Fractals in Economic and Financial Indicators

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Math 538: Discrete Dynamical Systems (Fall 2019)

Overview

Fractals in Economics

- Benoit Mandelbrot first found fractals in cotton-prices, noting the self-similarity in financial indicators.
- Many economists have incorporated fractals via some form of Brownian motion (Bachelier, Black, Scholes, Peters, Samuelson).
- Self-similarity in economics is prevalent enough to allow for theories around fractals (Fractal Market Hypothesis, Brownian Model of Markets)

Fractals in Economics

- Fractals in economics comes in many flavors, but usually they come in some variant of Brownian Motion.
- In the majority of financial indicators either fractal Brownian Motion, geometric Brownian motion, or multi-fractal Brownian motion is present.
- How do we calculate fractal dimension of economic indicators and how should we interpret them?

Standard Brownian Motion

We define standard Brownian motion (Random Walk) with the following properties:

- $B(t_0) = B(0) = 0$
- The increment $\Delta B(t_j, t_i) = B(t_j) - B(t_i)$ is an independent random variable for $0 \leq i \leq j$
- The increment $\Delta B(t_j, t_i)$ is normally distributed with mean of 0 and variance of increment. $\Delta B(t_j, t_i) \sim \mathcal{N}(0, t_j - t_i)$

Standard Brownian Motion

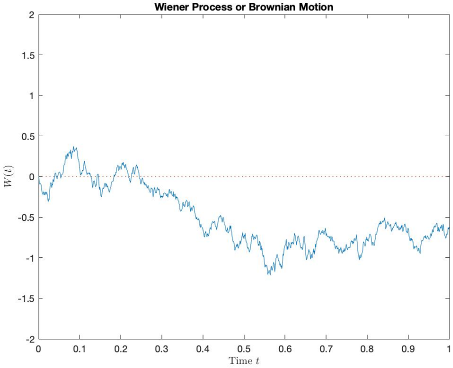

Figure: Brownian Motion

Fractal Brownian Motion

- Fractal Brownian motion has the following co-variance:

$$E(B_H(t_j)B_H(t_i)) = \frac{1}{2}(t_j^{2H} + t_i^{2H} - (t_j - t_i)^{2H})$$

- H is the Hurst exponent which is useful in finding the fractal dimension of stocks and other time-series.

Fractal Brownian Motion

- Random Walk ($H = 0.5$). Process exhibits perfect randomness.
- Persistence ($0.5 < H < 1$). Upward movement makes upward movement more probable and vice versa. Markets have long-run memory.
- Anti-Persistence ($0 < H < 0.5$). Upward movement makes downward movement more probable and vice versa. Markets have short-run memory.

Persistence

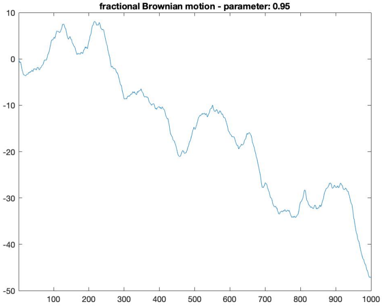

Figure: Fracatal Brownian Motion

Anti-Persistence

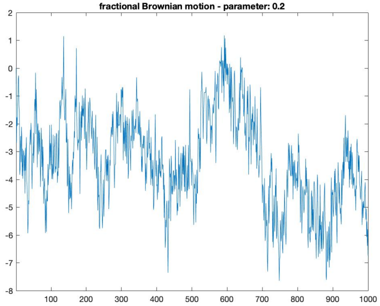

Figure: Fractal Brownian Motion

Why Fractal Brownian Motion in Economics?

- Markets have a degree of stochasticity (i.e. they are not perfectly deterministic).
- Most financial indicators show some degree of persistence.
- Standard Brownian motion does not capture the true volatility of the market which tends to trend generally upward for sometime (boom) and then downward for some time (bust).

