# Predicates and Quantifiers

Section 1.4

# Section Summary

- Predicates
- Variables
- Quantifiers
  - Universal Quantifier
  - Existential Quantifier
- Negating Quantifiers
  - De Morgan's Laws for Quantifiers
- Translating English to Logic

# Propositional Logic Not Enough

- If we have:

  "All men are mortal."

  "Socrates is a man."
- Does it follow that "Socrates is mortal?"
- Can't  be represented in propositional logic. Need a language that talks about objects, their properties, and their relations.
- Later we'll see how to draw inferences.

# Introducing Predicate Logic

- Predicate logic uses the following new features:
  - Variables: $x, y, z$
  - Predicates: $P(x), M(x)$
  - Quantifiers (*to be covered in a few slides*):
- *Propositional functions* are a generalization of propositions.
  - They contain variables and a predicate, e.g., $P(x)$
  - Variables can be replaced by elements from their *domain*.

# Propositional Functions

- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the *domain* (or *bound* by a quantifier, as we will see later).
- The statement $P(x)$ is said to be the value of the propositional function $P$ at $x$.
- For example, let $P(x)$ denote "$x > 0$" and the domain be the integers. Then:

  P(-3) is false.

  P(0) is false.

  P(3) is true.
- Often the domain is denoted by $U$. So in this example $U$ is the integers.

# Examples of Propositional Functions

- Let "$x + y = z$" be denoted by $R(x, y, z)$ and $U$ (for all three variables) be the integers. Find these truth values:

  R(2,-1,5)

  **Solution: F**

  R(3,4,7)

  **Solution: T**

  R($x$, 3, $z$)

  **Solution: Not a Proposition**

- Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with U as the integers. Find these truth values:

  Q(2,-1,3)

  **Solution: T**

  Q(3,4,7)

  **Solution: F**

  Q($x$, 3, $z$)

  **Solution: Not a Proposition**

# Compound Expressions

- Connectives from propositional logic carry over to predicate logic.
- If $P(x)$ denotes "$x > 0$," find these truth values:

  $P(3) \vee P(-1)$ **Solution**: T

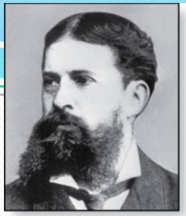
  $P(3) \wedge P(-1)$ **Solution**: F

  $P(3) \rightarrow P(-1)$ **Solution**: F

  $P(-1) \rightarrow P(3)$ **Solution**: T
- Expressions with variables are not propositions and therefore do not have truth values. For example,

  $P(3) \wedge P(y)$

  $P(x) \rightarrow P(y)$
- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.



# Quantifiers

Charles Peirce (1839-1914)

- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - *Universal Quantifier*, "For all," symbol: $\forall$
  - *Existential Quantifier*, "There exists," symbol: $\exists$
- We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.
- $\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> $x$ in the *domain*.
- $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> $x$ in the *domain*.
- The quantifiers are said to bind the variable $x$ in these expressions.

# Universal Quantifier

- $\forall x\, P(x)$ is read as "For all $x$, P($x$)" or "For every $x$, P($x$)"

  **Examples**:

  1) If $P(x)$ denotes "$x > 0$" and $U$ is the integers, then $\forall x\, P(x)$ is false.
  2) If $P(x)$ denotes "$x > 0$" and $U$ is the positive integers, then $\forall x\, P(x)$ is true.
  3) If $P(x)$ denotes "$x$ is even" and $U$ is the integers, then $\forall\, x\, P(x)$ is false.

# Existential Quantifier

- $\exists x\, P(x)$ is read as "For some $x$, P($x$)", or as "There is an $x$ such that P($x$)," or "For at least one $x$, P($x$)."

**Examples**:

1. If $P(x)$ denotes "$x > 0$" and $U$ is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and $U$ is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "$x$ is even" and $U$ is the integers, then $\exists x$ $P(x)$ is true.

