

INTRO TO DATA SCIENCE

LECTURE 7: PROBABILITY AND NAIVE BAYES CLASSIFICATION

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11/16/2015

LAST TIME:

- CLASSIFICATION**
- K-NEAREST NEIGHBORS CLASSIFICATION**

QUESTIONS?

I INTRO TO PROBABILITY

II. BAYESIAN INFERENCE

III. NAIVE BAYES CLASSIFICATION

HANDS-ON: NAIVE BAYES CLASSIFICATION

I. INTRO TO PROBABILITY

*Q: What is a **probability**?*

A: A number between 0 and 1 that characterizes the likelihood that some event will occur.

The probability of event A is denoted $P(A)$.

Q: What is the set of all possible events called?

*A: This set is called the **sample space** Ω . Event A is a member of the sample space, as is every other event.*

The probability of the sample space $P(\Omega)$ is 1.

Q: What is a probability distribution?

A: A function that assigns probability to each event in the sample space.

*A distribution can be **discrete** or **continuous***

Ex: Discrete – Uniform distribution

$$X \sim \{1, \dots, N\} \longrightarrow P(X = x) = 1/N$$

Continuous – Normal distribution – $N(\mu, \sigma^2)$

$$X \sim N(0, 1) \longrightarrow P(X = x) = 0$$

Discrete Probability Distributions:

- *These are **probability mass functions***
- *Each value $P(X=x)$ represents the probability of observing a given value x for variable X*

$$P(\Omega) = \sum_{X=x} P(X = x) = 1$$

Continuous Probability Distributions:

- *These are **probability density functions (PDF)***
- *Each value $P(X=x)$ represents the **relative probability** of observing a given value x for variable X*

$$P(x_0 < x < x_1) = \int_{x_0}^{x_1} P(x) dx$$

$$P(\Omega) = \int_{-\infty}^{+\infty} P(x) dx = 1$$

Q: What is a random variable?

A: Essentially, a measurable whose possible values have a probability distribution

Values of these are the “Events” for which we’re looking to measure the probabilities

Q: What is expected value?

A: It is the average value of a random variable

For discrete distributions

$$E(X) = \sum x * p(x)$$

For continuous distributions

$$E(X) = \int (x * p(x)) dx$$

Q: Consider two events A & B . How can we characterize the intersection of these events?

*A: With the **joint probability** of A and B , written $P(A, B)$.*

*Q: Suppose event B has occurred. What quantity represents the probability of A **given** this information about B ?*

A: The intersection of A & B divided by region B .

*This is called the **conditional probability** of A given B , written $P(A|B) = P(A \cap B) / P(B)$.*

Q: What does it mean for two events to be independent?

A: Information about one does not affect the probability of the other.

This can be written as $P(A|B) = P(A)$.

Using the definition of the conditional probability, we can also write:

$$P(A|B) = P(A \cap B) / P(B) = P(A) \rightarrow P(A \cap B) = P(A) * P(B)$$

II. BAYESIAN INFERENCE

This result is called Bayes' theorem. Here it is again:

$$P(A|B) = P(B|A) * P(A) / P(B)$$

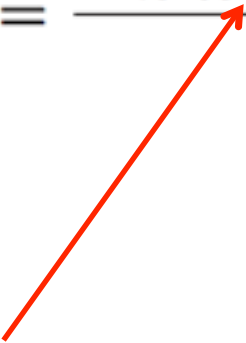
Some facts:

- This is a simple algebraic relationship using elementary definitions.*
- It's a very powerful computational tool.*

Each term in this relationship has a name, and each plays a distinct role in any probability calculation (including ours). Here's how it might look in the context of classification.

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$

*This term is the **likelihood function**. It represents the joint probability of observing features $\{x_i\}$ given that that record belongs to class C .*

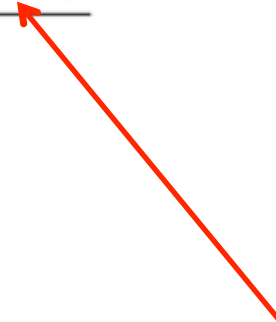
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We can observe the value of the likelihood function from the training data.

