Homework 5

References

- Lectures 17-20 (inclusive).

Instructions

- Type your name and email in the "Student details" section below.
- Develop the code and generate the figures you need to solve the problems using this notebook.
- For the answers that require a mathematical proof or derivation you should type them using latex. If you have never written latex before and you find it exceedingly difficult, we will likely accept handwritten solutions.
- The total homework points are 100. Please note that the problems are not weighed equally.

```
import matplotlib.pyplot as plt
%matplotlib inline
import matplotlib_inline
matplotlib_inline.backend_inline.set_matplotlib_formats('svg')
import seaborn as sns
sns.set_context("paper")
sns.set_style("ticks")

import scipy
import numpy as np
import scipy.stats as st
import urllib.request
import os

def download(
    url : str,
    local_filename : str = None
):
    """Download a file from a url.

    Arguments
    url            -- The url we want to download.
    local_filename -- The filemame to write on. If not
                      specified
    """
    if local_filename is None:
        local_filename = os.path.basename(url)
    urllib.request.urlretrieve(url, local_filename)
```

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Problem 1 - Clustering Uber Pickup Data

In this problem you will analyze Uber pickup data collected during April 2014 around New York City. The complete data are freely on Kaggle. The data consist of a timestamp (which we are going to ignore), the latitude and longitude of the Uber pickup, and a base code (which we are also ignoring). The data file we are going to use is uber-raw-data-apr14.csv. As usual, you have to make it visible to this Jupyter notebook. On Google Colab, just run this:

```
url = "https://github.com/PredictiveScienceLab/data-analytics-se/raw/master/lecturebook/data/uber-raw-data-apr14.csv"
download(url)
```

And you can load it using pandas:

```
import pandas as pd
p1_data = pd.read_csv('uber-raw-data-apr14.csv')
```

Here is how the data look like:

```
p1_data
```

	Date/Time	Lat	Lon	Base
0	4/1/2014 0:11:00	40.7690	-73.9549	B02512
1	4/1/2014 0:17:00	40.7267	-74.0345	B02512
2	4/1/2014 0:21:00	40.7316	-73.9873	B02512
3	4/1/2014 0:28:00	40.7588	-73.9776	B02512
4	4/1/2014 0:33:00	40.7594	-73.9722	B02512
...	...	...	...	...
564511	4/30/2014 23:22:00	40.7640	-73.9744	B02764
564512	4/30/2014 23:26:00	40.7629	-73.9672	B02764
564513	4/30/2014 23:31:00	40.7443	-73.9889	B02764
564514	4/30/2014 23:32:00	40.6756	-73.9405	B02764
564515	4/30/2014 23:48:00	40.6880	-73.9608	B02764

564516 rows × 4 columns

If you have never played before with pandas, you can find a nice tutorial here.

We have half a million data points. Let's extract the latitude and longitude and put them in a numpy array:

```
loc_data = p1_data[['Lon', 'Lat']]
loc_data
```

	Lon	Lat
0	-73.9549	40.7690
1	-74.0345	40.7267
2	-73.9873	40.7316
3	-73.9776	40.7588
4	-73.9722	40.7594
...	...	...
564511	-73.9744	40.7640
564512	-73.9672	40.7629
564513	-73.9889	40.7443
564514	-73.9405	40.6756
564515	-73.9608	40.6880

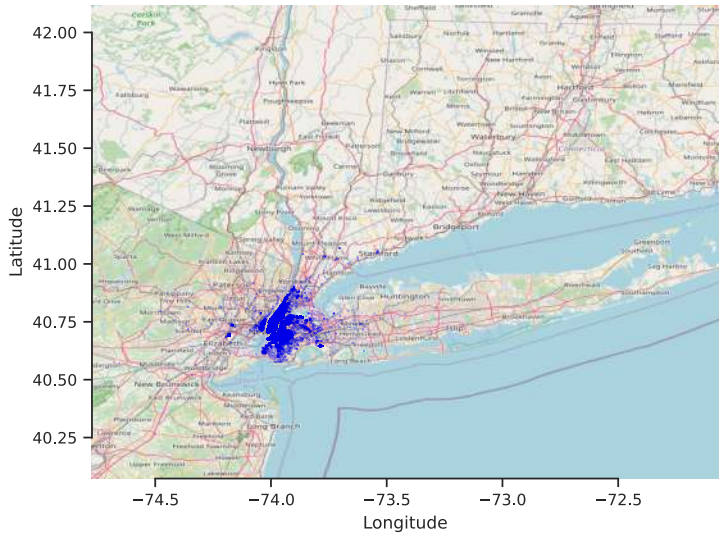
564516 rows × 2 columns

Let's visualize these points on the map of New York City:

```
url = "https://raw.githubusercontent.com/PredictiveScienceLab/data-analytics-se/master/lecturebook/images/ny_map.png"
!curl -O $url
ny_map = plt.imread('ny_map.png')
box = ((loc_data.Lon.min(), loc_data.Lon.max(),
        loc_data.Lat.min(), loc_data.Lat.max()))
fig, ax = plt.subplots(dpi=600)
ax.scatter(
    loc_data.Lon,
    loc_data.Lat,
    zorder=1,
    alpha= 0.5,
    c='b',
```

```
    s=0.001
)
ax.set_xlim(box[0],box[1])
ax.set_ylim(box[2],box[3])
ax.imshow(
    ny_map,
    zorder=0,
    extent=box,
    aspect= 'equal'
)
ax.set_xlabel('Longitude')
ax.set_ylabel('Latitude')
sns.despine(trim=True);
```

```
  % Total    % Received % Xferd  Average Speed   Time    Time     Time  Current
                                 Dload  Upload   Total   Spent    Left  Speed
100  884k  100  884k    0     0  2670k      0 --:--:-- --:--:-- --:--:-- 2673k
```



Machine learning algorithms will be a bit slow because we have over half a million data points. So, as you develop your code, use only 50K observations. Once you have a stable version of your code, modify the following code segment to use the entire dataset.

```
# As you develop your code, use this:
# p1_train_data = loc_data[:50_000]

# Once you have a stable version of your code, use this:
p1_train_data = loc_data
```

Part A - Splitting New York City into Subregions

Suppose you are assigned to split New York City into operating subregions with equal demand. When a pickup is requested in each subregion, only the drivers in that region are called. Note that this can become a challenging problem very quickly. We are not looking for the best possible answer here. We are looking for a data-informed heuristic solution that is good enough.

Do (at least) the following:

- Use Kmeans clustering on the pickup data with different numbers of clusters;
- Visualize the labels of the clusters on the map using different colors (see the hands-on activities);
- Visualize the centers of the discovered Kmeans clusters (in red color);
- Use common sense, e.g., ensure there are enough clusters so no region crosses the water. If it is impossible to get perfect results simply by Kmeans, feel free to ignore a small number of outliers as they could be handled manually;
- Use MiniBatchKMeans, which is a much faster version of Kmeans suitable for large datasets (>10K observations);

Answer with as many text and code blocks as you like below.

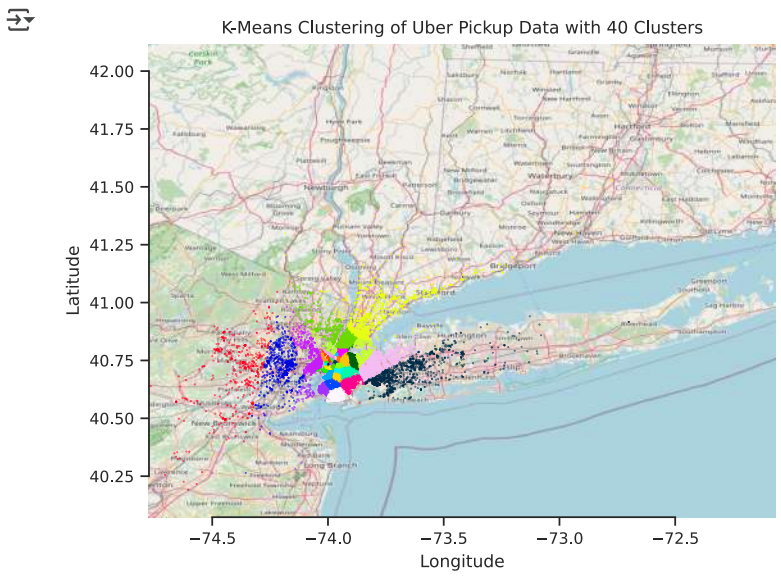
```
# Your code here
# Import MiniBatchKMeans
from sklearn.cluster import MiniBatchKMeans
```

```
# Apply K-means to the data
model = MiniBatchKMeans(n_clusters=40).fit(p1_train_data)
```

After experimenting with various cluster counts, I found that 40 clusters is the optimal amount. When I used fewer than 40 clusters, several regions spanned across bodies of water. Once I increased to 40 clusters, there were a minimal number of regions that crossed the water.

