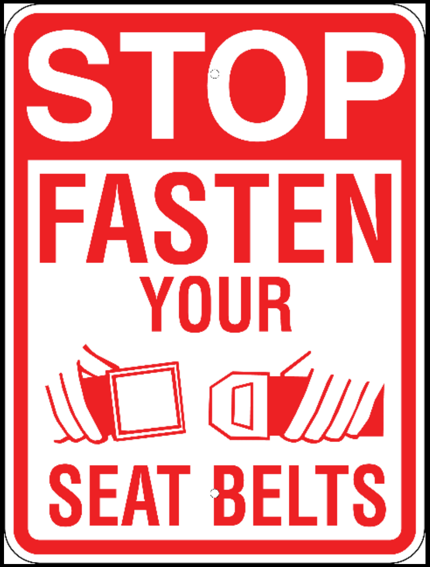
Introducing the Laplacian Operator

Justin Solomon

MIT, Spring 2017


WARNING
SIGN
MISTAKES
LIKELY



Lots of (sloppy) math!

http://www.wgnflag.com/xcart/images/P/G-50_StopSeatBeltsStockSign.jpg

Famous Motivation

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

> "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

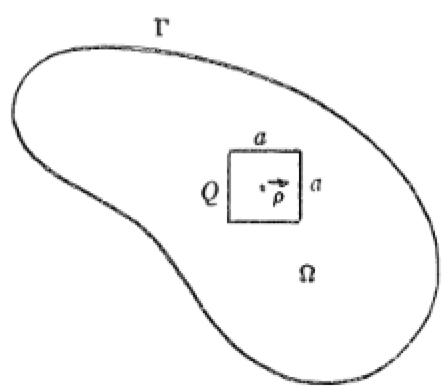
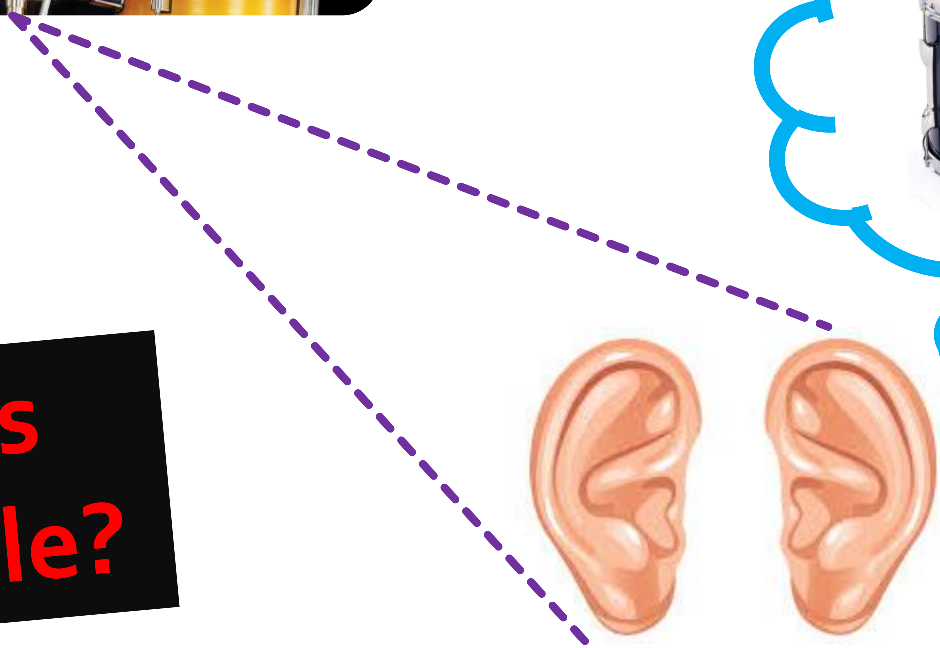


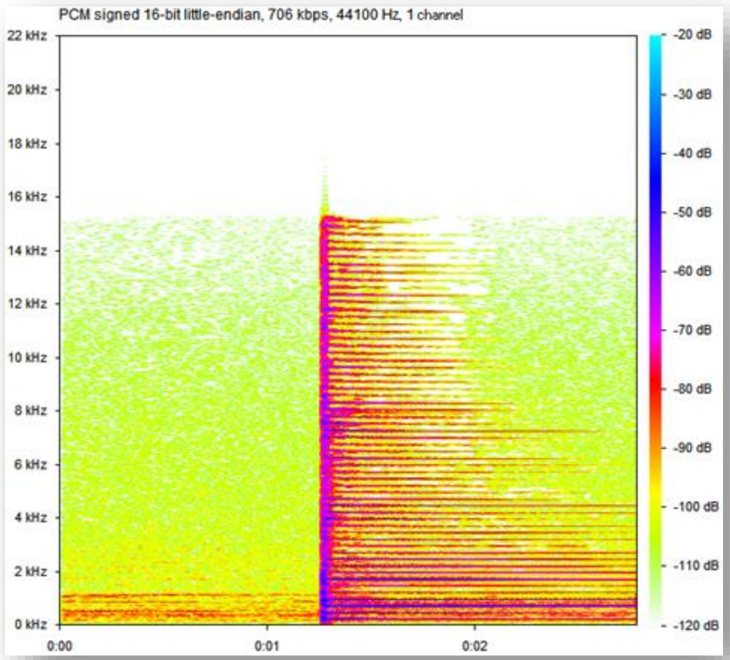
FIG. 1

An Experiment



Is this possible?

