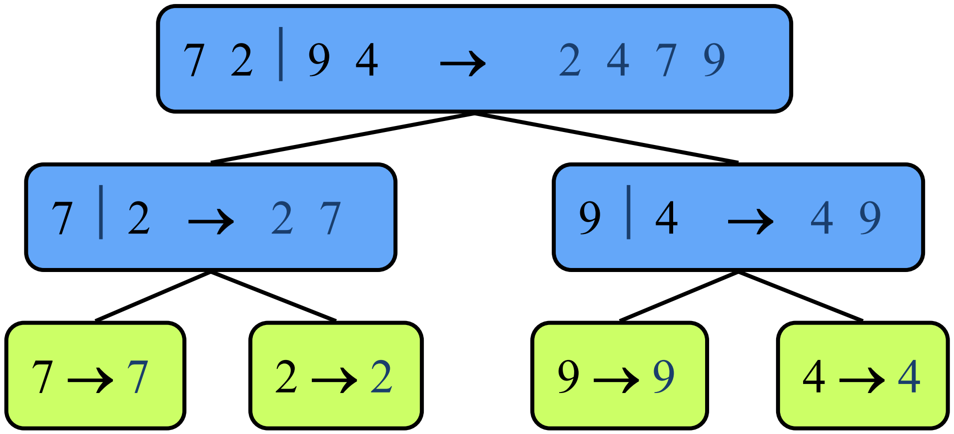
Divide and Conquer

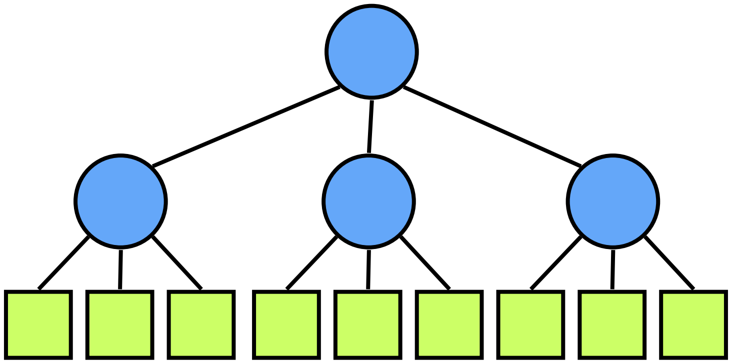


Outline / Reading

- Divide-and-conquer paradigm (5.2)
- Review Merge-sort (4.1.1)
- Recurrence Equations (5.2.1)
  - Recursion trees
  - Induction
    - Iterative substitution
    - Guess-and-test
  - The master method
- Integer Multiplication (5.2.2)

Divide-and-Conquer

- Divide-and conquer is a general algorithm design paradigm:
  - Divide: divide the input data in two or more disjoint subsets $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots$
  - Recur: solve the subproblems recursively
  - Conquer: combine the solutions for $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots,$ into a solution for $\boldsymbol{S}$
- The base case for the recursion are subproblems of constant size
- Analysis can be done using recurrence equations



Merge Sort Review

Merge-sort on an input sequence $\boldsymbol{S}$ with $\boldsymbol{n}$ elements consists of three steps:

- Divide: partition $\boldsymbol{S}$ into two sequences $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ of about $\boldsymbol{n}/2$ elements each
- Recur: recursively sort $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$
- Conquer: merge $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ into a unique sorted sequence

Algorithm ***mergeSort*(*S*, *C*)**
    Input sequence $\boldsymbol{S}$ with $\boldsymbol{n}$ elements, comparator $\boldsymbol{C}$
    Output sequence $\boldsymbol{S}$ sorted according to $\boldsymbol{C}$
    if $S.size() > 1$
        $(S_1, S_2) \leftarrow partition(S, n/2)$
        $mergeSort(S_1, C)$
        $mergeSort(S_2, C)$
        $S \leftarrow merge(S_1, S_2)$

Recurrence Equation Analysis

- The conquer step of merge-sort consists of merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes at most bn steps, for some constant b.
- Likewise, the basis case (n < 2) will take at most b steps.
- Therefore, if we let $T(n)$ denote the running time of merge-sort:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