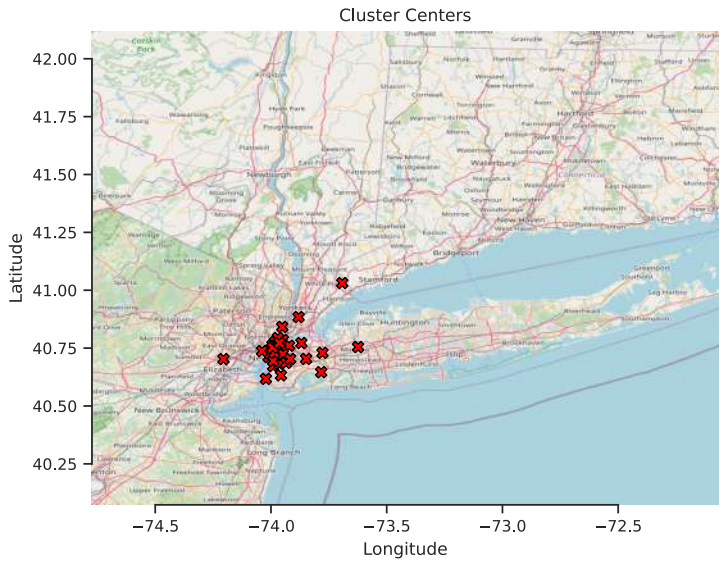
The plot below visualizes the 40 clusters on the map, where each cluster is shown in a different color.

```
# Visualize the clusters on the map using different colors
fig, ax = plt.subplots(dpi=600)
labels = model.predict(p1_train_data)
ax.scatter(
    p1_train_data.Lon,
    p1_train_data.Lat,
    zorder=1,
    alpha= 0.8,
    c=labels,
    s=0.1,
    cmap="gist_ncar")
ax.set_xlim(box[0],box[1])
ax.set_ylim(box[2],box[3])
ax.imshow(ny_map, zorder=0, extent=box, aspect='equal')
ax.set_xlabel('Longitude')
ax.set_ylabel('Latitude')
ax.set_title('K-Means Clustering of Uber Pickup Data with 40 Clusters')
sns.despine(trim=True);
```



The plot below visualizes the cluster centers on the map, which are denoted by the red X's.

```
# Visualize the centers of the K-means clusters (in red)
fig, ax = plt.subplots(dpi=600)
x_centers = model.cluster_centers_[:, 0]
y_centers = model.cluster_centers_[:, 1]
ax.scatter(x_centers, y_centers, marker='X', c='r', edgecolor='k', s=40)
ax.set_xlim(box[0],box[1])
ax.set_ylim(box[2],box[3])
ax.imshow(ny_map, zorder=0, extent=box, aspect='equal')
ax.set_xlabel('Longitude')
ax.set_ylabel('Latitude')
ax.set_title('Cluster Centers');
sns.despine(trim=True);
```



Part B - Create a Stochastic Model of Pickups

One of the key ingredients for a more sophisticated approach to optimizing the operations of Uber is the construction of a stochastic model of the demand for pickups. The ideal model for this problem is the Poisson Point Process. However, we will do something more straightforward, using the Gaussian mixture model and a Poisson random variable. The model will not have a time component, but it will allow us to sample the number and locations of pickups during a typical month. We will guide you through the process of constructing this model.

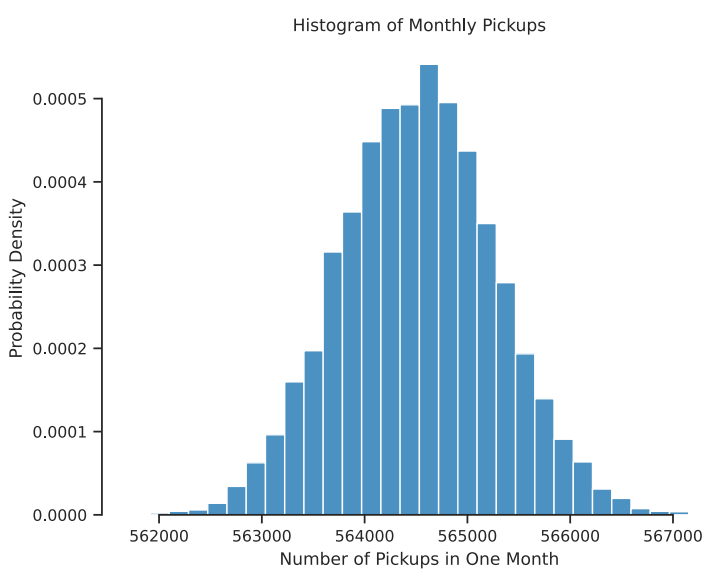
Subpart B.I - Random variable capturing the number of monthly pickups

Find the rate of monthly pickups (ignore the fact that months may differ by a few days) and use it to define a Poisson random variable corresponding to the monthly number of pickups. Use `scipy.stats.poisson` to initialize this random variable. Sample from it 10,000 times and plot the histogram of the samples to get a feeling about the corresponding probability mass function.

```
# Your code here
# Find the rate of monthly pickups
monthly_rate = len(p1_train_data)
print(f"The rate of monthly pickups is {monthly_rate}.")
```

```
The rate of monthly pickups is 564516.
```

```
# Define a Poisson distribution for the number of monthly pickups
monthly_pickup_dist = st.poisson(mu=monthly_rate)

# Take 10,000 samples from the Poisson distribution
samples = monthly_pickup_dist.rvs(size=10000)

# Plot the histogram of the samples
fig, ax = plt.subplots()
ax.hist(samples, density=True, bins=30, alpha=0.8)
ax.set_xlabel('Number of Pickups in One Month')
ax.set_ylabel('Probability Density')
ax.set_title('Histogram of Monthly Pickups')
sns.despine(trim=True);
```



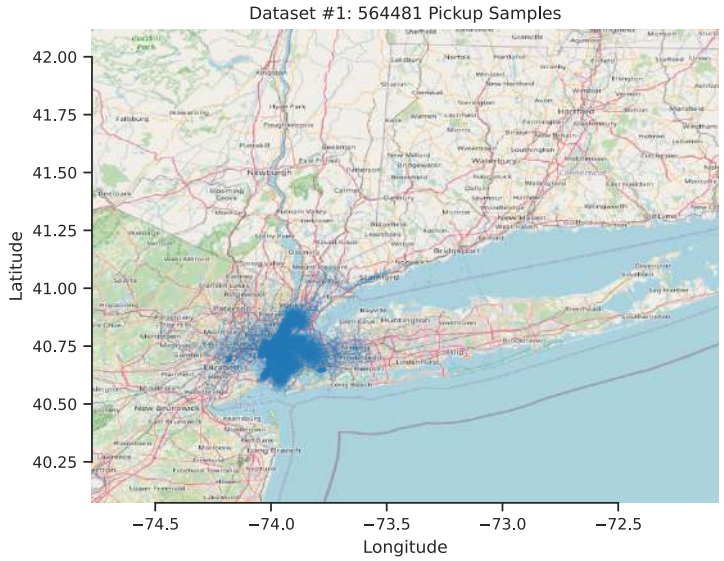
Subpart B.II - Sample some random monthly pickup numbers

