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Projective Invariants for Vision

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Abstract

This paper studies the invariant theory for its applications to computer vision. In the first part of the paper, three methods of invariant calculation are presented:

1. the infinitesimal method which gives a precise enumeration of the maximal number of independent invariants existing in a given problem;

2. the generalization and constraint method;

3. the use of canonical invariants associated with some transformations, which is, in our opinion, the method to be used primarily, for all problems where only simple elements (points, straight lines, conics) are involved.

In the last part, invariants are applied to 3D vision problems: 3D invariant computation from multiple images in investigated in detail in the case of uncalibrated cameras. Experimental results are given on artificial data: this allows to study the robustness of invariants to increasing levels of noise. Some results are also given on real data to show that 3D invariants are usable in practice.

1 Introduction

Invariant theory is a very classical mathematical theory. Some of its results date from antiquity and the major contributions are from the last century. The first use of this theory in computer vision was for the development of robust pattern recognition systems widely used in industry [BdF86, Hu62]. These algorithms needed good conditions of luminosity and contrast (black objects on white backgrounds for example), only one totally visible object per image and the use of invariants like the ratio of the surface of a plane object to the square of its perimeter (such a ratio is invariant under similarity).

More recent works involve the use of local invariants which are associated with small topological features. These invariants may be angles and length ratios associated with two concurrent lines [GM92], affine coordinates associated with four points [LSW90, Wol90], cross ratios associated with 5 lines or with 2 conics [RZFM92], differential quantities computed at every point of a curve [Wei88, Wei91] or a combination of algebraic and differential quantities [RZFM92, GKO91, GMPO92]. Thus it is now possible to find metrical and quantitative tokens in images like distances, length ratios or cross ratios, tokens which are natural complements for topological and qualitative ones used in many algorithms of vision: graph isomorphisms, research of facets, polygons, patterns...

There are two major fields of work within the new considerations on invariants in computer vision:

- precise study of the invariants: clarification of the invariant theory, robustness and numerical stability of the invariants, computation of new invariants associated with curves or with 3D configurations of points, lines and conics [GQ91];

- practical use of these invariants to solve vision problems; an interest clearly shown by Mohr *et al.* [MM91, MMG91], Forsyth *et al.* [FMZ90] and Kapur *et al.* [KM91].

More generally, one will find a state of the art approach in [ZM91, MZ92] for both mathematical aspects and applications. These applications concern difficult basic problems of vision and seem to be a new profitable way to solve them. As invariants are numerical quantities associated with topological features, they can be used to index models or images in a large data base and to recognize an object represented on an image using this data base [Wol90, RZFM92]. Given a model of the transformation induceded by a camera, invariants allow robust measurements to be made in images. These measures are useful for point matching [GM92], relative positioning of 3D features with respect to reference points and predicting the position of points in a new image when their projection is known in other images [MMIQ91, MM91, MMG91], shape reconstruction [KvD89] and epipolar geometry reconstruction (see section 5 and [Fau92]). They can be integrated in 2D or 3D models and give additional constraints during the verification stage of a recognition system [FMZ$^+$91]. These examples of applications clearly show the interest of using invariants in computer vision.

The first part of this paper deals with basic theoretical considerations. Given a geometrical configuration α of points, lines and curves, an invariant is a numerical quantity $f(\alpha)$ computed from this configuration, which remains the same when the configuration is transformed by a geometric function. If Φ is such a function, the invariant function should verify $f(\alpha) = f(\Phi(\alpha))$.

In this context, a configuration α may be indexed in a data-base by the value $f(\alpha)$. Given another configuration β, it is potentially the same as α if $f(\beta) = f(\alpha)$. This is a necessary condition. If the configuration is composed of sub-configurations $\alpha = (\alpha_1, \ldots, \alpha_n)$, an invariant function may be used as prediction function. If one knows $\alpha_1, \ldots, \alpha_{n-1}$ and $f(\alpha)$, the value of α_n may be computed if f is invertible.

The nature of the invariants existing in a problem mainly depends on the geometric function set. Angles and length ratios are the basic invariants associated with similarities, length ratios of collinear points play the same role with affine transformations, as do the cross ratio of collinear points or concurrent lines with collineations.

The main problem is to find the invariant functions existing in a given problem. It is clear that if f is an invariant function, $g \circ f$ is another one for any real function g. How many independent invariants exist for a given problem and how to find them are the two main questions. This problem is addressed in section 2. This section provides an analytical method to compute invariants. Section 3 gives another method of computation which is less general but easier to handle.

The second part of this paper studies a particular invariant: the cross ratio. Knowing that cross ratios are the basic invariants of projective geometry leads therefore to discover where they are hidden in the image configuration. Sections 4 and 5 show that such an approach is powerful and above all much easier to use than the systematic methods. Section 4 gives basic properties of the cross ratio. Section 5 deals with more advanced topics, like the computation of 3D invariants from two 2D images. It provides invariants for many configurations of 3D lines, conics and points. Section 6 provides experimental results on both artificial and real data and especially shows the robustness of 3D invariants with respect to image noise. Experiments are conducted for testing coplanarity and testing the discrimination capacities of invariants with noisy data. Experiments on real data prove the validity of the simulation results.

2 Invariant theory

This section introduces the invariant theory: after a few examples, a general formulation of the problem is given which leads to an invariant computation method. This method provides a way of counting the number of functionally independent invariants of a given problem. Finally, the case where the transformation set is not a group is discussed.

2.1 Introduction to invariants

2.1.1 A first example

Suppose you have a few geometric plane images called configurations, each of them containing five points. You have a camera and take several photographs of these images from different view points. Is there a method to know which image is represented by each photo?

The aim of invariant theory is to find numerical quantities that characterize each image and its photos. If the camera performs an affine projection of the image onto the photo, angles and lengths between points vary; but the affine coordinates of two of the points with respect to the basis constituted by three others are invariant. If the camera is not affine but is a true perspective projection, affine coordinates are no longer invariant and we have to use cross ratios.

These invariants allow you to sort the photos unless one image can be deduced from another one by an affine or projective transformation.

2.1.2 A second example

Consider a plane as the configuration set. Each configuration is a point. The transformations are the rotations around the origin. The questions to be answered are:

- what configurations can you reach if you apply all the possible transformations to a given configuration C? This configuration set is called the *orbit* of C.

- Is a configuration reachable from another one, i.e. are two configurations on the same orbit?

- How many orbits are there? Can they be characterized by functions invariant on each of them, the values of which vary from one orbit to another?

In this case the answers are very simple: each point may reach every other point on a circle centered at the origin, and a characterization of these circles is their radius (see fig. 1).
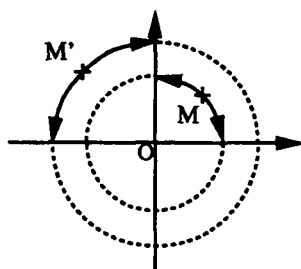

Figure 1: Case of points in a plane and rotations with centers at the origin

2.1.3 General formulation

Let E be a set called the space of configurations and let G be a set of transformations on that space E: the elements of G are functions from E into E.

Consider a particular configuration $x \in E$, to which we apply different transformations of G (cf. figure 2).

We would like to answer these questions:

- can we reach all the configurations of E from the configuration x?

- if not, can we characterize the set of configurations reachable from a given configuration?
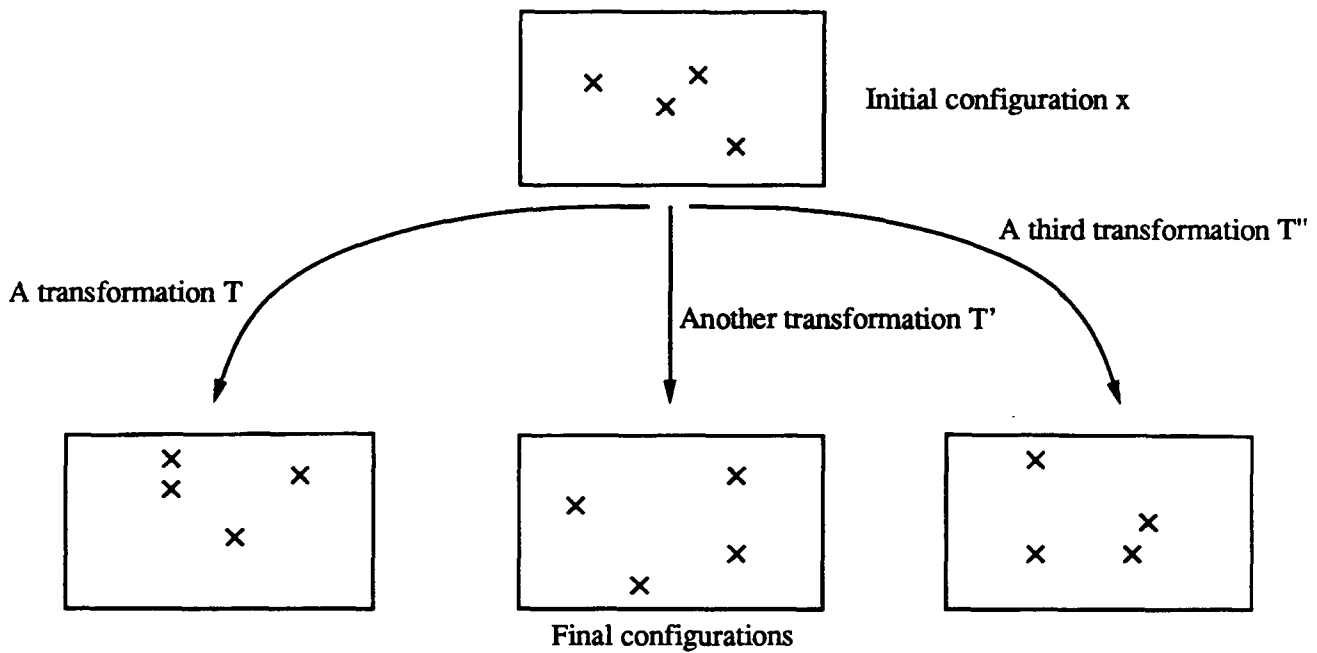
3

Figure 2: Invariant problem

- and if such a characterization is possible, how many independent parameters are needed?

These parameters are called invariants for the set of transformations G. They may be expressed as functions from the configuration space into the set of real numbers $I\!R$. In this paper, we will not consider the case of differential invariants, i.e. when derivatives of curves or surfaces are taken into account. Such a case may lead to an infinity of invariants as the derivatives of a curve can be taken an infinite number of times or at an infinite number of points.

2.1.4 Two more examples

1. Let an affine plane be the set of configurations and let the translations be the transformations. In this case, any point of the plane can reach any other point (see figure 3). Thus there is no invariant for this problem.
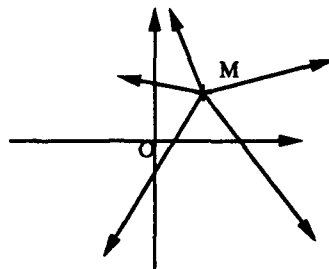

Figure 3: Case of points in a plane and translations

2. Consider all pairs of points (thus their four coordinates form a set isomorphic to $I\!R^4$) and the isometries (i.e. translations, rotations, and orthogonal symmetries) as the transformations. The situation can be represented either in the plane, as it is in figure 4, or in a four-dimension space.

4

All pairs cannot be reached: there exists an invariant which is the distance between the two points M and N.
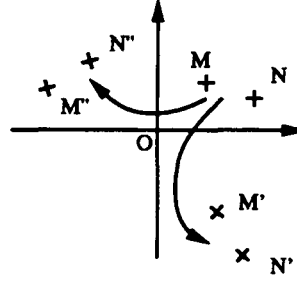
Figure 4: Case of points in a plane and isometries

2.2 Infinitesimal method of invariants calculation

Two theoretical methods of new invariant calculation are presented by Forsyth et al. in [FMZ$^+$91]. The first one, called the infinitesimal method, requires the concept of Lie groups. It is presented here, because it allows us to count the number of invariants existing in a given problem. The second one, called the symbolic method, is based on formal manipulations. As it does not provide more theoretical results than the infinitesimal method, we do not study it in this paper.

2.2.1 Groups acting on a set

First hypothesis. As we would like to analytically calculate the invariants of a given problem, we make a few hypotheses. The first one is that the set G is a **group** acting on E.

Group definition. Elements of G are functions from E into itself, and G can be provided with the function composition operation, which will be denoted either multiplicatively or by the symbol ∘. The axioms which define a group can be expressed as follows:

- composition is associative;

$$\forall (g, g', g'') \in G^3 \quad g \circ (g' \circ g'') = (g \circ g') \circ g'' \tag{1}$$

- there exists an identity element;

$$\exists e \in G \quad \forall g \in G \quad e \circ g = g \circ e = g \tag{2}$$

If there exists a surjective element in G, or , more precisely, if:

$$\forall x \in E \quad \exists g \in G \quad \exists y \in E \quad x = g(y)$$

then the neutral element of G is the identity function of E. This is the most frequent case, and we will suppose from now on that we are in this case.

- each element admits an inverse:

$$\forall g \in G \quad \exists g' \in G \quad g \circ g' = g' \circ g = e \tag{3}$$

This inverse is unique and will be denoted g^{-1}.

Intuitively, a group acting on a set E is a set of functions from E to E which has the three basic properties of all groups (1), (2) and (3): the set contains the identity function and every function of the set is invertible. In the example of paragraph 2.1.2 the group is that of rotations around the origin. This group acts on any affine plane.

Definition and properties of orbits. The orbit of an element is the set of all the images of this element under the transformation group of functions. They can be mathematically defined as follows. Relation $\exists g \in G, \; y = g(x)$ is an equivalence relation on E. The equivalence class of $x \in E$ is called the orbit of x according to G. We will denote it $O(x)$ or Gx.

$$O(x) = \{x' \in E \mid \exists g \in G \;\; x' = g(x)\}$$

The set of orbits of elements of E is a partition of E, and the following properties are deduced from the properties (1), (2) and (3):

$$\begin{cases} \forall(x,x') \in E^2 & x' \in O(x) & \implies & O(x) = O(x') \\ \forall(x,x') \in E^2 & x' \notin O(x) & \implies & O(x) \cap O(x') = \emptyset \\ \forall x \in E & x \in O(x) \end{cases}$$

With these hypotheses, invariants are functions whose value is constant on each orbit but varies across different orbits. In the example of paragraph 2.1.2, the orbits are circles centered at the plane origin. In the example of paragraph 2.1.4, the orbits are sets of pairs of points a given length distant.

2.2.2 Manifolds and Lie groups

New hypothesis. Since we want to calculate invariants analytically, we would like to have real parameters on the configuration set. Then we make a new hypothesis: let space E be a **manifold** of $\mathbb{R}^n$, and let G be a **Lie group**, which acts analytically on E.

Intuitively, a manifold is a set which admits a local regular parameterization at each one of its points. This definition has to be more precise to avoid degenerate cases. In an affine 3D space, most curves and surfaces are manifolds insofar as they do not contain singular points. A Lie group is a group which has a parameterization adapted to its group structure. More formal definitions follow.

A few words about manifolds. This paragraph is due to [Dem87]. Let F be a vector space of finite dimension n, V be a subset of F and a be a point of V.

Non-degenerate system of local equations. A set of functions $\phi_1, \ldots, \phi_d$ of class C^∞, defined in a neighborhood of a, form a *non-degenerate system of local equations* of V at a if it satisfies the following two conditions:

- there exists an open subset U of F with $a \in U$ on which the functions ϕ_i are defined, which verifies:
$$V \cap U = \{x \in U | \phi_1(x) = 0, \ldots, \phi_d(x) = 0\}$$

- the linear forms $d\phi_i(a)$ are linearly independent.

The number d is uniquely defined. It is called the *codimension* of V at a. We call the number $m = n - d$ *dimension* of V at a and we denote it $\dim_a V$.

