Wall Crossings of Stability Functions of Root Systems
S. Moore

Background

Let L be a finite dimensional semisimple Lie algebra. A subset $H \subset L$ is said to be a *Cartan subalgebra* if H is a maximal toral subalgebra (a subalgebra in which all elements are ad-diagonalizable). In particular, H will be abelian, implying that every $h \in H$ is simultaneously ad-diagonalizable. We call $\alpha \in H^*$ a *root* of L if $\alpha \neq 0$ and there exists nonzero $v \in L$ such that $[h, v] = \alpha(h)v \; \forall \; h \in H$. The set of roots, R_ϕ, is finite. Let $S_\phi = \{\alpha_1, \alpha_2, ..., \alpha_n\} \subset R_\phi$ be a basis of H^* such that any $\alpha \in R_\phi$ can be written as $\alpha = \sum_{i=1}^{n} c_i\alpha_i$ with all c_i either nonpositive or nonnegative integers. We call elements of S_ϕ *simple roots*. If $\alpha \in R_\phi$ has all c_i nonnegative, then α is said to be a *positive root*. We denote the set of positive roots by P_ϕ. These sets $S_\phi \subset P_\phi \subset R_\phi$ (together with some more data) are called the *root system* ϕ of L. For example, the root system of $sl_3(\mathbb{C})$ (denoted A_2) has simple roots $\{\alpha_1, \alpha_2\}$ and positive roots $\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$.

A *(Bridgeland) stability function* on a root system is a map

$$Z : P_\phi \to \mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

satisfying $Z(\alpha + \beta) = Z(\alpha) + Z(\beta)$ for all $\alpha, \beta \in P_\phi$. As such, Z is uniquely determined by $Z \mid_{S_\phi}$. We also typically require that Z be *generic*, meaning that $Z(\alpha) \neq cZ(\beta)$ for any $c \in \mathbb{R}$ whenever $\alpha \neq \beta \in P_\phi$. See Fig. 1 for examples of stability functions on A_2.

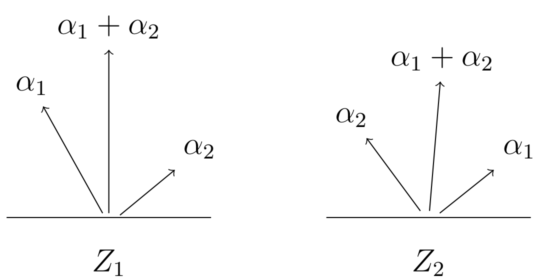

Figure 1: Two stability functions on A_2.

For $\alpha \in P_\phi$, the *phase of* α *under* Z is the angle from the positive real axis to $Z(\alpha)$. For generic Z, this induces a combinatorial ordering of the elements of P_ϕ via decreasing phase. Two stability functions $Z, Y : P_\phi \to \mathbb{H}$ are said to be *combinatorially different* if Z and Y induce different orderings of P_ϕ. In particular, Z_1 and Z_2 (see Fig. 1) are combinatorially different stability functions on A_2.

Our goal is to understand simple wall crossings of stability functions, which are defined as follows: Let Z and Y be stability functions of a root system ϕ. If the induced combinatorial orderings of P_ϕ under Z and Y are the same except that a consecutive triple $\tau, \tau + \omega, \omega$ under Z is rearranged to $\omega, \tau + \omega, \tau$ under Y, then Y is said to be obtained from Z by a *simple wall crossing*. For example, Z_2 is obtained from Z_1 by a simple wall crossing. Intuitively, a simple wall crossing arises from taking a path that connects Z and Y in the space of stability functions. Such a path will necessarily cross through a non-generic stability function in which $\tau, \omega,$ and $\tau + \omega$ have the same phase. Note that the space of stability functions is simply connected, so any two stability functions may be obtained from one another via a finite number of wall crossings.