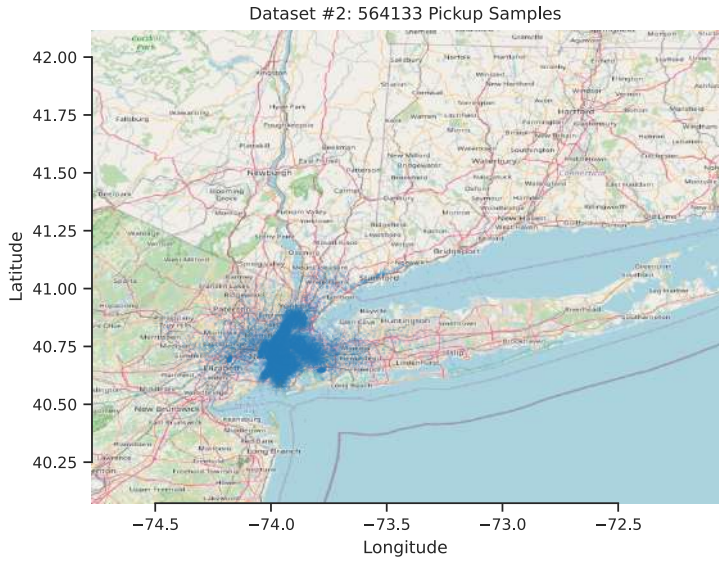
Now that you have a model that gives you the number of pickups and a model that allows you to sample a pickup location, sample five different datasets (number of pickups and location of each pick) from the combined model and visualize them on the New York map.

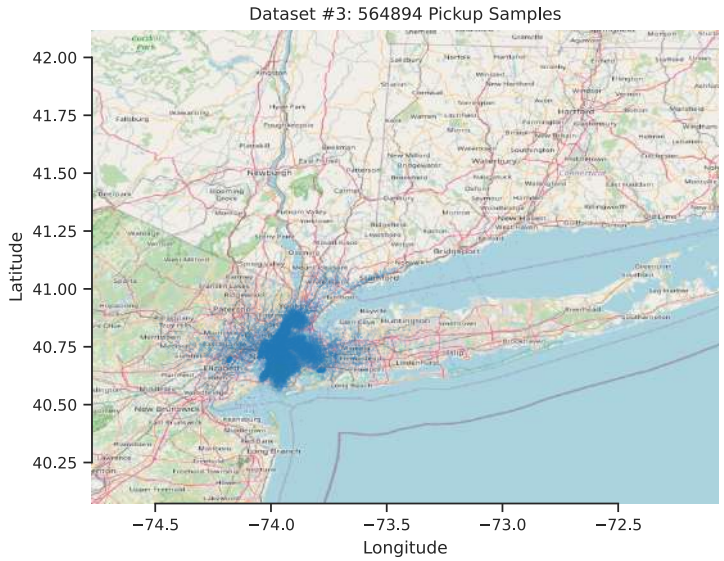
Hint: Don't get obsessed with making the model perfect. It's okay if a few of the pickups are on water.

```
# Your code here
# Set up the Gaussian mixture model for the number of pickups per month
from sklearn.mixture import GaussianMixture
gaussian_mixture = GaussianMixture(n_components=40).fit(p1_train_data)
```

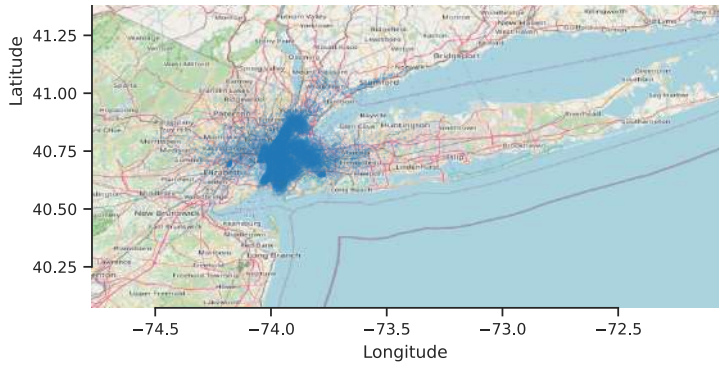
```
for i in range(5):
    # Sample the number of pickups
    num_pickups = monthly_pickup_dist.rvs()

    # Sample the locations of the pickups
    locs = gaussian_mixture.sample(num_pickups)[0]

    # Plot the samples on the map
    fig, ax = plt.subplots(dpi=150)
    ax.set_xlim(box[0],box[1])
    ax.set_ylim(box[2],box[3])
    ax.imshow(ny_map, zorder=0, extent=box, aspect='equal')
    ax.scatter(locs[:, 0], locs[:, 1], zorder=1, s=0.01,
               alpha=0.8, rasterized=True)
    ax.set_xlabel('Longitude')
    ax.set_ylabel('Latitude')
    ax.set_title(f'Dataset #{i + 1}: {num_pickups} Pickup Samples')
    sns.despine(trim=True)
    plt.show()
```


Dataset #1: 564481 Pickup Samples
Latitude
Longitude
42.00
41.75
41.50
41.25
41.00
40.75
40.50
40.25
−74.5
−74.0
−73.5
−73.0
−72.5

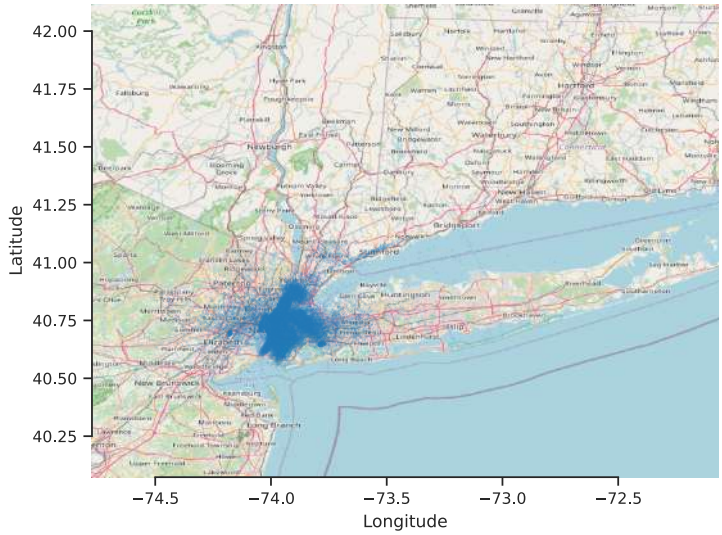

Dataset #2: 564133 Pickup Samples
Latitude
Longitude
42.00
41.75
41.50
41.25
41.00
40.75
40.50
40.25
−74.5
−74.0
−73.5
−73.0
−72.5


Dataset #3: 564894 Pickup Samples
Latitude
Longitude
42.00
41.75
41.50
41.25
41.00
40.75
40.50
40.25
−74.5
−74.0
−73.5
−73.0
−72.5


Dataset #4: 564384 Pickup Samples
42.00
41.75
41.50



Dataset #5: 562968 Pickup Samples



Problem 2 - Counting Celestial Objects

Consider this picture of a patch of sky taken by the Hubble Space Telescope.