Unreasonable to Ask?

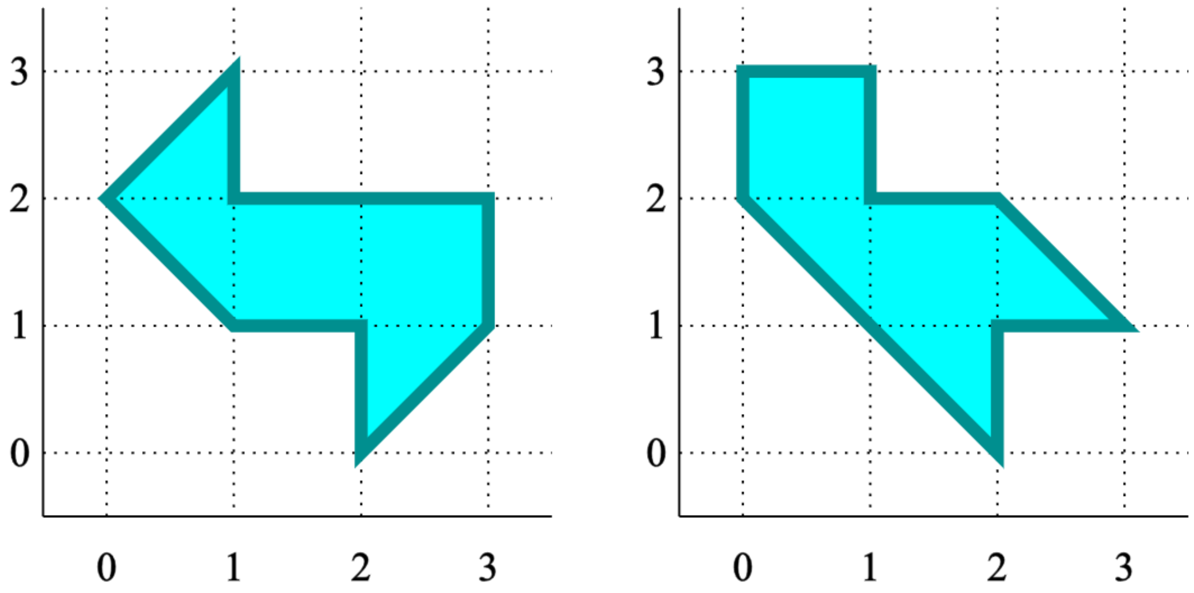


Length of string

http://www.takamine.com/templates/default/images/gclassical.png
http://firsttimeprogrammer.blogspot.com/2015/05/spectrogram-representation-of-b-note.html

Spoiler Alert

Extra credit: Make these!



"No, but…"

- Has to be a weird drum
- Spectrum tells you a lot!

https://en.wikipedia.org/wiki/Hearing_the_shape_of_a_drum

Rough Intuition

http://pngimg.com/upload/hammer_PNG3886.png

You can learn a lot about a shape by **hitting it** (lightly) **with a hammer**!

Spectral Geometry

What can you learn about its shape from vibration frequencies and oscillation patterns?

$$\Delta f = \lambda f$$

Objectives

- Make "vibration modes" more precise
- Progressively more complicated domains
  - Line segments
  - Regions in $\mathbb{R}^2$
  - Graphs
  - Surfaces/manifolds
- Next time: Discretization, applications

Review:

Vector Spaces and Linear Operators

$$\mathcal{L}[\vec{x} + \vec{y}] = \mathcal{L}[\vec{x}] + \mathcal{L}[\vec{y}]$$

$$\mathcal{L}[c\vec{x}] = c\mathcal{L}[\vec{x}]$$

Review:

In Finite Dimensions

$$\underbrace{A}_{\text{matrix}}\ \underbrace{\vec{x}}_{\text{vector}}$$

$$\underbrace{\vec{x} \mapsto A\vec{x}}_{\text{linear operator}}$$

Recall: Spectral Theorems in $\mathbb{C}^n$

Theorem. Suppose $A \in \mathbb{C}^{n\times n}$ is Hermitian. Then, A has an orthogonal basis of n eigenvectors. If A is positive definite, the corresponding eigenvalues are nonnegative.

Our Progression

- Line segments
- Regions in $\mathbb{R}^2$
- Graphs
- Surfaces/manifolds

Minus Second Derivative Operator

"Dirichlet boundary conditions"

$$\{f(\cdot) \in C^\infty([a,b]) : f(0) = f(\ell) = 0\}$$

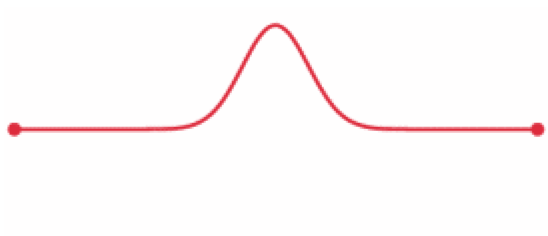
$$\mathcal{L}[f(\cdot)] := -f''(\cdot)$$

