# Hyperbolicity, injective hulls, and Helly graphs

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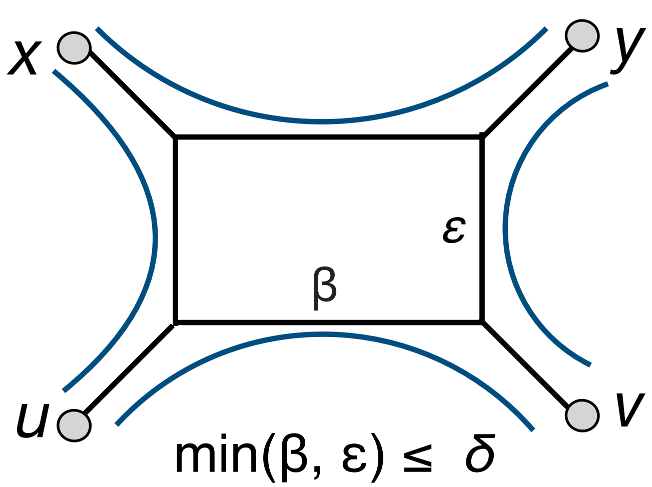
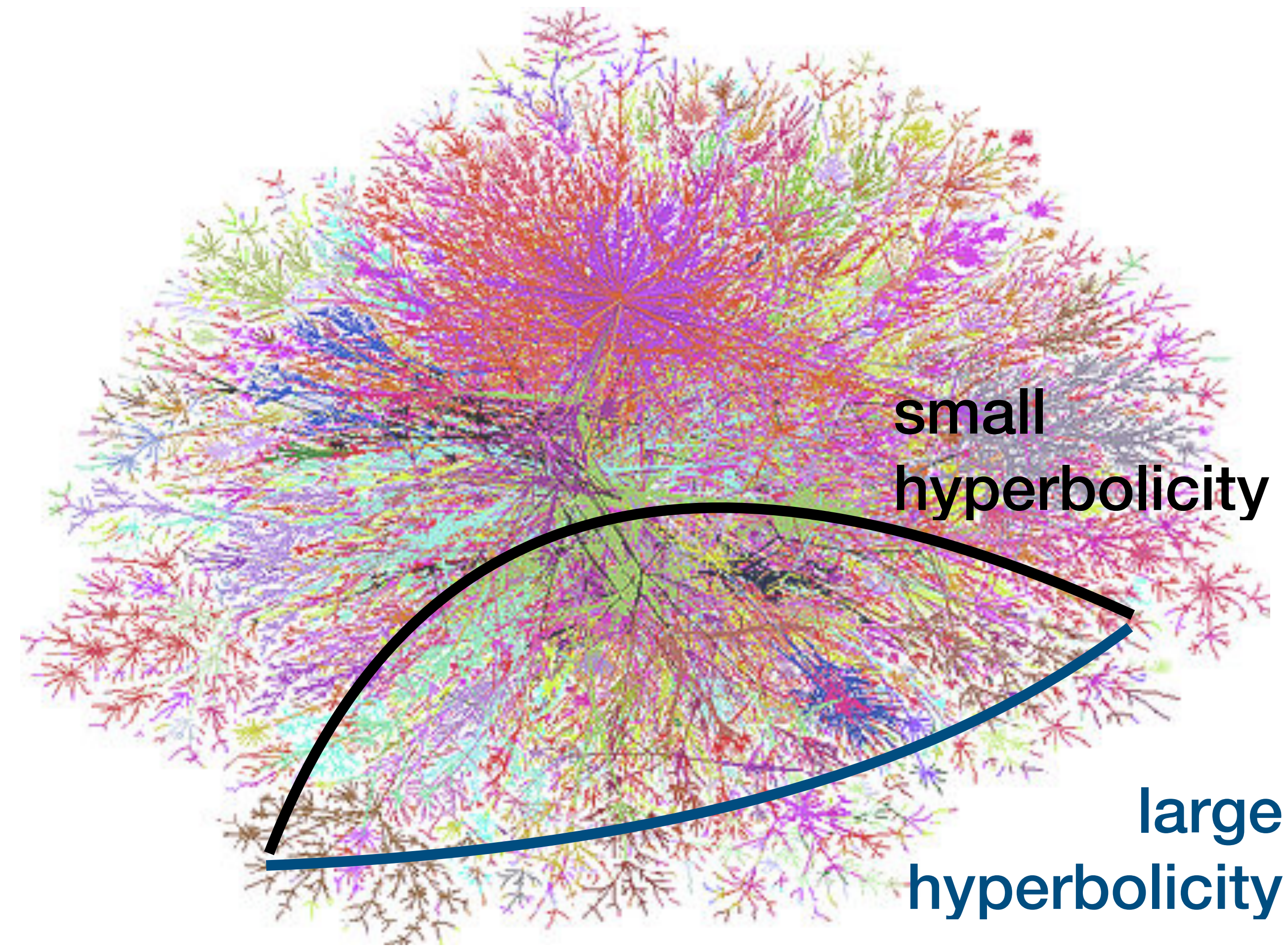
# Applications of Hyperbolicity

