Camera Calibration Using Principal-Axes Aligned Conics

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Abstract. The projective geometric properties of two principal-axes aligned (PAA) conics in a model plane are investigated in this paper by utilized the generalized eigenvalue decomposition (GED). We demonstrate that one constraint on the image of the absolute conic (IAC) can be obtained from a single image of two PAA conics even if their parameters are unknown. And if the eccentricity of one of the two conics is given, two constraints on the IAC can be obtained. An important merit of the algorithm using PAA is that it can be employed to avoid the ambiguities when estimating extrinsic parameters in the calibration algorithms using concentric circles. We evaluate the characteristics and robustness of the proposed algorithm in experiments with synthetic and real data.

Keywords: Camera calibration, Generalized eigenvalue decomposition, Principal-axes aligned conics, Image of the absolute conic.

1 Introduction

Conic is one of the most important image features like point and line in computer vision. The motivation to study the geometry of conics arises from the facts that conics have more geometric information, and can be more robustly and more exactly extracted from images than points and lines. In addition, conics are very easy to be produced and identified than general algebraic curves, though general algebraic curves may have more geometric information. Unlike a large number of researches have been developed on points and lines, there are just several algorithms proposed based on conics for pose estimation [2][10], structure recovery [11][15][7][17][13], object recognition [8] [14][5], and camera calibration [18][19][3]. Forsyth et al. [2] discovered the projective invariants for pairs of conics then developed an algorithm to determine the relative pose of a scene plane from two conic correspondences. However the algorithm requires solving quartics and has no closed form solutions. Ma [10] developed an analytical method based on conic correspondences for motion estimation and pose determination from stereo images. Quan [14] discovered two polynomial constraints from corresponding conics in two uncalibrated perspective images and applied them to object recognition. Weiss [18] demonstrated that two conics are sufficient for calibration under the affine projection and derived a nonlinear calibration algorithm. Kahl and Heyden [7] proposed an algorithm for epipolar geometry estimation from conic correspondences. They found that one conic correspondence gives two independent constraints on the fundamental matrix and a method to

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estimate the fundamental matrix from at least four corresponding conics was presented. Sugimoto [17] proposed a linear algorithm for solving the homography from conic correspondences, but it requires at least seven correspondences. Mudigonda et al. [13] shown that two conic correspondences are enough for solving the homography but requires solutions of polynomial equations.

The closest works to that proposed here are [19] and [3]. Yang et al. [19] presented a linear approach for camera calibration from concentric conics on a model plane. They showed that 2 constraints could be obtained from a single image of these concentric conics. However, it requires at least three concentric conics, and the equations of all these conics must be given in advance. Gurdjos et al. [3] utilized the projective and Euclidean properties of confocal conics to perform camera calibration. These properties are that the line conic consisted of the images of the circular points should belong to the conic range of these confocal conics. Two constraints on the IAC can be obtained from a single image of the confocal conics. Gurdjos et al. [3] claimed that the important reason to propose confocal conics for camera calibration is that there exist ambiguities in the calibration methods using concentric circles [9][6] when recovering the extrinsic parameters of the camera, and the algorithms using the confocal conics can avoid such ambiguities. In this paper, we discover a novel useful pattern, PAA. And the properties of two arbitrary PAA conics with unknown or known eccentricities are deeply investigated and discussed in this paper.

2 Basic Principles

2.1 Pinhole Camera Model

Let $\mathbf{X} = \begin{bmatrix} X & Y & Z & 1 \end{bmatrix}^T$ be a world point and $\tilde{\mathbf{x}} = \begin{bmatrix} u & v & 1 \end{bmatrix}^T$ be its image point, both in the homogeneous coordinates, they satisfy:

$$\mu \tilde{\mathbf{x}} = \mathbf{PX}\,, \tag{1}$$

where $\mathbf{P}$ is a 3×4 projection matrix describing the perspective projection process. μ is an unknown scale factor. The projection matrix can be decomposed as:

$$\mathbf{P} = \mathbf{K}[\mathbf{R} \mid \mathbf{t}], \tag{2}$$

where

$$\mathbf{K} = \begin{bmatrix} f_x & s & u_0 \\ 0 & f_y & v_0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

Here the matrix $\mathbf{K}$ is the matrix of the intrinsic parameters, and $(\mathbf{R}, \mathbf{t})$ denote a rigid transformation which indicate the orientation and position of the camera with respect to the world coordinate system.

