Monocular Pose Determination from Lines: Critical Sets and Maximum Number of Solutions

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Abstract

In this paper we consider a subpart of the following problem. We assume that we have a set of known three-dimensional lines that we observe with a camera with an unknown pose and orientation. The problem is to recover the position and orientation of the camera from the observed image lines assuming that the correspondence has been established between the 2D and the 3D lines. Numerical methods have already been proposed for solving this problem but the question of the uniqueness of the solution has not yet been addressed. We show that there exist infinite sets of three-dimensional lines such that no matter how many lines we observe in these sets, the solution to the orientation or pose determination problem is not unique. We also give the maximum number of possible solutions. These results are important because they clearly define the domain of validity of algorithms which solve the orientation or pose determination problem.

1 Introduction

We study the problem of determining the pose of a camera from the observation of a set of three-dimensional lines. We assume that we know the position and orientation of the three-dimensional lines in some coordinate system and also the *correspondence* between these lines and the image lines (two-dimensional). The problem is to compute the position and orientation of the camera, its pose, in the same coordinate system. This problem has already been studied by several authors who have derived the algebraic equations relating the unknown pose parameters to those of the known 2D and 3D lines and proposed numerical methods for solving them [3, 5, 2, 11].

The question we address in this paper is the following: given a number of 2D-3D line correspondences, is it possible to recover *uniquely* the pose of the camera? we show the answer to this question to be negative if the 3D lines are in some fairly large sets that we analyse in detail. These *critical* sets of lines are those for which the previous algebraic equations have several solutions for the pose parameters *independently* of the number of observations. We also give the maximum number of solutions for the pose parameters if the observed lines belong to the critical sets.

The problem of determining camera position from three images of lines in unknown positions in space has a similar statement to this pose determination problem, but mathematically it is much harder. Recently it has been found that critical sets of three-dimensional lines also exist for this problem [1, 10]. In this paper, we show that the same is true for the pose determination problem from 3D-2D line correspondences.

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2 Preliminaries

Vectors are represented in bold face, i.e $\mathbf{x}$. Transposition of vectors and matrices is indicated by ^{t}, i.e $\mathbf{x}^t$. For a given three-dimensional vector $\mathbf{x}$ we also use $\tilde{\mathbf{x}}$ to represent the 3×3 antisymmetric matrix such that $\tilde{\mathbf{x}}\mathbf{y} = \mathbf{x} \wedge \mathbf{y}$ for all vectors $\mathbf{y}$.

We model our camera with the standard pinhole model and assume that *everything is referred to the camera standard coordinate frame* $(Cxyz)$. In this paper, we call a camera and its optical center by the same letter. Therefore, a camera with C_i as optical center is called camera C_i. A 3D line going through the two points M_1 and M_2 is denoted by $\langle M_1, M_2 \rangle$.

2.1 The *Plücker* line representation

The *Plücker* coordinates is the lines canonical representation in the projective space. There is an equivalent Plücker representation in euclidean geometry. We use this Plücker representation in which a line is represented by two vectors $\mathbf{l}$ and $\mathbf{N}$. $\mathbf{l}$ is the line unit direction vector. $\mathbf{N} = h\mathbf{n}$, where $\mathbf{n}$ is the unit normal vector to the plane defining by the line and the origin of the coordinate system, and $h = \| \mathbf{N} \|$ equals to the

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distance of the line from the origin. Any point $\mathbf{M}$ on the line then verifies the following equation:

$$\mathbf{M} \wedge \mathbf{l} = \mathbf{N} \tag{1}$$

A 3D line is parameterized by four parameters. The set of all space lines has dimension 4. Subvarieties of this set of lines are: *ruled surface (dim. 1), line congruence (dim. 2), and line complex (dim. 3).*

If a 3D line D is represented by its Plücker coordinates $(\mathbf{N}, \mathbf{l})$ in the camera coordinate system its image d can be represented by the vector $\mathbf{N}$. Usually we have only the image of the 3D line hence the image lines are generally represented by unit vectors $\mathbf{n}$.

From the above definitions we can easily draw the two following equations:

$$\mathbf{n}^t\mathbf{l} = 0, \quad \mathbf{M}^t\mathbf{n} = 0 \tag{2}$$

where $\mathbf{M}$ is an arbitrary point on D.

