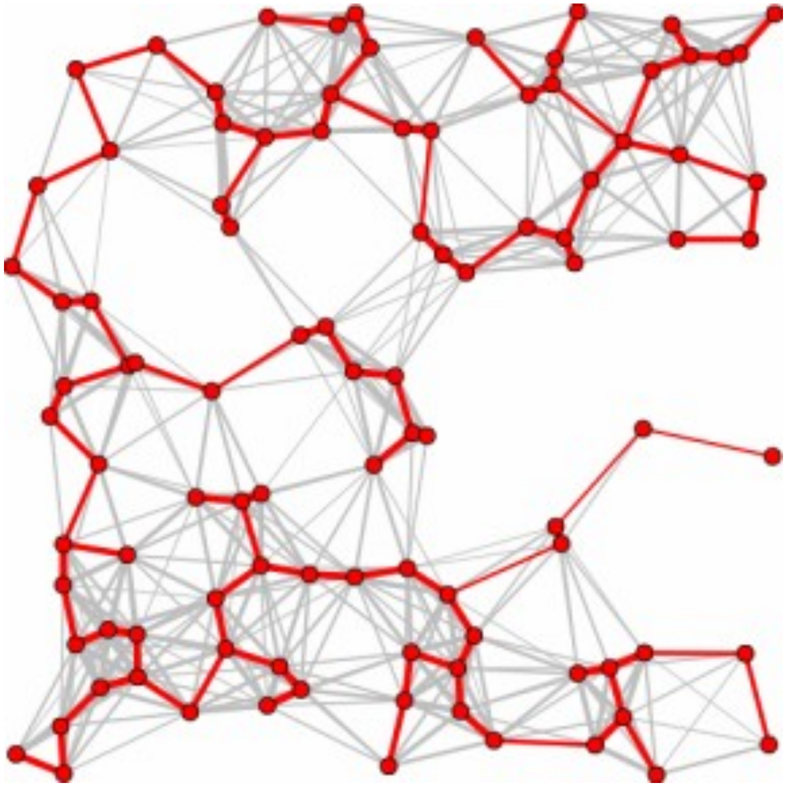
Minimum Spanning Trees



Outline and Reading

- Minimum Spanning Trees (7.3)
  - Definitions
  - A crucial fact
- The Prim-Jarnik Algorithm (7.3.2)
- Kruskal's Algorithm (7.3.1)
- Baruvka's Algorithm (7.3.3)

Minimum Spanning Tree

Spanning subgraph

- Subgraph of a graph ***G*** containing all the vertices of ***G***

Spanning tree

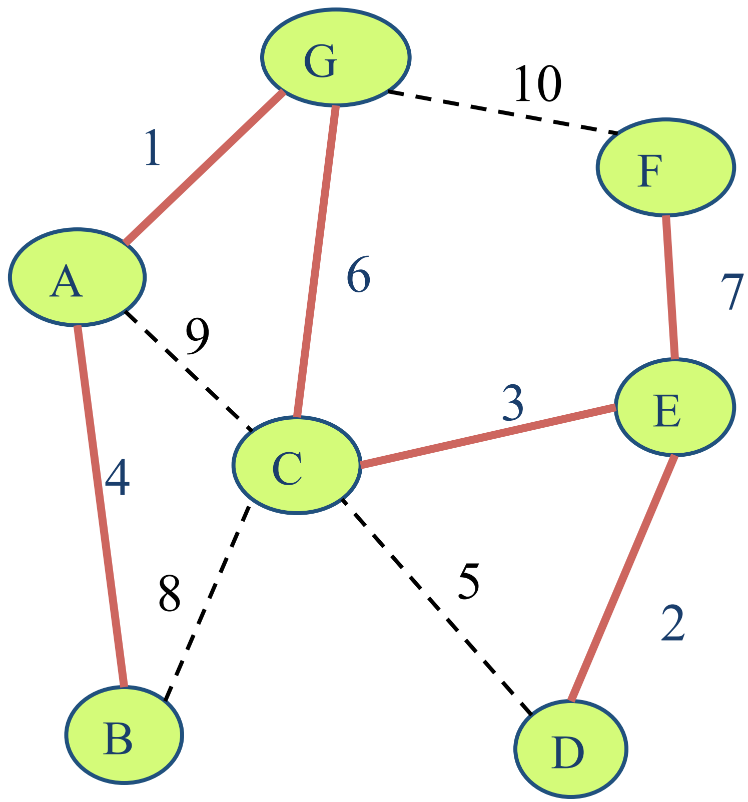
- Spanning subgraph that is itself a (free) tree

Minimum spanning tree (MST)

- Spanning tree of a weighted graph with minimum total edge weight

Applications

- Communications networks
- Transportation networks



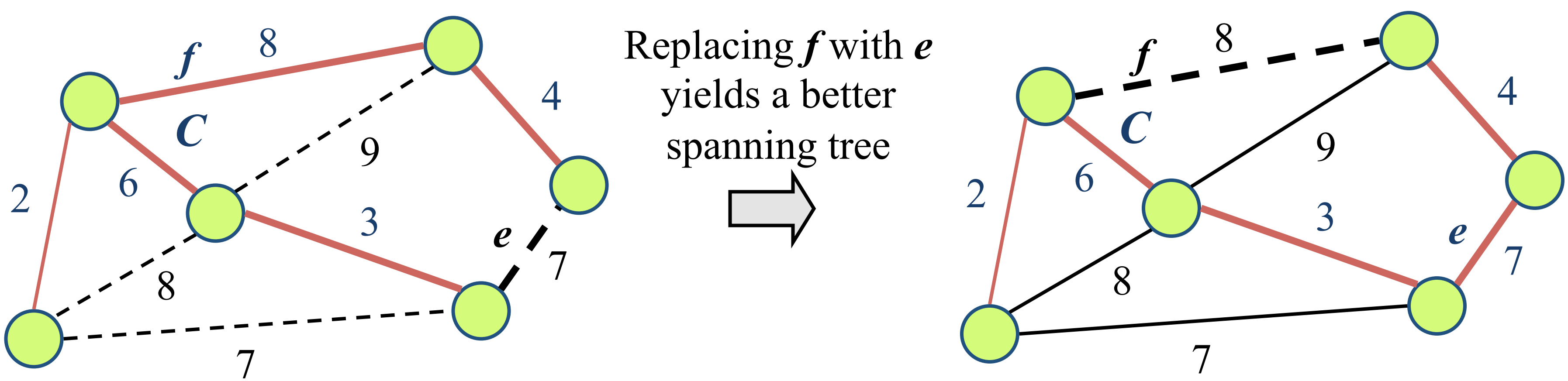
Cycle Property

Cycle Property:

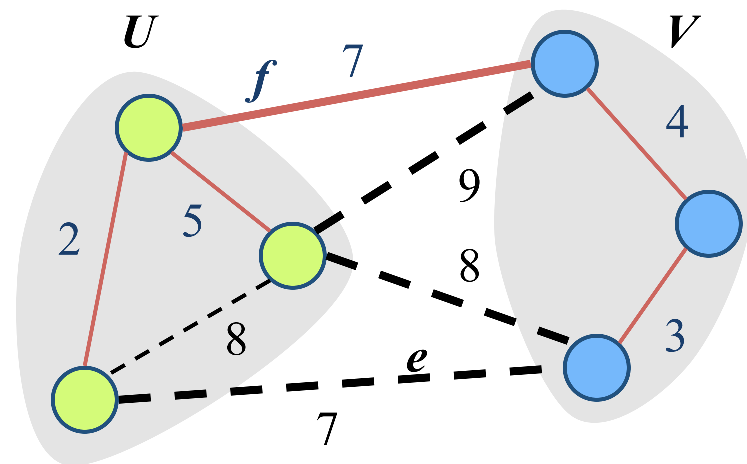
- Let ***T*** be a minimum spanning tree of a weighted graph ***G***
- Let ***e*** be an edge of ***G*** that is not in ***T*** and ***C*** let be the cycle formed by ***e*** with ***T***
- For every edge ***f*** of ***C***, ***weight*(*f*) ≤ *weight*(*e*)**

Proof:

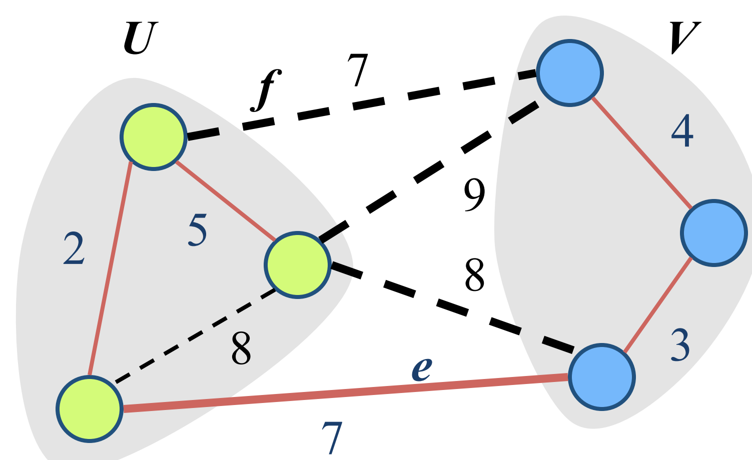
- By contradiction
- If ***weight*(*f*) > *weight*(*e*)** we can get a spanning tree of smaller weight by replacing ***e*** with ***f***



Partition Property



Replacing ***f*** with ***e*** yields another MST



Partition Property:

- Consider a partition of the vertices of ***G*** into subsets ***U*** and ***V***
- Let ***e*** be an edge of minimum weight across the partition
- There is a minimum spanning tree of ***G*** containing edge ***e***

Proof:

- Let ***T*** be an MST of ***G***
- If ***T*** does not contain ***e*,** consider the cycle ***C*** formed by ***e*** with ***T*** and let ***f*** be an edge of ***C*** across the partition
- By the cycle property, ***weight*(*f*) ≤ *weight*(*e*)**
- Thus, ***weight*(*f*) = *weight*(*e*)**
- We obtain another MST by replacing ***f*** with ***e***

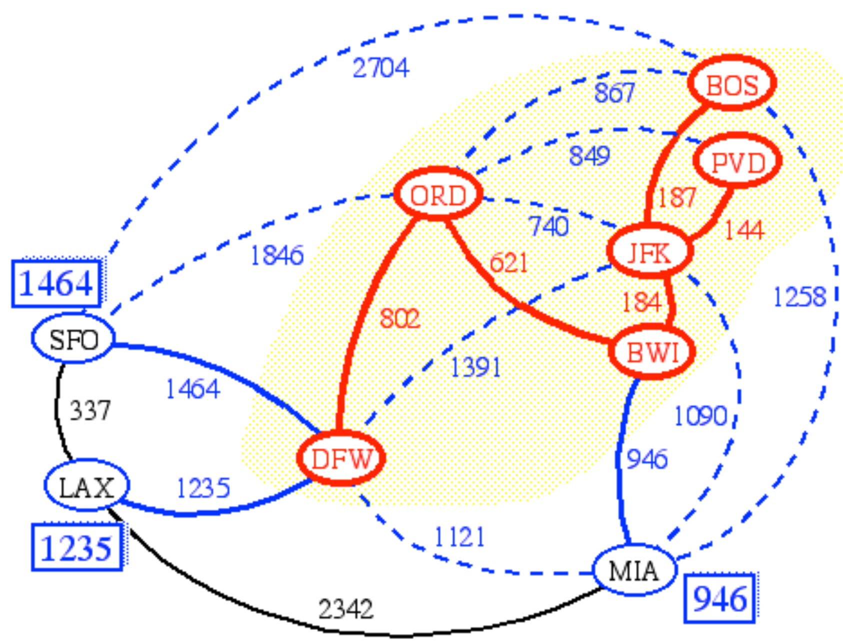
Prim-Jarnik's Algorithm

Idea:

- Similar to Dijkstra's algorithm (for a connected graph)
- Pick an arbitrary vertex ***s*** and we grow the MST as a cloud of vertices, starting from ***s***
- Store with each vertex ***v*** a label ***d***(***v***) = the smallest weight of an edge connecting ***v*** to a vertex in the cloud

At each step:

- Add to the cloud the vertex ***u*** outside the cloud with the smallest label
- Update the labels of the vertices adjacent to ***u***



Prim-Jarnik's Algorithm (cont.)

