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Exercise Sheet 4
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Exercises in 3D Computer Vision I

Exercise 1 Singular Value Decomposition Basics

Let $\mathbf{A}$ be a $(m \times n)$ real matrix with $m \geq n$. Then, the Singular Value Decomposition (SVD) of $\mathbf{A}$ can be written as

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^\top \tag{1}$$

Here $\mathbf{U}$ is an $(m \times m)$ matrix, $\mathbf{S}$ is an $(m \times n)$ matrix, and $\mathbf{V}$ an $(n \times n)$ matrix. The matrices $\mathbf{U}$ and $\mathbf{V}$ have orthonormal columns and $\mathbf{S}$ has entries only along the main diagonal.

The entries σ_i of the diagonal of $\mathbf{S}$ are called the singular values of the matrix $\mathbf{A}$. These entries are positive and always arranged in descending order: $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0$.

Note: *There is also another, equivalent notational convention where the matrix* $\mathbf{U}$ *is defined as an* $(m \times n)$ *matrix,* $\mathbf{S}$ *as an* $(n \times n)$ *matrix, and* $\mathbf{V}$ *as an* $(n \times n)$ *matrix. We will use the first convention, since it is also used by MATLAB.*

a) What similarities and differences are there between the SVD and the eigenvalue decomposition?

b) Give the relationship between the entries of $\mathbf{U}$, $\mathbf{S}$ and $\mathbf{V}$ and the eigenvalues and the eigenvectors of the matrices $\mathbf{A}\mathbf{A}^\top$ and $\mathbf{A}^\top\mathbf{A}$.

c) What do the entries of the diagonal matrix $\mathbf{S}$ tell about the rank of matrix $\mathbf{A}$?

d) How can the range and the null space of matrix $\mathbf{A}$ be expressed in terms of the columns of $\mathbf{U}$ and $\mathbf{V}$, respectively?

Exercise 2 Proof of Existence of the SVD

Prove the existence of the singular value decomposition for any $(m \times n)$ real matrix $\mathbf{A}$. You can take advantage of the basic theorems given below this exercise.

a) Start by assuming that there exist $\mathbf{u}_1 \in \mathbb{R}^m$ and $\mathbf{v}_1 \in \mathbb{R}^n$ with $\|\mathbf{u}_1\|_2 = \|\mathbf{v}_1\|_2 = 1$ and $\|\mathbf{A}\|_2 = \sigma_1$, such that $\mathbf{A}\mathbf{v}_1 = \sigma_1\mathbf{u}_1$. Interpret the equation and state why this assumption can be made.

b) Augment $\mathbf{v}_1$ to an $(n \times n)$ orthogonal matrix $\mathbf{V}_1 = [\mathbf{v}_1|\tilde{\mathbf{V}}_1]$ and $\mathbf{u}_1$ to an $(m \times m)$ orthogonal matrix $\mathbf{U}_1 = [\mathbf{u}_1|\tilde{\mathbf{U}}_1]$. Show that the product $\mathbf{U}_1^\top\mathbf{A}\mathbf{V}_1$ can be written as

$$\mathbf{U}_1^\top\mathbf{A}\mathbf{V}_1 = \begin{bmatrix} \sigma_1 & \mathbf{w}^\top \\ \mathbf{0} & \mathbf{A}_1 \end{bmatrix} =: \mathbf{X} \tag{2}$$

Hint: Write the equations in such a form that $\mathbf{w}^\top = \mathbf{u}_1^\top\mathbf{A}\tilde{\mathbf{V}}_1$. Why is the lower left part of $\mathbf{X}$ a vector of zeros? What does $\mathbf{A}_1$ explicitly look like?

c) Show that $\mathbf{w} = \mathbf{0}$. For this purpose, derive a lower and an upper bound on the term

$$\left\| \mathbf{X} \begin{bmatrix} \sigma_1 \\ \mathbf{w} \end{bmatrix} \right\|^2.$$

Hint: Observe that $\|\mathbf{X}\| = \|\mathbf{A}\| = \sigma_1$ (why?) and use the Cauchy-Schwarz inequality.

d) Proceed by induction on the size of $\mathbf{A}$ and show that $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^\top$. In doing so, you can conclude that there exists a singular value decomposition for any $m \times n$ real matrix $\mathbf{A}$. *Hint:* You can decompose $\mathbf{A}_1$ in the same way as you have done for $\mathbf{A}$.