The aim of this project can be broken into two main goals: ***First, given a root system ϕ, we aim to combinatorially describe the graph of cells of stability functions on ϕ separated by simple wall crossings. Second, we wish to determine the related identities among motivic characteristic classes of geometrically relevant spaces (see below).*** To accomplish these aims, we will use a combination of methods from combinatorics and rational function identities for neighboring cells (see [RR]).

Intellectual Merit

Such wall crossings have impacts beyond the study of Lie algebras. In particular, to a root system ϕ, we may associate a Cohomological Hall Algebra C_ϕ (defined by [KS]). This will be an infinite-dimensional, non-commutative, associative algebra whose product is denoted by $*$. Such algebras are motivated by string theory. To each $\alpha \in P_\phi$, [RR] assigns a motivic characteristic class $c^0_\alpha \in C_\phi$. The element c^0_α is a class in an equivariant cohomology (K-theory) algebra of a geometrically relevant space (such as the Grassmannian, or flag manifold). Such c^0_α have various interesting interpretations, such as motivic Chern classes, Chern-Schwartz-MacPherson classes, or stable envelopes in Okounkov's new theory relating geometry to physics (see [MO]).

Suppose that we have a simple wall crossing which permutes $\tau, \omega+\tau$, and ω. By [RR], this gives rise to an identity $c^0_{\omega+\tau} = [c^0_\tau, c^0_\omega]$ in the Cohomological Hall Algebra. Note that this result is similar in nature to wall-crossing formulas (also known as quantum dilogarithm identities) in Donaldson-Thomas theory [KS]. Such an identity gives a convenient way to calculate c^0_α for $\alpha \in P_\phi \setminus S_\phi$, as the commutator $[c^0_\tau, c^0_\omega]$ is well understood for $\tau, \omega \in S_\phi$. For example, in A_2 we can calculate $c^0_{\alpha_2+\alpha_1} = [c^0_{\alpha_1}, c^0_{\alpha_2}] = c^0_{\alpha_1} * c^0_{\alpha_2} - c^0_{\alpha_2} * c^0_{\alpha_1} = \left(1 + \frac{yb}{a}\right) - \left(1 - \frac{b}{a}\right) = (1+y)\frac{b}{a}$. That is, the class $c^0_{\alpha_2+\alpha_1}$ is obtained as the difference of the K-theoretic total Chern class $\left(1 + \frac{yb}{a}\right)$ and the K-theoretic Euler class $\left(1 - \frac{b}{a}\right)$.

Dissemination of Results

I will present the results of this project in a variety of settings. Locally, I plan to present at the Triangle's annual Assocation for Women in Mathematics conference (established last year) and at UNC's Graduate Student Seminar. On a larger level, I plan to return to a national conference to present as well. Furthermore, results will be published in a relevant mathematical journal.

Broader Impacts

After graduating, I plan to become a professor. As described in my personal statement, a large focus of my career will be in mentoring undergraduates in research projects. I have previously mentored a high school student in a research project where she explored various non-Euclidean geometries. If awarded the NSF GRFP, I will continue building my mentoring capabilities by creating a project for UNC's Directed Reading Program. This program allows graduate students to mentor undergraduates through a semester-long reading project. My project would be based in Lie algebra, culminating in an understanding of the root systems associated to each $sl_n(\mathbb{C})$. I will also become involved in the McNair Scholars Program at UNC by helping with their research program over a summer. This will allow me the opportunity to help minority undergraduate researchers in a variety of fields (not just mathematics) by providing them with critical feedback at various stages in the research process.

References

[RR] R. Rimanyi. Motivic characteristic classes in cohomological Hall algebras (Preprint). Available at `http://rimanyi.web.unc.edu/research1/`, 2018.

[KS] M. Kontsevich and Y. Soibelman. Stability structures, motivic Donaldson-Thomas invariants and cluster transformations. Available at `https://arxiv.org/abs/0811.2435`, 2008.

[MO] D. Maulik and A. Okounkov. Quantum Groups and Quantum Cohomology. Available at `https://arxiv.org/abs/1211.1287`, 2018.