- Many real world networks have small hyperbolicity (biological, social, collaboration, communication, etc.)
- Smaller value means the network
  - is metrically closer to a tree
  - has negative curvature




A graph is δ-hyperbolic provided for any vertices *x, y, u, v* in it, the two larger of the three sums
d(u,v) + d(x,y),
d(u,x) + d(v,y), and
d(u,y) + d(v,x) differ by at most 2δ.



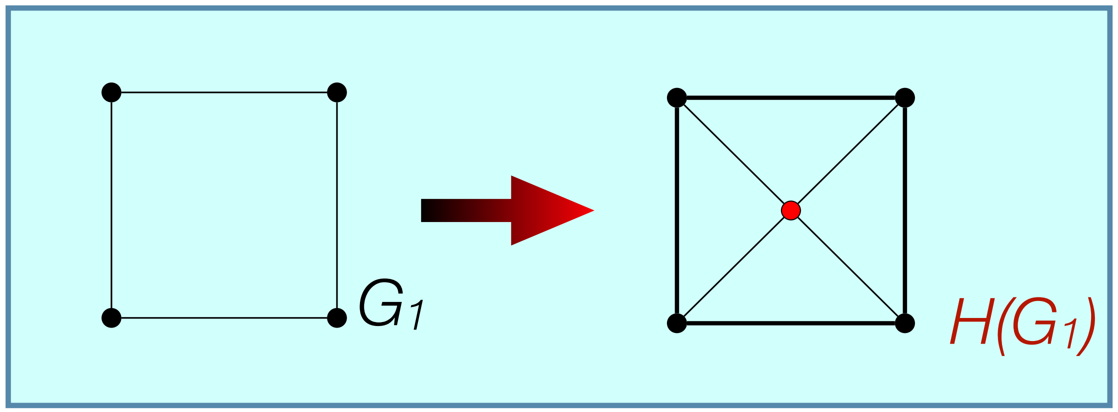

Small hyperbolicity implies that the shortest path between two points curves inward towards the core of the network.