For example, the unit circle in an affine 3D space admits the set $\{f_1, f_2\}$ as nondegenerate system of local equations with:

$$f_1\left(M \left| \begin{array}{c} x \\ y \\ z \end{array} \right.\right) = x^2 + y^2 - 1 \quad \text{and} \quad f_2\left(M \left| \begin{array}{c} x \\ y \\ z \end{array} \right.\right) = z$$

Thus the dimension of this circle is 1 in each one of its points. Similarly a regular surface in the 3D space has a dimension 2.

6

Manifold. The subset V is a manifold of F at a if it has a non-degenerate system of local equations at a; V is a *manifold* of F, if it is a manifold at each point of F, or if there exists a family of open subsets of F, whose union is V, such as V has a non-degenerate system of equations on each subset of this family.

If V is connected, V has the same dimension at each point. More generally, we define the dimension of V as:

$$\dim V = \sup_{a \in V}(\dim_a V)$$

Saying that V is a manifold of F at a is equivalent to saying that there exists an open subset U of F with $a \in U$ and a C^∞-diffeomorphism f from U into an open subset of $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^d$ such as $f(U \cap V) = f(U) \cap (\{0\} \times \mathbb{R}^m)$. In this situation, f provides a local system of m curvilinear coordinates, which we call group parameters.

Definition of Lie groups. A *Lie group* is a set which is at the same time a manifold of $\mathbb{R}^n$ and a group such that multiplication and inversion are continuous.

The *order* of a Lie group is its dimension as a manifold. For simplicity, we denote it $\dim G$. Here are a few examples:

- the order of the group of planar rotations whose center is the origin is 1, the parameter or curvilinear parameter of a rotation being its angle.

- the order of the planar displacement group is 3; each element of this group can decomposed into a rotation around the origin followed by a translation; the rotation has its angle as parameter; the translation has the coordinates of the translation vector as parameters;

- the order of the one-to-one affine planar transformation group, also called affine group, is 6. An element of this group is defined by the matrix of its linear part which is an invertible 2×2 matrix, and by the two coordinates of a translation vector;

- the order of the 3D displacement group is 6: 2 parameters to define the axis direction, 2 for the axis position, 1 for the rotation angle and 1 for the translation vector along the axis; this is the representation of a screw motion: a 3D rotation followed by a translation along the axis of rotation.

- the order of the one-to-one 3D affine transformation group is 12 (such a function is defined by an invertible 3×3 matrix and by a translation vector); the order of the planar collineation group is 8 (a 3×3 matrix defined up to a scale factor) and that of the 3D collineation group is 15 (a 4×4 matrix defined up to a scale factor).

As a manifold, a Lie group owns local systems of *coordinates*. We will suppose that all the coordinates of the identity element of the group are equal to zero.

The fact that G is a group acting analytically on E can be written as follows: let $(x_1, \ldots, x_n)$ be the coordinates of an element $x \in E$ and $(a_1, \ldots, a_m)$ be the parameters of an element $g \in G$, thus the coordinates $(x'_1, \ldots, x'_n)$ of the image of x by g verify:

$$\forall h \in \{1, \ldots, n\} \quad x'_h = \phi_h(x_1, \ldots, x_n; a_1, \ldots, a_m) \tag{4}$$

where the functions ϕ_h are analytical, and verify, according to our identity hypothesis:

$$\forall h \in \{1, \ldots, n\} \quad x_h = \phi_h(x_1, \ldots, x_n; 0, \ldots, 0)$$

The **orbits** are the surfaces defined by the previous equations (4) where the coordinates x_i are constant and the parameters a_j vary (see an example below).

7

2.2.3 Invariant computation

We search for invariants as analytical functions which are constant on orbits and can discriminate between orbits, that is, functions $f_1, \ldots, f_p$ that verify:

$$\begin{cases} \forall x \in E & \forall y \in O(x) & \forall j \in \{1, \ldots, p\} & f_j(x) = f_j(y) \\ \forall x \in E & \forall y \notin O(x) & \exists j \in \{1, \ldots, p\} & f_j(x) \neq f_j(y) \end{cases} \tag{5}$$

As the functions f_j are constant on every orbit, their gradient is orthogonal to the tangent space at every point of the orbit. Let f_j be an invariant function and let $(V_i(x))$ be some vectors forming a basis of the tangent sub-space to the orbit of x at x. With $\vec{\nabla} f_j = \overrightarrow{grad} f_j$, we have:

$$\forall j \quad \forall x \quad \forall i \quad \vec{\nabla} f_j . V_i(x) = 0 \tag{6}$$

There are still two problems to be solved: the effective calculation of the vectors $(V_i(x))_i$ and the resolution of a system of partial differential equations to obtain stable solutions.

A simple example. Consider again the example of paragraph 2.1.2. The configurations are the planar points and the transformations are the rotations whose center is the origin. The points are parametrized by their Cartesian coordinates (x_1, x_2) and the rotations by their angle θ. In this situation, equations (4) become:

$$\begin{cases} x'_1 = \phi_1(x_1, x_2; \theta) = x_1 \cos\theta - x_2 \sin\theta \\ x'_2 = \phi_2(x_1, x_2; \theta) = x_1 \sin\theta + x_2 \cos\theta \end{cases}$$

The orbits are circles centered at the origin. Their equation is:

$$\begin{aligned} x_1 &= \phi_1(a, b; t) = a\cos t - b\sin t \\ x_2 &= \phi_2(a, b; t) = a\sin t + b\cos t \end{aligned}$$

where a and b are fixed parameters and t varies.

The tangent subspace to the orbit of the point of coordinates (x_1, x_2) at this point is a straight line whose a director vector is $(-x_2, x_1)$. Then, we obtain the following partial differential equation:

$$\forall (x_1, x_2) \in \mathbb{R}^2 \quad -x_2 \frac{\partial f}{\partial x_1} + x_1 \frac{\partial f}{\partial x_2} = 0 \tag{7}$$

Solving equation (7) leads to the invariant: $f(x_1, x_2) = x_1^2 + x_2^2$.

Case of non-connected groups. Some groups are not connected. For example, the groups of 2D or 3D isometries have two connected components: the displacements and the antidisplacements; the groups of 2D or 3D one to one transformations also have two connected components: the transformations of strictly positive determinant, and those of strictly negative determinant. Finding a continuous real function, the determinant in the preceding cases, whose image is not an interval, is enough to prove non-connectedness.

In this situation, the orbits may be non-connected. The resolution of the established equations gives functions which are invariant on each connected component of the orbits. Then, we should search which are the components of each orbit. We will neglect this difficulty in the remainder of the paper.

2.2.4 Computation of the tangent subspace to an orbit

Before solving the equations (6), we have to compute the vectors $(V_i(x))$. As Forsyth et al. write it in their article [FMZ90], such a computation can be done by the general means of the concepts of Lie algebra and the exponential function. But, in our case, the introduction of these delicate concepts is not strictly necessary. There exists a derivation method whose only (small) drawback is that it provides vectors that are not independent from the group parameterization, which is however sufficient for our purpose.

8

Derivation method We have a group parameterization, and we can describe a group action by an equation similar to equation (4) of page 7. We derive these equations at the point whose coordinates are zero.

$$V_i(x) = \left(\left. \frac{\partial x'_h}{\partial a_i} \right|_{a_i=0} \right)_{h \in \{1,\dots,n\}}$$

$$= \left(\left. \frac{\partial \phi_h(x_1,\dots,x_n;0,\dots,0,a_i,0,\dots,0)}{\partial a_i} \right|_{a_i=0} \right)_{h \in \{1,\dots,n\}}$$

The vectors obtained by this method may not be independent (see Paragraph 2.3), but they generate the subspace tangent to the orbit. Another system of vectors generating the tangent subspace provides equivalent equations. Any system of vectors which is a basis of the tangent subspace gives a minimal number of equations.

The vectors obtained depend on the group parameterization. On the other hand, another local parameterization of a neighborhood of the identity element of the group gives equivalent results. When a group does not have a simple global parameterization, like the group of invertible square matrices of given size, one has to remember that a local parameterization is enough: for example, the nonsingular general linear group is an open set and we may use the parameterization of all the matrices, which is a local parameterization of the group in a neighborhood of the identity element.

2.3 Invariant enumeration

2.3.1 Interest of such an enumeration

First, we must remark that, if $f_1,\dots,f_p$ are invariant functions, every function $\Phi(f_1,\dots,f_p)$ where Φ is any non constant function, is also invariant. This is not due to the infinitesimal method, but this is intrinsic to the invariants. In this situation, the number of invariants is not the number of invariant functions existing for a given problem, since this number is infinite, but the maximal number of functionally independent invariant functions. Any functions $f_1,\dots,f_p$ are functionally independent at a point a if their differential forms $df_1(a),\dots,df_p(a)$ are linearly independent.

There is a related problem [Dix90]: does there exist a set of functions which generate all other invariant functions and how many functions are there in this set? This problem is more difficult and we will not answer it in this paper.

2.3.2 Enumeration

The number of invariants is related to the orbit dimension. If $f_1,\dots,f_p$ is a system of p functionally independent invariant functions, every orbit of dimension $n-p$, where n is the dimension of E, has a non-degenerate system of local equations of the form $f_1-r_1,\dots,f_p-r_p$, where $r_1,\dots,r_p$ are real numbers. The number of invariants for a given problem is equal to the minimum of the orbit codimensions; it gives the number of parameters necessary to characterize an orbit. If in a problem, we have only k invariant functions, when the theory predicts a greater number of them, we know that we will not be able to distinguish every orbit from the others.

Even in simple problems, all orbits do not have the same dimension. In the case of the planar rotation of given center, for example, the dimension of every orbit is 1, except that of the orbit of the rotation center which is 0. In such a case, 1 is the dimension to be considered.

More mathematically, the first part of the demonstration is easy. Let $f_1,\dots,f_p$ be p functionally independent invariant functions verifying the equations (5). Let a be any configuration. If we denote $r_i = f_i(a)$, we have:

$$\forall x \in O(a) \quad f_i(x) - r_i = f_i(x) - f_i(a) = 0$$

9

As the functions f_i are independent:

$$\dim O(a) \leq n - p \quad \text{and} \quad p \leq n - \dim O(a)$$

The converse proposition is more difficult. Its final formulation is: $p = n - \dim(O)$ where O is a generic orbit for the problem. See [For69, FM82] for more details.

2.3.3 Isotropy group of an element

In a first approximation, the dimension of the transformation group and the dimension of the orbits should be the same, except in a few cases. In such a case $p = n - \dim(G)$. However some transformations may sometimes leave some configurations invariant; they compose the isotropy group of those configurations. The dimension of the orbit of an element is equal to the dimension of the group minus the dimension of the isotropy group of this element.

Intuitively, the isotropy group is the subgroup of the transformations which have no action on a configuration. Thus the dimension of the effective transformation set for this configuration and the orbit dimension become smaller as the isotropy group dimension increases. An example of an isotropy group is shown in Paragraph 4.2.

Let x be any element of E. We call the isotropy group of x according to G, denoted G_x, the set of elements of G which leave x invariant.

$$G_x = \{g \in G \mid g(x) = x\}$$

G_x is a subgroup of G. First, it is not empty, because the identity element of G, which is the identity of E, is in G_x. Second, let g and h be two elements of G_x. If we have $h(x) = x$, then $x = h^{-1}(x)$ and:

$$x = g(x) = g \circ h^{-1}(x)$$

$g \circ h^{-1}$ belongs to G_x, which consequently is a subgroup.

The dimension $n - p$ of the orbit of a point x is $\dim G - \dim(G_x)$, and the number of invariants is:

$$\dim E - \left(\dim G - \min_{x \in E} \dim(G_x) \right) \tag{8}$$

Remark. The dimension of the orbit of a point has a formal similarity with the dimension of a quotient group: to find the dimension of an orbit, one has to consider the elements of G, modulo those of G_x. This is only a non mathematical analogy, since a quotient group would imply that G_x is distinguished, which we cannot suppose.

2.4 A few examples

- Let us consider the planar rotations whose center is the origin. We take the rotation angle as parameter for this group. In this situation, there is one invariant: the dimension of the transformation group is 1, the dimension of the configuration group is 2, and the isotropy group of an element of the plane different from the origin contains only the identity of E: its dimension is 0. The group action may be denoted by:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

We differentiate that formula with respect to θ:

$$\frac{\partial}{\partial\theta} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -x\sin\theta - y\cos\theta \\ x\cos\theta - y\sin\theta \end{pmatrix}$$

10

We evaluate this equation for $\theta = 0$ and we obtain $\frac{\partial x'}{\partial \theta} = -y$ and $\frac{\partial y'}{\partial \theta} = x$. The equation that an invariant function must verify is:

$$-y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0$$

This partial differential equation has as solutions the function $f(x,y) = x^2 + y^2$ and all the functions $G \circ f$ where g is any scalar function.

- We take the points of a 3D space as configurations and the translations of vector $(a, b, 0)$ with $a, b \in \mathbb{R}$ as transformations. The orbits are the planes parallel to the xy plane, therefore their dimension is 2, and there is one invariant. The group action may be denoted as follows:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x + a \\ y + b \\ z \end{pmatrix}$$

The derivation of this expression with respect to a and b shows that the vectors $(1, 0, 0)$ and $(0, 1, 0)$ generate the tangent plane to the orbit of a point. An invariant function has to verify:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

The solutions are the compound functions of $f(x, y, z) = z$.

- Let us consider again the points of a 3D space, but compel the translation to be parallel to the first axis of coordinates. There are two independent invariant functions which are solutions of the equation $\partial f / \partial x = 0$. We may take as functionally independent solutions $f(x, y, z) = y$ and $f(x, y, z) = z$, or $f(x, y, z) = y + z$ and $f(x, y, z) = yz$.

2.5   When the transformations are not a group

In many real problems, the transformations do not form a group, because their domain and codomain are not the same. This is the case of affine or projective 3D to 2D projections for example. There are two main differences with what we have already seen. First, orbits do not form a partition of the codomain, since they may cross each other. Second, all the points of an orbit are not equivalent, because you cannot pass from one to another with a transformation.

Burns et al. even demonstrate [BWR92] the following result: in the case of sets of n points, for any integer n, and of projective, weak or orthographic projections, the orbits cross each other in such a way that a function constant on each orbit is constant on the whole codomain. Therefore, there is no invariant for this problem. However, any additional information about the relative positions of the points would make the existence of an invariant possible.

In the case of groups, the theory is well established and date, for its main results, from the last century (by Desargues, Descartes, La Hire, Möbius, Laguerre, Cayley, Poncelet, Chasles, Pascal, Von Staudt, Klein...). On the other hand, there are few works on the case where the transformations only form a manifold but not a group. Among the precursors, two famous names are Poincaré, with his work on the qualitative study of differential equations and Hilbert.

In this general case, invariants are no longer functions which enable one to distinguish every orbit from the others, but only constant functions on the orbits.

Derivation method.   The fact that a function is constant on an orbit may be expressed in a differential way. We suppose that the group action is still expressed by analytical formulas like equation (4) of page 7. Let n, n' and m be the dimensions of the domain E, the codomain F, and

11

the transformation space G respectively. An element of G is a map from E into F. An invariant is a function $f : F \mapsto I\!R$ which satisfies:

$$\forall a_1, \ldots, a_m, a'_1, \ldots, a'_m \qquad f(\phi^1(x_1, \ldots, x_n; a_1, \ldots, a_m), \ldots, \phi^{n'}(x_1, \ldots, x_n; a_1, \ldots, a_m))$$
$$= \quad f(\phi^1(x_1, \ldots, x_n; a'_1, \ldots, a'_m), \ldots, \phi^{n'}(x_1, \ldots, x_n; a'_1, \ldots, a'_m))$$

Let us denote:

$$\Phi(x_1, \ldots, x_n; a_1, \ldots, a_m) = f(\phi^1(x_1, \ldots, x_n; a_1, \ldots, a_m), \ldots, \phi^{n'}(x_1, \ldots, x_n; a_1, \ldots, a_m))$$

The differentiation of this last equation with respect to the coordinates of the transformations leads to a differential system which consists of the following equations:

$$\forall j \in \{1, \ldots, m\} \qquad \frac{\partial \Phi}{\partial a_j}(x_1, \ldots, x_n; a_1, \ldots, a_m) =$$
$$\sum_{i=1}^{n'} \left(\frac{\partial f}{\partial x_i}(\phi^1(x_1, \ldots, x_n; a_1, \ldots, a_m), \ldots, \phi^{n'}(x_1, \ldots, x_n; a_1, \ldots, a_m)) \right.$$
$$\left. \frac{\partial \phi^i}{\partial a_j}(x_1, \ldots, x_n; a_1, \ldots, a_m) \right) = 0$$

In these equations, the difficulty comes from the derivatives of f which are composed with the functions ϕ. That makes these equations more difficult to solve than those obtained in the case of groups.