Benoit Mandelbrot

- Before developing the eponymous Mandelbrot set Benoit Mandelbrot examined fractals in financial indicators.
- Mandelbrot developed many processes which model qualitatively the movement of stocks over time (Iterative Generators, Stable Levy distributions, Fractal and Multi-Fractal Brownian motion).

Rescaled Range Analysis

- The rescaled range is a statistical measure of the variability of a time series first introduced by hydrologist Harold Edwin Hurst.
- Rescaled Range Analysis assesses the variability of a series with respect to the length of a given time interval.
- Rescaled Range Analysis gives the estimated Hurst Exponent.

Rescaled Range Analysis (1)

Expected Rescaled Range:

$$E(R/S)_T = c \cdot T^H$$

Mean:

$$E(x(t)) = \frac{1}{T} \sum_{t=1}^{T} x(t)$$

Cumulative Deviation from Average:

$$X(t, T) = \sum_{t=1}^{T} (x(t) - E(x(t)))$$

Rescaled Range Analysis (2)

Range of Deviation over time span T:

$$R(T) = max_{1 \leq t \leq T} X(t, T) - min_{1 \leq t \leq T} X(t, T)$$

Standard Deviation:

$$S(T) = \sqrt{\frac{1}{T} \sum_{t=1}^{T} (x(t) - E(x(t)))^2}$$

Log of Rescaled Range gives Hurst Exponent:

$$\log E(R/S)_T = H \log(T) + \log(c)$$

Companies and SP500

- We wanted to analyze the Hurst exponent and the fractal dimension of the S&P 500 and four other companies: Facebook, Amazon, Apple, and Microsoft.
- We can then compare them to one another and how the fractal dimension for each varies over time.

Results

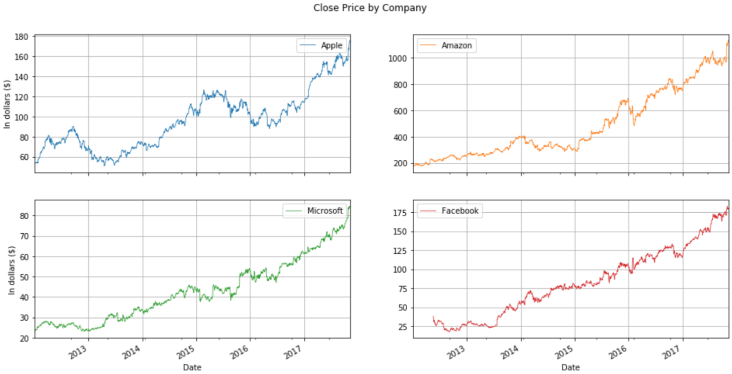

Figure: Closing prices of the companies over 2012-2017

Results

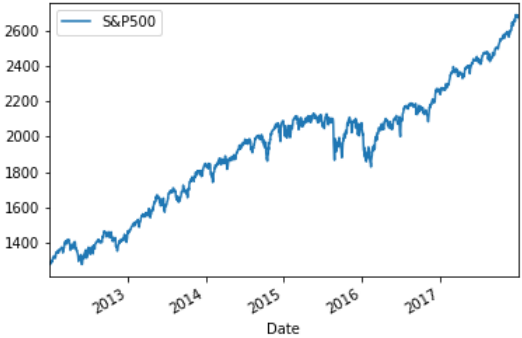

Figure: Closing prices of the SP500 over 2012-2017

Results

	Hurst Exponent
S&P500	0.557201
Apple	0.592904
Amazon	0.560250
Microsoft	0.539048
Facebook	0.582192