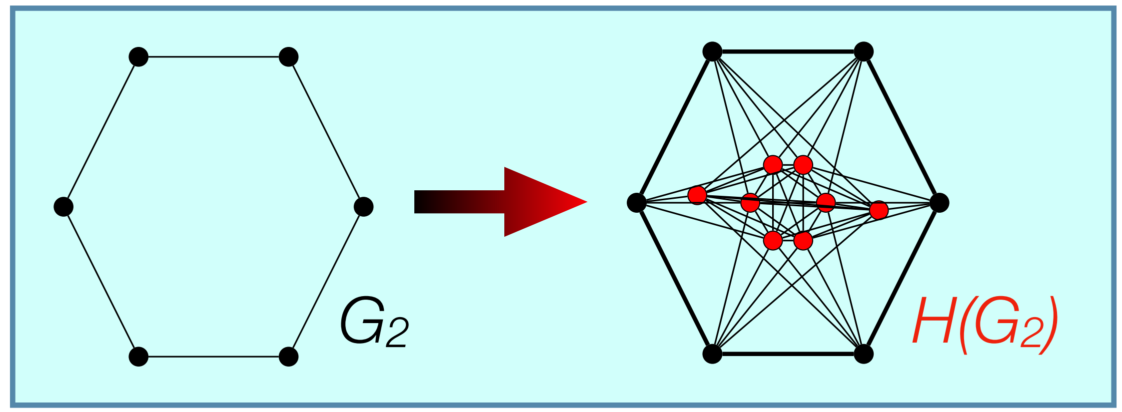
# How hyperbolicity relates to Injective Hulls

Every graph $G$ can be isometrically embedded into the smallest Helly graph $H(G)$ [1,2]

- $H(G)$ is called the injective hull of $G$
- $H(G)$ preserves hyperbolicity
- If $G$ is $\delta$-hyperbolic, any vertex in $H(G)$ is within $2\delta$ to a vertex in $G$ [3]

- A set $S$ of sets $S_i$ has the Helly property if for every subset $T$ of $S$ the following hold: if the elements of $T$ pairwise intersect, then the intersection of all elements of $T$ is also non-empty.
- A graph is called Helly if its family of disks satisfies the Helly property.




**We want to understand:**

- **(Q1) what governs hyperbolicity in Helly graphs in order to understand what governs hyperbolicity in regular graphs, and**
- **(Q2) how does the injective hull grow for various graph classes?**


[1] J. Isbell. *Six theorems about injective metric spaces*, Comment. Math. Helv (1964).

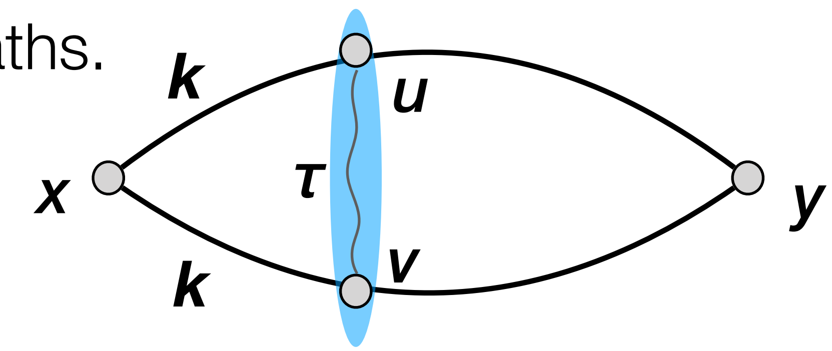
[2] A. Dress. *Trees, tight extensions of metric spaces, and the cohomological dimension of certain groups*, Adv. in Math (1984).

[3] U. Lang, *Injective hulls of certain discrete metric spaces and groups*, J. Topol. Anal. (2013)

# (**Q1**) Interval thinness governs hyperbolicity in Helly graphs

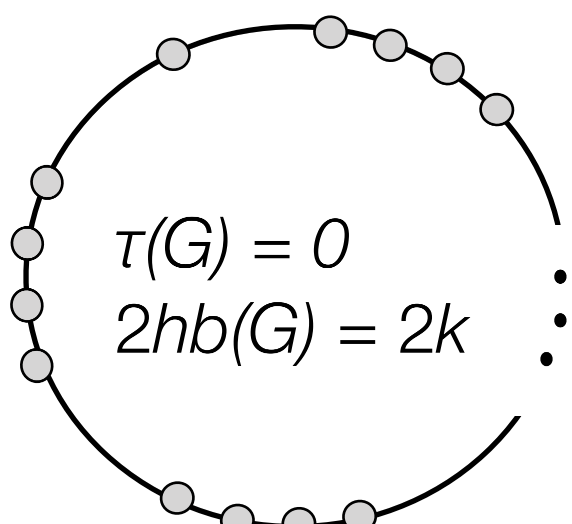
- An interval $I(x,y)$ is the set of all vertices from shortest (x,y)-paths.
- A slice of an interval at distance $k$ is defined as:

$$S_k(x,y) = \{z \in I(x,y) : d(z,x) = k\}$$




- An interval is $\tau$-thin if for any natural number $k$ and any two $u,v$ vertices of $S_k(x,y)$ are at most $\tau$ apart.
- A graph is $\tau$-thin if all of its intervals are at most $\tau$-thin.