Let's download it so that you have it here:

```
url = 'https://raw.githubusercontent.com/PredictiveScienceLab/data-analytics-se/master/lecturebook/images/galaxies.png'
download(url)
```

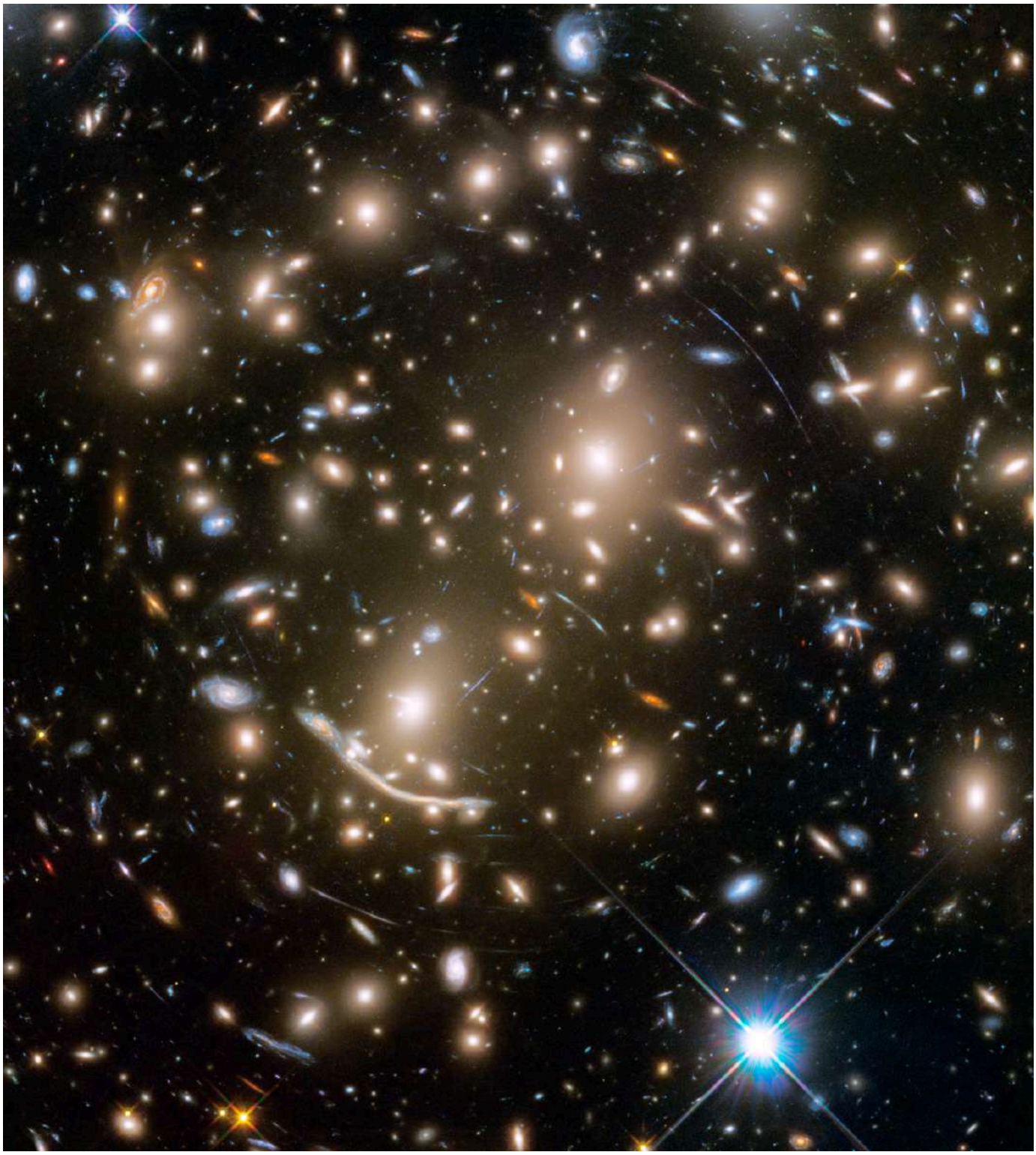
This picture includes many galaxies but also some stars. We will create a machine-learning model capable of counting the number of objects in such images. Our model will not be able to differentiate between the different types of objects and will not be very accurate. Still, it does form the basis of more sophisticated approaches. The idea is as follows:

- Convert the picture to points sampled according to the intensity of light.
- Apply Gaussian mixture on the resulting points.
- Use the Bayesian Information Criterion to identify the number of components in the picture.
- Associate the number of components with the actual number of celestial objects.

I will set you up with the first step. You will have to do the last three.

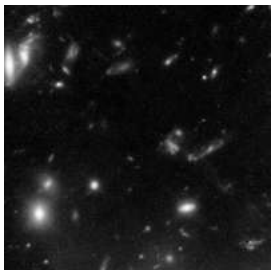
We are going to load the image with the Python Imaging Library (PIL), which allows us to apply a few basic transformations to the image:

```
from PIL import Image
hubble_image = Image.open('galaxies.png')
# here is how to see the image
hubble_image
```



Now, we are going to convert it to grayscale and crop it to make the problem a little bit easier:

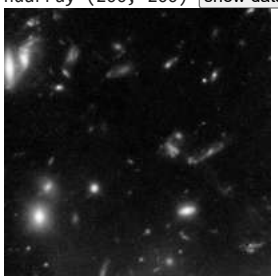
```
img = hubble_image.convert('L').crop((100, 100, 300, 300))
img
```



Remember that black-and white images are matrices:

```
img_ar = np.array(img)
print(img_ar)
img_ar
```

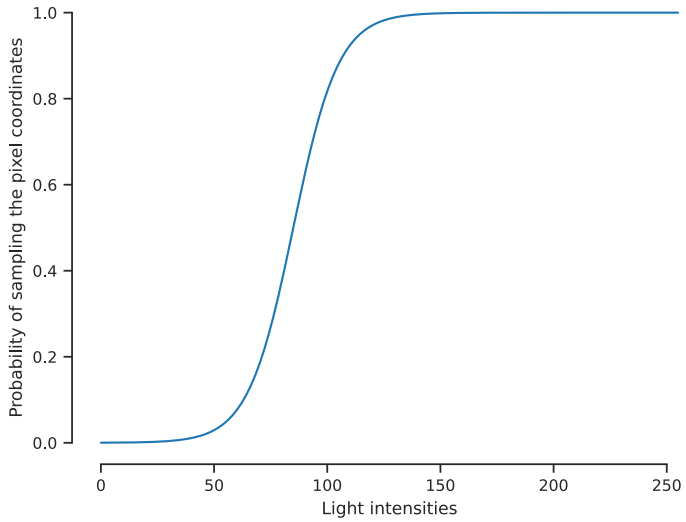
```
[[ 7 11 11 ... 28 62 88]
 [12 12 11 ... 29 47 86]
 [18 13 11 ... 24 34 87]
 ...
 [24 12 15 ... 43 47 40]
 [23 12 19 ... 48 49 40]
 [18 18 23 ... 50 49 41]]
ndarray (200, 200) show data
```



The minimum number is 0, corresponding to black, and the maximum is 255, corresponding to white. Anything in between is some shade of gray.

Now, imagine that each pixel is associated with some coordinates. Without loss of generality, let's assume that each pixel is some coordinate in $[0, 1]^2$. We will loop over each pixel and sample its coordinates in a way that increases with increasing light intensity. To achieve this, we will pass the intensity values of each pixel through a sigmoid with parameters that can be tuned. Here is this sigmoid:

```
intensities = np.linspace(0, 255, 255)
fig, ax = plt.subplots()
alpha = 0.1
beta = 255 / 3
ax.plot(
    intensities,
    1.0 / (1.0 + np.exp(-alpha * (intensities - beta)))
);
ax.set_xlabel('Light intensities')
ax.set_ylabel('Probability of sampling the pixel coordinates')
sns.despine(trim=True);
```

