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A Complete Linear 4-Point Algorithm for Camera Pose Determination[1)]

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Abstract. Camera pose estimation is the problem of determining the position and orientation of an internally calibrated camera from known 3D reference points and their images. We briefly survey several existing methods for pose estimation, then introduce our new improved linear algorithm. The advantage of the algorithm is that it is easy to implement and stable. We give the experimental results.

Key Words: Calibration, Camera Pose Estimation, Polynomial solving, Prolongation, SVD, Symbolic-Numeric Completion Algorithm, Partial differential Equation, Projected Elimination Test.

1. Introduction

Given a set of correspondence between 3D reference points and their images, *pose estimation* consists of determining the position and orientation of the camera with respect to the known reference points. It is also called *space resection* in the photogrammetry community. It is a classical and common problem in the computer vision and photogrammetry and has been studied in the past. It concerns many important fields, such as computer vision [3], automation, image analysis and automated cartography [2] and robotics [1], etc.

With three points, the problem generally has four possible solutions. Fischer and Boles give a biquadratic polynomial in one unknown. Haralick et al.[4] review many old and new variants of the basic 3-point method and carefully examine their numerical stability under different order of substitution and elimination. Gao et al.[6] use Wu-Ritt's zero decomposition method to obtain a complete solution classification. Three-point algorithms intrinsically give multiple solutions. If a unique solution is required, additional information must be given, a fourth point generally suffices. But there are certain degenerate cases for which no unique solution is possible. These *critical configurations* are known precisely and include the following notable degenerate case: a 3D line and a circle in an orthogonal plane touching the line. Many linear algorithms have been presented for finding the unique solution[9]. Horaud et al. [3] obtain a fourth degree polynomial equation and prove that the problem has at most four possible solutions. In [9], Quan and Lan give a special linear algorithm which finds the unique solution for the general case. In [17], the authors try to present an algorithm

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to solve the problem including the critical configuration, but the relative error and failure rate(backward error) are significantly higher than one would like. In paper[10], the authors give the triangular decomposition and find a close form solution for this problem. In this paper, we give a linear and unique-solution algorithm which has been implemented. From the discussion to the algorithm and experiments, we obtain the following results:

1. In general, our algorithm give a linear and unique solution. From the experiments we also know that the solution is stable and the algorithm is robust.

2. For the critical configuration, we firstly obtain the solution number, and then find the linear and unique solution according to the requirement.

3. From the discussion to the algorithm, it is obvious that the algorithm also adapts to the case $n > 4$.

The rest of the paper is organized as follows. In section 2, we introduce the basic geometry of the camera pose. In section 3, we present the algorithm and use an example to show the algorithm. In section 4, we give the simulated experimental results. Finally, section 5 summarizes the contributions and gives some conclusions.

2. Geometry of camera pose from four points

In the following part, we will mainly discuss the case: $n = 4$ which be called *4-point pose estimation problem*. Let P be the calibrated camera center, and A, B, C, D the control points. Let $p = 2\cos\angle(BPC), q = 2\cos\angle(APC), r = 2\cos\angle(APB), s = 2\cos\angle(CPE), t = 2\angle(APE), u = 2\cos\angle(BPE)$. (Figure 1).

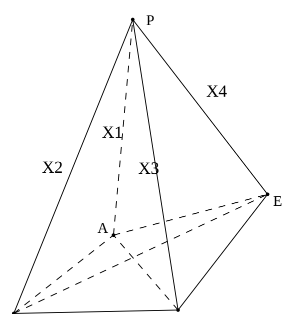

Fig. 1. The 4-point pose estimation problem

From triangles PAB,PAC,PBC,PAE,PBE and PCE, we obtain the $4-$*point pose estimation equation system*:

$$\begin{cases} X_1^2 + X_2^2 - X_1X_2r - |AB|^2 = 0 \\ X_1^2 + X_3^2 - X_1X_3q - |AC|^2 = 0 \\ X_2^2 + X_3^2 - X_2X_3p - |BC|^2 = 0 \\ X_1^2 + X_4^2 - X_1X_4s - |AE|^2 = 0 \\ X_4^2 + X_3^2 - X_3X_4t - |CE|^2 = 0 \\ X_2^2 + X_4^2 - X_2X_4u - |BE|^2 = 0 \end{cases} \tag{1}$$