2.2 Homography Between the Model Plane and Its Image

Without loss of generality, we assume the model plane is on $Z = 0$ of the world coordinate system. Let us denote the i^{th} column of the rotation matrix $\mathbf{R}$ by $\mathbf{r}_i$. From (1) and (2), we have,

$$\mu\tilde{\mathbf{x}} = \mathbf{K}\begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t}\end{bmatrix}\begin{bmatrix}X \\ Y \\ 0 \\ 1\end{bmatrix} = \mathbf{K}\begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t}\end{bmatrix}\begin{bmatrix}X \\ Y \\ 1\end{bmatrix}. \tag{4}$$

We denote $\mathbf{x} = \begin{bmatrix}X & Y & 1\end{bmatrix}^T$, then a model point $\mathbf{x}$ and its image $\tilde{\mathbf{x}}$ is related by a 2D homography $\mathbf{H}$:

$$\mu\tilde{\mathbf{x}} = \mathbf{H}\mathbf{x}, \tag{5}$$

where

$$\mathbf{H} = \mathbf{K}\begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \mathbf{t}\end{bmatrix}. \tag{6}$$

Obviously, $\mathbf{H}$ is defined up to a scale factor.

2.3 Standard Forms for Conics

All conics are projectively equivalent under the projective transformation [16]. This means that any conic can be converted into any anther conic by some projective transformations. A conic is an ellipse (including circle), a parabola or a hyperbola, respectively, if and only if its intersection with the line at infinity on the projective plane consists of 2 imaginary points, 2 repeated real points or 2 real points, respectively. In cases of central conics (ellipses and hyperbolas), by moving the coordinate origin to the center and choosing the directions of the coordinate axes coincident with the so-called principal axes (axes of symmetry) of the conic, we can obtain that the equation in standard form for an ellipse is $X^2/a^2 + Y^2/b^2 = 1$, where $a^2 \ge b^2$, and the equation in standard form for a hyperbola is $X^2/a^2 - Y^2/b^2 = 1$. These equations can be written in a simpler form:

$$AX^2 + BY^2 + C = 0, \tag{7}$$

and rewritten in matrix form, we obtain,

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = 0, \tag{8}$$

where

$$\mathbf{A} = \begin{bmatrix}A & & \\ & B & \\ & & C\end{bmatrix}. \tag{9}$$

For a parabola, let the unique axis of symmetry of the parabola coincident with the X-axis, and let the Y-axis pass through the vertex of the parabola, then the equation of the parabola is brought into the form:

$$Y^2 = 2pX \ , \tag{10}$$

or

$$\mathbf{x}^T \mathbf{B} \mathbf{x} = 0 \ , \tag{11}$$

where

$$\mathbf{B} = \begin{bmatrix} & & -p \\ & 1 & \\ -p & & \end{bmatrix} . \tag{12}$$

Equation (12) can be rewritten in a homogenous form:

$$\mathbf{B} = \begin{bmatrix} & & E \\ & D & \\ E & & \end{bmatrix} . \tag{13}$$

