Fractals in Economic and Financial Indicators

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Abstract

This paper takes a look into how fractals play a role in market analysis and the fractal properties in financial time series. In this project, the fractal dimension is calculated using the Hurst exponent with re-scaled range analysis or the R/S method. The results lead to interesting observations concerning the variation of price series and the relationships between fractal properties and correspondence trends. We explore possible explanations of this phenomena such as the Fractal Market Hypothesis and the Adaptive Market Hypothesis.

1 Introduction

1.1 Importance

Understanding price movement of financial assets has long been of interest to economists and mathematicians. The earliest noteworthy modern example is that of Louis Bachelier, a French mathematician who first posited that stocks behave like a 'random walk'. That is financial assets are equally likely to move either up or down at any given time. There is much evidence that stocks and financial assets move in such a way that it forms similar structures through varying time horizons. Financial assets are fractals and exhibit fractal behavior. How do we find the fractal dimension of a financial asset? What are the implications of a certain fractal dimension for the movement of that financial asset over time?

If markets follow a random walk then this is consistent with the prevailing market theory, the Efficient Market Hypothesis. The Efficient Market Hypothesis (EMH) was both largely developed and heavily criticized by those who expected markets to follow some type of Brownian Motion. Eugene Fama gave his famous review of standing literature in 1970. Fama concluded that markets generally price in available information and behave in a Martingale pattern [4].

However the research of Benoit Mandelbrot, the mathematician and coiner of the term 'fractal', believed that a random walk does not capture the richness of market behavior [8]

[9] [10]. Mandelbrot described the behavior of markets in several papers and his book 'The Misbehavior of Markets'. His book notes that movement in markets are not normally distributed, and thus do not follow a random walk. This does not mean that one must use a new method analysis to understand behavior. However for the most part prediction of market behavior becomes nigh impossible. This is due to the volatility in price movements. Mandelbrot's works on the application of R/S analysis shows that volatility clusters in intervals and they are subjected to long-memory dependencies in time series (i.e. big movements encourage further big movements)[9].

The fractional Brownian Motion makes use of the Hurst exponent H and equivalently, its fractal dimension D = 2 - H. When we examine the fractal dimension of stock behavior we can examine also the randomness of the stock over time. The fractal dimension heavily relates to the co-variance of the stock. Thus we can determine how much the history of the asset price affects the next movement, which classifies into 3 categories: persistent, anti-persistent, and Brownian (random walk).

1.2 Brownian Motion

Much of financial theory is based upon the supposition that prices adhere to Martingale behavior. This in turn implies that stock prices behave according to some type of Brownian motion. Brownian motion can appear in a manner of different ways, but behaves according to certain properties.

We briefly define the Hurst index. Which will be more thoroughly explained down below. The Hurst index can be used to show the fractal dimension of Brownian motion. Here H is taken to to be the Hurst index a real-valued number on the interval $I = (0, 1)$. H is directly related to the fractal dimension in self-similar time series.

$$D_H = D_B = 2 - H$$

Let B(t) be a one-dimensional real valued process Brownian Motion over discrete time values t_i where i = 0,1,2,... such that

1. $B(t_0) = B(0) = 0$
2. The increment $\Delta B(t_j, t_i) = B(t_j) - B(t_i)$ is an independent random variable for $0 \leq i \leq j$
3. The increment $\Delta B(t_j, t_i)$ is normally distributed with mean of 0 and variance of increment. $\Delta B(t_j, t_i) \sim \mathcal{N}(0, t_j - t_i)$

The probability of a move in any direction is given by the Gaussian Bell Curve:

$$P_{B_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}$$

where t here describes the variance at time t.

Brownian motion itself is fractal and exhibits self-similarity. Brownian motion is currently used in many models include the Black-Scholes Model, a partial differential equation describing option pricing. In that context geometric Brownian motion is used to describe the price of the stock at some time t. Economic time series are expected to exhibit fractal Brownian motion per Mandelbrot's fundamental work. The literature since Mandelbrot has largely validated the presence of Brownian motion in economic and financial indicators. We can compare their fractal dimension to truly random Brownian motion as a benchmark. The fractal dimension of the stock tells us which way the stock is likely to move.

Brownian motion has a fractal dimension of $D_H = D_B = \frac{3}{2}$ with a Hurst exponent of 0.5. Brownian motion is also often called a random walk since there is no correlation between the past values and future values. In this case the market is exhibiting qualities consistent with the Efficient Market Hypothesis (EMH). The properties of Brownian motion are related to Fractal Brownian motion below. If we model Brownian motion in higher dimensions (i.e. $(D > 2)$) then Brownian motion has a dimension of 2. An example of this would be a Brownian surface. However this paper will limit its scope to Brownian motion with a fractal dimension of $H_D < 2$.

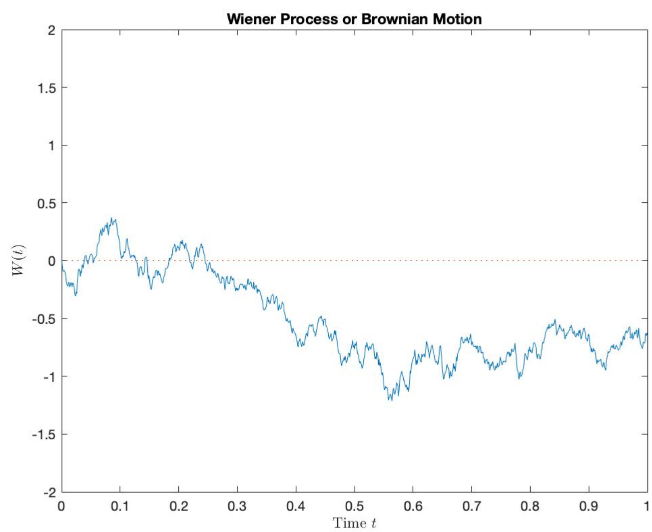

Figure 1: Brownian Motion with $H = 0.5$

1.3 Fractal Brownian Motion

In order to better understand the behavior of financial indices Benoit Mandelbrot explored the idea of using fractal Brownian Motion to model stock changes.