A priority queue stores the vertices outside the cloud

- Key: distance
- Element: vertex

Locator-based methods

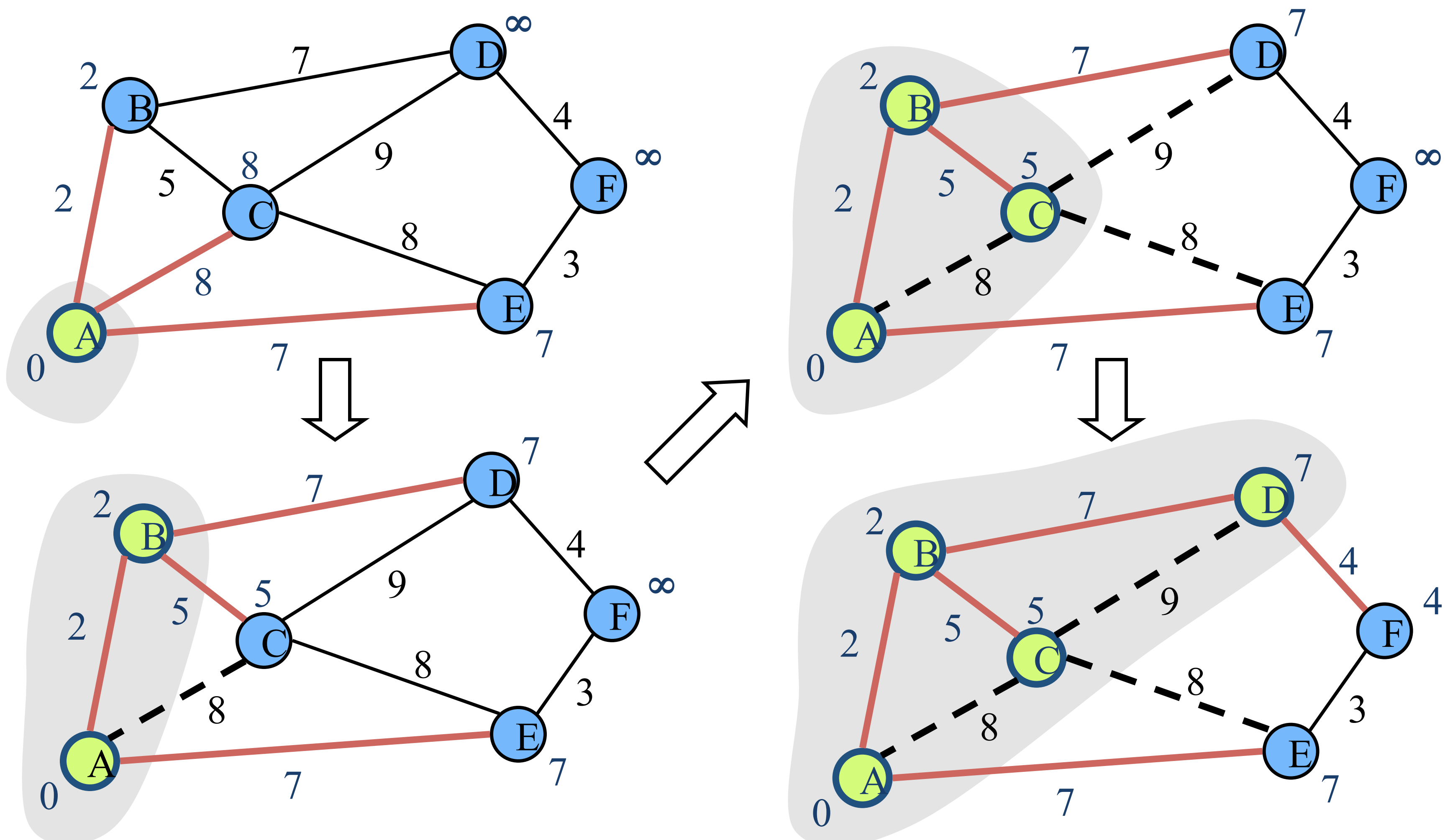
- *insert*(*k*,*e*) returns a locator
- *replaceKey*(*l*,*k*) changes the key of an item

We store three labels with each vertex:

- Distance
- Parent edge in MST
- Locator in priority queue

```
Algorithm PrimJarnikMST(G)
  Q ← new heap-based priority queue
  s ← a vertex of G
  for all v ∈ G.vertices()
    if v = s
      setDistance(v, 0)
    else
      setDistance(v, ∞)
    setParent(v, ∅)
    l ← Q.insert(getDistance(v), v)
    setLocator(v,l)
  while ¬Q.isEmpty()
    u ← Q.removeMin()
    for all e ∈ G.incidentEdges(u)
      z ← G.opposite(u,e)
      r ← weight(e)
      if r < getDistance(z)
        setDistance(z,r)
        setParent(z,e)
        Q.replaceKey(getLocator(z),r)
```

Example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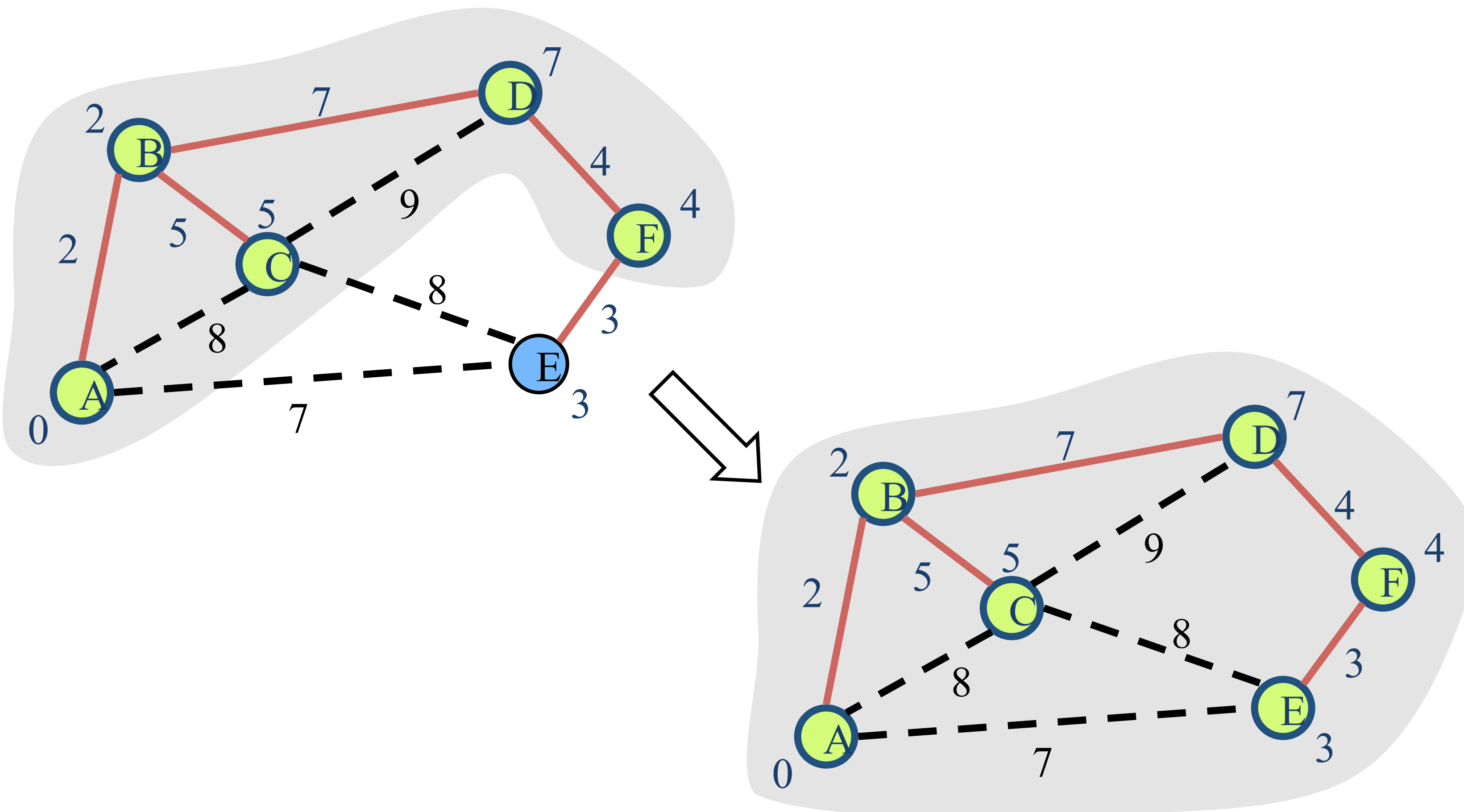
Example (contd.)



Analysis

- Graph operations
  - Method incidentEdges is called once for each vertex

- Label operations
  - We set/get the distance, parent and locator labels of vertex z $O(\deg(z))$ times
  - Setting/getting a label takes $O(1)$ time

- Priority queue operations
  - Each vertex is inserted once into and removed once from the priority queue, where each insertion or removal takes $O(\log n)$ time
  - The key of a vertex w in the priority queue is modified at most $\deg(w)$ times, where each key change takes $O(\log n)$ time

- Prim-Jarnik's algorithm runs in $O((n + m) \log n)$ time provided the graph is represented by the adjacency list structure
  - Recall that $\Sigma_v \deg(v) = 2m$

- The running time is $O(m \log n)$ since the graph is connected

Kruskal's Algorithm

A priority queue stores the edges outside the cloud

- Key: weight
- Element: edge

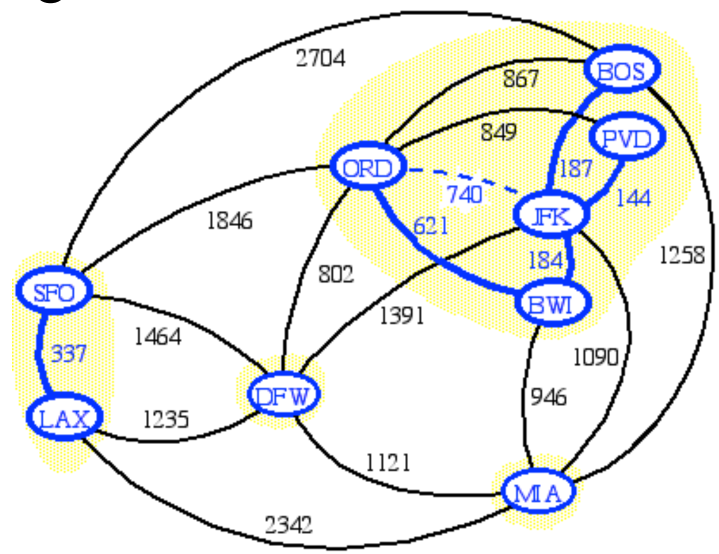
At the end of the algorithm

- We are left with one cloud that encompasses the MST
- A tree *T* which is our MST