2.4 Equations for the Images of Conics in Standard Form

Given the homography $\mathbf{H}$ between the model plane and its image, from (5) and (8), we can obtain the image of a central conic in standard form satisfies:

$$\tilde{\mathbf{x}}^T \tilde{\mathbf{A}} \tilde{\mathbf{x}} = 0 \ , \tag{14}$$

where

$$\tilde{\mathbf{A}} = \mathbf{H}^{-T} \mathbf{A} \mathbf{H}^{-1} . \tag{15}$$

Similarly, the image of a parabola in standard form satisfies,

$$\tilde{\mathbf{x}}^T \tilde{\mathbf{B}} \tilde{\mathbf{x}} = 0 \ , \tag{16}$$

where

$$\tilde{\mathbf{B}} = \mathbf{H}^{-T} \mathbf{B} \mathbf{H}^{-1} . \tag{17}$$

3 Properties of PAA Conics

3.1 Properties of Two Conics Via the GED

Conics are still conics under an arbitrary 2D projective transformation [16]. An interesting property of two conics is that the GED of the two conics is projectively invariant [12]. This property is interpreted in details as follows: Given two point conic pairs $(\mathbf{A}_1, \mathbf{A}_2)$ and $(\tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_2)$, they are related by a plane homography $\mathbf{H}$, i.e., $\tilde{\mathbf{A}}_i \sim \mathbf{H}^{-T} \mathbf{A}_i \mathbf{H}^{-1}$, $i = 1,2$. If $\mathbf{x}$ is the generalized eigenvector of $(\mathbf{A}_1, \mathbf{A}_2)$ i.e.,

$\mathbf{A}_1\mathbf{x} = \lambda\mathbf{A}_2\mathbf{x}$, then $\tilde{\mathbf{x}} = \mathbf{H}\mathbf{x}$ must be the generalized eigenvector of $(\tilde{\mathbf{A}}_1, \tilde{\mathbf{A}}_2)$, i.e., $\tilde{\mathbf{A}}_1\tilde{\mathbf{x}} = \tilde{\lambda}\tilde{\mathbf{A}}_2\tilde{\mathbf{x}}$. In general, there are 3 generalized eigenvectors for two 3×3 matrixes. Therefore, for a point conic pair, we may obtain three points (i.e., the three generalized eigenvectors of a point conic pair), which are projectively invariant under the 2D projective transformation in the projective plane. Similarly, for a line conic pair, we may obtain three lines (i.e., the three generalized eigenvectors of a line conic pair), which are projectively invariant under the 2D projective transformation in the projective plane.

3.2 Properties of Two PAA Central Conics

Two PAA central conics (point conics) in standard form are:

$$\mathbf{A}_1 = \begin{bmatrix} A_1 & & \\ & B_1 & \\ & & C_1 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} A_2 & & \\ & B_2 & \\ & & C_2 \end{bmatrix}. \tag{18}$$

The GED of the two conics is:

$$\mathbf{A}_1\mathbf{x} = \lambda\mathbf{A}_2\mathbf{x}. \tag{19}$$

It is not difficult to find that, the generalized eigenvalues and the generalized eigenvectors of $\mathbf{A}_1$ and $\mathbf{A}_2$ are as follows:

$$\lambda_1 = \frac{A_1}{A_2}, \ \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \lambda_2 = \frac{B_1}{B_2}, \ \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \lambda_3 = \frac{C_1}{C_2}, \ \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{20}$$

where $\mathbf{x}_1$ is the directional vector in the X-axis, $\mathbf{x}_2$ is the directional vector in the Y-axis, and $\mathbf{x}_3$ is the homogeneous coordinates of the common center of the two central conics.

From the projective geometric properties of two point conics via the GED as presented in Section 3.1, we obtain,

Proposition 1. *From the images of two PAA central conics, we can obtain the image of the directional vector in the X-axis, the image of the directional vector in the Y-axis, and the image of the common center of the two central conics via the GED.*

3.3 Properties and Ambiguities in Concentric Circles

Two concentric circles in standard form are:

$$\mathbf{A}_1 = \begin{bmatrix} A_1 & & \\ & A_1 & \\ & & C_1 \end{bmatrix}, \ \mathbf{A}_2 = \begin{bmatrix} A_2 & & \\ & A_2 & \\ & & C_2 \end{bmatrix}. \tag{21}$$

