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Exercise Sheet 1
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Exercises in 3D Computer Vision I

Exercise 1 Homogeneous Coordinates

Short reminder on projective geometry
We call $\mathbb{P}^n$ the projective space of dimension n. For any point in $\mathbb{P}^n$, we define its corresponding vector in homogeneous coordinates:

$$\mathbf{x}_h = [x_1, x_2, \ldots, x_{n+1}]^\top \in \mathbb{R}^{n+1}$$

We introduce on $\mathbb{P}^n$ a new equivalence operator $\propto$ such that: $\forall \mathbf{x}_h, \mathbf{y}_h \in \mathbb{R}^{n+1}$, we have

$$\mathbf{x}_h \propto \mathbf{y}_h \Leftrightarrow \exists \alpha \in \mathbb{R}^* \text{ where } \mathbf{x}_h = \alpha \mathbf{y}_h$$

In the literatur you will also sometimes find the sign $\equiv$ for the equivalence operator.

If $x_{n+1} = 0$, then $\mathbf{x}_h$ is a *point at infinity*. If $x_{n+1} \neq 0$, then we can obtain the Euclidean coordinates $\mathbf{x}_e \in \mathbb{R}^n$ from the homogeneous coordinates $\mathbf{x}_h$.
For $\mathbf{x}_h = [x_1, x_2, \ldots, x_{n+1}]^\top$ and $x_{n+1} \neq 0$, then $\mathbf{x}_e = \left[\frac{x_1}{x_{n+1}}, \frac{x_2}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right]^\top$.

a) In the lecture, the homogeneous representation of points and lines was introduced. Now, we will consider its geometrical interpretation. In the following, we will always operate in 2D ($\mathbb{R}^2$ and $\mathbb{P}^2$) since it is easier to visualize and get a grip on projective geometry. As a starter, consider the following questions.

   (i) How can points and lines be represented in $\mathbb{R}^2$ (*inhomogeneously*)?

   (ii) How are points and lines represented in $\mathbb{P}^2$ (*homogeneously*)?

   (iii) How are points and planes represented in $\mathbb{R}^3$ (*inhomogeneously*)?

b) Now think of the geometric meaning of 2D projective space for representing points and lines. As a hint, take the homogeneous representation of 2D points and lines as 3D entities. Then think of how to project these 3D entities into a suitable 2D space. Draw a schematic that displays your geometric interpretation of projective space $\mathbb{P}^2$ for points and lines.

   (i) How can the direction of a line and its distance to the origin be determined?

   (ii) Explain geometrically that a point $\mathbf{p}$ that lies on a line $\mathbf{l}$ satisfies $\mathbf{p}^\top \mathbf{l} = 0$.

   (iii) Intersecting two lines gives a point $\mathbf{p}$ in 2D. How can $\mathbf{p}$ be determined in $\mathbb{P}^2$?

   (iv) How can points at infinity $\{(x, y, z) \in \mathbb{P}^2 | z = 0\}$ and the line at infinity $\mathbf{l}_\infty$ be explained geometrically?

Exercise 2 Hierarchy of transformations in 2D

a) List all classes of 2D transformations you can recall. Show the containment relation of the different classes in a schematic way.

b) Name the invariants of each class of transformations.

c) Explain algebraically why a line is transformed under a projective transformation $\mathbf{H}$ as $\mathbf{l}' \equiv \mathbf{H}^{-\top}\mathbf{l}$, if the points of this line are transformed as $\mathbf{p}' \equiv \mathbf{H}\mathbf{p}$.

   Hint: Think of the equation that holds when a point lies on a line: $\mathbf{p}^\top\mathbf{l} = 0$. Write the equation with an arbitrary projective transformation applied to the point $\mathbf{p}$ and then decide how you need to transform the line.

d) Prove that projective transformations preserve intersection of lines.

   Hint: You need to show that if two lines $\mathbf{l}_1$ and $\mathbf{l}_2$ are intersecting in a point $\mathbf{p}$, the two lines arising under a transformation by an arbitrary homography $\mathbf{H}$ are still intersecting. By the help of the previous exercise, think of why it is sufficient to prove

$$\mathbf{H}^{-\top}(\mathbf{l}_1 \times \mathbf{l}_2) \equiv \mathbf{H}\mathbf{l}_1 \times \mathbf{H}\mathbf{l}_2.$$

   The equation states that the left-hand side and the right-hand side are meant to be equal up to a scaling factor. Simplify the equation a bit and then try to interpret both sides geometrically (what is the result of the cross product of two vectors?).

<u>Exercise 3</u> (H) Euclidean Transformations

The following exercises should be solved using MATLAB. All commands you used to answer the questions should be handed in in one file called `exercise1-3.m` (except for the function `plotHomo.m`). We take the point set

```
p = [0 0;1 0;1 1;0 1;0.5 1.6;1 1;0 0;0 1;1 0]'.
```

The points can be changed to homogeneous coordinates by `p_h = [p; ones(1,9)]` and can be displayed by `plot(p(1,:),p(2,:)); axis equal`. Note that this is just possible because the w-value of a point $[x\,y\,w]^\top$ is normalized to 1.

Just to remind you, if $\mathbf{T}$ is a transformation matrix and $\mathbf{P}$ is a matrix that was generated by concatenated vectors, then $\mathbf{T} \cdot \mathbf{P}$ does the same as transforming the points first and concatenating them after.

All of the exercises must be calculated in homogeneous coordinates.

a) Write the function `plotHomo(p)` that normalizes each point separately and plots the points afterwards.

b) Create a transformation matrix $\mathbf{T}_1$ that translates the points by $\mathbf{t} = [3, -4]^\top$.

c) Create a transformation matrix $\mathbf{T}_2$ that rotates the points by 20 degrees. Mind that the function `sin()` just accepts radian values.

d) Create a transformation matrix $\mathbf{T}_3$ by multiplication that applies the rotation first and the translation after.

e) Create a transformation matrix $\mathbf{T}_4$ by multiplication that applies the translation first and the rotation after.

f) Express $\mathbf{T}_3$ and $\mathbf{T}_4$ by the rotation matrix $\mathbf{R}$, translation vector $\mathbf{t}$. Do this calculation manually.

Exercise 4 **(H) Affine Transformations**

a) Apply the matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0.5 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ to the set of points and plot the resulting points.

b) What would the matrix look like if the bottom right point of the house was to be projected to $[2, -0.5]^\top$ and the left top point to $[-1, 1]^\top$?

Exercise 5 **(H) Projective Transformations**

a) Apply the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.3 & 0.5 & 1 \end{bmatrix}$ to the set of points and plot the resulting points.

b) Now apply the matrix $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ and display. Can you explain what you see?