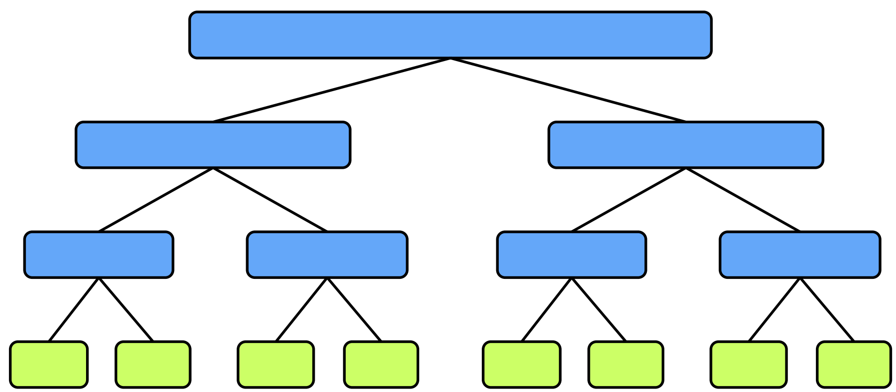
- We can analyze the running time of merge-sort by finding a closed form solution to the above equation.
  - That is, a solution that has $T(n)$ only on the left-hand side.

Recursion Tree

Draw the recursion tree for the recurrence relation and look for a pattern:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn & \text{if } n \geq 2 \end{cases}$$

depth	T's	size	time
0	1	$\boldsymbol{n}$	$\boldsymbol{bn}$
1	2	$\boldsymbol{n}/2$	$\boldsymbol{bn}$
$\boldsymbol{i}$	2^i	$\boldsymbol{n}/2^i$	$\boldsymbol{bn}$
…	…	…	…



Total time = $\boldsymbol{bn} + \boldsymbol{bn} \log \boldsymbol{n}$

(last level plus all previous levels)

Iterative Substitution

In the iterative substitution, or "plug-and-chug," technique, we iteratively apply the recurrence equation to itself and see if we can find a pattern, then prove it is true by induction:

$$
\begin{aligned}
T(n) &= 2T(n/2) + bn \\
&= 2(2T(n/2^2)) + b(n/2)) + bn \\
&= 2^2 T(n/2^2) + 2bn \\
&= 2^3 T(n/2^3) + 3bn \\
&= 2^4 T(n/2^4) + 4bn \\
&= \ldots \\
&= 2^i T(n/2^i) + ibn
\end{aligned}
$$

- Note that the base case, $T(n) = b$, case occurs when $2^i = n$. That is, $i = \log n$. So we have: $T(n) = bn + bn\log n$
- Once we prove this by induction, then $T(n)$ is $O(n\log n)$.

Guess-and-Test Method

In the guess-and-test method, we guess a closed form solution and then try to prove it is true by induction:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn\log n & \text{if } n \geq 2 \end{cases}$$

- Guess #1: $T(n) \leq cn\log n$.

$$\begin{aligned} T(n) &= 2T(n/2) + bn\log n \\ &\leq 2(c(n/2)\log(n/2)) + bn\log n \\ &= cn(\log n - \log 2) + bn\log n \\ &= cn\log n - cn + bn\log n \end{aligned}$$

- Wrong: we cannot make this last line be less than $cn\log n$

Guess-and-Test Method (2)

Recall the recurrence equation:

$$T(n) = \begin{cases} b & \text{if } n < 2 \\ 2T(n/2) + bn\log n & \text{if } n \geq 2 \end{cases}$$

- <u>Guess #2</u>: T(n) $\leq$ $cn\log^2 n$.

$$\begin{aligned} T(n) &= 2T(n/2) + bn\log n \\ &\leq 2(c(n/2)\log^2(n/2)) + bn\log n \\ &= cn(\log n - \log 2)^2 + bn\log n \\ &= cn\log^2 n - 2cn\log n + cn + bn\log n \\ &\leq cn\log^2 n \qquad \text{if } c > b. \end{aligned}$$

- So, T(n) is $O(n\log^2 n)$.

In general, to use this method, you need to have a good guess.

Master Method

Many divide-and-conquer recurrence equations have the form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,

   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Master Method: Ex.1

The form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a}\log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a}\log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = 4T(n/2) + n$$

Solution: $\log_b a=2$, so case 1 says T(n) is $\Theta(n^2)$.

Master Method: Ex. 2

The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = 2T(n/2) + n \log n$$

Solution: $\log_b a=1$, so case 2 says T(n) is $\Theta(n \log^2 n)$.

Master Method: Ex. 3

The form: $T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a}\log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a}\log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = T(n/3) + n\log n$$

Solution: $\log_b a=0$, so case 3 says T(n) is Θ(n log n).