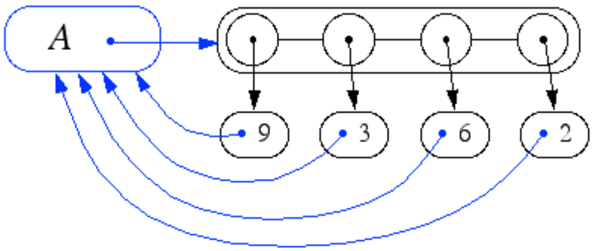
Algorithm ***KruskalMST*(*G*)**
    for each vertex *V* in ***G*** **do**
        define a ***Cloud(v)*** of $\leftarrow \{v\}$
    let ***Q*** be a priority queue
    Insert all edges into ***Q*** using their weights as the key
    $T \leftarrow \emptyset$
    while *T* has fewer than ***n***-1 edges **do**
        edge ***e*** = ***T.removeMin()***
        Let ***u*, *v*** be the endpoints of *e*
        { check if edge is necessary to connect two clouds }
        if ***Cloud(v)*** ≠ ***Cloud(u)*** **then**
            Add edge ***e*** to ***T***
            Merge ***Cloud(v)*** and ***Cloud(u)***
    return ***T***

Data Structure for Kruskal Algortihm

- The algorithm maintains a forest of trees
- An edge is accepted it if connects distinct trees
- We need a data structure that maintains a **partition**, i.e., a collection of disjoint sets, with the operations:
  - **find**(u): return the set storing *u*
  - **union**(u,v): replace the sets storing *u* and *v* with their union



Representation of a Partition



- Each set is stored in a sequence

- Each element has a reference back to the set
  - operation find(u) takes O(1) time, and returns the set of which u is a member.
  - in operation union(u,v), we move the elements of the smaller set to the sequence of the larger set and update their references
  - the time for operation union(u,v) is min(n_u, n_v), where n_u and n_v are the sizes of the sets storing u and v

- Whenever an element is processed, it goes into a set of size at least double, hence each element is processed at most log$n$ times

Partition-Based Implementation

A partition-based version of Kruskal's Algorithm performs cloud merges as unions and tests as finds.

Algorithm **Kruskal**(***G***):

  Input: A weighted graph ***G***.

  Output: An MST ***T*** for ***G***.

Let ***P*** be a partition of the vertices of ***G***, where each vertex forms a separate set.

Let ***Q*** be a priority queue storing the edges of ***G***, sorted by their weights

Let ***T*** be an initially-empty tree

while ***Q*** is not empty **do**

  (***u***,***v***) ← ***Q***.removeMinElement()

  if ***P***.find(***u***) != ***P***.find(***v***) **then**

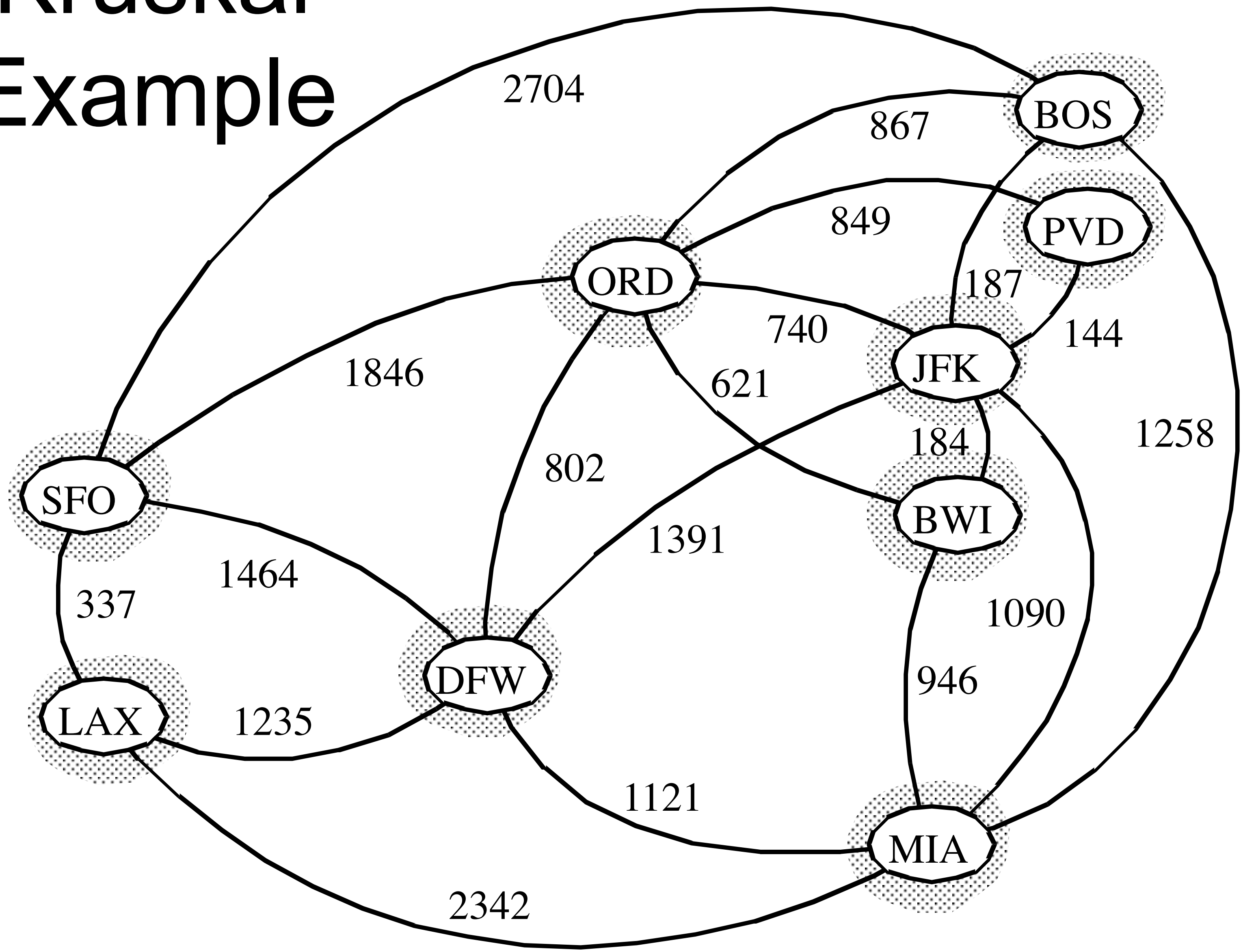
    Add (***u***,***v***) to ***T***

    P.union(***u***,***v***)

return ***T***

Running time: $O((n+m)\log n)$

Kruskal Example



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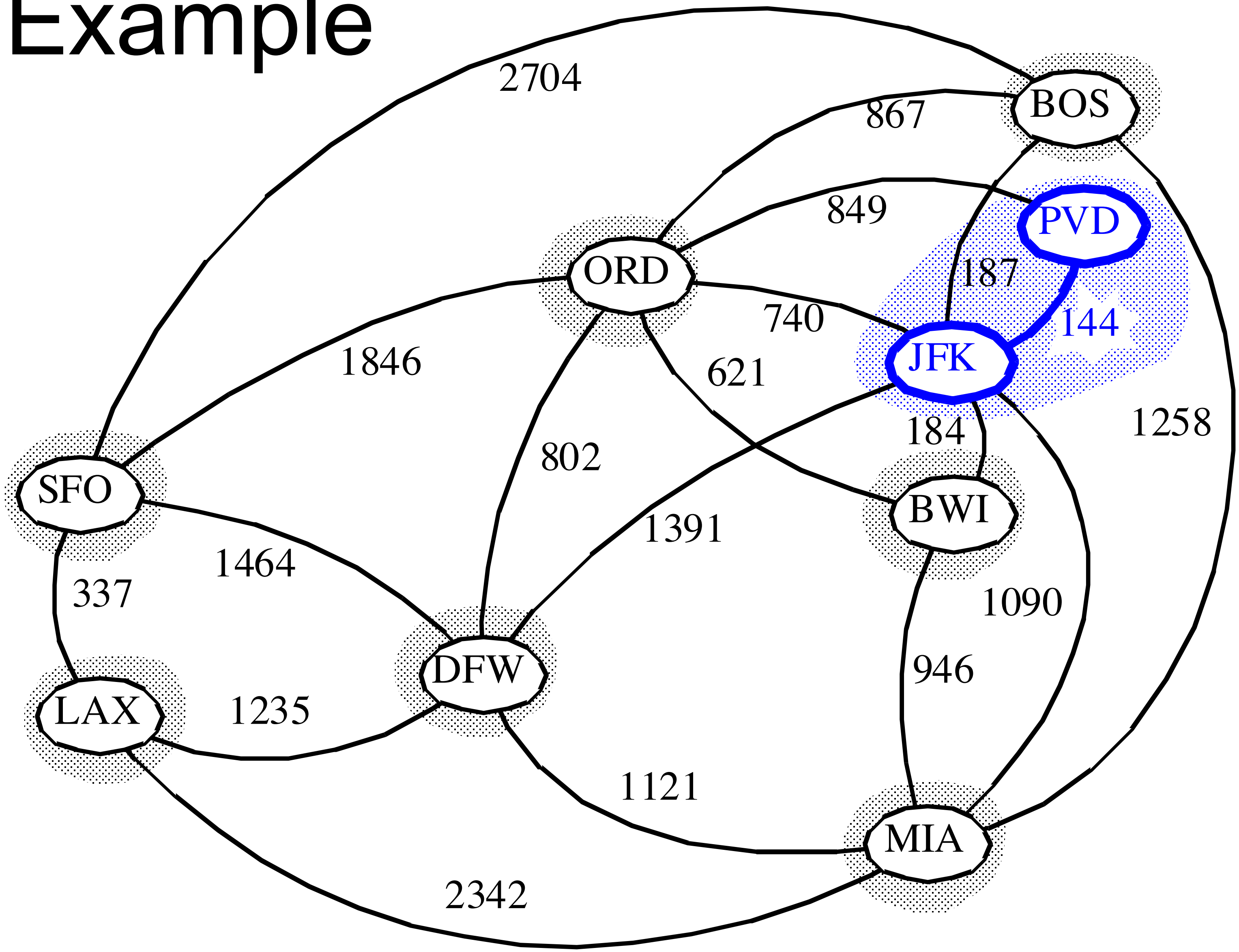
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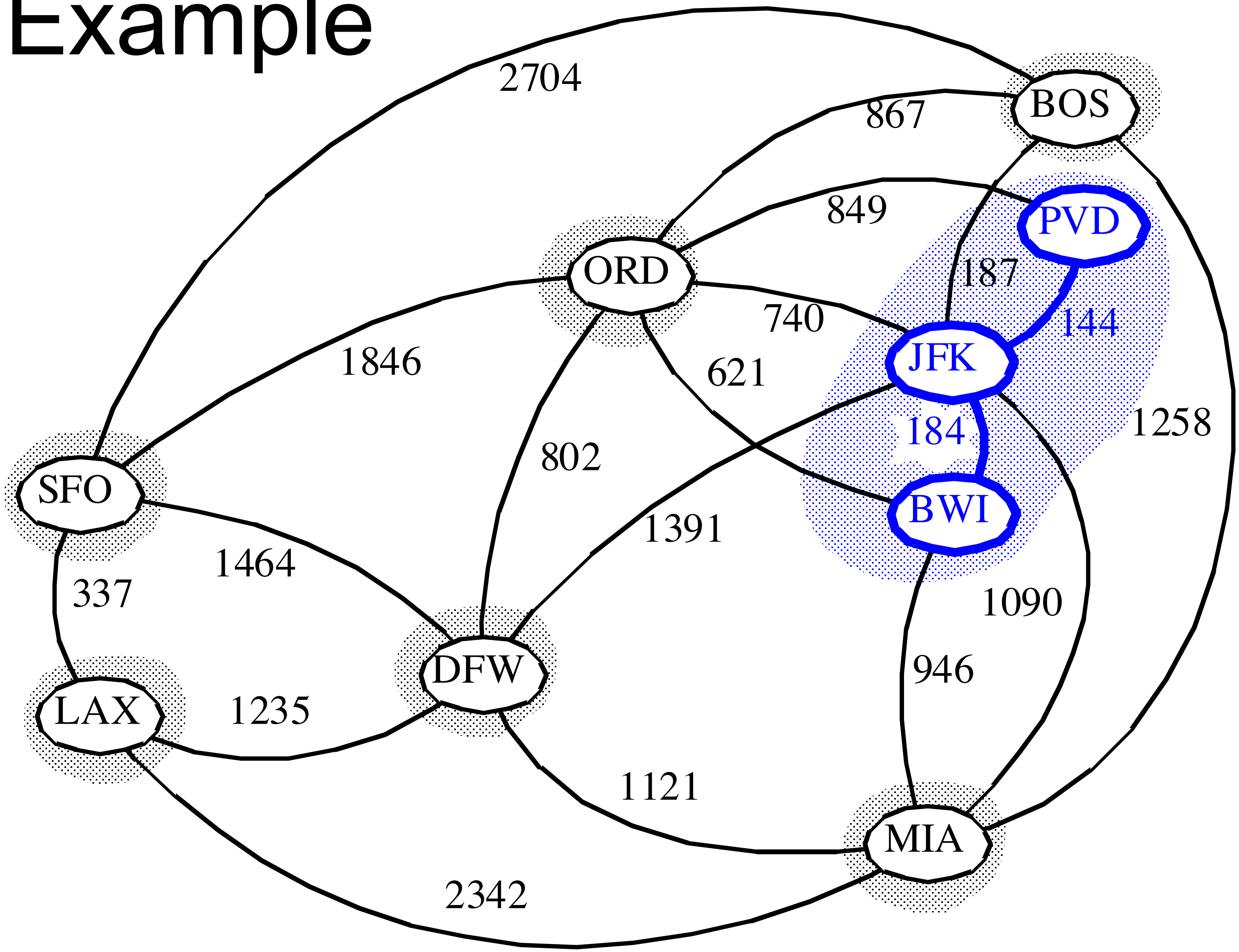