**For general graphs $\tau(G) \leq 2hb(G)$, but $\tau(G)$ and $hb(G)$ can be far apart.**

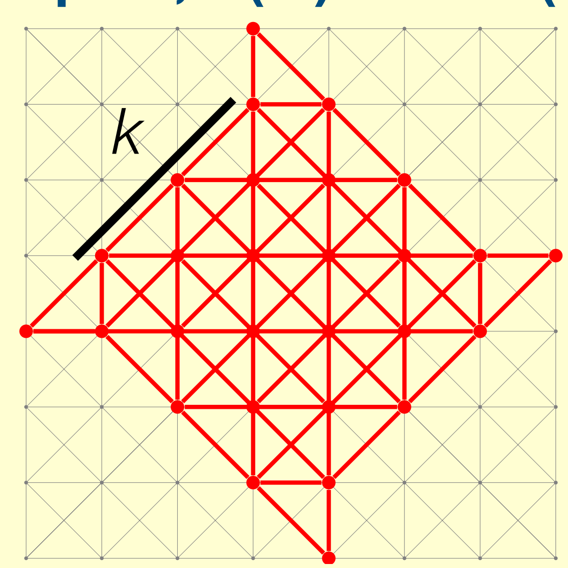



example: odd cycle with 4k+1 vertices

Our Contribution

**Theorem [4]:**

**For Helly graphs, $\tau(G) \leq 2hb(G) \leq \tau(G) + 1$.**




Example when $2hb(G) = \tau(G) + 1$

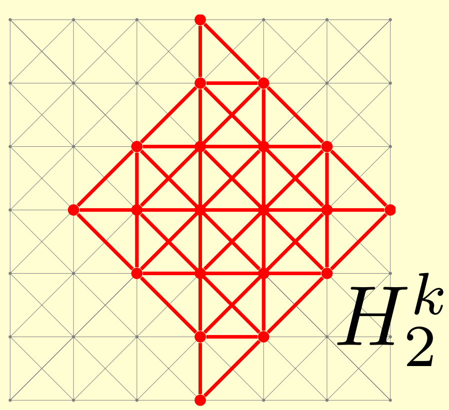
[4] F. Dragan and H. Guarnera, *Obstructions to a small hyperbolicity in Helly graphs*, under review.

# (**Q1**) Special subgraphs of a chess grid govern hyperbolicity in Helly graphs

**Our Contribution**

**Theorem [4]: We show that for Helly graphs and any integer *k*,**

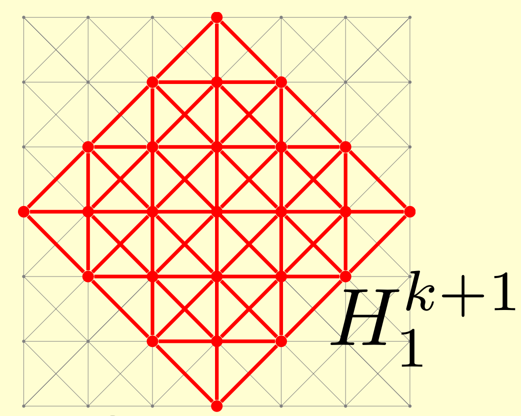
- ***hb*(*G*) ≤ *k* if and only if *G* has no isometric $H_2^k$**