A discrete fractal Brownian Motion(fBM) is a real valued stochastic process $t_i \longrightarrow B_H(t)$ over discrete time values t_i for i = 0,1,2,... such that

1. $B_H(t_0) = B_H(0) = 0$
2. The increment $\Delta B(t_j, t_i)$ follows the Normal distribution with mean of 0 and variance of $(t_j - t_i)^{2H}$, where H is the Hurst exponent satisfies $0 < H < 1$. $\Delta B_H(t_j, t_i) \sim \mathcal{N}(0, (t_j - t_i)^{2H})$

The primary modification made to Brownian motion is to have the following co-variance function:

$$E(B_H(t_j)B_H(t_i)) = \frac{1}{2}(t_j^{2H} + t_i^{2H} - (t_j - t_i)^{2H})$$

We take the Hurst index defined below to be $H = \frac{1}{2}$ then it becomes a Brownian Motion and the process is still perfectly random. Suppose $0 < i < j$ without any loss of generality, the co-variance in increments is taken to be:

$$E(B_{\frac{1}{2}}(t_j)B_{\frac{1}{2}}(t_i)) = \frac{t_j + t_i - (t_j - t_i)}{2} = \frac{2t_i}{2} = t_i = E(B_{\frac{1}{2}}(t_i))$$

Therefore $B_{\frac{1}{2}}(t_j)$ is independent of $B_{\frac{1}{2}}(t_i)$ and we have a truly random process.

Furthered we can deduce from values of the Hurst index what sort of behavior the time series is exhibiting. If $H = \frac{1}{2}$ then we know the time series is perfectly random. If $H > \frac{1}{2}$ the the increments are positively correlated or categorized as persistent, meaning that a rise in value is expected to be followed by another rise in value. For $H < \frac{1}{2}$ we expect the opposite, that is for a rise in value we expect a subsequent fall in value, which is categorized as anti-persistent. We are interested in showing for empirical data what the Hurst exponent is and how it changes over time.

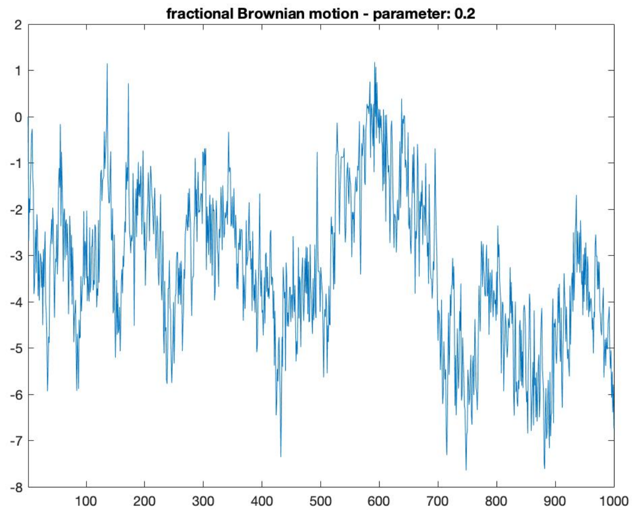

Figure 2: Fractal Brownian Motion with $H = 0.2$

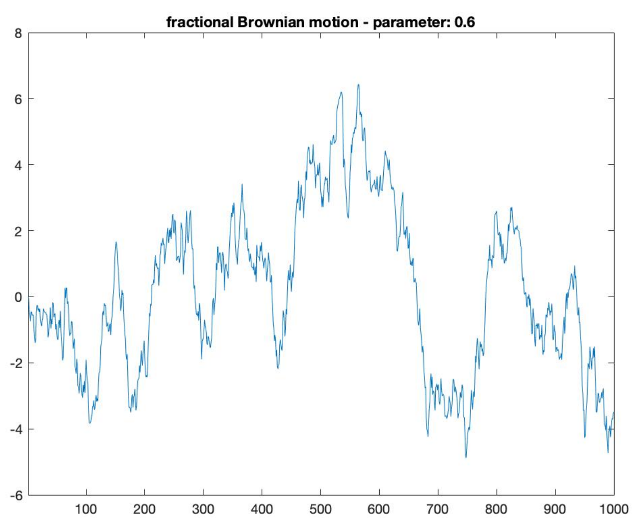

Figure 3: Fractal Brownian Motion with $H = 0.6$

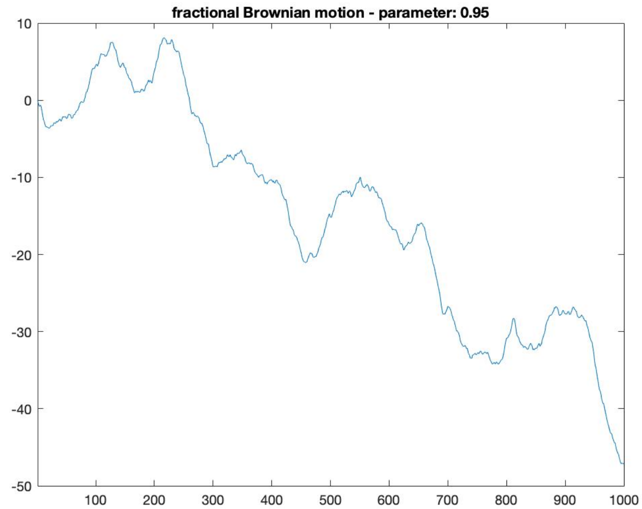

Figure 4: Fractal Brownian Motion with $H = 0.95$

We know from the above relationship $D_H = D_B = 2 - H$ that the respective fractal dimensions are 1.8, 1.4, 1.05. We see that as the Hurst index increases the motion exhibited becomes more and more like a line. If the Hurst index is low (i.e. $0 < H << 1$) then the motion should look more like a two dimensional object at least int terms of the Hausdorff of Minkowski dimension.