In this method, the recovered camera-point distances X_i, are used to estimate the coordinates of the 3D reference points in a camera-centered 3D frame: $\mathbf{\bar{P}_i = X_i K^{-1} u_i}$. The final step is the absolute orientation determination[14]. The determination of the translation and the scale follow immediately from the estimation of the rotation.

3. Linear algorithms for pose determination from 4 points

3.1. Symbolic-Numeric completion of polynomial systems

Consider a general polynomial system $PS = \{p_1(x_1, \ldots, x_n) = 0, \ldots, p_m(x_1, \ldots, x_n) = 0\}$, we can form its coefficient matrix M_0 w.r.t the monomial vector

$$M_0 \cdot \begin{pmatrix} x_1^q \\ x_1^{q-1} x_2 \\ \vdots \\ x_n^2 \\ x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

Suppose that $[\xi_1, \xi_2, \ldots, \xi_n]$ is one of the solutions of the polynomial system, then

$$[\xi_1^q, \xi_1^{q-1}\xi_2, \ldots, \xi_n^2, \xi_1, \ldots, \xi_n, 1]$$

is a null vector of the coefficient matrix. Since the number of monomials is usually bigger than the number of polynomials, the dimension of the null space is much bigger than one. We need to include additional polynomials belong to the ideal generated by PS. There are many ways to include new polynomials. For example, in [17], they suggest to compute the null space of the coefficient matrix corresponding to $\{x_i p_j,\ 1 \le i \le n,\ 1 \le j \le m\}$. If the null space is of dimension 1, then the solution of PS can be easily recovered from the null vector. Otherwise, their method fails to retrieve the roots from the null vectors. Although they point out that for the *4-point pose estimation problem*, the enlarged matrix generically should has one dimension null space, they also show that the failure rate of their method is significantly higher than one would like in near singular cases.

In the following context, we show how the method based on symbolic-numeric completion algorithms [24] [11] [25] can be used to solve polynomial systems without limit on the dimension of null space. Under the bijection

$$\phi : x_i \leftrightarrow \frac{\partial}{\partial x_i}, \quad 1 \le i \le n,$$

the system PS is equivalent to linear partial differential equations denoted as R. The linear partial differential equations can also be written in the matrix form:

$$M_0 \cdot \begin{pmatrix} \frac{\partial^q u}{\partial x_1^q} \\ \frac{\partial^q u}{\partial x_1^{q-1} x_2} \\ \vdots \\ \frac{\partial^2 u}{\partial x_n^2} \\ \frac{\partial u}{\partial x_1} \\ \vdots \\ \frac{\partial u}{\partial x_n} \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

One prolongation(differentiation) of the system R yields DR and a constant matrix $M_1 = \begin{pmatrix} * & * \\ \mathbf{0} & M_0 \end{pmatrix}$ satisfies

$$M_1 \cdot \begin{pmatrix} \frac{\partial^{q+1} u}{\partial x_1^{q+1}} \\ \frac{\partial^{q+1} u}{\partial x_1^q x_2} \\ \vdots \\ \frac{\partial^3 u}{\partial x_n^3} \\ \frac{\partial^2 u}{\partial x_1^2} \\ \vdots \\ \frac{\partial u}{\partial x_n} \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

For the polynomial system PS, the prolongation equivalents to add degree one monomial multiplies each equation of system, i.e, add $\{x_1p_1, x_1p_2, \ldots, x_np_m\}$ to PS. The monomial vectors are updated to $[x_1^{q+1}, x_1^q x_2, \ldots, x_n^3, x_1^2, \ldots, x_n, 1]^T$. Successive prolongations $D^2R, \ldots$ corresponds to enlarged constant matrix systems $M_2, \ldots, etc.$