Eigenfunctions:

$$f_k(x) = \sin\left(\frac{\pi k x}{\ell}\right), \quad \lambda_k = \left(\frac{\pi k}{\ell}\right)^2$$

Physical Intuition: Wave Equation

$$\frac{\partial^2 u}{\partial t^2} \overbrace{- \frac{\partial^2 u}{\partial x^2}}^{\text{Minus second derivative operator!}} = 0$$



Observation

$$\{f(\cdot) \in C^\infty([a,b]) : f(0) = f(\ell) = 0\}$$

$$\begin{aligned}
\langle f, \mathcal{L}[f] \rangle &= -\int_0^\ell f(x) f''(x)\, dx \\
&= -[f(x) f'(x)]_0^\ell + \underbrace{\int_0^\ell f'(x)^2\, dx}_{} \\
&\geq 0
\end{aligned}$$

Hilbert-Schmidt Theorem

Theorem. **Let $H \neq 0$ be an infinite-dimensional, separable Hilbert space and let $K \in L(H)$ be compact and self-adjoint. Then, there exists a countable orthonormal basis of H consisting of eigenvectors of K.**

Hilbert space: Space with inner product
Separable: Admits countable, dense subset
Compact operator: Bounded sets to relatively compact sets
Self-adjoint: $\langle Kv, w\rangle = \langle v, Kw\rangle$

https://www.math.ku.edu/~matjohn/Teaching/S16/Math951/HilbertSchmidt.pdf

Can you hear the length of an interval?

$$\lambda_k = \left(\frac{\pi k}{\ell}\right)^2$$

Yes!

Homework (?)

for a curve $\gamma(u) = (x[u], y[u]) : \mathbb{R} \to \mathbb{R}^2$

- $\Delta\gamma = (\Delta x, \Delta y)$ is gradient of arc length
- $\Delta\gamma$ is the *curvature normal* $\kappa\hat{n}$
- minimal curves are harmonic (straight lines)

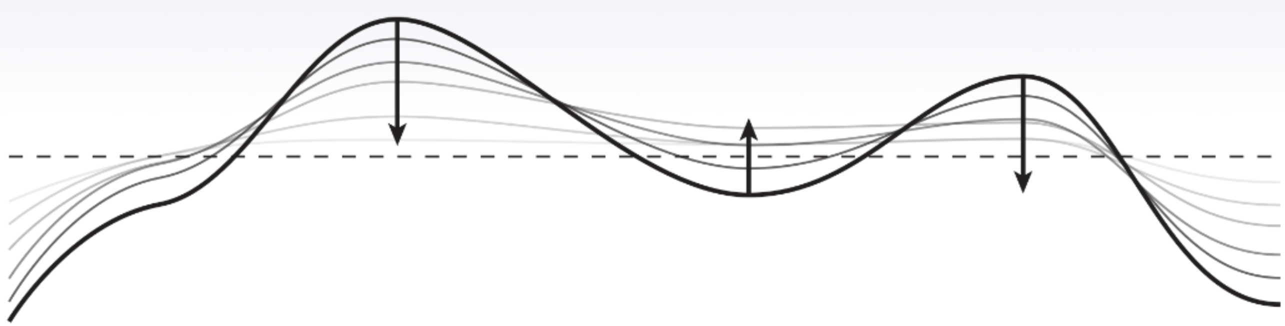
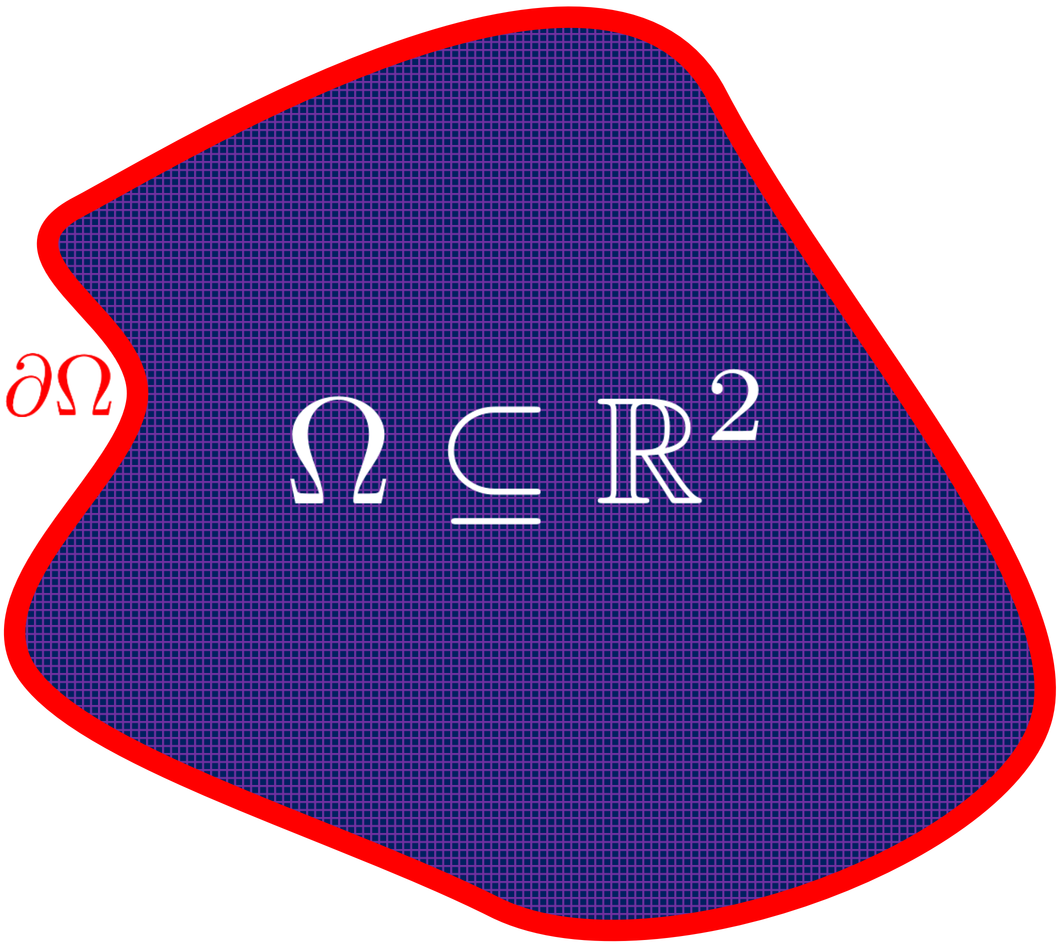