1.4 Measure of Variability

The most notable method in the subject of long-term dependency analysis of time series is the rescaled range analysis method (R/S). The Hurst exponent H can be computed using rescaled range analysis as follows:

$$E(R/S)_T = c \cdot T^H$$

where c is a positive constant, T is time span and H is the Hurst exponent. For any given time t as the stochastic process $x(t) \in (x(1), ..., x(T)$ where $t \in (1, T)$:

$$E(x(t)) = \frac{1}{T} \sum_{t=1}^{T} x(t)$$

Then let X(t,T) be the accumulated deviation for x(t) form the average:

$$X(t, T) = \sum_{t=1}^{T} (x(t) - E(x(t)))$$

And let $R(T)$ be the range for $X(t, T)$ over the time span T:

$$R(T) = max_{1 \le t \le T} X(t,T) - min_{1 \le t \le T} X(t,T)$$

$$S(T) = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(x(t) - E(x(t)))^2}$$

From here we can calculate the Hurst Exponent the slope of the line fitted to the log of the $(R/S)_t$ against the log of the time span as follows:

$$logE(R/S)_T = H\log(T) + \log(c)$$

1.5 Interpretation of Hurst exponent

Due to the noise that exists in the market data system, it is nearly impossible for the Hurst exponent to be exact of 0.5 so we introduce a certain thresh-hold of 0.02 such that the Hurst exponent values are distinguished into 3 different categories [8]:

1. Antipersistence ($0 \le H < 0.48$): The series has a mean reversion tendency. In any given period the value of the series increased then the following period will most likely to decrease and vice versa. As $H \longrightarrow 0$, the more noise is introduced into the system.

2. Random walk ($0.48 \le H \le 0.52$): The time series is random and acts like Brownian motion. On average, the behaviors of the past stocks have no effects on the present series. The series itself is unpredictable.

3. Persistence ($0.52 < H \le 1$): The series has a persistent tendency and is caused by the presence of long-term data dependency. As $H \longrightarrow 1$, the trend gets stronger. A special occasion may arise when a strong enough kick is applied to the time series such that a previously increasing trend may suddenly become a decreasing trend in the future. Persistence is generally more descriptive of economic boom-times.

2 Applications in Finance and Economics

Brownian motion has been applied in various areas of both theoretical and empirical economics. Geometric Brownian motion is often used as a stochastic differential equation to model stock growth over time. Fractal Brownian motion is another method to model stock growth which does not have a deterministic factor.

2.1 Geometric Brownian Motion

Geometric Brownian Motion differentiates itself in that it incorporates some drift μ. Thus the process tends up or down in a deterministic function, but a stochastic process still occurs along that path. We define Geometric Brownian motion by:

$$dS_t = \mu S_t dt + \sigma dS_t dW_t$$

μ describes the percentage drift. σ describes the percentage volatility. W_t is the Wiener process, another term for Brownian motion.

Geometric Brownian motion is known to have the solution via Ito's calculus:

$$S = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}$$

This solution is often used to model stock and options prices and is used to model stock price in the Black-Scholes differential equation..

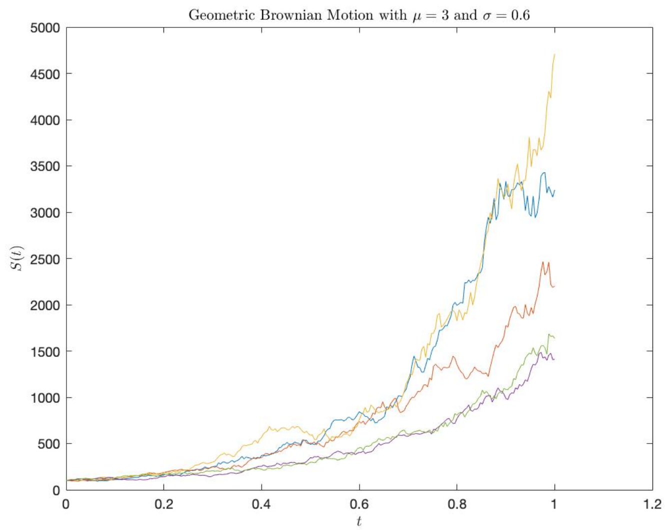

Figure 5: Five Sample Paths of Geometric Brownian Motion

Geometric Brownian motion is frequently used in economic models as the upward drift is assumed to be accounted for by the parameter μ. This parameter can be estimated, but this assumes that only the upward drift is deterministic, while falls in value are the consequence of stochastic behavior. There is, however, no reason apriori or aposteriori to assume that only deterministic forces drive the stock price upwards. There are certainly deterministic effects at play, but these remain difficult to estimate given the complexity of the economy. Stocks generally, or at least stocks that survive, trend upwards, but the volatility of the stock does not follow a random walk. Rather the volatility is much stronger. The distribution of changes in value has fat-tails. Furthermore the covariance function does generally have higher Hurst exponents. Actual stock behavior lies somewhere in between fractal Brownian motion and geometric Brownian motion.

