Projectively Invariant Decomposition and Recognition of Planar Shapes

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Abstract

An algorithm is presented for computing a decomposition of planar shapes into convex subparts represented by ellipses. The method is invariant to projective transformations of the shape, and thus the ellipses can be used for matching and definition of invariants in the same way as points and lines. The method works for arbitrary planar shapes admitting at least four distinct tangents and it is based on finding ellipses with four points of contact to the given shape. The cross ratio computed from the four points on the ellipse can then be used as a projectively invariant index. For a given shape, each pair of ellipses can be used to compute two independent projective invariants. The set of invariants computed for each ellipse pair can be used as indexes to a hash table from which model hypothesis can be generated

1 Introduction

The desire to achieve efficient viewpoint independent object recognition has recently led to an increased interest in projective invariance for object representation [1, 7, 9, 12]. Projectively invariant descriptors of objects can be computed from relations between points, lines and conics that are coplanar on object surfaces in 3-D. For arbitrary curved objects, projectively invariant point and line descriptions are more complex. In this case invariant points and lines can be extracted from inflexions or bitangents (e.g. [9, 10]). Related to this is the use of combined algebraic and differential invariant descriptors [11].

These methods of point and line descriptions have limitations for arbitrary curved shapes. They cannot for example be used for convex shapes. For complex shapes the number of invariants grow very rapidly with the number of points and lines used in the representation which leads to problems when the invariants are used for indexing.

It is therefore desirable to look for more complex and global primitives for projectively invariant representation of planar shape. A most natural extension of the use of lines is to use homogeneous polynomials. A homogeneous polynomial is transformed projectively into a homogeneous polynomial with the same degree. The parameters of the homogeneous polynomial can therefore be used in the same way as the coordinates of points and lines as projectively invariant shape descriptors.

The problem lies in the association of the homogeneous polynomial curve with the given shape. This association must commute with the transformation. Methods of affine invariant associations of homogeneous polynomials to point sets were developed in [3]. and [7]. These methods cannot be extended to the projective case and continuous curves. In the projective case, the association has to be based on projectively invariant properties. Two curves are said to be in contact of order n if they coincide at a certain point and their derivatives up to order $n-1$ are the same. Contact is a property that is invariant over projective transformations. For a given shape we can consider the class of homogeneous polynomials and a certain number of contact points with a specified order of contact. A member of this class will then project to a member of the corresponding class given by the projective transform of the shape.

The method that will be presented, [4], [5], is based on using ellipses, which are a subset of second-order homogeneous polynomials. For a given we will study the class of ellipses with four contact points with the shape. The order of contact is two, that is, we consider ellipses where the tangents of the ellipses at the four contact points coincide with the tangents of the given

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shape. As will be explained in the next section, the choice of four contact points and second-order contact provides a method of identifying single members of this family of ellipses over projective transforms using the cross ratio of four points on a conic. It is thus possible to extract a finite set of ellipses from a shape in two projectively corresponding images in such a way that the ellipses in the two frames are in projective correspondence.

The proposed method of shape representation has similarities with the medial axis transform [2] and can in certain respects be seen as a generalization. Using ellipses as primitives, which are convex, recalls shape decomposition methods [8] . It will be seen that the ellipses extracted with this method, will in general correspond to a perceptual decomposition of the object into its convex subparts.

2 Projective Invariance of Contact Point Ellipses

To discuss the invariance properties of ellipses with various contact points we start with the cross ratio, the fundamental invariant for points and lines in the plane. The cross ratio can be expressed as a ratio involving determinants. Given three column vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ in R^3 we will use the bracket notation [ ] for the determinant of the 3×3 matrix formed by these vectors: For 5 points in the plane, with homogeneous coordinates $\mathbf{x}_a, \mathbf{x}_b, \mathbf{x}_c, \mathbf{x}_d, \mathbf{x}_e$, the cross ratio is defined as:

$$\frac{[\mathbf{x}_a\ \mathbf{x}_b\ \mathbf{x}_e]\ [\mathbf{x}_c\ \mathbf{x}_d\ \mathbf{x}_e]}{[\mathbf{x}_a\ \mathbf{x}_c\ \mathbf{x}_e]\ [\mathbf{x}_b\ \mathbf{x}_d\ \mathbf{x}_e]} = \sigma \tag{1}$$

The invariance of the cross ratio over projective transformations $\mathbf{x}_i^{'} = T\ \mathbf{x}_i$, where T is a nonsingular 3×3 matrix, follows easily from the product rule for determinants. If we consider the fifth point as a variable $\mathbf{x}$ and denote the cross ratio $\sigma = \lambda_1/\lambda_2$ we have

$$\lambda_1[\mathbf{x}_a\ \mathbf{x}_c\ \mathbf{x}]\ [\mathbf{x}_b\ \mathbf{x}_d\ \mathbf{x}] - \lambda_2[\mathbf{x}_a\ \mathbf{x}_b\ \mathbf{x}]\ [\mathbf{x}_c\ \mathbf{x}_d\ \mathbf{x}] = 0 \tag{2}$$

