Invariant of a Pair of Non-coplanar Conics in Space: Definition, Geometric interpretation and Computation

Long QUAN
LIFIA - CNRS - INRIA,46, avenue Felix Viallet, 38031 Grenoble, France

Abstract

The joint invariants of a pair of **coplanar** *conics has been widely used in recent vision literature. In this paper, the algebraic invariant of a pair of* **non-coplanar** *conics in space is concerned.*

The algebraic invariant of a pair of non-coplanar conics is first derived from the invariant algebra of a pair of quaternary quadratic forms by using the dual representation of space conics. Then, this algebraic invariant is geometrically interpreted in terms of cross-ratios. Finally, an analytical procedure for projective reconstruction of a space conic from two uncalibrated images is developed and the correspondence conditions of the conics between two views are also explicited.

1 Introduction

The study of invariants has recently provoked much interest in the computer vision community, since it is crucial for the development of efficient recognition systems for model based vision, *cf.* the collection book [16]. Most of the invariants are derived for planar objects using geometric entities such as points, lines and conics from one single image. One of the most used planar invariants is the joint invariants of a pair of planar conics [5]. Some of the algebraic and geometric properties of these invariants are further clarified in [15, 19].

Following the important results of Faugeras [2] and Hartley *et al.* [9] concerning the projective reconstruction of point sets from the epipolar geometry of the two uncalibrated images, the invariants for the configurations of points and lines in space from two uncalibrated images have been investigated in [1, 8, 6, 7]. The invariants from three or more uncalibrated images have also been studied in [18] and [14].

In this paper, we propose to study the invariants of a pair of *non-coplanar* conics in space from two uncalibrated images. The key idea for computing invariants from two images is the use of the epipolar geometry of the two uncalibrated images. This assumption is still maintained in this paper.

The geometric invariant of a pair of non-coplanar conics has first been mentioned in [7] as illustrated in Figure 1. This geometric invariant is very simply, but unfortunately is not directly computable. In this paper, an algebraic invariant of a pair of non-coplanar conics in space will be defined from the algebra of invariants of quaternary quadratic forms, then its relationship with the geometric invariant will also be established, *i.e.* the algebraic invariant is geometrically interpreted in terms of cross-ratios which define the geometric invariant. This study is inspired by the last century's mathematical developement on invariants of the quadratic forms [21, 22, 10]. As the algebraic structure of absolute invariants is much more complicated than that of relative invariants, the mathematicians have mainly been concerned with the relative invariants and have paid little attention for the absolute invariants which are the most useful for computer vision.

After the definition of the algebraic invariant, it remains to compute it effectively from two uncalibrated images. At this stage, the key operation is the projective reconstruction of the space conic from two uncalibrated images. Once the projective reconstruction is done, the invariant can be computed in a straightforward way. A very simple analytical method for projective reconstruction of conics in space will be developed in this paper. The reconstruction procedure is essentially linear in that the two solutions of reconstruction are solved together with only linear computation. Only the extraction of the two different solutions needs to solve a quadratic equation. It is also clarified that the solutions to conic reconstruction are generally ambiguous up to two solutions and is unique only for non transparent objects.

2 Invariants of a pair of non-coplanar conics in space

2.1 Number of invariants

Given a geometric configuration, the number of invariants is roughly speaking the difference of the dimension of the configuration group and the dimension of the transformation group (if the dimension of the isotropy group of the configuration is null). For a pair of non-coplanar conics in $\mathcal{P}^3$, there is $1 = 2 \times (3+5) - (16-1)$ absolute invariant under the action of the general linear group $GL(3)$ in $\mathcal{P}^3$. Since each space conic has $8 = 3+5$ degrees of freedom (5 for a conic in a given plane and 3 for the plane in which the conic lies), the dimension of a pair of space conics is $2 \times 8 = 16$. The transformation group $GL(3)$ is represented by a 4×4 matrix up to a scaling factor, its dimension is $4 \times 4 - 1 = 15$.