2.2 Mandelbrot Cartoon Fractals

Benoit Mandelbrot used the fractional Brownian motion shown above to run economic simulations. Below we have shown the differences in movement between Brownian motion and fractal Brownian motion:

Mandelbrot showed that fractal Brownian motion better describes market movement over time. However Mandlebrot extended this to multi-fractal systems. The fractal dimension of these systems cannot be captured by a single Hurst exponent, but rely upon a spectrum of exponents to describe their behavior. This generates power-laws as Mandelbrot notes. This is completely contradictory to the Efficient Markets Hypothesis. Mandelbrot generated many what he called cartoon fractals to better understand the behavior of markets in general. He used the word cartoon to mean fractals that look like stocks, but are not necessarily modeled from real world stocks. The cartoon fractals were good enough to be indistinguishable from real world stocks. Mandelbrot found that markets move in correspondence with multi-fractal behavior that is they often have a spectrum of Hurst exponents to describe the fractal behavior in varying time periods.

2.3 Fractal Brownian Motion Since Mandelbrot

Since Mandelbrot fractals have been explored in a variety of economic domains. Fractal Brownian motion has has been shown to be present in many financial indicators although there are some that still model economic processes as a random walk as is the case with the Black-Scholes equation. Dr. C-Ren Dominique and her co-authors in 2011 extended the idea of fractal Brownian motion to Mixed fractal Brownian motion, what Mandelbrot referred to as a Multi-fractal system. Dr. Dominique however believed that Multi-fractal systems ultimately resolved the question of short versus long run memory in financial indicators. She argued they exhibited both due to a variety of causes. In her view the market alternates between persistence and anti-persistence in accordance with the boom-bust cycle [2]. She argues this a consequence of investor outlook, ultimately affirming the Economist John Maynard Keynes in his attribution of the boom-bust to psychological considerations.

3 Analysis of Hurst Exponent and Fractal Dimension

3.1 Data

The Hurst exponent, which was originally used to study natural phenomena of river water flow, is now extended to study behavior of financial markets and other phenomena. The data has a multitude of stock prices, but this paper will only examine four stock prices and one index: Facebook, Microsoft, Apple, Amazon, and the SP500. These companies are large, well-known and carry a significant amount of market power. This means that they are not only subject to market forces, but affect the market.

Using R/S analysis we are able to estimate the Hurst exponent both over five-year periods and over 60 day periods. It was found that H exponent value of most of the index series are close to 0.50 or somewhat above as expected from an quasi-independent process. But when the H exponent is estimated over smaller window size of 60 days, the value is found to vary widely, reflecting departure from normality.

This analysis does not account for the possibility of drift as shown above in geometric Brownian motion. We only analyze the market movements as movement. Drift would need to account for a wide variety of parameters that would make the model unwieldy. Mandelbrot's theory suggests there is no single μ that would make the geometric Brownian accurate in the face of large shocks which could crash companies. For this reason we do a simple R/S analysis and examine the Hurst exponent and fractal dimension.

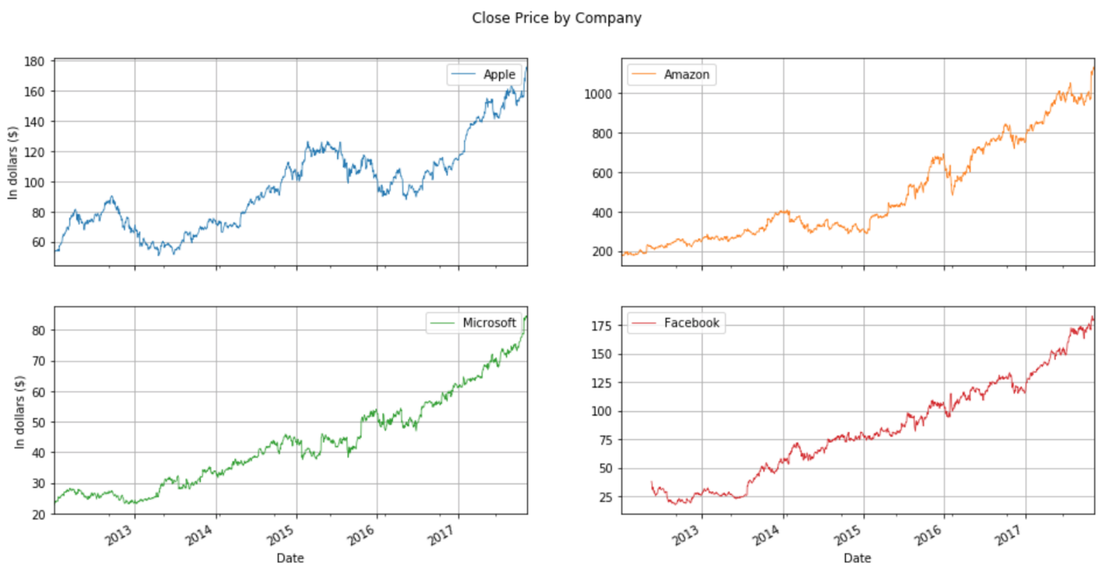

Figure 6: Closing price for four companies.

Furthermore, to analyse the dynamics of market persistence, we briefly introduce the rolling mean or the simple moving average approach for time series to smooth the price data to form a trend following indicator. A simple moving average is formed by computing the average price of a security over a specific number of periods. Here it is based on the closing prices. It takes the sum of the past closing prices over the chosen time period and divides the result by the number of data used in the calculation.

3.2 Analysis

	Hurst Exponent
S&P500	0.557201
Apple	0.592904
Amazon	0.560250
Microsoft	0.539048
Facebook	0.582192

Figure 7: Hurst Exponent for four companies and one index over the whole timeline.