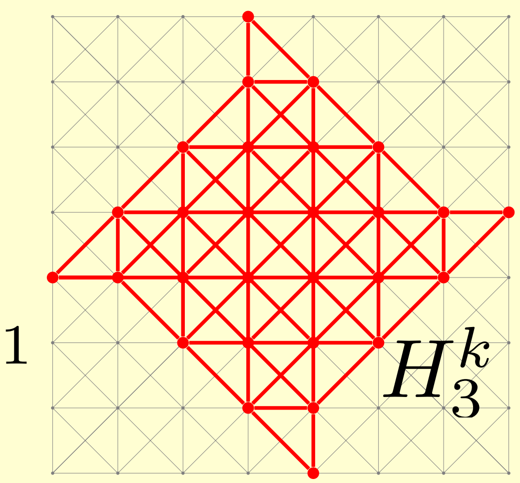

δ = *k* + 1/2

hb(*G*) is an integer

- ***hb*(*G*) ≤ *k*+1/2 if and only if *G* has neither isometric $H_1^{k+1}$ nor $H_3^k$**



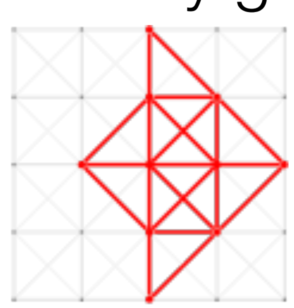

δ = *k* + 1



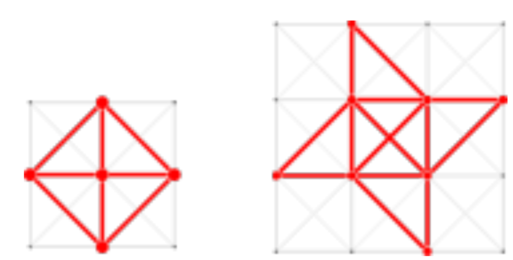

δ = *k* + 1

hb(*G*) is a half-integer

**Example:** forbidden isometric subgraphs for **1-hyperbolic** Helly graphs.



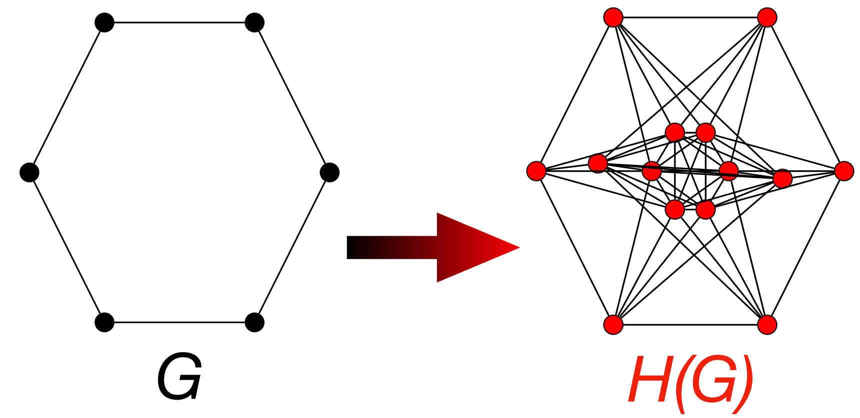
**Example:** forbidden isometric subgraphs for **1/2-hyperbolic** Helly graphs.




[4] F. Dragan and H. Guarnera, *Obstructions to a small hyperbolicity in Helly graphs*, under review.

# (**Q2**) How big is *H*(*G*) with respect to *G*?

Every graph *G* can be isometrically embedded into the smallest Helly graph *H(G)*, called the injective hull of *G*.