This difficulty is illustrated by the figure 5. Let M be a configuration and M' its image under a
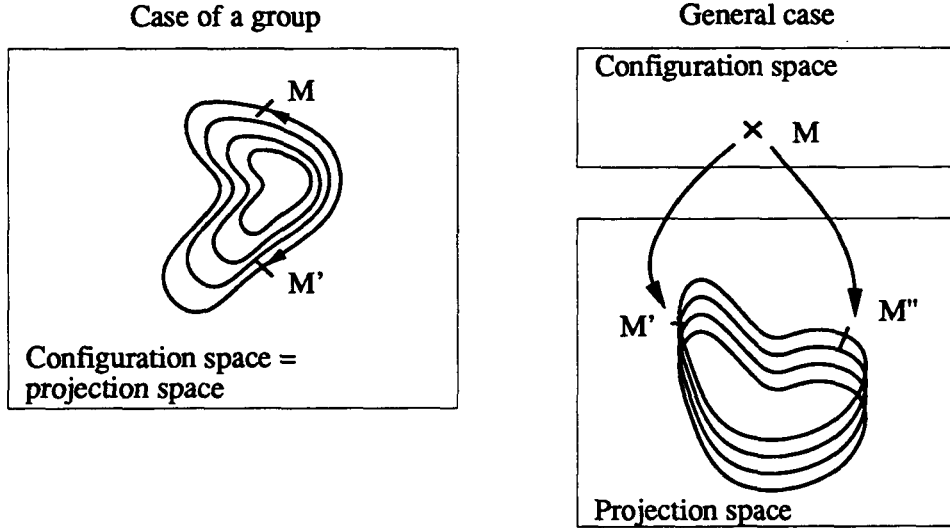
Figure 5: One orbit in the case of groups and in the general case

given transformation. In the case of a group of transformations (on the left of the figure 5), the orbits of M and M' are the same and we may express their equation in terms of any of its points, either M or M'. All the elements of the orbit are reachable from one to another by a transformation of the group. On the other hand, when the transformations do not form a group, the points M and M' do not belong to the same space. M' may belong to several orbits. It is not possible, then, to obtain the equation of the orbit of M if we know only M'. After all, if M'' is another point of the orbit of M, there is no transformation such that M' would be the image of M'' by this transformation.

12

2.6 Conclusion about the infinitesimal method

The main interest of this method is its exactness, which allows a precise enumeration of the invariants. As the invariants are expressed as the set of solutions of a partial differential equation, we can apply to the invariants the results valid for the solutions of these equations.

On the other hand, the practical use of this method in order to calculate new invariants in nontrivial cases is difficult. Firstly, it hides the geometrical aspects of the problem. Secondly, the equations obtained with this method may be hard to solve if we have no idea of the form of the solution. That may be sometimes compensated by the search of particular solutions, polynomials for example.

3 Generalization and constraint method

The infinitesimal method is hard to use in real cases. In this section and the next one, we present two practical methods for calculating invariants.

3.1 Method outline

This method is very classical among mathematicians who work on invariant problems [Dix90]. To find the invariants associated with a given problem, we will generalize this problem, find the invariants of this general problem and express that our real problem is a particular restriction of it. The difficult point of this method is to find a problem which we can solve, and such that our problem is a particular case of it. Here is an example of this method.

3.2 An example

3.2.1 Introduction and generalization of the problem

Let us consider, in a projective plane, pairs of conics and linear changes of variables. We generalize this problem to quadratic forms, and we finally come back to conics.

Let $\mathcal{C}$ be a conic whose homogeneous equation is:

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz = 0$$

This equation may be denoted ${}^t XAX = 0$, where ${}^t X = (x, y, z)$ and A is a symmetric matrix:

$$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$$

The equation of the conic $\mathcal{C}$ is defined up to a scale factor and so is the matrix A. To matrix A we associate the quadratic form whose matrix is A.

3.2.2 The invariants of pairs of quadratic forms

Concerning this paragraph, one may read [SK52, page 194]. Let the pairs of 3-variable quadratic forms be the configurations. We denote them by their 3×3 symmetric associated matrix $A = (a_{ij})_{i,j \in \{1,2,3\}}$ and $B = (b_{ij})_{i,j \in \{1,2,3\}}$. Let us consider the following transformations:

$$(A, B) \mapsto ({}^t PAP, {}^t PBP)$$

where the determinant of P is equal to 1; these transformations are the linear changes of variables of determinant 1. As a matter of fact, this transformation transforms the matrix A of a quadratic form into ${}^t PAP$. We denote by A_{ij} and B_{ij} the cofactors of a_{ij} and b_{ij}.

13

We know two elementary invariants for this problem: $\det(A)$ and $\det(B)$; they are homogeneous functions of degree 3: $\det(tA) = t^3 \det(A)$. More generally, we have the following invariant $\det(\lambda A + \mu B)$ which verifies:

$$\det(\lambda A + \mu B) = \lambda^3 I_1 + \lambda^2 \mu I_2 + \lambda \mu^2 I_3 + \mu^3 I_4 \tag{9}$$

with

$$
\begin{aligned}
I_1 &= \frac{1}{3}(a_{11}A_{11} + a_{22}A_{22} + a_{33}A_{33} + 2a_{23}A_{23} + 2a_{13}A_{13} + 2a_{12}A_{12}) = \det A \\
I_2 &= b_{11}A_{11} + b_{22}A_{22} + b_{33}A_{33} + 2b_{23}A_{23} + 2b_{13}A_{13} + 2b_{12}A_{12} \\
I_3 &= a_{11}B_{11} + a_{22}B_{22} + a_{33}B_{33} + 2a_{23}B_{23} + 2a_{13}B_{13} + 2a_{12}B_{12} \\
I_4 &= \frac{1}{3}(b_{11}B_{11} + b_{22}B_{22} + b_{33}B_{33} + 2b_{23}B_{23} + 2b_{13}B_{13} + 2b_{12}B_{12}) = \det B
\end{aligned}
$$

As the polynomial (9) is constant so are its coefficients I_1, I_2, I_3 and I_4 which are therefore four invariants. There is a more powerful result [Dix90]: every polynomial invariant of A and B is a polynomial function of I_1, I_2, I_3 and I_4.

If we relax the constraint $\det P = 1$, the I_i are no longer invariants, because they become $(\det P)^2 I_i$ after the transformation. The true invariants in this case are the ratios I_i/I_j.

3.2.3 The invariants of the pairs of conics

Let us come back to the case of conics. A conic defines a quadratic form up to a scale factor. We can no longer suppose that the matrix associated with a conic has its determinant equal to 1. The invariants of conics are thus those of quadratic forms which are also invariant when the matrices are multiplied by a scalar. In our situation, we have:

$$
\begin{aligned}
I_1(\lambda A, \mu B) &= \lambda^3 I_1(A, B) \\
I_2(\lambda A, \mu B) &= \lambda^2 \mu I_2(A, B) \\
I_3(\lambda A, \mu B) &= \lambda \mu^2 I_3(A, B) \\
I_4(\lambda A, \mu B) &= \mu^3 I_4(A, B)
\end{aligned}
$$

We search for monomials $I_1^{n_1} I_2^{n_2} I_3^{n_3} I_4^{n_4}$ which are true invariants. As:

$$\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda^{3n_1} I_1^{n_1} . \lambda^{2n_2} \mu^{n_2} I_2^{n_2} . \lambda^{n_3} \mu^{2n_3} I_3^{n_3} . \mu^{3n_4} I_4^{n_4} = I_1^{n_1} I_2^{n_2} I_3^{n_3} I_4^{n_4}$$

this implies:

$$
\begin{cases}
3n_1 + 2n_2 + n_3 = 0 \\
n_2 + 2n_3 + 3n_4 = 0
\end{cases}
$$

We can obtain two independent solutions with $n_1 = 1$, $n_2 = -2$, $n_3 = 1$, $n_4 = 0$, and $n_1 = 0$, $n_2 = 1$, $n_3 = -2$, $n_4 = 1$. They provide the two invariants:

$$\alpha = \frac{I_1 I_3}{I_2{}^2}, \quad \beta = \frac{I_2 I_4}{I_3{}^2}$$

3.3 Conclusion about this method

This method has the double advantage to be mathematically very classical and very powerful. It may be a **very good practical approach** in many cases when the infinitesimal method leads to equations we do not know how to solve. However it implies the discovery of the generalized problem which leads to the solution. As there is no systematic way to do this generalization, the applications of this method may be quite limited.

4 Use of the cross ratio

The main idea of this section is that there exists canonical invariants associated with some transformations. In the case of Euclidean transformations i.e. rotations, translations and reflections, the basic invariant is the distance between two points. Angles and distance ratios are the basic invariants for similarities (i.e. Euclidean transformations and scale factor). So are ratios of distances between collinear points for affine transformations and cross ratios of collinear points for collineations.

We would like to express every problem of invariants as a search for distances, ratios and cross ratios. This method may fail if the objects of the problem are not elementary, cubic curves or surfaces for example. On the other hand, this method allows a simple search and an easy geometric interpretation of the invariants.

We consider in this section the case of planar projective geometry. The first part gives basic properties of cross ratio, conjugate elements and self-polar triangles. The two last parts show two detailed examples: the case of two lines and two points, and the case of two conics.

4.1 Elementary properties of the cross ratio

For an introduction to projective geometry, one could read [SK52]. In this paragraph, we will only introduce the few notions necessary for the construction of invariants; these notions are well known and date mostly from the ancient greek mathematical school and from the last century. We will work with spaces based on complex numbers.

4.1.1 Definitions of the cross ratio

- We call the cross ratio of four numbers a, b, c and d the following ratio, when it exists:

$$\{a, b; c, d\} = \frac{(a - c)(b - d)}{(b - c)(a - d)}$$

By permutation of these four numbers, we can form 24 cross ratios; they are not all distinct one from another. They actually define only 6 different values. If one of these values is ρ, the five other ones are:

$$\frac{1}{\rho}, 1 - \rho, \frac{1}{1 - \rho}, \frac{\rho}{\rho - 1}, \frac{\rho - 1}{\rho} \tag{10}$$

This first definition may be generalized by:

$$\{\infty, b; c, d\} = \frac{b - d}{b - c} \; ; \; \{a, \infty; c, d\} = \frac{a - c}{a - d} \; ; \; \{a, b; \infty, d\} = \frac{b - d}{a - d} \; ; \; \{a, b; c, \infty\} = \frac{a - c}{b - c}$$

- Four points on a projective line define a cross ratio. Let $(P_i)_{i \in \{1,...,4\}}$ be four points of homogeneous coordinates (λ_i, μ_i). We have:

$$\{P_1, P_2; P_3, P_4\} = \frac{(\theta_1 - \theta_3)(\theta_2 - \theta_4)}{(\theta_2 - \theta_3)(\theta_1 - \theta_4)}, \text{ with } \theta_i = \frac{\lambda_i}{\mu_i}$$

- A pencil of lines is formed by several lines which intersect one another at only one point. For such a pencil of four lines of the plane intersecting at one point a, the cross ratio of their intersection points with a variable line not passing through a is independent of this last line. This cross ratio is called the cross ratio of the four lines. The cross ratio of four planes of a 3D space, which have a line in common, may be defined similarly. This cross ratio is dual to the previous one.

- The cross ratio of four lines which join four points θ_1, θ_2, θ_3 and θ_4 of a conic to any point a of this conic (see Fig. 6), is independent of this point a: it is called the cross ratio of the four points of the conic (Chasles's theorem).

15

● Let T_1, T_2, T_3, T_4 be four tangents of a given conic. The cross ratio of the intersection points of these tangents with any tangent Δ to that conic, is independent from Δ: it is called the cross ratio of the four tangents T_1, T_2, T_3, T_4 of the conic and it is dual to the cross ratio of four points. This cross ratio is equal to that of the four contact points of the tangents with the conic (see Fig. 6):

$$\{T_1, T_2; T_3, T_4\} = \{m_1, m_2; m_3, m_4\} = \{\theta_1, \theta_2; \theta_3, \theta_4\}$$
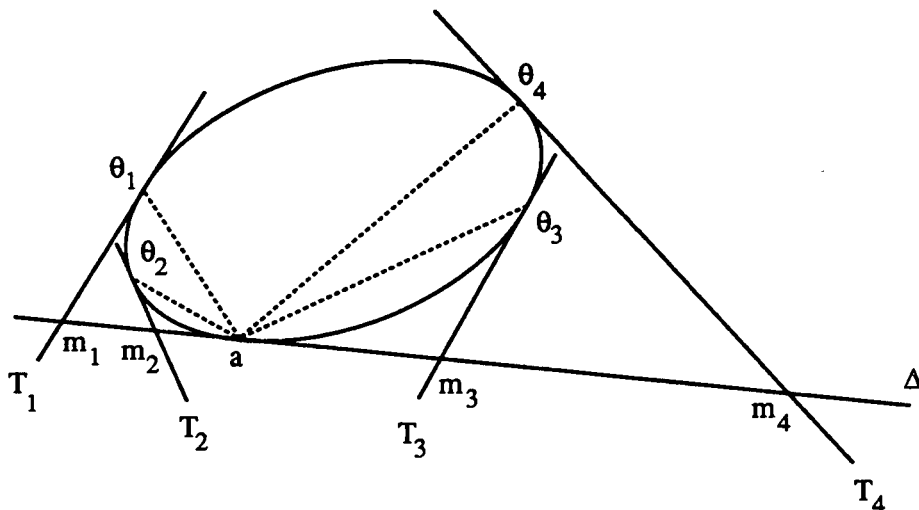
Figure 6: Definition of cross ratios from a conic

Other cross ratios are derived from these ones. Let us consider for example 5 points of the plane such that no three of them lay on the same straight line, $(P_i)_{i \in \{1,\dots,5\}}$; they define two independent cross ratios, that of the line pencils $(P_1 P_2), (P_1 P_3), (P_1 P_4), (P_1 P_5)$ and $(P_2 P_1), (P_2 P_3), (P_2 P_4), (P_2 P_5)$ (see Fig. 7).
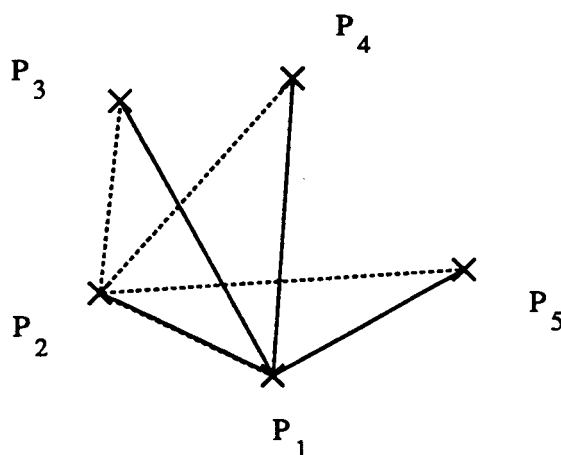
Figure 7: Definition of two cross ratios from 5 points

The given formulas are not symmetric and depend on the order of the considered points or lines. We can always have symmetric formulas using symmetric polynomials. For example, we may form $a + b$ and ab from a and b, to have expressions which no longer depend on the order.

16

Polynomials sometimes have to be more complex. For instance, if ρ is a cross ratio of four collinear points, the possible values obtained when considering the different orders of the four points are given in (10). With these values, the symmetric polynomials $\sum x_i$ or $\prod x_i$ always have the same value, 3 and 1 respectively. Thus we have to use polynomials like $\sum x_i x_j$ or $\sum x_i x_j x_k$ or functions like $\min(x_i)$.