2.2 Camera Displacement

In this paper very often we have to represent the lines in different cameras coordinates. It is thus necessary to establish the relationship between the Plücker coordinates of a line in two different camera systems. Here, the quantities with subscripts 0 are expressed in the coordinate system of the camera before displacement. A camera displacement is defined by a rotation $\mathbf{R}$, followed by a translation $\mathbf{T}$. We take a 3D line D represented in the camera coordinate system by its Plücker coordinates $(\mathbf{N}_0, \mathbf{l}_0)$. Then there is the following relationship between the line coordinates in camera systems before and after the camera displacement [8] (see [4, 9] for the continuous approach):

$$\begin{bmatrix} \mathbf{l} \\ \mathbf{N} \end{bmatrix} = \mathbf{D} \begin{bmatrix} \mathbf{l}_0 \\ \mathbf{N}_0 \end{bmatrix} \tag{3}$$

where the matrix $\mathbf{D}$ is defined as follows:

$$\mathbf{D} = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{E} & \mathbf{R} \end{bmatrix}$$

and $\mathbf{E} = \tilde{\mathbf{T}}\mathbf{R}$.

Using the line motion equation (3), and after simple algebraic manipulations, we obtain:

$$\mathbf{N}_0 \wedge \mathbf{R}^t\mathbf{N} = -(\mathbf{N}^t\mathbf{T})\mathbf{l}_0 \tag{4}$$

If a line D is also observed by a third camera defined by the displacement $(\mathbf{R}', \mathbf{T}')$, let $\mathbf{N}', \mathbf{l}'$ be its Plücker coordinates in the third camera, $\mathbf{R} = [\mathbf{R}_1 \; \mathbf{R}_2 \; \mathbf{R}_3]$, and $\mathbf{R}' = [\mathbf{R}'_1 \; \mathbf{R}'_2 \; \mathbf{R}'_3]$. Then, when $\mathbf{N}^t\mathbf{T} \neq 0$ and $\mathbf{N}'^t\mathbf{T}' \neq 0$, equation (4) yields, see [6, 12]:

$$\tilde{\mathbf{N}}_0 \begin{bmatrix} \mathbf{N}^t\mathbf{E}_1\mathbf{N}' \\ \mathbf{N}^t\mathbf{E}_2\mathbf{N}' \\ \mathbf{N}^t\mathbf{E}_3\mathbf{N}' \end{bmatrix} = 0 \tag{5}$$

where $\mathbf{E}_i = \mathbf{R}_i\mathbf{T}'^t - \mathbf{T}\mathbf{R}'^t_i$, $i = 1..3$.

3 Pose determination from 2D to 3D line correspondences

The pose determination problem can be considered as estimating the three dimensional location and orientation of a camera from the image of a set of known landmarks, in our case 3D lines. We assume that the camera coordinate system is related to the space coordinate system, in which the 3D lines are defined, by a rotation $\mathbf{R}$ followed by a translation $\mathbf{T}$. Thus the problem is, given a set of correspondences between 3D lines D_i and their images d_i, $i = 1..N$, where N is the number of available line correspondences, to estimate the rotation matrix $\mathbf{R}$ and the translation vector $\mathbf{T}$.

3.1 The fundamental equations

The solution is obtained by using the equations (2) of section 2.1, and equation (3) of section 2.2. Equation (2), is written in the camera coordinate system. Its interpretation is that the camera optical center, the 3D line and its image are coplanar. Equation (3) tells us how to express the 3D line direction $\mathbf{l}_0$, in the camera coordinate system. Using these two equations we obtain the following:

$$\mathbf{n}^t\mathbf{R}\mathbf{l}_0 = 0 \tag{6}$$

This equation relates the observation, $\mathbf{n}$, the known direction of the observed 3D line, $\mathbf{l}_0$, and the unknown rotation matrix, $\mathbf{R}$.

A rotation can be defined by three parameters. Therefore equation (6) can be considered as a set of non-linear equations in three variables. Three 3D to 2D line correspondences are at least needed to find the solution. Many researchers tried to use this minimum number of correspondences to solve the problem. Dhome et al. [3] came up with a nice formulation of the problem. They decompose the camera (or object) rotation into three consecutive rotations of angles α, β and γ. Using particular world- and viewer-coordinate systems they fix the rotation angle γ, eliminate β in the remaining equations and succeed in obtaining an eighth degree equation in $\tan(\alpha/2)$. Lutton and Maitre [7] noticed that there is an obvious symmetry between the obtained results.

Once the rotation matrix is estimated using equation (6), the translation vector $\mathbf{T}$ can be obtained through the following linear equation:

$$\mathbf{n}^t(\mathbf{R}\mathbf{M}_0 + \mathbf{T}) = 0 \tag{7}$$

where $\mathbf{M}_0$ is an arbitrary point on the 3D line D.

