Design of Digital Filters using Window Functions

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Usually we want to design filters in the frequency domain to obtain some specifications required by our design. For example, while designing a lowpass filter, we desgin it in the frequency domain first and then consider the time domain representation. A low pass filter looks like this.

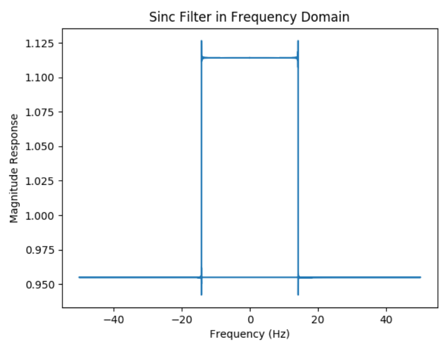

Figure 1: Frequency Domain

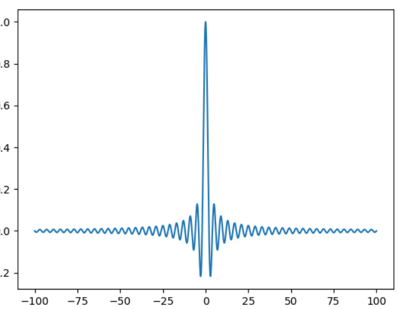

Figure 2: Time Domain

Clearly, if we want to start designing a LPF, we start by frequency domain representation and then derive the time domain representation. To our surprise, it is a IIR filter, which cannot be implemented directly. For this reason we put a window function with a multiplication operator in front of the time domain signal (here sinc function, sweeping to infinity time). However, there are some unusual effects on the frequency domain by this approach. The same can be seen from the figure. There are ripples at the ends of the high level of the square wave. Why this is so, is a part of further discussion. The above frequency representation is generated by the window function itself. Suppose we have a IIR representation of the lowpass filter with impulse response represented by $h_{IIR}(n)$, we can obtain a window of sinc by multiplying $h_{IIR}(n)$ by a rectangular wave of length N. Hence, $w(n) = u(n) - u(n-N)$. To obtain a FIR filter we do this: $h_{FIR}(n) = h(n) = h_{IIR}(n)w(n)$. In frequency domain, we get

$$H(e^{j\omega}) = \int_{-\infty}^{\infty} H_{IIR}(e^{j\theta})W(e^{\omega-\theta})d\theta$$

Another thing to note down about this window function is that it is a sinc function in the frequency domain. So, if this sinc function has large center lobe, then it's height must be small. This is governed by the following equation.

$$|W(e^{j\omega})| = \left|\frac{\sin(\frac{\omega N}{2})}{\sin(\frac{\omega}{2})}\right|$$

By L'Hopital Rule, the height of the central lobe is N. The first minima occur at $\pm\frac{2\pi}{N}$. If we want good approximation of the LPF frequency response, we must make the central lobe as thin as possible

(just like the dirac pulse, because the convolution of dirac pulse with the $H_{IIR}(e^{j\omega})$ will give $H(e^{j\omega}) = H_{IIR}(e^{j\omega})$). Since it is not possible to make the central lobe as dirac pulse because for that N has to be ∞ which is the same problem with which we started. This is like saying that if you want $H(e^{j\omega}) = H_{IIR}(e^{j\omega})$, you need a window which will pass all of the $H_{IIR}(e^{j\omega})$ and hence it must have ∞ sample points. Hence there must be a trade off between the number of sample points N and the width of the central lobe. And because of this trade off, there are ripples at the edges of the LPF frequency response.
The solution to this problem is chosing a window function in which there is a smooth decrease of value to zero. Lets see what is the difference. Consider a recatangular and a hanning window shown below with their frequency domain representations.