It is not difficult to find that, the generalized eigenvalues and the generalized eigenvectors of $\mathbf{A}_1$ and $\mathbf{A}_2$ are as follows:

$$\lambda_1=\lambda_2=\frac{A_1}{A_2},\ \mathbf{x}_1=\rho_1\begin{bmatrix}1\\0\\0\end{bmatrix}+\mu_1\begin{bmatrix}0\\1\\0\end{bmatrix},\ \mathbf{x}_2=\rho_2\begin{bmatrix}1\\0\\0\end{bmatrix}+\mu_2\begin{bmatrix}0\\1\\0\end{bmatrix},\ \lambda_3=\frac{C_1}{C_2},\ \mathbf{x}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}, \tag{22}$$

where $\rho_1,\mu_1,\rho_2,\mu_2$ are four real constants which are only required to satisfy that $\mathbf{x}_1 \neq \mathbf{x}_2$ up to a scale factor. This means $\rho_1,\mu_1,\rho_2,\mu_2$ cannot be determined uniquely. There are infinitely many solutions for $\rho_1,\mu_1,\rho_2,\mu_2$, thus infinitely many solutions for $\mathbf{x}_1$ and $\mathbf{x}_2$. $\mathbf{x}_1$ and $\mathbf{x}_2$ are two points at infinity, and $\mathbf{x}_3$ is the homogeneous coordinates of the common center of the two central conics. The ambiguities in $\mathbf{x}_1$ and $\mathbf{x}_2$ can be comprehended from the facts that we cannot establish a unique XY coordinate system from the two concentric circles on the model plane because there exists a degree of freedom in the 2D rotation around the common center. However for two general central PAA conics, it is very easy to establish a XY coordinate system in the supporting plane without any ambiguities because we can choose the coordinate axes coincident with the principal axes of two PAA conics.

Proposition 2. *From the images of two concentric circles, we can obtain the image of the common center, and the image of the line at infinity of the supporting plane via the GED.*

4 Calibration

4.1 Dual Conic of the Absolute Points from Conics in Standard Form

The eccentricity e is one of the very important parameters in a conic. If $e=0$, the conic is a circle. If $0<e<1$, the conic section is an ellipse. If $e=1$, it is a parabola. If $e>1$, it is a hyperbola. The equation in standard form for an ellipse is: $X^2/a^2+Y^2/b^2=1$, then $e=c/a$, where $c^2=a^2-b^2$, thus, $b^2=(1-e^2)a^2$. Therefore, we can obtain that the line at infinity $l_\infty=(0,0,1)^T$ of the supporting plane intersects the ellipse at two imaginary points:

$$\mathbf{I}_E=\begin{bmatrix}1\\ \sqrt{1-e^2}i\\0\end{bmatrix},\mathbf{J}_E=\begin{bmatrix}1\\ -\sqrt{1-e^2}i\\0\end{bmatrix}. \tag{23}$$

The equation in standard form for a hyperbola is $X^2/a^2-Y^2/b^2=1$, then $e=c/a$, where $c^2=a^2+b^2$, thus, $b^2=(e^2-1)a^2$. Therefore, we can obtain that the line at infinity $l_\infty=(0,0,1)^T$ of the supporting plane intersects the hyperbola at two real points:

$$\mathbf{I}_H=\begin{bmatrix}1\\ \sqrt{e^2-1}\\0\end{bmatrix},\mathbf{J}_H=\begin{bmatrix}1\\ -\sqrt{e^2-1}\\0\end{bmatrix}. \tag{24}$$

The equation in standard form for a parabola is $Y^2=2pX$, it is not difficult to obtain that the line at infinity $l_\infty=(0,0,1)^T$ of the supporting plane intersects the parabola at two repeated real points, or say that the line at infinity is tangent to the parabola at one real point:

$$\mathbf{I}_P = \mathbf{J}_P = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{25}$$

From discussions above, we obtain:

Definition 1. *The line at infinity intersects a conic in standard form at two points, which are called the absolute points of a conic in standard form:*

$$\mathbf{I}_A = \begin{bmatrix} 1 \\ \sqrt{e^2-1} \\ 0 \end{bmatrix}, \mathbf{J}_A = \begin{bmatrix} 1 \\ -\sqrt{e^2-1} \\ 0 \end{bmatrix}. \tag{26}$$

For a circle ($e=0$), the two absolute points is the well-known circular points, $\mathbf{I} = [1 \quad i \quad 0]^T$ and $\mathbf{J} = [1 \quad -i \quad 0]^T$.