2.2 Geometric invariant

The unique invariant can be geometrically constructed as illustrated in Figure 1. If the 4 intersection points of the conics with the common line of the conic planes are respectively P_1, P_2, P_3 and P_4 whose projective parameters are θ_1, θ_2, ψ_1 and ψ_2. The cross-ratio

$$\rho = (P_1, P_2; P_3, P_4) = (\theta_1, \theta_2; \psi_1, \psi_2) = \frac{(\theta_1 - \psi_1)/(\theta_2 - \psi_1)}{(\theta_1 - \psi_2)/(\theta_2 - \psi_2)}$$

is the geometric invariant.

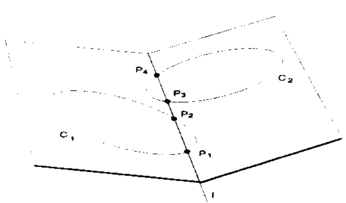

Figure 1: Geometric invariant of a pair of non-coplanar conics: the cross-ratio of the 4 intersection points of the conics with the common line of the conic planes.

2.3 Space conic as a quadric envelope

The principle of duality is a fundamental concept in projective geometry. In $\mathcal{P}^3$, points and planes are dual to each other. The space dual of a plane curve is a cone. The conic is a quadric envelope of rank 3 which is called a disk quadric [21]

The plane equation (in plane coordinates instead of usual point coordinates) of the conic, represented as the complete intersection of a proper cone $x^T Q x = 0$ where Q is a symmetric 4×4 matrix and a plane $\pi = p^T x = 0$ in point coordinates, is

$$\begin{vmatrix} 0 & 0 & u^T \\ 0 & 0 & p^T \\ u & p & Q \end{vmatrix} = 0 \qquad (1)$$

where $u = (u_1, u_2, u_3, u_4)^T$ represents a variable plane.

2.4 Review of invariants of quaternary quadratic forms

Since a space conic is a quadric envelope, we first review invariants of a pair of quaternary quadratic forms following Semple and Kneebone [21], Todd [22] and Johnson [10]. Given two quaternary quadratic forms, $\mathcal{Q}_1 \equiv x^T A x$ and $\mathcal{Q}_2 \equiv x^T B x$ in x where A and B are both symmetric 4×4 matrices, consider the one dimensional family of quadratic forms, *i.e.* linear combination of $\mathcal{Q}_1$ and $\mathcal{Q}_2$, $\mathcal{Q}(\lambda, \mu) \equiv \lambda \mathcal{Q}_1 + \mu \mathcal{Q}_2$. The vanishing of the determinant of $\mathcal{Q}(\lambda, \mu)$ is given by $|\lambda A + \mu B| = 0$, that is, $I_1 \lambda^4 + I_2 \lambda^3 \mu + I_3 \lambda^2 \mu^2 + I_4 \lambda \mu^3 + I_5 \mu^4 = 0$,

where $I_1 = |A|$, $I_2 = \sum_{i,j=1}^4 b_{ij} A_{ij}$, $I_4 = \sum_{i,j=1}^4 a_{ij} B_{ij}$, $I_5 = |B|$, and $I_3 = |b_1\, b_2\, a_3\, a_4| + |b_1\, a_2\, b_3\, a_4| + |b_1\, a_2\, a_3\, b_4| + |a_1\, b_2\, b_3\, a_4| + |a_1\, b_2\, a_3\, b_4| + |a_1\, a_2\, b_3\, b_4|$, a_{ij} and b_{ij} are entries of A and B, A_{ij} and B_{ij} cofactors of a_{ij} and b_{ij} in A and B. $A = (a_1\, a_2\, a_3\, a_4)$ and $B = (b_1\, b_2\, b_3\, b_4)$ are the column partition of A and B.

When x transforms into $x' = Tx$ by any non singular space collineation T, I_j are transformed into $I'_j = |T|^2 I_j$. I_j $(j = 1, \ldots, 5)$ are relative invariants of weight 2 of the two quaternary quadratic forms $\mathcal{Q}_1$ and $\mathcal{Q}_2$.