Image by E. Vouga

Our Progression

- Line segments
- Regions in $\mathbb{R}^2$
- Graphs
- Surfaces/manifolds

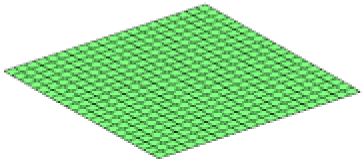
Planar Region



Wave equation:

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

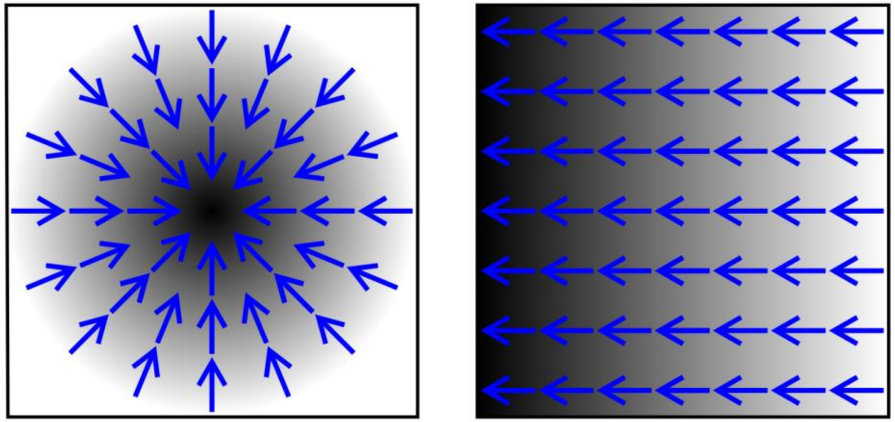
$$\Delta := \sum_i \frac{\partial^2}{\partial x_i^2}$$



Typical Notation

$$\text{``}\Delta = \underbrace{\nabla \cdot}_{\text{divergence}} \underbrace{\nabla}_{\text{gradient}}\text{''}$$

More later...



Gradient operator:

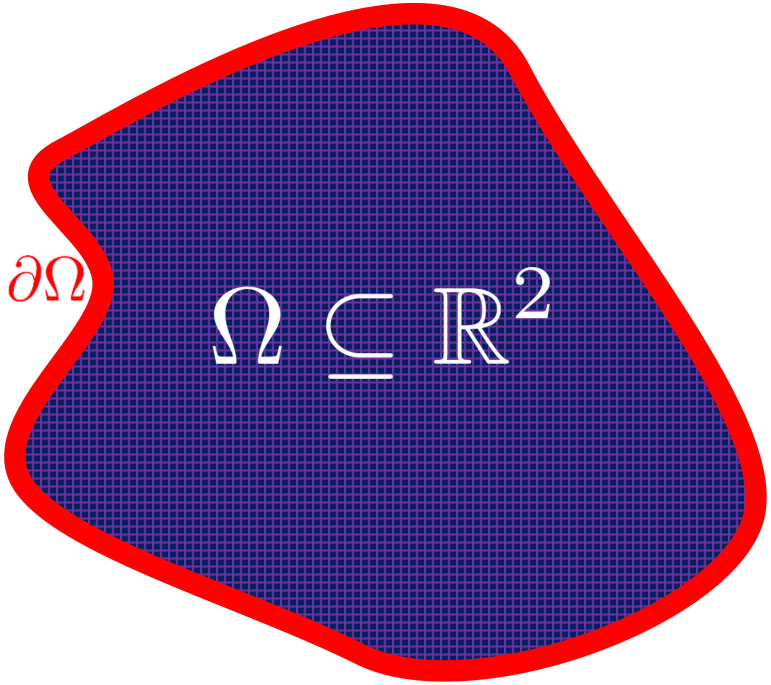
$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right)$$

http://www.gamasutra.com/db_area/images/feature/4164/figy.png, https://en.wikipedia.org/wiki/Gradient