4.1.2 Conjugate elements with respect to a conic

In this subsection and in the next one, standard properties on conic geometry are introduced. These results are used in paragraph 4.3.1. They allow to interpret geometrically the algebraic invariants of a pair of conic (for example those given by Forsyth *et al.* [FMZ+91]). Given a straight line L of equation $\sum_{i=1}^{3} a_i x_i = 0$, the vector of the coefficients of this equation ${}^t(a_1, a_2, a_3)$ is called the homogeneous coordinates of L.

Four numbers, points or lines are said to be in harmonic relation when one of their cross ratios is equal to -1. We say that two of these points are harmonic conjugates with the two others [SK52, page 48].

Two points P and Q are conjugate with respect to a conic C if they are conjugate with the two intersection points M_1 and M_2 of the line (P, Q) and the conic C. In this situation, we have $\{P, Q; M_1, M_2\} = -1$. If we denote $\mathcal{M}_C$ one of the matrix associated with the conic C, X and Y the coordinate vectors of P and Q, we have ${}^t X \mathcal{M}_C Y = 0$. This last formula may be taken as a definition of conjugation [SK52, page 108].

The locus of the points conjugate with a given point P with respect to a conic is a straight line, called the polar of this point. With our notations, if l is the vector of homogeneous coordinates of the line, we have:

$$\lambda l = \mathcal{M}_C X \tag{11}$$

where λ is a real number. Conversely, all the polars of points on the same straight line with respect to a conic intersect a fixed point, called the pole of this line.

If a point P is not in the concavity of the conic, there are two lines which pass through this point and which are tangential to the conic at two points P_1 and P_2. By definition, the polar of P is the line $(P_1 P_2)$ (see Fig. 4.1.2). If the point is in the concavity of the conic, the same construction is possible, but the two tangent lines will be imaginary.
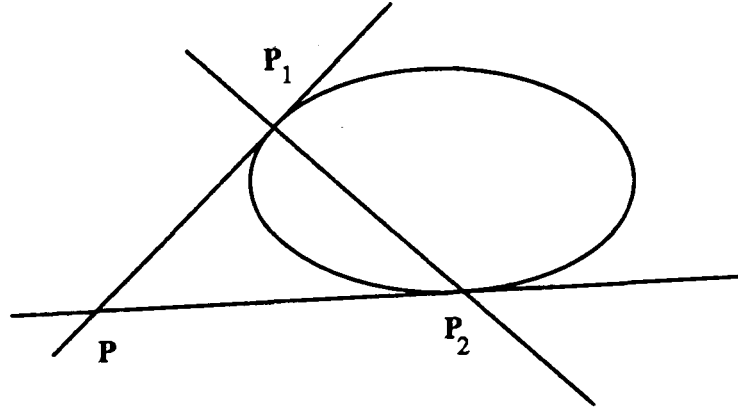

Figure 8: Construction of the polar of a point.

4.1.3 Study of self-polar triangles

Three points P, Q, R form a self-polar triangle with respect to a conic, if the polar of each of the three vertices of the triangle is the straight line defined by the two other vertices.

Construction of the self-polar triangles. Let C be a conic and $\mathcal{M}_C$ be one of its associated matrices. Let A, B, C, D be any four points of this conic. We define the following points: $P = (AB) \cap (CD)$, $Q = (AC) \cap (BD)$ and $R = (AD) \cap (BC)$ (see Fig. 9). The triangle (P, Q, R) is self-polar.
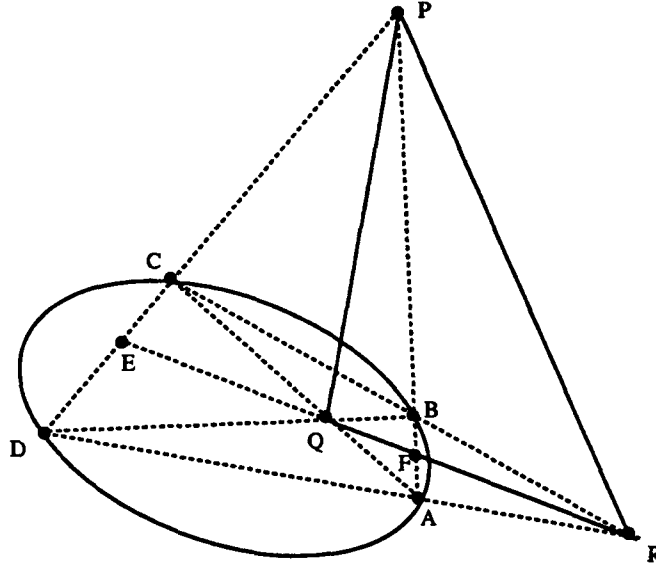
Figure 9: Construction of a self-polar triangle

To demonstrate the self-polarity of the triangle, notice that the polar of a point with respect to a conic reduced to two straight lines passes through the intersection points of these two lines. Let E and F be the point such that P, E and P, F are the harmonical conjugates of C, D and A, B respectively. E and F belong to the polars of P with respect to the two lines (AC) and (BD) which intersect each other in Q, and with respect to the two lines (AD) and (BC) which intersect each other in R. The points E, F, Q and R are consequently collinear; Q and R belong to (EF), the polar of P with respect to the conic C. The same reasoning with the two other vertices leads to the result.

Equation of a conic in relation to a self-polar triangle. Let take as points of reference the three vertices P, Q, R of a self-polar triangle with respect to a conic C of matrix $\mathcal{M}_C$: P, Q, R have respectively as homogeneous coordinates $X_P = (1, 0, 0)$, $X_Q = (0, 1, 0)$, $X_R = (0, 0, 1)$, which are also the homogeneous coordinates $l_{(QR)}$, $l_{(PR)}$ and $l_{(PQ)}$ of the lines $(QR), (PR), (PQ)$. The self-polarity of the triangle implies that there exists three real numbers λ_1, λ_2 and λ_3 such that:

$$\begin{cases} \lambda_1 l_{(QR)} &= \mathcal{M}_C X_P \\ \lambda_2 l_{(PR)} &= \mathcal{M}_C X_Q \\ \lambda_3 l_{(PQ)} &= \mathcal{M}_C X_R \end{cases} \tag{12}$$

The matrix $\mathcal{M}_C$ is therefore diagonal, and in this reference system, the equation of the conic is reduced to:

$$ax^2 + by^2 + cz^2 = 0 \tag{13}$$

18

Conversely, if a conic C has an equation like (13) in a projective basis P, Q, R, S, then the system (12) is satisfied and P, Q, R is a self polar triangle with respect to the conic C.

Eigenvectors of a matrix $\mathcal{M}_C$ associated with a nondegenerate conic. The matrix associated with any conic C is symmetric (cf. 3.2.1). When the conic is not degenerate, this matrix is defined. In this case, the matrix $\mathcal{M}_C$ has three orthogonal eigenvectors P, Q, R.

The polar of P with respect to C has for homogeneous coordinates $l = \mathcal{M}_C X_P = \lambda X_P$ because P is an eigenvector of $\mathcal{M}_C$. Because of orthogonality, Q and R are on this polar ($X_P.X_Q = 0$, $X_P.X_R = 0$). The same reasoning with Q and R shows that P, Q, R is a self-polar triangle.

4.2 Invariant of two straight lines and two points

In a projective plane, we consider the configurations made of two points and two straight lines. We look for an invariant for these configurations, with the set of all planar collineations as transformation space.

It is very easy to find an invariant. The construction of one or more cross ratios, with the help of stable constructions for collineations, is enough: these cross ratios are invariant. In the present case, we construct the straight line passing through the two points of the configuration. This line intersects the two straight lines of the configuration in two points, which can be at finite or infinite distance. We thus have four points on the same straight line: the two points of the configuration, and the two intersection points; these points define a cross ratio which is an invariant for the problem (see Fig. 10.
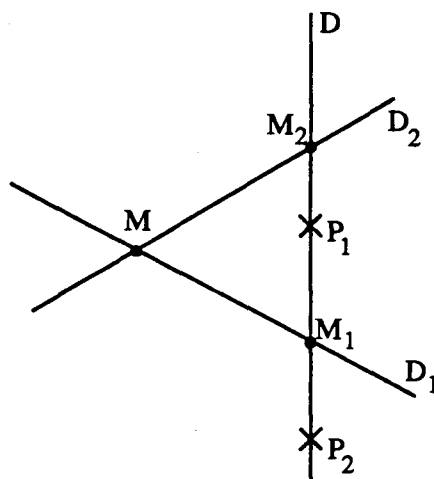
Figure 10: A configuration of 2 points and 2 straight lines

Let us try to count the number of invariants in this problem. The set of planar collineations which are defined by a 3×3 matrix up to a scale factor, is a manifold of dimension 8. The constraint (determinant not equal to zero) removes a submanifold of transformations, but leaves the dimension unchanged. A straight line or a point are defined by 3 parameters, but up to a scale factor: thus it belongs to a manifold of dimension 2. The configurations contain two lines and two points and also form a manifold of dimension 8.

Although the dimensions of the transformation and configuration spaces are equal, the existence of an invariant is possible: according to formula (8) page 10, it can be the consequence of an isotropy group whose dimension is not zero. We specify this isotropy group in the next p[201zaragraph.

Looking for an isotropy group. We consider only non-degenerate configurations, i.e. configurations such that the straight lines are not the same and the two points are not the same and are not lying on the two straight lines. (see fig. 10). Let P_1 and P_2 be the two points and D_1 and D_2 the two straight lines for a configuration. We denote D the straight line $(P_1 P_2)$, M_1 and M_2 the intersection points of the straight line D with D_1 and D_2 respectively. We denote M the intersection of D_1 and D_2.

We take a system of homogeneous coordinate such that M is the origin and D is the line at infinity.

Let F be a collineation which leaves this configuration invariant. The points P_1 and P_2 are invariant and the line D is globally invariant: consequently, the point M_1 is invariant. (P_1, P_2, M_1) form a projective basis of the line D, and this line is point invariant (every point of D is invariant). After all, as the lines D_1 and D_2 are globally invariant, their intersection, the point M, is invariant.

The line at infinity being globally invariant by F, F is an affine function. As the origin is invariant, F is reduced to its linear part. All the points of the line at infinity, especially the cocyclic points, are invariant. According to Laguerre's formula [SK52, page 95], F preserves the angles: it is a similarity (i.e. the compound of an isometry and of a scaling).

Every straight line which passes through the origin has two invariant points: the origin and its point at infinity. These lines are thus globally invariant, and every vector direction is an eigendirection of F. The scalings are the only similarities that verify this condition: F is a scaling whose center is the origin. The set of such transformations is a group of order 1, which is the isotropy group.

Conclusion. In this example concerning configurations of two points and two lines, the transformation manifold and the configuration manifold have the same dimension. There is one invariant which has been found, along with an isotropy group which explains the existence of such an invariant.

4.3 Invariants of a pair of conics

Let C_1 and C_2 be two conics, associated with some matrices $\mathcal{M}_1$ and $\mathcal{M}_2$. We look for the invariants of these two conics, the transformations being the planar collineations. These transformations form a manifold of dimension 8, the pairs of conics one of dimension 10. We should find two independent invariants, and only two, if the isotropy group is of dimension 0.

The problem of research of the invariants of two conics has already been approached in different manners. Maybank [May91] calculates the cross ratios of the four points of intersection of the two conics. Forsyth et al. [FMZ+91] give as invariants $Trace(\mathcal{M}_2{}^{-1}\mathcal{M}_1)$ and $Trace(\mathcal{M}_1{}^{-1}\mathcal{M}_2)$. Actually these traces are invariants only for the quadratic forms associated with $\mathcal{M}_1$ and $\mathcal{M}_2$, and not for the conic. In order that they become invariants for the conics, the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ must have their determinant normalized to 1. That is what Forsyth et al. [FMZ+91] suppose. From a practical point of view, this constraint should be included in the invariant, as we do in this paper. In their article [KM91], Kapur et al. interpret these invariants in terms of the common self-polar triangle of the two conics, without explicitly introducing this triangle.

For our part, we show how to interpret these invariants in terms of cross ratios.

4.3.1 Interpretation of the matrix $\mathcal{M}_2{}^{-1}\mathcal{M}_1$ in terms of cross ratios

Let C_1 and C_2 be two conics with associated matrices $\mathcal{M}_1$ and $\mathcal{M}_2$. These two conics intersect one another in four points A, B, C, D, distinct or not, real or imaginary. From these four points, we construct the triangle (P, Q, R) which is self-polar for both conics. It is the only common self-polar triangle.

Trace of the matrix $\mathcal{M}_2{}^{-1}\mathcal{M}_1$. Let $\mathcal{N}$ be an invertible 3×3 matrix. The collineation which map corresponding points X' to X such that $X = \mathcal{N}X'$, transforms $\mathcal{M}_1$ and $\mathcal{M}_2$ in ${}^t\mathcal{N}\mathcal{M}_1\mathcal{N}$ and

$^t\mathcal{N}\mathcal{M}_2\mathcal{N}$. We have:

$$Trace((^t\mathcal{N}\mathcal{M}_2\mathcal{N})^{-1}\,{}^t\mathcal{N}\mathcal{M}_1\mathcal{N}) \;=\; Trace(\mathcal{N}^{-1}\mathcal{M}_2{}^{-1}\mathcal{M}_1\mathcal{N})$$
$$=\; Trace(\mathcal{M}_2{}^{-1}\mathcal{M}_1)$$

The trace of the matrix $\mathcal{M}_2{}^{-1}\mathcal{M}_1$ is then an invariant for this matrix, but not for the associated conics; as a matter of fact, the conics define their matrices up to a scale factor, and the trace is not invariant to scaling of one of the matrices.

As the trace is not a real invariant of the conics, we cannot interpret it in terms of cross ratios.

Eigenvectors of the matrix $\mathcal{M}_2{}^{-1}\mathcal{M}_1$. Let l_P be the homogeneous coordinates of the straight line (QR). As the triangle (P, Q, R) is self-polar for both conics, we have: $\lambda l_P = \mathcal{M}_1 P$ and $\mu l_P = \mathcal{M}_2 P$. Thus $\mathcal{M}_2{}^{-1}\mathcal{M}_1 P = \lambda/\mu P$. We do the same for Q and R. The vertices of the common self-polar triangle are the eigenvectors of the matrix $\mathcal{M}_2{}^{-1}\mathcal{M}_1$.

In conclusion, the eigenvectors of $\mathcal{M}_2{}^{-1}\mathcal{M}_1$ are related to the common self-polar triangle, which is related to the conics by cross ratios.

Eigenvalues of the matrix $\mathcal{M}_2{}^{-1}\mathcal{M}_1$. Let us take the points P, Q and R as references. With these references, the two conics have reduced equations:

$$
\begin{array}{llll}
C_1 & : & a_1 x^2 + b_1 y^2 + c_1 z^2 & = & 0 \\
C_2 & : & a_2 x^2 + b_2 y^2 + c_2 z^2 & = & 0
\end{array}
$$

As the eigenvalues of a matrix do not depend on the basis, a_1, b_1, c_1 are the eigenvalues of $\mathcal{M}_1$ and a_2, b_2, c_2 are those of $\mathcal{M}_2$. We must remark that it is not the eigenvalues which are characteristics of the conics, but their ratios (due to the scale factor) [KM91].

The eigenvalues of $\mathcal{M}_2{}^{-1}\mathcal{M}_1$ are:

$$\lambda_1 = \frac{a_1}{a_2}, \quad \lambda_2 = \frac{b_1}{b_2}, \quad \lambda_3 = \frac{c_1}{c_2}$$

For these values also, only the ratios of eigenvalues are characteristics of the conics and this provides only 2 independent invariants, λ_1/λ_3 and λ_2/λ_3. These ratios may be interpreted geometrically.