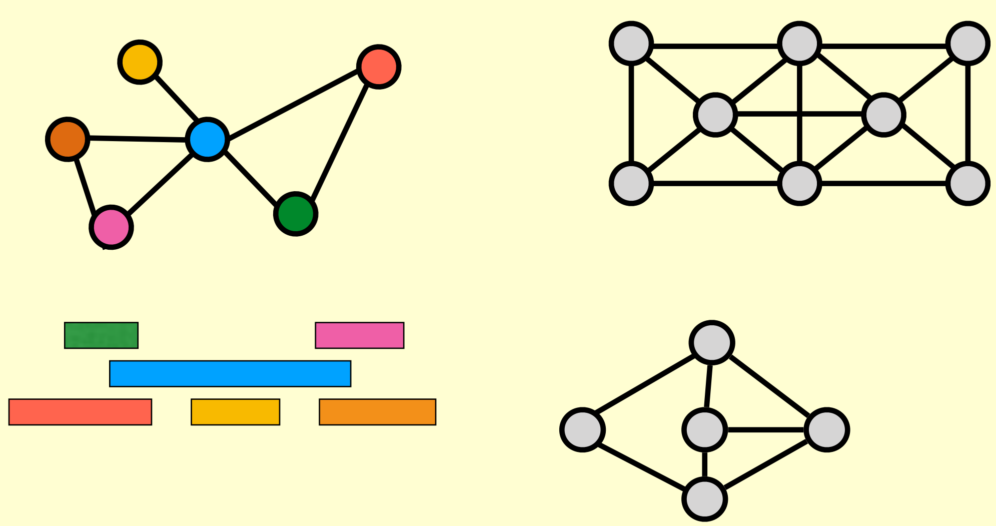



**Our Contribution**

**Theorem [5]: There are some graph classes for which *H*(*G*) …**

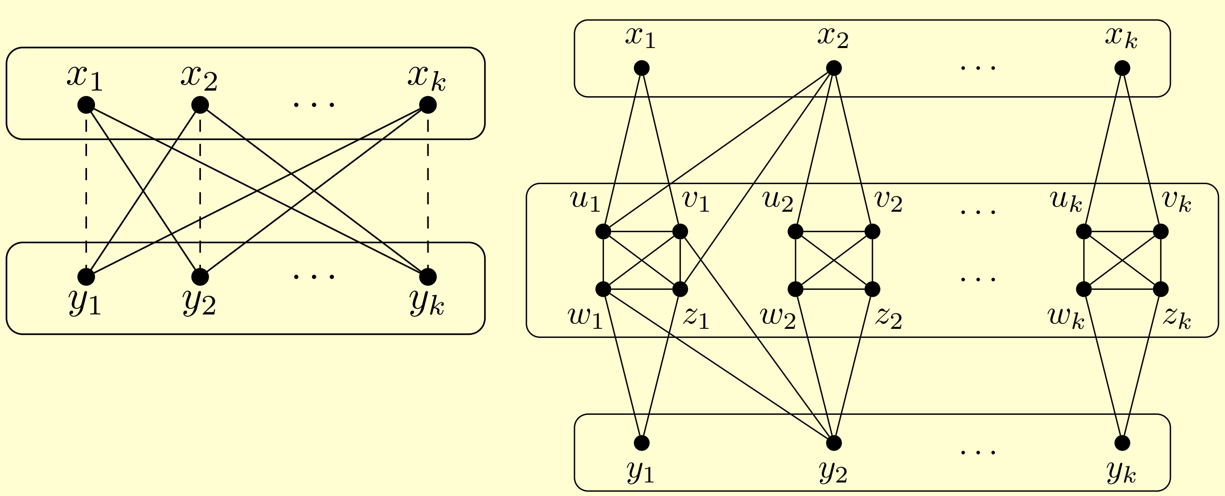
**contains at most 2*n* vertices.**

- Interval graphs
- Helly graphs
- Distance hereditary graphs



**can contain at least $2^n$ vertices.**

- Chordal bipartite graphs
- Chordal graphs
- Cocomparability graphs

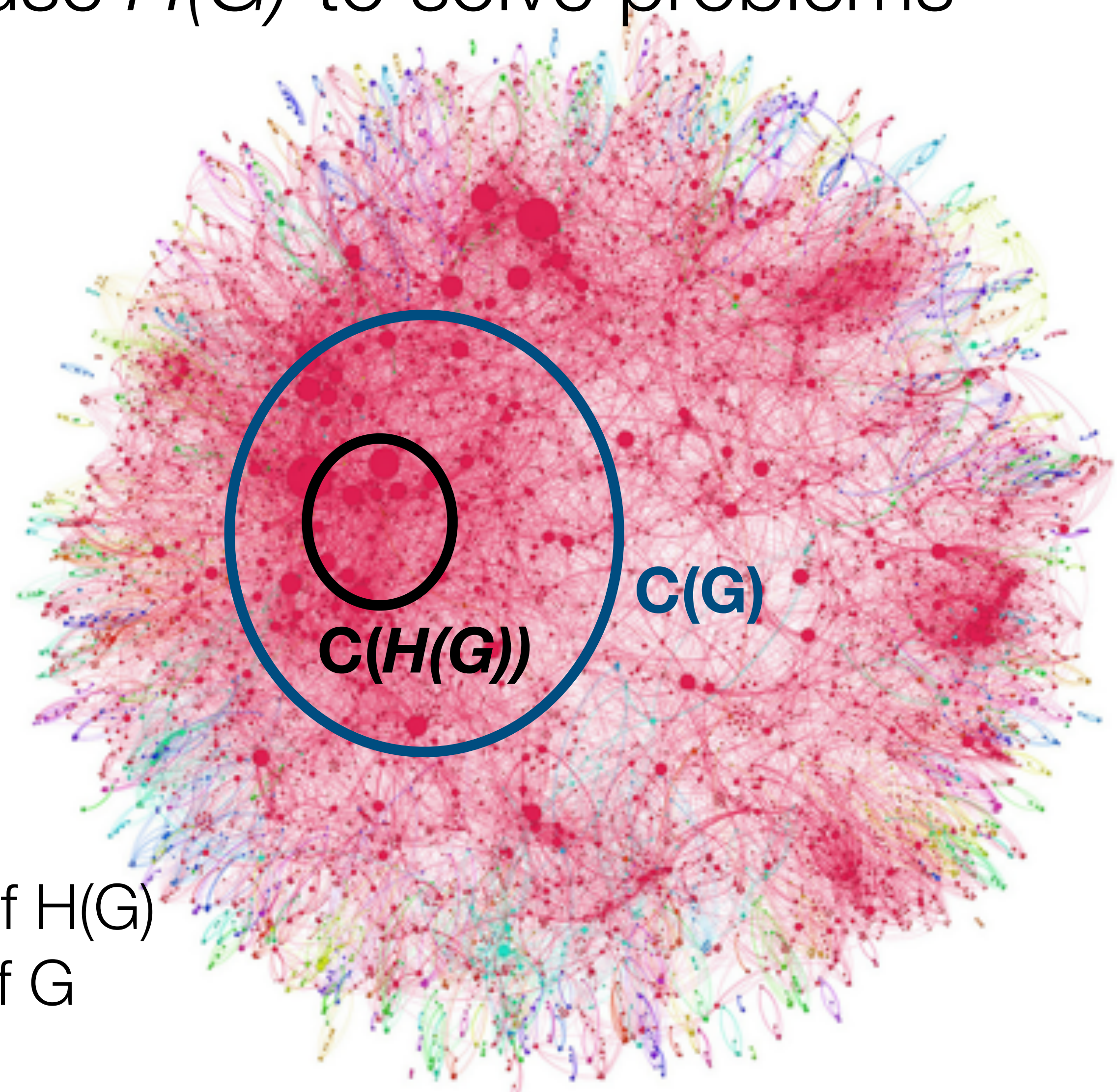




[5] F. Dragan, H. Guarnera, A. Leitert, *Hellification of special graph classes*, manuscript in preparation.

# Open Questions and Future Work

- What other graph classes can be Hellified efficiently?
- What other graph classes require exponentially many Helly vertices?
- What kind of problems can use *H(G)* to solve problems efficiently on *G?*
  - Diameter
  - Radius
  - Center




**idea:** use center of H(G) to find center of G