Above we've estimated the Hurst exponent for each of the assets when examining the entire temporal range. We see that each of the indicators appear to move slightly less than random. Using these estimates the fractal dimensions respectively are: 1.442799, 1.407096, 1.43975, 1.460952, 1.417808. This is however only with respect to the whole picture. We may also want to examine smaller time intervals. Below we have estimated the Hurst exponent for 500 days intervals:

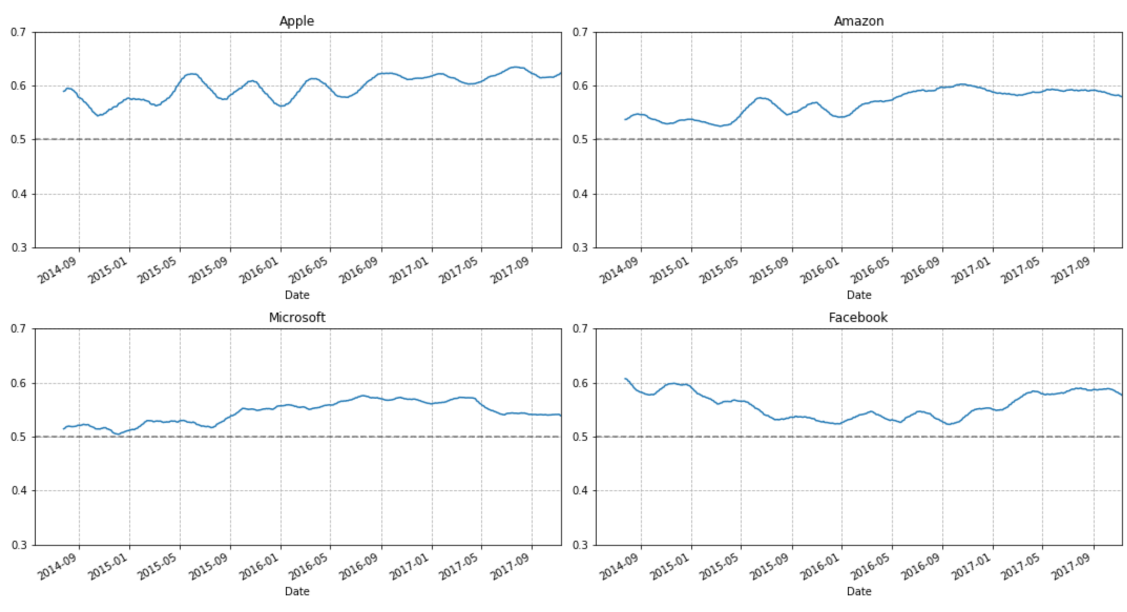

Figure 8: 50 day average 500-day interval for 4 companies

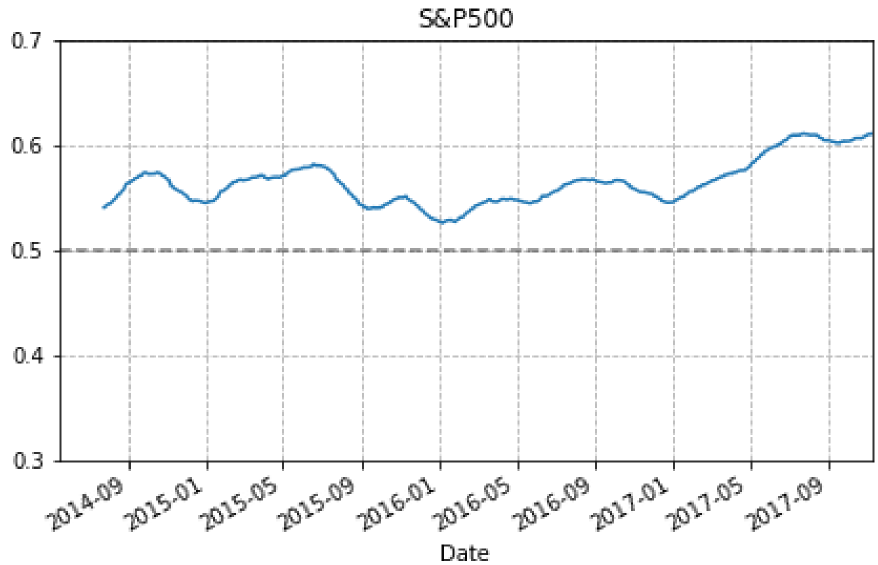

Figure 9: 50 day average 500-day interval for SP500 index.

We see above that the estimated Hurst Exponent varies over time depending upon the interval we take. Generally the Hurst Exponent remains above 0.5 and below 0.65. Even though there are some slight variations, in general they display a persistent behavior, especially the Apple company which has stronger trends than the rest. But we also want to estimate the Hurst exponent for smaller intervals to see how the trends behave.