Let M_1, N_1 and M_2, N_2 be the intersections of the straight line (PQ) with C_1 and C_2 respectively (see Fig. 11). Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be real or imaginary square roots of respectively a_1, a_2, b_1, b_2 and let $i^2 = -1$. The equation of (PQ) is: $z = 0$. M_1, N_1, M_2 and N_2 are defined by the following systems:

$$
\left\{
\begin{array}{rcl}
a_1 x^2 + b_1 y^2 + c_1 z^2 & = & 0 \\
z & = & 0
\end{array}
\right.
\quad \text{and} \quad
\left\{
\begin{array}{rcl}
a_2 x^2 + b_2 y^2 + c_2 z^2 & = & 0 \\
z & = & 0
\end{array}
\right.
$$

We obtain:

$$
\begin{array}{rcl}
M_1 & = & (\beta_1, i\alpha_1, 0) \\
N_1 & = & (\beta_1, -i\alpha_1, 0) \\
M_2 & = & (\beta_2, i\alpha_2, 0) \\
N_2 & = & (\beta_2, -i\alpha_2, 0)
\end{array}
$$

Thus:

$$\{M_1, M_2; P, Q\} = \{-i\frac{\beta_1}{\alpha_1}, -i\frac{\beta_2}{\alpha_2}; \infty, 0\} = \frac{\alpha_1\beta_2}{\beta_1\alpha_2}$$

and:

$$\frac{\lambda_1}{\lambda_2} = \{M_1, M_2; P, Q\}^2$$

We can interpret the other ratios of eigenvalues in the same way.

In the case where the two conics have a double point of intersection, the same construction is possible: if $A = B$, one has just to replace the straight line (AB) by the tangent to the conic through A.
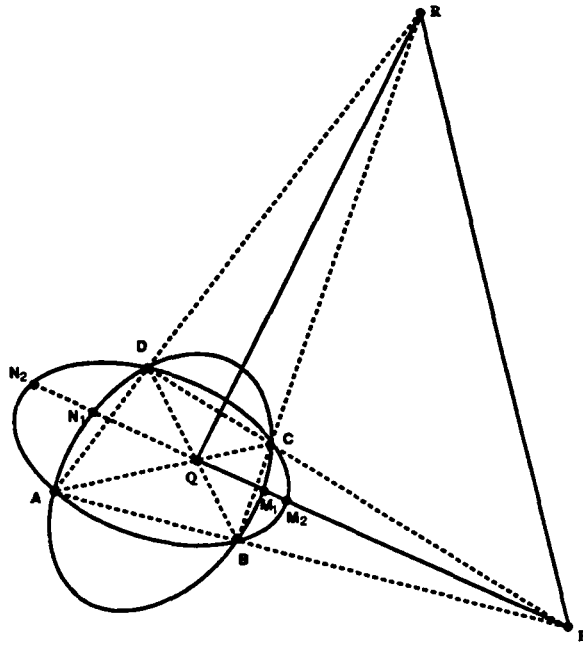
21

Figure 11: Interpretation of the conic invariants in terms of cross ratios

4.3.2 New interpretation of the results of the paragraph 3.2

In this paragraph, we interpret these invariants in terms of cross ratios. Let us consider the vertices of the common self-polar triangle as points of reference, and calculate I_1, I_2, I_3 and I_4 with the reduced equations of the conics:

$$
\begin{aligned}
I_1 &= a_1 b_1 c_1 \\
I_2 &= a_2 b_1 c_1 + b_2 a_1 c_1 + c_2 a_1 b_1 \\
I_3 &= a_1 b_2 c_2 + b_1 a_2 c_2 + c_1 a_2 b_2 \\
I_4 &= a_2 b_2 c_2
\end{aligned}
$$

We can express the ratios of these numbers as functions of the eigenvalues of the matrix $\mathcal{M}_2{}^{-1}\mathcal{M}_1$:

$$
\begin{aligned}
\frac{I_1}{I_4} &= \lambda_1 \lambda_2 \lambda_3 \\
\frac{I_2}{I_4} &= \lambda_2 \lambda_3 + \lambda_1 \lambda_3 + \lambda_1 \lambda_2 \\
\frac{I_3}{I_4} &= \lambda_1 + \lambda_2 + \lambda_3
\end{aligned}
$$

Obviously:

$$
\begin{aligned}
Trace(\mathcal{M}_2{}^{-1}\mathcal{M}_1) &= \frac{I_3}{I_4} \\
Trace(\mathcal{M}_1{}^{-1}\mathcal{M}_2) &= \frac{I_2}{I_1} \\
\frac{\det \mathcal{M}_1}{\det \mathcal{M}_2} &= = \frac{I_1}{I_4}
\end{aligned}
$$

22

To obtain the invariants of the conics, we must use the cross ratios, i.e. the ratios of the eigenvalues as we have already seen it. For example, we can take:

$$\alpha = \frac{I_1 I_3}{I_2^2} = \frac{(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 \lambda_2 \lambda_3)}{(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3)^2} = \lambda_3 / \lambda_2 \frac{1 + \lambda_2 / \lambda_1 + \lambda_3 / \lambda_1}{(1 + \lambda_3 / \lambda_1 + \lambda_3 / \lambda_2)^2},$$

$$\beta = \frac{I_4 I_2}{I_3^2} = \frac{\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3}{(\lambda_1 + \lambda_2 + \lambda_3)^2} = \lambda_2 / \lambda_1 \frac{1 + \lambda_3 / \lambda_1 + \lambda_3 / \lambda_2}{(1 + \lambda_2 / \lambda_1 + \lambda_3 / \lambda_1)^2}.$$

The invariants given at paragraph 3.2 are then related to the cross ratios.

Note that we have also:

$$\alpha = \frac{I_1 I_3}{I_2^2} = \frac{Trace(\mathcal{M}_2{}^{-1}\mathcal{M}_1)}{(Trace(\mathcal{M}_1{}^{-1}\mathcal{M}_2))^2} \frac{\det \mathcal{M}_2}{\det \mathcal{M}_1}$$

$$\beta = \frac{I_4 I_2}{I_3^2} = \frac{Trace(\mathcal{M}_1{}^{-1}\mathcal{M}_2)}{(Trace(\mathcal{M}_2{}^{-1}\mathcal{M}_1))^2} \frac{\det \mathcal{M}_1}{\det \mathcal{M}_2}$$

So, the invariants produced by Forsyth and al. [FMZ⁺91] are related to some cross ratios, as soon as they are combined in such a way that they form real invariants of the conics and not only of the quadratic forms.

4.3.3   The cross ratios and the intersections of conics

Two conics intersect one another in four points. According to the paragraph 4.1.1, four points on a conic define a cross ratio. The four points of intersection, which belong to each of the two conics, define two cross ratios.

In this paragraph, we make the relation between these cross ratios and those produced in the previous paragraph. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two conics of matrices $\mathcal{M}_1$ and $\mathcal{M}_2$, A, B, C, D be their points of intersection and k_1 and k_2 be the two cross ratios defined by these points. In his paper [May91], Maybank shows that these cross ratios may be expressed as functions of the roots of the equation:

$$\det(\mathcal{M}_1 + \xi \mathcal{M}_2) = 0 \tag{14}$$

With this aim in view, he chooses the coordinates in such a way that the four points of intersection are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$. With these coordinates, the conics have as equations:

$$\begin{aligned} \mathcal{C}_1 &: & \phi_1 xy + \psi_1 yz - (\phi_1 + \psi_1)xz &= 0 \\ \mathcal{C}_2 &: & \phi_2 xy + \psi_2 yz - (\phi_2 + \psi_2)xz &= 0 \end{aligned}$$

The three roots of the equation (14) are:

$$\xi_1 = -\frac{\psi_1}{\psi_2}$$

$$\xi_2 = -\frac{\phi_1}{\phi_2}$$

$$\xi_3 = -\frac{\phi_1 + \psi_1}{\phi_2 + \psi_2}$$

On the other hand, in the following parameterization of $\mathcal{C}_1$:

$$s \mapsto (s[\psi_1(1-s) - s\phi_1], \psi_1(1-s) - s\phi_1, -s\phi_1)$$

the four intersection points have as parameters: $\infty, 0, \psi_1/(\psi_1 + \phi_1), 1$. Therefore:

$$\begin{aligned} k_1 &= \{A, B; C, D\}_{\mathcal{C}_1} \\ &= \{\infty, 0; \psi_1/(\psi_1 + \phi_1), 1\} \\ &= 1 + \frac{\phi_1}{\psi_1} \\ &= \{0, \xi_2; \xi_3, \xi_1\} \end{aligned}$$

23

An identical calculation gives:

$$k_2 = \{\infty, \xi_2; \xi_3, \xi_1\}$$

With the notations of the previous paragraph, we have:

$$\det(\mathcal{M}_1 + \xi\mathcal{M}_2) = (a_1 + \xi a_2)(b_1 + \xi b_2)(c_1 + \xi c_2)$$

and thus:

$$\xi_1 = -\lambda_1, \quad \xi_2 = -\lambda_2, \quad \xi_3 = -\lambda_3$$

We may then conclude:

$$k_1 = \frac{\lambda_3(\lambda_2 - \lambda_1)}{\lambda_1(\lambda_2 - \lambda_3)} = \frac{\frac{\lambda_2}{\lambda_1} - 1}{\frac{\lambda_2}{\lambda_3} - 1}$$

$$k_2 = \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_3} = \frac{1 - \frac{\lambda_1}{\lambda_2}}{1 - \frac{\lambda_3}{\lambda_2}}$$

This makes the connection.

4.4   Conclusion about the cross ratio

The idea developed in this section, is that the invariants of some classes of problems may be found easily. These problems are those where the transformation space is classical (Euclidean or affine transformations or collineations), and the tokens are elementary (points, straight lines, and conics). In this case, the invariants may be searched for in the form of angles, distances, ratios or cross ratios according to the kind of transformations used. The link between these simple invariants and other invariant forms used when one manipulates less elementary elements, may be more complex.

In the light of what was said, the following results are obvious:

- the configuration formed by two straight lines and a conic has one invariant which is the cross ratio of the four points of intersection between the lines and the conic;

- two concentric conics and a straight line have as invariant the cross ratio of the four points of intersection of the line and the conics;

- a conic and two points have an invariant, the cross ratio of these two points and the two points of intersection between the straight line they define, and the conic;

- any five straight lines have two invariants: four of the lines intersect the fifth one in four collinear points and that gives an invariant; only two of these invariants are independent;

- four straight lines and a point on one of them have one invariant, the cross ratio of this point with the three intersections which are on the same line;

- ...

Due to equation (8), these are the maximum number of invariants that can be found.

5   Applications of invariance to computer vision

5.1   References to other applications

In 2D vision, i.e. from a single image, the matching of plane curves using invariants has been studied by Van Gool et al. [GKO91] and by Zisserman et al. [RZFM92]. In 3D vision, i.e. from multiple images, the invariants were used to do relative reconstruction and localization [MMIQ91, MM91, MMG91, MQVB92]. Koenderink et al.[KvD89] and also Lee et al. [LH90] use affine 3D invariants to compute the shape of objects. Quan et al. [QM91] extended this work to 3D projective invariants.

24

5.2 Basic tools for 3D vision

It should be noted that in the general case 7 point matches are necessary to compute the epipolar geometry[FLM92]. Using 8 point matches, it is possible to compute it linearly without the coplanarity assumption made here [Fau92, HGC92].

5.2.1 Reconstruction of the epipolar geometry from 6 point matches

Let A, B, C, D, E and F be six 3D points such that A, B, C and D lie in the same plane P (see Fig. 12). From two points O' and O'', we take two images of these points. Let us denote $E' = P \cap (O'E)$, $E'' = P \cap (O''E)$, $F' = P \cap (O'F)$ and $F'' = P \cap (O''F)$. The images of all these points are denoted with lower–case letters and an index 1 in the first image, 2 in the second image.

As O', E and E' are collinear, e_1 and e_1' are the same point. The projective coordinates of e_1' in the basis (a_1, b_1, c_1, d_1) can be computed. They are also the coordinates of E' in the basis (A, B, C, D) and those of e_2' in the basis (a_2, b_2, c_2, d_2). The points e_1'', f_2' and f_1'' are constructed the same way.

By construction, the points O', O'', E, E' and E'' are coplanar. So are O', O'', F, F' and F''. The intersection of these two planes is the line (O', O'') and the intersection of this line with the plane P is the intersection of the two lines (E', E'') and (F', F''). Let us denote this point G.

The projection of G can be computed in each image:

$$\begin{cases} g_1 &= (e_1'e_1'') \cap (f_1'f_1'') \\ g_2 &= (e_2'e_2'') \cap (f_2'f_2'') \end{cases}$$

As the three points O', O'' and G are collinear, g_1 and g_2 are the epipoles of the two images.

A new point h_1 in the first image can be interpreted as the projection of a 3D point H belonging to plane P. The epipolar line of h_1 in the second image is the line (g_2h_2) where h_2 is the point having the same projective coordinates in the basis (a_2, b_2, c_2, d_2) as h_1 in (a_1, b_1, c_1, d_1) and H in (A, B, C, D).

5.2.2 A coplanarity test

Let us suppose we have two images and that we know the epipolar geometry between these two images. Given the projections of any four points onto these images, it is possible to determine if the four 3D points are coplanar.

Let A, B, C and D be the four 3D points and a_1, b_1, c_1 and d_1, and a_2, b_2, c_2 and d_2 be their projections onto the two images (see Fig. 13). Let e_1 and e_2 be the epipoles in the images. We suppose that the projections are not merged with the epipoles and that they are labeled in such a way that (a_1, c_1, e_1), (b_1, d_1, e_1), (a_2, c_2, e_2) and (b_2, d_2, e_2) are not collinear.

We denote $f_1 = (a_1c_1) \cap (b_1d_1)$ and $f_2 = (a_2c_2) \cap (b_2c_2)$. A, B, C, and D are coplanar if and only if the two pencils of lines $(e_1a_1, e_1b_1, e_1c_1, e_1d_1, e_1f_1)$ and $(e_2a_2, e_2b_2, e_2c_2, e_2d_2, e_2f_2)$ define the same cross ratios (the definition of a pencil of lines is given on page 15).

As a matter of fact, if they are coplanar, the two lines (AC) and (BD) intersect one another at a point F and f_1 and f_2 are the projections of that point F onto the images. According to the epipolar lines properties, the two pencils of lines $(e_1a_1, e_1b_1, e_1c_1, e_1d_1, e_1f_1)$ and $(e_2a_2, e_2b_2, e_2c_2, e_2d_2, e_2f_2)$ are homographic and define the same cross ratios.