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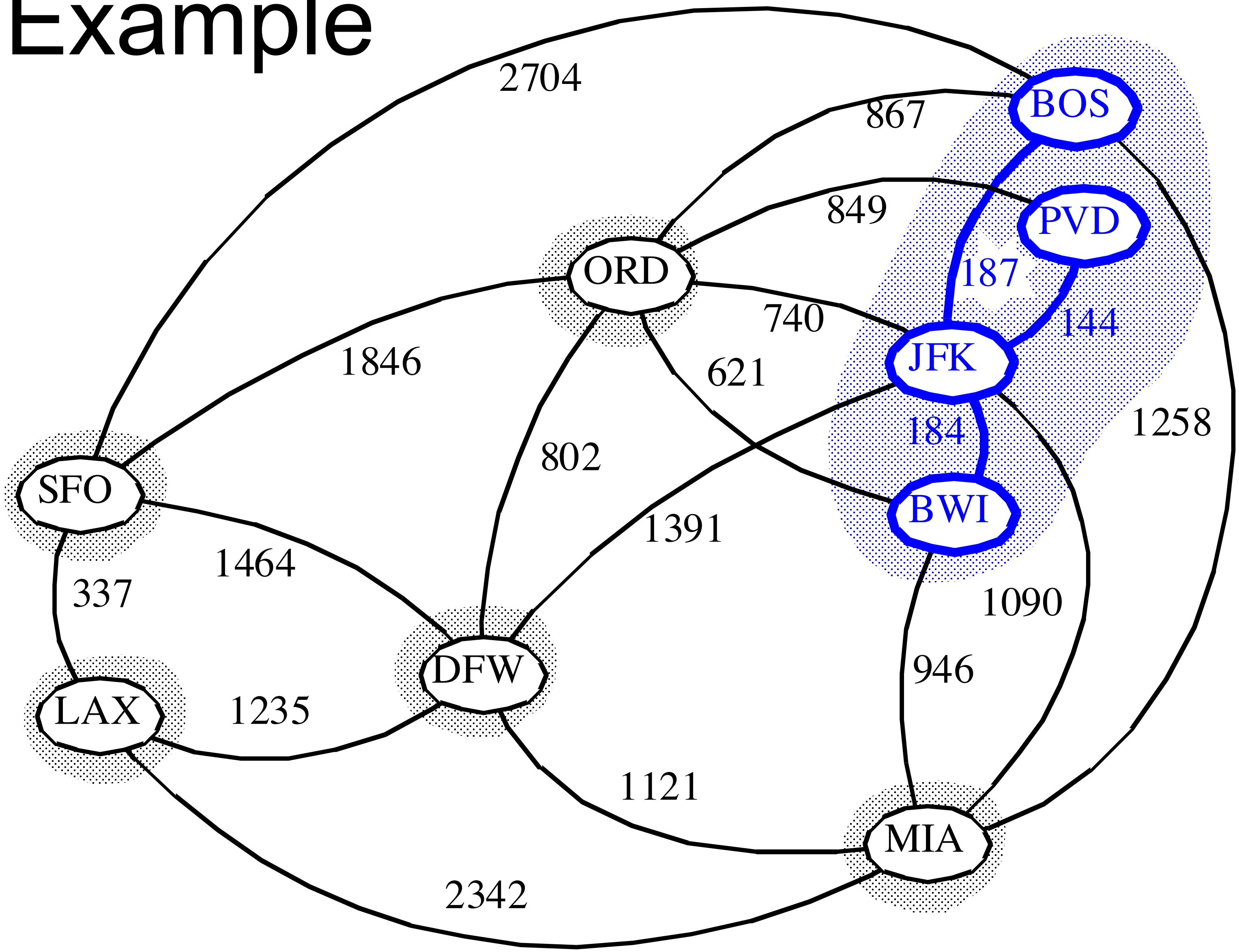

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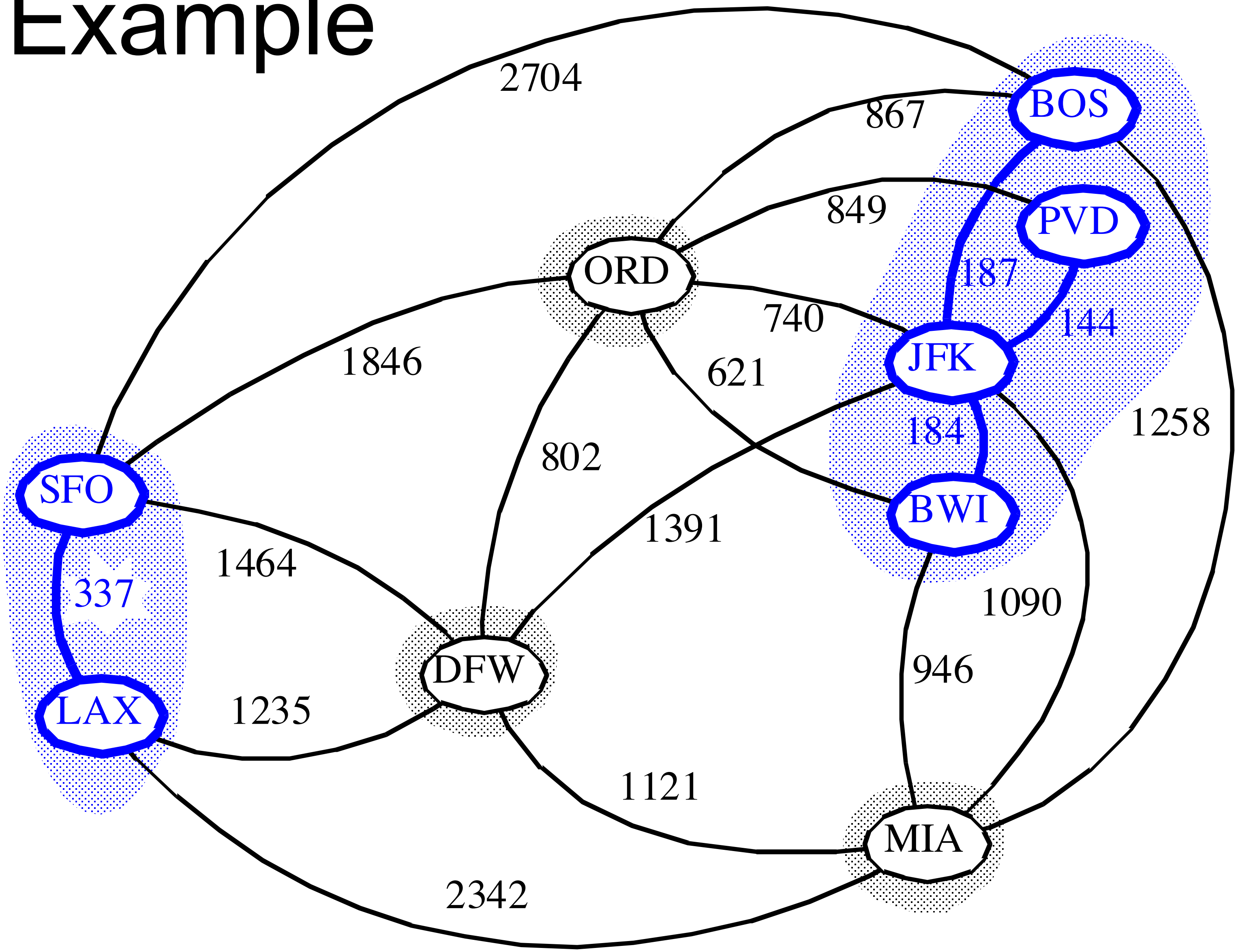