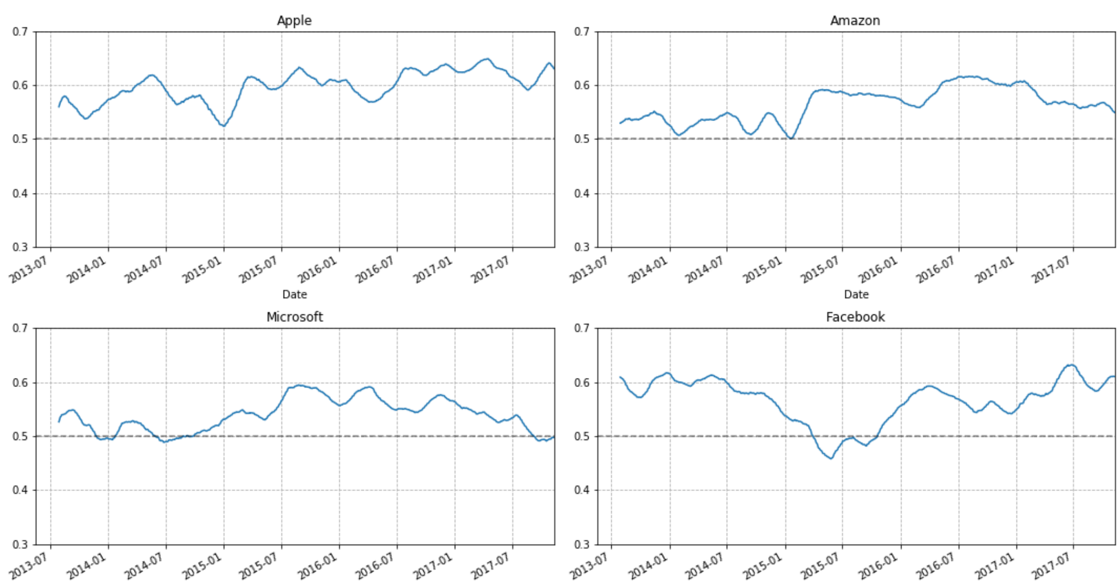

Figure 10: 50 day average 250-day interval for 4 companies

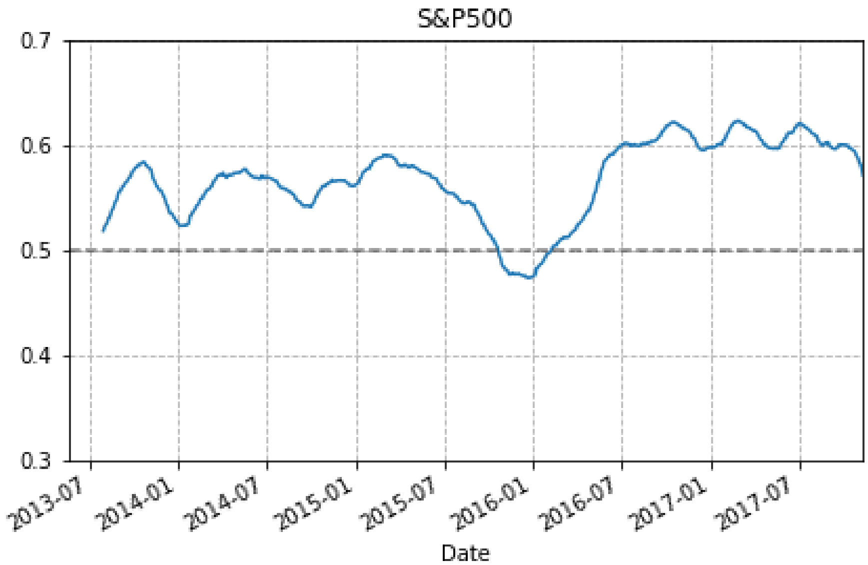

Figure 11: 50 day average 250-day interval for SP500 index.

Above we have further estimated the Hurst exponent for 250 days interval and we can see that as the interval is halved, the trends aren't as smooth as in the previous timescale interval as variations before took place in the overall scale. We also see a time interval where the Hurst exponent is below 0.5, showing anti-persistent behavior for a short time. But then the behaviors change abruptly over time as the time scale intervals are getting smaller and smaller.