2.5 Algebraic invariant of a pair of space conics

Let C_1 and C_2 be two space conics, represented by their dual quadric envelopes $\mathcal{C}_1 \equiv u^T A u = 0$ and $\mathcal{C}_2 \equiv u^T B u = 0$. Consider the invariant algebra of quadratic forms $\mathcal{C}_1$ and $\mathcal{C}_2$ in u. As $\mathcal{C}_1 = 0$ and $\mathcal{C}_2 = 0$ represent conics, so their associated quadric envelopes are of rank 3, therefore $I_1 = I_5 = |A| = |B| = 0$.

We are left with I_2, I_4 and I_5 non vanishing. Two absolute invariants $I_2 : I_3 : I_4$ can be defined for their associated quadratic forms, however they are not yet the invariants of the conics, since each conic $\mathcal{C}_i = 0$

is associated with a family of quadratic forms $\lambda_i \mathcal{C}_i$ for any scalar $\lambda_i \neq 0$.

To obtain the absolute invariants of two conics, the power degrees of λ and μ in the family of quadratic forms should also taken into account. When this is done, the unique and simplest absolute invariant is

$$I = \frac{I_3^2}{I_2 I_4}. \tag{2}$$

3 Geometric interpretation of the algebraic invariant

As we mentioned earlier, the geometric invariant ρ of two non-planar conics is easily defined as the cross ratio of two pairs of points, each pair of which is the intersection of a conic with the common line of the two conic planes. In this section, we want to interpret geometrically the algebraic invariant I in terms of this geometric invariant ρ. The basic idea is to choose particular coordinate representations for quadrics in order to bring them into the simplest forms, [21, 10]. The detailed development of this part is omitted due to page limitation, one can consult [17][1] The relation between the geometric invariant ρ and the algebraic invariant I is established as follows:

$$I = 4(\frac{\rho+1}{\rho-1})^2 \tag{3}$$

4 Projective reconstruction of the conics in space

It remains now to reconstruct the space conic from image conics which are in $\mathcal{P}^2$. This section will give an algorithm for projective reconstruction of a space conic from its two uncalibrated images.

4.1 Preliminaries

If we assume a perspective projection for camera model, represented by a 3×4 matrix P, the relation between an image point in its homogeneous coordinates $\tilde{x} = (u, v, w)^T$ in $\mathcal{P}^2$ and a space point in its homogeneous coordinates as well $x = (x_1, x_2, x_3, x_4)^T$ in $\mathcal{P}^3$ are linearly related by $\lambda(u, v, w)^T = P(x_1, x_2, x_3, x_4)^T$.

[1] Available at ftp.imag.fr, /pub/MOVI/Quan-iccv95-ext.ps.gz.

We assume that the epipolar geometry is always given for the pair of uncalibrated images. The epipolar geometry can be nicely coded by a 3×3 rank 2 matrix F, called fundamental matrix [3, 9]. According to Hartley [9], one possible choice of projection matrices for two cameras consistent with a decomposition of $F = M[t]_x$ is given by

$$P = (I_3 \;\; 0_3) \quad \text{and} \quad P' = (M^* \;\; -M^* t) \tag{4}$$

where M is a non singular 3×3 matrix, $t = (t_1, t_2, t_3)^T$. M^* is the adjoint matrix of M, and $[t]_x$ is the antisymmetric 3×3 matrix associated to the vector t. For more details, see [9].

The reconstruction in $\mathcal{P}^3$ is therefore defined up to the projective transformation of the placement of the first camera.

4.2 Formulation of projective reconstruction

Given a corresponding pair of conics in two distinct images, $\mathcal{C} \equiv \tilde{x}^T C \tilde{x} = 0 \leftrightarrow \mathcal{C}' \equiv \tilde{x}'^T C' \tilde{x}' = 0$,

we require to find a conic in space which has been projected respectively into $\mathcal{C}$ and $\mathcal{C}'$. A conic in space is generally represented as the complete intersection of a quadric surface and a plane. The reconstruction is therefore equivalent to find the plane in which the conic lies, as we can take any one of the two cones associated to two conics in images as the quadric surface.

