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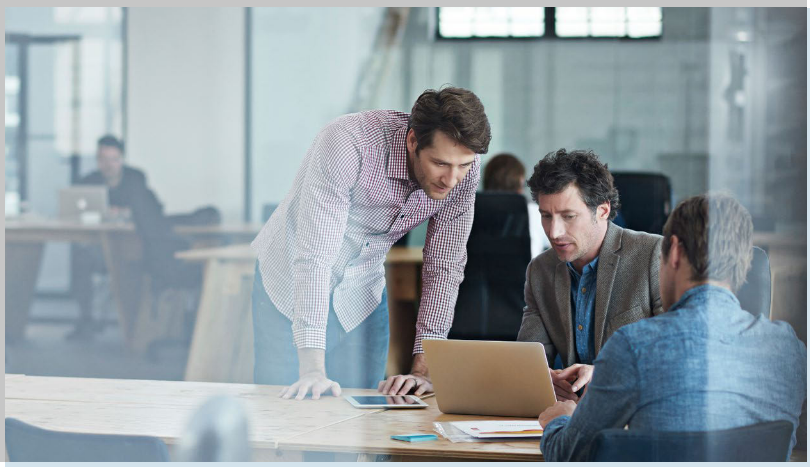

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Abstract

Repo Rates and the Collateral Spread Puzzle

Repo rates frequently exceed unsecured rates in practice. As an explanation, this paper derives a constrained-arbitrage relation between the unsecured rate, the repo rate, and the illiquidity adjusted expected rate of return of the underlying collateral. The theory is based on unsecured borrowing constraints in the market for liquidity. Repos and security cash-market trades are alternative means to get liquidity. Collateral spreads (unsecured less repo rate) can turn negative if borrowing constraints tighten, unsecured rates spike down, or from a depressed and illiquid security market. The constrained-arbitrage theory sheds light on the evolution of collateral spreads over time.

Keywords: collateral spread, constrained-arbitrage, liquidity, market linkages, repo rate, unsecured rate, general collateral
JEL: G01, G12, G21

1 Introduction

Repurchase agreements (repos) are often characterized as being, in effect, a type of collateralized loan (Duffie, 1996). Repo rates would, therefore, be expected to be lower than unsecured rates, and they typically are. However, a puzzling feature of the market for liquidity is that repo rates frequently exceed unsecured rates, often for extended periods of time. This is illustrated in Figure 1. The figure plots the overnight *collateral spread*, defined as the unsecured rate less the repo rate, for the two most active general collateral baskets on Eurex Repo. Across the two baskets, the collateral spread is negative on approximately 25% of days, using the Eonia as the unsecured rate.[1] The long periods of negative collateral spreads in Figure 1 are all the more surprising because the repos are from a sound central counterparty (CCP), which should all but eliminate credit risk. Negative collateral spreads can also be seen in US data under a different market and contractual structure (Bartolini, Hilton, Sundaresan, and Tonetti, 2011)[2] and at longer maturities. For example, the spread between the historical benchmark rates three-month Eonia swap and Eurepo is negative on 21.8% of days.[3] In this paper, I seek to understand the puzzle of negative collateral spreads.

[insert Figure 1 about here]

The paper develops a constrained-arbitrage theory that links repo and unsecured rates with the cost of raising liquidity in the security cash market. The theory offers a fresh perspective on the determinants of collateral spreads and gives rise to a rich set of predictions. Unsecured borrowing constraints and the focus on raising liquidity differentiates the model from Duffie (1996). The emphasis on repo versus unsecured borrowing also

[1] The two collateral baskets in Figure 1 are the GC Pooling ECB basket and the more expansive (lower quality collateral) GC Pooling ECB Extended basket. Discussions of the two baskets can be found on www.eurexrepo.com and in Mancini, Ranaldo, and Wrampelmeyer (2016) and Nyborg and Rösler (2019). In Figure 1, the collateral spread is negative 22.9% and 28.7% of the time for the higher and lower quality baskets, respectively.

[2] These authors focus on the spread between repo rates on different baskets, which they refer to interchangeably as the collateral spread or the repo spread. What is referred to as the collateral spread in the current paper is referred to as collateral rent by Bartolini et al. (2011).

[3] The Eonia swap is an overnight index swap that was a benchmark in the euro area from June 20, 2005 until June 30, 2014. The Eurepo was in existence from March 25, 2002 to December 31, 2014. Details and historical data are available on www.emmi-benchmarks.eu/eoniaswap-org/eoniaswap-history.html and www.emmi-benchmarks.eu/eurepo-org/eurepo-history.html, respectively.

differentiates the analysis from Bartolini, Hilton, Sundaresan, and Tonetti (2011), whose focus is on the spread between different general collateral repo rates. According to the constrained-arbitrage theory, negative collateral spreads may arise if conditions in the unsecured market tighten, unsecured rates spike down (e.g., overnight rates at the end of reserve maintenance periods), or the security market is depressed in terms of prices, illiquidity, or volatility.

These results, which will be explained more fully below, suggest that liquidity-easing central bank policies should put upward pressure on collateral spreads and could cause collateral spreads to reverse from negative to positive. This is consistent with what one can see in Figure 1. Toward the end of the paper, I use the theory to provide a narrative of collateral spreads in the euro area over time and will, in that context, come back to the empirical effects of some key European Central Bank (ECB) unconventional monetary policies.

Improving our understanding of repo and unsecured rates is important for a number of reasons. First, in many countries and currency areas, these rates are important vehicles for the implementation of monetary policy. Second, the market for liquidity is central to the financial system and frictions in it can spill over to other markets and potentially precipitate financial instability (see, among others, Bhattacharya and Gale, 1987; Allen, Carletti, and Gale, 2009; Brunnermeier and Pedersen, 2009; Afonso, Kovner, and Schoar, 2011; Gorton and Metrick, 2011; Nyborg and Östberg, 2014). Third, interbank rates are used as reference rates in mortgages, various credit agreements, and in derivatives markets. The significance of these rates to society as a whole is emphasized by the public and regulatory outrage at the manipulation of Libor by a number of banks.[4] In the wake of the Libor scandal, the Financial Stability Board (FSB) in Basel and others have called for Libor to be replaced by a repo rate benchmark.[5] This points to the importance of rates set in the market for liquidity, especially in the post-crisis landscape, and Figure 1

[4]Disputes and court investigations continue long time after the detection of the manipulation (Bloomberg, April 04, 2016, *Five Ex-Barclays traders plead not guilty to Libor manipulation*).

[5]See Financial Times, April 25, 2014, *US regulators urge quick Libor replacement*, retrieved from FT.com on April 29, 2014. In 2017, the Financial Conduct Authority announced that Libor will be phased out by the end of 2021 (Bloomberg, *Libor Funeral Set for 2021 as FCA Abandons Scandal-Tarred Rate*, July 27, 2017).

underscores that we have much to learn about them. This paper studies the relation between repo rates and unsecured rates theoretically. Some of the key predictions of the constrained-arbitrage theory developed here are tested in separate work by Nyborg and Rösler (2019) with positive results.

As discussed by Duffie (1996), repos are often used as vehicles by cash takers to finance the purchase of the underlying collateral. They can also be driven by cash providers' objectives of obtaining particular securities. Duffie's theoretical focus is on such *special* repos.

In contrast, the theory in this paper is driven by the use of repos to obtain liquidity. It is, therefore, most relevant for general collateral (GC) repos, which are typically thought of as being driven by this objective. In GC repos, the cash taker (borrower) may deliver one of several securities to the cash provider (lender) from a prescribed basket, or list, of eligible collateral. For example, in the popular GC Pooling ECB basket on Eurex Repo (Figure 1, green line), the list of eligible collateral stood at approximately 7,500 ISINs in August 2013. It is the feature of a repo that the underlying collateral is made available to the cash provider that makes a repo different from a plain collateralized loan. This is also a key ingredient in the theory, but for a different reason than in a special repo. In particular, in the model, the cash provider needs the underlying collateral to (partially) finance the reverse repo. To explain this, it is useful to first summarize the basic theoretical framework.

The perspective in this paper is that, apart from borrowing unsecured, the alternative to raising liquidity through a repo is to sell the underlying security in the cash market and buy it back later. One can think of this as a "home-made" repo. This may involve additional transaction costs, for example due to illiquidity. In addition, in a home-made repo, the interest cost of raising the liquidity is a function of the security's future price, which is stochastic. So the relative attractiveness of doing a regular or a home-made repo is a function of the repo rate in comparison to the illiquidity and risk-adjusted cost of engaging in cash market trades. Nyborg and Östberg (2014) provide empirical support for the idea that the security cash market is an important alternative source of liquidity to banks, especially when the unsecured market is tight.

An important ingredient in the model is that both the cash taker and the cash provider face borrowing constraints in the unsecured market. Without this, arbitrage would equate the repo and unsecured rates, since the model abstracts from credit risk. The exclusion of credit risk is for both theoretical and empirical reasons. From a theoretical perspective, credit risk cannot explain negative collateral spreads (except possibly through a segmentation argument). From an empirical perspective, as seen in Figure 1, negative collateral spreads are common in overnight Eurex CCP repos, where credit risk should not be a concern. The model allows for the cash provider and taker to face different borrowing constraints. While it may seem natural to think of a cash provider as relatively liquidity rich, the analysis covers both the case that he is less, and the case that he is more, constrained than the cash taker. This adds to the richness of the predictions of the theory.

In practice, unsecured borrowing constraints may arise for a number of reasons. Central bank money (liquidity) has no substitute and, at any point in time, there is a fixed quantity of it, with much of this held by banks that are not active in the market. So banks that seek liquidity may face significant search costs, e.g., as in Duffie, Gârleanu, and Pedersen (2005). Consistent with this perspective, Bindseil, Nyborg, and Strebulaev (2009) provide evidence that the market for liquidity is allocationally inefficient, even during times of normalcy. Regulation can also create constraints. Credit risk may another reason, however, as mentioned, this is abstracted from in this paper. Additionally, in stressed conditions, liquidity hoarding may contribute toward a tighter interbank market (Acharya and Merrouche, 2013). In the model, the cash provider may be thought of as a bank acting as an intermediary in the market for liquidity.

Another feature of the model is that it allows for the possibility that the price of the collateral that is used in the repo is different from the price that would be realized in the security cash market. Such "pricing errors" may occur in practice, for example, because the security is not perfectly liquid. This affects the relative interest costs of a regular versus home-made repo.

Constraints in the unsecured market and the need to trade in the security cash market gives rise to a constrained-arbitrage relation between the repo rate, the unsecured rate, and the collateral's cash market adjusted (for risk and illiquidity) rate of return. For the cash taker, the weighted average cost of liquidity from combining a repo with unsecured borrowing must be at most the cost of liquidity from combining a home-made repo with unsecured borrowing. For a cash provider, the situation is the reverse. So conditions in the security market contribute toward putting upper and lower bounds on the repo rate. The higher is the cost of liquidity in the security market, the higher can the repo rate be in equilibrium. Thus, the collateral spread is decreasing in the expected rate of return, illiquidity, and volatility of the underlying security. This is consistent with the evidence in Bartolini et al. (2011) and Figure 1 that repo rates are higher for lower quality baskets. The collateral spread is predicted to be increasing in haircuts (conditional on a positive collateral spread) and the unsecured rate, ceteris paribus.

Collateral spreads can go from the normal positive situation to negative if (1) the unsecured rate drops sufficiently, (2) security markets are sufficiently depressed in terms of prices, liquidity, or volatility, or (3) conditions in the unsecured market are sufficiently tight. Scenario (1) helps explain the spikes seen in Figure 1. Unsecured rates usually spike either up or down at the end of reserve maintenance periods and up at the end of calendar months (see, e.g., Hamilton, 1996 for US evidence and Perez-Quiros and Mendizabal, 2006; Nautz and Offermanns, 2008; Fecht, Nyborg, and Rocholl, 2008, for evidence from the euro area). This causes a move in the same direction in the collateral spread, since the securities market is not similarly affected by these calendar effects. Scenarios (2) and (3) help explain the prolonged periods of negative collateral spreads shown in Figure 1 as periods where securities markets or the interbank market were highly stressed.

In the theory, negative collateral spreads are only possible if the cash provider is less constrained in the unsecured market than the cash taker. Thus, the fact that collateral spreads are often negative may be viewed as empirical confirmation of the commonly held notion that cash providers are players who have a comparative advantage in the unsecured market. This result arises because of the basic mechanism of a repo with constrained players. If the collateral spread is negative, then it is advantageous to borrow as much as possible in the unsecured market. If the potential cash provider is less constrained than the player that seeks liquidity, then the latter can, essentially, relax her unsecured borrowing constraint by raising funds from the former. If the potential cash provider is more constrained, however, there are no such gains from trade. So in this case, a negative collateral spread would not be consistent with equilibrium.

Nyborg and Rösler (2019) use data provided by Eurex Repo to test three predictions of the constrained-arbitrage theory. All three tests use overnight trades in the two most active GC baskets, namely the GC Pooling ECB basket and the GC Pooling Extended basket that are plotted in Figure 1. The first test examines the effects of end-of-maintenance period spikes in the unsecured rate and tests the prediction that the collateral spread should move in the same direction, but less. The second test uses an exogenous change in haircuts to test the prediction that the collateral spread is increasing in haircuts (conditional on being positive). The third test addresses the prediction that collateral spreads are increasing in volatility. The findings are all consistent with the constrained-arbitrage theory and, thereby, provide support for the explanation for negative collateral spreads in this paper.

The rest of the paper is organized as follows. Section 2 lays out the theoretical framework. Section 3 contains the theoretical analysis of positive and negative collateral spreads. Section 4 draws out empirical implications and predictions. Section 5 discusses the collateral spread over time and the impact of central bank policies. Section 6 concludes.

2 Theoretical framework

This section provides the theoretical framework used to study collateral spreads. Important features in the setup include liquidity constrained players and collateral "pricing errors." To provide context for the model, I start by reviewing a generic repo.

2.1 Generic repurchase agreement

A generic repurchase agreement between two counterparties, a cash taker (borrower) and a cash provider (lender), has five main ingredients; the underlying collateral (e.g. a security), the price of the underlying collateral, the haircut that is applied to this price, the repo rate, and the maturity (tenor) of the repo agreement. For example, if the price (for the purpose of a repo) of the collateral is P, the haircut is h, and the repo rate is r, then the cash taker delivers the underlying collateral to the cash provider and receives cash of

Table 1: *Basic cash flows in a repo/reverse repo*[*]

	Date 0	Date 1 (maturity)
Repo (cash taker)	$1-h$	$-(1-h)(1+r)$
Reverse (cash provider)	$-(1-h)$	$(1-h)(1+r)$

[*] The price, P, of the collateral is normalized to 1 and the probability of default is assumed to be 0.

$P(1-h)$. At maturity, the cash taker buys back the underlying collateral at a price equal to P plus the accumulated interest at the repo rate. The cash taker is said to be doing a repo while the cash provider is said to be doing a reverse repo.

In addition, the cash provider typically obtains the use of the collateral until maturity. Ignoring this feature as well as the possibility of default, the cash flows arising from a repo that starts at date 0 and runs until date 1 are described in Table 1. These flows make the repo look like a simple (collateralized) loan of $1-h$ at the repo rate, r (with the price normalized to 1). However, this ignores that the cash provider has use of the collateral until maturity. In the model, this feature will play an important role.