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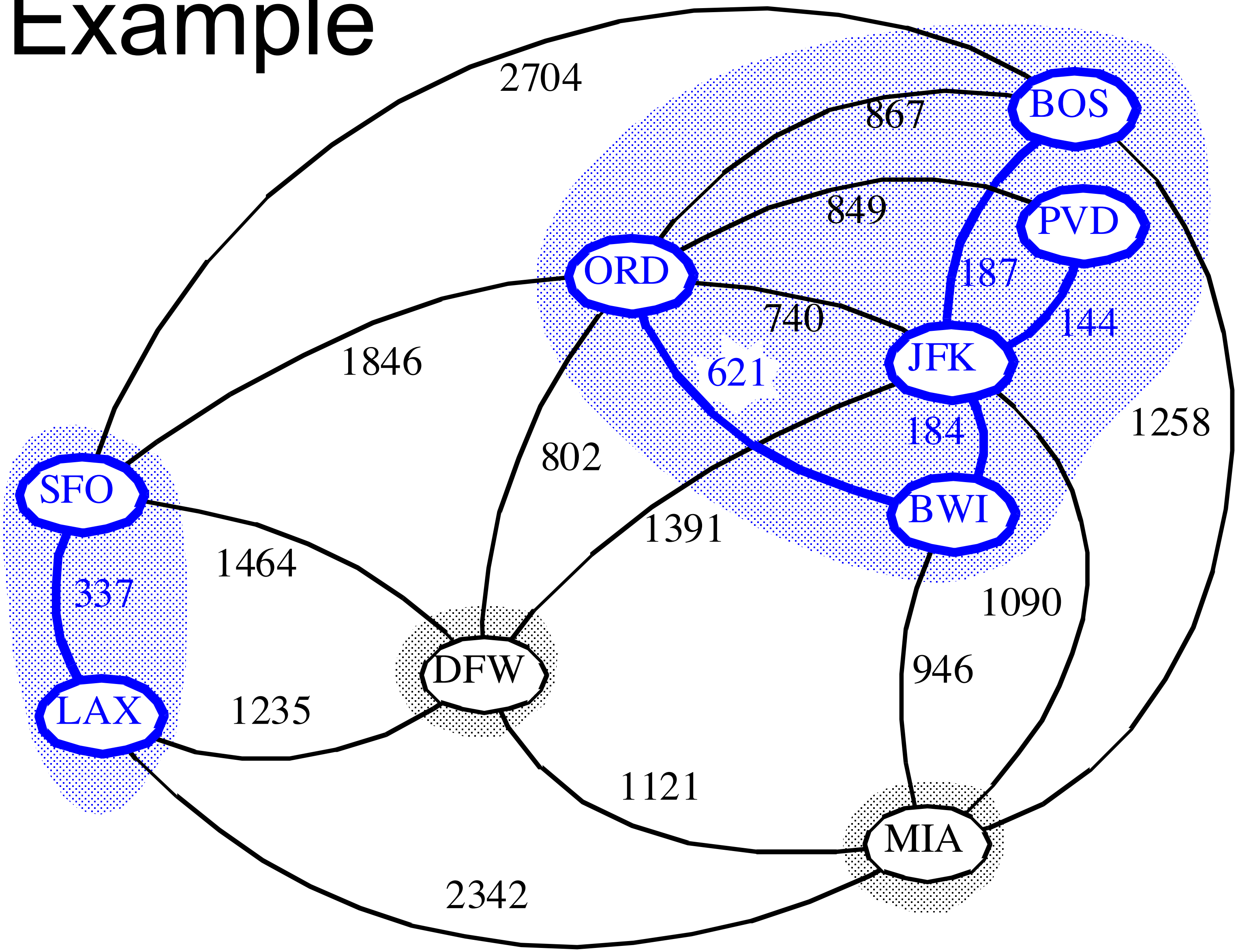

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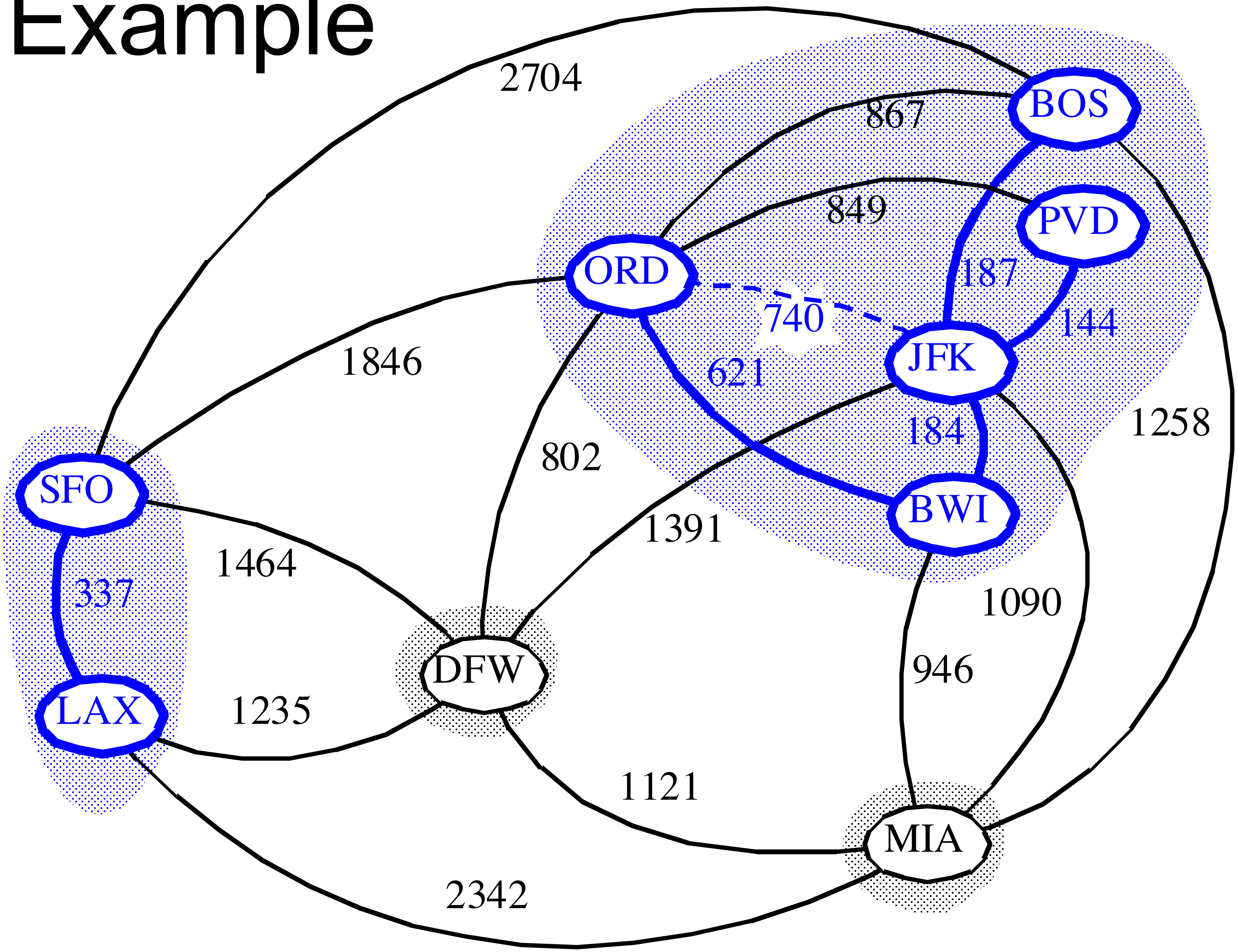

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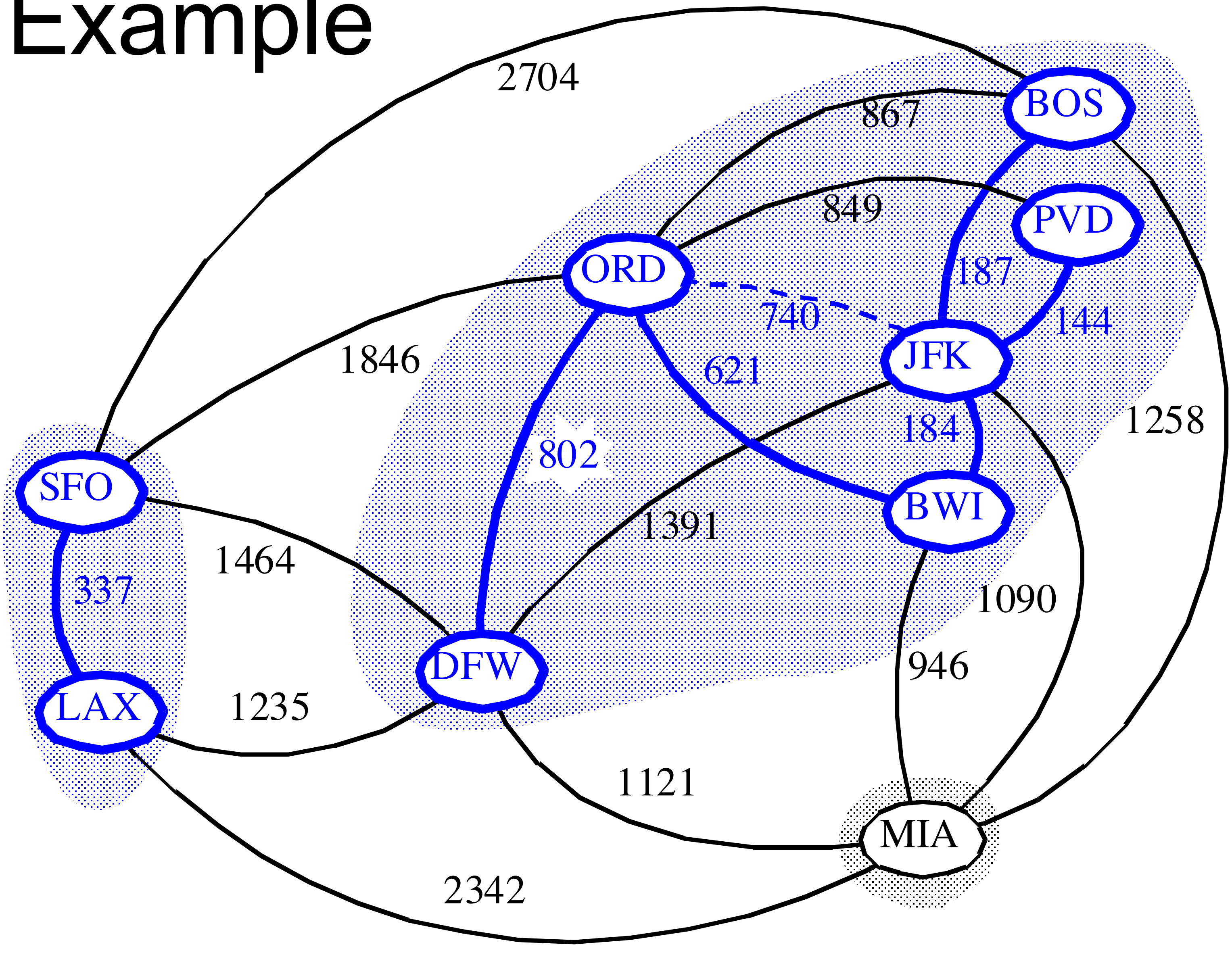
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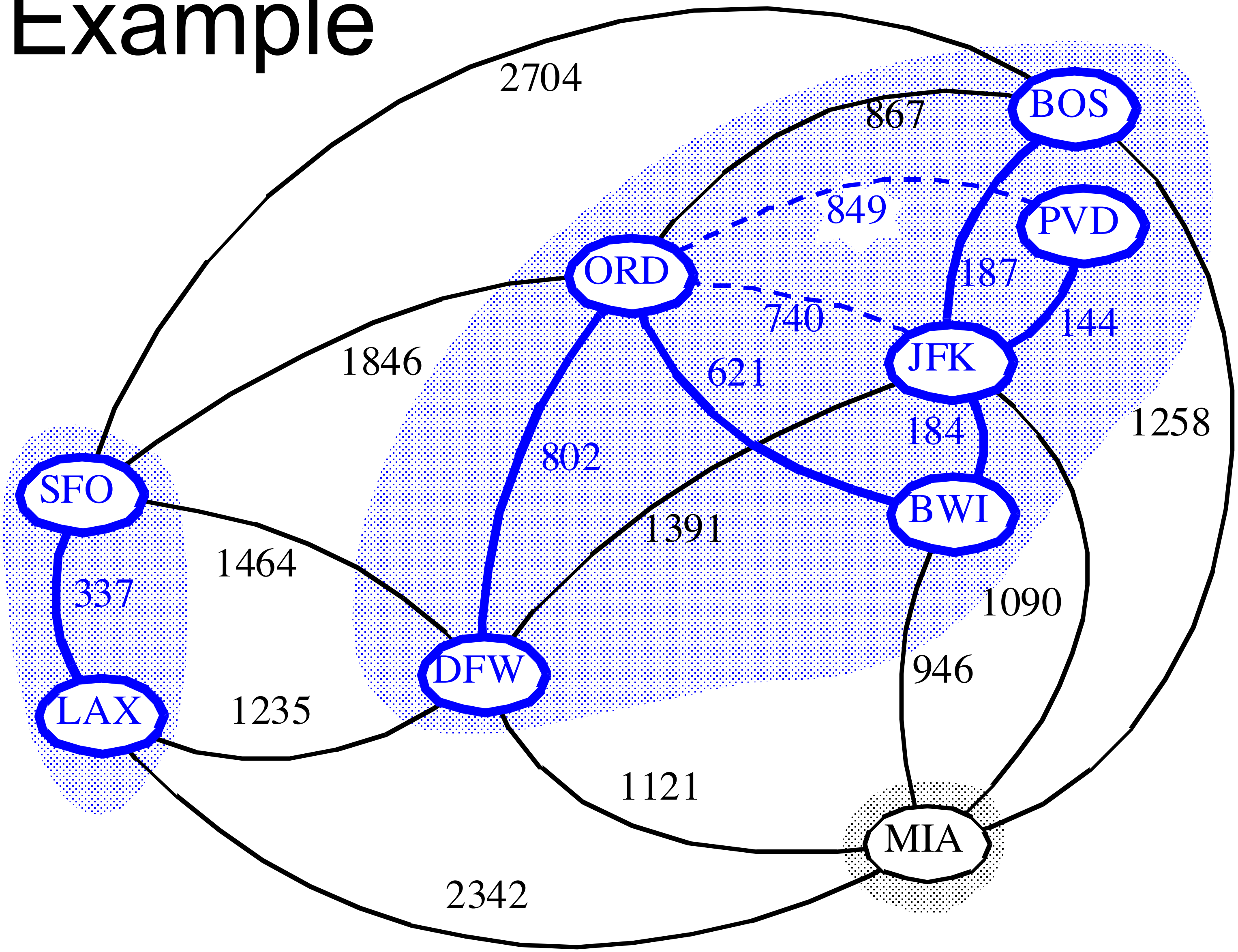