The cone equation associated to a given conic and a given camera is obtained as follows.

Given a projection matrix P of a camera, the equation of the cone which joins the conic $\tilde{x}^T C \tilde{x} = 0$ in the image plane to the projection center of the camera is $x^T Q x = x^T P^T C P x = 0$.

This can be easily proved by substituting $\lambda \tilde{x}^T = Px$ into the conic equation $\tilde{x}^T C \tilde{x} = 0$. $x^T Q x = 0$ is effectively a proper cone, for rank(Q) = rank(P) = rank(C) = 3 and Ker(Q) = Ker(P) which is meant that the vertex of the cone is the projection center of the camera.

The cones corresponding to the pair of conics are respectively $\mathcal{Q} \equiv x^T A x = x^T P^T C P x = 0$ and $\mathcal{Q}' \equiv x^T B x = x^T P'^T C' P' x = 0$ in $\mathcal{P}^3$.

Consider the pencil of quadric surfaces $\mathcal{Q} + \lambda \mathcal{Q}' = 0$, for every value of λ the equation $\mathcal{Q} + \lambda \mathcal{Q}' = 0$ represents a quadric surface which passes through all the common points of $\mathcal{Q}$ and $\mathcal{Q}'$. The points common to all quadric surfaces of the pencil are simply the points which make up the curve of intersection of $\mathcal{Q}$ and $\mathcal{Q}'$, and this curve is the base curve of the pencil.

The base curve of two quadric surfaces is generally a quartic curve. In our context, the reconstruction constraints impose that the corresponding cones intersect in a conic in space. As this conic in space should be part of the base curve, thus the base curve of the pencil should break up and one of the components is a conic in space. Even more, if one of the components of the base curve is a conic, the residual component should also be a conic. As a pair of planes can be considered as a degenerate quadric surface of rank 2, according to the results of projective geometry (cf. [21]) on pencils of quadric surfaces, the degenerate quadric surface composed of the pair of planes belongs to the pencil of quadric surfaces in consideration. We are therefore led to examine a special pencil of quadric surfaces which contains a degenerated member of rank 2. Based on this, we can reformulate the problem of conic reconstruction as follows:

The reconstruction of a conic in space from two images is equivalent to find a λ such that the λ-matrix $C(\lambda) = A + \lambda B$ has rank 2. $x^T Ax = 0$ and $x^T Bx = 0$ are the proper cones corresponding to the two images of the conic in space.

4.3 Correspondence conditions

Unlike points and lines, two images of a conic in space contain sufficient information to impose correspondence conditions. The number of the independent conditions which can be derived is established as follows:

There exist two independent polynomial conditions for a corresponding pair of conics.

To prove it, we need only to count the degrees of freedom of the rank 2 matrix and those of the matrix pencil. A 4×4 symmetric matrix up to a scaling factor counts for $10 - 1 = 9$ degrees of freedom, thus a general matrix pencil counts for $9 - 1 = 8$ degrees of freedom. A rank 2 symmetric matrix C of order 4 counts for 6 degrees of freedom, so there remain $2 = 8 - 6$ independent conditions.

We will now derive these two polynomial conditions.

Consider the characteristic polynomial of λ−matrix $C(\lambda) = A + \lambda B$,

$$|C(\lambda)-\mu I| = \mu^4+a_1(\lambda)\mu^3+a_2(\lambda)\mu^2+a_3(\lambda)\mu+a_4(\lambda) = 0.$$

$C(\lambda)$ is a real symmetric 4×4 matrix. For it to have rank 2, it must have two distinct nonzero eigenvalues and a double zero eigenvalue. The conditions we are looking for are equivalent to have

$$\begin{cases} a_3(\lambda) = 0, \\ a_4(\lambda) = 0. \end{cases} \tag{5}$$

By definition,

$$a_4(\lambda) = |C(\lambda)| = |A+\lambda B| = I_1\lambda^4+I_2\lambda^3+I_3\lambda^2+I_4\lambda+I_5,$$

where the coefficients I_j are polynomials in the entries of A and B.