Figure: Estimated Hurst Exponent

Results

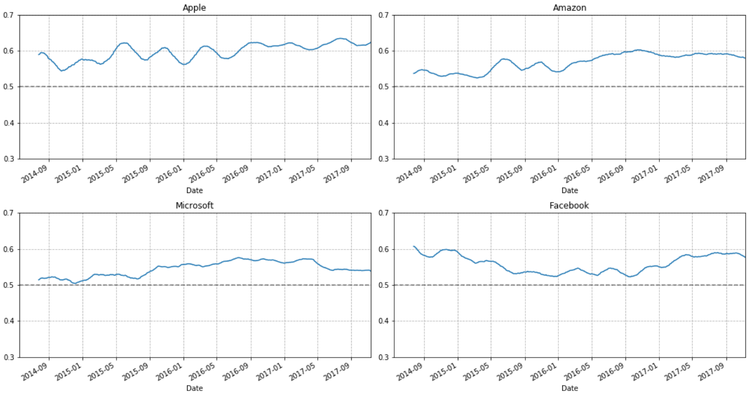

Figure: 50 days average 500-day interval

Results

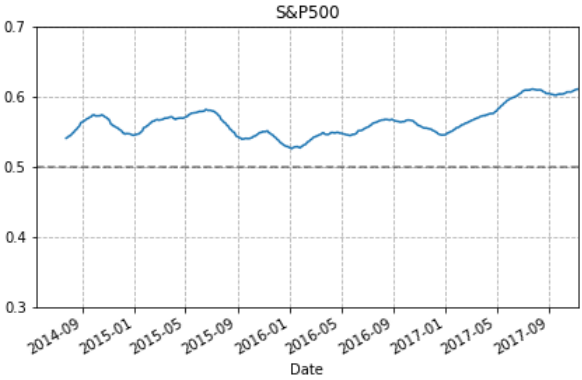

Figure: 50 days average 500-day interval

Results

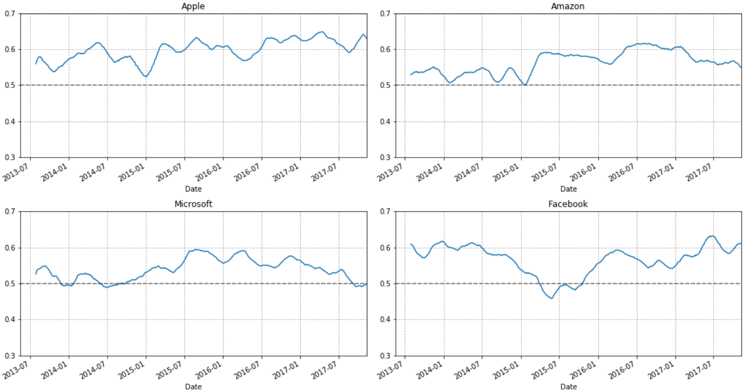

Figure: 50 days average 250-day interval

Results

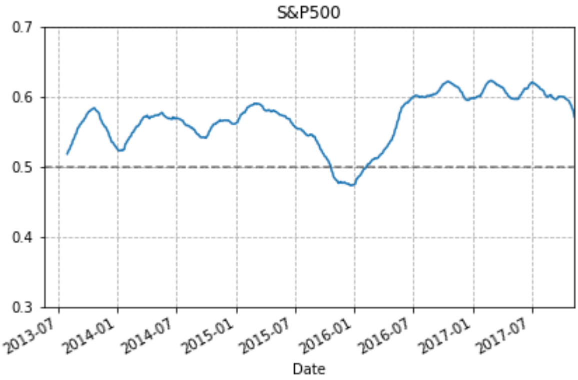

Figure: 50 days average 250-day interval

Results

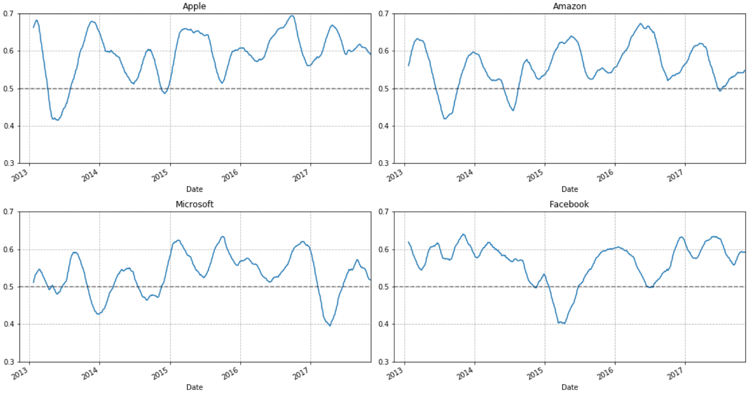

Figure: 50 days average 120-day interval

Results

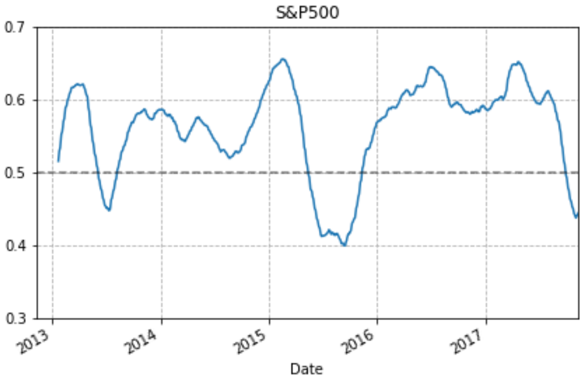

Figure: 50 days average 120-day interval

Results

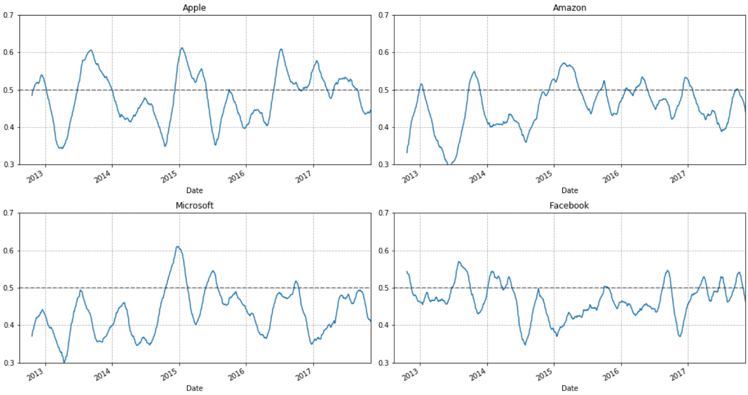

Figure: 50 days average 60-day interval

Results

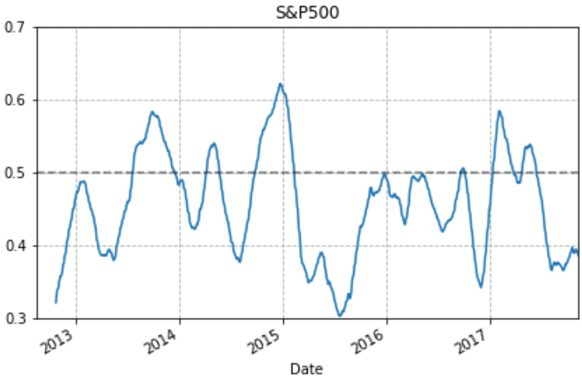

Figure: 50 days average 60-day interval

Fractal Dimensions

- We see often persistence describes the stock quality over most time intervals and over the larger time interval especially.
- The estimated fractal dimension over the whole time interval are as follows:
  S&P500: 1.442799. Apple: 1.407096. Amazon: 1.43975.
  Microsoft: 1.460952. Facebook: 1.417808.

Implications of Results

- Stocks and Financial Indicators generally have long-term dependence (i.e. memory). Past history affects future movement.
- Stocks generally behave more like lines than random walks. This is a consequence of the long-term dependence in the asset.

Economic Theory: Market Hypotheses

- Stocks do not appear to follow a random walk (i.e. do not price in all information (Efficient Markets Hypothesis)
- One explanation for this is the Fractal Market Hypothesis which says that markets exhibit self-similar structures due to differing investment horizons and liquidity changes.
- Another explanation is the Adaptive Market's Hypothesis which states that markets ebb with irrational behavior.

Conclusions

- The company stock examined exhibit fractal structure and long-term memory. The fractal dimension is generally less than 1.5, but greater than 1.3
- This suggests that market movement is less than truly random and arbitrage is possible.
- The evidence for this is well documented, but the why such behavior occurs is unclear.

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