When the rotation matrix has been obtained and we have the images $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$ of three 3D lines;

the translation vector $\mathbf{T}$ can be uniquely determined if the matrix $[\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{n}_3]$ is of rank 3, i.e. its determinant is non zero. This determinant is zero if the three image lines intersect in one point. This means that their corresponding 3D lines are parallel or that the optical center is located on a special quadric surface defined by those three 3D lines. In fact this surface is the locus of the lines (optical rays) intersecting these three 3D lines and this is by definition a *hyperboloid of one sheet* [8]. Usually more than three line correspondences are available, and equation (7) becomes overdetermined and is solved by for example the least-squares method.

Equations (6) and (7) are the fundamental equations of the pose determination problem from 2D to 3D line correspondences. They relate the image measurements, the 3D model, and the unknown pose parameters.

4 Existence conditions

Let us now suppose that we have three pairs of 3D to 2D line correspondences. Some necessary conditions have to be satisfied by the three 3D lines to enable us to compute a unique solution for the camera location, $\mathbf{T}$, and orientation, $\mathbf{R}$. Chen [2], considers the problem of line to plane correspondences and gives the existence conditions for equation (6).

Our problem can be considered as a particular case of line to plane correspondences in which all planes pass through the camera optical center and are represented by the vectors $\mathbf{n}_i$ normal to them. If three 3D lines with unit directions $\mathbf{l}_i$ ($i = 1..3$), and their corresponding planes with their respective normals $\mathbf{n}_i$ ($i = 1..3$) are given, a solution to the equation (6) exists unless some conditions described in [2] are satisfied. These conditions are not satisfactory for the pose determination problem for two main reasons:

a) They are given both for 3D lines and their images. The images are only the perspective projections of the 3D lines. Therefore it is much more interesting and useful to discuss only in terms of 3D lines configuration.
b) They are only necessary conditions which guarantee that there is not an infinite number of solutions, but they do not guarantee the uniqueness of the solution either.

In section 5, we introduce the critical set of lines Γ which defeats the equation (6).

Finally, in section 7, we introduce the critical set of lines Ψ for pose determination. These are the lines which yield at least two solutions for pose determination and that whatever equation or algorithm we use.

5 The critical set of lines for orientation determination

In this section, we introduce the critical set of lines Γ which defeat the equation (6). By this we mean that taking any number of lines belonging to Γ always yields more than one solution for the rotation matrix $\mathbf{R}$ satisfying the equation (6).

Let us consider two cameras C_0 and C_1 with different orientations (we do not care about translation since it does not appear in equation (6)). The orientation of C_1 is obtained from that of C_0 by applying a rotation defined by the rotation matrix $\mathbf{R}$ to it ($\mathbf{R} \neq \mathbf{I}_3$). We call Γ_R the set of lines D which yield two solutions $\mathbf{R}_{w0}$ and $\mathbf{R}_{w1}$ to the pose determination problem, where $\mathbf{R}_{w0}$ defines the orientation of the first camera, C_0, and $\mathbf{R}_{w1}$ the orientation of the second camera, C_1, in that coordinate system. Without loss of generality we express Γ_R in the first camera coordinate system. In this coordinate system the two solutions to the equation (6) are the identity matrix $\mathbf{I}_3$ and the rotation matrix $\mathbf{R}$.

5.1 Algebraic equation of the line complex Γ_R

Proposition 1 *When two distinct cameras*[1] C_0 *and* C_1 *are given, a 3D line* $D : (\mathbf{N}, \mathbf{l})$ *not going through* C_0, $\mathbf{N} \neq \mathbf{0}$, *expressed in the first camera coordinate, belongs to the critical set of lines* Γ_R, *iff it verifies the equation:*

$$\mathbf{N}^t\mathbf{R}\mathbf{l} = 0 \tag{8}$$

Proof. We represent the line D and its image in C_0 in the first camera coordinate system by $(\mathbf{N}, \mathbf{l})$ and $\mathbf{N}$ respectively, see section 2.1. Therefore, the identity matrix is a solution of equation (6). If the line D satisfies also the equation (8), then there are at least two solutions to equation (6), the identity matrix $\mathbf{I}_3$ and the rotation matrix $\mathbf{R}$. Any line passing through the optical center, $\mathbf{N} = \mathbf{0}$, is excluded because its image is reduced to a single point. Therefore any line $D \notin Star(C_0)$[2] verifying the equation (8) belongs to Γ_R. If we take now a line $D \in \Gamma_R$ by its definition it verifies the solution $\mathbf{R}$. The identity matrix $\mathbf{I}_3$ is also a solution to (8) because the image at C_0 is defined to be compatible with the lines in space. □

The equation (8) represents one constraint on the Plücker coordinates of the line D. Therefore it defines a line complex Γ_R. At each *general point* $\mathbf{M}$ there

[1] Here two cameras with different orientations are considered as two distinct cameras.

