Parametricity and semi-cubical types

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Summary

Introduction to type theory

Introduction to parametricity

Constructing semi-cubical models

A semantical point of view [Dybjer 95]

Definition

A model of type theory consists of:

- A collection of contexts.
- For any context Γ, a collection of types over Γ.
- For any type A over Γ, a collection of terms in A.

with a lot of structure (substitutions, Π, Σ, $\top$ and $\mathcal{U}$).

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In type theory	In set theory	In logic	Notation
Context Γ	Set Γ	Hypothesis	$\Gamma \vdash$
Type A	Family of sets $(A_\gamma)_{\gamma \in \Gamma}$	Proposition	$\Gamma \vdash A$
Term a	Sections $(a_\gamma \in A_\gamma)_{\gamma \in \Gamma}$	Proof	$\Gamma \vdash a : A$

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Such a model can be considered as a mathematical universe.

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- Homotopic models (e.g. Kan cubical sets).
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Slogan

The abundance of models makes the strength of type theory.

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Proof assistants like Coq and Agda implement an initial model.

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So there is a rich interaction between models and proof assistants:

A formal proof $\Rightarrow$ One theorem per model.

A model $\Rightarrow$ An extension of the proof assistant.

Summary

Parametricity for the initial model [Bernardy et al. 2010]

We can define operations $\_{}_*$ in the initial model:

$$\begin{array}{ccc} \Gamma \vdash & \text{gives} & \Gamma_0, \Gamma_1 \vdash \Gamma_* \\ \Gamma \vdash A & \text{gives} & \Gamma_0, \Gamma_1, \Gamma_*, A_0, A_1 \vdash A_* \\ \Gamma \vdash a : A & \text{gives} & \Gamma_0, \Gamma_1, \Gamma_* \vdash a_* : A_*(a_0, a_1) \end{array}$$

by induction using equations (E) summarized next slide.

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An extension of type theory by unary operations defined inductively in the initial model is called an interpretation.

Equations (E) summarized

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\begin{aligned}
(A \times B)_*((x_0, y_0), (x_1, y_1)) &= A_*(x_0, x_1) \times B_*(y_0, y_1) \\
(s.1)_* &= s_*.1 \\
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(s, t)_* &= (s_*, t_*)
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(A \to B)_*(f_0, f_1) &= \Pi\ (x_0, x_1 : A).\ A_*(x_0, x_1) \to B_*(f_0(x_0), f_1(x_1)) \\
(s(t))_* &= s_*(t_0, t_1, t_*) \\
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$$U_*(A_0, A_1) \quad = \quad A_0 \to A_1 \to U$$

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Goal

We want to build models with parametricity from arbitrary ones.

Summary

Parametricity and cubes

When defining internal parametricity, cubical structures arise:

- [Bernardy, Coquand, Moulin 2015]
- [Cavallo, Harper 2018]

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Claim

There is a general procedure:

$$\{\textit{Interpretations of type theory}\} \to \{\textit{Structures on types}\}$$

sending (external) parametricity to semi-cubical structures.

A semi-cubical set consists of:

A set of points

For any two points
a set of paths between them

For any square S
a set of surfaces with border S

...

A semi-cubical set consists of:	Starting from a context and applying parametricty we get:
A set of points	$\Gamma \vdash$
For any two points a set of paths between them	$\Gamma_0, \Gamma_1 \vdash \Gamma_*$
For any square S a set of surfaces with border S	$\Gamma_{00}, \Gamma_{01}, \Gamma_{0*}, \Gamma_{10}, \Gamma_{11}, \Gamma_{1*}, \Gamma_{*0}, \Gamma_{*1} \vdash \Gamma_{**}$
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So we guess semi-cubes model parametricity.

Main result

Theorem

The functor forgetting parametricity:

$$U : \{\textit{Models with parametricity}\} \to \{\textit{Models of type theory}\}$$

has a right adjoint:

$$\textit{Cube} : \{\textit{Models of type theory}\} \to \{\textit{Models with parametricity}\}$$

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Indeed $\mathit{Cube}(\mathcal{C})$ is the model of semi-cubes in $\mathcal{C}$.

Sketch of proof

Let T be a finitary essentially algebraic theory, I_T its initial algebra.

Lemma

Let O a set of unary operations inductively defined on I_T by equations E. Then the forgetful functor:

$$U : Alg_{T,O,E} \to Alg_T$$

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has a right adjoint.

We use colimits in Alg_T defined as QIITs. Then U commutes with:

- Initial objects almost by hypothesis.
- Pushouts because O is unary.
- Filtered colimits as T, O, E are finitary.

So U has a right adjoint.

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The forgetful functor from $\{X : Set \mid f : X \to X\}$ to sets has a right adjoint:

$$Cube \;:\; X \;\mapsto\; \big(\, \mathbb{N} \to X \ \mathit{with}\ (u_n) \mapsto (u_{n+1}) \,\big)$$

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Many other right adjoints can be constructed the same way.

Semi-cubes

Let $\mathcal{C}$ be a model of type theory.

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Adjunction equation

$$Ctx_{Cube(\mathcal{C})} = Hom_{param}(I_X, Cube(\mathcal{C})) = Hom(U(I_X), \mathcal{C})$$

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Let $\mathcal{C}$ be a model of type theory.
Let I_X be the parametric model freely generated by X a context.

Adjunction equation

$$Ctx_{Cube(\mathcal{C})} = Hom_{param}(I_X, Cube(\mathcal{C})) = Hom(U(I_X), \mathcal{C})$$

But $U(I_X)$ is freely generated by:

$$\begin{array}{rcl}
X & \vdash & \\
X_0, X_1 & \vdash & X_* \\
X_{00}, X_{01}, X_{0*}, & & \\
X_{10}, X_{11}, X_{1*}, & \vdash & X_{**} \\
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& \vdots &
\end{array}$$

Conclusion and further work

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- We build semi-cubical models from parametricity.
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- Applications to other interpretations for type theory.

For specialists, I intend to:

- Find an interpretation giving Kan cubical types, starting in low dimension (i.e. with setoids).
- Build definitionally univalent models from univalent ones using [Tabareau, Tanter, Sozeau 2017].