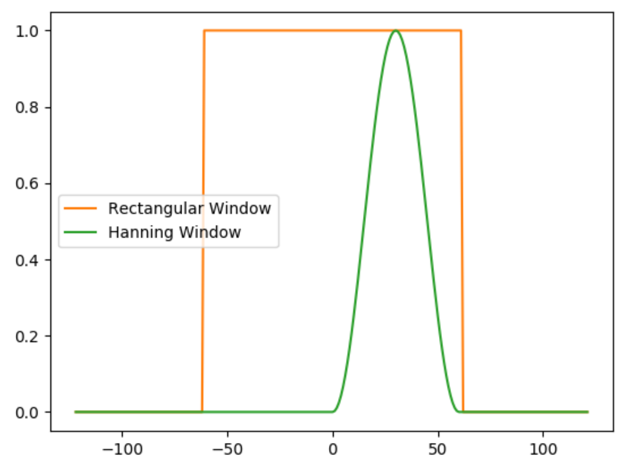

Figure 3: Time Domain

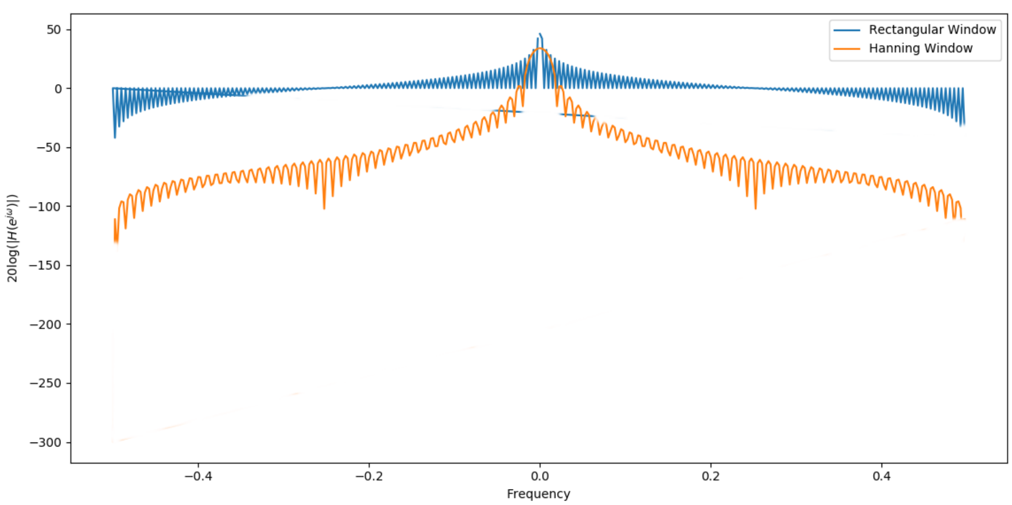

Figure 4: Frequency Domain

Clearly, we can see that at high frequency, a hanning window is much more efficient in reducing the effects of side lobes. The problem with hanning window however is that its width in time domain has decreased or equivalently, the central lobe width has increased which was not desired from the beginning. So, although a hanning widow reduces the ripples in the square wave (see Figure 6), it has a wider central lobe or less width in time doimain, and so we have wider transitions at the discontinuities of $H_{IIR}(e^{j\omega})$. The fact about ripples is evident from the plots given below.

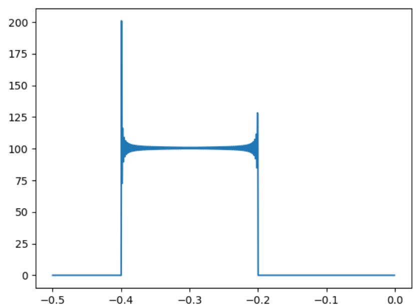

Figure 5: Filter generated using Rectangular Window

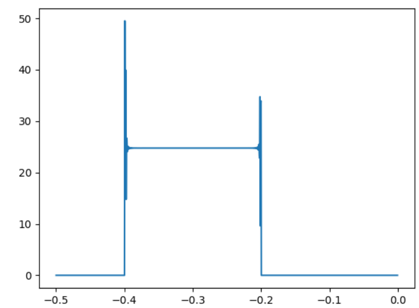

Figure 6: Filter generated using Hanning Window (Reduced oscillations)

Comparison of transition at discontinuities is shown below.

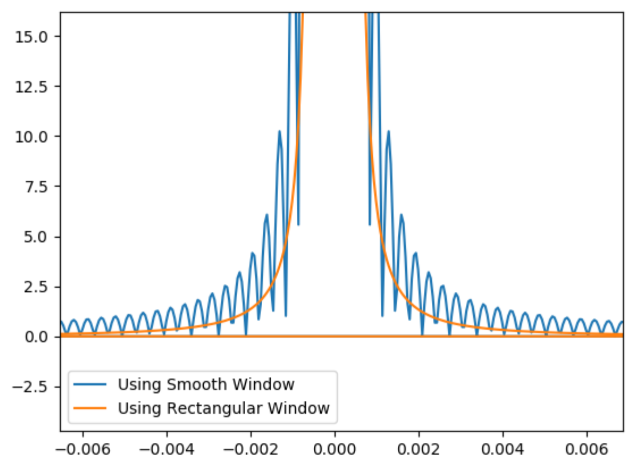

Figure 7: Comparison of transitions at discontinuities of $H_{IIR}(e^{j\omega})$

It is evident from the figure that a hanning window has a wider transition than a rectangular window. So more the window is smooth in time domain, more wider are the transitions in the filter. This is a disadvantage, the advantage being less ripples. Now it totally depends on the type of window required for the design upon the designer. A suitable choice could be a Kaiser window.

References

[1] Digital Signal Processing, Alan V. Oppenheim, Ronald W. Schafer, 2nd Edition