Technische Universität München
Fakultät für Informatik
Prof. Dr. N. Navab

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Exercise Sheet 2
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Exercises in 3D Computer Vision I

Exercise 1 3D Rigid Transformations

We will study how to represent 3D rigid transformations, and in particular 3D rotations, as they are very essential for many application areas including computer vision, computer graphics, robotics etc.

A rigid transformation $\mathbf{T}$ on homogeneous coordinates in the 3D space can be described with the following matrix:

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{t} \\ \mathbf{0} & 1 \end{bmatrix},$$

where $\mathbf{R}$ is an orthogonal 3×3 rotation matrix and $\mathbf{t}$ a 3×1 translation vector. A (3×3) rotation matrix can be denoted equally by its row or column vectors:

$$\mathbf{R} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1^\top \\ \mathbf{r}_2^\top \\ \mathbf{r}_3^\top \end{bmatrix}^\top . \tag{1}$$

Thus, the rigid transformation can be expressed in terms of the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, or in terms of the elements of the matrix, as follows:

$$\mathbf{T} = \begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 & \mathbf{t} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

a) Rotation matrices belong to the so called SO(3) group. What characterizes this group? In other words, how can you identify if a matrix is a rotation matrix (give at least two conditions)? How are the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ et $\mathbf{r}_3$ related? What is the the relationship between a rotation matrix and an orthogonal matrix?

b) Apply the rotation $\mathbf{R}$ to a vector $\mathbf{v} = [x_1, x_2, x_3]^\top$ in non-homogeneous coordinates and interpret the result as a linear combination of vectors $\mathbf{r}_i$, with $i \in \{1, 2, 3\}$.

c) Write the inverse of a rotation matrix $\mathbf{R}$ using the vectors $\mathbf{r}_1$, $\mathbf{r}_2$ et $\mathbf{r}_3$.

d) Derive algebraically the inverse $\mathbf{T}$. Hint: apply the transformation to a point $\mathbf{v} = [\mathbf{x}^\top, 1]^\top$ to obtain the expression for the transformed vector $\mathbf{v}' = \mathbf{T}\mathbf{v} = [\mathbf{x}'^\top, 1]^\top$. Recover $\mathbf{v}$ in terms of $\mathbf{v}'$.

e) Give three different possibilities to parameterize a rotation matrix.

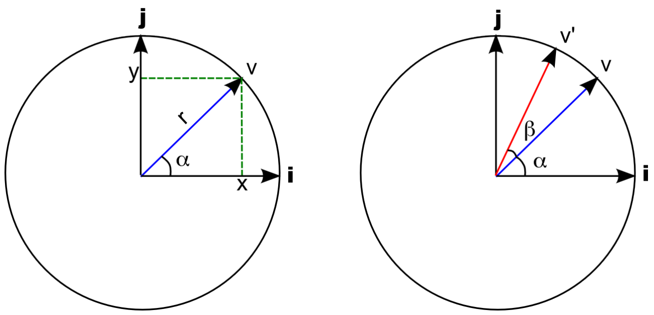
f) Why is it convenient to use these parameterizations over the rotation matrices, for example when we want to optimize over a rotation? Hint: think of the degrees of freedom of a matrix and those of a rotation. Also remember the properties in a).

Exercise 2 Rotation representations

There are several ways in which the rotation can be represented. It is common then to get confronted with converting one representation into another. Here, we will explore the transformation between Rotation matrices, Euler Angles and the Axis-Angle representation, also known as Rodrigues Angles.

a) Derive the 2D rotation matrix as a function of the rotation angle β.

(i) Take a 2D vector $\mathbf{v}$ with non-homogeneous coordinates $\mathbf{v} = [x, y]$ as illustrated in the figure on the left. How are the coordinates x and y expressed in terms of the angle α and distance r?