[2] We call $Star(C)$ the set of all the lines passing through the point C.

are infinitely many lines belonging to Γ_R. These lines form a quadratic cone χ_M of vertex $\mathbf{M}$.

A 3D line passing through the point $\mathbf{M}$ can be defined by its Plücker coordinates $(\mathbf{M} \wedge \mathbf{l}, \mathbf{l})$. Replacing its coordinates in the equation (8) we obtain:

$$\mathbf{l}^t \tilde{\mathbf{M}} \mathbf{R} \mathbf{l} = |\mathbf{l}\ \mathbf{M}\ \mathbf{R}\mathbf{l}| = 0 \qquad (9)$$

Therefore χ_M is a quadratic cone of vertex $\mathbf{M}$. Then, by definition, Γ_R is a line complex of degree 2. Note that the cones are translated versions of each other for all points of the line $\langle C_0, M\rangle$.

If the point $\mathbf{M}$ is on the rotation axis $\mathbf{r}$, corresponding to the rotation defined by $\mathbf{R}$, then the quadratic cone is reduced to the line $\langle C_0, M\rangle$ which belongs to $Star(C_0)$ and should therefore be eliminated. The way to see this is to decompose $\mathbf{l}$ as the sum $\mathbf{l}_r + \mathbf{l}_\perp$ of a vector parallel to $\mathbf{r}$ and a vector perpendicular to it. We then have

$$\mathbf{l} \wedge \mathbf{R}\mathbf{l} = \mathbf{l}_r \wedge \mathbf{R}\mathbf{l}_\perp + \mathbf{l}_\perp \wedge \mathbf{l}_r + \mathbf{l}_\perp \wedge \mathbf{R}\mathbf{l}_\perp$$

taking the inner product with $\mathbf{r}$ yields $\mathbf{r}^t(\mathbf{l}_\perp \wedge \mathbf{R}\mathbf{l}_\perp)$ which is zero if and only if $\mathbf{l}_\perp = \mathbf{0}$.

We now proceed to give a geometric description of Γ_R which helps building up our intuition.

5.2 Geometrical description of Γ_R

To give a geometrical description of Γ_R, we consider two different cases:.

a) $\mathbf{l}$ is not parallel to the axis of rotation $\mathbf{r}$,
b) $\mathbf{l}$ is parallel to the axis of rotation $\mathbf{r}$,

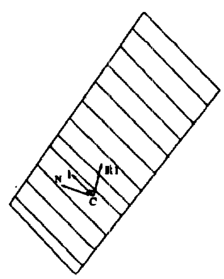

Figure 1: The set of lines Γ_l.

In the fist case, we have $\mathbf{N}^t\mathbf{l} = \mathbf{N}^t\mathbf{R}\mathbf{l} = 0$ and therefore $\mathbf{N}$ is parallel to $\mathbf{l} \wedge \mathbf{R}\mathbf{l}$. Any point $\mathbf{M}$ of the line D then verifies:

$$\mathbf{M}^t(\mathbf{l} \wedge \mathbf{R}\mathbf{l}) = 0$$

Therefore the lines $D \in \Gamma_R$ of direction $\mathbf{l}$ are all included in the plane going through C_0 and defined by the two vectors $\mathbf{l}$ and $\mathbf{R}\mathbf{l}$ as shown in figure 1. We call such a set of lines Γ_l. If we consider $\mathbf{l}$ as a point in the plane at infinity, Γ_l can be considered as a pencil of lines passing defined at this point and lying in the plane defined by $\mathbf{l}$ and $\mathbf{R}\mathbf{l}$[3]. The line complex Γ_R contains all the elements of the union of those sets of parallel lines Γ_l, defined on the star of plane passing through C_0, except those passing through C_0 which belong to $Star(C_0)$.

In the second case $\mathbf{l}$ is parallel to the axis of rotation $\mathbf{r}$ and therefore it is also parallel to $\mathbf{R}\mathbf{l}$ and the equation (6) is therefore always verified. The set of all the lines parallel to the vector $\mathbf{r}$ is a line congruence of order 1 and class 0. We call this line congruence Ψ_r. If we consider $\mathbf{r}$ as a point in the plane at infinity, Ψ_r is then a star of lines defined at this point.