A single geometric projection is defined as

$$E(R) := \{(x, u, \underset{1}{u}, \ldots, \underset{q}{u} \in J^q : R(x, u, \underset{1}{u}, \ldots, \underset{q}{u}, \underset{q+1}{u}) = 0\}$$

The projection operator E maps a point in J^{q+1}(the jet spaces of order $q+1$) to one in J^q by simply removing the jet variables of order $q+1$. Corresponding to polynomial system, the projection means eliminating the monomials of the highest degree $q+1$. Gröbner basis algorithms [21] or Ritt-Wu's characteristic algorithms [23] can be used to carry out the symbolic projection. But due to the instability of symbolic elimination method for polynomial system with inexact coefficients, in [25], we proposed a numeric projection operator $\hat{E}$. First, a singular value decomposition [13] of M_k is computed to find numeric rank of M_k and a basis for its null space. Then, the components corresponding to the highest order derivatives are deleted, which yields a projected basis. This projected basis generates an approximate spanning set for $\hat{E}(D^kR)$. Application of the singular value decomposition to the spanning set will yield the approximate null space of $\hat{E}(D^k R)$. The system $R = 0$ is said to be symbolically involutive [12] at order k and projected order l, if $\hat{E}^l(D^k(R))$ satisfies the projected elimination test

$$\dim \hat{E}^l(D^k R) = \dim \hat{E}^{l+1}(D^{k+1} R)$$

and Symbol $E^l(D^k R)$ is involutive(see [24][11] [12] for detail on Symbol matrix). By the famous Cartan-Kuranishi Theorem [18] [19], after application of a finite number of prolongations and projections, we can tell whether any system of partial differential algebraic equations is involutive or inconsistent. The involutive systems are locally solvable and contain all their integrability conditions. They allow an existence and uniqueness theorem for local analytic solutions of the original system R by Cartan-Kähler Theorem [20]. Corresponding to polynomial system PS, if R is involutive at order k and l, then the dimension of $\hat{E}^l(D^k R)$ tells us the number of solutions of PS and all these solutions form the null space of $\hat{E}^l(D^k R)$. We can compute eigenvalues and eigenvectors to find these solutions. For the details, see the following example.

3.2. An example for pose estimation from 4 points

Consider the following example which corresponding to the third singular case as pointed in [17], where the coordinate of the camera point is $(1, 1, 1)$, and the coordinates of the four control points are $(-1, 1, 0)$, $(-1, -1, 0)$, $(1, -1, 0)$ and $(1, 1, 0)$ respectively. The detailed *4-point pose estimation equation system* is:

$$\begin{aligned}
p_1 &= x_1^2 + x_2^2 - 1.490711985 x_1 x_2 - 4. \\
p_2 &= x_1^2 + x_3^2 - .4000000002 x_1 x_3 - 7.999999996 \\
p_3 &= x_1^2 + x_4^2 - .8944271903 x_1 x_4 - 4. \\
p_4 &= x_2^2 + x_3^2 - 1.490711985 x_2 x_3 - 4. \\
p_5 &= x_2^2 + x_4^2 - .6666666680 x_2 x_4 - 7.999999996 \\
p_6 &= x_3^2 + x_4^2 - .8944271903 x_3 x_4 - 4
\end{aligned} \tag{2}$$

For the four variables x_1, x_2, x_3, x_4,we show how our symbolic-numeric method can be used to find the solutions of the polynomial system (2).