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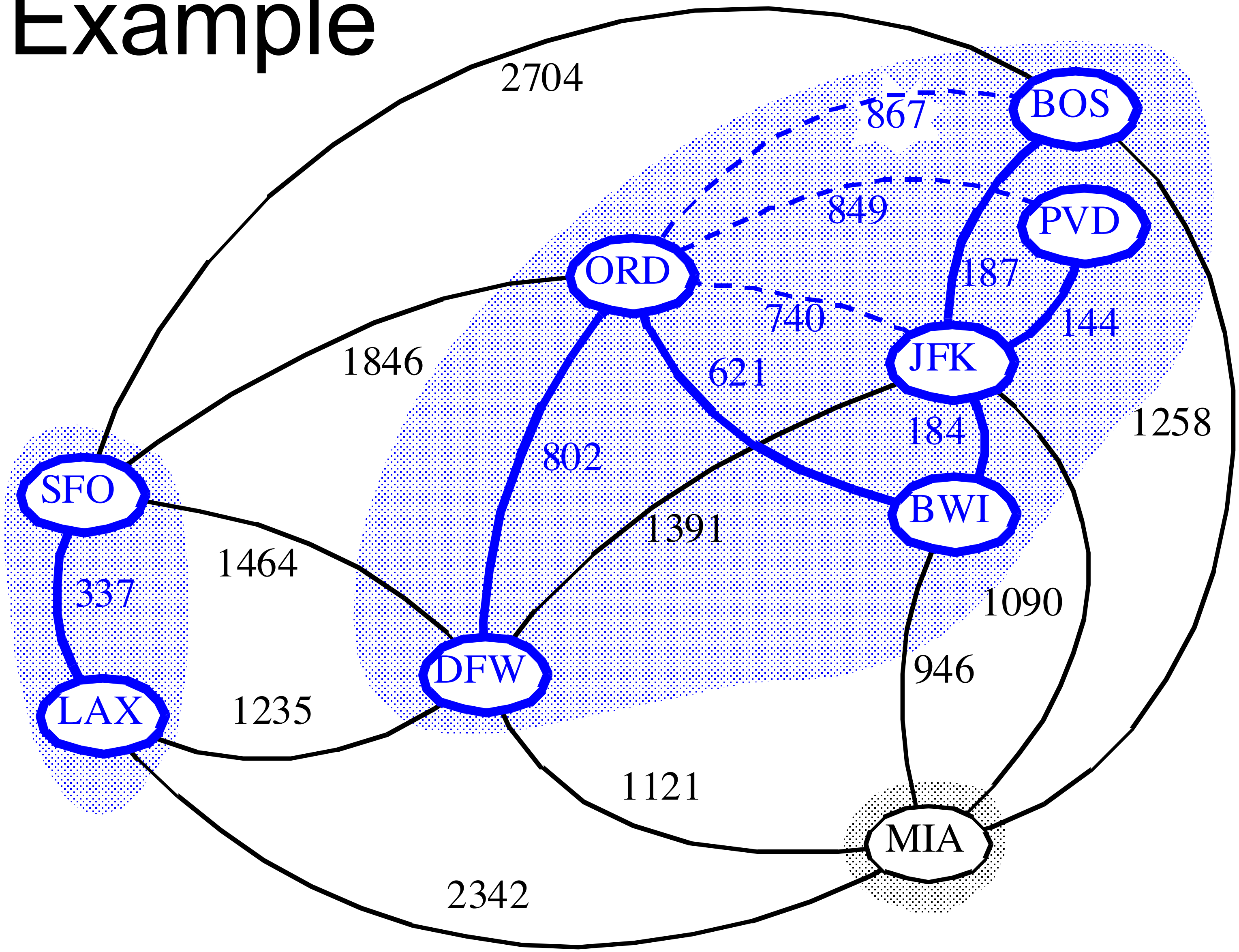
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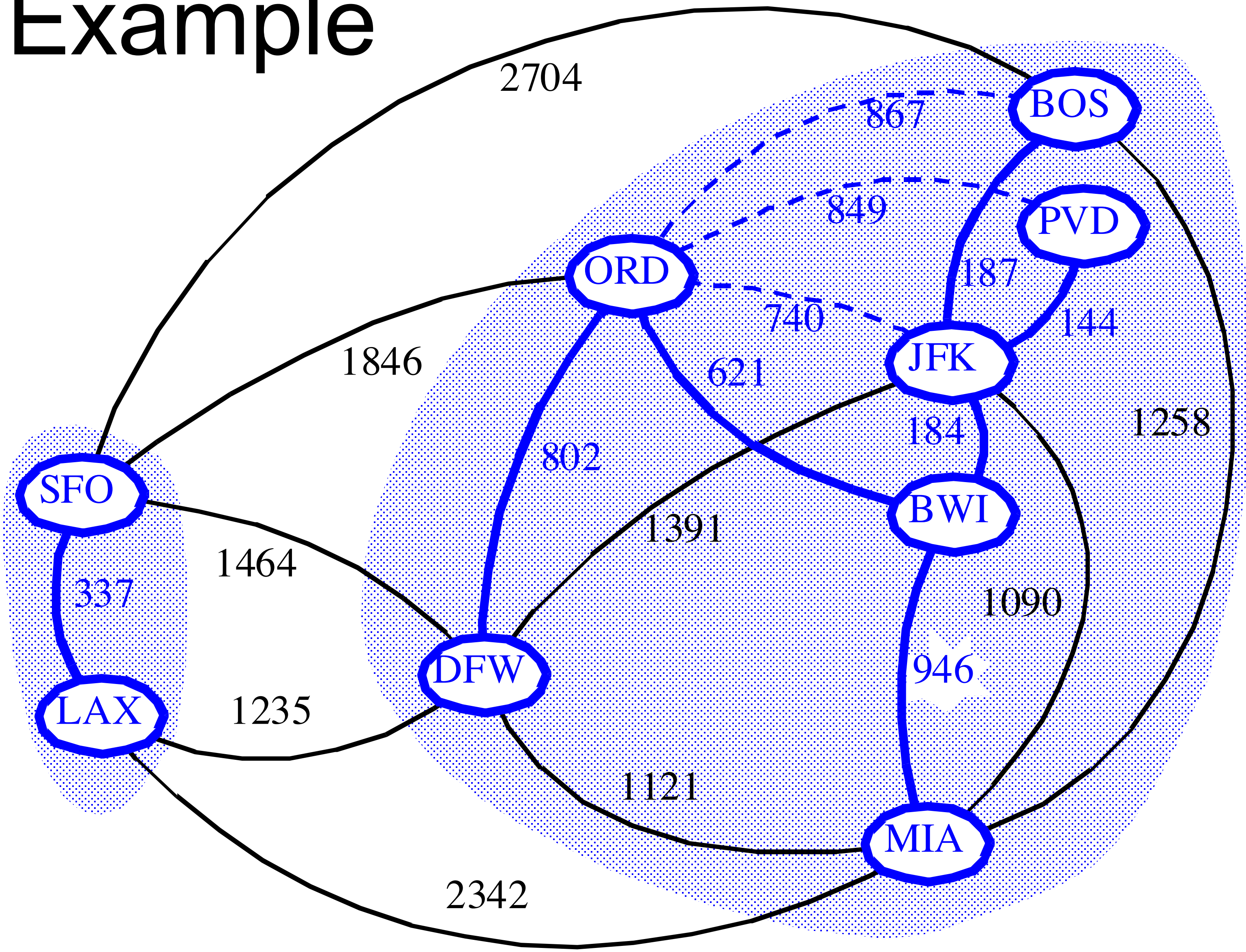