We conclude that:

Proposition 2 *The critical line complex Γ_R can be defined as:*

$$\Gamma_R = \cup_{\mathbf{l} \neq \mathbf{r}} \Gamma_l \cup \Psi_r \setminus Star(C_0)$$

Note that the equations $\mathbf{N}^t\mathbf{l} = 0$ and $\mathbf{N}^t\mathbf{R}\mathbf{l} = 0$ are verified for all the lines $D \in \Gamma_R \cup Star(C_0)$.

In the next section we use this geometrical description of Γ_R to obtain the maximum number of solutions of the equation (6).

6 Maximum number of solutions for the orientation determination

Let us now suppose that for a given set of 3D lines and their corresponding image lines there are three solutions for camera orientation, equation (6). Once again we express everything in the first camera coordinates and assume that the three *distinct* solutions are $\mathbf{I}_3$, $\mathbf{R}$ and $\mathbf{R}'$ corresponding respectively to the orientations of three cameras C_0, C_1 and C_2. If there exists a set of lines which satisfy equation (6) for the three solutions they must belong to $\Gamma_R \cap \Gamma_{R'}$. Let us characterize this intersection. With the notations of the previous paragraph we have:

$$\Gamma_R = \cup_{\mathbf{l} \neq \mathbf{r}} \Gamma_l \cup \Psi_r \setminus Star(C_0) \qquad (10)$$

$$\Gamma_{R'} = \cup_{\mathbf{l} \neq \mathbf{r}'} \Gamma'_l \cup \Psi_{r'} \setminus Star(C_0) \qquad (11)$$

[3] In fact fixing $\mathbf{l}$ means choosing a point in the plane at infinity. We have shown in section 5.1 that the line complex Γ_R is quadratic, thus either $\mathbf{l}$ is the vertex of a quadric cone that splits into a plane pair, or every line through $\mathbf{l}$ is in the line complex (thanks to Steve Maybank who mentioned this point). In this case $\mathbf{l}$ is the vertex of a quadric cone that splits into a plane pair. One of the plane pair is defined by Γ_l and the other one is the plane at infinity. Any line in the plane at infinity satisfies (8) and belongs to Γ_R, but for the sake of clarity we do not discuss these lines in our Euclidean descriptions. This can be better done in terms of projective geometry and does not much influence our future conclusions.

Proposition 3 $\Gamma_R \cap \Gamma_{R'} = (\Gamma_{r'} \cup \Gamma'_r \cup \Psi_{rr'} \setminus \{\langle C_0, r\rangle, \langle C_0, r'\rangle\}$ *where $\Psi_{rr'}$ is defined in the proof.*

Proof. Equations (10) and (11) show that $\Gamma_{R'} \cap \Gamma_{R'}$ is the union four sets of lines from which we exclude $Star(C_0)$:

$$\Gamma_R \cap \Gamma_{R'} = (S_1 \cup S_2 \cup S_3 \cup S_4) \setminus Star(C_0)$$

where:

$$\begin{aligned} S_1 &= \Psi_r \cap \Psi_{r'} \\ S_2 &= \cup_{\mathbf{l}\neq\mathbf{r}}\Gamma_l \cap \cup_{\mathbf{l}\neq\mathbf{r}'}\Gamma'_l \\ S_3 &= \cup_{\mathbf{l}\neq\mathbf{r}}\Gamma_l \cap \Psi_{r'} \\ S_4 &= \cup_{\mathbf{l}\neq\mathbf{r}'}\Gamma'_l \cup \Psi_r \end{aligned}$$

As we have assumed that the rotations $\mathbf{R}$ and $\mathbf{R}'$ are distinct, $\mathbf{r}$ and $\mathbf{r}'$ are not parallel. Ψ_r and $\Psi_{r'}$ are defined as the set of the lines parallel respectively to $\mathbf{r}$ and $\mathbf{r}'$. Therefore S_1 is a empty set, $S_1 = \emptyset$.

For all $\mathbf{l}$, $\mathbf{l} \neq \mathbf{r}$ and $\mathbf{l} \neq \mathbf{r}'$, the lines belonging to Γ_l and Γ'_l are included in two planes both containing the optical center C_0. These two planes are either identical or their intersection is a line belonging to $Star(C_0)$. If the two planes are identical then we have $|\mathbf{l}\ \mathbf{R}\mathbf{l}\ \mathbf{R}'\mathbf{l}| = 0$, which defines a cubic curve. Each point of this cubic curve defines a line direction $\mathbf{l}$ and therefore a set of lines $\Gamma_l = \Gamma'_l$. This defines a line congruence $\Psi_{rr'}$. It has been shown [8] that $\Psi_{rr'}$ is of order 3 and class 3.

Therefore $S_2 = \Psi_{rr'} \cup Star(C_0) \setminus \{\langle C_0, r\rangle, \langle C_0, r'\rangle\}$.