This is a second-order polynomial in $\mathbf{x}$ representing a conic through the points $\mathbf{x}_a, \mathbf{x}_b, \mathbf{x}_c, \mathbf{x}_d$. Varying the cross ratio we get a pencil of conics through the four points. This pencil of conics can be expressed using the homogeneous symmetric matrices $P(\sigma), P_1, P_2$ as:

$$\mathbf{x}^T P(\sigma)\mathbf{x} = \mathbf{x}^T(\lambda_1 P_1 - \lambda_2 P_2)\mathbf{x} = 0 \tag{3}$$

Due to the duality between points and lines, a conic can be expressed as a quadratic form in line coordinates $\mathbf{u}$. The pencil of conics tangential to four lines $\mathbf{u}_a, \mathbf{u}_b, \mathbf{u}_c, \mathbf{u}_d$ expressed in line coordinates can be written:

$$\lambda_1[\mathbf{u}_a\ \mathbf{u}_c\ \mathbf{u}]\ [\mathbf{u}_b\ \mathbf{u}_d\ \mathbf{u}] - \lambda_2[\mathbf{u}_a\ \mathbf{u}_b\ \mathbf{u}]\ [\mathbf{u}_c\ \mathbf{u}_d\ \mathbf{u}] = 0 \tag{4}$$

Just as in the point case this can be expressed using homogeneous symmetric matrices $Q(\sigma), Q_1, Q_2$ as:

$$\mathbf{u}^T Q(\sigma)\mathbf{u} = \mathbf{u}^T(\lambda_1 Q_1 - \lambda_2 Q_2)\mathbf{u} = 0 \tag{5}$$

A projective transformation T that maps points $\mathbf{x}_i$ into $\mathbf{x}_i^{'} = T\ \mathbf{x}_i$ will map the matrices $P(\sigma)$ and $Q(\sigma)$ to matrices $P^{'}(\sigma)$ and $Q^{'}(\sigma)$. If we apply this transformation to the coordinates in the pencils (2) and (4) the conic matrices can be shown to be related as:

$$P(\sigma) = T^T P^{'}(\sigma) T \qquad Q^{'}(\sigma) = T Q(\sigma) T^T \tag{6}$$

That is, the conic with cross ratio $\sigma = \lambda_1/\lambda_2$ maps into a conic with the same cross ratio.

For a given curve, we will consider the class of ellipses having four second order contact points. Line coordinates of the tangents at the contact points can then be computed from the conic matrix P as:

$$\mathbf{u}_a = P\mathbf{x}_a \quad \mathbf{u}_b = P\mathbf{x}_b \quad \mathbf{u}_c = P\mathbf{x}_c \quad \mathbf{u}_d = P\mathbf{x}_d \tag{7}$$

For four points on a conic the cross ratio will equal that computed from the tangents of the points. This follows easily from the fact that

$$[\mathbf{u}_a\ \mathbf{u}_b\ \mathbf{u}] = [P\mathbf{x}_a\ P\mathbf{x}_b\ P\mathbf{x}] = [P]\ [\mathbf{x}_a\ \mathbf{x}_b\ \mathbf{x}] \tag{8}$$

which is applied to all the brackets in (4). An ellipse with four contact points can therefore be expressed in point and line coordinates as:

$$\mathbf{x}^T P(\sigma)\mathbf{x} = 0 \qquad \mathbf{u}^T Q(\sigma)\mathbf{u} = 0 \tag{9}$$

where P and Q are related to the point and line coordinates of the contact points according to (3) and (5).

Since line and point coordinates of a conic are related by $\mathbf{u} = P\mathbf{x}$ the relation $\mathbf{u}^T Q\mathbf{u} = 0$ can be written as $\mathbf{x}^T P^T Q P\mathbf{x} = 0$ from which we see that $P = P^T Q P$; that is, we have the important relation:

$$P(\sigma) = Q^{-1}(\sigma) \tag{10}$$

This equation relates point and line coordinates of contact points and will play an important part in the design of an algorithm for actually locating contact points.

The cross ratio computed from the contact points is invariant and can be used to identify corresponding ellipses in projective transform pairs. In the general case, for a given shape the class of ellipses with four contact points will be infinite. It is, however, a one-parameter infinite family as can be seen from the following crude equation-counting argument. Suppose that the curve can be parameterized with a parameter s. The contact points are then represented by parameters s_a, s_b, s_c, and s_d. Together with the five parameters for representing the ellipse, this gives us a total of nine parameters to be determined. These parameters are subject to the constraint that the ellipse should be in contact with the shape at the four points. This gives four constraints for coincidence of points and four constraints for coincidence of tangents, making up a total of eight constraints. Nine parameters and eight constraints implies that the class of four contact point ellipses will be a one-parameter family. The important consequence of this is that in the generic case there will only be a finite number of ellipses with a *specific* cross ratio; that is, we will have at most a finite ambiguity when identifying ellipses in projective transform pairs. By choosing ellipses with unit cross ratio = 1, we get an even further reduction of the ambiguity.

