match, in correct order, four digits selected by a random mechanical process (where the same digit could be picked more than once). What is the probability that a player wins the prize?

Solution: 1/10,000 = 0.0001

- There is only 1 way to pick the correct digits
- By the product rule there are 10^4= 10,000 ways to pick four digits.

Applying Laplace’s Definition

Ex: (*continued*) A smaller prize is won if only 3 digits are matched. What is the probability that a player wins the small prize?

Solution: 36/10,000 = 9/2500 = 0.0036

- If exactly 3 digits are matched, one of the four digits must be incorrect and the other 3 digits must be correct. For the digit that is incorrect, there are 9 possible choices. Hence, by the sum rule, there a total of 36 possible ways to choose four digits that match exactly 3 of the winning four digits.
- By the product rule there are 10^4= 10,000 ways to pick four digits.

08 27 34 04 19 10

Applying Laplace’s Definition

Ex: There are many lotteries that award prizes to people who correctly choose a set of six numbers out of the first n positive integers, where n is usually between 30 and 60. What is the probability that a person picks the correct six numbers out of 40 (numbers range from [1,40])?

Solution: 1/ 3,838,380 ≈ 0.00000026

- The number of ways to choose six numbers out of 40 is

$$C(40,6) = 40!/(34!6!) = 3{,}838{,}380.$$

Can you work out the probability of winning the lottery with the biggest prize where you live?

Applying Laplace’s Definition

Ex: What is the probability that the numbers 11, 4, 17, 39, and 23 are drawn in that order from a bin with 50 balls labeled with the numbers 1,2, …, 50 if

a) The ball selected is not returned to the bin.

b) The ball selected is returned to the bin before the next ball is selected.

Solution: Use the product rule in each case.

a) *Sampling without replacement*: The probability is 1/254,251,200 since there are $50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 = 254{,}251{,}200$ ways to choose the five balls.

b) *Sampling with replacement*: The probability is $1/50^5 = 1/312{,}500{,}000$ since $50^5 = 312{,}500{,}000$.

The Probability of Complements and Unions of Events

Theorem 1: Let E be an event in sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E).$$

Proof: Using the fact that $|\overline{E}| = |S| - |E|$,

$$p(\overline{E}) = \frac{|S| - |E|}{|S|} = 1 - \frac{|E|}{|S|} = 1 - p(E).$$ ◀

The Probability of Complements and Unions of Events

Ex: Throw two dice. What's the probability that the two scores are **different**?

Solution:

- E = the event that two scores are different
- $\overline{E}$ = the event that two scores are the same
- |S| = 36 ways for two dice to land

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{6}{36} = 5/6$$

The Probability of Complements and Unions of Events

Ex: A sequence of 10 bits is chosen randomly. What is the probability that at least one of these bits is 0?

Solution: Let E be the event that at least one of the 10 bits is 0. Then $\overline{E}$ is the event that all of the bits are 1s. The size of the sample space S is 2^{10}. Hence,

$$p(E) = 1 - p(\overline{E}) = 1 - \frac{|\overline{E}|}{|S|} = 1 - \frac{1}{2^{10}} = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

The Probability of Complements and Unions of Events

Theorem 2: Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$

Proof: Given the *inclusion-exclusion* formula

$|A \cup B| = |A| + |B| - |A \cap B|$, it follows that

$$p(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1| + |E_2| - |E_1 \cap E_2|}{|S|}$$

$$= \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|} - \frac{|E_1 \cap E_2|}{|S|}$$

$$= p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

◀

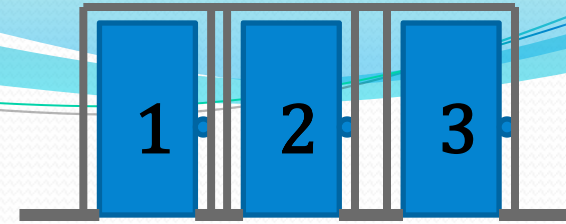
The Probability of Complements and Unions of Events

Ex: What is the probability that a randomly selected positive integer between 1 and 100 is divisible by either 2 **or** 5?

Solution:

- Let E_1 be the event that the integer is divisible by 2
- Let E_2 be the event that it is divisible 5
- Then, $E_1 \cup E_2$ is the event that the integer is divisible by 2 or 5
- And, $E_1 \cap E_2$ is the event that it is divisible by 2 and 5.

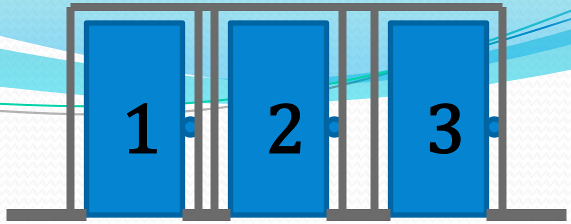
$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$
$$= 50/100 + 20/100 - 10/100 = 3/5.$$



Monty Hall Puzzle

Ex: You are asked to select one of three doors to open. Behind one of the doors is a car; behind the others, goats. If you select the door with a car, you win the car. After you select a door, the game show host opens one of the other doors (which he knows is not the winning door). The prize is not behind the door and he gives you the opportunity to switch your selection. Should you switch?

(*This is a notoriously confusing problem that has been the subject of much discussion. Do a web search to see why!)*



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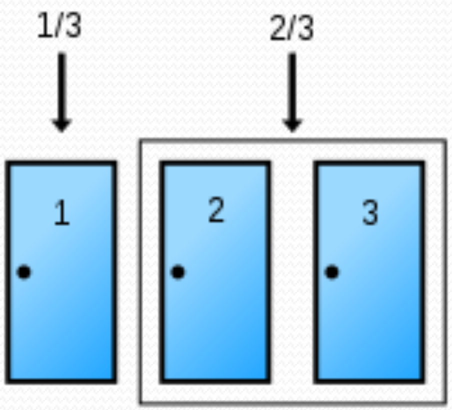
Solution: You should switch. The probability that your initial pick is correct is 1/3. This is the same whether or not you switch doors. But since the game show host always opens a door that does not have the prize, if you switch the probability of winning will be 2/3, because you win if your initial pick was not the correct door and the probability your initial pick was wrong is 2/3.

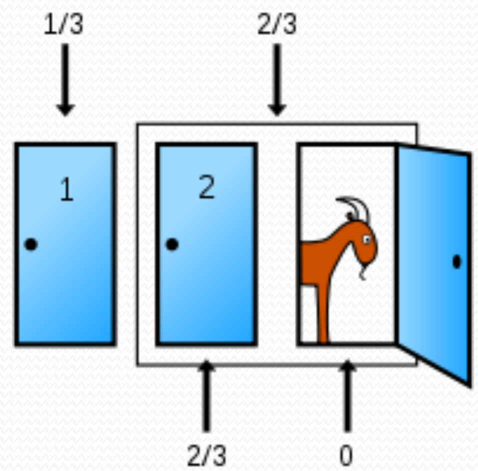
Monty Hall Puzzle

Before the host picks:



Car has ⅓ chance of being behind the player's pick and a ⅔ chance of being behind one of the other two doors.

After the host pick a door with a goat:



The odds for the two sets don't change, but the odds move to 0 for the open door and ⅔ for the closed door.