From the definitions of Γ_l (resp. Γ'_l) and Ψ_r (resp. $\Psi_{r'}$) we can easily deduce that:

$$S_3 = \Gamma_{r'}, \quad and \quad S_4 = \Gamma'_r$$

see figure 2.

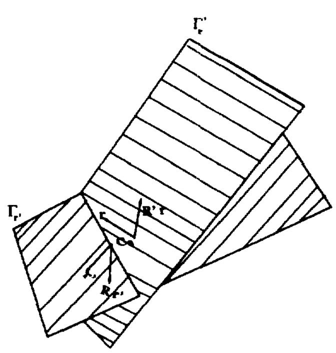

Figure 2: The regular surface $S(R, R')$.

Therefore, the set of lines $S(R, R')$ yielding three independent solutions $\mathbf{I}$, $\mathbf{R}$ and $\mathbf{R}'$ for the equation (6), is the union of $\Gamma_r \cup \Gamma_{r'}$ and $\Psi_{rr'}$.

If a fourth rotation matrix $\mathbf{R}''$ is also a solution to equation (6), then it should be included in the intersection of $S(R, R')$ and $S(R, R'')$. $S(R, R') \cap S(R, R'')$ is a ruled surface defined by the intersections of the two cubic curves, $|\mathbf{l}\ \mathbf{R}\mathbf{l}\ \mathbf{R}'\mathbf{l}| = 0$ and $|\mathbf{l}\ \mathbf{R}\mathbf{l}\ \mathbf{R}''\mathbf{l}| = 0$. That is in general 9 points $\mathbf{l}_i$, $i = 1..9$, which define a union of ruled surfaces $\Phi(\mathbf{R}, \mathbf{R}', \mathbf{R}'') = \cup_{l_1}^{l_9}\Gamma_{l_i}$.

Proposition 4 *Four is the maximum number of solutions to equation (6).*

This proposition falls directly from the previous results. In fact, if there exists a fifth solution $\mathbf{R}_5$, then the direction vectors $\mathbf{l}$ of the lines satisfying the five solutions have to be at the intersection of three cubic curves defined by $|\mathbf{l}\ \mathbf{R}\mathbf{l}\ \mathbf{R}'\mathbf{l}| = 0$, $|\mathbf{l}\ \mathbf{R}\mathbf{l}\ \mathbf{R}''\mathbf{l}| = 0$ and $|\mathbf{l}\ \mathbf{R}\mathbf{l}\ \mathbf{R}_5\mathbf{l}| = 0$. The intersection of three cubic curves, in general, is an empty set of points. Therefore the maximim number of solution to equation (6) is four. □

Note that the choice of the rotation matrix $\mathbf{R}$ or that of the independent triplets of rotation matrices $(\mathbf{R}, \mathbf{R}', \mathbf{R}'')$ in this section is quite arbitrary, and for any choice of $\mathbf{R}$ or $(\mathbf{R}, \mathbf{R}')$ or $(\mathbf{R}, \mathbf{R}', \mathbf{R}'')$ there exist associated line complexes, Γ_R, $\Gamma_{R'}$, and $\Gamma_{R''}$, and line congruences $\Psi_{rr'}$, and $\Psi_{rr''}$ and union of ruled surfaces $\Phi(\mathbf{R}, \mathbf{R}', \mathbf{R}'')$.

7 The critical set of lines for pose determination

Definition 1 *Let us consider two independent cameras (C_0) and (C_1). A 3D line D belongs to the critical set of lines Ψ for the pose determination problem, if this 3D line D has the same images on (C_0) and (C_1). It means that there are two solutions $(\mathbf{R}_1, \mathbf{T}_1)$ and $(\mathbf{R}_2, \mathbf{T}_2)$ to equations (6) and (7).*

We now characterize this critical set.

Proposition 5 *The critical set of lines for pose determination Ψ is a line congruence of order 1 and class 3.*

Proof. In fact it is easy to give a proof by simple geometrical reasoning. Consider the two stars of planes $Starp(C_0)$ and $Starp(C_1)$ (the sets of all planes going through C_0 and C_1, respectively). We call homologous planes those planes of $Starp(C_0)$ and $Star(C_1)$ which define the same image lines on these cameras. The line congruence Ψ is defined as the lines which are intersections of pairs of homologous planes. If the two cameras are parallel the homologous planes are parallel and intersect on the plane at infinity. Then the congruence is the set of all lines in the plane at infinity. If they are not parallel the intersection of two homologous plane is a line which by construction has the same images on the first and the second camera, see figure 3. Taking