3 Iterative Algorithm for Ellipse Fitting

The algorithm for finding four and five contact works iteratively, starting with an initial ellipse and updating it. It is designed to have a fixed point for ellipses with four contact points and unit cross ratio, and to break whenever five contact points are encountered. There are two main steps in the algorithm.

1. Given an ellipse, find four points on the curve as candidates for contact points
2. Given four points on the curve, find an ellipse as a candidate for four contact point ellipse.

The first step is based on constructing a four sided polygon the sides of which are tangents to the curve and scaled versions of the ellipse. The second step uses the pencils $P(\sigma)$ and $Q(\sigma)$ of ellipses constructed from the four tangent points and tangent lines. A unique ellipse is found by minimizing the criterion: $||P(\sigma)\ Q(\sigma) - I||^2$, with P and Q properly normalized. For a four contact point ellipse this was shown to be identically 0 eq. 10 For a full description of the algorithm see [5], [6].

4 Experimental Results: Conic Pair Invariants

4.1 Edge Detection and Ellipse Fitting

In order to test the performance of viewpoint invariant recognition, images of 2 planar objects were acquired in two different viewpoint. The edges of the objects were found with a standard edge detector and the linked edge coordinates were used to compute ellipses with 4 contact points and unit cross ratio or 5 contact points. The initial ellipses for the iterative algorithm were in this case chosen as circles, regularly spaced over the object with distances 20 pixels apart.

The result of edge detection and ellipse shape decomposition for each object is shown in the figures 1 and 2. Although edge detection on real world images will seldom result in perfect contours, this is not critical for the ellipse fitting algorithm to work. The linked edge sequences only have to be long enough in order that the tangent can be computed using the point of local extrema of algebraic distance and the matrix of the ellipse. Important to note is however that if breaks occur in the edge sequences at contact points of unit cross ratio ellipses, this will of course affect the outcome. Another serious problem is the presence of spurious edges, leading to unwanted fittings. An example of this can be seen in the second view of object A in the upper right part of the edge picture. Since there is a threshold in determining whether an ellipse has converged to unit cross ratio and since different initial ellipses will have different trajectories of convergence, there is a slight spread of converged ellipses in some cases. The threshold for deciding unit cross ratio was in these examples ± 0.01.

4.2 Invariants From Pairs of Ellipses

In order to compute projective invariants we have to use at least two ellipses. Give two coplanar conics described by the symmetric matrices P_1 ad P_2 normalized so that:

$$det(P_1) = det(P_2) = 1 \tag{11}$$

it is possible to define two projective invariants [7]:

$$I_1 = Trace(P_1^{-1}P_2) \quad I_2 = Trace(P_2^{-1}P_1) \tag{12}$$

In order to test the stability and robustness of the conic pair invariants, they were computed from ellipses from both views in the figures 1 and 2, and the result is shown in these figures too. The axis here represent the signed logarithms of the absolute values of