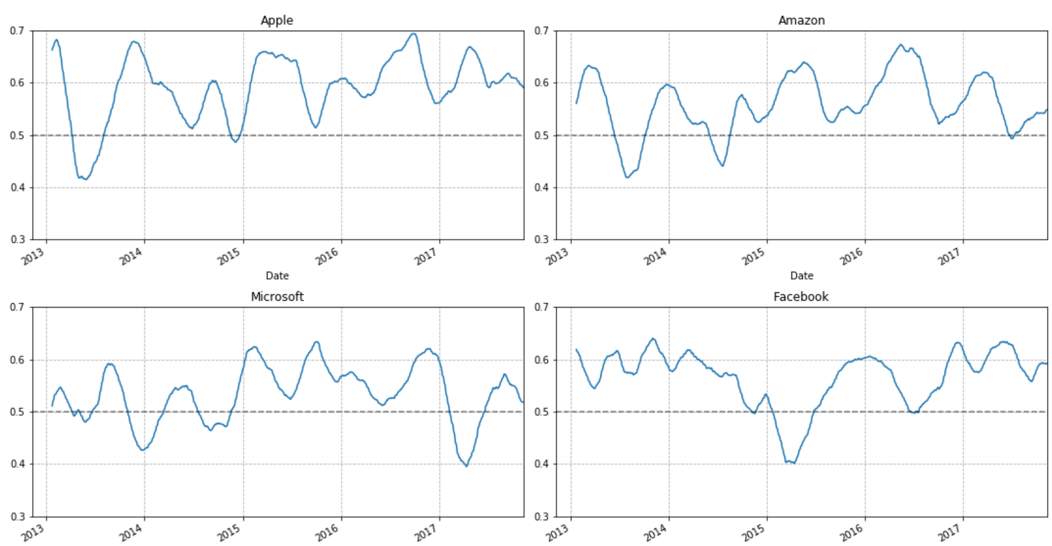

Figure 12: 50 day average 120-day interval for 4 companies

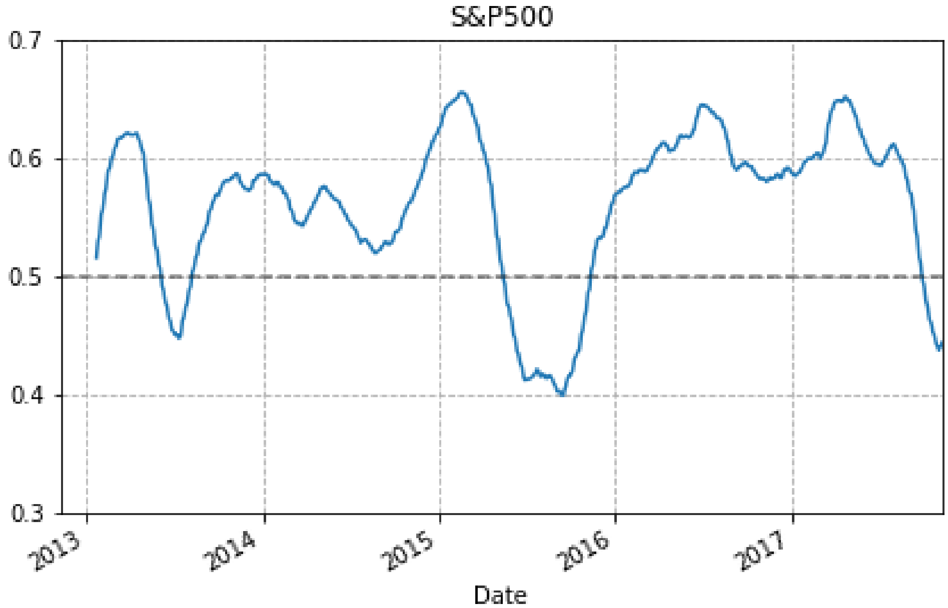

Figure 13: 50 day average 120-day interval for SP500 index.

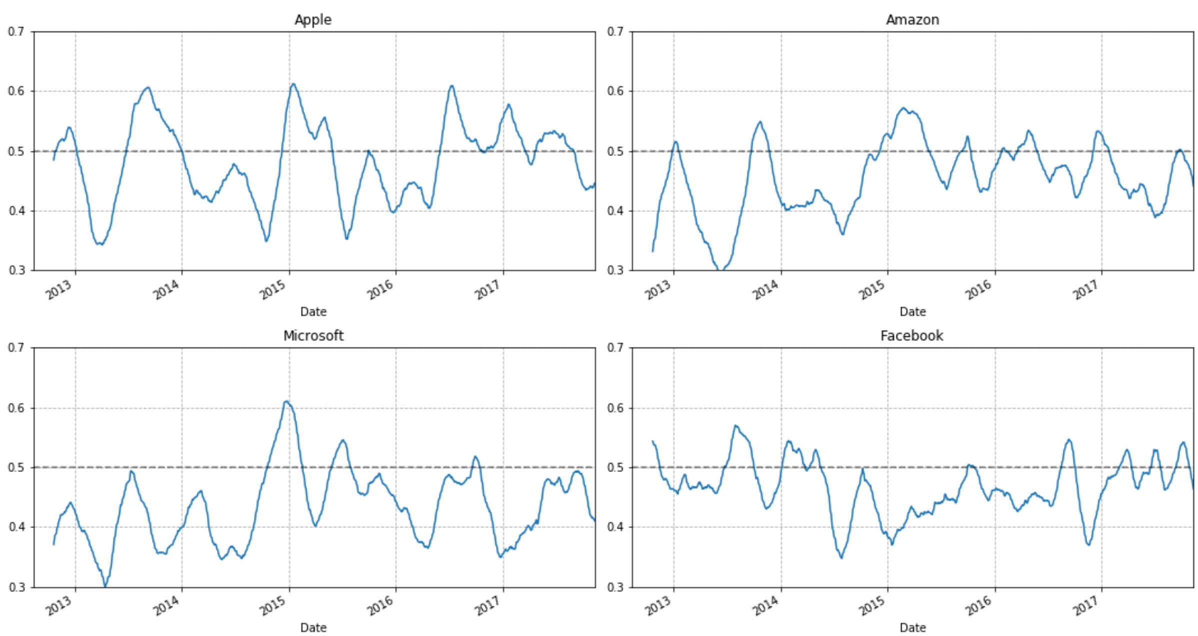

Figure 14: 50 day average 60-day interval for 4 companies

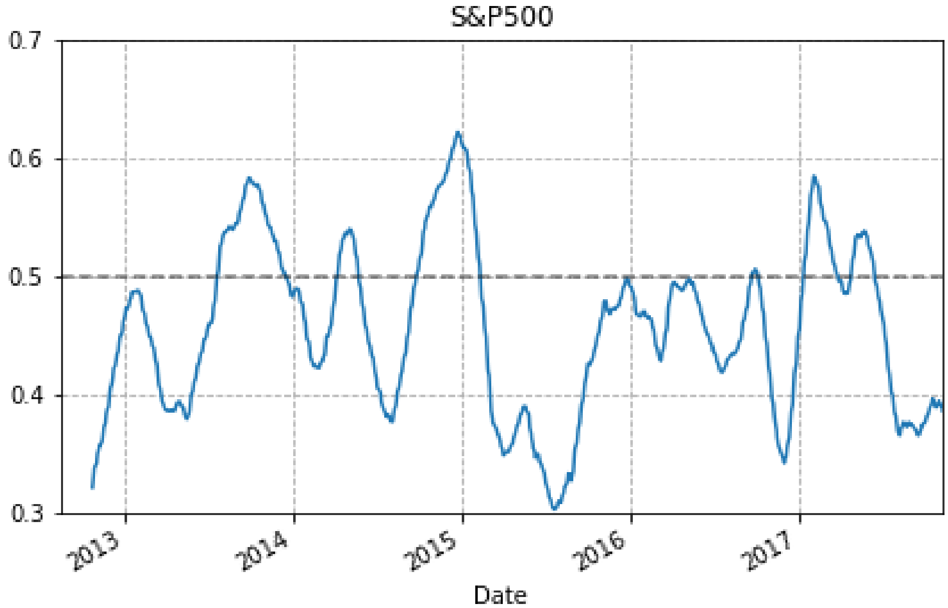

Figure 15: 50 day average 60-day interval for SP500 index.