Master Method: Ex. 4

The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = 8T(n/2) + n^2$$

Solution: $\log_b a=3$, so case 1 says T(n) is $\Theta(n^3)$.

Master Method: Ex. 5

The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \ge d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a - \varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a} \log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a} \log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a + \varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
   provided $af(n/b) \le \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = 9T(n/3) + n^3$$

Solution: $\log_b a=2$, so case 3 says T(n) is $\Theta(n^3)$.

Master Method: Ex. 6

The form:

$$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

1. if $f(n)$ is $O(n^{\log_b a-\varepsilon})$, then $T(n)$ is $\Theta(n^{\log_b a})$
2. if $f(n)$ is $\Theta(n^{\log_b a}\log^k n)$, then $T(n)$ is $\Theta(n^{\log_b a}\log^{k+1} n)$
3. if $f(n)$ is $\Omega(n^{\log_b a+\varepsilon})$, then $T(n)$ is $\Theta(f(n))$,
   provided $af(n/b) \leq \delta f(n)$ for some $\delta < 1$.

Example:

$$T(n) = T(n/2) + 1 \qquad \text{(binary search)}$$

Solution: $\log_b a=0$, so case 2 says T(n) is Θ(log n).

Master Method: Ex. 7

The form: $$T(n) = \begin{cases} c & \text{if } n < d \\ aT(n/b) + f(n) & \text{if } n \geq d \end{cases}$$

The Master Theorem:

$$1.\ \text{if } f(n) \text{ is } O(n^{\log_b a - \varepsilon}), \text{then } T(n) \text{ is } \Theta(n^{\log_b a})$$

$$2.\ \text{if } f(n) \text{ is } \Theta(n^{\log_b a} \log^k n), \text{then } T(n) \text{ is } \Theta(n^{\log_b a} \log^{k+1} n)$$

$$3.\ \text{if } f(n) \text{ is } \Omega(n^{\log_b a + \varepsilon}), \text{then } T(n) \text{ is } \Theta(f(n)),$$

$$\text{provided } af(n/b) \leq \delta f(n) \text{ for some } \delta < 1.$$

Example:

$$T(n) = 2T(n/2) + \log n$$ (heap construction)

Solution: $\log_b a=1$, so case 1 says T(n) is Θ(n).

Iterative Justification of the Master Theorem

Use iterative substitution to find a pattern:

$$
\begin{aligned}
T(n) &= aT(n/b) + f(n) \\
&= a(aT(n/b^2)) + f(n/b)) + f(n) \\
&= a^2T(n/b^2) + af(n/b) + f(n) \\
&= a^3T(n/b^3) + a^2 f(n/b^2) + af(n/b) + f(n) \\
&= \ldots \\
&= a^{\log_b n}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i) \\
&= n^{\log_b a}T(1) + \sum_{i=0}^{(\log_b n)-1} a^i f(n/b^i)
\end{aligned}
$$

We then distinguish the three cases as

- Case 1: The first term is dominant
- Case 2: Each part of the summation is equally dominant
- Case 3: The second term is dominant

Integer Multiplication

Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

$$J = J_h 2^{n/2} + J_l$$

- We can then define I*J by multiplying the parts and adding:

$$\begin{aligned} I * J &= (I_h 2^{n/2} + I_l) * (J_h 2^{n/2} + J_l) \\ &= I_h J_h 2^n + I_h J_l 2^{n/2} + I_l J_h 2^{n/2} + I_l J_l \end{aligned}$$

- So, $T(n) = 4T(n/2) + n$, which implies $T(n)$ is $\Theta(n^2)$.
- But that is no better than the algorithm we learned in grade school.

Improved Integer Multiplication

Algorithm: Multiply two n-bit integers I and J.

- Divide step: Split I and J into high-order and low-order bits

$$I = I_h 2^{n/2} + I_l$$

$$J = J_h 2^{n/2} + J_l$$

- Observe that there is a different way to multiply parts:

$$\begin{aligned} I * J &= I_h J_h 2^n + [(I_h - I_l)(J_l - J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + [(I_h J_l - I_l J_l - I_h J_h + I_l J_h) + I_h J_h + I_l J_l] 2^{n/2} + I_l J_l \\ &= I_h J_h 2^n + (I_h J_l + I_l J_h) 2^{n/2} + I_l J_l \end{aligned}$$

- So, $T(n) = 3T(n/2) + n$, which implies $T(n)$ is $\Theta(n^{\log_2 3})$.
- Thus, $T(n)$ is $O(n^{1.585})$.