Since A and B both have rank 3, $I_1 = |A| = 0$ and $I_5 = |B| = 0$. The characteristic polynomial of the pencil is factorized as $a_4(\lambda) = \lambda(I_2\lambda^2 + I_3\lambda + I_4) = 0$.

There are generally four singular matrices of the pencil, each corresponds to one of the four generalized eigenvalues of the pencil, the roots of $a_4(\lambda) = 0$. Two generalized eigenvalues of the pencil are easily read out as $\lambda = 0$ and $\lambda = \infty$ which corresponds respectively to A and B. The two others are the solutions of the quadratic equation

$$I_2\lambda^2 + I_3\lambda + I_4 = 0. \tag{6}$$

In order to have a rank 2 matrix in the pencil, we should at least have a generalized eigenvalue of multiplicity 2, hence the above quadratic equation (6) must have two equal roots. This gives the first condition for correspondence. The second one is derived by computing the resultant of $a_3(\lambda)$ and $a_4(\lambda)$ with respect to λ. The final results can be summarized in

The two polynomial correspondence conditions for a pair of corresponding conics are respectively $\Delta = 0$ and $\Theta = 0$:

$$\begin{aligned} &\Delta \equiv I_3^2 - 4I_2I_4 = 0, \\ &\Theta \equiv -J_1I_3^3 + 2J_2I_3^2I_2 - 4J_3I_3I_2^2 + 8J_4I_2^3 = 0. \end{aligned}$$

where J_i are polynomials in the entries of A and B.

4.4 Closed-form solutions of reconstruction

The degenerate quadric surface Since we must have two equal roots for the quadratic equation (6), the double generalized eigenvalue is directly obtained by $\lambda = -\frac{I_3}{2I_2}$.

Then we obtain the matrix $C = A + \lambda B$ of the degenerate quadric surface.

The remaining effort for conic reconstruction requires only to extract the two planes from this rank 2 matrix C.

Extraction of the plane pair Going back to the characteristic polynomial of the matrix $C(\lambda)$, it is simplified by the second condition $\Theta = 0$ as $\mu^2(\mu^2 + a_1(\lambda)\mu + a_2(\lambda)) = 0$.

The remaining two nonzero eigenvalues μ_1 and μ_2 are the roots of the quadratic equation:

$$\mu^2 + a_1(\lambda)\mu + a_2(\lambda) = 0. \quad (7)$$

As C is a real symmetric matrix, there exists a non singular transformation T such that C is diagonalized: $T^T C T = \mathrm{diag}(\mu_1, \mu_2, 0, 0)$.

The quadric surface $x^T C x = 0$ is therefore transformed by $x = Tx'$ into $x'^T \mathrm{diag}(\mu_1, \mu_2, 0, 0)x' = 0$, i.e. $\mu_1 x_1'^2 + \mu_2 x_2'^2 = 0$.

The pair of planes $\pi_i' \equiv p_i'^T x' = 0, i = 1, 2$ in the transformed reference frame is $(\sqrt{\mu_1}, \pm\sqrt{-\mu_2}, 0, 0)^T x' = \sqrt{\mu_1}x_1' \pm \sqrt{-\mu_2}x_2' = 0$.

It is obvious that to obtain real planes, we must have $a_2(\lambda) = \mu_1\mu_2 < 0$.

Let v_1 and v_2 be the eigenvectors corresponding to the eigenvalues μ_1 and μ_2 of C. The plane pair $\pi_i \equiv p_i^T x = 0, i = 1, 2$ in the original reference frame are obtained by $(Tp_i')^T x = (\sqrt{\mu_1}v_1 \pm \sqrt{-\mu_2}v_2)^T x = 0$.

Then the conic in space is defined as the intersection of one of the two cones with the plane recovered above.

4.5 Related work on conic reconstruction

It is also important to note that several authors have remarked the importance of conics as basic image features and developed some procedures for pose estimation, stereo and motion based on conics, for instance [11, 12, 20, 16, 4, 13]. The conic reconstruction algorithm proposed in this paper is related to but different from those of Ma [12] and Safaee-Rad *et. al* [20]. They both have been interested in the Euclidean reconstruction of space conic and proposed different solutions to the problem.