Definition 2. *The conic*

$$\mathbf{C}_\infty^* = \mathbf{I}_A \mathbf{J}_A^T + \mathbf{J}_A \mathbf{I}_A^T \tag{27}$$

is the conic dual to the absolute points.

The conic $\mathbf{C}_\infty^*$ is a degenerate (rank 2 or 1) line conic, which consists of the two absolute points. In a Euclidean coordinate system it is given by

$$\mathbf{C}_\infty^* = \mathbf{I}_A \mathbf{J}_A^T + \mathbf{J}_A \mathbf{I}_A^T = \begin{bmatrix} 1 & & \\ & 1-e^2 & \\ & & 0 \end{bmatrix}. \tag{28}$$

The conic $\mathbf{C}_\infty^*$ is fixed under scale and translation transformation. The reasons are as follows: Under the point transformation $\tilde{\mathbf{x}} = \mathbf{H}\mathbf{x}$, where $\mathbf{H}$ is a scale and translation transformation, one can easily verify that,

$$\tilde{\mathbf{C}}_\infty^* = \mathbf{H}\mathbf{C}_\infty^*\mathbf{H}^T = \mathbf{C}_\infty^* . \tag{29}$$

The converse is also true, and we have,

Proposition 3. *The dual conic* $\mathbf{C}_\infty^*$ *is fixed under the projective transformation* $\mathbf{H}$ *if and only if* $\mathbf{H}$ *is a scale and translation transformation.*

For circles, $\mathbf{C}_\infty^*$ is fixed not only under scale and translation transformation, but also fixed under rotation transformation [4].

4.2 Calibration from Unknown PAA Central Conics

Given the images of two PAA central conics, from Proposition 1, we can determine the images of the directional vectors in the X-axis and Y-axis, then denote them as $\tilde{\mathbf{x}}_1$

and $\tilde{\mathbf{x}}_2$, respectively. From [4] we know, the vanishing points of lines with perpendicular directions satisfy:

$$\tilde{\mathbf{x}}_1^T \boldsymbol{\omega} \tilde{\mathbf{x}}_2 = 0 , \tag{30}$$

where $\boldsymbol{\omega} = \mathbf{K}^{-T}\mathbf{K}^{-1}$ is the IAC [4]. Therefore, we have:

Proposition 4. *From a single image of two PAA conics, if the parameters of the two conics are both unknown, one constraint can be obtained on the IAC.*

Given 5 images taken in general positions, we can linearly recover the IAC $\boldsymbol{\omega}$. The intrinsic parameter matrix $\mathbf{K}$ can be obtained by the Cholesky factorization of the IAC $\boldsymbol{\omega}$. After the intrinsic parameters are known, it is not difficult to obtain the images of the circular points for each image by intersecting the image of the line at infinity and the IAC $\boldsymbol{\omega}$. From the images of the circular points, the image of the common center, and the images of the directional vectors in the X-axis and Y-axis, we can obtain the extrinsic parameters without ambiguities [4].

4.3 Calibration from Eccentricity-Known PAA Central Conics

Assume that the eccentricity of one of the PAA central conics is known, from Proposition 2, we can determine the image of the line at infinity from the images of the two PAA conics. Then we can obtain the images of the absolute points of the conic with known eccentricity by intersecting the image of the line at infinity and the image of this conic. Thus we can obtain the image of the conic dual to the absolute points, $\tilde{\mathbf{C}}_\infty^*$. Actually, a suitable rectifying homography may be obtained directly from the identified $\tilde{\mathbf{C}}_\infty^*$ in an image using the eigenvalue decomposition, and after some manipulation, we can obtain,

$$\tilde{\mathbf{C}}_\infty^* = \mathbf{U}\begin{bmatrix} 1 & & \\ & 1-e^2 & \\ & & 0 \end{bmatrix}\mathbf{U}^T . \tag{31}$$

The rectifying projectivity is $\mathbf{H} = \mathbf{U}$ up to a scale and translation transformation.