2.2 Further structure and assumptions

I consider a setup where one agent, that I think of as a bank and refer to as "the short," is short one unit of liquidity. The shortage may arise from a need to fulfill reserve requirements, satisfy regulatory liquidity constraints, buy securities, or fulfill some obligation. The need to obtain the liquidity is modeled as a hard constraint. The short is assumed to be endowed with one unit of a security that can be used as collateral in a repo. Besides doing a repo, she can obtain liquidity by selling the security in the cash market or borrowing in the unsecured market. In the case the short does a repo, she is referred to as the cash taker, in line with standard terminology. Her potential counterparty is referred to as the cash provider.

The unsecured rate is denoted by u and is assumed to be the same for the short and the potential cash provider. However, each player face individual unsecured borrowing constraints. The combined unsecured sum the cash taker and provider can raise is less than the unit the short needs. In addition, as a normalization, the cash provider is assumed to have no cash on hand at the start of date 0. This means that it is not feasible for the cash provider to finance a reverse repo, that will provide the short with the liquidity she needs, in the unsecured market or out of his own pocket. A fraction of the collateral held by the short will have to be sold in the cash market, either by the short herself or, in the case she engages in a repo, by the cash provider. This will lead to a link between the unsecured rate, the repo rate, and the cash market rate of return of the underlying collateral. A repo rate is an *equilibrium* repo rate if the short and the cash provider are willing to undertake a repo at that rate.

Assumption 1. *The combined amount the short and the cash provider can obtain in the unsecured market is strictly less than one (i.e., the quantity of liquidity the short needs). That is,* $0 < \eta + \kappa < 1$*, where* $\eta \in (0, 1)$ *and* $\kappa \in (0, 1)$ *are the short and cash provider's borrowing capacities in the unsecured market, respectively.*

The cash provider can be thought of as an intermediary between the short and banks with excess liquidity. It could be another bank. Unsecured capacity constraints are consistent with anecdotal evidence of interbank credit limits. While these are sometimes thought of as arising from credit risk, they may also have their origin in other frictions, for example, relating to search costs and the fact that there is a fixed quantity of central bank money (liquidity) in the system at any point in time, as discussed in the Introduction. Capacity constraints can also be thought of as a simple way to model increasing marginal unsecured borrowing rates.

To put further structure on the analysis, several additional assumptions are made, as listed below.

Assumption 2. *There is no default risk.*

This is assumed, in part, to keep the analysis as simple as possible and to put the focus on the effects of borrowing constraints in the unsecured market.

Assumption 3. *Haircuts are exogenous to the repo itself. That is, the haircut,* $h \in [0, 1)$*, is not subject to negotiation between the counterparties.*

This reflects the factual situation in many repo agreements that haircuts are set in advance and often by a third party. For example, in the case of Eurex Repo, a list of haircuts for each day is made available (weekly) on the Eurex website and (daily) in their system, Xemac. This is not updated during the day, except in special circumstances. As also noted by Mancini, Ranaldo, and Wrampelmeyer (2016) and Nyborg (2016), haircuts in Eurex' GC Pooling contracts (as used in Figure 1) are based on those set by the ECB for Eurosystem repos. These are updated only every three to four years (Nyborg, 2016).[6] For US triparty repo agreements, Krishnamurthy, Nagel, and Orlov (2014) have observed that haircuts are not managed actively either and, according to Copeland, Martin, and Walker (2010), " ...haircuts are not negotiated at the trade level but are instead written into the appendix of the tri-party repo custodial agreement between the cash investor, the collateral provider, and the clearing bank. While it is possible to change the appendix containing the haircuts, the change may not apply until the next day. Such changes are only made occasionally."[7] In Europe, Clearstream and Euroclear, two triparty repo agents, ask banks to set their own haircuts according to certain security criteria (e.g. type of security, maturity, rating etc.) before they start trading repo in their systems. This list of haircuts can be amended and banks can apply additional margins in individual contracts. However, haircuts are not normally updated on a daily basis. Thus, I study repo rates and collateral spreads as a function of haircuts.

Assumption 4. *The date 0 price, for the purpose of the repo, of the underlying security (collateral) is normalized to 1. The actual (security) cash market price that a seller could obtain is* $1 - \varepsilon_0$*, where* $\varepsilon_0 < 1$ *is a constant. At date 1, the cash market price of the underlying security is* $1 + \tilde{x} - \varepsilon_1$*, where* ε_1 *is a constant and* $\tilde{x}$ *is a random variable*

The parameter ε_t is the collateral pricing error arising from a lack of perfect liquidity or, at date 0, because the price used in the repo is based on a model. Even if a market price were used in a repo agreement, in practice, it is not clear that if additional securities were sold into the market, that the cash taker could actually achieve that price. The less liquid the underlying security is, the larger would the price discrepancy be expected to be. Because the ε's reflect illiquidity, it may be most natural to think of them as positive,

[6]Nyborg (2016) also documents that Eurex deviates from the ECB haircuts in about 10% of cases.

[7]It is possible that haircuts in the bilateral repo market may be updated more frequently than for other repos, see e.g., (Gorton and Metrick, 2011).

but, formally, they could also be negative. Settlement in the cash market is immediate.[8]

Unsecured borrowing capacities

I make two further assumptions regarding the players' unsecured borrowing constraints.

Assumption 5. *The unsecured borrowing capacities of both the short and the cash provider, η and κ, respectively, exceed the haircut, h, and the date 0 pricing error, ε_0.*

This ensures that the cash provider is able to finance a reverse repo and that the short is able to raise the requisite unit of liquidity either through a regular or home-made repo.

A final issue with respect to the players' borrowing capacities is whether these are linked or not. The baseline assumption is that they are linked:

Assumption 6. *If the short draws down on her unsecured borrowing capacity, the unsecured borrowing capacity of the cash provider is reduced by the same amount.*

This can be thought of as reflecting (unmodeled) linkages in the unsecured market for liquidity. It is analytically equivalent to assuming that, in a regular repo (combined with unsecured borrowing), the short obtains the full unit of liquidity she needs from the single, modeled cash provider. If $\eta > \kappa$, the possibility arises that the short borrows more than the borrowing capacity of the potential cash provider, turning his effective capacity negative. The interpretation would be that the banks outside the model call in loans that the cash provider has outstanding or reduce credit lines. An alternative to Assumption 6 is that the two players' borrowing capacities are not linked. How this affects the analysis and the results is discussed in Subsection 4.4.

Returns and preferences

Let $1+\bar{x}$ denote the expected date 1 price of the underlying collateral in a perfectly liquid market without pricing errors. Since the date 0 price is normalized to 1, $\bar{x}$ is the expected rate of return in these perfect conditions. In contrast, the actual expected rate of return to an agent that buys in the cash market at date 0 and sells at date 1 is

$$\bar{y} = \frac{1+\bar{x}-\varepsilon_1}{1-\varepsilon_0} - 1 = \frac{\bar{x}-(\varepsilon_1-\varepsilon_0)}{1-\varepsilon_0}. \tag{1}$$

[8] If settlement is instead the next day, for example, the repo studied here could be thought of as a tomorrow-next transaction.

Similarly, define the random variable

$$\tilde{y} \equiv \frac{\tilde{x} - (\varepsilon_1 - \varepsilon_0)}{1 - \varepsilon_0}. \tag{2}$$

This is the underlying collateral's rate of return in the cash market.

The short's objective is to raise one unit of liquidity at date 0 in the way that yields the maximum date 1 utility. The cash provider in a repo seeks to provide liquidity while also maximizing date 1 utility.

Assumption 7. *The short and the cash provider have CARA utility with risk aversion parameter $\rho \geq 0$, and $\tilde{x}$ is normally distributed with mean $\bar{x}$ and variance σ_x^2.*

The cash market rate of return, $\tilde{y}$, is, therefore, normally distributed with mean $\bar{y}$ and variance $\sigma_y^2 = \sigma_x^2/(1-\varepsilon_0)^2$. As shown by Grossman (1976), Assumption 7 leads to mean-variance preferences. For the analysis in the next section, it is useful to make some observations regarding the certainty equivalents of the returns obtained from various positions or trades of the security held by the short.

Position certainty equivalents

Using Grossman's (1976) arguments, one can establish that the certainty equivalent of receiving $\omega\tilde{x}$, $\omega \in [0,1]$, is $\omega\bar{x} - \frac{\rho}{2}\sigma_x^2\omega^2$, while the certainty equivalent of an outflow of $\omega\tilde{x}$ is[9]

$$\hat{x}(\omega) \equiv \bar{x}\omega + \frac{\rho}{2}\sigma_x^2\omega^2. \tag{3}$$

That is to say, an agent who has to pay $\omega\tilde{x}$ would be indifferent between paying this random sum or the fixed sum $\hat{x}(\omega)$. In general, given Assumption 7, the certainty equivalent

[9] Let $\tilde{z} = \omega\tilde{x}$. Thus, $\tilde{z}$ is normally distributed with mean $\bar{z} = \omega\bar{x}$ and variance $\sigma_z^2 = \omega^2\sigma_z^2$. The expected utility of receiving $\omega\tilde{x}$ is given by,

$$E[U(\omega\tilde{x})] = E[U(\tilde{z})] = \frac{-1}{\sqrt{2\pi}\sigma_z}\int_{-\infty}^{\infty} \exp(-\rho z)\exp\left(\frac{-(z-\bar{z})^2}{2\sigma_z^2}\right)dz = -\exp\left(-\rho(\bar{z} - \frac{\rho}{2}\sigma_z^2)\right),$$

where $E[\cdot]$ is the expectation operator, $U(\cdot)$ denotes the negative exponential (CARA) utility function, i.e. $U(z) = -\exp(-\rho z)$, and $\exp(z) \equiv e^z$. Thus the certainty equivalent of receiving $\omega\tilde{x}$ is $\omega\bar{x} - \frac{\rho}{2}\omega^2\sigma_x^2$. Similarly, $E[U(-\omega\tilde{x})] = -\exp\left(-\rho(\omega\bar{x} + \frac{\rho}{2}\omega^2\sigma_x^2)\right)$, implying that the certainty equivalent of an outflow of $\omega\tilde{x}$ is $\hat{x}(\omega)$ as defined in (3).

of an outflow of $\omega\tilde{x} + a$, where a is a constant, is $\hat{x}(\omega) + a$.

Cash market certainty equivalents

The alternative to raising one unit of liquidity by doing a repo and borrowing h in the unsecured market is to sell ω units of the security in the security cash market, borrow $1 - \omega(1 - \varepsilon_0)$ in the unsecured market, and then buying back ω units of the security at date 1. An agent that follows this strategy, has a net *outflow* of $\omega\,[\tilde{x} - (\varepsilon_1 - \varepsilon_0)]$ from the two cash market trades. The certainty equivalent of this is $\hat{x}(\omega) - \omega(\varepsilon_1 - \varepsilon_0)$. In other words, *per unit of cash raised at date 0*, the certainty equivalent of the net cash *outflow* from selling fraction ω of the security at date 0 and buying this back at date 1 is[10]

$$\hat{y}(\omega) \equiv \frac{\hat{x}(\omega) - \omega(\varepsilon_1 - \varepsilon_0)}{\omega(1 - \varepsilon_0)} = \frac{\bar{x} - (\varepsilon_1 - \varepsilon_0)}{1 - \varepsilon_0} + \frac{\rho}{2}\frac{\omega\sigma_x^2}{1 - \varepsilon_0} = \bar{y} + \frac{\rho}{2}\omega\sigma_y^2(1 - \varepsilon_0). \tag{4}$$

The quantity $\hat{y}(\omega)$ is referred to as the "adjusted rate of return," or "cost," from selling the fraction ω of the security at date 0 and buying it back at date 1. This terminology reflects that $\hat{y}(\omega)$ is adjusted relative to the fundamental expected rate of return of the security, $\bar{x}$. As seen, $\hat{y}(\omega)$, is increasing in the fraction traded, ω. The adjustments are, in part, due to risk aversion and volatility and, in part, to illiquidity. In particular, the cash market adjusted rate of return from selling ω shares at date 0 and buying this back at date 1 is increasing in risk aversion (ρ), volatility (σ_x^2), and illiquidity (ε_0). As seen in (4), $\hat{y}(\omega)$ is also increasing in $\bar{x}$. Finally, note that u and $\bar{y}$ are assumed to strictly exceed -1.

3 Analysis

In the model, constraints in the unsecured market mean that at least a fraction of the security held by the short will have to be sold in the cash market. This may be done by the short herself or, in the case that she raises liquidity through a repo, by the cash provider in that repo. Because the cash provider is also constrained in the unsecured market and does not have excess liquidity, he needs to finance the reverse position by selling securities

[10] The second step in (4) uses (3) and the final step uses (1) and the expression for σ_y^2 above.

at date 0. Thus, the theory considers reverse repos where the cash provider needs to sell at least a fraction of the underlying repoed collateral to finance his position. The existence of constraints in the unsecured market for both players and the need to trade in the cash market creates a link between the repo rate, the unsecured rate, and the collateral's cash market adjusted rate of return. In this section, I work out the nature of this constrained-arbitrage link.

3.1 Alternative trades

Regular and home-made repos are considered as discrete alternatives to raise liquidity. Thus, the basic tradeoff between repo and cash market trades is studied on a discrete margin. The same basic tradeoff can be studied on an infinitesimal margin by allowing continuous mixtures of repos, cash market trades, and unsecured borrowing. Since the tradeoff is fundamentally the same, the results would be similar to what is derived here. The discrete approach has an advantage with respect to clarity of exposition and realism. For both the regular and home-made repo alternatives, the analysis considers the optimal mixture between repo or cash-market trading, on the one hand, and unsecured borrowing, on the other.

The short, then, has two alternative sets of trades. She can either combine unsecured borrowing with a repo (Alternative 1) or with direct cash market trades (Alternative 2). In Alternative 1, the short clearly prefers minimizing her unsecured borrowings if $r \leq u$ and maximizing them if $r > u$.[11] In Alternative 2 (home-made repo), the optimal cash market trade and unsecured borrowings will have to be determined.

By way of illustration, the cash flows from the short's two alternatives are laid out in Table 2, assuming a positive collateral spread. Given $r \leq u$, under Alternative 1, the short optimally chooses to borrow the minimum amount, h, in the unsecured market and raise $1-h$ by repoing her security. In Alternative 2, the short sells ω units in the cash market, yielding a cash inflow of $\omega(1-\varepsilon_0)$, and borrows $1-\omega(1-\varepsilon_0)$ at the unsecured rate. At date 1, she buys back ω shares of the security and repays her loan, to yield the cash flows

[11] If $r = u$, the short is indifferent between raising liquidity through a repo or unsecured borrowing. It is assumed that the short borrows as little as possible in the unsecured market if $r = u$. Thus, there are two scenarios: $r \leq u$ and $r > u$.