Under the bijection $\phi : x_i \leftrightarrow \frac{\partial}{\partial x_i}$ for $i = 1, 2, 3, 4$, the system is equivalent to linear partial differential equations denoted as R.

$$\begin{aligned}
p_1\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right) u &= \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_2}^2} - 1.490711985 \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2} - 4.u \\
p_2\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right) u &= \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_3}^2} - .4000000002 \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_3} - 7.999999996 u \\
p_3\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right) u &= \frac{\partial^2 u}{\partial {x_1}^2} + \frac{\partial^2 u}{\partial {x_4}^2} - .8944271903 \frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_4} - 4.u \\
p_4\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right) u &= \frac{\partial^2 u}{\partial {x_2}^2} + \frac{\partial^2 u}{\partial {x_3}^2} - 1.490711985 \frac{\partial u}{\partial x_2}\frac{\partial u}{\partial x_3} - 4.u
\end{aligned}$$

$$p_5\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4}\right)u=\frac{\partial^2 u}{\partial {x_2}^2}+\frac{\partial^2 u}{\partial {x_4}^2}-.666666680\frac{\partial u}{\partial x_2}\frac{\partial u}{\partial x_4}-7.999999996u$$
$$p_6\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3},\frac{\partial}{\partial x_4}\right)u=\frac{\partial^2 u}{\partial {x_3}^2}+\frac{\partial^2 u}{\partial {x_4}^2}-.8944271903\frac{\partial u}{\partial x_3}\frac{\partial u}{\partial x_4}-4.u \qquad (3)$$

Apply the symbolic prolongation and numerical projection to the partial differential equations R with tolerance 10^{-9}, we obtain the dimensions of null spaces and ranks of the symbol matrix:

$$\mathrm{DimMtx}=\begin{bmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ 1 & 5 & 9 & * & * & * & * \\ 1 & 3 & 7 & 7 & * & * & * \\ 1 & 3 & 4 & \underline{\mathbf{4}} & 4 & * & * \\ 1 & 3 & 4 & 4 & 4 & 4 & * \\ 1 & 3 & 4 & 4 & 4 & 4 & 4 \end{bmatrix} \quad \mathrm{RankSymbolMtx}=\begin{bmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ 0 & 0 & 6 & * & * & * & * \\ 0 & 1 & 6 & 20 & * & * & * \\ 0 & 1 & 9 & 20 & 35 & * & * \\ 0 & 1 & 9 & 20 & 35 & 56 & * \\ 0 & 1 & 9 & 20 & 35 & 56 & 84 \end{bmatrix}$$

From the table of dimensions and the table of ranks of the symbol matrix, we can tell that the system becomes involutive after twice prolongations and one projection, and the elimination test is satisfied as shown in bold in the DimMtx. From the dimension table, we can also see that sequenced prolongation and projections do not yield new relations. The dimensions of $D^2R, \Pi D^2R$, and $\Pi^2 D^2 R$ are all equal to 4 but the dimension of $\Pi^3 D^2 R$ is 3. Equivalently, this tells us the number of solutions of system (2) is 4, and the monomial bases should include the second degree monomials in order to recover the solutions of (2). In the following, we use eigenvalue algorithm to solve (2).

1. Compute an approximate basis of the null space of D^2R, denoted by a 4×70 matrix B. Since $\dim(D^2R)=\dim(\Pi D^2R)=\dim(\Pi^2D^2R)=4$, the 4×15 submatrix B_1 and 4×35 submatrix B_2 of B by deleting entries corresponding to the third and fourth degree monomials are bases of null spaces of Π^2D^2R and ΠD^2R.

2. For numerical stability, we choose 4 random linear combinations of monomials

$$Mons=[x_1^2,x_1x_2,x_1x_3,x_1x_4,x_2^2,x_2x_3,x_2x_4,x_3^2,x_3x_4,x_4^2,x_1,x_2,x_3,x_4,1]$$

as monomial bases, i.e, multiply these monomials by an 15×4 random matrix RM. we obtain the polynomial basis $Pbasis=\{pb_1,pb_2,pb_3,pb_4\}$:

$$\begin{aligned} pb_1 &= -3/10\,{x_1}^2+1/5\,x_1\,x_3+1/5\,x_1\,x_2-1/10\,x_1\,x_4-1/10\,{x_3}^2-2/5\,x_3\,x_2+1/2\,x_3\,x_4\\ &\quad -1/10\,{x_2}^2+1/2\,x_2\,x_4+1/2\,{x_4}^2+1/5\,x_1-1/5\,x_3-1/2\,x_4-1/2\\ pb_2 &= 1/5\,{x_1}^2+1/5\,x_1\,x_3+1/5\,x_1\,x_2+1/2\,x_1\,x_4-1/10\,{x_3}^2+1/10\,x_3\,x_2+2/5\,x_3\,x_4\\ &\quad -3/10\,{x_2}^2+2/5\,x_2\,x_4-2/5\,{x_4}^2-1/5\,x_1+1/5\,x_3+3/10\,x_4\\ pb_3 &= -1/2\,{x_1}^2+2/5\,x_1\,x_3+1/2\,x_1\,x_2+3/10\,x_1\,x_4+1/5\,{x_3}^2-1/10\,x_3\,x_2-1/10\,x_3\,x_4\\ &\quad -1/2\,{x_2}^2+1/2\,x_2\,x_4-1/5\,{x_4}^2-1/10\,x_1+3/10\,x_3-2/5\,x_2-1/5\,x_4+1/5\\ pb_4 &= -2/5\,{x_1}^2+1/5\,x_1\,x_3+1/10\,x_1\,x_2+2/5\,x_1\,x_4+3/10\,{x_3}^2-1/10\,x_3\,x_2+2/5\,x_3\,x_4\\ &\quad -2/5\,{x_2}^2-2/5\,x_2\,x_4+3/10\,x_1-3/10\,x_3+2/5\,x_2+1/10\,x_4+1/2 \end{aligned}$$

3. The multiplication matrix of x_1 with respect to $Pbasis$ can be formed as

$$\begin{aligned} M_{x_1} &= (B_1 \cdot RM)^{-1} B_{21} \cdot RM \\ &= \begin{bmatrix} -36.13728149 & -16.04635701 & -14.85556376 & -9.251955047 \\ -1.373880912 & 2.113219987 & -0.3083031459 & -0.7513204462 \\ 67.57537796 & 25.41489056 & 26.98954731 & 18.93657186 \\ 34.48769057 & 18.20260808 & 15.22293297 & 7.034514241 \end{bmatrix} \end{aligned}$$

Where B_{21} is the submatrix of B_2 with columns corresponding to monomials $x_1 \cdot Mons$:

$$B_{21} := \text{SubMatrix}(B2, 1..4, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 21, 22, 23, 24, 31]).$$

The eigenvalues are:

$$2.236068008, 2.236067977, -2.236067954, -2.236067977 \qquad (*)$$

It shows that two positive(negative) real eigenvalues are coincident up to digits 6. Form the multiplication tables w.r.t. x_2, x_3, x_4 independently, and compute the eigenvalues. The same phenomena as (*) can be observed. We choose one set of positive real eigenvalues

$$\begin{aligned} \xi_1 &= 2.236068008 \\ \xi_2 &= 3.000000001 \\ \xi_3 &= 2.236068014 \\ \xi_4 &= 1.000029429 \end{aligned}$$

Substitute the solution to (2), $|p_i(\xi_1, \xi_2, \xi_3, \xi_4)| < 10^{-6}$ for $i = 1, 2, \ldots, 6$.