If A, B, C and D are not coplanar, let us suppose that A, B and C are stationary and that D is moving along its view line with respect to the first image. a_1, b_1, c_1 and d_1 are stationary, so is f_1. In the second image, a_2, b_2 and c_2 are stationary and d_2 is moving along its epipolar line. As a_2, c_2 and e_2 are not collinear, (a_2, c_2) is not an epipolar line and f_2 does not remain on the same epipolar line. The two pencils of lines are no longer homographic: they do not define the same cross ratios.
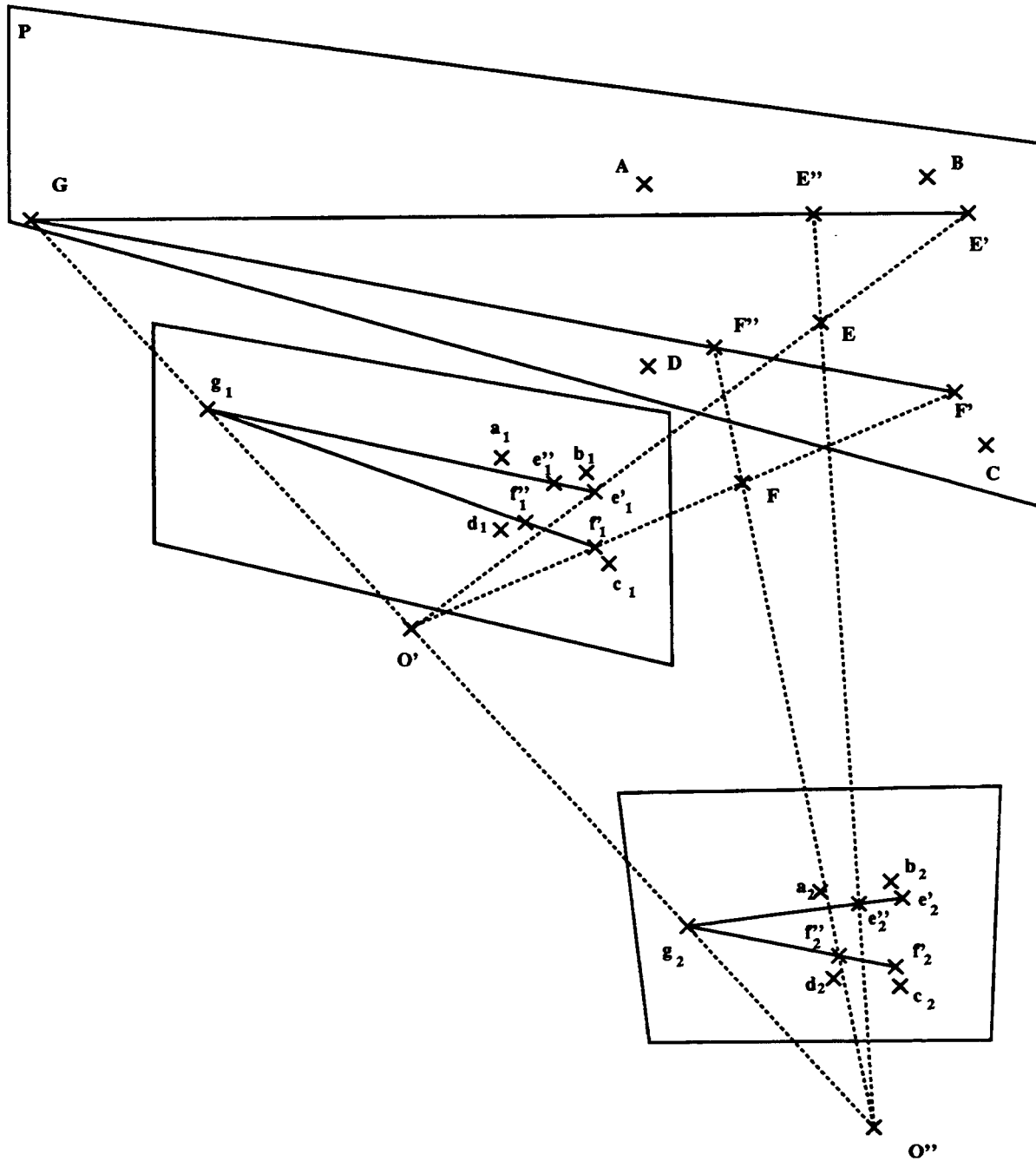
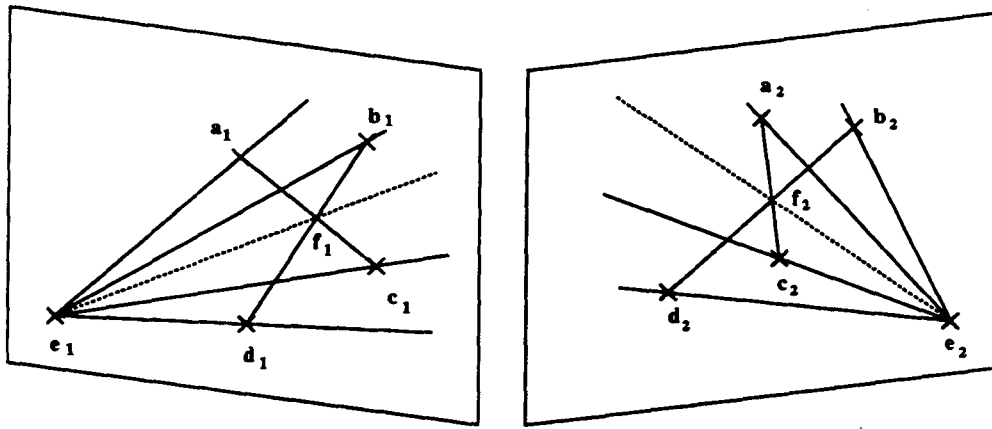Figure 12: Reconstruction of the epipolar geometry from 6 points.

Figure 13: A coplanarity test.

Applications to vision problems. There are two ways to use this test. Firstly, to test if four points are coplanar or not when only their projections onto two images are known. One has just to construct the points f_1 and f_2 and to verify that the two pencils of lines define the same cross ratios.

Secondly, it permits the construction of coplanar points. Suppose that we know the projections onto two images of three points A, B and C and of one straight line L. Let $a_1, b_1, c_1, l_1, a_2, b_2, c_2$ and l_2 be these projections. If the epipolar geometry is known, if (a_1, c_1, e_1), (b_1, d_1, e_1), (a_2, c_2, e_2) and (b_2, d_2, e_2) are not collinear and if l_1 and l_2 do not pass through e_1 and e_2 respectively, it is therefore possible to compute the projections of the point D of L which is coplanar with A, B and C. The method is the following:

1. Let e_1 and e_2 be the two epipoles and d_1 be any point of l_1 (see Fig. 1). We can compute the
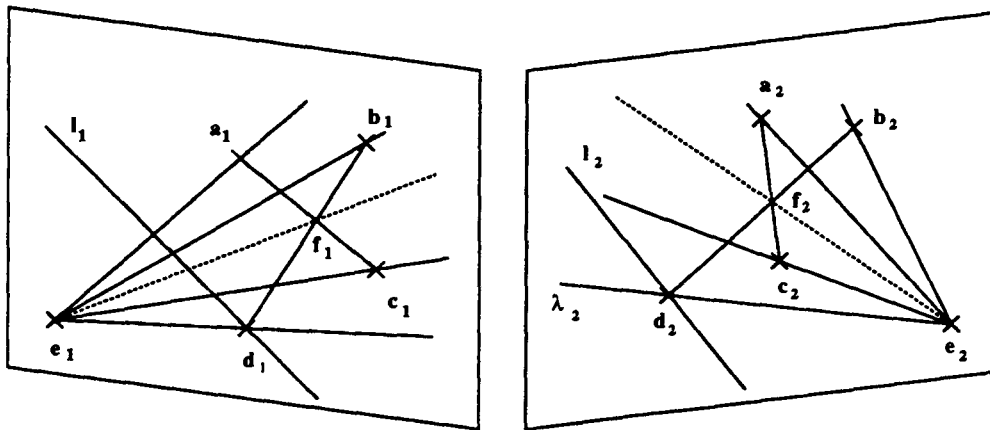
Figure 14: Construction of coplanar points.

cross ratio k of the pencil of lines $(e_1a_1, e_1b_1, e_1c_1, e_1f_1)$.

2. This cross ratio define a single line λ_2 passing through e_2 such that the cross ratio of the pencil $(e_2a_2, e_2b_2, e_2c_2, \lambda_2)$ is k.

3. Let d_2 be the intersection of λ_2 and l_2, and f_1, f_2 be those of (a_1c_1) and (b_1d_1), and (a_2c_2) and (b_2d_2) respectively.

4. According to the coplanarity test, d_1 and d_2 are the projection of a point D of L coplanar with

27

A, B and C if and only if the cross ratios of $(e_1a_1, e_1b_1, e_1c_1, e_1f_1)$ and $(e_2a_2, e_2b_2, e_2c_2, e_2f_2)$ are equal.

We therefore have to solve one equation with one unknown: the coordinate of d_1 on l_1. Unfortunately, this equation is quadratic. One solution is the true solution, the other one corresponds to the case where (b_1l_1) passes through the epipole. To avoid this nonlinearity, the following constructive method can be used.

Second method. Given the projections of five 3D points A, B, C, D and E into two images, we look for the projections into these two images of the point J which belongs to the line (DE) and which is coplanar with A, B and C (see Fig. 15). We suppose that we know the epipolar geometry
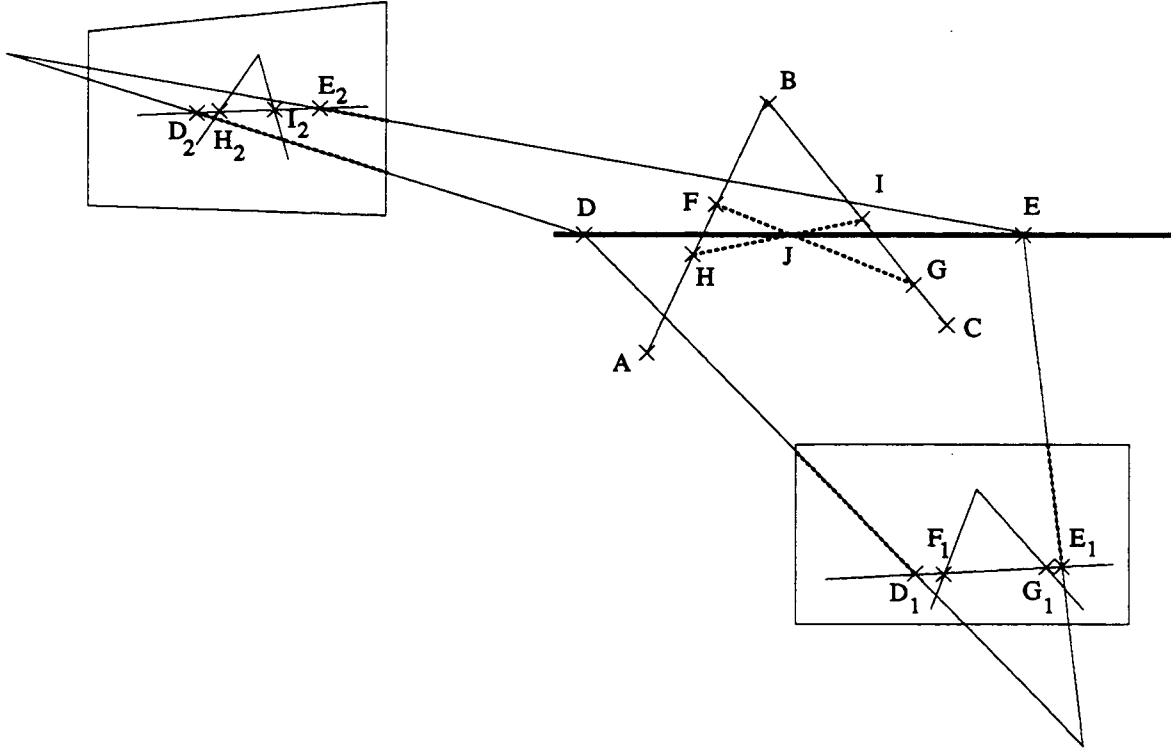
Figure 15: Reconstruction of the projections of 3D points: 3D view.

between the two images. The view plane of (DE) with respect to the first image respectively cuts (AB) and (BC) in F and G. The lines (FG) and (DE) intersect each other at a point coplanar with A, B and C: it is J. In the first image, we can construct the two projections of F and G: F_1 and G_1 (see Fig. 16).

In a similar way we can construct H_2 and I_2 the projections in the second image of H and I the intersections of (AB) and (BC) with the view plane of (DE) with respect to the second image. With the help of epipolar geometry, we can construct H_1, I_1, F_2 and G_2. Then we have $(FG) \cap (HI) = J$, $(F_1G_1) \cap (H_1I_1) = J_1$ and $(F_2G_2) \cap (H_2I_2) = J_2$.

This method will be used intensively in the next paragraphs.

5.3 3D vision invariants

In Section 4, we show how to use the cross ratio to find invariants in planar problems. The same methodology is usable in 3D. For some configurations of points and straight lines in 3D, we can

28
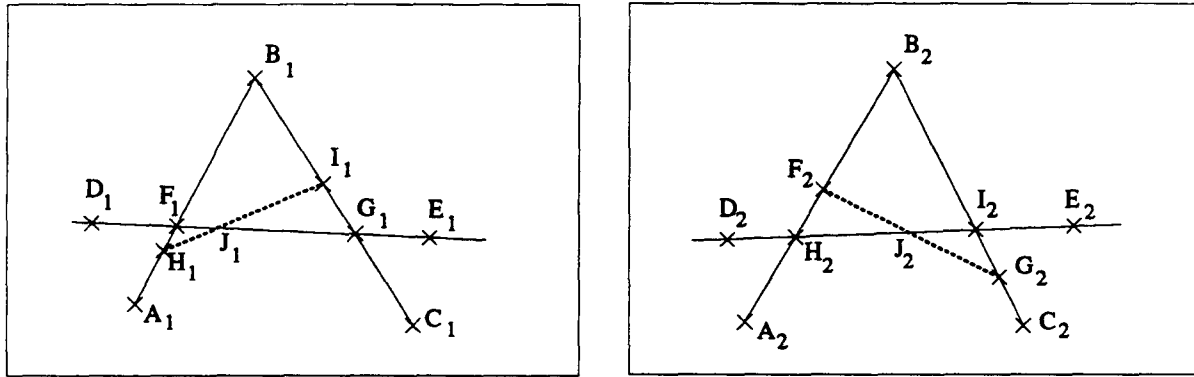
Figure 16: Reconstruction of the projections of 3D points: the two images.

interpret the existing invariants with respect to 3D collineations in terms of cross ratios. Furthermore, if we do not know the position of these 3D elements but only their projections onto two images and if the epipolar geometry is known, it is often possible to compute the cross ratios from these images.

The possible applications in vision are very large: when one disposes of two or more images of the same object, if these images have already been matched, one can compute the epipolar geometry, find the image points which correspond to 3D coplanar points using the coplanarity test and compute 3D invariants. These invariants are useful for projective reconstruction of the 3D points, to modelize geometrically and not only topologically an object, to index these models in a large data base, to match such a model with a new image...It should be noticed that all this can be done with images taken from one or more uncalibrated cameras. The only assumption is that these cameras realize a perspective projection.

In this section, we consider only points, lines and conics, because they can be easily observed in images, unlike planes. We remind the reader that the 3D collineation group is a manifold of dimension 15 (a collineation is defined by a 4×4 matrix up to a scale factor).

5.3.1 Case of six points

Consider 6 points in a 3D space. Each point is characterized by four homogeneous coordinates defined up to a scale factor: it therefore has three degrees of freedom, and the 6 point configuration has 18 degrees of freedom. Thus 18 is the dimension of configuration space. As the transformation group has a dimension 15, there are 3 invariants for this problem.

Let A, B, C, D, E and F be 6 points (see Fig. 17). We can construct three new points in a projectively invariant way: $D' = (EF) \cap (ABC)$, $E' = (DF) \cap (ABC)$ and $F' = (DE) \cap (ABC)$, where (ABC) represents the plane containing the three points A, B and C. The points A, B, C, D', E' and F' are coplanar; four of them form a projective basis and the two others have two projective coordinates in this basis, i.e. define four cross ratios (see Fig. 7). Three of them are independent: they are the invariants of the problem.

In the case where we know only the projections of the six points A, B, C, D, E and F onto two images, computing the projections onto these images of points D', E' and F' is possible using the coplanarity test. The cross ratios defined by coplanar points are invariant under projective projections. The three independent invariants of the problem can therefore be computed from the two images.
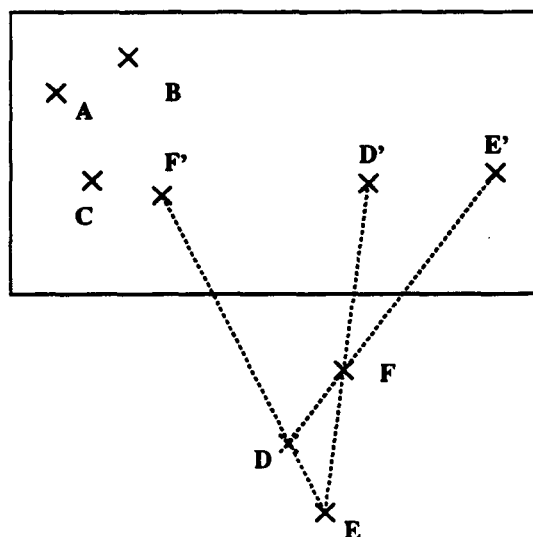
Figure 17: Case of 6 points in 3D space

5.3.2 Case of one straight line and four points

A straight line in 3D has four degrees of freedom. A configuration of one line and 4 points has 16 degrees of freedom, and one invariant under the collineation group action.