Table 2: *Cash flows from alternatives for raising one unit of liquidity*

	Date 0	Date 1
Alternative 1 (when $r \leq u$)		
Repo	$1-h$	$-(1-h)(1+r)$
Borrow unsecured	h	$-h(1+u)$
Sum	1	$-[1+(1-h)r+hu]$
Alternative 2 ("home-made repo")		
Sell	$\omega(1-\varepsilon_0)$	–
Buy	–	$-\omega(1+\tilde{x}-\varepsilon_1)$
Borrow unsecured	$1-\omega(1-\varepsilon_0)$	$-(1-\omega(1-\varepsilon_0))(1+u)$
Sum	1	$-\omega(1+\tilde{x}-\varepsilon_1)-(1-\omega(1-\varepsilon_0))(1+u)$

shown. In order to compare the two alternatives, it is necessary to derive the optimal ω, subject to constraints, which I turn to next.

3.2 Alternative 2: Raising liquidity in the cash market

As seen in Table 2, the outflow at date 1 from the short's Alternative 2 is

$$\omega(1+\tilde{x}-\varepsilon_1)+(1-\omega(1-\varepsilon_0))(1+u).$$

Using (3), the certainty equivalent of this is

$$1+\hat{x}(\omega)-\omega(\varepsilon_1-\varepsilon_0)+(1-\omega(1-\varepsilon_0))u. \tag{5}$$

Using (4), this can be written as $1+c(\omega)$, where

$$c(\omega) \equiv \omega(1-\varepsilon_0)\hat{y}(\omega)+(1-\omega(1-\varepsilon_0))u. \tag{6}$$

$c(\omega)$ has the intuitive interpretation as the (adjusted) weighted average cost of liquidity under Alternative 2 when ω shares are sold in the cash market at date 0 and the remaining liquidity of $1-\omega(1-\varepsilon_0)$ is obtained in the unsecured market.

Thus, under Alternative 2, maximizing date 1 utility for the short is equivalent to choosing ω so as to minimize the weighted average cost of liquidity. In doing so, the short faces two constraints. First, it is not feasible for her to sell more of the security than the one unit she is endowed with. Second, since she cannot borrow more than η unsecured, she must sell at least $(1-\eta)/(1-\varepsilon_0)$ units. Hence, the short's problem is to solve the following constrained minimization problem:

$$\begin{aligned} &\min_\omega c(\omega) \quad \text{subject to} \\ &\text{Feasibility:} \qquad\qquad\quad \omega \leq 1 \\ &\text{Unsecured borrowing:} \quad \omega \geq \tfrac{1-\eta}{1-\varepsilon_0}. \end{aligned} \tag{7}$$

The first-order condition of the *unconstrained* problem is[12]

$$\omega^* \hat{y}'(\omega^*) + \hat{y}(\omega^*) - u = 0. \tag{8}$$

Thus, using (4), the unconstrained optimal cash market trade is

$$\omega^* = \frac{u - \bar{y}}{\rho \sigma_y^2 (1 - \varepsilon_0)}. \tag{9}$$

Hence, with respect to feasibility, $\omega^* \leq 1$ if and only if

$$u - \bar{y} \leq \rho \sigma_y^2 (1 - \varepsilon_0). \tag{10}$$

With respect to the unsecured borrowing constraint, $\omega^* \geq (1-\eta)/(1-\varepsilon_0)$ if and only if

$$u - \bar{y} \geq \rho \sigma_y^2 (1 - \eta). \tag{11}$$

The upper bound on $u - \bar{y}$ (for an unconstrained solution) in (10) is larger than the lower bound in (11) since, by assumption, $\eta \geq \varepsilon_0$ (feasibility of Alternative 2). Thus, the constrained optimal cash market trade is

[12]From (4), it is straightforward that $c''(\omega^*) > 0$ (assuming $\rho > 0$ and $\sigma_y^2 > 0$). So the second order condition for a minimum is satisfied. If either ρ or σ_y^2 equal zero, it is straightforward that $\omega^* = 1$ if $\bar{y} \leq u$ and $\frac{1-\eta}{1-\epsilon_0}$ otherwise. In other words, either the feasibility or the borrowing constraint binds. The analysis below goes through, but without the intermediate case where neither of these constraints bind.

$$\Omega = \begin{cases} 1 & \text{if } u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0) \\ \omega^* & \text{if } \rho\sigma_y^2(1-\eta) \leq u - \bar{y} < \rho\sigma_y^2(1-\varepsilon_0) \\ \frac{1-\eta}{1-\varepsilon_0} & \text{if } u - \bar{y} < \rho\sigma_y^2(1-\eta). \end{cases} \tag{12}$$

This says that if the unsecured rate is "very large" relative to the cost of cash market trades, the short optimally trades her whole unit. On the other hand, if the unsecured rate is "very low," the short trades as little as possible, preferring instead to borrow as much as she can in the unsecured market. Between these extremes, the unconstrained optimum obtains. In the limiting case that $\rho = 0$, $\Omega = 1$ if and only if $\bar{y} \leq u$, which is intuitive.

3.3 Positive collateral spread

This subsection derives necessary and sufficient conditions for a positive collateral spread. I assume initially that $r \leq u$. Given this, the short's two alternative trades are as laid out in Table 2, with $\omega = \Omega$ as given by (12). Using the analysis in Subsection 3.2, it follows from Table 2 that for the short to engage in a repo, we must have,

$$(1-h)r + hu \leq \Omega(1-\varepsilon_0)\hat{y}(\Omega) + (1-\Omega(1-\varepsilon_0))u. \tag{13}$$

This intuitive condition says that the interest cost of doing a repo (in combination with unsecured borrowing) must be no larger than the cost of trading in the cash market (in combination with unsecured borrowing). In turn, this implies that the maximum repo rate the short is willing to accept is

$$r \leq \overline{r} \equiv \frac{\Omega(1-\varepsilon_0)\hat{y}(\Omega) + (1-\Omega(1-\varepsilon_0))u - hu}{1-h}. \tag{14}$$

This expression is derived under the assumption that $r \leq u$. Thus, an upper bound on the repo rate is $\min\{\overline{r}, u\}$.

Turning now to the cash provider, recall that he needs to finance the reverse repo by selling a fraction of the collateral at date 0. This may be combined with a loan in the

Table 3: *Cash flows to the cash provider (when $r \leq u$)*

	Date 0	Date 1
Reverse	$-(1-h)$	$(1-h)(1+r)$
Sell security	$\alpha(1-\varepsilon_0)$	–
Unsecured loan	$1-h-\alpha(1-\varepsilon_0)$	$-(1-h-\alpha(1-\varepsilon_0))(1+u)$
Buy back security	–	$-\alpha(1+\tilde{x}-\varepsilon_1)$
Sum	0	$(1-h)(1+r)-(1-h-\alpha(1-\varepsilon_0))(1+u)$ $-\alpha(1+\tilde{x}-\varepsilon_1)$

unsecured market. In order to be able to return the collateral to the short at date 1, the cash provider has to buy it back in the market at that time. At date 0, it is possible that the cash provider generates excess liquidity through the sale of the security in the cash market. If so, this is assumed to be placed in the unsecured market at u. The cash provider, therefore, faces cash flows shown in Table 3, where α denotes the fraction of the underlying security he sells in the cash market to generate liquidity and finance the reverse repo.

The cash flows in Table 3 imply that for the (potential) cash provider to be willing to enter into the reverse repo, we must have (using (3) and (4)),

$$(1-h)r + hu \geq \alpha(1-\varepsilon_0)\hat{y}(\alpha) + (1-\alpha(1-\varepsilon_0))u. \tag{15}$$

This is the reverse of the condition for which the short is willing to enter a repurchase agreement, but with α substituting for Ω. To derive the optimal fraction to sell in the cash market for the cash provider, note that the problem he faces is identical to the problem faced by the cash taker under her Alternative 2, except that the cash provider's unsecured borrowing cap is κ rather than η. Thus, using the same argument as in Subsection 3.2, the cash provider's constrained optimal cash market trade is[13]

[13] Assumption 6 implies that the cash provider's unsecured borrowing capacity is reduced to $\kappa - h$ in the case of a repo, since the short draws h of unsecured borrowing. Thus, the unsecured borrowing constraint in the cash provider's problem becomes $\alpha(1-\varepsilon_0) \geq 1-h-(\kappa-h) = 1-\kappa$. The term $1-h$ reflects that the cash provider only has to provide $1-h$ units of liquidity through the reverse repo. Using (9) gives the associated expression for A in (16). With non-linked borrowing capacities, the unsecured borrowing constraint of the cash provider would be given by $\alpha(1-\varepsilon_0) \geq 1-h-\kappa$. See Subsection 4.4 for a discussion of the implications.

$$A = \begin{cases} 1 & \text{if } u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0) \\ \omega^* & \text{if } \rho\sigma_y^2(1-\kappa) \leq u - \bar{y} < \rho\sigma_y^2(1-\varepsilon_0) \\ \frac{1-\kappa}{1-\varepsilon_0} & \text{if } u - \bar{y} < \rho\sigma_y^2(1-\kappa). \end{cases} \tag{16}$$

Equation (15) now implies that the minimum repo rate the cash provider is willing to accept is given by

$$r \geq \underline{r} \equiv \frac{A(1-\varepsilon_0)\hat{y}(A) + (1 - A(1-\varepsilon_0))u - hu}{1-h}. \tag{17}$$

Comparing (17) to (14) shows that the minimum repo rate the cash provider is willing to accept, $\underline{r}$, coincides with the maximum the short is willing to pay, $\overline{r}$, if their respective unsecured borrowing capacities, κ and η, are identical. For in this case, their respective security cash market trades, A and Ω would be identical for all values of $u - \bar{y}$. More generally, if $\eta \neq \kappa$, $\underline{r}$ and $\overline{r}$ diverge if at least one of the players' borrowing constraints bind.

The expressions above have been derived under the assumption of a positive collateral spread, that is, $r \leq u$. So necessary conditions for a positive collateral spread are (i) $\underline{r} \leq u$ and (ii) $\underline{r} \leq \overline{r}$. By construction, it is clear that these two conditions are also sufficient for the existence of equilibrium $r \leq u$. Lemma 1 addresses the implications of the analysis thus far in the simple case that neither the short's nor the cash provider's unsecured borrowing constraints are binding.

Lemma 1. *If $u - \bar{y} \geq \rho\sigma_y^2 \max\{1-\eta, 1-\kappa\}$ then $\Omega = A$, $\overline{r} = \underline{r} \leq u$, and there is a unique equilibrium repo rate $r \leq u$ (as a function of parameter values). Specifically:*

1. *If $u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0)$ then*

$$r = \frac{(1-\varepsilon_0)\hat{y}(1) + \varepsilon_0 u - hu}{1-h} \leq u. \tag{18}$$

2. *If $\rho\sigma_y^2 \max\{1-\eta, 1-\kappa\} \leq u - \bar{y} < \rho\sigma_y^2(1-\varepsilon_0)$ then*

$$r = \frac{\omega^*(1-\varepsilon_0)\hat{y}(\omega^*) + (1-\omega^*(1-\varepsilon_0))u - hu}{1-h} = u - \frac{1}{2}\frac{(u-\bar{y})^2}{\rho\sigma_y^2(1-h)} \leq u. \tag{19}$$

Proof: See the Appendix.

The lemma shows that when borrowing constraints are not binding, the maximum repo rate the short is willing to pay, $\overline{r}$, and the minimum rate the cash provider is willing to accept, $\underline{r}$, coincide and are bounded above by u. Hence, this common rate is an equilibrium repo rate $r \leq u$. This also means that $u - \bar{y} \geq \rho\sigma_y^2 \max\{1-\eta, 1-\kappa\}$ is a sufficient condition for a positive equilibrium collateral spread. While the lemma is incomplete in that it does not consider binding borrowing constraints or the possibility of negative collateral spreads, it is suggestive of a positive collateral spread being associated with a "large" u relative to $\bar{y}$. As seen, what "large" means here depends on volatility, risk aversion, and the players' unsecured borrowing capacities.

A general point of the analysis is that there is a constrained arbitrage relation between a (potential) equilibrium repo rate, r, the unsecured rate, u, and the cash market adjusted rate of return of the underlying security, $\hat{y}$. If $\underline{r}$ and $\overline{r}$ (and A and Ω) coincide, as they do in Lemma 1, this can be expressed as

$$(1-h)r + hu = \Omega(1-\varepsilon_0)\hat{y}(\Omega) + (1 - \Omega(1-\varepsilon_0))u. \tag{20}$$

In words: the weighted average cost of liquidity raised via a regular repo equals the weighted average cost of liquidity raised via security cash market trading (home-made repo). Thus, when there is a unique repo rate, a positive collateral spread ($r \leq u$) requires that the risk and illiquidity adjusted expected rate of return of the underlying security (cost of liquidity in the security cash market) is less than the unsecured rate ($\hat{y}(\Omega) \leq u$). So, from the perspective of the analysis, one may interpret a positive collateral spread as indicative of unsecured borrowing being relatively expensive, or, equivalently, security prices being relatively high.

In general, movements in the unsecured rate or the price of the underlying security imply that the repo rate must also change so that the balance in (20) is maintained. This basic insight and its generalization (below), gives rise to a rich set of empirical predictions, as discussed further in Section 4.

When the unsecured borrowing constraint is binding for either player, the repo rate is indeterminate within the upper and lower bounds derived above, if $\underline{r} < \overline{r}$, or there is no equilibrium repo rate less than or equal to u. The exact conditions for a positive collateral spread depend on the players' unsecured borrowing capacities, as laid out in the following theorem.

Theorem 1.

1. *Suppose the short has a larger unsecured borrowing capacity than the cash provider, that is, $\eta > \kappa$. There is equilibrium $r \leq u$ if and only if borrowing constraints are not binding, that is,*

$$u - \bar{y} \geq \rho\sigma_y^2(1-\kappa). \tag{21}$$

   If this condition is met, there is a unique equilibrium repo rate $r \leq u$ (as a function of parameter values), as given in Lemma 1.

2. *Suppose the short has a smaller unsecured borrowing capacity than the cash provider, that is, $\eta \leq \kappa$. There is equilibrium $r \leq u$ if and only if the risk and illiquidity adjusted expected rate of return of the underlying security at the smallest possible volume to finance a reverse repo is at most equal to the unsecured rate, that is, if and only if*

$$\hat{y}\left(\frac{1-\kappa}{1-\varepsilon_0}\right) = \bar{y} + \frac{\rho}{2}\sigma_y^2(1-\kappa) \leq u. \tag{22}$$

   In this case:

   (a) *If $u - \bar{y} \geq \rho\sigma_y^2(1-\eta)$, there is a unique equilibrium repo rate (as a function of parameter values) $r \leq u$, as given in Lemma 1.*

   (b) *If $u - \bar{y} < \rho\sigma_y^2(1-\eta)$ then equilibrium $r \leq u$ is any $r \in [\underline{r}, R]$, where*[14]

$$R = \min\{u, \overline{r}\} = \begin{cases} \overline{r} & \text{if } \rho\sigma_y^2(1-\eta) > u - \bar{y} > \rho\sigma_y^2(1-\eta)/2 \\ u & \text{if } u - \bar{y} \leq \rho\sigma_y^2(1-\eta)/2, \end{cases} \tag{23}$$

$$\overline{r} = \frac{(1-\eta)\hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right) + (\eta - h)u}{1-h} = \frac{(1-\eta)\left(\bar{y} + \frac{1}{2}\sigma_y^2(1-\eta)\right) + (\eta-h)u}{1-h}, \tag{24}$$

[14]If $\eta = \kappa$ then 2(b) of Theorem 1 can be stated as: $\overline{r} = \underline{r}$ and there is a unique equilibrium repo rate (as a function of parameter values) given by the lower expression in (25).

and

$$\underline{r} = \begin{cases} u - \frac{1}{2}\frac{(u-\bar{y})^2}{\rho\sigma_y^2(1-h)} & \textit{if } u-\bar{y} \geq \rho\sigma_y^2(1-\kappa) \\ \frac{(1-\kappa)\hat{y}\left(\frac{1-\kappa}{1-\varepsilon_0}\right)+(\kappa-h)u}{1-h} = \frac{(1-\kappa)\left(\bar{y}+\frac{1}{2}\rho\sigma_y^2(1-\kappa)\right)+(\kappa-h)u}{1-h} & \textit{if } u-\bar{y} < \rho\sigma_y^2(1-\kappa). \end{cases} \tag{25}$$

Proof: See the Appendix.