Positivity, Self-Adjointness

$$\{f(\cdot) \in C^\infty(\Omega) : f|_{\partial\Omega} \equiv 0\}$$

"Dirichlet boundary conditions"



$$\mathcal{L}[f] := -\Delta f$$

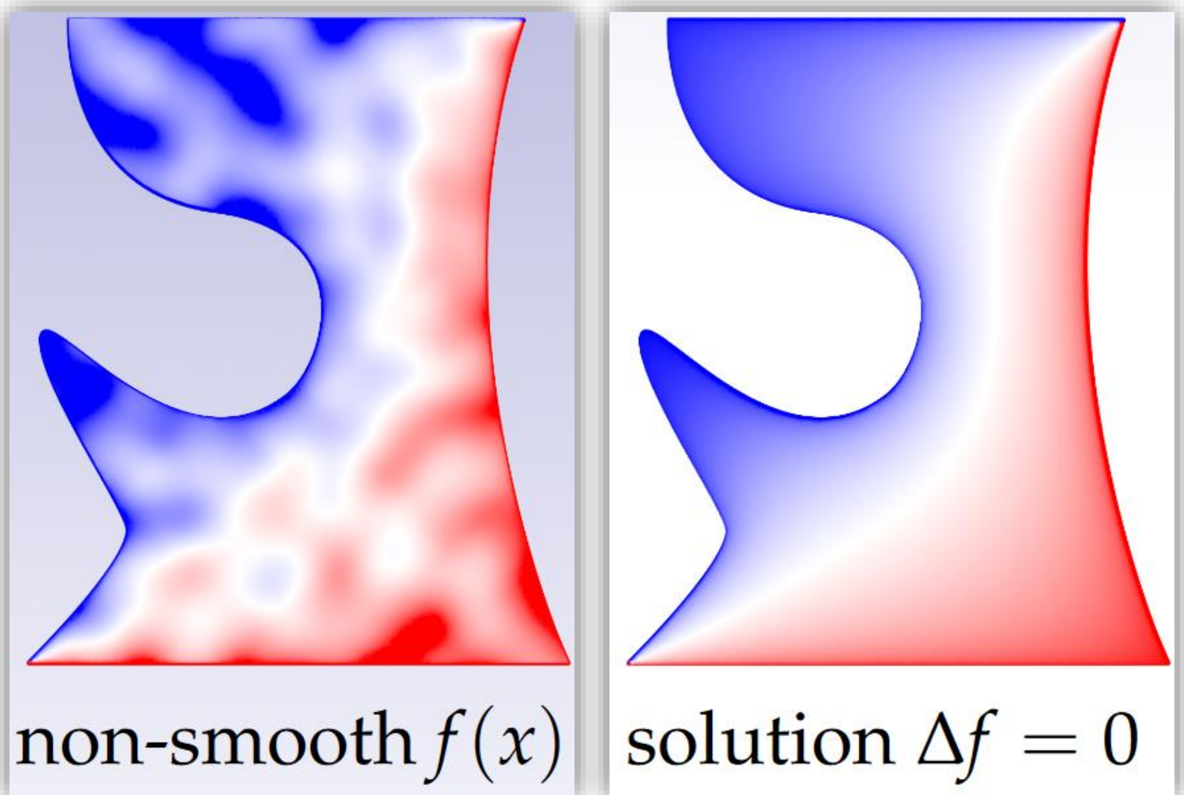
$$\langle f, g \rangle := \int_\Omega f(x) g(x)\, dx$$

On board:

1. **Positive:** $\langle f, \mathcal{L}[f] \rangle \geq 0$
2. **Self-adjoint:** $\langle f, \mathcal{L}[g] \rangle = \langle \mathcal{L}[f], g \rangle$

Dirichlet Energy

$$E[f] := \int_\Omega \langle \nabla f, \nabla f \rangle \, dA$$



Images made by E. Vouga

On board:

$$\min_f E[f]$$

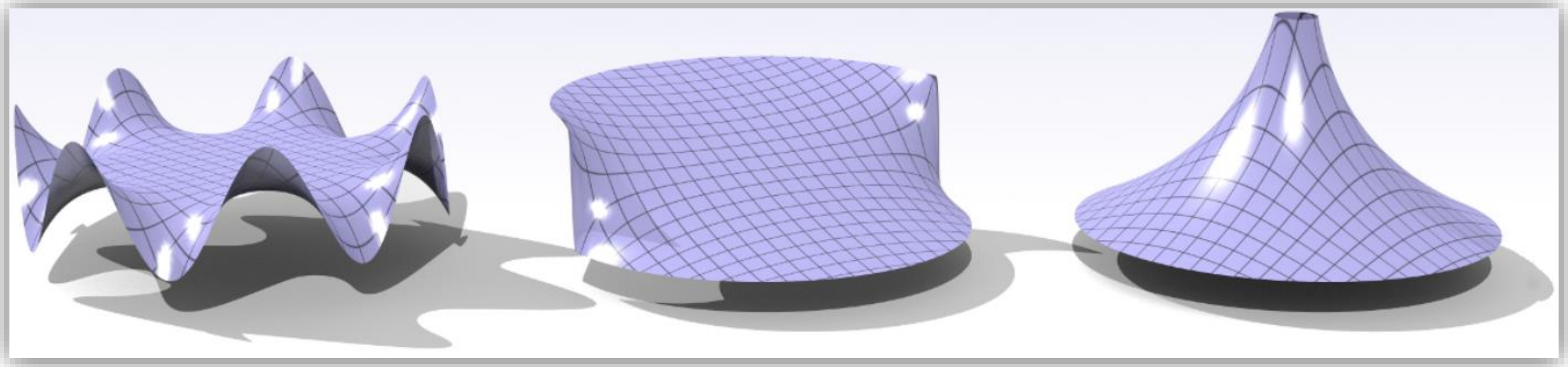
$$\text{s.t. } f|_{\partial\Omega} = g$$