Given any configuration, three among the four points of the configuration define a plane which intersect the straight line of the configuration in one point. As there are four different subsets of three points, we can construct four points on the line: these four points define a cross ratio.

Like in the previous case, the coplanarity test allows us to compute this cross ratio from two images of the configuration.

5.3.3 Case of two straight lines and three points

A configuration of two straight lines and three points has $2 \times 4 + 3 \times 3 = 17$ degrees of freedom, and thus two invariants. The three points define a plane which intersects the two lines: that gives 5 coplanar points which define 2 independent cross ratios. These cross ratios may be computed from two images of the configuration using the coplanarity test.

5.3.4 Case of three straight lines and two points

This kind of configuration has $3 \times 4 + 2 \times 3 = 18$ degrees of freedom, that is to say three invariants. Let P_1, P_2, D_1, D_2 and D_3 be the two points and the three lines of a configuration.

We construct four planes π_1, π_2, π_3 and π_4 as follows: they contain respectively D_1 and P_1, D_1 and P_2, D_2 and P_1, and D_2 and P_2. These four planes intersect D_3 in four points which define a cross ratio.

By circular permutation on the indices of the lines, we obtain three cross ratios, which can, once more, be obtained using the coplanarity test. That implies that none of the projections of the three lines pass through one of the epipoles. With this assumption, we can determine the projections of each point of the three straight lines, and the construction of the coplanarity test is possible.

30

5.3.5 Case of four straight lines, two of them being coplanar

In this case, a configuration is composed of 2 coplanar straight lines and any two other lines. Two coplanar lines have 7 degrees of freedom, and the full configuration has 15.

An invariant may be easily constructed: the two coplanar lines define a plane which intersects the two other lines. We obtain two lines and two points in the same plane, and there is an invariant (see 4.2). As was the case in section 4.2, the transformation group and the configuration set have the same dimension. The existence of an invariant is due to an isotropy group.

Like the previous cases, the invariant cross ratio of this problem is computable from two images of the configuration if the epipolar geometry is known.

A particular subcase consists of two pairs of coplanar straight lines. Each pair defines a plane. The intersection of these planes is a line which meets the four lines in four points, which are therefore collinear: this gives a cross ratio.

5.3.6 Case of any four straight lines

This case is much more complex than the previous one. As any four lines have 16 degrees of freedom, there should be at least one invariant. A geometric construction of invariant cross ratios is possible. Here are the different steps of construction.

1. The fact that a line is included in a quadric gives 3 constraints that must be verified by that quadric. Let A and B be any two points and $\mathcal{Q}$ a quadric of equation:

$$ax^2 + by^2 + cy^2 + dyz + exz + fxy + gx + hy + iz + j = 0$$

Suppose the straight line (AB) is included in $\mathcal{Q}$: any point of (AB) may be written $P = \lambda A + B$ ($P = A$ if λ is infinite). We have:

$$(\forall \lambda \quad P \in \mathcal{Q}) \implies$$
$$a(\lambda x_A + x_B)^2 + b(\lambda y_A + y_B)^2 + c(\lambda z_A + z_B)^2$$
$$+ d(\lambda y_A + y_B)(\lambda z_A + z_B) + e(\lambda x_A + x_B)(\lambda z_A + z_B) + f(\lambda x_A + x_B)(\lambda y_A + y_B)$$
$$+ g(\lambda x_A + x_B) + h(\lambda y_A + y_B) + i(\lambda z_A + z_B) + j = 0$$

Writing that the three terms in λ^2, λ^1 and λ^0 are vanishing gives the three constraints.

2. As a quadric is define by 10 coefficients up to a scale factor, the fact that three lines belong to a quadric gives 9 constraints and totally define this quadric. As this quadric contains straight lines, it is a ruled quadric, i.e. an hyperboloid of one sheet or a hyperbolic paraboloid.

3. A ruled quadric contains two families of generators (i.e. straight lines included in this quadric). A single line of each family passes through each point of the quadric. Two lines of the same family never intersect one another, but a line of one of the families intersects all the lines of the other family.

4. Four lines of the same family of generators on a quadric define a cross ratio. As a matter of fact, any line D of the other family intersects these four lines at four collinear points. These points define a cross ratio which does not depend on D, but only on the four initial lines.

5. We consider any four straight lines in a 3D space such that no two of them are coplanar (see Section 5.3.5 otherwise). Three of them define a ruled quadric. As these lines do not intersect each other (they are not coplanar), they belong to the same family F_1 of generators. The fourth line intersects the quadric in two points P_1 and P_2. Through each of these points pass a line belonging to the generator family F_1. We obtain five lines of the same generator family, i.e. two cross ratios.

31

6. We can make permutations on the lines used to define the quadric, and we obtain 8 cross ratios. Two questions remain: how many of these cross ratios are independent? Are these cross ratios computable from two images of the four lines, the epipolar geometry being known?

This demonstration shows that there exist two lines L_1 and L_2 which meet any four lines. This defines four points of intersection on each of these two lines and then two cross ratios.

Conversely, there exist only two such lines. If a third one L_3 also meet the four lines, it would meet the quadric Q in three points, and then would be included in Q; it would met the fourth line in P_1 or P_2 and would be equal to one of the two first lines L_1 or L_2 according to the properties of generator families.

Hartley [1] has independently investigated the invariants of four lines and he demonstrates a few more results: the two cross ratios found are independent and completely characterize the set of four lines up to a 3D collineation. Intuitively, the independence is quite clear. The fourth line may move in such a way that P_1 remains in the same place and P_2 moves on the quadric: one of the cross ratio remains unchanged, and the other one varies.

As four lines have 16 degrees of freedom and as we found 2 invariants, there should exist an isotropy subgroup of dimension 1. With the reference points chosen such that the equations of L_1 and L_2 are $x = y = 0$ and $z = t = 0$, the isotropy subgroup contains the transformations represented by matrices of the form $diag(\alpha, \alpha, \beta, \beta)$ where α and β are any two constants.

5.3.7 Case of conics, points and lines

Case of one conic and three points. A conic in 3D is defined by 8 parameters: 3 to define the plane in which the conic lies, and 5 to define the conic in that plane. Each points has 3 parameters and the whole configuration has 17 of them. That is $\dim E = 8 + 3 \times 3 = 17$, $\dim G = 15$. There exist at least two invariants.

The three points define three lines which meet the plane defined by the conic in three collinear points. The line, which these points belong to, meets the conic in two points and we obtain thus five collinear points that allow us to compute two cross ratios (see Fig. 18).
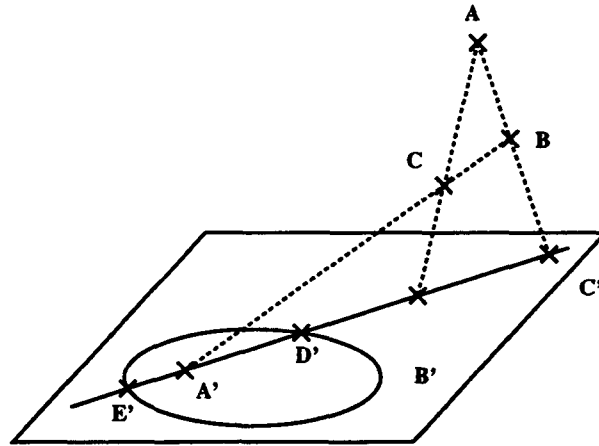
Figure 18: Case of one conic and three points

Other cases. In the case of one conic, two points and one line, the two points define a line, and thus we have two lines which intersect the conic plane in two points. That defines a line which intersects

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[1] He spoke about his results during the workshop day of ECCV92, but no paper has been published yet.

32

the conic in two points: we therefore obtain four collinear points and we can compute a cross ratio. To find the two other invariants, we consider the pencil of planes defined by the configuration line. We can get five particular planes in this pencil: two planes are tangent to the conic, two go through the two configuration points and one goes through the point of intersection between the conic plane and the line defined by the two points. Five planes in a pencil define two cross ratios.

In the same way, we can respectively find one invariant for one conic and two lines ($1 = (8 + 2 \times 4) - 15$), one for two conics ($1 = 2 \times 8 - 15$)...

6 Experimental results

6.1 Introduction

In this section we show results on both simulated and real data. The simulated data allow to study quantitatively the effect of noise on the numerical results. As simulated noise is not exactly similar to real noise, some results on real images are also shown.

For the simulated images, noise is uniform. This choice was done to control precisely noise amplitude. With a Gaussian noise, the variance may change, but any value of noise is always possible since its probability never vanishes. Furthermore some of the image noise is uniform such as discretization noise.

Many of the tests done lead to compare cross ratios. To do so, we used the method of the random cross ratios [ÅM92]. Four points are uniformly chosen on a line and their cross ratio is computed. That allow to compute the density of the cross ratio values on the real line, and then the corresponding partition function F. To compare two cross ratios x and y, we compute their "difference", i.e. $|F(x) - F(y)|$, their distance with respect to the repartition function. As the point at infinity plays the same role as all the other real points, we have to consider the two possible ways to go from x to y: the real way, and the way going through the point at infinity. In conclusion, the "difference" between cross ratios is $\min(|F(x) - F(y)|, 1 - |F(x) - F(y)|)$, where:

$$F(x) = \begin{cases} F1(x) + F3(x) & \text{if } x < 0 \\ 1/3 & \text{if } x = 0 \\ 1/2 + F2(x) + F3(x) & \text{if } 0 < x < 1 \\ 2/3 & \text{if } x = 1 \\ 1 + F1(x) + F2(x) & \text{if } 1 < x \end{cases}$$

$$F1(x) = \frac{1}{3}\left(x(1-x)\ln\left(\frac{x-1}{x}\right) - x + \frac{1}{2} \right)$$

$$F2(x) = \frac{1}{3}\left(\frac{x - x\ln(x) - 1)}{(x-1)^2} \right)$$

$$F3(x) = \frac{1}{3}\left(\frac{(1-x)\ln(1-x) + x}{x^2} \right)$$

For more details, see [ÅM92].

This method is absolutely necessary to compare cross ratio values. For example 0 is as far from 1 as ∞ due to the symmetric properties of the cross ratio.

The computation of the epipolar geometry, i.e. the fundamental matrix and the two epipoles of a pair of images, can be efficiently done if we consider each matrix entries as unknown with no other constraints. This leads to a system of linear homogeneous equations, that can be solved using a singular value decomposition method (SVD, see [PFTW88]) in the least squares sense. Unfortunately, this linear solution without constraints is sometimes far away from the real solution in practice. Nonlinear iterative method is necessary to obtain better estimation, as it has been reported in [FLM92].

Our method to compute the epipolar geometry is to get a first solution by the SVD linear method and to improve it by a nonlinear least squares method, based on Levenberg-Marquardt's algorithm.

33

The error function is the same as that proposed in [FLM92]: it is defined as the sum of the squares of the distance of a point to its epipolar line. The constraint of rank 2 of the matrix is directly imposed to the matrix representation, as it is done in [FLM92].

6.2 The coplanarity test

The coplanarity test is studied with simulated data. This test gives a way to detect image points which correspond to 3D coplanar points. The search for such points may be used to compute 2D invariants associated with some point particular configurations, or to choose a projective basis for uncalibrated projective reconstruction [Fau92, MQVB92]. In this last case, the test needs only to be a necessary test.

The experiment is set as follows:

1. Nine points are uniformly chosen in the cube $[10, 246]^3$.

2. Three other points are chosen such that they are respectively coplanar with three of the previous points. Therefore we have 3 quadruplets of coplanar points, and 492 quadruplets of non coplanar points.

3. The points are projected into two images. Between the two images, the motion of the camera is a rotation of 30 degrees.

4. Some noise is added to the the position of these projected points.

5. The epipolar geometry is computed.

6. For all quadruplets of image points corresponding to 3D coplanar points, two cross ratios are computed in each image. With the notations of the paragraph 5.2.2, these cross ratios are $(e_1 a_1, e_1 b_1, e_1 c_1, e_1 f_1)$, $(e_1 b_1, e_1 c_1, e_1 d_1, e_1 f_1)$ and the equivalent ones in the second image.

7. The differences between the corresponding cross ratios is computed.

8. For each level of noise, twenty such image pairs are drawn. That gives Fig. 19 which shows the maximum, minimum and mean difference between the cross ratios.

   For one of the noise levels (0.5 pixels) we show on Fig. 20, the histogram of the cross ratio differences.

9. The coplanarity test is then applied to all the quadruplets of points corresponding to 3D non coplanar points. For each noise level, Fig. 21 shows the percentage of these quadruplets that have two pairs of cross ratios more similar than the mean value or the maximum value obtained in Fig. 19 for the same noise level.

Fig. 19 shows that the coplanarity test works well in general. The mean value obtained for the cross ratio differences computed from the projections of coplanar points does not grow too fast with respect to image noise. For a noise level inferior to 2.5 pixels, the mean value is smaller than 0.05. this value is 0.075 for a three-pixel noise.

With comparison to these good results with mean values, the maximum values seem to be very bad. Even with low levels of noise, 0.5 pixels for example, the projections of coplanar points give very different cross ratios, i.e. are not recognized as corresponding to 3D coplanar points. The Fig. 20 shows that these degenerate cases are rare. They correspond to coplanar points whose plane almost go through the projection center, and for which the noise can have catastrophic effects. The order of the lines in the line pencils may change for example. In the histogram, almost 80% of the values are smaller than 0.01 and 97.5% are smaller than 0.075.

Fig. 21 shows the results of the test in terms of discrimination between projections of coplanar points and projections of non coplanar points. If we take into account the worst values obtained to
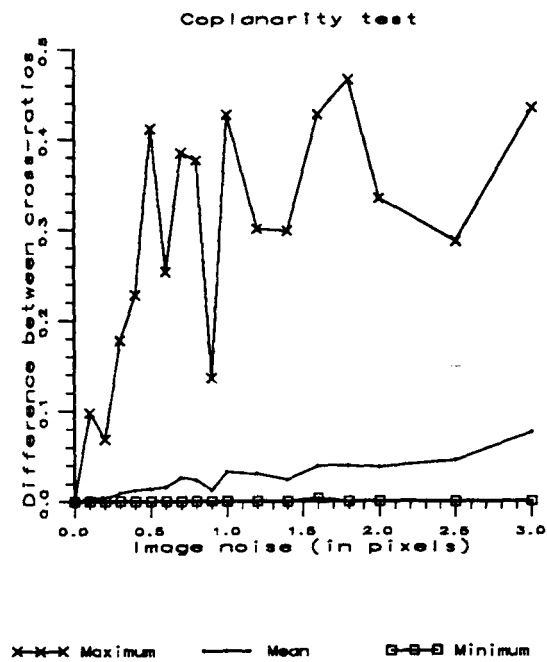
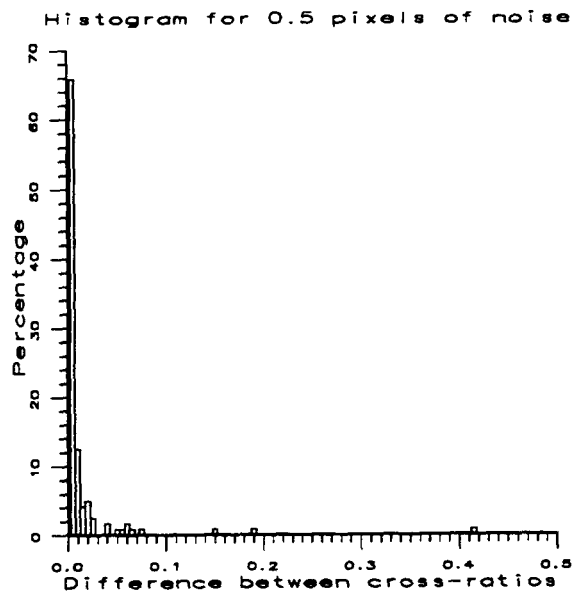Figure 19: Coplanarity test applied to coplanar point projections.