The theorem establishes that there is an equilibrium repo rate below the unsecured rate if and only if $u - \bar{y}$ is positive and bounded away from zero. The exact bound depends on which player is more constrained in the unsecured market. There is a discrete drop as we go from the short being less constrained ($\eta > \kappa$) to more constrained ($\eta \leq \kappa$). In other words, the range of $u - \bar{y}$ for which there is a positive collateral spread expands if the short's borrowing capacity falls below that of the cash provider.

The discontinuity in the lower bound on $u - \bar{y}$ for a positive collateral spread and, more generally, the link between the repo rate, the unsecured rate, and the security's expected rate of return arise partly because, for the short, the alternative to a regular repo is a home-made repo. The latter combines trade in the security cash market with unsecured borrowing. In addition, the cash provider in a regular repo needs to transact in the cash market in order to help finance the reverse position. Thus, the terms at which the players are willing to do a repo depends on conditions in the security cash market and the unsecured market. Unsecured borrowing capacities matter because they affect the players' constrained-optimal mixtures of unsecured borrowing and security trading.

If neither player's unsecured borrowing constraint is binding in their respective security cash market problems, there is a unique equilibrium repo rate which, as is intuitive, is independent of the players' unsecured borrowing capacities (Lemma 1).

However, if unsecured borrowing is so cheap that either of the borrowing constraints bind, the maximum repo rate the short is willing to pay, $\bar{r}$, and the minimum rate the cash provider requires, $\underline{r}$ diverge. For example, if the short has a larger unsecured borrowing capacity, she is able to get closer to the unconstrained optimum (in the cash market problem) than the constrained cash provider. As a result, the maximum rate the short is willing to pay is lower than the minimum rate required by the cash provider. In other words, there is no equilibrium repo rate. Thus, when the short has the larger unsecured borrowing capacity, there is a positive equilibrium collateral spread if and only if the cash provider's borrowing constraint is not binding, that is, (21) in Theorem 1 holds. In this case, there is a unique equilibrium repo rate.

On the other hand, if the short has a smaller unsecured borrowing capacity, her borrowing constraint binds first and, when it does, we have $\overline{r} > \underline{r}$. The short is essentially willing to pay a premium over what the cash provider requires because her demand for unsecured funds is unsatisfied (the constraint is binding) in the cash market alternative. The cash provider can now funnel additional unsecured borrowings to the short through the device of the repo. In a way, the repo allows the players to arbitrage their borrowing constraints. This is mutually beneficial at $r \leq u$ as long as $\underline{r} \leq u$. The condition for this is just (22) in Theorem 1 and any rate in the interval $[\underline{r}, u]$ is equilibrium.

The condition (22) also represents the point where the risk and illiquidity adjusted cost of liquidity in the security cash market equals the unsecured rate when the cash provider borrows up to his capacity in the unsecured market. So this is an intuitive condition for a positive equilibrium collateral spread.

A way to think about what happens in the model is that a repo potentially allows a constrained short to expand her unsecured borrowing capacity through repoing with another player. This can reduce her total cost of liquidity if unsecured borrowing is cheap and if the counterparty can take advantage of this to a larger extent than the short. If the counterparty faces a tighter constraint in the unsecured market, the repo has no added value, and a repo is only acceptable to both parties if the borrowing constraint of the counterparty does not bind. On the other hand, if the counterparty is less constrained in the unsecured market, then the repo can add value by allowing the short to implicitly expand her borrowing capacity. This also provides a simple intuition as to why the range over which there is a positive equilibrium collateral spread expands when the short is more constrained.

Under risk neutrality, the conditions for a positive equilibrium collateral spread collapse to $\bar{y} \leq u$. This can be understood by noting that if $\bar{y} > u$, the cash provider's cost of funding the reverse repo would exceed the unsecured rate. To break even, he therefore needs a repo rate above the unsecured rate. The short would be willing to pay such a rate because her home-made repo would also cost her more than u.

The analysis provides constrained arbitrage relations between unsecured, security cash market, and repo rates. It is predicated on the idea that two of these rates may diverge, for reasons outside of the model. As they diverge, this then has implications for the third rate. The unsecured and the security cash markets are, in a sense, the fundamental markets, with the repo market being derived from those. The analysis reflects that the two fundamental markets have different participants and are subject to different shocks. In practice, the participants in the unsecured markets are banks trading reserves (central bank money). The participants in the security cash market is much broader. As is well known, the unsecured market is subject to up and down spikes that relate to the shift from an outgoing to a new reserve maintenance period (see, e.g., Hamilton (1996), for the US, or Nautz and Offermanns (2008), for the euro area) or particular calendar dates, e.g., (Fecht, Nyborg, and Rocholl, 2008; Perez-Quiros and Mendizabal, 2006). Securities markets do not experience the same extreme movements around the same dates and are subject to a different set of issues such as investors' rebalancing portfolios, information flows, market uncertainty, etc. Given that the unsecured rate does not move in lock-step with the security cash market, the expressions for the repo rate in Theorem 1 will give rise to testable empirical predictions. However, it is first necessary to study the case of negative collateral spreads.

3.4 Negative collateral spread

This subsection derives necessary and sufficient conditions for a negative equilibrium collateral spread. Suppose initially that $r > u$. In this case, the short optimally borrows her maximum of η in the unsecured market under Alternative 1 (repo) and repos the fraction

$$\phi = \frac{1-\eta}{1-h} \tag{26}$$

of her security. Thus, under the repo alternative, the short's cost of liquidity is

$$(1-\eta)r+\eta u. \tag{27}$$

Under the home-made repo alternative, the short faces the same situation in the cash market as before. Thus, her optimal transaction is still Ω as given in (12). So, by definition, the cost of one unit of liquidity under any other trade is weakly larger. Consider $\omega = (1-\eta)/(1-\epsilon_0)$ and borrowing η at the unsecured rate. This will cost $(1-\eta)\hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right)+\eta u$. Comparing this to (27) implies that if $u<r$ then $r \leq \hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right)$. This establishes:

Lemma 2. *There is equilibrium $r>u$ only if*

$$u < \hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right) = \bar{y} + \frac{\rho}{2}\sigma_y^2(1-\eta). \tag{28}$$

In particular, this implies that the short's unsecured borrowing constraint (under Alternative 2) is strictly binding, that is, $u-\bar{y}<\rho\sigma_y^2(1-\eta)$.

Proof: Follows immediately from the analysis above and the definition of $\hat{y}(\cdot)$. □

The lemma says that a negative collateral spread does not only mean that the unsecured rate is low relative to the repo rate, but also that it is low relative to the expected rate of return of the underlying security. In fact, the unsecured rate is so low that the short's borrowing constraint is binding. Put differently, unsecured borrowing is cheap relative to all alternatives.

Since the short's borrowing constraint must be binding, under Alternative 2 she transacts $\Omega=(1-\eta)/(1-\varepsilon_0)$ in the security cash market and borrows η at the unsecured rate. Hence, using (27), the maximum repo rate the short is willing to pay is

$$r \leq \overline{r}_{neg} \equiv \hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right) = \bar{y} + \frac{\rho}{2}\sigma_y^2(1-\eta). \tag{29}$$

The short is willing to do a repo at any $r \leq \overline{r}_{neg}$.

The cash provider's problem is essentially the same as when $r \leq u$, except that he now needs to provide $1-\eta$ of liquidity against $\phi<1$ units of the underlying security.

The cash flows from his trades are identical to those in Table 3, with η in place of h. The borrowing constraint remains the same as before, by Assumption 6. However, the feasibility constraint changes because the cash provider cannot sell more than the ϕ units he gets in the repo. Thus, the cash provider's optimal trade is now

$$A = \begin{cases} \phi & \text{if } u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0)\phi \\ \omega^* & \text{if } \rho\sigma_y^2(1-\kappa) \leq u - \bar{y} \leq \rho\sigma_y^2(1-\varepsilon_0)\phi \\ \frac{1-\kappa}{1-\varepsilon_0} & \text{if } u - \bar{y} \leq \rho\sigma_y^2(1-\kappa). \end{cases} \tag{30}$$

As written, (30) assumes that the feasibility constraint does not trivially bind, that is, $(1-\varepsilon_0)\phi \geq (1-\kappa)$. This is, in fact, a necessary condition for equilibrium $r > u$:

Lemma 3. *Equilibrium $r > u$ implies that the cash provider's feasibility constrained does not trivially bind, that is, $(1-\varepsilon_0)\phi \geq (1-\kappa)$.*

Proof: See the Appendix.

The intuition is as follows: The feasibility constraint gives the maximum units the cash provider can sell. The borrowing constraint gives the smallest quantity he needs to sell in order to finance the reverse repo. Thus, equilibrium requires the borrowing constraint to be below the feasibility constraint. One can also think about this result as follows: If $r > u$, it is desirable to borrow as much as possible unsecured and, therefore, sell as little of the underlying security as possible so that the feasibility constrained does not bind.

Since the cash provider extends η in liquidity to the short in the repo (rather than h as when $r \leq u$), the condition for him to be willing to do a repo becomes

$$(1-\eta)r + \eta u \geq A(1-\varepsilon_0)\hat{y}(A) + (1 - A(1-\varepsilon_0))u. \tag{31}$$

Thus, the lowest acceptable repo rate to the cash provider is

$$r \geq \underline{r}_{neg} \equiv \frac{A(1-\varepsilon_0)\hat{y}(A) + (1 - A(1-\varepsilon_0))u - \eta u}{1-\eta}. \tag{32}$$

To check whether $r > u$ is consistent with equilibrium, it is necessary to compare this lower bound on r with the maximum the short is willing to pay, as derived above. It is also necessary to check that $\overline{r}_{neg} > u$.

Theorem 2.

1. *Suppose the short has a smaller unsecured borrowing capacity than the cash provider, that is, $\eta \leq \kappa$. Suppose also that $(1-\varepsilon_0)\phi \geq 1-\kappa$. There is equilibrium $r > u$ if and only if (28) holds, that is, $u - \bar{y} < \frac{1}{2}\rho\sigma_y^2(1-\eta)$. When this holds, $\overline{r}_{\text{neg}} > u$. Furthermore, if $\eta < \kappa$, $\underline{r}_{\text{neg}} < \overline{r}_{\text{neg}}$ and equilibrium $r > u$ may take on any value in the interval $(R_{\text{neg}}, \overline{r}_{\text{neg}}]$, where $R_{\text{neg}} = \max\{u, \underline{r}_{\text{neg}}\}$, and*

$$\underline{r}_{\text{neg}} = \begin{cases} \frac{\phi(1-\varepsilon_0)\hat{y}(\phi)+(1-\phi(1-\varepsilon_0))u-\eta u}{1-\eta} & \text{if } u-\bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0)\phi \\ \\ u - \frac{1}{2}\frac{(u-\bar{y})^2}{\rho\sigma_y^2(1-\eta)} & \text{if } \rho\sigma_y^2(1-\kappa) \leq u-\bar{y} < \rho\sigma_y^2(1-\varepsilon_0)\phi \\ \\ \frac{(1-\kappa)\hat{y}\left(\frac{1-\kappa}{1-\varepsilon_0}\right)+(\kappa-\eta)u}{1-\eta} & \text{if } u-\bar{y} < \rho\sigma_y^2(1-\kappa). \end{cases} \tag{33}$$

   If (28) holds and $\eta = \kappa$ then $\overline{r}_{\text{neg}} = \underline{r}_{\text{neg}}$ and there is a unique equilibrium $r > u$, namely, $r = \hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right) = \bar{y} + \frac{1}{2}\rho\sigma_y^2(1-\eta)$.

2. *Suppose the short has a larger unsecured borrowing capacity than the cash provider, that is, $\eta > \kappa$. There does not exist equilibrium $r > u$.*

Proof: See the Appendix.

The theorem establishes that there is a negative equilibrium collateral spread only if the short has a smaller unsecured borrowing capacity than the cash provider, ($\eta \leq \kappa$) and the latter's feasibility constrained does not trivially bind (Lemma 3). In this case, there is a negative equilibrium collateral spread if and only if the unsecured rate is less than the risk and illiquidity adjusted cost of liquidity in the security cash market when the short borrows up to her capacity in a home-made repo, that is, $u < \hat{y}\left((1-\eta)/(1-\varepsilon_0)\right)$. When this holds, the cost of a home-made repo exceeds the unsecured rate. Thus, the short is willing to pay a repo rate above the unsecured rate. This is equilibrium since the cash provider requires a lower repo rate than the maximum the short is willing to pay, because of the cash provider's larger unsecured borrowing capacity.

Comparing this condition with the condition for a positive equilibrium collateral spread in Theorem 1 shows that positive (negative) collateral spreads are associated with large (small) $u - \bar{y}$. There is a region of overlap, where both negative and positive collateral spreads may occur. The overlap region is

$$(u - \bar{y})/(\rho\sigma_y^2) \in \left[\frac{1-\kappa}{2}, \frac{1-\eta}{2}\right), \quad \text{i.e.,} \quad \hat{y}\left(\frac{1-\kappa}{1-\varepsilon_0}\right) \leq u < \hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right). \qquad (34)$$

In words, both positive and negative collateral spreads are equilibrium if the unsecured rate falls between the risk and illiquidity adjusted costs of liquidity for the short and cash provider when they borrow to their full capacities in their respective cash market problems. To the left of this region, equilibrium collateral spreads are negative. To the right, they are positive.

If the short is less constrained than the cash provider ($\eta > \kappa$), Theorem 2 says that a negative collateral spread is not consistent with equilibrium. One may therefore interpret the fact that negative collateral spreads are common as being consistent with the intuitive idea that cash providers are typically less constrained, or put differently, have "easier" access to liquidity, than cash takers.

The reason a negative collateral spread cannot be equilibrium when $\eta > \kappa$ relates to the fact that it implies that the potential cash provider's unsecured borrowing constraint is binding (Lemma 2). But, as discussed after Theorem 1, when the cash provider faces tighter borrowing constraints and these are binding, the short can do better through a home-made repo than going through the repo market. The maximum rate the short is willing to do a repo at falls below the minimum rate that is acceptable to the constrained cash provider. Thus, there is no equilibrium repo rate (either positive or negative).