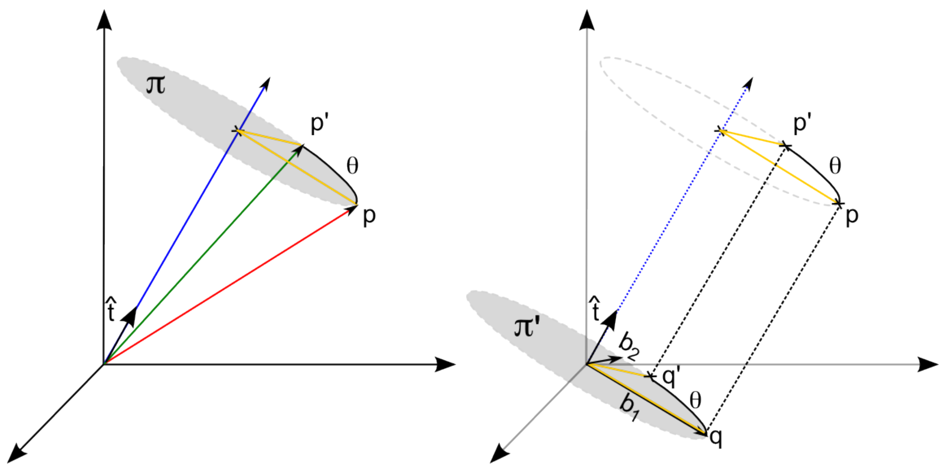


(ii) Rotate the vector $\mathbf{v}$ by an angle β, as illustrated in the figure on the right. The rotation leads to a transformed vector $\mathbf{v}' = [x', y']$. What are the new coordinates of $\mathbf{v}'$ in terms of the angle β and the previous coordinates x and y? Express this result as the product of a 2×2 matrix with vector $\mathbf{v}$.

Hint: remember the trigonometric identities,

$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$
$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

b) Convert the axis-angle representation of a rotation into a matrix. As indicated by the name, the axis-angle representation is formed by a rotation axis $\widehat{\mathbf{t}} = (x, y, z)^\top$ with $\|\widehat{\mathbf{t}}\|_2 = 1$ and an angle θ. Intuitively, this representation can be used to rotate a point $\mathbf{p}$ around the axis $\widehat{\mathbf{t}}$ by an angle θ as illustrated in the figure below (left). We want to derive a matrix $\mathbf{R}$ such that $\mathbf{p}' = \mathbf{R}\mathbf{p}$ yields the point rotated around the given axis by the angle θ.



In order to find this rotation matrix we will try to find the geometric relation between points $\mathbf{p}$ and $\mathbf{p}'$. The overall procedure is illustrated in the figure on the right. First, we project the original point $\mathbf{p}$ onto a plane π' perpendicular to the rotation vector and passing trough the point $[0, 0, 0]^\top$, we name this projected point $\mathbf{q}$. Then, we rotate $\mathbf{q}$ in the plane π' to obtain $\mathbf{q}'$. Finally, we rise the result to the plane π along the rotation axis, to go from $\mathbf{q}'$ to the final location of the rotated point $\mathbf{p}'$. Follow these steps:

(i) Find the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ (see the figure above) in terms of the vectors $\widehat{\mathbf{t}}$ and $\mathbf{p}$. We will use this vectors to form a basis to represent the points on the plane π. Hint: use the cross-product of $\mathbf{p}$ and $\widehat{\mathbf{t}}$. Just try to create an orthogonal basis. Can you prove, why $\mathbf{b}_1$ is a projection of $\mathbf{p}$ onto the plane π?

(ii) Use the results of exercise a) to represent the position of $\mathbf{q}'$ in the projected plane π', in terms of θ, $\mathbf{b}_1$ and $\mathbf{b}_2$. ($\mathbf{b}_1$ and $\mathbf{b}_2$ work like $\mathbf{i}$ and $\mathbf{j}$ in the 2D rotation problem above).

(iii) Translate the rotated point $\mathbf{q}'$ by the appropriate vector $\mathbf{o}$ to obtain the final point $\mathbf{p}'$. Hint: Express $\mathbf{p}'$ in terms of $\mathbf{q}'$ and $\mathbf{o}$. How can $\mathbf{o}$ be calculated?

(iv) Use the results of (i), (ii) and (iii) to find the final expression for $\mathbf{p}'$ in terms of $\widehat{\mathbf{t}}$, θ, and $\mathbf{p}$.

(v) Use the following cross-product properties to simplify the expression and convert it to a matrix.

- anticommutative: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- The cross product can be expressed as the multiplication with a skew-symmetric matrix:

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \mathbf{b}$$

(vi) What is the expression for the 3×3 rotation matrix $\mathbf{R}$ in terms of θ and $\widehat{\mathbf{t}}$: $\mathbf{R}(\theta, \widehat{\mathbf{t}})$?.

(vii) Expand the expression above to obtain the values of each element in the 3×3 rotation matrix $\mathbf{R}(\theta, \widehat{\mathbf{t}})$. Hint: Since $\widehat{\mathbf{t}}$ is normalized, you can use the equation $x^2 + y^2 + z^2 = 1$ to simplify your solution.

c) Use the derivation of b) to determine rotation matrices around the axes of the Euclidean coordinate frame with unit vector basis.

Exercise 3 (H) Rodrigues Angles

Write two matlab scripts `angle2matrix.m` and `matrix2angle.m` to convert from one representation to the other.