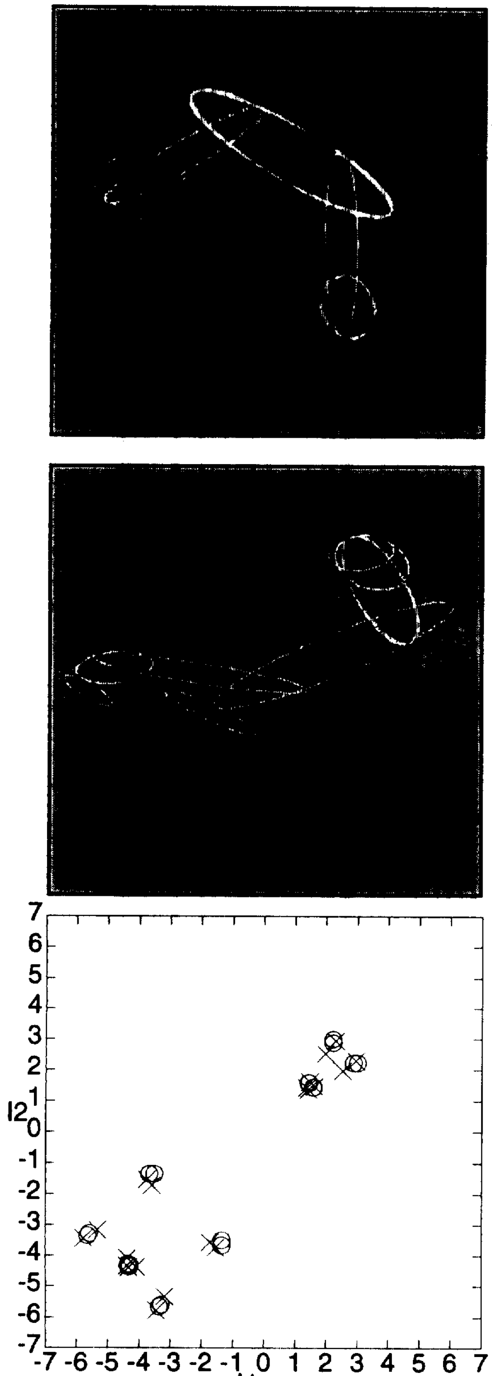

Figure 1: Invariants I1, I2 computed for view 1 (circles) and view 2 (crosses) of object A.

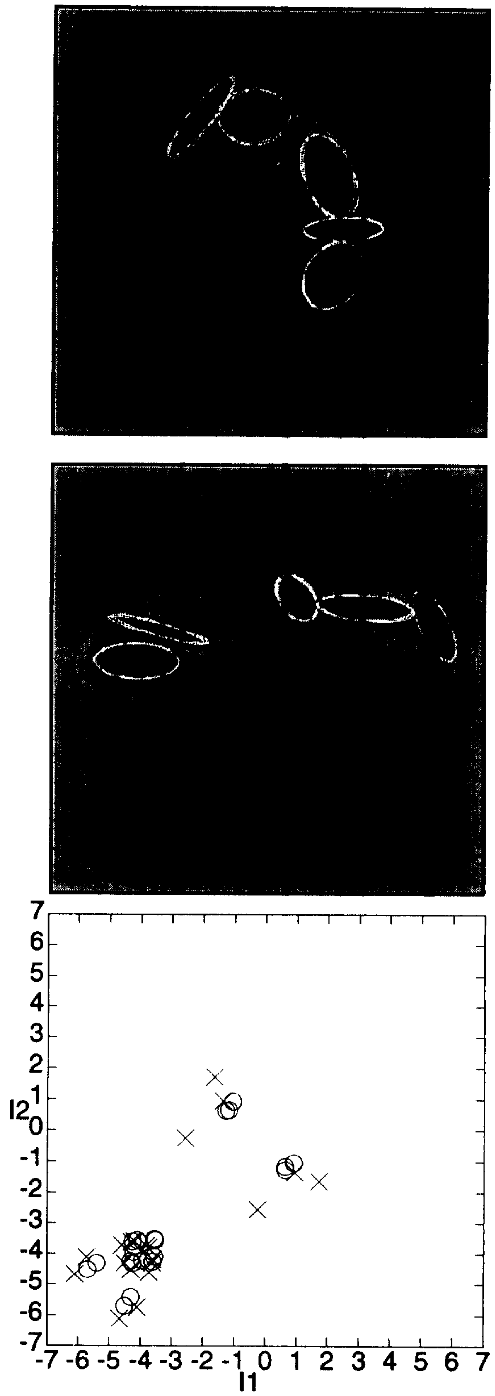

Figure 2: Invariants I1, I2 computed for view 1 (circles) and view 2 (crosses) of object B.

the invariants I_1, I_2 In order to reduce the effects of the spread in ellipses that converge to the same four contact points, a simple clustering of ellipses was performed so that all ellipses that differ in center with less than 5 pixels were grouped together. Only ellipses that were judged to have a counterpart in the opposite viewpoint image were selected in this figure. Ideally ,therefore for a certain object the values of the invariants from the two viewpoints should coincide. As can be seen from figs. 1 and 2. this is most often the case for all objects with a few notable exceptions. Reasons for deviation of invariants in different viewpoints are imperfect edge detection, discrete contour point representation and termination criteria of the ellipse extraction algorithm. The most important reason for deviation can be seen if we study the edge images of objects A and B. Note that since the objects have a height of 2-3 mm, edges will be obtained both from the top and bottom part off the objects. Due to occlusion and imperfect edge detections, ellipses will i some cases match to contact points that project both from the top and bottom part. These ellipses will not be perfectly coplanar with the object surface which of course will affect the invariants.

By using the invariants extracted from the images in figs. 1 and 2 as indexes into a hash table similar to that in [9] very few (1 - 5) false hypothesis were generated from a library of 100 random models [5]. The threshold hypothesis generation was set just in order for the correct model hypothesis to be generated. Although more exact comparisons with other invariant indexing systems based on lower level geometric features remains to be established, it seems that using invariant ellipses would lead to a more robust recognition system due to its more global properties. The negative side compared to using simpler features is the increase in complexity of the feature extraction process.

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