Proposition 5. *Once the dual conic* $\mathbf{C}_\infty^*$ *is identified on the projective plane then projective distortion may be rectified up to a scale and translation transformation.*

After performing the rectification, we can translate the image so that the coordinate origin is coincident with the common center. Thus we obtain the 2D homography between the supporting plane and its image while the coordinate system in the supporting plane is established whose axes are coincident with the principal axes of the central PAA conics. Let us denote $\mathbf{H} = [\mathbf{h}_1 \quad \mathbf{h}_2 \quad \mathbf{h}_3]$, from (6), we have,

$$\mathbf{H} = [\mathbf{h}_1 \quad \mathbf{h}_2 \quad \mathbf{h}_3] = \mathbf{K}[\mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{t}] . \tag{32}$$

Using the fact that $\mathbf{r}_1$ and $\mathbf{r}_2$ are orthonormal, we have [20],

$$\mathbf{h}_1^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_2 = 0 \text{, i.e., } \mathbf{h}_1^T \boldsymbol{\omega} \mathbf{h}_2 = 0 \text{,} \tag{33}$$

$$\mathbf{h}_1^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_1 = \mathbf{h}_2^T \mathbf{K}^{-T} \mathbf{K}^{-1} \mathbf{h}_2 \text{, i.e., } \mathbf{h}_1^T \boldsymbol{\omega} \mathbf{h}_1 = \mathbf{h}_2^T \boldsymbol{\omega} \mathbf{h}_2 \text{.} \tag{34}$$

These are 2 constraints on the intrinsic parameters from one homography. If the eccentricities of two PAA central conics are both known, we can obtain a least squares solution for the homography. From discussions above, we have,

Proposition 6. *From a single image of two PAA conics, if the eccentricity of one of the two conics is known, two constraints can be obtained on the IAC.*

Given 3 images taken in general positions, we can obtain the IAC $\boldsymbol{\omega}$. The intrinsic parameter matrix $\mathbf{K}$ can be obtained by the Cholesky factorization of the IAC $\boldsymbol{\omega}$. Once the intrinsic parameter matrix $\mathbf{K}$ is obtained, the extrinsic parameters for each image can be recovered without ambiguity as proposed in [20].

5 Experiments

We perform a number of experiments, both simulated and real, to test our algorithms with respect to noise sensitivity. Due to lack of space, the simulated experimental results are not shown here. In order to demonstrate the performance of our algorithm, we capture an image sequence of 209 real images, with resolution 800×600, to perform augmented reality. Edges were extracted using Canny's edge detector and the ellipses were obtained using a least squares ellipse fitting algorithm [3]. Some augmented realities examples are shown in Fig. 1 to illustrate the calibration results.

Fig. 1. Some augmented realities results

6 Conclusion

A very deep investigation in the projective geometric properties of the principal-axes aligned conics is given in this paper. These properties are obtained by utilizing the generalized eigenvalue decomposition of two PAA conics. We define the absolute points of a conic in standard form, which is analogy of the circular points of a circle. Furthermore, we define the dual conic consisted of the two absolute points, which is analogy of the dual conic consisted of the circular points. By using the dual conic consisted of the two absolute points, we propose a linear algorithm to obtain the extrinsic parameters of the camera. We also discovered a novel example of the PAA conics, which is consisted of a circle and a conic concentric with each other while the parameters of the circle and the conic are both unknown, and two constraints on the IAC can be obtained from a single image of this pattern. Due to lack of space, these are not discussed in this paper. To explore more novel patterns containing conics is our ongoing work.

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