If one substitutes the positive solution to the Jacobian matrix

$$\begin{bmatrix} \frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} & \frac{\partial p_1}{\partial x_3} & \frac{\partial p_1}{\partial x_4} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2} & \frac{\partial p_2}{\partial x_3} & \frac{\partial p_2}{\partial x_4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial p_6}{\partial x_1} & \frac{\partial p_6}{\partial x_2} & \frac{\partial p_6}{\partial x_3} & \frac{\partial p_6}{\partial x_4} \end{bmatrix}$$

then the singular values of the Jacobian matrix is

$$6.531956647, 6.196758256, 3.577682509, 0.00007595224342, 0, 0$$

The Jacobian matrix is near singular. This tells us that the solution is quite unstable for any small perturbation. But our method can deal with this singular case well. Suppose we perturb the (2) by errors of order 10^{-6}, the number of solutions read from the dimension table will generally become 2. The system becomes involutive after three prolongations and three projections.

$$\text{DimMtx} = \begin{bmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ 1 & 5 & 9 & * & * & * & * \\ 1 & 3 & 7 & 7 & * & * & * \\ 1 & 3 & 3 & 3 & 3 & * & * \\ 1 & 2 & \underline{\mathbf{2}} & 2 & 2 & 2 & * \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix} \quad \text{RankSymbolMtx} = \begin{bmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ 0 & 0 & 6 & * & * & * & * \\ 0 & 0 & 6 & 20 & * & * & * \\ 0 & 2 & 10 & 20 & 35 & * & * \\ 0 & 3 & 10 & 20 & 35 & 56 & * \\ 0 & 3 & 10 & 20 & 35 & 56 & 84 \end{bmatrix}$$

The computed positive root has backward error of order $10^{-5} \sim 10^{-8}$ in general.

For the same example, the linear algorithm in [17] fails since the null space of the 24×24 matrix has dimension 2 and their algorithm only deals with one dimension null space case. As we can see from the dimension table and Symbol rank table, the system becomes involutive after at least two prolongations. While the method in [17] only make use of one prolongation. That is the main reason we think that their method fails in some cases, especially for singular cases. On the other hand, the matrix we used to solve for the example is of order 70×90 which is quite bigger compared with the matrix in [17].

4. Experimental Results

We first demonstrate the accuracy and stability of the linear algorithm for the general case. Then, we also check the the linear algorithm at the *critical configurations*. The following experiments are done with Maple.

The first experiment is to show the stability of the algorithm about the number of solutions.

The optical center is located at the origin and the matrix of camera's intrinsic parameters is assumed to be the identity matrix. At each trial, four noncoplanar control points are generated at random within a cube centered at $(0, 0, 50)$ and of dimension $60 \times 60 \times 60$. The orientation Euler angles of the camera are positioned randomly. The control points are projected onto an image plane using the camera pose and internal parameters. One hundred trials are carried out and 100 sets of control points are generated for each trial. For each set of control points, two results are computed: one with the original control points; the other with the control points perturbed by random noises in certain level. In trial i, let n_i be the number of the control points such that the two results are the same and let $\frac{\|n_i - n\|}{n}$ (here $n = 100$) be the relative error. The following table 4. gives the relative errors w.r.t. varying noises. We observe that the computation is robust.

Noise Level	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
Relative error	0.01	0.03	0.03	0.04	0.06	0.05	0.05	0.06	0.07	0.07

Table 1

The second experiment is to show the accuracy and the stability of the algorithm about the solution for the general case. In this experiment, we test the *4-point pose estimation equations system 1* like the first experiment. For a set of solutions, we substitute them into (1) and check the backward error. Like the first experiment, we take 100 sets of control points randomly. From the experimental results, the maximal backward error is $0.3 * 10^{-9}$ for the *4-point pose estimation equation system.* For the stability of the algorithm about the solution, we carry out one hundred trials and generate 100 sets of control points for each trial. For each set of control points, two results are computed: one with the original control points X; the other with the control points perturbed by random noises in certain level $\tilde{X}$. In trial i, let $s_i = |\frac{\tilde{X}-X\|}{X}$ be the relative solution error and Fig. 3 gives the relative solution errors w.r.t. varying noises which support our statements. We also check the failure rate defined as the percentage of total trials where the absolute solution error is over 0.5(Figure 2).