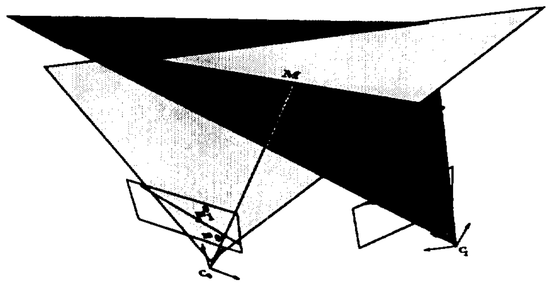

Figure 3: The intersection of two homologous plane has the same images in both cameras.

an arbitrary general point $\mathbf{M}$ in space, there is only one line $D \in \Psi$ which passes through this point and therefore Ψ is of order 1. The proof is very easy. Let $\mathbf{m}_0$ and $\mathbf{m}_1$ be respectively the images of the point $\mathbf{M}$ on the first and the second camera. The line $D \in \Psi$ passing through $\mathbf{M}$ has by definition the same images on both cameras, therefore the image line on the first camera lies not only on the optical ray $\langle C_0, m_0 \rangle$ but also on the optical ray $\langle C_0, m_1^0 \rangle$ the homologous ray of $\langle C_1, m_1 \rangle$ (m_1^0 is the pixel in C_0 with the same coordinates as m_1 in C_1). In the same way, the image line on the next camera lies on the optical rays $\langle C_1, m_1 \rangle$ and $\langle C_1, m_0^1 \rangle$ the homologous ray of $\langle C_0, m_0 \rangle$. The intersection of the two planes defined by these two pairs of optical rays belongs to Ψ and passes through the points $\mathbf{M}$, see figure 3. Note that as we take a general point we do not consider the case where $\mathbf{M}$ has the same images on C_0 and C_1. This special case is discussed later. Therefore the order of Ψ is 1.

As in the section 5, without loss of generality we express Ψ in the first camera coordinate system. In this coordinate system the two solutions to the equations (6) and (7) are $(\mathbf{I}_3, \mathbf{0})$, and $(\mathbf{R}, \mathbf{T})$ which defines the displacement of the camera from C_0 to C_1. The line $D \in \Psi$ passing through the point $\mathbf{M}$ can then be defined by its Plücker coordinates $(\mathbf{M} \wedge \mathbf{l}, \mathbf{l})$, where:

$$\mathbf{l} = (\mathbf{M} \wedge \mathbf{M}_2) \wedge \mathbf{R}^t(\mathbf{M} \wedge \mathbf{M}_2) \tag{12}$$

where $\mathbf{M}_2 = \mathbf{RM} + \mathbf{T}$.

If the line $(\mathbf{N}, \mathbf{l})$ belongs to a general plane Π defined by a vector $\mathbf{n}_\Pi$ orthogonal to it such that any point $X \in \Pi$ satisfies the following equation:

$$X^t \mathbf{n}_\Pi + 1 = 0$$

A line $D : (\mathbf{N}, \mathbf{l})$ belonging to the critical set of line for pose determination Ψ_P, has the same images in C_0 and C_1. This line is presented in the second camera coordinate system as: $(\mathbf{RN} + \mathbf{T} \wedge \mathbf{Rl}, \mathbf{l})$. Therefore if its image is the same as in C_0, we have:

$$\mathbf{N} \wedge (\mathbf{RN} + \mathbf{T} \wedge \mathbf{Rl}) = 0 \tag{13}$$

Then $\mathbf{l} = \mathbf{n}_\Pi \wedge \mathbf{N}$, and equation (13) can be written as

$$\mathbf{FN} = \lambda \mathbf{N}, \quad \lambda \in \Re$$

where $\mathbf{F} = \mathbf{R}^t + \tilde{\mathbf{T}}\mathbf{R}\tilde{\mathbf{n}}_\Pi$. Therefore it yields in general 3 solutions (real or complex) which are the three eigenvectors of the matrix $\mathbf{F}$. Therefore Ψ is of class 3. □

If the point $\mathbf{M}$ has the same images on both cameras (C_0) and (C_1) it satisfies the following equation:

$$\mathbf{M} \wedge (\mathbf{RM} + \mathbf{T}) = 0 \tag{14}$$

The points satisfying this equation lie on the intersection of two quadrics which is in general a space quartic Σ, a space curve of degree 4. Each point along this curve is the vertex of a quadratic cone of lines belonging to the critical set. This quadratic cone is defined by equation (9). Indeed, equation (6) can be written as:

$$\mathbf{N}^t\mathbf{Rl} = (\tilde{\mathbf{M}}\mathbf{l})^t\mathbf{Rl} = \mathbf{l}^t\tilde{\mathbf{M}}\mathbf{Rl} = 0$$

which defines the quadratic cone, and because of equation (14), equation (7) is satisfied. However as we have shown before, in general there is only one line of Ψ passing through each point in space.