Figure 20: Histogram of the cross ratio differences
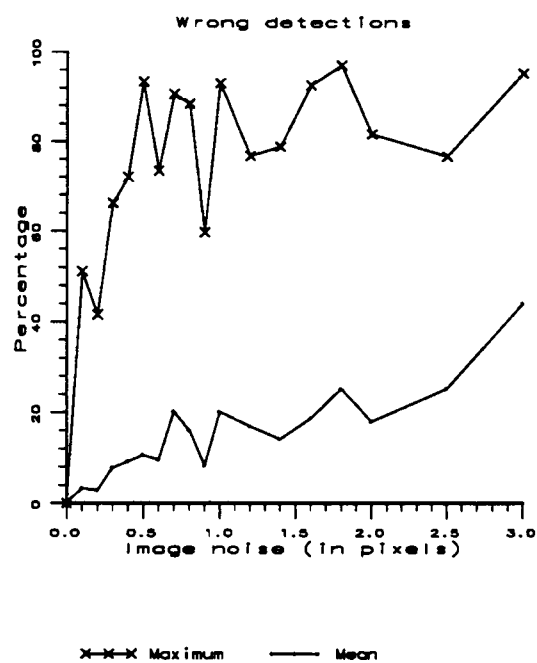
35

Wrong detections

Figure 21: Percentage of image point quadruplets giving a better answer to the coplanarity test than coplanar point projections. The maxima correspond to degenerate cases.

36

the test by the projections of coplanar points, the results are bad: 93% of the projections of non coplanar points give a better answer to the test than the worst projections of coplanar points in the case of 0.5 pixel noise. If we consider the mean values given by the Fig. 19, the results are not so bad: for a noise smaller than 0.5 pixels, 90% of the quadruplets are eliminated, and for a noise smaller than 2.5 pixels, 80% of the quadruplets are eliminated.

In conclusion one can see that the test works well if one eliminates the degenerate cases of points almost lying on the line going through the epipole. This is not surprising since it is a case for which the test is not valid (see paragraph 5.2.2). Otherwise the test is necessary but not sufficient to detect the projections of coplanar points. This insufficiency is due to the image noise. Nevertheless it allows a efficient reduction of the potential candidates to coplanarity.

6.3   The invariant of two pairs of coplanar lines with simulated data

This paragraph deals with the invariant defined by 2 pairs of coplanar lines. It is a particular case of the invariant studied in 5.3.5 page 31. The method used to obtain data is:

1. 48 points are uniformly chosen in $[10, 236]^3$. They are organized in 16 angles, i.e. there is a segment between the points 1 and 2, 2 and 3, 4 and 5, 5 and 6, 7 and 8...

2. All these points and lines are projected into two images. Between the two images, the motion of the camera is a rotation of the camera of 30 degrees around the points. Some noise is added to the position of the projected points.

3. For each pair of angles, an invariant is computed using the method of coplanar point reconstruction described in 5.2.2. Because of noise, the cross ratios computed in the two images may differ. Fig. 22 (left) shows the mean differences between these invariants.

4. To be able to evaluate these results, we use another method. We reconstruct the 3D points from the noisy 2D points using Faugeras's method [Fau92]. Then we compute from these 3D points the invariants of each pair of angles. This invariant is compared to the invariant computed from the original 3D points. Fig. 22 (right) shows the mean value of their difference.

5. With our method, the invariants computed in the two images are different because of noise. For each invariant of one image, we can count the number of invariants of the other image which are nearer to itself than the corresponding invariant. As a matter of fact, we count the the number of invariants of the second image which gives a difference between cross ratios inferior to the thresholds determined in Fig. 22. That gives the results of the left of Fig. 23.

6. The same thing is done for the invariants computed from the reconstructed points with respect to the invariants computed from the exact 3D points. This gives the result of Fig. 23 (right).

Fig. 22 is easily interpretable. It shows the stability of invariant computation. The mean stability is good, but even with no noise, some degenerated cases give bad results due to numerical errors. These degenerate cases are rare as shown on the histogram of the cross ratios differences for 0.5 pixels of noise (Fig. 24).

Fig. 23 is rather more difficult to interpret. It shows a statistical measure of the discriminant character of the invariants. Next paragraph gives more easily interpretable results.

Both figures show better results for our method than for the one using reconstruction. This is due to the errors added by the reconstruction stage.

6.4   The invariant of two pairs of coplanar lines with real data

This paragraph presents results on real data. Fig. 25 represents three images of the same object: it is a dihedron with triangles painted on both sides. Contours were extracted from these three images
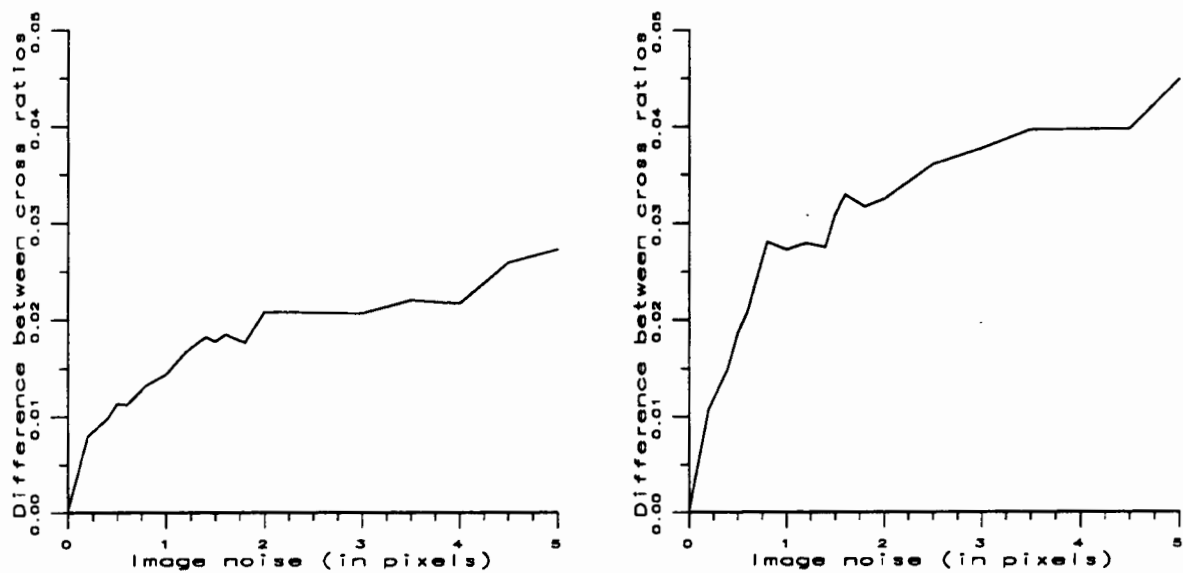
Figure 22: Robustness of the 3D invariant of two pairs of coplanar lines. Left: our method. Right: using 3D reconstruction.
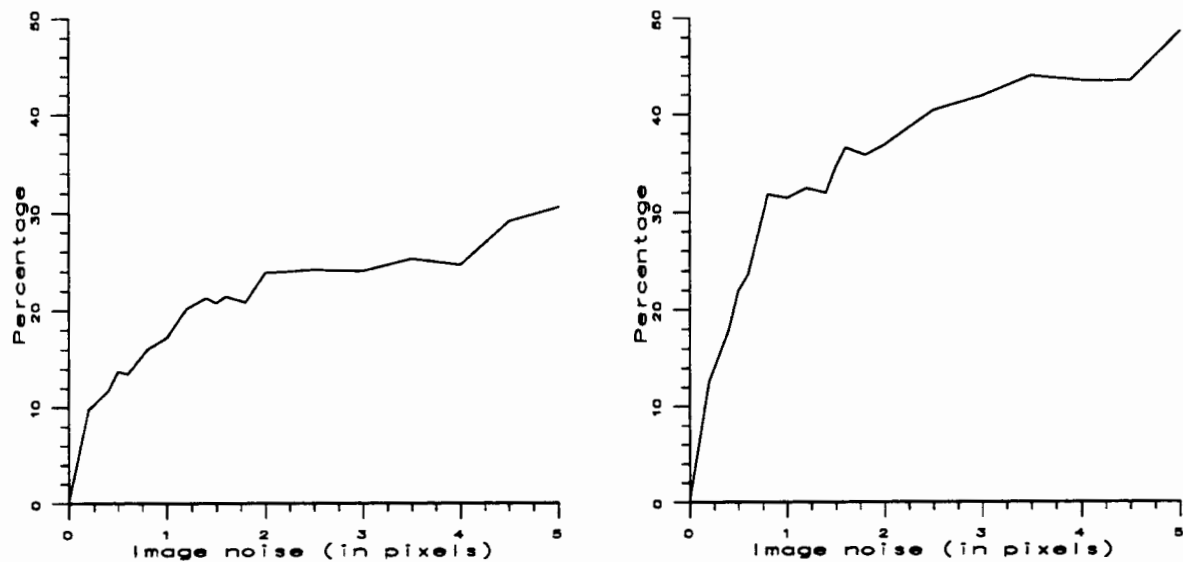

Figure 23: Percentage of wrong matches giving a better result than correctly matched angles. Left: our method. Right: using 3D reconstruction.
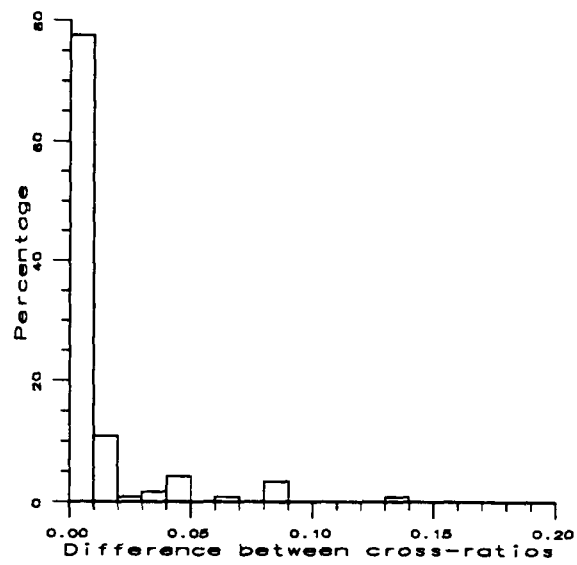
38

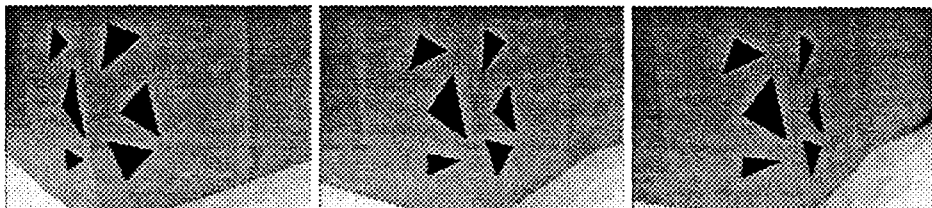Figure 24: Histogram of the cross ratio differences for 0.5 pixels of noise.

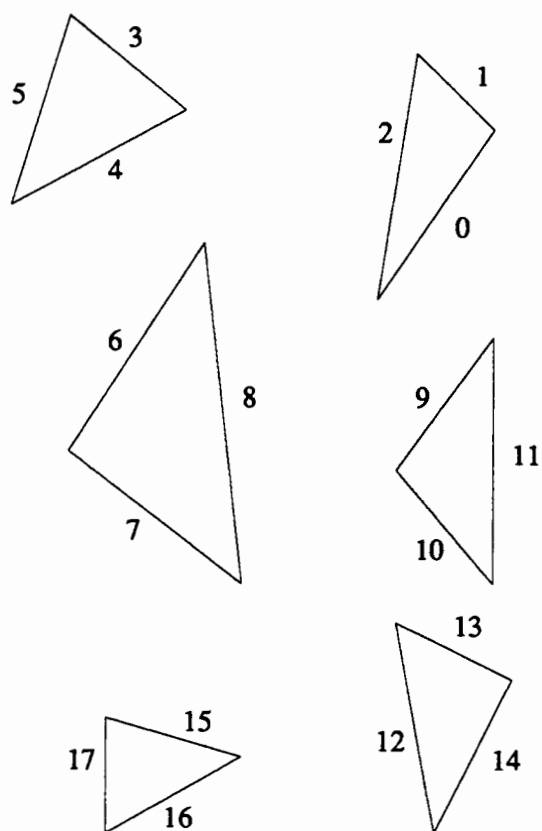Figure 25: Three images of a dihedron

39

Figure 26: Segment numbering.

and were approximated by segments numbered as in Fig. 26.

The two first image were matched, the epipolar geometry was computed, and some invariants were computed. Like in the previous paragraph, these invariants are those of two pairs of coplanar lines. The same operations were done with the last two images.

Table 27 shows the results: the (i, j) element of the table gives the difference of the invariants of S_i computed from the images 1 and 2, and S_j computed from the images 2 and 3. This difference is computed as mentioned in the beginning of the section. We use the following sets of lines:

$$
\begin{aligned}
S_1 &= \{0,2,3,5\} \\
S_2 &= \{1,2,15,16\} \\
S_3 &= \{3,5,9,11\} \\
S_4 &= \{3,5,12,14\} \\
S_5 &= \{7,8,12,14\} \\
S_6 &= \{9,11,15,17\} \\
S_7 &= \{12,13,15,17\}
\end{aligned}
$$

	S_1	S_2	S_3	S_4	S_5	S_6	S_7
S_1	**0.000766**	0.035241	0.046054	0.011993	0.082378	0.097035	0.007931
S_2	0.034279	**0.000195**	0.081099	0.047038	0.117423	0.132080	0.027115
S_3	0.037669	0.072144	**0.00915**	0.024910	0.045474	0.060131	0.044834
S_4	0.018515	0.052989	0.028305	**0.005756**	0.064629	0.079286	0.025679
S_5	0.079718	0.114193	0.032898	0.066960	**0.003425**	0.018082	0.086883
S_6	0.106032	0.140507	0.059212	0.093273	0.022888	**0.008231**	0.113197
S_7	0.007940	0.026535	0.054760	0.020699	0.091083	0.105573	**0.000775**

Figure 27: Differences between invariants.

The invariants appear to be a good discriminator. The only ambiguities are the entry (1, 7) and (7, 1) of the table. As in the two last paragraphs, invariants constitute a necessary but non sufficient test of recognition. Non sufficiency is due to image noise.

Two points have to be noticed: firstly the test only real data is better than that with simulated data. This is due to the simulated data which contain degenerate or almost degenerate cases. Because of the noise and of numerical approximations, such degenerate cases are probable with a random generator of data. These cases do not occur so frequently in practice.

Secondly the invariants of 2 pairs of coplanar lines do not depend from dihedron angle. Then these invariants could be used for the recognition of articulated objects.

7 Conclusion

In this paper, we have shown 3 ways to approach the problem of finding invariants. The first one, based on the concept of manifolds and on the properties of these manifolds, mainly produces an enumeration of the independent invariants existing in a given problem; except in very simple cases, it leads to very difficult equations whose resolution is often impossible. The second method initiates a generalization of the problem into a form that may be solved, and then specialized back to the initial problem. This is helpful to calculate the invariants of the subproblems of a given problem that can be solved. The third method looks for particular invariants related to the canonical invariant of the considered transformation.

Even if this last method is not as general as the other two, it easily finds the invariants of a problem in many cases, and it provides an immediate geometrical interpretation of these invariants.

The use of this method should therefore be given first priority, even if the results are possibly to be completed with other methods.

In the last part, we have shown how this last method can be used to solve 3D vision problems like recognition. These applications are due to the fact that 3D invariants like cross ratios can be computed directly from 2D images without any reconstruction, if point matching and epipolar geometry between the two images are known. As they may be computed on local features, they are usable in case or partial occlusion. Some of them are also usable for articulated objects.

The experimental results show that the cross ratio is not too sensitive to noise if the degenerate cases are eliminated. In any case these degenerate cases are not very frequent. The discriminating character of the cross ratio is dependent on image noise and in a practical use, the test of coplanarity presented in the paper is only necessary. The experiments on real data prove the validity of the results on simulated data.

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Contents