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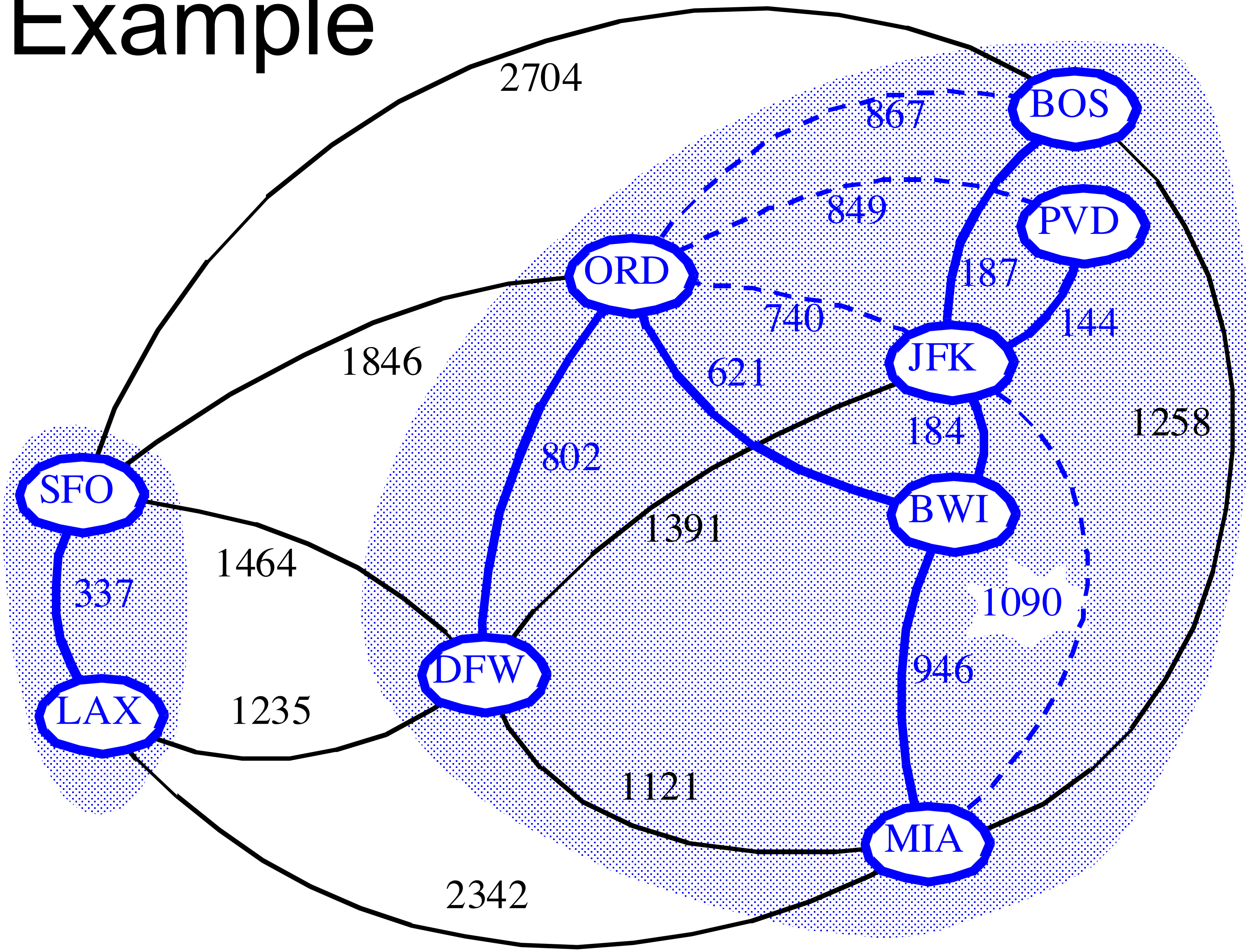
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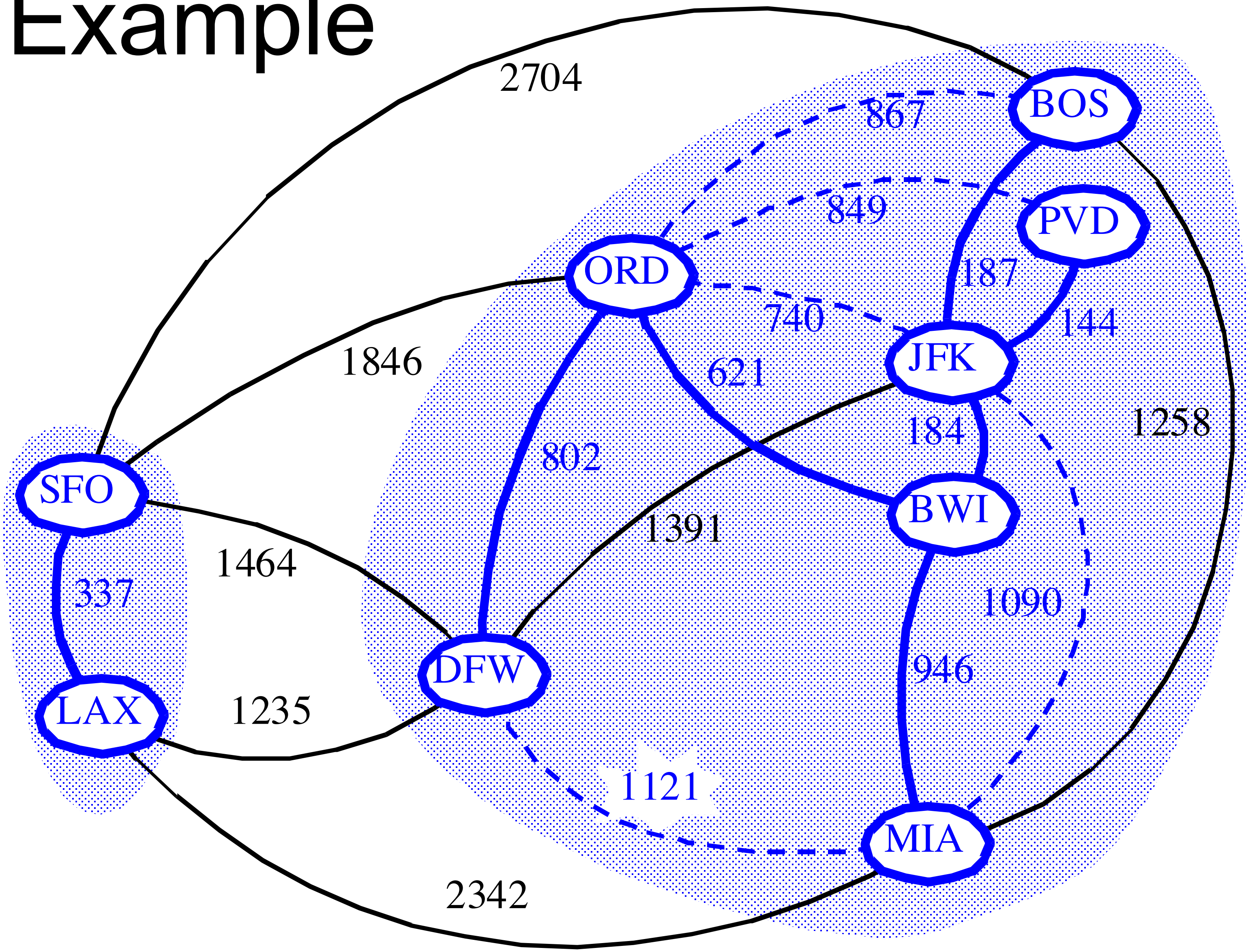

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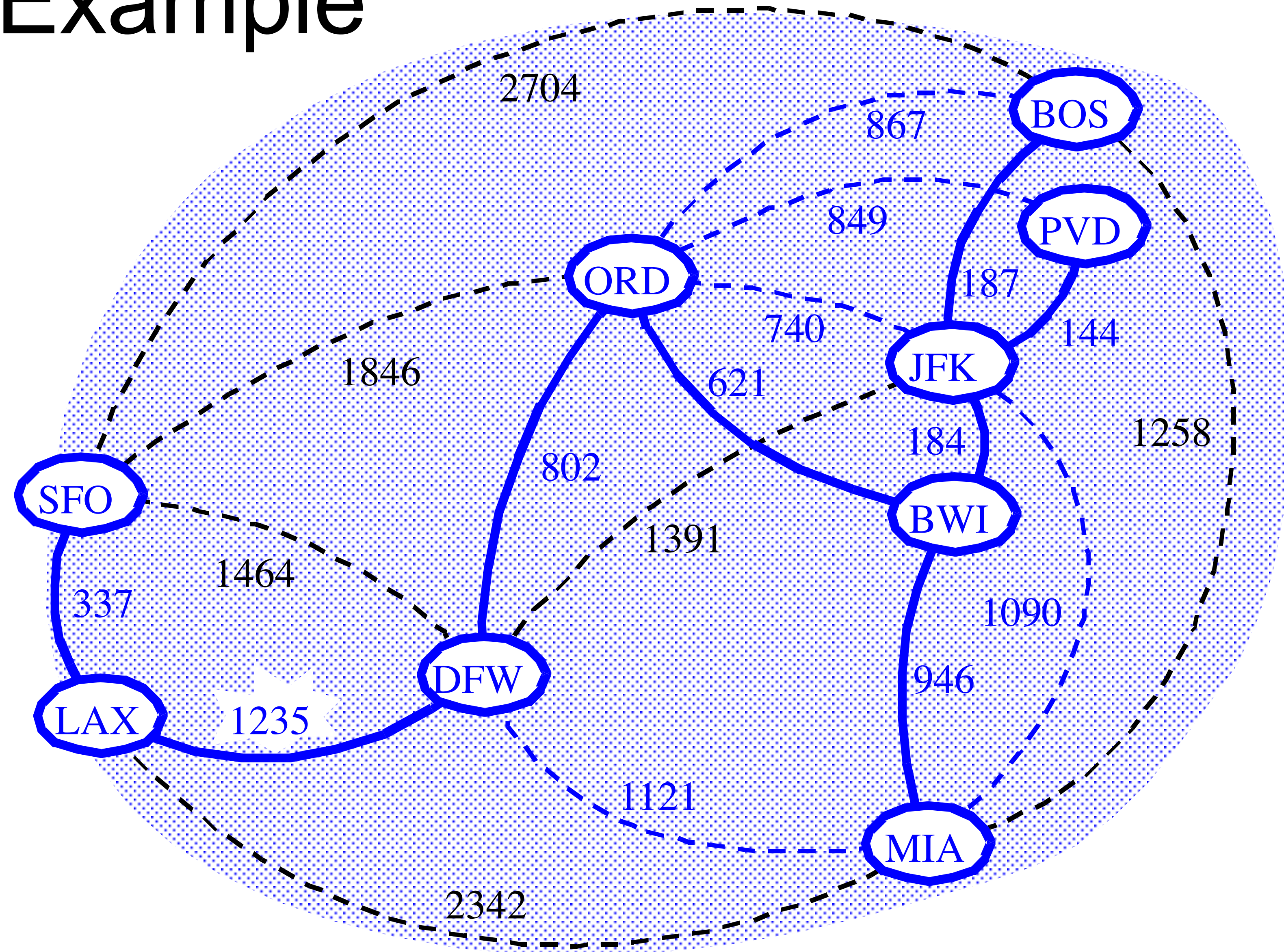
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Baruvka's Algorithm