Exercise 4 (H) Representing Projective Geometry in $\mathbb{R}^3$

The projective space $\mathbb{P}^2$ is a generalization of the Euclidean space $\mathbb{R}^2$. To get a better understanding of the projective geometry it is useful to think of the points and lines in homogeneous coordinates as vectors in $\mathbb{R}^3$ with coordinates x, y, z and plot them. In this exercise we will explore how to represent $\mathbb{P}^2$ in $\mathbb{R}^3$ using MATLAB, assuming the projection center is at $[0, 0, 0]^T$.

Hint: when drawing lines and planes use a regular sampling of the coordinates; for example if a line is defined as $y = 3x + 10$, use values of x at regular intervals, varying between -100 and 100, with intervals of 10 (in matlab notation: x = -100:10:100).

Hint 2: make sure that all the input vectors are column vectors.

a) Write a matlab function `draw_points.m` that takes as input the non-homogeneous representation of a point in a vector form $\mathbf{x} = [a, b]^T$ and draws its $\mathbb{P}^2$ representation in $\mathbb{R}^3$. Use the option `'r'` in the `'plot3'` function. Hint: draw the equivalence class.

b) What is the shape of a point when using $\mathbb{R}^3$ to represent the projective space $\mathbb{P}^2$?

c) Write a matlab function `draw_lines.m` that takes as input the parameters of a line in $\mathbb{P}^2$ ($\mathbf{l} = [a, b, c]^\top$) or two points in $\mathbb{P}^2$ passing trough a line ($\mathbf{p}_1 = [x_1, y_1, z_1]^\top$ and $\mathbf{p}_2 = [x_2, y_2, z_2]^\top$) and plots samples of the $\mathbb{P}^2$ line in $\mathbb{R}^3$. Use the matlab commands `[X,Y]=meshgrid(-100:10:10)` and `surf(X,Y,Z)` to draw planes. Use the `'nargin'` matlab command to determine if the input is $\mathbf{l}$, or $\mathbf{p}_1$ and $\mathbf{p}_2$.

d) What is the shape of a line when using $\mathbb{R}^3$ to represent the projective space $\mathbb{P}^2$?

e) Write a matlab function `draw_ideal.m` that plots in green (option `'gx'`) ideal points (or points at infinity) in $\mathbb{P}^2$ (Hint: sample the two free coordinates of the ideal points). Superposed to the ideal points, plot the $\mathbb{P}^2$ ideal line $\mathbf{l}_\infty$. Hint: use the matlab command `'hold on'` to superpose the two plots. Use the function `draw_lines.m`.

f) What is the relationship between the ideal points and the ideal line? What is the homogeneous $\mathbb{P}^2$ representation of the ideal line $\mathbf{l}_\infty$? What is the corresponding equation in $\mathbb{R}^3$?. What is the shape of the line at infinity when using $\mathbb{R}^3$ to represent the projective space $\mathbb{P}^2$?

g) Write a matlab function `draw_infinity_intersection.m` that takes as input the homogeneous representation of a line $\mathbf{l} = [a, b, c]^T$ and:

- draws the lines $\mathbf{l}$ and $\mathbf{l}_\infty$ using the `draw_lines.m` function.
- finds the intersection point $\mathbf{p}$ of $\mathbf{l}$ with the infinity line $\mathbf{l}_\infty$.
- draws the intersection point $\mathbf{p}$.
- draws other 5 lines intersecting $\mathbf{l}_\infty$ at the same point. Hint: Where do parallel lines intersect?. How can we tell two lines are parallel?

h) Assume that the projection center is at $(0, 0, 0)$ and that the image plane is parallel to the xy plane. Write a matlab function `draw_image_plane.m`, that takes as input the distance of the image plane f to the origin and a vector with three coordinates ($\mathbf{v} = [a, b, c]^T$) and:

- Draws the image plane. Use the matlab commands `[X,Y]=meshgrid(-100:10:10)` and `surf(X,Y,Z)`
- In the same figure. Think of $\mathbf{v}$ as a $\mathbb{P}^2$ point and draw it using `draw_points`. Find the intersection of point $\mathbf{v}$ with the image plane and draw it with the option `'gx'`. Hint: use the inhomogeneous representation for the point and the distance to the image plane f.
- Still in the same figure. Interpret $\mathbf{v}$ as a line and draw it using `draw_lines`. Find the intersection of line $\mathbf{v}$ with the image plane and draw it with the option `'r'`. Hint: Do not try to solve for a general formula, instead use the fact that the intersection line should lie on the plane $z = f$.
- Put the commands `colormap(gray);` and `shading flat;` at the end of the function to enhance the visualization.