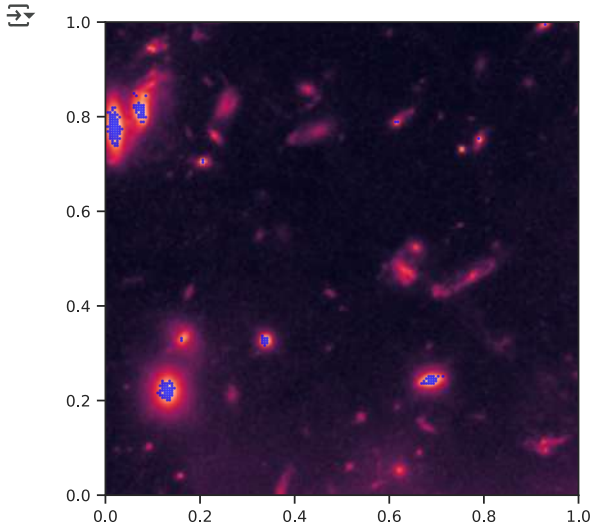


And here is the code that samples the pixel coordinates. I am organizing it into a function because we may want to use it with different pictures:

```
def sample_pixel_coords(img, alpha, beta):
    """
    Samples pixel coordinates based on a probability defined as the sigmoid of the intensity.

    Arguments:

        img    -     The gray scale pixture from which we sample as an array
        alpha     -     The scale of the sigmoid
        beta      -     The offset of the sigmoid
    """
    img_ar = np.array(img)
    img_ar = img_ar.astype(np.float32)
    x = np.linspace(0, 1, img_ar.shape[0])
    y = np.linspace(0, 1, img_ar.shape[1])
    X, Y = np.meshgrid(x, y)
    img_to_locs = []
    # Loop over pixels
    for i in range(img_ar.shape[1]):
        for j in range(img_ar.shape[0]):
            # Calculate the probability of the pixel by looking at each
            # light intensity
            prob = 1.0 / (1.0 + np.exp(-alpha * (img_ar[j, i] - beta)))
            # Pick a uniform random number
            u = np.random.rand()
            # If u is smaller than the desired probability,
            # the consider the coordinates of the pixel sampled
            if u <= prob:
                img_to_locs.append((Y[i, j], X[-i-1, -j-1]))
    # Turn img_to_locs into a numpy array
    img_to_locs = np.array(img_to_locs)
    return img_to_locs
```

Let's test it:

```
locs = sample_pixel_coords(img, alpha=0.1, beta=200)
fig, ax = plt.subplots(dpi=150)
ax.imshow(img, extent=((0, 1, 0, 1)), zorder=0)
ax.scatter(
    locs[:, 0],
    locs[:, 1],
    zorder=1,
    alpha=0.5,
    c='b',
    s=1
);
```



Note that playing with α and β makes the whole thing more or less sensitive to the light intensity.

Complete the following function:

```
from sklearn.mixture import GaussianMixture

def count_objs(img, alpha, beta, nc_min=1, nc_max=50):
    """Count objects in image.

    Arguments:
        img       -     The image
        alpha     -     The scale of the sigmoid
        beta      -     The offset of the sigmoid
        nc_min    -     The minimum number of components to consider
        nc_max    -     The maximum number of components to consider
    """
    locs = sample_pixel_coords(img, alpha, beta)
    # **** YOUR CODE HERE ****
    # Use BIC to search for the best GaussianMixture model
    # with components between nc_min and nc_max
    # YOU CAN PULL THIS OFF BY COPY-PASTING MATERIAL FROM
    # LECTURE 17

    bics = np.ndarray((nc_max - 1,))
    models = []
    for nc in range(nc_min, nc_max):
      m = GaussianMixture(n_components=nc).fit(locs)
      bics[nc - nc_min] = m.bic(locs)
      models.append(m)

    # Set the following variables
    # best_nc = NotImplemented('Set equal to the # components of best model.')
    # best_model = NotImplemented('Set this equal to the best model.')

    best_idx = np.argmin(bics)
    best_nc = best_idx + nc_min
    best_model = models[best_idx]

    return best_nc, best_model, locs
```

Once you have completed the code, try out the following images. Feel free to play with α and β to improve the performance. **Do not try to make a perfect model. We would have to go beyond the Gaussian mixture model to do so. This is just a homework problem.**

Here is a helpful function that you can use to visualize the results:

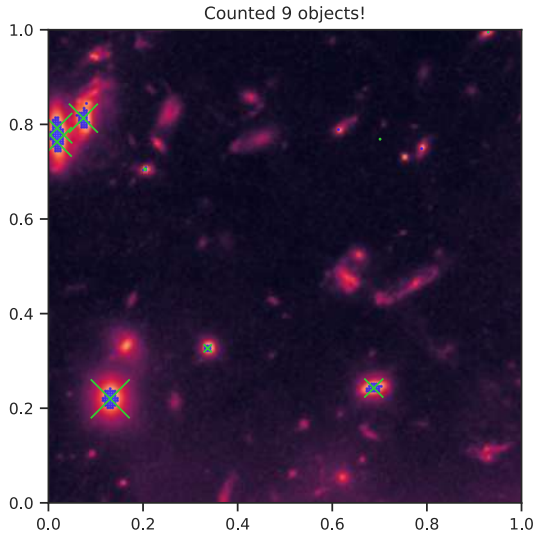
```
def visualize_counts(img, objs, model, locs):
    """Visualize the counts.

    Arguments
    img    --  The image.
    objs   --  Returned by count_objs()
```

```
    model  --  Returned by count_objs()
    locs   --  Returned by count_objs()
    """
    fig, ax = plt.subplots(dpi=150)
    ax.imshow(img, extent=((0, 1, 0, 1)))
    for i in range(model.means_.shape[0]):
        ax.plot(
            model.means_[i, 0],
            model.means_[i, 1],
            'x',
            color='limegreen',
            markersize=(
                10.0 * model.weights_.shape[0]
                * model.weights_[i]
            )
        )
    ax.scatter(
        locs[:, 0],
        locs[:, 1],
        zorder=1,
        alpha=0.5,
        c='b',
        s=1
    )
    ax.set_title('Counted {0:d} objects!'.format(objs));
```

Here is how to use it:

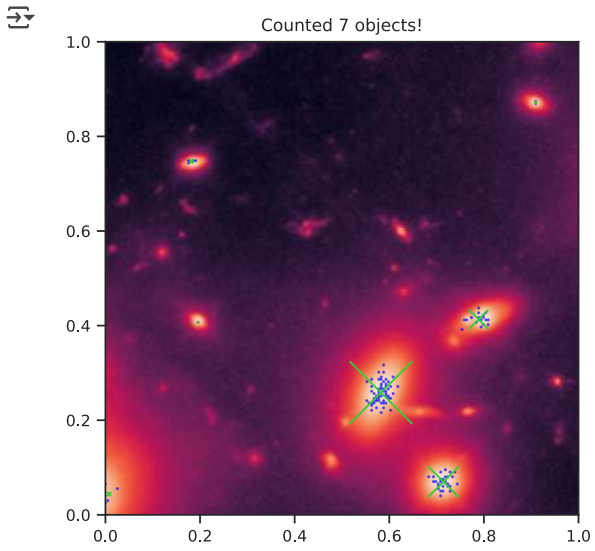
```
objs, model, locs = count_objs(img, alpha=1.0, beta=200)
visualize_counts(img, objs, model, locs)
```

```
/tmp/ipython-input-76-3750215251.py:22: RuntimeWarning: overflow encountered in exp
  prob = 1.0 / (1.0 + np.exp(-alpha * (img_ar[j, i] - beta)))
```



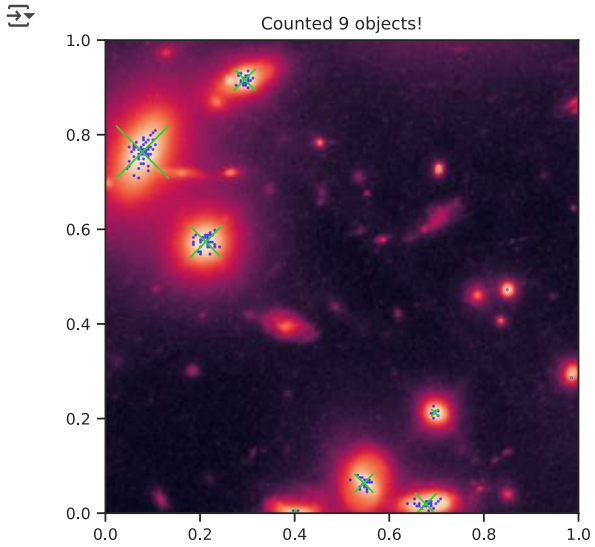
Try this image:

```
img = hubble_image.convert('L').crop((200, 200, 400, 400))
objs, model, locs = count_objs(img, alpha=.1, beta=250)
visualize_counts(img, objs, model, locs)
```