This paper is motivated with reference to the puzzle of negative collateral spreads. The theory says that this may occur when the unsecured rate drops sufficiently below the expected rate of return of the underlying security. The exact distance depends on risk aversion, volatility, and unsecured borrowing caps. Such a drop may occur as a result of conditions in the unsecured market. It may also occur if securities prices drop so that their expected rates of return rise. In other words, negative collateral spreads may be symptoms of especially high unsecured rates (for example, relating to the reserve maintenance period cycle), tight unsecured borrowing capacities, or depressed securities prices.

3.5 Remark 1: Role of unsecured borrowing constraints

In the model, there are potentially two riskfree rates, namely, repo and unsecured rates, since the setup excludes credit risk by Assumption 2. This does not give rise to arbitrage because of the assumption that both the short (cash taker) and the cash provider are constrained in the unsecured market. If Assumption 1 were dropped, the short could borrow the unit of liquidity she needs at a rate of u, implying that r cannot exceed u. Furthermore, from a cash provider's perspective, it is clear that r could not be less than u. Thus, with no constraints in the unsecured market, we must have $r = u$. The fact that these rates are rarely equivalent in practice suggests that there are constraints in the unsecured market.[15] This makes sense since there is a limited quantity of reserves in the economy. The expressions for the repo rate in Theorems 1 and 2 also show that variations in these constraints can contribute to volatility in the collateral spread.

The results in this paper bear some relation to those in Duffie (1996), although his focus is on special repo rates. In particular, he shows that the special repo rate is at most equal to the riskfree rate (in his notation, $R \leq i$). Duffie's result is driven by trading asymmetries whereby one needs to hold the specific collateral in order to short it. In the current model, it is also not possible to sell more than what is owned. However, in Duffie's model, unlike the current one, a player can always borrow at the riskfree rate. This makes the specific security more expensive relative to the riskfree bond, resulting in a special repo rate lower than the riskfree rate. The current, constrained-arbitrage model yields an analogous result in that $r \leq u$ as long as the unsecured rate is above the cash market adjusted rate of return of the security, which one can think of as the underlying security being relatively "expensive." However, in the current model, the reverse is also possible when the underlying security is relatively "cheap."

[15] Repo and unsecured rates could also differ due to differential trading costs.

4 Predictions and implications

This section draws out empirical predictions and implications. I first provide explicit expressions for the collateral spread and present the comparative statics when the spread is positive and then when it is negative. This considers the effects on the collateral spread of the unsecured rate, u; the (liquidity adjusted) expected rate of return of the underlying security, $\bar{y}$; the haircut, h; volatility, σ_y^2; and illiquidity, ε_0. The effect of risk aversion, ρ, is the same as that of volatility. I then discuss the determinants of the sign of the collateral spread. Finally, I analyze the implications of switching from an assumption of linked (Assumption 6) to non-linked borrowing capacities.

4.1 Positive collateral spread

Theorem 1 shows that there are three cases when the collateral spread is positive. The first two give rise to a unique repo rate so that comparative statics are easily derived. However, in the third case, the repo rate is indeterminate between $\underline{r}$ and $\overline{r}$. In this case, I analyze the expressions for $u - \underline{r}$ and $u - \overline{r}$ and present comparative statics for these.

Proposition 1. *If the collateral spread is positive, that is $u - r \geq 0$, expressions for the collateral spread and associated comparative statics are as in Table 4. In particular, the collateral spread is increasing in the unsecured rate, u, and the haircut, h, and decreasing in risk aversion, ρ, and the following properties of the underlying security: the expected rate of return, $\bar{y}$, volatility, σ_y^2, and illiquidity, ε_0.*

Proof: See the Appendix.

The intuition for the comparative statics in Proposition 1 relates to the tradeoff between doing a regular repo and the alternative of a home-made repo. To clarify this, it is useful to focus on the relatively simple case where the short's feasibility constraint in the home-made repo problem, (7), binds. That is, in the home-made repo, the short optimally chooses to sell the entire holding of her underlying security and, thus, minimize her unsecured borrowings. This is listed as Case 1 in Table 4. In this case, as seen in Theorem 1, there

Table 4: Collateral spread and comparative statics when $u - r \geq 0$

Case	Collateral spread	Effect of change in					
		u	$\bar{y}$	h	σ_y^2	ε_0	ρ
1. $u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0)$	$u - r = \frac{1-\varepsilon_0}{1-h}\left[u - \bar{y} - \frac{1}{2}\rho\sigma_y^2(1-\varepsilon_0)\right]$	+	−	+	−	−	−
2. $\rho\sigma_y^2 max\left\{1-\eta, 1-\kappa\right\} \leq u - \bar{y} < \rho\sigma_y^2(1-\varepsilon_0)$	$u - r = \frac{1}{2}\frac{(u-\bar{y})^2}{\rho\sigma_y^2(1-h)}$	+	−	+	−	−	−
3. $\eta \leq \kappa$ and $u - \bar{y} < \rho\sigma_y^2(1-\eta)$							
a. $\overline{r}$: $u - \bar{y} \geq \frac{1}{2}\rho\sigma_y^2(1-\eta)$	$u - \overline{r} = \frac{1-\eta}{1-h}\left[u - \bar{y} - \frac{1}{2}\rho\sigma_y^2(1-\eta)\right]$	+	−	+	−	−	−
b. $\underline{r}$: $u - \bar{y} \geq \rho\sigma_y^2(1-\kappa)$	$u - \underline{r} = \frac{1}{2}\frac{(u-\bar{y})^2}{\rho\sigma_y^2(1-h)}$	+	−	+	−	−	−
c. $\underline{r}$: $\frac{1}{2}\rho\sigma_y^2(1-\kappa) \leq u - \bar{y} < \rho\sigma_y^2(1-\kappa)$	$u - \underline{r} = \frac{1-\kappa}{1-h}\left[u - \bar{y} - \frac{1}{2}\rho\sigma_y^2(1-\kappa)\right]$	+	−	+	−	−	−

Notes. Case 3: If $\eta = \kappa$, $\underline{r} = \overline{r}$. Case 3(a): The condition here ensures $\overline{r} \leq u$.

is a unique equilibrium repo rate as given by (18). This can also be written as

$$\begin{aligned} (1-h)r + hu &= (1-\varepsilon_0)\hat{y}(1) + \varepsilon_0 u, \\ \text{where} \qquad \hat{y}(1) &= \bar{y} + \tfrac{\rho}{2}\sigma_y^2(1-\varepsilon_0). \end{aligned} \tag{35}$$

This says that, in equilibrium, the weighted average cost of liquidity from doing a repo equals the weighted average cost from doing a home-made repo. Recall that $\hat{y}(1)$ is the risk and illiquidity adjusted cost of liquidity from selling the entire security in the cash market and repurchasing it at date 1. As an initial observation, note that $\hat{y}(1) < u$ because $r < u$ (by assumption).

Given (35), the comparative statics of the collateral spread, $u-r$, are relatively straightforward. First, it is clear from (35) that an increase in the unsecured rate, u, implies that the repo rate must change by less than the unsecured rate in order to keep the costs of the two alternatives equal. Thus, the collateral spread is increasing in the unsecured rate.

Second, if $\hat{y}(1)$ increases, the repo rate must increase relative to the unsecured rate in order to keep the two alternatives equally costly, implying a drop in the collateral spread. In other words, an increase in the security's expected rate of return, $\bar{y}$, or its volatility, σ_y^2, cause a decrease in the collateral spread, *ceteris paribus*.

Third, an increase in the haircut, h, increases the cost of liquidity from the repo alternative, since $r < u$. This makes the home-made alternative more attractive. Thus, the repo rate must drop relative to the unsecured rate in order to keep the costs of the two alternatives the same.

Fourth, the effects of an increase in illiquidity, ε_0, can also be understood intuitively from (35). Note that illiquidity affects both $\hat{y}(1)$ and the weights on the security cash trade and unsecured borrowing in the home-made repo case. As is intuitive, an increase in illiquidity, increases σ_y^2. Thus, an increase in ε_0 raises the cost of the home-made repo since it also puts more weight on unsecured borrowing. As a result, the repo rate must increase relative to the unsecured rate in order to keep the cost of the regular repo alternative equal to that of the home-made repo. In short, the collateral spread must tighten.

Equation (35) also emphasizes a key point of the analysis, namely that when agents in the market for liquidity face constraints in the unsecured market, the repo rate, the unsecured rate, and the security cash market expected rate of return must be linked. Case 1 in Table 4 gives a clear-cut link because, in this case, the largest repo rate at which the short is willing to trade coincides with the smallest acceptable rate for the cash provider. Proposition 1 can be thought of as showing that the intuition above carries through when these bounds do not coincide.

4.2 Negative collateral spread

As seen in Theorem 2, a negative collateral spread is an equilibrium phenomenon only if the short has a smaller unsecured borrowing capacity than the cash provider, $\eta \leq \kappa$. When this inequality is strict, the repo rate is not uniquely determined, but lies in an interval $(R_{neg}, \overline{r}_{neg})$, where $R_{neg} = \max\{u, \underline{r}_{neg}\}$. If $\eta = \kappa$ the upper and lower bounds coincide and define a unique equilibrium $r > u$. As above, I provide comparative statics for the collateral spread using the lower and upper bounds of the repo rate when a unique rate does not exist.

Proposition 2. *If the collateral spread is negative, that is $u - r < 0$, expressions for the collateral spread and associated comparative statics are as in Table 5. In particular, the collateral spread is increasing in the unsecured rate, u and decreasing in risk aversion, ρ, and the following properties of the underlying security: the liquidity adjusted expected rate of return, $\bar{y}$, volatility, σ_y^2, and illiquidity, ε_0. An increase in the haircut, h, has a nonnegative effect on the collateral spread.*

The comparative statics are, with one exception, the same as for the case of a positive collateral spread. The basic intuition for the similarity is that the fundamental tradeoffs between the regular repo and the home-made repo do not change when collateral spreads are negative. The focus here is, therefore, on the exception, namely the haircut. In four out of the six scenarios in Table 5, this is seen to have no effect on the collateral spread.

The irrelevance of the haircut can be understood by first noting that when $r > u$, the short borrows as much as possible in the unsecured market under the regular repo alternative. Therefore, the haircut is not relevant for the short's liquidity cost. However, it affects the quantity that she repos, ϕ as defined in (26). Thus, it affects the cash

Table 5: Collateral spread and comparative statics when $u - r < 0$

Case			Collateral spread	Effect of change in					
				u	$\bar{y}$	h	σ_y^2	ε_0	ρ
1.	$\eta < \kappa$								
a.	$\overline{r}_{neg}$:		$u - \overline{r}_{neg} = u - \bar{y} - \frac{1}{2}\rho\sigma_y^2(1-\eta)$	$+$	$-$	0	$-$	$-$	$-$
b.	$\underline{r}_{neg}$:	$u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0)\phi$	$u - \underline{r}_{neg} = \frac{1-\varepsilon_0}{1-h}\left[u - \bar{y} - \frac{1}{2}\rho\sigma_y^2(1-\varepsilon_0)\phi\right]$	$+$	$-$	$+$	$-$	$-$	$-$
c.	$\underline{r}_{neg}$:	$\rho\sigma_y^2(1-\kappa) \leq u - \bar{y} \leq \rho\sigma_y^2(1-\varepsilon_0)\phi$	$u - \underline{r}_{neg} = \frac{1}{2}\frac{(u-\bar{y})^2}{\rho\sigma_y^2(1-\eta)}$	$+$	$-$	0	$-$	$-$	$-$
d.	$\underline{r}_{neg}$:	$u - \bar{y} \leq \rho\sigma_y^2(1-\kappa)$	$u - \underline{r}_{neg} = \frac{1-\kappa}{1-\eta}\left[u - \bar{y} - \frac{1}{2}\rho\sigma_y^2(1-\kappa)\right]$	$+$	$-$	0	$-$	$-$	$-$
2.	$\eta = \kappa$		$u - r = u - \bar{y} - \frac{1}{2}\rho\sigma^2(1-\eta)$	$+$	$-$	0	$-$	$-$	$-$

Note: $\phi \equiv (1-\eta)/(1-h)$.

provider's cost of providing liquidity *if* the cash provider would optimally raise liquidity by selling the entire quantity, ϕ, of the underlying security he gets from the reverse repo. This happens in one of the five cases in Table 5 (Case 1b). More generally, the cash provider's feasibility constraint does not bind and, therefore, the haircut generally does not affect the collateral spread when this is negative.

Corollary

As a corollary of Propositions 1 and 2, the collateral spread is decreasing in the expected rate of return, volatility, and illiquidity of the underlying collateral. The implication is that baskets that contain relatively low quality collateral should have lower collateral spreads. This is consistent with Figure 1 (see Section 5 for details) and also with the findings in Bartolini, Hilton, Sundaresan, and Tonetti (2011).

4.3 The sign of the collateral spread

The following is a direct corollary of Theorems 1 and 2.

Proposition 3. *Ceteris paribus, the collateral spread can switch from positive to negative if (i) borrowing constraints tighten (η and κ decrease), (ii) the expected rate of return or volatility of the underlying assets increase, (iii) risk aversion increases, (iv) illiquidity increases, (v) the unsecured rate decreases.*

Proof: This follows directly from the conditions in Theorems 1 and 2 for a positive or negative collateral spread. With respect to the effect of illiquidity, note that $\bar{y}$ and σ_y^2 are functions of ε_0 as seen in (1) and the discussion that follows that equation. $\square$

The intuition is essentially the same as to the intuitions behind the results in Propositions 1 and 2. The only new element here is the role of the borrowing constraints. Intuitively, when these tighten, more of the underlying security will need to be sold in order to raise the required unit of liquidity. This increases the overall risk taken by the cash taker and provider and, therefore, the cost of raising liquidity. This is tantamount to saying that the repo rate must rise relative to the unsecured rate.

4.4 Remark 2: Linked versus non-linked borrowing capacities

This subsection addresses the robustness of the results to Assumption 6 by changing it to:

Assumption $6'$. *If the short draws down on her unsecured borrowing capacity, the unsecured borrowing capacity of the cash provider is unchanged.*

Consider first a positive collateral spread. Assumption $6'$ does not change that, in the case of Alternative 1, the short optimally repos the entire security she holds. It also does not affect her optimal trade size, Ω, under Alternative 2. So from the short's direct perspective, Assumption $6'$ is immaterial. However, Assumption $6'$ changes the borrowing constraint of the cash provider. As already noted in footnote 13, this now becomes $\alpha(1-\varepsilon) \geq 1-h-\kappa$. This is because the cash provider's obligation to the short is only to provide $1-h$ in cash. Intuitively, because the cash provider's funding requirement is h less than that of the short, he can be thought of as effectively increasing his unsecured borrowing capacity by h. Defining $\kappa' \equiv \kappa + h$, it is clear that the analysis for a positive collateral spread under Assumption $6'$ is identical to the analysis under Assumption 6, but with κ' replacing κ everywhere.

Consider next a negative collateral spread. Again, only the cash provider's borrowing constraint is directly affected. Now this becomes $\alpha(1-\varepsilon) \geq 1-\eta-\kappa$. Defining $\kappa'' \equiv \kappa+\eta$, the analysis is the same as under Assumption 6, but with κ'' in place of κ. However, there is no analogue to the scenario $\eta > \kappa$ because η is always strictly less than κ''. Thus, under Assumption $6'$, there is equilibrium $r > u$ if and only if $u - \bar{y} < \frac{1}{2}\rho\sigma_y^2(1-\eta)$, regardless of the relation between η and κ.