As we can see from Figure 12, Figure 13, Figure 14 and Figure 15 that on a smaller timescale, the trends can exhibit persistent behavior at some range while exhibiting some anti-persistent behavior. Most of the time, however, we can see that they have mean-reversion trends for the 60-day interval with strong sense of predictability for the 120-day interval. It can be interpreted that the predictability of the method worsens if we consider a smaller and smaller time interval. We can then conclude that the Hurst exponent for these stocks suggests the stocks are acting randomly in short time intervals, but then the Hurst Exponent is above 0.5 for very long time scales. This suggest over long time trends one is more likely to make a return on investment as opposed to short time scales if one is using a bullish trading strategy.

3.3 Further considerations

- Certain biases may encourage short-term dependence in smaller intervals. This is consistent with the Fractal Market Hypothesis discussed below which asserts that the fractal structure arises due to differences investment horizon and available liquidity. For example, daily trading relies heavily on short-term dependence.

- The most troublesome problem in applying this method is deciding the length of time that the pattern should repeat in market prediction. It could be repeated on a daily, weekly, monthly, or even longer basis. Furthermore, the pattern would likely not repeat in identical fashion due to randomness.

- As we have noted before, the method might work sometimes, but it doesn't work all the time. A strong enough kick can get a increasing persistent trend to be decreasing.

- In reality, even though these calculations are based on the past data, they aren't necessarily intuitive nor predictive. It is still up to the trader, statistician, or mathematician to interpret the information the method is providing.

- We haven't looked at special time points like recession periods in 2000 or 2008 to learn more about the dynamical behavior, which might exhibit some interesting behaviors about the Hurst exponent even though such events aren't necessarily predictable.

4 Discussion

4.1 Results and Possible Explanations

Fractal Brownian motion has serious implications for economic theory concerning the fluctuations of stock prices. If the stock prices are truly not independent of one another (as appears to be the case) then we should not expect the Hurst index to be $H = \frac{1}{2}$ and the parameter may need to be estimated when modeling financial asset prices over time. The fractal dimension of the stocks has implications for the best manner in which to profit from the buying and selling of assets. If a Hurst exponent is less than 0.5, $H < 0.5$ then we would not expect great gains over time. However we did not see any stock which consistently had a Hurst exponent less than 0.5 for a large time horizon.

We did however consistently see Hurst exponents which were slightly above 0.5 for a large timescale interval. This has implications for trading strategy as our stocks are not following a random walk. It is not equally likely at any given point that the stock will move up or down rather it depends upon the current value of the stock. Markets may not be efficient due to psychological bias, uncertainty, or even market manipulation. However the behavior of the market is clear. All information is not priced in at any given time and the markets carry some momentum being more likely to increase of decrease in accordance with the previous jump. Trading strategies must be appropriate to the investment horizon.

There are two existing hypotheses which may explain this behavior: the Fractal Market Hypothesis, and the Adaptive Market Hypothesis. These two hypothesis are not exclusive of one another, but they do offer different explanations for the fractal behavior present in economic indicators. First the Fractal Market Hypothesis was developed by investor Edgar Peters to explain the volatility of the economic indicators [16]. Peters argues that differences in investment horizons and available liquidity create 'self-similar statistical structures' in financial indicators. The example he gives in his work 'Fractal Market Analysis' states that a six-sigma event is a very different phenomena to a short-term investor as opposed to the long-term investor. The long term investor is therefore willing to provide liquidity to that short-term investor. If there is suddenly an absence of liquidity the structure begins to break and in Peters' view this leads to economic bust.

The Adaptive Market Hypothesis was developed by Andrew Lo, another successful investor. The Adaptive Market Hypothesis attempts to explain market behavior through behavioral economics and psychology. Human psychology is developed to handle the physical world not the fluctuating lines that represent the stock market. As a consequence human irrationality and bias, especially those explored in the research of Amen Tversky and Daniel Kahneman, generate volatility, opportunity for arbitrage, and prevent the pricing in of all available information [8]. This idea is not necessarily new, but extends all the way back John Maynard Keynes who once stated that most of our 'positive decisions' are the consequence of our 'animal spirits.'

As stated these theories are compatible. Benoit Mandelbrot stated in 'The Mis-Behavior of Markets' that we can understand qualitatively how markets move, but attempting to derive cause and effect would yield misleading results [12]. So much is clear: financial indicators show persistence. The rescaled range analysis conducted in this paper supports this, but does not indicate why or how the market exhibits this behavior only that it does. It will require much more work to give a causal explanation of such behavior although some have already begun to speculate why such behavior arises.

5 Conclusion

The evidence suggests stock price movement is slightly less than random. Stock prices do not follow a random walk, which contradicts the efficient market hypothesis. Markets are much more volatile than a normal bell curve would suggest and as a consequence the opportunity for arbitrage is possible. The R/S Analysis above shows that financial assets can exhibit persistence over large time intervals, but may exhibit anti-persistence over small time intervals. Some economists and investors have given explanations for why the Market behaves this way, but its not obvious that these explanations account for the whole picture or account for any part of it well. Further analysis will need to be conducted to offer a strong explanation of such behavior.

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