And this one:

```
img = hubble_image.convert('L').crop((300, 300, 500, 500))
objs, model, locs = count_objs(img, alpha=.1, beta=250)
visualize_counts(img, objs, model, locs)
```



Problem 3 - Filtering of an Oscillator with Damping

Assume that you are dealing with a one-degree-of-freedom system which follows the equation:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = u_0\cos(\omega t),$$

where $x = x(t)$ is the generalized coordinate of the oscillator at time t, and the parameters ζ, ω_0, u_0, and ω are known to you (we will give them specific values later). Furthermore, assume that you are making noisy observations of the *absolute acceleration* at discrete timesteps Δt (also known):

$$y_j = \ddot{x}(j\Delta t) - u_0\cos(\omega t) + w_j,$$

for $j = 1, \ldots, n$, where $w_j \sim N(0, \sigma^2)$ with σ^2 also known. Finally, assume that the initial conditions for the position and the velocity (you need both to get a unique solution) are given by:

$$x_0 = x(0) \sim N(0, \sigma_x^2),$$

and

$$v_0 = \dot{x} \sim N(0, \sigma_v^2).$$

Of course, assume that σ_x^2 and σ_v^2 are specific numbers we will specify below.

Before we go over the questions, let's write code that generates the actual trajectory of the system at some random initial conditions and some observations. We will use the code to generate a synthetic dataset with known ground truth, which you will use in your filtering analysis.

The first step we need to do is to turn the problem into a first-order differential equation. We set:

$$\mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}.$$

Assuming $\mathbf{x} = (x_1, x_2)$, then the dynamics are described by:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} x_2 \\ -2\zeta\omega_0\dot{x} - \omega_0^2 x + u_0\cos(\omega t) \end{bmatrix} = \begin{bmatrix} x_2 \\ -2\zeta\omega_0 x_2 - \omega_0^2 x_1 + u_0\cos(\omega t) \end{bmatrix}$$

The initial conditions are of course, just:

$$\mathbf{x}_0 = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}.$$

This first-order system can solved using [scipy.integrate.solve_ivp](). Here is how:

```
from scipy.integrate import solve_ivp

# You need to define the right hand side of the equation
def rhs(t, x, omega0, zeta, u0, omega):
    """Return the right hand side of the dynamical system.

    Arguments
    t       -    Time
    x       -    The state
    omega0  -    Natural frequency
    zeta    -    Dumping factor (0<=zeta)
    u0      -    External force amplitude
    omega   -    Excitation frequency
    """
    res = np.ndarray((2,))
    res[0] = x[1]
    res[1] = -2.0 * zeta * omega0 * x[1] - omega0 ** 2 * x[0] + u0 * np.cos(omega * t)
    return res
```

And here is how you solve it for given initial conditions and parameters:

```
# Initial conditions
x0 = np.array([0.0, 1.0])
# Natural frequency
omega0 = 2.0
# Dumping factor
zeta = 0.4
# External forcing amplitude
u0 = 0.5
# Excitation frequency
omega = 2.1
# Timestep
dt = 0.1
# The final time
final_time = 10.0
# The number of timesteps to get the final time
n_steps = int(final_time / dt)
# The times on which you want the solution
t_eval = np.linspace(0, final_time, n_steps)
# The solution
sol = solve_ivp(rhs, (0, final_time), x0, t_eval=t_eval, args=(omega0, zeta, u0, omega))
```

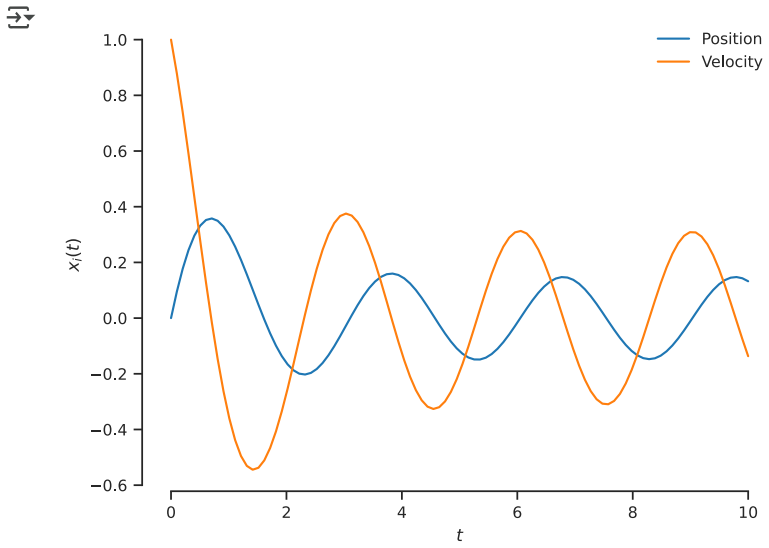
The solution is stored in the `sol` variable:

```
sol.y.shape
```

```
(2, 100)
```

The shape of `sol.y` is (2, 100), which means that we have 100 timesteps and two variables (position and velocity). Let's plot the position and the velocity:

```
fig, ax = plt.subplots(dpi=150)
ax.plot(t_eval, sol.y[0, :], label='Position')
ax.plot(t_eval, sol.y[1, :], label='Velocity')
ax.set_xlabel('$t$')
ax.set_ylabel('$x_i(t)$')
plt.legend(loc='best', frameon=False)
sns.despine(trim=True);
```

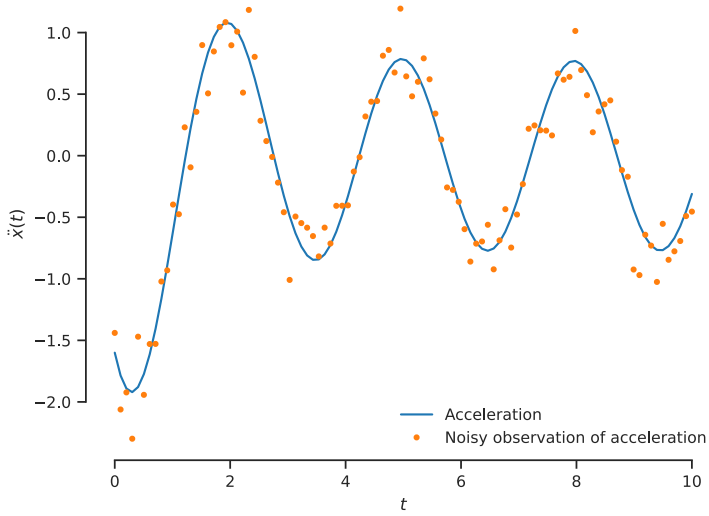


Let's now generate some synthetic observations of the acceleration with some given Gaussian noise. To get the acceleration, you can do this:

```
# Compute external excitation.
us = u0 * np.cos(omega * t_eval)
# Subtract us from \ddot{x}
true_acc = np.array([rhs(t, x, omega0, zeta, u0, omega)[1] for (t, x) in zip(t_eval, sol.y.T)])-us
```

Let's add some noise:

```
sigma_r = 0.2
observations = true_acc + sigma_r * np.random.randn(true_acc.shape[0])

fig, ax = plt.subplots(dpi=150)
ax.plot(t_eval, true_acc, label='Acceleration')
ax.plot(
    t_eval,
    observations,
    '.',
    label='Noisy observation of acceleration'
)
ax.set_xlabel('$t$')
ax.set_ylabel(r'$\ddot{x}(t)$')
plt.legend(loc='best', frameon=False)
sns.despine(trim=True);
```



Okay. Now, imagine that you only see the noisy observations of the acceleration. The filtering goal is to recover the state of the underlying system (as well as its acceleration). I am going to guide you through the steps you need to follow.