The key differences between the analyses under Assumptions 6 and $6'$, as implied by the above discussion, are summarized in Table 6. *Ceteris paribus*, the range over which an equilibrium repo rate exists expands when borrowing capacities are not linked. One may interpret this as saying that linked unsecured credit lines reduce the ability of repos to serve as an alternative source of liquidity to banks. The range over which there is a positive collateral spread also expands under unlinked borrowing capacities and when haircuts increase. Intuitively, this is because unlinking borrowing capacities and increasing haircuts reduce the effective funding burden of the reverse position.

Table 6: Positive vs negative equilibrium collateral spreads.

Panel a: Assumption 6. Borrowing capacities are linked.				
	$\eta \leq \kappa$		$\eta > \kappa$	
Spread:	Positive	Negative	Positive	Negative
if $\frac{u-\bar{y}}{\rho\sigma_y^2}$	$\geq \frac{1-\kappa}{2}$	$\leq \frac{1-\eta}{2}$	$\geq 1-\kappa$	Not equilibrium
Panel b: Assumption 6$'$. Borrowing capacities are not linked.				
	$\eta \leq \kappa + h$		$\eta > \kappa + h$	
Spread:	Positive	Negative	Positive	Negative
if $\frac{u-\bar{y}}{\rho\sigma_y^2}$	$\geq \frac{1-\kappa-h}{2}$	$\leq \frac{1-\eta}{2}$	$\geq 1-\kappa-h$	$\leq \frac{1-\eta}{2}$

Examination of the expressions for the collateral spread derived in Propositions 1 and 2 (as presented in Tables 4 and 5) and keeping in mind the above discussion shows that the comparative statics are the same as before except possibly with respect to h in Case 3c in Table 4, because κ should here be replaced by κ', which is a function of h. However, given the conditions on $u - \bar{y}$ in Case 3c, it is straightforward that the effect of an increase in h remains positive. Thus, the effects of switching from Assumption 6 to Assumption 6$'$ are relatively minor. This is not all that surprising since the basic forces in the model are not affected by this switch.

5 Evolution of collateral spreads

In this section, I use the theory to comment on the developments of the two collateral spreads in Figure 1 over time. I also examine more closely the theoretical prediction that the higher quality basket should have a larger spread.

With respect to the evolution of collateral spreads, my focus is on the overall, non-spike trends within and across the time-periods indicated by the vertical lines in the figure. The frequent spikes in the collateral spreads relate to end-of-maintenance period spikes in the unsecured rate (as discussed above and tested more rigorously by Nyborg and Rösler, 2019). After the introduction of the full-allotment policy, the spikes are predominantly downward. This is because the full-allotment policy is associated with an excess quantity of liquidity in the system (see, e.g., Nyborg, 2016). Ignoring the spikes, Figure 1 shows that collateral spreads may be negative or positive over extended periods of time.

As noted in the Introduction, the collateral spreads in Figure 1 are for the two most active baskets on Eurex Repo, namely the GC Pooling ECB and GC Pooling ECB Extended baskets. These comprise approximately 99.5% of all overnight GC repo transactions (Nyborg and Rösler, 2019). The GC Pooling ECB basket represents higher quality collateral than the GC Pooling ECB Extended basket. Both baskets contain a subset of Eurosystem eligible collateral, but with the ECB Extended basket including lower rated and substantially more unsecured bank bonds than the ECB basket.[16] The data series in Figure 1 run from January 2, 2007 to June 30, 2015 for the ECB basket and from November 24, 2008 to June 30, 2015 for the ECB Extended basket. The shorter time period for the lower quality basket reflects that this was created after the ECB introduced the full-allotment policy and lowered collateral eligibility criteria in Eurosystem repos in the fall of 2008. The time series of spreads are from Nyborg and Rösler (2019) and are based on complete transactions data provided by Eurex. For each basket, collateral spreads are calculated on a daily basis as the Eonia less the volume-weighted average overnight repo rate.

5.1 Time periods

As indicated by vertical bars in Figure 1, the ECB implemented several significant liquidity-easing policies after the onset of the financial crisis. Besides the full-allotment policy, these include the first one-year LTRO, the two three-year LTROs, and quantitative easing. In total, Figure 1 divides the sample period into eight subperiods:[17]

1. *Pre-crisis, Jan 2, 2007 – Jul 31, 2007.* Signs of stress in the interbank market started to show in the beginning of August 2007.
2. *Early crisis, Aug 1, 2007 – Oct 8, 2008.* Markets experienced substantial turmoil after Lehman Brother's bankruptcy on September 15, 2008. The full-allotment policy was subsequently introduced by the ECB as a measure to ease banks' access to liquidity and calm markets.

[16] See Table 8 in Nyborg and Rösler (2019) or www.eurexrepo.com for details.

[17] The periods are based on standard accounts of the crisis and the ECB's unconventional monetary policies. See, for example, European Central Bank (2010), Szczerbowicz (2015), or Nyborg (2016) and the references therein. See also the ECB's webpage, www.ecb.europa.eu.

3. *Start of full allotment, Oct 9, 2008 – Jun 24 2009.* The full-allotment policy relates to the ECB's refinancing operations, the main and the longer term refinancing operations (MROs and LTROs). This policy give banks the quantity of liquidity they ask for (subject to posting sufficient collateral) at a fixed rate. It contrasts with the former policy of liquidity-neutrality under which liquidity was rationed in the refinancing operations.

4. *First one-year LTRO by the ECB, Jun 25, 2009 – Jul 1, 2010.* The LTROs originally had terms of around three months. During the course of the financial crisis, the Eurosystem increased the tenor of several LTROs to one and three years to combat the crisis. The first one-year LTRO settled on June 25, 2009 (it was held the day before) and matured July 1, 2010. It was held under the full-allotment policy and injected EUR 442 billion of one-year money into the banking system.

5. *Growing euro area sovereign crisis, Jul 2, 2010 – Dec 21, 2011.*

6. *Three-year LTROs, Dec 22, 2011 – Jun 30, 2013.* The ECB held two three-year LTROs, both under the full-allotment policy. The first settled on Dec 22, 2011 and the second on March 1, 2012. Combined, these injected more than EUR one trillion of three-year money into the banking system.

7. *One-third of LTROs repaid early, Jul 1, 2013 – Jan 21, 2015.*

8. *Quantitative easing, Jan 22, 2015 – end of dataset.* Quantitative easing in the euro area was announced September 4, 2014, initially targeting covered bonds and asset-backed securities. On January 22, 2015, the ECB announced the Public Sector Purchase Programme (PSPP), which is by far the largest program within its quantitative easing program.

[**insert Table 7 about here**]

Table 7 reports means, standard errors, and the percentage of negative days for overnight collateral spreads for the two GC Pooling baskets for the eight subperiods above (last six for the GC Pooling ECB Extended basket). The collateral spread is consistently larger for the higher quality ECB basket across subperiods and the incidence of negative spreads is lower. The spread is larger for the ECB basket on 97.5% of the 1,604 overlapping days. That repo rates decrease in the quality of the underlying collateral is as predicted by the constrained-arbitrage theory in this paper. The more classical no-arbitrage argument in Bartolini, Hilton, Sundaresan, and Tonetti (2011) generates a similar result, namely that repo rates increase in the yield of the underlying collateral. Their argument assumes no default risk and does not address the spread between unsecured and repo rates.

5.2 Negative collateral spread, Eonia–GC Pooling ECB basket

As seen in Table 7, the spread based on the GC Pooling ECB basket is predominantly negative only in the early crisis period that ends with the introduction of full allotment (Subperiod 2). Ignoring spikes, the collateral spread initially turned negative in the Spring of 2008. This was a time when sovereign yields in the euro area started to diverge (Cline, 2014). In terms of the model, this can be thought of as $\bar{y}$ and σ_y^2 starting to rise. Propositions 1, 2, and 3 tell us that a rise in these parameters could contribute to collateral spreads turning negative. Stress in the interbank market is also well documented during this time. Proposition 3 shows that this can lead collateral spreads to turn negative (in terms of the model, unsecured borrowing capacities fall). So, by way of a narrative, the constrained-arbitrage theory would say that the negative spreads in 2008 reflect stress in the security market as well as the unsecured market for liquidity.

The spread is especially large negative immediately before full allotment – around the time of Lehman's default. The narrative based on the theory is that this reflects that securities lost substantial market value around this time and the interbank market experienced increased levels of stress.

The introduction of full allotment on October 25, 2008, sees the collateral spread immediately turn positive. The increase in collateral spreads is what one would expect based on the theory, under the view that liquidity-easing policies soften constraints in the interbank market and put upward pressure on prices and liquidity in the security market. That the spread flipped from negative to positive speaks to the effectiveness of the full-allotment policy in the short term. However, as seen in Figure 1 and Table 7, the collateral spread subsequently fell again. Based on the theory, the interpretation is that the the full-allotment policy was not entirely effective over the longer term. It did not resolve the underlying problems in euro area markets. The initial boon faded over time as reflected in falling collateral spreads.

5.3 Both collateral spreads

The collateral spread based on the Extended basket is negative, on average, in Subperiods 3 (full allotment) and 5 (sovereign crisis). In contrast, the overnight collateral spread for the ECB basket is positive, on average, over both subperiods. This suggests especially strong downward pressure on the lower quality collateral in the Extended basket. While full allotment may have eased constraints in the market for liquidity, at least for a time, it may have been less effective with respect to lifting security prices, especially at the lower end of the spectrum.

The two periods of negative spreads for the Extended basket end with the introductions of the first one-year and three-year LTROs, respectively. This is consistent with the theory given the empirical evidence that yields fell with the introduction of the first three-year LTRO (Woschitz, 2017). However, as seen in the graph, the initial uplifts provided by the one- and three-year LTROs subside over time. The pattern of collateral spreads over time is the same as after the introduction of full allotment. The narrative based on the theory is, therefore, also the same: Liquidity-easing policies gave an initial lift to markets and collateral spreads, but could not solve the underlying problems in the euro area. As stress re-emerged, collateral spreads fell.

Just as with the other liquidity-easing policies, the introduction of quantitative easing increased collateral spreads for both baskets. In the quantitative-easing subperiod, the incidence of negative collateral spreads is also very low. There are no days with negative spreads over this period for the higher quality basket. This suggests that quantitative easing put strong upward pressure on security prices, at least initially. The sample period does not include the end of quantitative easing, but there is an indication in Figure 1 that the initial uplift in collateral spreads also reverses after a while under quantitative easing.

To summarize, the broad fluctuations in collateral spread over time can be understood in terms of the theory. Periods of negative collateral spreads represent periods of stressed unsecured markets and depressed security markets. Liquidity-easing central bank policies lift collateral spreads in the short run. However, over time, their effects subside as the underlying problems in the markets re-emerge. This then gives rise to stronger central bank liquidity-easing policies (full allotment, one-year LTRO, three-year LTRO, quantitative easing). Using the theory, the collateral spread can be seen to be a useful lens through which to evaluate the state of the markets and the effectiveness of central bank liquidity policies.

6 Concluding remarks

This paper is motivated by the puzzle of negative collateral spreads, that is, repo rates that are larger than unsecured rates. This is a regular phenomenon in the euro overnight market and for longer-term benchmark rates. It is also common in the US. This is at odds with the paradigm of credit risk as the main driver of interest rate spreads in the market for liquidity. It suggests that other forces are at work. In this paper, I have developed a theory of repo rates and collateral spreads that is capable of explaining negative collateral spreads as well as overall trends in spreads over time.

The fundamental contribution of the paper lies in deriving a constrained-arbitrage relation between the repo rate, the unsecured rate, and the expected rate of return of the underlying asset. Unsecured borrowing constraints imply that these three rates are jointly determined. According to the theory, negative collateral spreads are symptoms of tight conditions in the interbank market, down-spikes in unsecured rates, or depressed security markets in terms of prices, illiquidity, or volatility. Roughly speaking, negative collateral spreads occur when the (adjusted) expected rate of return on the underlying collateral exceeds the unsecured rate. Capacity constraints in the unsecured market limit the ability to arbitrage the negative spread. Such constraints are consistent with the fact that at any point in time there is a fixed quantity of liquidity in the system. Furthermore, there is evidence that the market for liquidity is allocationally inefficient (Bindseil, Nyborg, and Strebulaev, 2009).

The theory in this paper is based on the simple observation that the alternative to raising liquidity through a repo is to sell the underlying security in the cash market and buy it back later (home-made repo). The specific setup analyzed assumes that the unsecured market is needed to raise the requisite quantity of liquidity regardless of which alternative is used, regular or home-made repo. (This can also be thought of as a collateral constraint). The potential cash provider in a regular repo is also liquidity constrained so that he needs to sell a portion of the underlying security to finance the reverse repo. This cements the constrained-arbitrage relation between the three rates in the model. Different capacity constraints for the cash taker and provider mean that there may be a range of equilibrium repo rates. The collateral spread can go from positive to negative when unsecured borrowing constraints tighten as this increases the overall risk of the transaction, since a larger portion of the underlying security will have to be sold to generate the quantity of liquidity that the short player needs.

The theory yields a number of other implications and predictions. For example: (i) Negative collateral spreads require the cash provider to be less constrained in the unsecured market than the cash taker. (ii) The collateral spread is decreasing in the following properties of the underlying security: expected rate of return, volatility, and illiquidity (pricing errors). (iii) It is also decreasing in risk aversion. (iv) The collateral spread is increasing in the unsecured rate and may, therefore, spike with it (but less). (v) If the collateral spread is positive (negative) it is increasing (non-decreasing) in the haircut. Nyborg and Rösler (2019) test some of these predictions using a comprehensive transactions dataset from Eurex Repo with supportive results.

This is the first paper to highlight the puzzle of negative collateral spreads and, more generally, develop a theory to explain the determinants of the collateral spread. The fundamental insight of the theory is that the repo rate, the unsecured rate, and the (adjusted) expected rate of return of the underlying security are linked. Given unsecured and repo rates, the expressions in this paper may be used to estimate expected rates of return of securities. In more broad terms, the theory provides a fresh perspective on the view that liquidity drives security prices and may serve as a useful starting point for further research into that idea.

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Appendix: Proofs

Proof of Lemma 1

It follows directly from the expressions (12) for Ω, (14) for $\overline{r}$, (16) for A, and (17) for $\underline{r}$, that if $u - \bar{y} \geq \rho\sigma_y^2 \max\{1-\eta, 1-\kappa\}$ then $\Omega = A$ and $\overline{r} = \underline{r}$. The expression (18) in the case that $u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0)$ then follows directly from (17). The expression (19) in the case that $\rho\sigma_y^2 \max\{1-\eta, 1-\kappa\} \leq u - \bar{y} < \rho\sigma_y^2(1-\varepsilon_0)$ also follows from (17) after substituting in the expression for ω^* from (9). To complete the proof, it is necessary to show that the expressions for r in (18) and (19) are less than or equal to u.

Suppose first that $u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0)$. Then (18) applies. Now, (18) implies that $r \leq u$ if and only if $(1-\varepsilon_0)\hat{y}(1) + \varepsilon_0 u \leq u$, which, in turn, holds if and only if $\hat{y}(1) \leq u$ (since $\varepsilon_0 < 1$). Thus, using (4), we have that $r \leq u$ if and only if $\bar{y} + \rho\sigma_y^2(1-\varepsilon_0)/2 \leq u$, or $u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0)/2$, which holds by assumption.