# Uniqueness Quantifier (*optional*)

- $\exists! x\, P(x)$ means that $P(x)$ is true for <u>one and only one</u> $x$ in the universe of discourse.
- This is commonly expressed in English in the following equivalent ways:
  - "There is a unique $x$ such that $P(x)$."
  - "There is one and only one $x$ such that $P(x)$"
- Examples:
  1. If $P(x)$ denotes "$x + 1 = 0$" and U is the integers, then $\exists! x\, P(x)$ is true.
  2. But if $P(x)$ denotes "$x > 0$," then $\exists! x\, P(x)$ is false.
- The uniqueness quantifier is not really needed as the restriction that there is a unique $x$ such that $P(x)$ can be expressed as:

$$\exists x\, (P(x) \land \forall y\, (P(y) \rightarrow y = x))$$

# Thinking about Quantifiers

- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\, P(x)$ loop through all $x$ in the domain.
  - If at every step $P(x)$ is true, then $\forall x\, P(x)$ is true.
  - If at a step $P(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.
- To evaluate $\exists x\, P(x)$ loop through all $x$ in the domain.
  - If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
  - If the loop ends without finding an $x$ for which $P(x)$ is true, then $\exists x\, P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

# Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain $U$.
- **Examples**:
  1. If $U$ is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
  2. If $U$ is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
  3. If $U$ consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true. But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

# Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$
- $\forall x\, (P(x) \vee Q(x))$ means something different.
- Unfortunately, often people write $\forall x\, P(x) \vee Q(x)$ when they mean $\forall x\, (P(x) \vee Q(x))$.

# Translating from English to Logic

**Example 1**: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

**Solution**:

First decide on the domain *U*.

**Solution 1**: If *U* is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

**Solution 2**: But if *U* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \land J(x))$ is not correct. What does it mean?

# Translating from English to Logic

**Example 2**: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

**Solution**:

First decide on the domain *U*.

**Solution 1**: If *U* is all students in this class, translate as

$$\exists x\, J(x)$$

**Solution 2**: But if *U* is all people, then translate as

$\exists x\, (S(x) \land J(x))$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

# Returning to the Socrates Example

- Introduce the propositional functions *Man(x)* denoting "*x* is a man" and *Mortal(x)* denoting "*x* is mortal." Specify the domain as all people.
- The two premises are: $\forall x Man(x) \rightarrow Mortal(x)$

  $Man(Socrates)$
- The conclusion is: $Mortal(Socrates)$
- Later we will show how to prove that the conclusion follows from the premises.

# Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that $S$ and $T$ are logically equivalent.
- **Example:** $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

# Thinking about Quantifiers as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.
- If $U$ consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

- Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

# Negating Quantified Expressions

- Consider $\forall x\, J(x)$

  "Every student in your class has taken a course in Java."

  Here $J(x)$ is "x has taken a course in Java" and

  the domain is students in your class.

- Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

  Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

# Negating Quantified Expressions (*continued*)

- Now Consider $\exists x\, J(x)$

  "There is a student in this class who has taken a course in Java."

  Where $J(x)$ is "x has taken a course in Java."

- Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

  Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

# De Morgan's Laws for Quantifiers

- The rules for negating quantifiers are:

**TABLE 2** De Morgan's Laws for Quantifiers.

| *Negation* | *Equivalent Statement* | *When Is Negation True?* | *When False?* |
|---|---|---|---|
| $\neg\exists x P(x)$ | $\forall x \neg P(x)$ | For every $x$, $P(x)$ is false. | There is an $x$ for which $P(x)$ is true. |
| $\neg\forall x P(x)$ | $\exists x \neg P(x)$ | There is an $x$ for which $P(x)$ is false. | $P(x)$ is true for every $x$. |

- The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

- These are important. You will use these.

# Translation from English to Logic

**Examples**:

1. "Some student in this class has visited Mexico."

   **Solution**: Let $M(x)$ denote "$x$ has visited Mexico" and $S(x)$ denote "$x$ is a student in this class," and $U$ be all people.

   $$\exists x \ (S(x) \land M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   **Solution**: Add $C(x)$ denoting "$x$ has visited Canada."

   $$\forall x \ (S(x) \rightarrow (M(x) \lor C(x)))$$