You can use the following properties for your proof:

- If a $(m \times n)$ matrix $\mathbf{V}_1$ has orthonormal columns $(m > n)$, then there exists a $(m \times (m-n))$ matrix $\mathbf{V}_2$ such that $\mathbf{V} = [\mathbf{V}_1 \ \mathbf{V}_2]$ is orthogonal, where the orthogonal complement of the span of column vectors of the matrix $\mathbf{V}_1$ is equal to the span of the column vectors of the matrix $\mathbf{V}_2$.
- Let $\|.\|_2$ be the 2-norm in $\mathbb{R}^n$, i.e. $\forall \mathbf{x} \in \mathbb{R}^n$, we have:
$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x}^\top \mathbf{x}}$$
- Let $\mathbf{M}$ be a $(m \times n)$ matrix, the 2-norm of $\mathbf{M}$ is defined as:
$$\|\mathbf{M}\|_2 = \max_{\|\mathbf{x}\|_2 = 1} \|\mathbf{M}\mathbf{x}\|$$
- A $(m \times n)$ matrix $\mathbf{M}$ has orthonormal columns, if $\mathbf{M}^\top \mathbf{M} = \mathbf{I}$.
- One can easily prove that $\forall \mathbf{x} \in \mathbb{R}^n$, if $\mathbf{M}$ is a $(m \times n)$ matrix with orthonormal columns, then:
$$\|\mathbf{M}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$$
- Let $\mathbf{M}$ be a $(m \times n)$ matrix. If the $(m \times m)$ matrix $\mathbf{A}_1$ and the $(n \times n)$ matrix $\mathbf{A}_2$ are orthogonal matrices, then one can also prove that:
$$\|\mathbf{A}_1 \mathbf{M} \mathbf{A}_2\|_2 = \|\mathbf{M}\|_2$$

Exercise 3 Pseudo-inverse

In this exercise, we will solve the following problem:

$$\mathbf{Q} = \mathbf{M}\mathbf{P}$$

where $\mathbf{Q}$ and $\mathbf{P}$ are known and the matrix $\mathbf{M}$ is unknown. In our case, $\mathbf{Q}$ is a matrix with N columns, where each column contains a measured image point (in homogeneous coordinates), and $\mathbf{P}$ is a matrix containing the corresponding N points in the 3D scene in each column (also in homogeneous coordinates). One way for solving this problem is provided by the use of the pseudo-inverse.

Let $\mathbf{A}$ be a full-rank real matrix that has the dimensions $(m \times n)$, where $m \leq n$. The pseudo-inverse of $\mathbf{A}$ can be written as:

$$\mathbf{A}^+ = \mathbf{A}^\top \left(\mathbf{A}\mathbf{A}^\top\right)^{-1} \tag{3}$$

Let $\mathbf{A}$ be a full-rank real matrix that has the dimensions $(m \times n)$, where $m \geq n$. The pseudo-inverse of $\mathbf{A}$ can be written as:

$$\mathbf{A}^+ = \left(\mathbf{A}^\top \mathbf{A}\right)^{-1} \mathbf{A}^\top \tag{4}$$

a) What does the matrix $\mathbf{M}$ represent and what are its dimensions?

b) Show that $\mathbf{M} = \mathbf{Q}\mathbf{P}^{+}$.

c) If $\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^{\top}$ is the SVD of the matrix $\mathbf{A}$, prove that the pseudo-inverse of $\mathbf{A}$ can be written as $\mathbf{A}^{+} = \mathbf{V}\mathbf{S}^{+}\mathbf{U}^{\top}$. Express the matrix $\mathbf{S}^{+}$ using the singular values σ_i of the matrix $\mathbf{A}$.

d) Explain why the pseudo-inverse can be defined by equation (3) only if $rank(\mathbf{A}) = m$, i.e. if $\mathbf{A}$ has full rank.

Exercise 4 (H) Line Fitting

The Singular Value Decomposition can be used to determine solutions for minimization problems, i.e. minimize $||\mathbf{A}\mathbf{x}||$ with $||\mathbf{x}|| = 1$.

In this exercise, we want to use the SVD for line fitting. Given a set of coordinate data (u_i, v_i) and the model of a line, we want to find the parameters that fit the line best to the data.

d) Reformulate the problem such that it becomes one of minimizing $\|\mathbf{A}\mathbf{x}\|$, with $\mathbf{x} = [a, b, c]^{\top}$ containing the line parameters.

e) Set up an array in MATLAB for the matrix $\mathbf{A}$ that contains points of the line $v = 2u - 1$. Apply Gaussian noise to it to simulate measurement errors (use the function `randn`).

f) Solve the resulting linear system of equations using the SVD and plot the line and the measurements array.

g) Apply the algorithm with different measurements (apply `randn` 100 times) and plot the results again.

h) For each measurement, state what the standard deviation is between the measured points and the computed line. In other words, determine how well you fitted the line to the data.