Suppose next that $\rho\sigma_y^2 \max\{1-\eta, 1-\kappa\} \leq u - \bar{y} < \rho\sigma_y^2(1-\varepsilon_0)$. Then (19) applies, and $r \leq u$ follows immediately (because $1 - h > 0$). □

Proof of Theorem 1

Case 1, $\eta > \kappa$: If (21) holds, that is, $u - \bar{y} \geq \rho\sigma_y^2(1-\kappa)$, by Lemma 1, there is a unique equilibrium repo rate $r \leq u$ (as a function of parameter values). Completing the proof requires showing that the bound (21) is also a necessary condition for equilibrium $r \leq u$.

Suppose, therefore, that $u - \bar{y} < \rho\sigma_y^2(1-\kappa)$. By (16), we have $A = (1-\kappa)/(1-\varepsilon_0)$. Suppose, by contradiction, that there is equilibrium $r \leq u$.

Consider first the case that $u - \bar{y} < \rho\sigma_y^2(1-\eta)$ so that $\Omega = (1-\eta)/(1-\varepsilon_0)$, by (12). Equilibrium $r \leq u$ requires $\overline{r}$, as given by (14), to be at least large as $\underline{r}$, as given by (17). Using these equations and simplifying, this is equivalent to saying that

$$(1-\eta)\hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right) + \eta u \geq (1-\kappa)\hat{y}\left(\frac{1-\kappa}{1-\varepsilon_0}\right) + \kappa u. \tag{36}$$

This can be written as $f(\eta) \geq f(\kappa)$, where $f(x) = (1-x)\hat{y}((1-x)/(1-\varepsilon_0)) + xu$. From (4),

$$\hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right) = \bar{y} + \frac{\rho}{2}\sigma_y^2(1-\eta). \tag{37}$$

Thus, $f'(x) = -\bar{y} - \rho\sigma_y^2(1-x) + u$. Hence, $f'(x) < 0$ if and only if $u - \bar{y} < \rho\sigma_y^2(1-x)$. This holds by assumption on the interval $[\kappa, \eta]$. Thus, $f(\eta) < f(\kappa)$. This is equivalent to saying that (36) does not hold and contradicts that there is equilibrium $r \leq u$.

Consider next the case that $u - \bar{y} \geq \rho\sigma_y^2(1-\eta)$ so that $\Omega = \omega^*$ as given by (9). Thus,

$$\overline{r} = u - \frac{1}{2}\frac{(u-\bar{y})^2}{\rho\sigma_y^2(1-h)},$$

and the equilibrium condition $\overline{r} \geq \underline{r}$ becomes

$$u - \frac{1}{2}\frac{(u-\bar{y})^2}{\rho\sigma_y^2(1-h)} \geq \frac{(1-\kappa)\left(\bar{y} + \frac{1}{2}\rho\sigma_y^2(1-\kappa)\right) + (\kappa - h)u}{1-h}. \tag{38}$$

This can be written as

$$(u-\bar{y})^2 - 2\rho\sigma_y^2(1-\kappa)(u-\bar{y}) + \rho^2\sigma_y^4(1-\kappa)^2 \leq 0, \tag{39}$$

or,

$$((u-\bar{y}) - \rho\sigma_y^2(1-\kappa))^2 \leq 0, \tag{40}$$

which is not possible (because $\rho\sigma_y^2(1-\kappa) \neq (u-\bar{y})$). Hence, there is no equilibrium $r \leq u$ when $\kappa < \eta$ and $u - \bar{y} < \rho\sigma_y^2(1-\kappa)$. Thus, the bound (21) is also a necessary condition for equilibrium $r \leq u$ when $\eta > \kappa$.

Case 2, $\boldsymbol{\eta \leq \kappa}$: If $u - \bar{y} \geq \rho\sigma_y^2(1-\eta)$, by Lemma 1, there is a unique equilibrium repo rate $r \leq u$ (as a function of parameter values). This takes care of Case 2a in the statement of the theorem.

The remainder of the proof concerns Case 2b, that is, $u-\bar{y} < \rho\sigma_y^2(1-\eta)$. In dealing with this case, the bound (22) will also be established as a necessary and sufficient condition for equilibrium $r \leq u$. (The bound is automatically satisfied in Case 2a). The proof proceeds by first deriving the equation for $\underline{r}$ in the statement of the theorem, namely (25). (22) is then established as a necessary condition for $r \leq u$ by checking the condition $\underline{r} \leq u$. Following this, the equations for the upper bound, R, (24) and (23), in the statement of the theorem are derived, in that order. Sufficiency of (22) follows thereafter by checking that $\underline{r} \leq R$.

Suppose in the remainder of this proof that $u - \bar{y} < \rho\sigma_y^2(1-\eta)$. Recall that $\underline{r}$ is given by (17). Substituting in the values of A from (16), we get the expressions for $\underline{r}$ in (25). It is immediate that $\underline{r} < u$ if $u - \bar{y} \geq \rho\sigma_y^2(1-\kappa)$. Suppose next that $u - \bar{y} < \rho\sigma_y^2(1-\kappa)$. From the associated expression for $\underline{r}$ in (25), we get, after some algebra, that $\underline{r} \leq u$ if and only if $\hat{y}\left(\frac{1-\kappa}{1-\varepsilon_0}\right) = \bar{y} + \frac{\rho}{2}\sigma_y^2(1-\kappa) \leq u$. In other words, (22) is a necessary condition for $r \leq u$.

Next, consider $\overline{r}$. By (12), we have $\Omega = (1-\eta)/(1-\varepsilon_0)$. Substituting this into the expression for $\overline{r}$ in (14), we get the first expression for $\overline{r}$ in (24). The second expression follows from (37). Using this expression, it can be seen that $\overline{r} < u$ if and only if $\bar{y} + \rho\sigma_y^2(1-\eta)/2 < u$. Thus, we get (23).

Suppose now (22) holds so that $\underline{r} \leq u$. We need to show that $\underline{r} \leq R$, where $R = \min\{u, \overline{r}\}$. If so, any $r \in [\underline{r}, R]$ would be an equilibrium repo rate (of at most u), which is what we want to show. Note first that by (23), $\underline{r} \leq u$ implies $\underline{r} \leq R$ for $u - \bar{y} \leq \frac{\rho}{2}\rho\sigma_y^2(1-\eta)$. Suppose, therefore, that $u - \bar{y} \geq \frac{\rho}{2}\sigma_y^2(1-\eta)$. By (25) and since $\eta \leq \kappa$, there are two cases:

Case (a): $1 - \kappa \leq (1-\eta)/2$.

We need to show that $\underline{r} \leq \overline{r}$. Since $\rho\sigma_y^2(1-\eta) > u - \bar{y} \geq \frac{\rho}{2}\sigma_y^2(1-\eta)$, this is, using (23)–(25),[18]

$$u - \frac{1}{2}\frac{(u-\bar{y})^2}{\rho\sigma_y^2(1-h)} \leq \frac{(1-\eta)\left(\bar{y} + \frac{1}{2}\rho\sigma_y^2(1-\eta)\right) + (\eta - h)u}{1-h}. \tag{41}$$

This is just the reverse of (38) in the proof of Case 1 above, with η in place of κ, and, therefore, holds.

Case (b):$1 - \kappa > (1-\eta)/2$.

We need to show that $\underline{r} \leq \overline{r}$. There are two subcases. First, $\rho\sigma_y^2(1-\eta) > u - \bar{y} \geq \rho\sigma_y^2(1-\kappa)$. In this case, $\underline{r} \leq \overline{r}$ can once again be written as (41), which holds.

Second,

[18]The left hand side of (41) is $\underline{r}$ when $u - \bar{y} \geq \rho\sigma_y^2(1-\kappa)$ as given in (25). The right hand side is $\overline{r}$ for $\rho\sigma_y^2(1-\eta) > u - \bar{y} > \rho\sigma_y^2(1-\eta)/2$ as given in (24).

$$\frac{u - \bar{y}}{\rho \sigma_y^2} \in \left[\frac{1-\eta}{2}, 1-\kappa \right). \tag{42}$$

Using (12), (14), (16), and (17)), the condition $\underline{r} \leq \overline{r}$ can now be written

$$(1-\kappa)\hat{y}\left(\frac{1-\kappa}{1-\varepsilon_0}\right) + \kappa u \leq (1-\eta)\hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right) + \eta u. \tag{43}$$

This is just (36) in the proof of Case 1 above, but with η and κ having switched places (now $\eta \leq \kappa$). Thus, (43) holds. In other words, $\underline{r} \leq \overline{r}$ given (42).

The above establishes that $\underline{r} \leq R$ if (22) holds. It follows immediately that if (22) holds, then any $r \in [\underline{r}, R]$ is an equilibrium repo rate, $r \leq u$. This completes the proof. □

Proof of Lemma 2

Under Alternative 2, the short's optimal trade cannot lead to a higher cost of liquidity than if she chooses any arbitrary trade that gives her the requisite unit of liquidity. Consider $\omega = (1-\eta)/(1-\epsilon_0)$ and borrowings of η at the unsecured rate. Thus, using the same line of argument as in the derivation of (13), we get that for the short to be willing to do a repo, we must have

$$(1-\eta)r + \eta u \leq (1-\eta)\hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right) + \eta u, \tag{44}$$

or

$$r \leq \hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right) = \bar{y} + \frac{\rho}{2}\sigma_y^2(1-\eta). \tag{45}$$

(28) follows from $u < r$. Statements (i) and (ii) follow immediately. □

Proof of Lemma 3

By Assumption 6, the unsecured borrowing constraint of the cash provider is $\alpha(1-\varepsilon_0) \geq 1 - \eta - (\kappa - \eta) = 1 - \kappa$. Since, we must also have $\phi \geq \alpha$ (feasibility), it follows that $\phi(1-\varepsilon_0) \geq 1-\kappa$, which is what we wanted to show. □

Proof of Theorem 2

By Lemma 2, (28) is a necessary condition for equilibrium $r > u$. Suppose, henceforth, that this holds, that is, $u - \bar{y} < \frac{1}{2}\rho\sigma_y^2(1-\eta)$. By (29), this is equivalent to saying that

$u < \overline{r}_{neg}$. To show that equilibrium $r > u$ exists, therefore, we only need to show that $\underline{r}_{neg} \leq \overline{r}_{neg}$. If this holds as a strict inequality, any repo rate in the interval $(R_{neg}, \overline{r}_{neg}]$, where $R_{neg} = \max\{u, \underline{r}_{neg}\}$, is an equilibrium $r > u$. If $\underline{r}_{neg} = \overline{r}_{neg}$, then this common rate is a unique equilibrium $r > u$.

Case 1, $\boldsymbol{\eta \leq \kappa}$:

Claim: $\underline{r}_{neg} < \overline{r}_{neg}$ if $\eta < \kappa$ and $\underline{r}_{neg} = \overline{r}_{neg} = \hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right)$ if $\eta = \kappa$.

To prove the claim, suppose first that $\eta < \kappa$. There are two cases to consider.

Case (a): $1 - \kappa \geq (1-\eta)/2$. In this case, by (30), $A = (1-\kappa)/(1-\varepsilon_0)$. By (32), the minimum acceptable rate to the cash provider is, therefore,

$$\underline{r}_{neg} = \frac{(1-\kappa)\hat{y}\left(\frac{1-\kappa}{1-\varepsilon_0}\right) + \kappa u - \eta u}{1-\eta} \tag{46}$$

Thus, using (29), we have that $\underline{r}_{neg} < \overline{r}_{neg}$ if and only if (43) [from the proof of Theorem 1)] holds as a strict inequality, which it does here because $\eta < \kappa$.

Case (b): $1 - \kappa < (1-\eta)/2$.

We need only consider $u - \bar{y} > \rho\sigma_y^2(1-\kappa)$, as the reverse is covered by the argument in Case (a). Suppose first that $u - \bar{y} < \rho\sigma_y^2(1-\varepsilon_0)\phi$ so that $A = \omega^*$. Thus, using (4), (9), and (32), we have

$$\underline{r}_{neg} = u - \frac{1}{2}\frac{(u-\bar{y})^2}{\rho\sigma_y^2(1-\eta)}. \tag{47}$$

Hence, $\underline{r}_{neg} < u < \overline{r}_{neg}$.

Suppose next that $u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0)\phi$ so that $A = \phi = (1-\eta)/(1-h)$. Using (32), we have

$$\underline{r}_{neg} = \frac{\phi(1-\varepsilon_0)\hat{y}(\phi) + (1-\phi(1-\varepsilon_0))\,u - \eta u}{1-\eta}. \tag{48}$$

Hence, $\underline{r}_{neg} < u$ if and only if $\hat{y}(\phi) < u$, which is equivalent to $\bar{y} + \frac{1}{2}\rho\sigma_y^2\phi(1-\varepsilon_0) < u$. This holds since $u - \bar{y} \geq \rho\sigma_y^2\phi(1-\varepsilon_0)$ (by assumption). Hence, $\underline{r}_{neg} < u < \overline{r}_{neg}$. Thus, we have proved the claim above when $\eta < \kappa$.

Suppose next that $\eta = \kappa$. Now the same algebra as in Case (a) above establishes that $\underline{r}_{neg} = \overline{r}_{neg} = \hat{y}\left(\frac{1-\eta}{1-\varepsilon_0}\right)$, as claimed.

Finally, note that the expressions for $\underline{r}_{neg}$ in the different cases above jointly establish

(33).

Case 2, $\eta > \kappa$: By Lemma 3, we only need to consider the case that $(1-\varepsilon_0)\phi \geq (1-\kappa)$. By Lemma 2, a necessary condition for equilibrium $r > u$ is that $u - \bar{y} < \rho\sigma_y^2(1-\kappa)$ (since $\kappa < \eta$). Suppose, this holds. Thus, by (30), $A = (1-\kappa)/(1-\varepsilon_0)$. By (28), we also have $u - \bar{y} < \rho\sigma_y^2(1-\eta)$ so that $\overline{r}_{neg}$ is given by (29). The same algebra as in Case 1(a) above now shows that $\underline{r}_{neg} > \overline{r}_{neg}$, since now $\eta > \kappa$. Thus, there is no equilibrium $r > u$. $\square$

Remark

This remark relates to the derivation of the comparative statics of the collateral spread with respect to ε_0 that will be carried out in the proofs of Propositions 1 and 2 below. In particular, note that $\bar{y}$ and σ_y^2 are functions of ε_0. Using the expression for $\bar{y}$ in terms of $\bar{x}$, σ_x^2, ε_1, and ε_0 in (1), we have $\bar{y}'(\varepsilon_0) = (1+\bar{x}-\epsilon_1)/(1-\varepsilon_0)^2 = (1+\bar{y})/(1-\varepsilon_0)$. Let $f(\varepsilon_0) = \sigma_y^2(\varepsilon_0)(1-\varepsilon_0) = \sigma_x^2/(1-\varepsilon_0)$. Hence $f'(\varepsilon_0) = \sigma_x^2/(1-\varepsilon_0)^2 = \sigma_y^2$. Furthermore, $\sigma_y^{2\prime}(\varepsilon_0) = 2\sigma_y^2/(1-\varepsilon_0)$.