Part A - Discretize time (Transitions)

Use the Euler time discretization scheme to turn the continuous dynamical system into a discrete-time dynamical system like this:

$$\mathbf{x}_{j+1} = \mathbf{A}\mathbf{x}_j + Bu_j + \mathbf{z}_j,$$

where

$$\mathbf{x}_j = \mathbf{x}(j\Delta t),$$
$$u_j = u(j\Delta t),$$

and $\mathbf{z}_j$ is properly chosen process noise term. You should derive and provide mathematical expressions for the following:

- The 2×2 transition matrix $\mathbf{A}$.
- The 2×1 control "matrix" B.
- The process covariance $\mathbf{Q}$. For the process covariance, you may choose your values by hand.

Answer:

Transition Matrix $\mathbf{A}$:

First, we must derive the mathematical expression for $\mathbf{A}$. From the problem statement, we know that:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} x_2 \\ -2\zeta\omega_0\dot{x} - \omega_0^2 x + u_0\cos(\omega t) \end{bmatrix} = \begin{bmatrix} x_2 \\ -2\zeta\omega_0 x_2 - \omega_0^2 x_1 + u_0\cos(\omega t) \end{bmatrix}$$

We can decompose the above equation by separating the x-terms and t-terms:

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ -2\zeta\omega_0 x_2 - \omega_0^2 x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ u_0 cos(\omega t) \end{bmatrix}$$
$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_0 cos(\omega t)$$

For continuous time, we know that:

$$\dot{\mathbf{x}} = \mathbf{A}_{cont}\mathbf{x} + B_{cont}u(t)$$

By comparing the two equations for $\dot{\mathbf{x}}$, we find that $\mathbf{A}_{cont}$ and B_{cont} are as follows:

$$\mathbf{A}_{cont} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}$$
$$B_{cont} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now we can apply the Euler discretization scheme:

$$\mathbf{x}(t+\Delta t) \simeq \mathbf{x}(t) + \Delta t \cdot \dot{\mathbf{x}}$$

$$\mathbf{x}(t+\Delta t) \simeq \mathbf{x}(t) + \Delta t \cdot (\mathbf{A}_{cont}\mathbf{x}(t) + B_{cont}u(t))$$

Therefore, the transition matrix is:

$$\mathbf{A} = \mathbf{I} + \Delta t \cdot \mathbf{A}_{cont}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \Delta t \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}$$

$$\boxed{\mathbf{A} = \begin{bmatrix} 1 & \Delta t \\ -\omega_0^2 \Delta t & 1 - 2\zeta\omega_0 \Delta t \end{bmatrix}}$$

Control Matrix B:

Next, we must find the mathematical expression for B. Using the Euler discretization scheme:

$$B = \Delta t \cdot B_{cont}$$

$$\boxed{B = \begin{bmatrix} 0 \\ \Delta t \end{bmatrix}}$$

Process Covariance $\mathbf{Q}$:

Lastly, we would would like to define matrix $\mathbf{Q}$ of the form:

$$\mathbf{Q} = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$$

Choosing $q_1 = 0.0001$ and $q_2 = 0.0001$ gives the following:

$$\boxed{\mathbf{Q} = \begin{bmatrix} 0.0001 & 0 \\ 0 & 0.0001 \end{bmatrix}}$$

```
# You should be using the parameters dt, omega0, zeta, etc.
# from above
A = np.array(
    [
        [1, dt],
        [-(omega0 ** 2) * dt, 1 - 2 * zeta * omega0 * dt]
    ]
)

B = np.array(
    [
        [0],
        [dt]
    ]
)

Q = np.array(
    [
        [0.0001, 0.0],
        [0.0, 0.0001]
    ]
)
```

Part B - Discretize time (Emissions)

Establish the map that takes you from the states to the accelerations at each timestep. That is, specify:

$$y_j = \mathbf{C}\mathbf{x}_j + w_j,$$

where

$$y_j = \ddot{x}(j\Delta t) - u_0 \cos(\omega t) + w_j,$$

and w_j is a measurement noise. You should derive and provide mathematical expressions for the following:

- The 1×2 emission matrix $\mathbf{C}$.
- The 1×1 covariance "matrix" R of the measurement noise.

Answer:

Emission Matrix $\mathbf{C}$:

From the problem statement, we know that:

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = u_0\cos(\omega t)$$

Rearrange to solve for $\ddot{x}$:

$$\ddot{x} = u_0\cos(\omega t) - 2\zeta\omega_0\dot{x} - \omega_0^2 x$$

Next, we can substite the state vector $\mathbf{x}_j = \begin{bmatrix} x(j\Delta t) \\ \dot{x}(j\Delta t) \end{bmatrix}$:

$$\ddot{x}(j\Delta t) = u_0\cos(\omega t) - 2\zeta\omega_0\dot{x}(j\Delta t) - \omega_0^2 x(j\Delta t)$$

Substitute the above equation into $y_j = \ddot{x}(j\Delta t) - u_0\cos(\omega t) + w_j$:

$$y_j = u_0\cos(\omega t) - 2\zeta\omega_0\dot{x}(j\Delta t) - \omega_0^2 x(j\Delta t) - u_0\cos(\omega t) + w_j$$

$$y_j = -2\zeta\omega_0\dot{x}(j\Delta t) - \omega_0^2 x(j\Delta t) + w_j$$

$$y_j = \begin{bmatrix} -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} \begin{bmatrix} x(j\Delta t) \\ \dot{x}(j\Delta t) \end{bmatrix} + w_j$$

By comparing the above equation to $y_i = \mathbf{C}\mathbf{x}_j + w_j$, we will find that the emission matrix equals:

$$\boxed{\mathbf{C} = \begin{bmatrix} -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}}$$

Covariance Matrix R:

Since we know the standard deviation for the measurement noise (σ_r), the covariance matrix is:

$$\boxed{R = \sigma_r^2}$$

```
C = np.array(
    [
        [-omega0 ** 2, -2 * zeta * omega0]
    ]
)

R = np.array(
    [
        [sigma_r ** 2]
    ]
)
```

Part C - Apply the Kalman filter

Use `FilterPy` (see the hands-on activity of Lecture 20) to infer the unobserved states given the noisy observations of the accelerations. Plot time-evolving 95% credible intervals for the position and the velocity along with the true unobserved values of these quantities (in two separate plots).

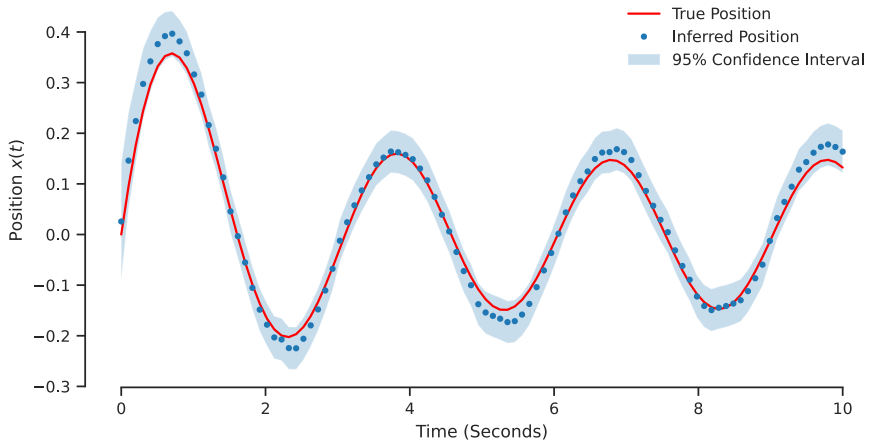
```
# Your answer here (as many code and text blocks as you want)
!pip install filterpy
```