8 Maximum number of solutions for the pose determination problem

From the above discussion about the critical set of line Ψ the first interesting conclusion we may draw is the maximum number of solutions. To exclude the simple and easily detected particular configuration of lines, such as coplanar or parallel lines, here we suppose that the given lines are not all in such configuration or at least a minimum number of three skew lines is given. In such general cases, we showed in the section 7 that if a set of lines yields two solutions for the pose determination problem then they are included in a line congruence Ψ.

The question is: "if a set of lines is included in a critical line set Ψ and therefore produce the same images on two cameras, for example (C_0) and (C_1), is there any other independent camera pose (C_2) which also yields the same image lines?". *We give a positive answer to this question and show that 3 is the maximal number of solutions to the pose determination problem when all observed lines belong to the critical set Ψ.*

Suppose that a line D yields two solutions $\mathbf{R}_{w0}$ and $\mathbf{R}_{w1}$ to the pose determination problem. Once again we express everything in the first camera coordinate system. In this coordinate system the line D is supposed to have the same image lines on C_0 and C_1 and yield two solutions $(\mathbf{I}, \mathbf{0})$ and $(\mathbf{R}, \mathbf{T})$ to the pose determination problem. If the line D has also the same image on a third camera C' independent from the other cameras, it satisfies also another solution $(\mathbf{R}', \mathbf{T}')$ for pose determination. It means that the line $(\mathbf{N}, \mathbf{l})$ has the same images on the three cameras. Therefore the equation (5) can be written in this particular case as:

$$\tilde{\mathbf{N}} \begin{bmatrix} \mathbf{N}^t\mathbf{E}_1\mathbf{N} \\ \mathbf{N}^t\mathbf{E}_2\mathbf{N} \\ \mathbf{N}^t\mathbf{E}_3\mathbf{N} \end{bmatrix} = 0 \qquad (15)$$

where $\mathbf{E}_i$, $i = 1..3$, are defined as in equation (5).

We have shown [10] that the above equation yiels 7 solutions and there are only 6 of them which satisfy the constraints of general positions of the three cameras[4].

For each solution $\mathbf{N}_i$, $i = 1..6$, of equation (15), the corresponding lines lie by definition of the Plücker coordinates in the plane Π passing by the origin C_0 and defined by the normal vector $\mathbf{N}_i$. In this plane the set of lines Π_i which satisfy equation (8) are parallel to $\mathbf{N}_i \wedge \mathbf{R}\mathbf{N}_i$. These lines are the only lines in this plane which have the same images in the three cameras C_0, C_1 and C_2.

Therefore the set of lines which provide the above three solutions for pose determination problem is $\Pi(R, R') = \cup_{i=1}^{6} \Pi_i$.

Proposition 6 *Three is the maximum number of solutions for pose determination problem, i.e. satisfying both equations (6) and (7).*

This proposition falls directly from the previous results. If there exists a set of lines $D"$: $(\mathbf{N}", \mathbf{l}")$ yielding a fourth solution $(\mathbf{R}", \mathbf{T}")$ other than the previous solutions the the vector $\mathbf{N}"$ has to be a commun solutions of equation (15) and a similar equation written for the first, second and the fourth cameras. These two sets of equations have in general no common roots. Therefore the maximum number of solutions for pose determination problem is *three*.

9 Conclusion

In this paper we introduced the critical set of lines for camera orientation and pose determination problem. We used it to obtain the maximum number of solutions in each case. It has been shown that the maximum number of solutions for orientation determination is 4. That is in agreement with the four independent solutions we can obtain from the eighth degree equation obtained by Dhome et al [3] in the case of orientation determination using 3 lines. We have also shown that 3 is the maximum number of solutions for the pose determination problem. These are some of our preliminary theoretical results. It is interesting to see what happens when 3D lines do not belong but are not far from belonging to a critical set. A complete analysis of the relationship between the sets of critical lines and the stability of the algorithms is an important task that is worth considering.

[4] The seventh solution corresponds to $\mathbf{N}_7 = \mathbf{T} \wedge \mathbf{T}'$ which implies the constraints $|\mathbf{N}_7 \ \mathbf{R}^t\mathbf{N}_7 \ \mathbf{R}'^t\mathbf{N}_7|$ on our cameras configuration.

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