Proof of Proposition 1

Suppose $u - r \geq 0$. The cases for the collateral spread in Table 4 are taken directly from Theorem 1. Associated expressions for r, $\underline{r}$, and $\overline{r}$ are also in Theorem 1. These have been subtracted from u to give the expressions for the collateral spread in Table 4. From these expressions, it is immediately obvious that the partial derivatives with respect to σ_y^2 and ρ are identical. The comparative statics are verified below. Note first that, by assumption, $1-\varepsilon_0 > 0$, $1-h > 0$, $1-\eta > 0$, and $1-\kappa > 0$.

Case 1. $u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0)$.

Let $B \equiv u - \bar{y} - \frac{1}{2}\rho\sigma_y^2(1-\varepsilon_0)$. Note that in Case 1, $B > 0$. Now, using the expression for $u - r$ in Table 4, we have: $\frac{\partial(u-r)}{\partial u} = \frac{1-\varepsilon_0}{1-h} > 0$; $\frac{\partial(u-r)}{\partial \bar{y}} = -\frac{1-\varepsilon_0}{1-h} < 0$; $\frac{\partial(u-r)}{\partial h} = \frac{1-\varepsilon_0}{(1-h)^2}B > 0$; $\frac{\partial(u-r)}{\partial \sigma_y^2} = -\frac{1}{2}\frac{(1-\varepsilon_0)^2}{1-h} < 0$; and, using the expressions in the remark above,

$$\frac{\partial(u-r)}{\partial\varepsilon_0} = \frac{-B}{1-h} + \frac{1-\varepsilon_0}{1-h}\left[-\bar{y}'(\varepsilon_0) - f'(\varepsilon_0)\right] = \frac{-u-1}{1-h} < 0$$

since $u > -1$. This concludes Case 1.

Case 2: $\rho\sigma_y^2 max\{1-\eta, 1-\kappa\} \leq u - \bar{y} < \rho\sigma_y^2(1-\varepsilon_0)$.

Note that in Case 2, $u-\bar{y} > 0$. Using the expression for $u-r$ in Table 4, we have: $\frac{\partial(u-r)}{\partial u} = \frac{u-\bar{y}}{\rho\sigma_y^2(1-h)} > 0$; $\frac{\partial(u-r)}{\partial\bar{y}} = -\frac{u-\bar{y}}{\rho\sigma_y^2(1-h)} < 0$; $\frac{\partial(u-r)}{\partial h} = \frac{1}{2}\frac{(u-\bar{y})^2}{\rho\sigma_y^2(1-h)^2} > 0$; $\frac{\partial(u-r)}{\partial\sigma_y^2} = -\frac{(u-\bar{y})^2}{\rho\sigma_y^4(1-h)} < 0$; and, using the expressions in the remark above

$$\frac{\partial(u-r)}{\partial\varepsilon_0} = \frac{1}{\sigma_y^2\rho(1-h)}\left[-(u-\bar{y})\bar{y}'(\varepsilon_0) - \frac{(u-\bar{y})^2}{1-\varepsilon_0}\right] = \frac{u-\bar{y}}{\sigma_y^2(1-\varepsilon_0)\rho(1-h)}[-u-1] < 0.$$

This concludes Case 2.

Case 3a: $\eta \leq \kappa$ and $\frac{1}{2}\rho\sigma_y^2(1-\eta) \leq u-\bar{y} < \rho\sigma_y^2(1-\eta)$.

Using the expression for $u-\bar{r}$ in Table 4, we have $\frac{\partial(u-\bar{r})}{\partial u} = \frac{1-\eta}{1-h} > 0$; $\frac{\partial(u-\bar{r})}{\partial\bar{y}} = -\frac{1-\eta}{1-h} < 0$; $\frac{\partial(u-\bar{r})}{\partial h} = \frac{1-\eta}{(1-h)^2}\left[u-\bar{y}-\frac{1}{2}\rho\sigma_y^2(1-\eta)\right] > 0$; $\frac{\partial(u-\bar{r})}{\partial\sigma_y^2} = -\frac{1}{2}\rho\frac{(1-\eta)^2}{1-h} < 0$; and, using the expressions in the remark above

$$\frac{\partial(u-\bar{r})}{\partial\varepsilon_0} = \frac{1-\eta}{1-h}\left[-\frac{1+\bar{y}}{1-\varepsilon_0} - \frac{\rho\sigma_y^2(1-\eta)}{1-\varepsilon_0}\right] < 0,$$

since $\bar{y} > -1$. This concludes Case 3a.

Observe now that the expression, and therefore also the comparative statics, for $u-\underline{r}$ in Case 3b is identical to that for $u-r$ in Case 2. Finally, the expression for $u-\underline{r}$ in Case 3c is the same as for $u-\bar{r}$ in Case 3a, but with κ in place of η. The comparative statics are, therefore, analogous. □

Proof of Proposition 2

Suppose $u-r < 0$. The cases for the collateral spread in Table 5 are taken directly from Theorem 2. Associated expressions for r, $\underline{r}_{neg}$, and $\bar{r}_{neg}$ are also in Theorem 2. These have been subtracted from u to give the expressions for the collateral spread in Table 5. From these expressions, it is immediately obvious that the partial derivatives with respect to σ_y^2 and ρ are identical. The comparative statics are verified below. Note first that, by assumption, $1-\varepsilon_0 > 0$, $1-h > 0$, $1-\eta > 0$, and $1-\kappa > 0$.

Case 1a: $\eta \leq \kappa$.

Using the expression for $u-\bar{r}_{neg}$ in Table 5, we have: $\frac{\partial(u-\bar{r}_{neg})}{\partial u} = 1$; $\frac{\partial(u-\bar{r}_{neg})}{\partial\bar{y}} = -1$;

$\frac{\partial(u-\overline{r}_{neg})}{\partial h} = 0$; $\frac{\partial(u-\overline{r}_{neg})}{\partial \sigma_y^2} = -\frac{1}{2}\rho(1-\eta) < 0$; and using the expressions in the remark above $\frac{\partial(u-\overline{r}_{neg})}{\partial \varepsilon_0} = -\sigma_y^2(1-\eta)/(1-\varepsilon_0) < 0$. This concludes Case 1a.

Case 1b: $\boldsymbol{\eta \leq \kappa}$ and $\boldsymbol{u - \bar{y} \geq \rho\sigma_y^2(1-\varepsilon_0)\phi}$.

Let $B \equiv u - \bar{y} - \frac{1}{2}\rho\sigma_y^2(1-\varepsilon_0)\phi$. Note that $B > 0$ in Case 1b. Using the expression for $u - \underline{r}_{neg}$ in Table 5, we have: $\frac{\partial(u-\underline{r}_{neg})}{\partial u} = \frac{1-\varepsilon_0}{1-h} > 0$ and $\frac{\partial(u-\underline{r}_{neg})}{\partial \bar{y}} = -\frac{1-\varepsilon_0}{1-h} < 0$. Recalling that $\phi = (1-\eta)/(1-h)$, we also have

$$\frac{\partial(u-\underline{r}_{neg})}{\partial h} = \frac{1-\varepsilon_0}{(1-h)^2}B + \frac{1-\varepsilon_0}{1-h}\left[-\frac{1}{2}\rho\sigma_y^2\frac{(1-\varepsilon_0)(1-\eta)}{(1-h)^2}\right] = \frac{1-\varepsilon_0}{(1-h)^2}\left[u - \bar{y} - \rho\sigma_y^2(1-\varepsilon_0)\phi\right] \geq 0.$$

Now, $\frac{\partial(u-\underline{r}_{neg})}{\partial \sigma_y^2} = -\frac{\rho}{2}\frac{(1-\varepsilon_0)^2}{(1-h)^2}\phi < 0$. Finally, using the expressions in the remark above, we have

$$\frac{\partial(u-\underline{r}_{neg})}{\partial \varepsilon_0} = -\frac{B}{1-h} + \frac{1-\varepsilon_0}{1-h}\left[-\frac{1+\bar{y}}{1-\varepsilon_0} - \frac{\rho}{2}\sigma_y^2\phi\right] < 0.$$

This concludes Case 1b.

Case 1c: $\boldsymbol{\eta \leq \kappa}$, $\boldsymbol{\rho\sigma_y^2(1-\kappa) \leq u - \bar{y} \leq \rho\sigma_y^2(1-\varepsilon)\phi}$.

This is the same as Case 2 in Proposition 1, but with $(1-\eta)$ in the denominator instead of $1-h$. All comparative statics are, therefore, the same, except for with respect to h, where the partial derivative is now equal to zero.

Case 1d: $\boldsymbol{\eta \leq \kappa}$, $\boldsymbol{u - \bar{y} \leq \rho\sigma_y^2(1-\kappa)}$.

This is the same as Case 2 in Proposition 1, but with $(1-\eta)$ in the denominator instead of $1-h$. All comparative statics are, therefore, the same, except for with respect to h, where the partial derivative is now equal to zero.

Case 2: $\boldsymbol{\eta = \kappa}$

The collateral spread has the same expression as in Case 1a above, and so the comparative statics are identical.

Case 3: $\boldsymbol{\eta > \kappa}$ and $\boldsymbol{(1-\varepsilon_0)\phi \leq (1-\kappa)}$.

The collateral spread has the same expression as in Case 1b above, and so the comparative statics are identical. □

Appendix: Figure 1 and Table 7.

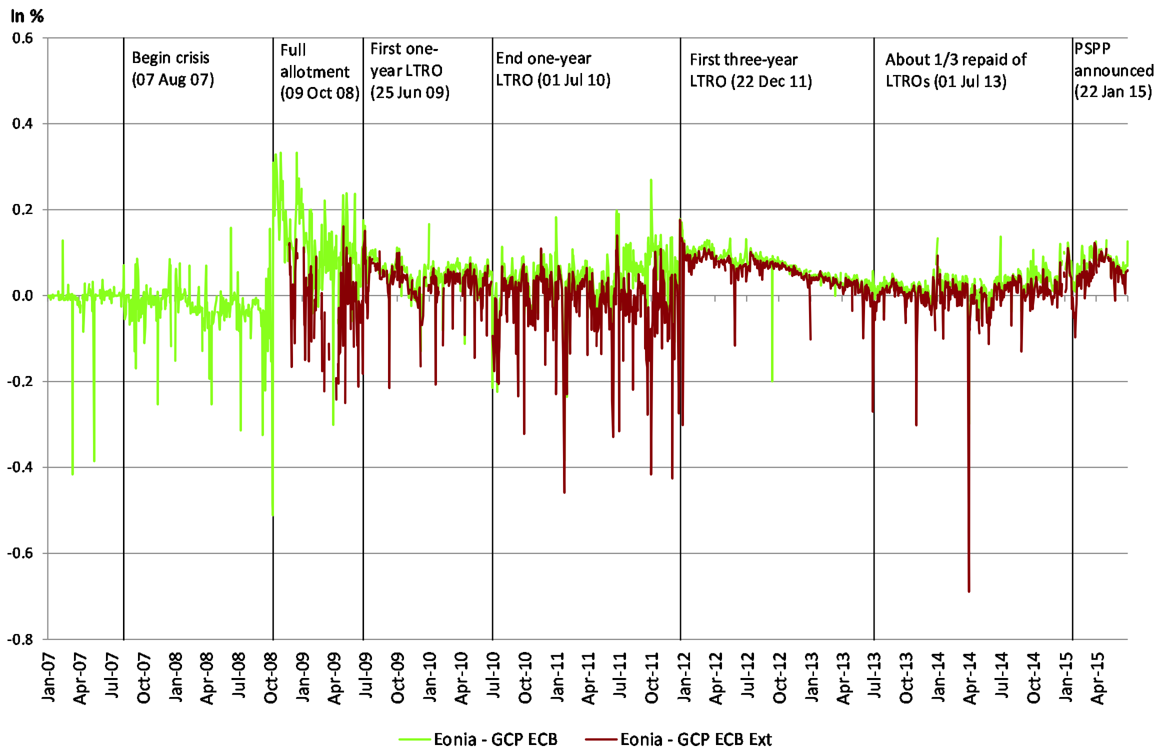

Figure 1: Overnight collateral spreads (in percent). (i) Eonia−GCP ECB basket, January 2, 2007 to June 30, 2015. (ii) Eonia−GCP ECB Extended basket, November 25, 2017 to June 30, 2015. Eonia is a volume-weighted average of overnight unsecured euro transactions by reporting panel banks. Eonia is an acronym for Euro Overnight Index Average. See http://www.euribor-rates.eu/eonia.asp. The repo rates are volume-weighted averages of all overnight transactions in Eurex Repo's GC Pooling ECB and GC Pooling ECB Extended baskets, respectively. Vertical bars relate to the financial crisis and ECB unconventional monetary policies (see Section 5). The underlying data for the figure is described in detail in Nyborg and Rösler (2019), which also contains a slightly different version of the figure.

Table 7: Collateral spreads over time

This table provides descriptive statistics of the collateral spreads Eonia - GCP ECB and Eonia-GCP ECB Ext. (both in bps) in eight different time periods determined by central bank policies. The unsecured rate, Eonia, and the secured rates, GCP ECB, and GCP ECB Ext., are daily volume-weighted overnight averages. The sample period for the spread Eonia – GCP ECB is January 02, 2007 to June 30, 2015, and for the spread Eonia – GCP ECB Ext. it is November 24, 2008 to June 30, 2015. p-values are under the null hypothesis that means are zero (two-tailed test). "Negative" provides the percentage of days with negative collateral spreads. N is the number of days when trades occur in the relevant basket under the relevant time period.

	Eonia−GC Pooling ECB basket					Eonia−GC Pooling ECB Extended basket				
Period	N	Mean	st. error	p-value	Negative (%)	N	Mean	st. error	p-value	Negative (%)
1. Pre-crisis, Jan 2, 2007–Jul 31, 2007										
	147	−0.480	0.408	0.241	47.177			n.a.		
2. Early crisis, Aug 1, 2007–Oct 8, 2008										
	303	−3.085	0.378	0.000	78.218			n.a.		
3. Start of full allotment Oct 9, 2008–June 24, 2009										
	178	11.699	0.664	0.000	7.263	101	−2.822	0.936	0.003	57.426
4. First one-year LTRO, Jun 25, 2009–Jul 1, 2010										
	262	5.317	0.265	0.000	6.870	227	2.902	0.303	0.000	15.419
5. Growing euro area sovereign crisis, Jul 2, 2010–Dec 21, 2011										
	382	3.502	0.355	0.000	22.251	381	−2.284	0.413	0.000	50.131
6. Three-year LTROs, Dec 22, 2011–Jun 30, 2013										
	387	7.145	0.172	0.000	0.517	387	5.602	0.207	0.000	3.876
7. One-third of LTROs repaid early, Jul 1, 2013–Jan 21, 2015										
	397	2.485	0.174	0.000	14.106	397	0.091	0.245	0.711	36.272
8. Quantitative easing (PSPP), Jan 22, 2015–June 30, 2015										
	111	7.321	0.255	0.000	0.000	111	5.133	0.364	0.000	9.910
All periods combined										
	2,167	3.773	0.145	0.000	22.889	1,604	1.362	0.167	0.000	28.741

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