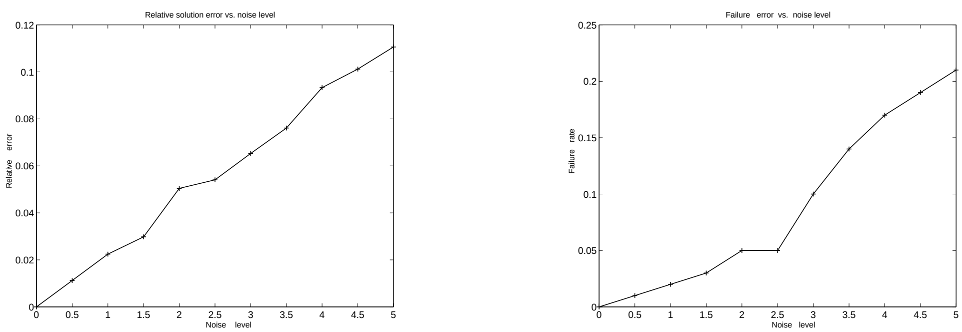

Fig. 2. Experimental results for the general case

The third experiment is to show the accuracy and the stability of the algorithm about the solution for the critical configuration. As mention in the introduction, the pose problem has some obtrusive singular cases which often cause troubles in computations. Most of algorithms such as in [9, 10] does not consider with the singular cases. The only algorithm we can find which deals with the singular case is [17], but the relative error and the failure rate(backward error) reported in the paper are significantly higher than one would like. Figure 3 shows the relative error and the failure rate for one such configuration using our symbolic-numeric linear method. The data is 4 coplanar points in a square $[-1, 1] \times [-1, 1]$ and the camera starts at position=0, at a singular point directly above their center($0.5 < h < 1.5$), where h is the hight of the camera. The camera then moves sideways parallel to one edge of the square. At position=$\sqrt{2}$ units it crosses the side of the vertical circular cylinder through the 4 data points, where another singularity occurs. From the following Figure 4, the relative error and especially the failure rate of the algorithm are significantly lower comparing with the former methods[17]. The failure rate defined as the percentage of total trials where the solution error was over 10^{-8}. The relative error and the failure rate also satisfied what one would like despite the error and failure rate at the position 0 and $\sqrt{2}$ are still a little higher than at other positions.

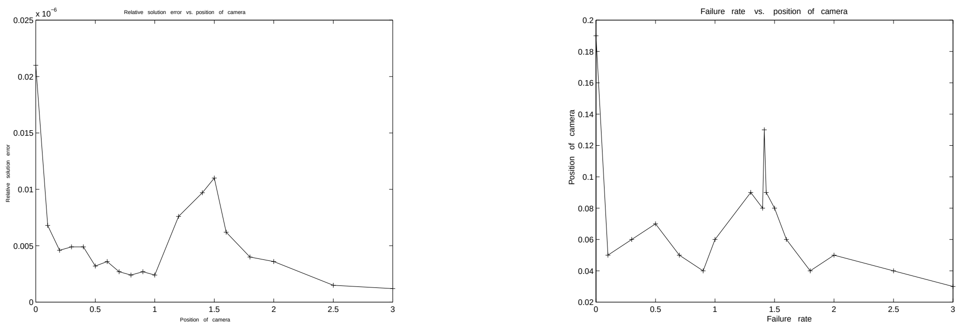

Fig. 3. Experimental results for the critical configuration case

5. Conclusion

In this paper, we presented a linear and robust algorithm to find the numeric solutions for pose estimation determination. The linear algorithm gives a linear and unique solution whenever the control points are not sitting on one of the known critical configuration. When the control points are sitting the some known critical configuration, the algorithm also can obtain the solutions and find the required solution. The simulated experiments also support our view to the algorithm. Compared with other algorithm, the main advantages of the linear algorithm are: it is more stable and for the some critical configuration, the algorithm still can find the solution. Finally, the method developed in this paper can easily be applied to other problem in computer vision with overconstrained system of polynomial equations.

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