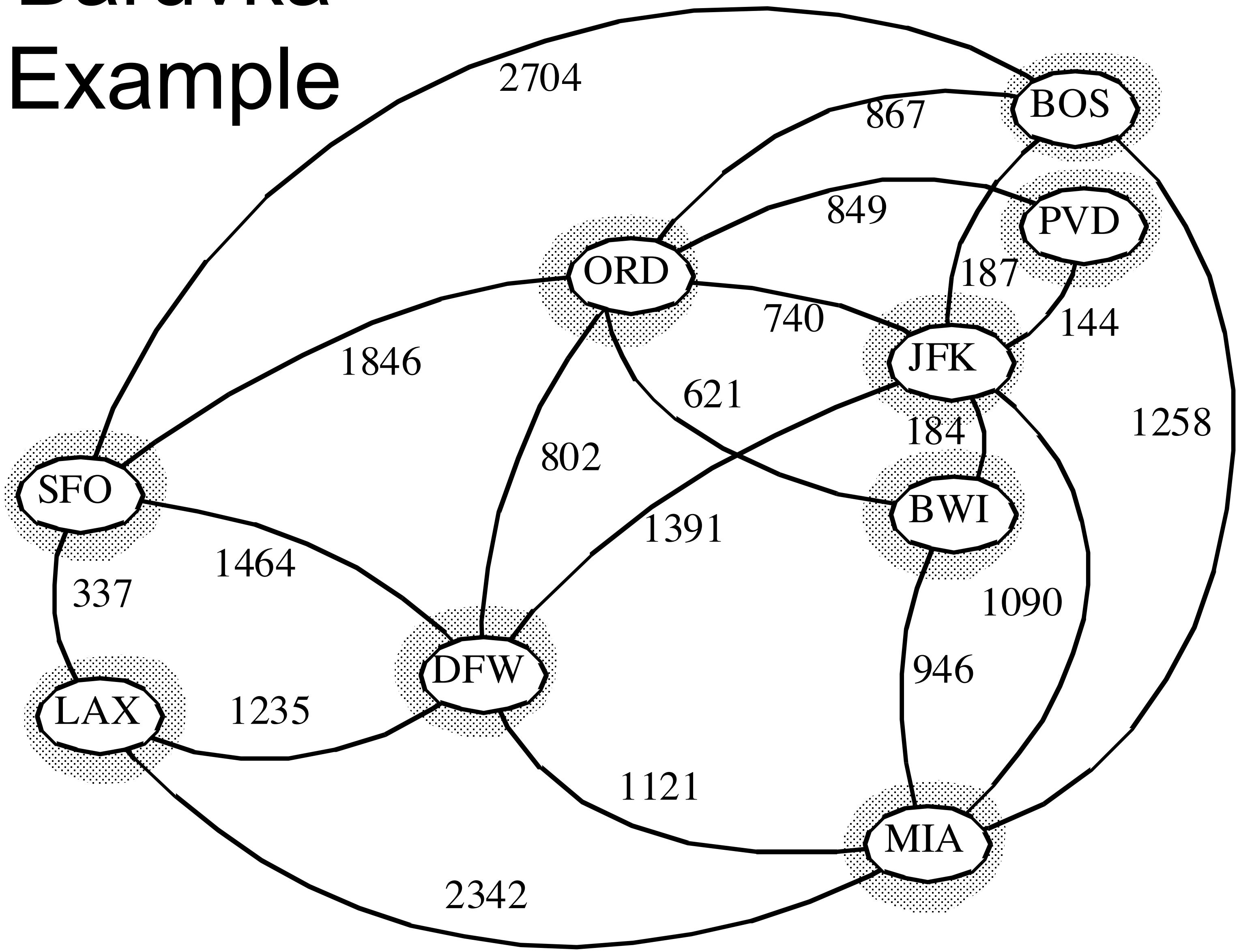
Like Kruskal's Algorithm, Baruvka's algorithm grows many "clouds" at once.

Algorithm ***BaruvkaMST(G)***
  $T \leftarrow V$ {just the vertices of G}
  while T has fewer than n-1 edges **do**
    for each connected component C in T **do**
      Let edge e be the smallest-weight edge from C to another component in T.
      if e is not already in T **then**
        Add edge e to T
  return T

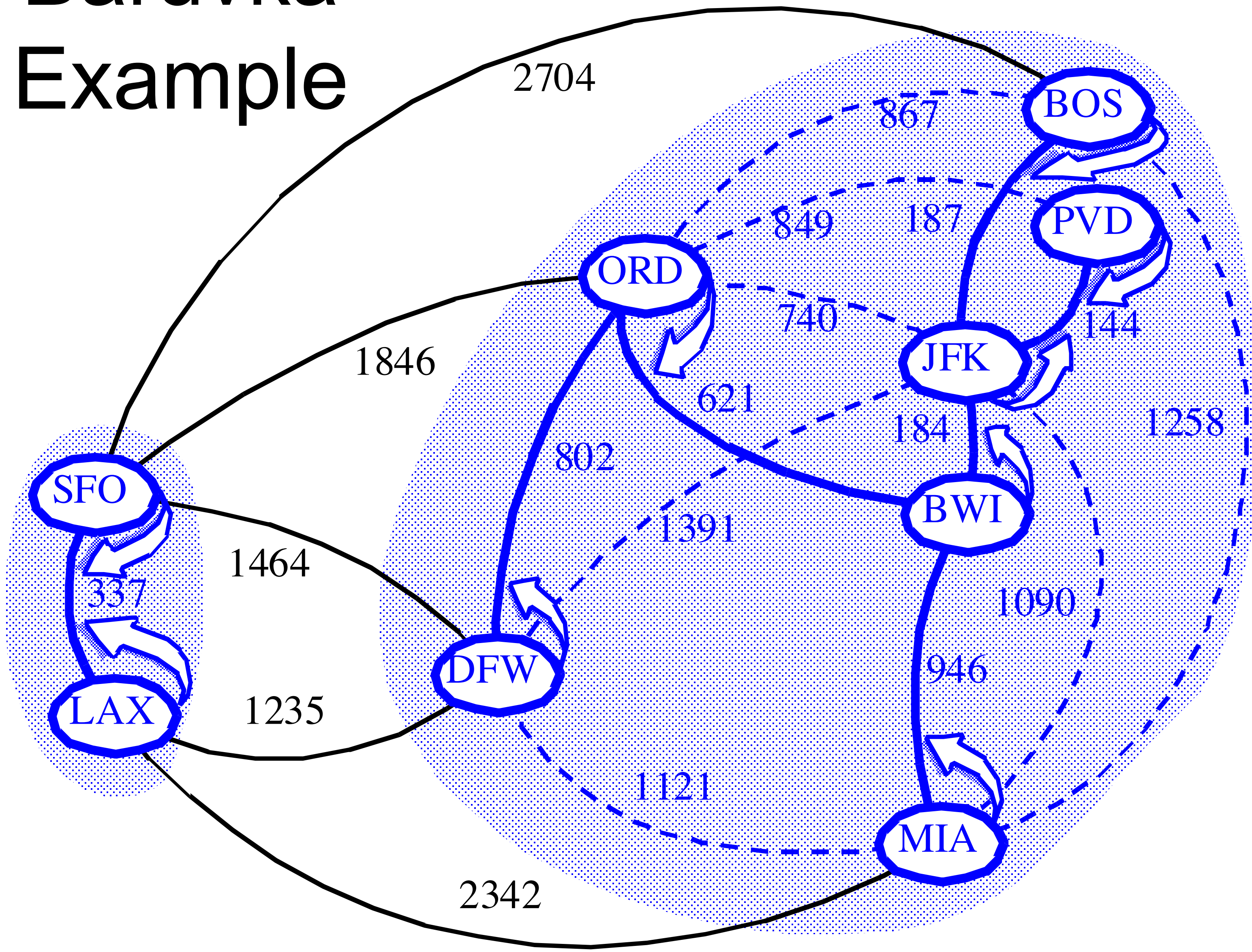
Each iteration of the while-loop halves the number of connected components in T.

- The running time is O(m log n).

Baruvka Example



Baruvka Example



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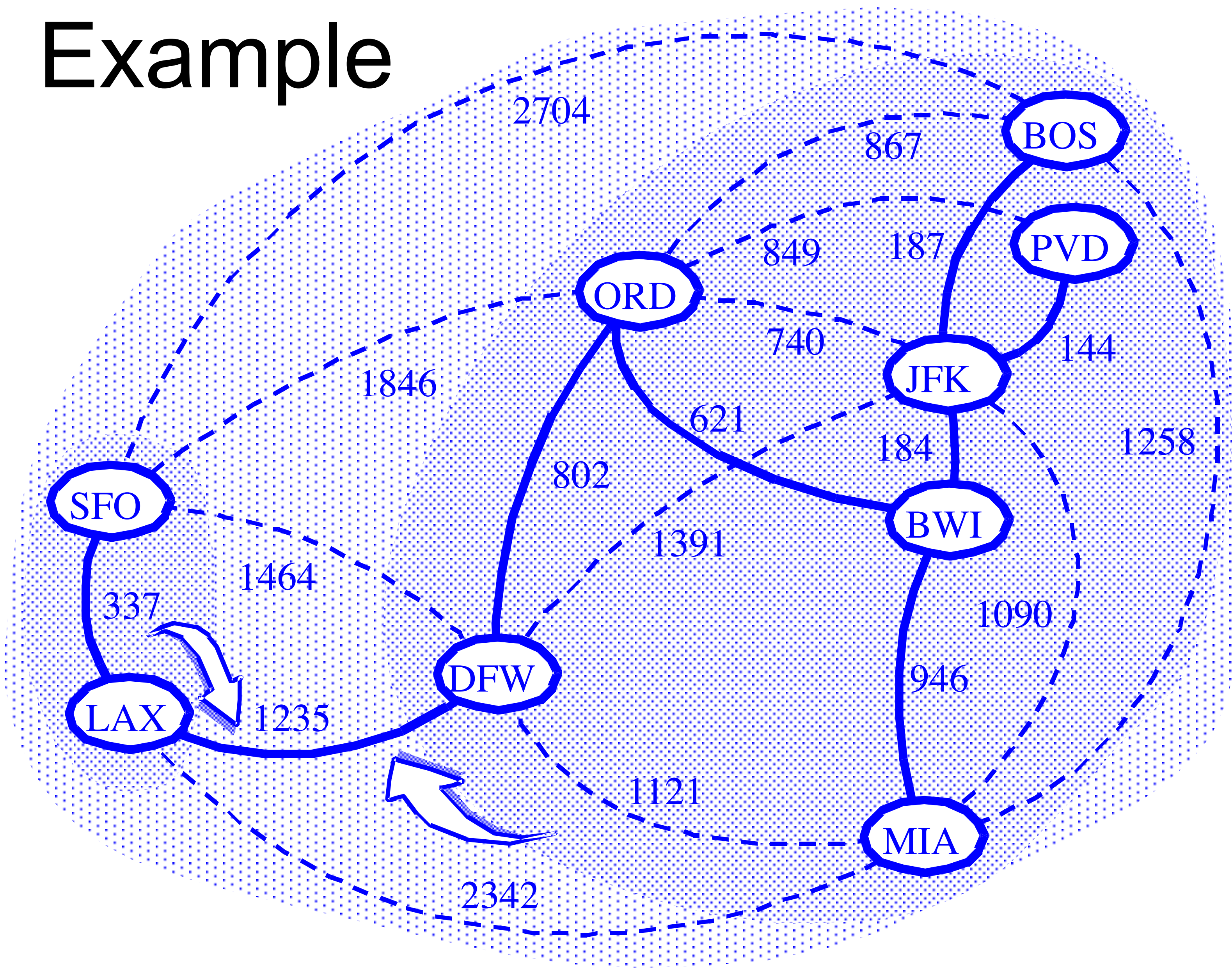

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Other – TSP

There's a long list of cities that Santa Claus needs to visit, and he only has from now until Christmas to figure out a good route to take.

Give an efficient algorithm that will come up with a guaranteed short route starting from the North Pole that will visit every city and come back to the North Pole again.

Note: You might not find the shortest route, but you can guarantee that you will always come *close* to the shortest route. This is known as an approximation algorithm. How closely does your algorithm approximate the optimal route?