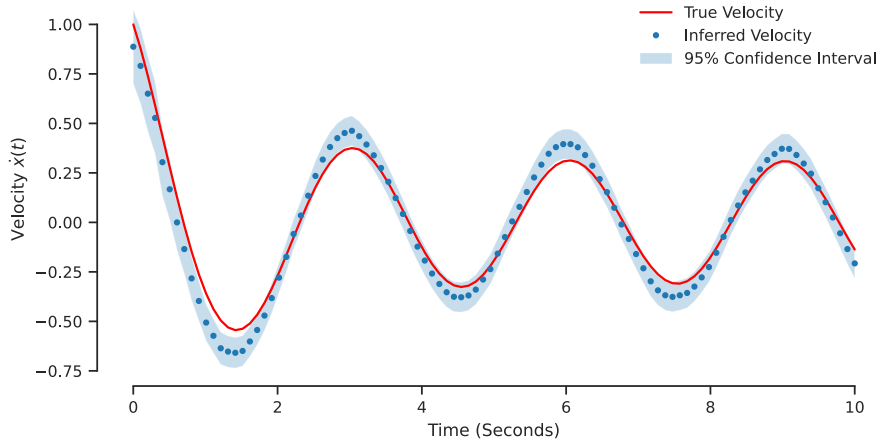
```
Collecting filterpy
  Downloading filterpy-1.4.5.zip (177 kB)
     ━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━━ 178.0/178.0 kB 4.0 MB/s eta 0:00:00
  Preparing metadata (setup.py) ... done
Requirement already satisfied: numpy in /usr/local/lib/python3.11/dist-packages (from filterpy) (2.0.2)
Requirement already satisfied: scipy in /usr/local/lib/python3.11/dist-packages (from filterpy) (1.15.3)
Requirement already satisfied: matplotlib in /usr/local/lib/python3.11/dist-packages (from filterpy) (3.10.0)
Requirement already satisfied: contourpy>=1.0.1 in /usr/local/lib/python3.11/dist-packages (from matplotlib->filterpy) (1.3.2)
Requirement already satisfied: cycler>=0.10 in /usr/local/lib/python3.11/dist-packages (from matplotlib->filterpy) (0.12.1)
Requirement already satisfied: fonttools>=4.22.0 in /usr/local/lib/python3.11/dist-packages (from matplotlib->filterpy) (4.58.5)
Requirement already satisfied: kiwisolver>=1.3.1 in /usr/local/lib/python3.11/dist-packages (from matplotlib->filterpy) (1.4.8)
Requirement already satisfied: packaging>=20.0 in /usr/local/lib/python3.11/dist-packages (from matplotlib->filterpy) (25.0)
Requirement already satisfied: pillow>=8 in /usr/local/lib/python3.11/dist-packages (from matplotlib->filterpy) (11.2.1)
Requirement already satisfied: pyparsing>=2.3.1 in /usr/local/lib/python3.11/dist-packages (from matplotlib->filterpy) (3.2.3)
Requirement already satisfied: python-dateutil>=2.7 in /usr/local/lib/python3.11/dist-packages (from matplotlib->filterpy) (2.9.0.post0)
Requirement already satisfied: six>=1.5 in /usr/local/lib/python3.11/dist-packages (from python-dateutil>=2.7->matplotlib->filterpy) (1.
Building wheels for collected packages: filterpy
  Building wheel for filterpy (setup.py) ... done
```

```
  Created wheel for filterpy: filename=filterpy-1.4.5-py3-none-any.whl size=110460 sha256=39b425d83d968b3206cdfcde3c6f7d7919d6615a09dd35
  Stored in directory: /root/.cache/pip/wheels/12/dc/3c/e12983eac132d00f82a20c6cbe7b42ce6e96190ef8fa2d15e1
Successfully built filterpy
Installing collected packages: filterpy
Successfully installed filterpy-1.4.5
```

```
# Apply the Kalman filter
from filterpy.kalman import KalmanFilter
kf = KalmanFilter(dim_x=2, dim_z=1)

# Mean of the initial state vector
x = x0.reshape(2,1)

# Standard deviations of the initial position and velocity
sigma_x = 0.1
sigma_v = 0.1

# Covariance of the initial state vector
V0 = np.array([[sigma_x ** 2, 0.0],
               [0.0, sigma_v ** 2]])

# Assign the various matrices
kf.x = x
kf.P = V0
kf.Q = Q
kf.R = R
kf.H = C
kf.F = A
kf.B = B

# Run the batch
means, covs, _, _ = kf.batch_filter(observations, us=us)
```

```
# True unobserved values
true_position = sol.y[0, :]
true_velocity = sol.y[1, :]

# Lower and upper bounds for the 95% confidence intervals
position_lb = means[:, 0, 0] - 2 * np.sqrt(covs[:, 0, 0])
position_ub = means[:, 0, 0] + 2 * np.sqrt(covs[:, 0, 0])
velocity_lb = means[:, 1, 0] - 2 * np.sqrt(covs[:, 1, 1])
velocity_ub = means[:, 1, 0] + 2 * np.sqrt(covs[:, 1, 1])

# Create the plot for position
fig, ax = plt.subplots(figsize=(8, 4))
ax.plot(t_eval, true_position, 'r', lw=1.25, label='True Position')
ax.plot(t_eval, means[:, 0], '.', ms=5, label='Inferred Position')
ax.fill_between(t_eval, position_lb, position_ub,
                alpha=0.25, label='95% Confidence Interval')
ax.set_xlabel('Time (Seconds)')
ax.set_ylabel('Position $x(t)$')
ax.legend(frameon=False)
sns.despine(trim=True)

# Create the plot for velocity
fig, ax = plt.subplots(figsize=(8, 4))
ax.plot(t_eval, true_velocity, 'r', lw=1.25, label='True Velocity')
ax.plot(t_eval, means[:, 1], '.', ms=5, label='Inferred Velocity')
ax.fill_between(t_eval, velocity_lb, velocity_ub,
                alpha=0.25, label='95% Confidence Interval')
ax.set_xlabel('Time (Seconds)')
ax.set_ylabel('Velocity $\dot{x}(t)$')
ax.legend(frameon=False)
sns.despine(trim=True)
```





Part D - Quantify and visualize your uncertainty about the actual acceleration value

Use standard uncertainty propagation techniques to quantify your epistemic uncertainty about the true acceleration value. You will have to use the inferred states of the system and the dynamical model. This can be done either analytically or by Monte Carlo. It's your choice. In any case, plot time-evolving 95% credible intervals for the acceleration (epistemic only), the true unobserved values, and the noisy measurements.

Answer:

We know that $\mathbf{x}_j$ follows a normal distribution:

$$\mathbf{x}_j \sim N(\mu_j, \mathbf{V}_j)$$

Since $y_j = \mathbf{C}\mathbf{x}_j$, we know that y_i must also follow a normal distribution:

$$y_j \sim N(\mathbf{C}\mu_j, \mathbf{C}\mathbf{V}_j\mathbf{C}^T)$$

From the above distribution, we know the mean and variance of the inferred acceleration. These values can be used to compute the 95% credible interval:

$$\mathbb{E}[y_i] = \mathbf{C}\mu_j$$
$$\mathbb{V}[y_i] = \mathbf{C}\mathbf{V}_j\mathbf{C}^T$$

```
# Your answer here (as many code and text blocks as you want)
# Compute the inferred acceleration mean (C * mu)
inferred_acc_mean = (C @ means)[:, 0, 0]

# Compute the inferred acceleration variance (C * V * C^T)
inferred_acc_cov = (C @ covs @ C.T)[:, 0, 0]

# Compute the lower and upper bounds for the 95% confidence interval
acc_lb = inferred_acc_mean - 2 * np.sqrt(inferred_acc_cov)
acc_ub = inferred_acc_mean + 2 * np.sqrt(inferred_acc_cov)
```

```
# Plot true acceleration, observations, inferred acceleration and 95% interval
fig, ax = plt.subplots()
ax.plot(t_eval, true_acc, 'r-', label='True Acceleration')

ax.plot(t_eval,
        observations,
        '.',
        color='forestgreen',
        label='Observed Acceleration')

ax.plot(t_eval,
        inferred_acc_mean,
        '.',
        c='royalblue',
        label='Inferred Acceleration')

ax.fill_between(t_eval,
                acc_lb,
                acc_ub,
                alpha=0.25,
                label='95% Confidence Interval')

ax.set_xlabel('Time (Seconds)')
ax.set_ylabel('Acceleration $\ddot{x}(t)$')
plt.legend(loc='best', frameon=False)
sns.despine(trim=True)
```