*This term is the **prior probability** of C . It represents the probability of a record belonging to class C before the data is taken into account.*

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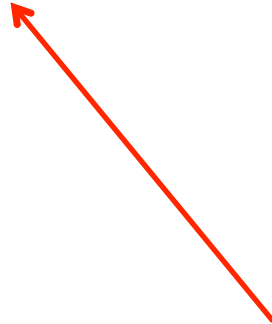
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$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$

The value of the prior is also observed from the data.

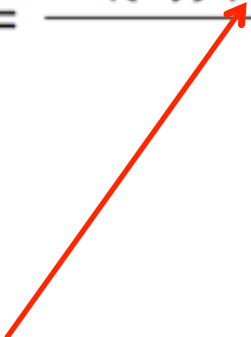
*This term is the **normalization constant**. It doesn't depend on C , and is generally ignored until the end of the computation.*

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$



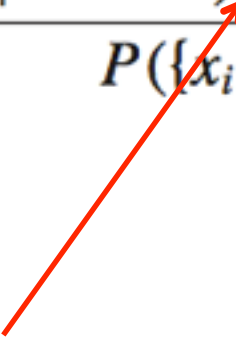
Maximum likelihood estimator (MLE):

*What parameters **maximize** the likelihood function?*

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$


Maximum a posteriori estimate (MAP):

*What parameters **maximize** the likelihood function **AND** prior?*

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$


III. NAIVE BAYES CLASSIFICATION

	<i>continuous</i>	<i>categorical</i>
<i>supervised</i>	<i>regression</i>	<i>classification</i>
<i>unsupervised</i>	<i>dim reduction</i>	<i>clustering</i>

*This term is the **posterior probability** of C . It represents the probability of a record belonging to class C after the data is taken into account.*

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$

*The goal of any Bayesian computation is to find (“learn”) the posterior distribution of a particular variable given new **evidence**.*

Problem:

We observe the following coin flips:

HTHH

What is $P(X = \text{Heads})$?

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What is $P(X = \text{Heads})$? $3/4$, Why?

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We observe the following coin flips:

HTHHTHT

Maximum likelihood estimator (MLE):

*What parameters **maximize** the likelihood function?*

Let $P(X = \text{Heads}) = q$, and write Bayes Theorem

$$P(q \mid \text{observations}) = P(\text{observations} \mid q) * P(q) / \text{constant}$$

Maximum likelihood estimator (MLE):

*What parameters **maximize** the likelihood function?*

Let $P(X = \text{Heads}) = q$, and write Bayes Theorem

$$P(q \mid \text{observations}) = \frac{P(\text{observations} \mid q) * P(q)}{\text{constant}}$$

$$P(\text{observations} \mid q) = ?$$

$$P(q) = ?$$

Binomial Distribution:

$$\Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$\begin{aligned} P(4 \text{ heads, } 3 \text{ tails} \mid q) &= P(X = 4, n = 7) \\ &= (7 \text{ choose } 4) * q^4 * (1-q)^3 \end{aligned}$$

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Optimize w.r.t. $q \longrightarrow$ **MLE:** $q = 4/7$

A prior distribution is known as **conjugate prior** if its from the same family as the posterior for a certain likelihood function

For the binomial distribution, the conjugate prior is the **Beta distribution**

$$\begin{aligned} &= \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} \\ &= \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \end{aligned}$$

The **MAP estimate** is the value that maximizes both the likelihood function and prior – the product of the two.

In the coin flip setting is the value that optimizes

$$P(4H, 3T \mid q) * P(q)$$

The **MAP estimate** is the value that maximizes both the likelihood function and prior – the product of the two.

In the coin flip setting with a Beta distribution it's the value that optimizes:

$$\begin{aligned} & P(4H, 3T \mid q) * P(q) \\ &= \binom{7}{4} q^4 (1-q)^3 q^{(\alpha-1)} (1-q)^{(\beta-1)} \end{aligned}$$

Why do you care?

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Ex. 1:

Sample 100 people and ask if they support a politician?

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Ex. 1:

Sample 100 people and ask if they support a politician?

23 say Yes – Is the correct prediction 23/100?

What's the prior?

*Suppose we have a dataset with features x_1, \dots, x_n and a class label C .
What can we say about classification using Bayes' theorem?*

Suppose we have a dataset with features x_1, \dots, x_n and a class label C . What can we say about classification using Bayes' theorem?