The projective reconstruction of space conic can be easily extended to Euclidean reconstruction when the stereo system is strongly calibrated. Also, the ambiguity of double solution can be removed if we suppose that the conic in space is a non transparent object The details of these are given in [17].

5 Experimental results

The experimental results are mainly presented in [17], due to page limitation, we are content with one example here.

We used a real stereo system coupled to a robot, the stereo system is off-line calibrated with a special calibration objet. The image pair of Figure 2 is taken by this stereo system. The process from edge detection to conic fitting is the same as in the above example.

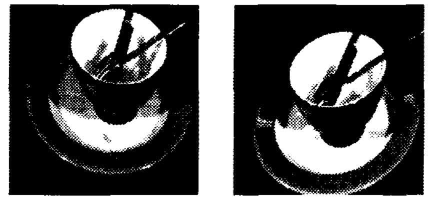

Figure 2: The initial stereo pair of breakfast images.

The correspondences are unambiguously established based on the pairwise Δ, shown in Table 1.

	bowl	dish inside	dish outside
bowl	**-0.001**	−15.0	−3.7
dish inside	−6.0	**-0.0001**	0.54
dish outside	−9.4	3.8	**-0.0005**

Table 1: The computation of Δ for each pair of conics of two images. The raw entries correspond to the conics of the first image and the column entries to those of the second image.

Table 2 shows Euclidean reconstruction results. To have a rough idea of the reconstruction quality, the heights of the conics from the ground, measured with a ruler, are respectively $8.5cm$ for bowl, $3.0cm$ for dish outside and $2.3cm$ dish inside. That makes a difference of $5.5cm$ between bowl and dish outside border and $0.7cm$ between dish inside and outside. Obviously the planes on which conics lie should be all parallel to the ground. The computed difference of the heights are $5cm$ for $5.5cm$ and $0.8cm$ for $0.7cm$. The difference of plane orientations are 2.6^o between bowl and inside and 1.7^o between inside and outside border.

	Plane pair $(n^T, -d)$
bowl	**(0.9181, -0.05092, -0.3930, 11.85)**
	$(-0.1349, -0.9518, 0.2753, 8.492)$
dish inside	**(0.9210, -0.09307, -0.3782, 6.093)**
	$(-0.1248, -0.9425, 0.3100, 7.172)$
dish outside	**(0.9131, -0.07843, -0.4001, 6.880)**
	$(-0.1430, -0.9431, 0.3002, 7.377)$

Table 2: The reconstruction results of the three conics of the breakfast images.

As the fundamental matrix is first extracted from the two projection matrices provided by the stereo calibration. Then, the two projection matrices up to a collineation are realised by the formule 4. The conics are then projectively reconstructed. It follows the invariants are computed from this projective reconstruction.

	bowl/dish-out	bowl/dish-in	dish-in/dish-out
1	-17.9643	13.6358	11.1248
2	2.1267	0.6532	4.42365
3	2.1136	0.676919	4.42377
4	**4.0192**	**4.01421**	**4.0008167**

Table 3: The results of the computed invariants from the projective reconstruction of the conics.

As we know that three conics are three circles all parallel to the ground, for a pair of parallel circles, the invariant equals 4 according to the formule (3).

6 Discussion

This paper proposed an algebraic invariant for a pair of non-coplanar conics in space with the help of projective geometry and classical invariant theory. The relationship between the geometric invariant (in terms of cross ratios) and the algebraic invariant is also established. Algebraic invariants of other configuration composed of conics can be developed in a similar way. Instead of considering the system of a pair of quadratic forms, we can consider the simultaneous invariants of linear forms and quadratic forms. In order to compute this invariant from two uncalibrated images of the conics, we have proposed a solution to conic reconstruction from two images and conic correspondence between two images within a unified framework for both projective and Euclidean case. The conic reconstruction method shown is simpler and more stable in comparison with existing methods, as the intrinsic properties of the problem are fully exploited.

Acknowledgements

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