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$

Bayes' theorem can help us to determine the probability of a record belonging to a class, given the data we observe.

*The idea of Bayesian inference, then, is to **update** our beliefs about the distribution of C using the data (“evidence”) at our disposal.*

$$P(\text{class } C \mid \{x_i\}) = \frac{P(\{x_i\} \mid \text{class } C) \cdot P(\text{class } C)}{P(\{x_i\})}$$

Then we can use the posterior for prediction.

Q: What piece of the puzzle we've seen so far looks like it could be intractably difficult in practice?

Remember the likelihood function?

$$P(\{x_i\} | C) = P(\{x_1, x_2, \dots, x_n\} | C)$$

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Observing this exactly would require us to have enough data for every possible combination of features to make a reasonable estimate.

Q: What piece of the puzzle we've seen so far looks like it could intractably difficult in practice?

A: Estimating the full likelihood function.

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*This “**naïve**” assumption simplifies the likelihood function to make it tractable.*

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A: In our training phase, we ‘learn’ the probability of seeing our training examples under each class.

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Q: Given that we can compute this value, what do we do with it?

A: In our training phase, we 'learn' the probability of seeing our training examples under each class.

Then we use Bayes Theorem to compute $P(\text{class} | \text{inputs})$

Example: Text Classification

Does this news article talk about politics?

Training Set: Collection of New Articles

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Does this news article talk about politics?

Training Set: Collection of New Articles

Article 1: The computer contractor who exposed....

Article 2: The parents of a missing U.S. journalist in Syria...

Q: What are my features?

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A: The text in the documents.

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Q: How do I represent them?

A: Binary occurrence? Word counts?

the, computer, contractor, exposed, parents, missing, Syria, U.S.

<i>1</i>	<i>1</i>	<i>1</i>	<i>1</i>	<i>0</i>	<i>0</i>	<i>0</i>	<i>0</i>
<i>1</i>	<i>0</i>	<i>0</i>	<i>0</i>	<i>1</i>	<i>1</i>	<i>1</i>	<i>1</i>

he, computer, contractor, exposed, parents, missing, Syria, U.S.

1 1 1 1 0 0 0 0

1 0 0 0 1 1 1 1

We can make some alterations

1) Drop stop words (commonly occurring words that don't have meaning)

the, computer, contractor, exposed, parents, missing, Syria, U.S., POL

1 1 1 1 0 0 0 0

1 0 0 0 1 1 1 1

Our goal is to compute compute

$P (POL = T \mid \text{words in the text})$

*We need to **learn** $P(\text{word} \mid POL)$*

i.e. $P (Syria \mid POL)$

*the, computer, contractor, exposed, parents, missing, Syria, U.S., **POL***

1 1 1 1 0 0 0 0

1 0 0 0 1 1 1 1

Once we've learned $P(\text{computer} \mid \text{POL})$, $P(\text{U.S.} \mid \text{POL})$ etc. on our training set, we want to label our test set

the, computer, contractor, exposed, parents, missing, Syria, U.S., **POL**

1 1 1 1 0 0 0 0

1 0 0 0 1 1 1 1

The predicted label, POL = True or

POL = False is the one that maximizes our posterior.

the, computer, contractor, exposed, parents, missing, Syria, U.S., **POL**

1 1 1 1 0 0 0 0

1 0 0 0 1 1 1 1

Compute probability in each class:

$$P (POL = T \mid \{x\}) = c * P (\{x\} \mid POL = T) * P(POL=T)$$

$$P (POL = F \mid \{x\}) = c * P (\{x\} \mid POL = F) * P(POL=F)$$

the, computer, contractor, exposed, parents, missing, Syria, U.S., POL

1 1 1 1 0 0 0 0

1 0 0 0 1 1 1 1

Article 2: The parents of a missing U.S. journalist in Syria...

$$P (POL = T \mid \{x\}) = P (\{x\} \mid POL = T) * P(POL=T)$$

$$= P(Syria \mid POL=T) * P(journalist \mid POL=T) * P(parents \mid POL=T) ... * P(POL=T)$$

INTRO TO DATA SCIENCE

HANDS-ON: NAIVE BAYES