$$\Delta f \equiv 0$$

"Laplace equation"
"Harmonic function"

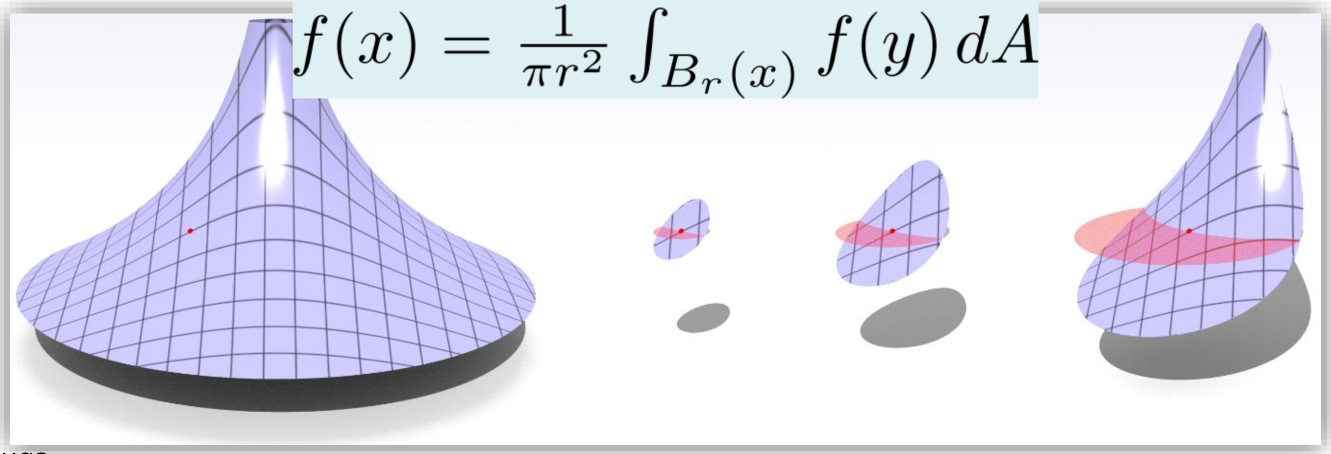
Harmonic Functions

$$\Delta f \equiv 0$$



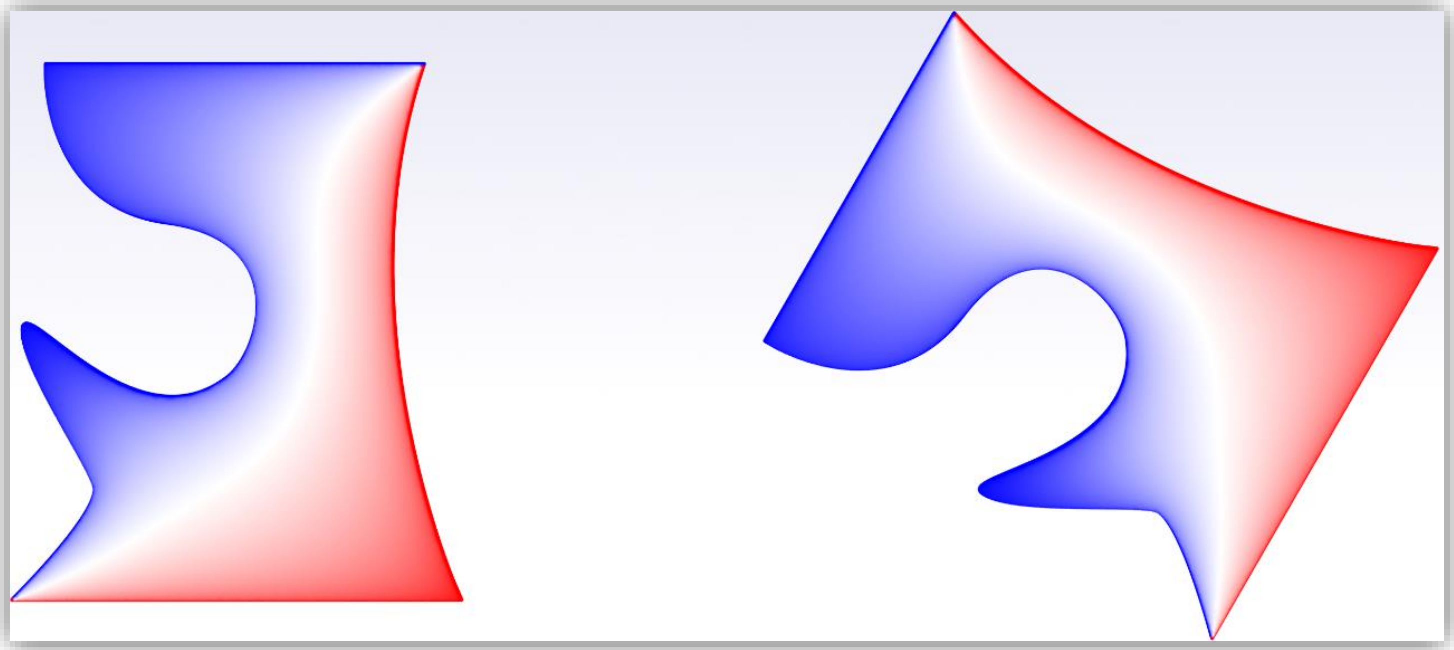
Mean value property:

$$f(x) = \frac{1}{\pi r^2} \int_{B_r(x)} f(y)\, dA$$



Images made by E. Vouga

Intrinsic Operator



Images made by E. Vouga

Coordinate-independent (important!)

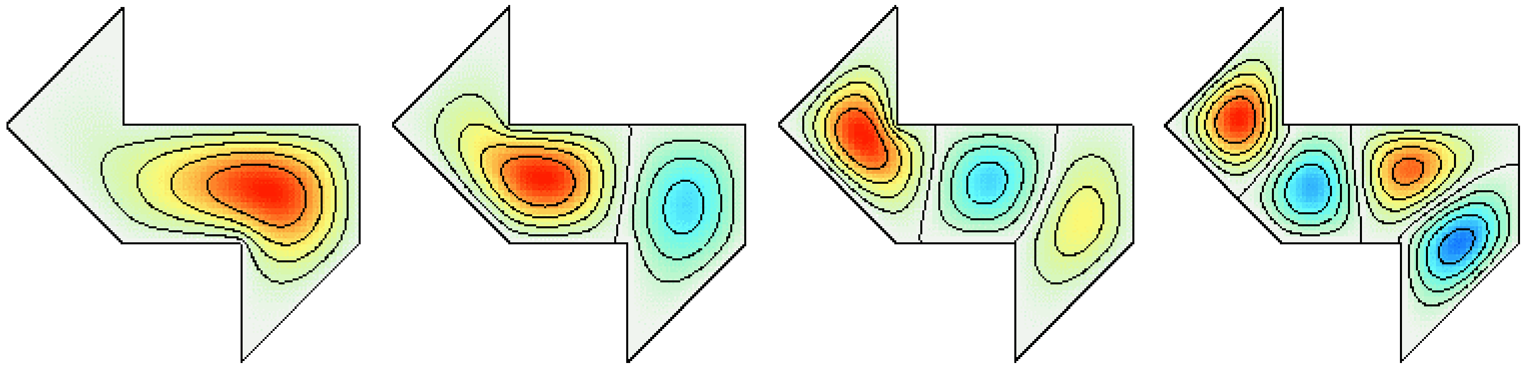
Aside:

Common Misconception

$$\min_f \; E[f] \text{ s.t. } f(p) = \text{const.}$$

Point constraints are ill-advised

Another Interpretation of Eigenfunctions



$$\text{Find critical points of } E[f]$$

$$\text{s.t.} \int_\Omega f^2 = 1$$

http://www.math.udel.edu/~driscoll/research/gww1-4.gif

Small eigenvalue: Small Dirichlet Energy

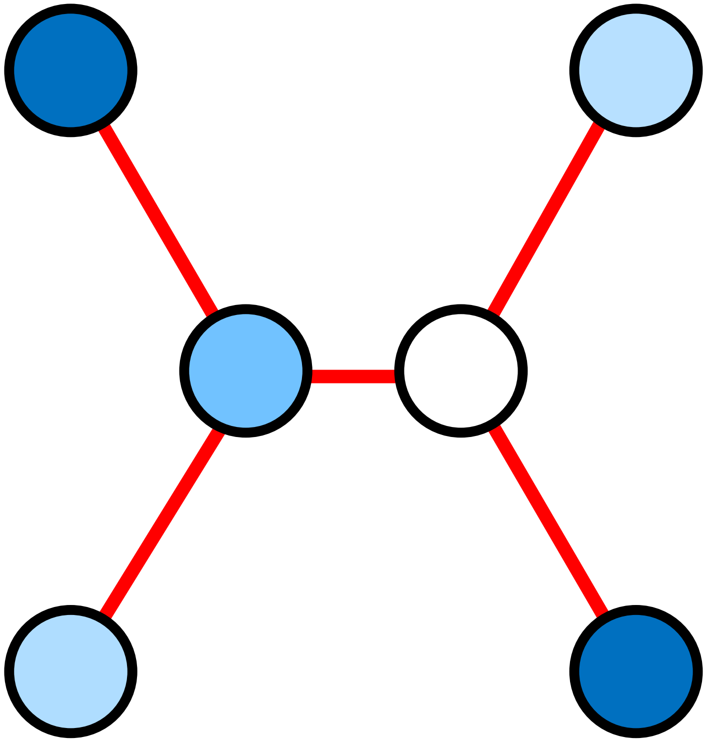
Our Progression

- Line segments
- Regions in $\mathbb{R}^2$
- Graphs
- Surfaces/manifolds

Basic Setup

- **Function:**

  One value per vertex



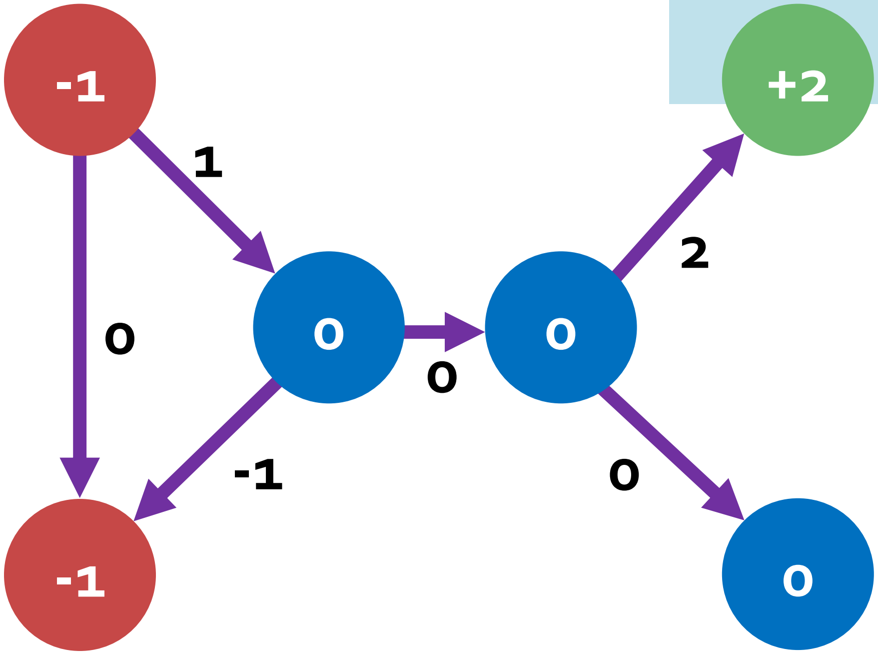
Q:

What is the **Dirichlet energy** of a function on a graph?

Differencing Operator

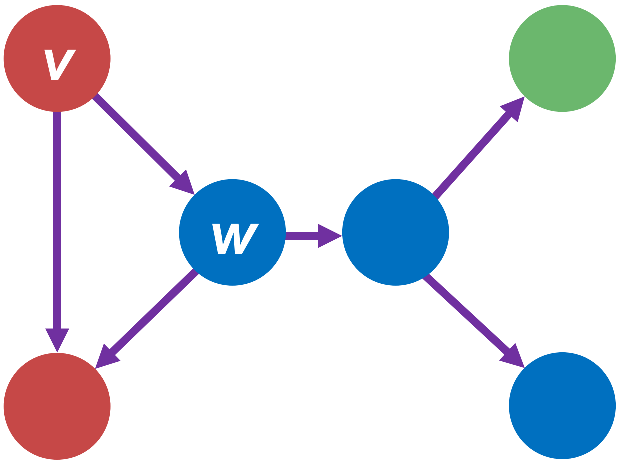
$$D_{ev} := \begin{cases} -1 & \text{if } E_{e1} = v \\ 1 & \text{if } E_{e2} = v \\ 0 & \text{otherwise} \end{cases}$$

$$D \in \{-1, 0, 1\}^{|E| \times |V|}$$



Orient edges arbitrarily

Dirichlet Energy on a Graph



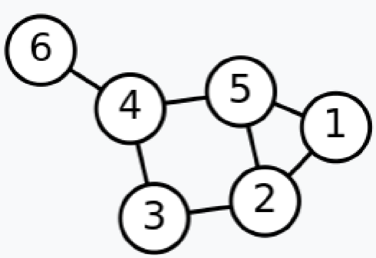
$$D_{ev} := \begin{cases} -1 & \text{if } E_{e1} = v \\ 1 & \text{if } E_{e2} = v \\ 0 & \text{otherwise} \end{cases}$$

$$E[f] := \|Df\|_2^2 = \sum_{(v,w)\in E} (f_v - f_w)^2$$

(Unweighted) Graph Laplacian

$$E[f] = \|Df\|_2^2 = f^\top (D^\top D) f := f^\top L f$$

$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

Labeled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2&0&0&0&0&0 \\ 0&3&0&0&0&0 \\ 0&0&2&0&0&0 \\ 0&0&0&3&0&0 \\ 0&0&0&0&3&0 \\ 0&0&0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0&1&0 \\ 1&0&1&0&1&0 \\ 0&1&0&1&0&0 \\ 0&0&1&0&1&1 \\ 1&1&0&1&0&0 \\ 0&0&0&1&0&0 \end{pmatrix}$	$\begin{pmatrix} 2&-1&0&0&-1&0 \\ -1&3&-1&0&-1&0 \\ 0&-1&2&-1&0&0 \\ 0&0&-1&3&-1&-1 \\ -1&-1&0&-1&3&0 \\ 0&0&0&-1&0&1 \end{pmatrix}$

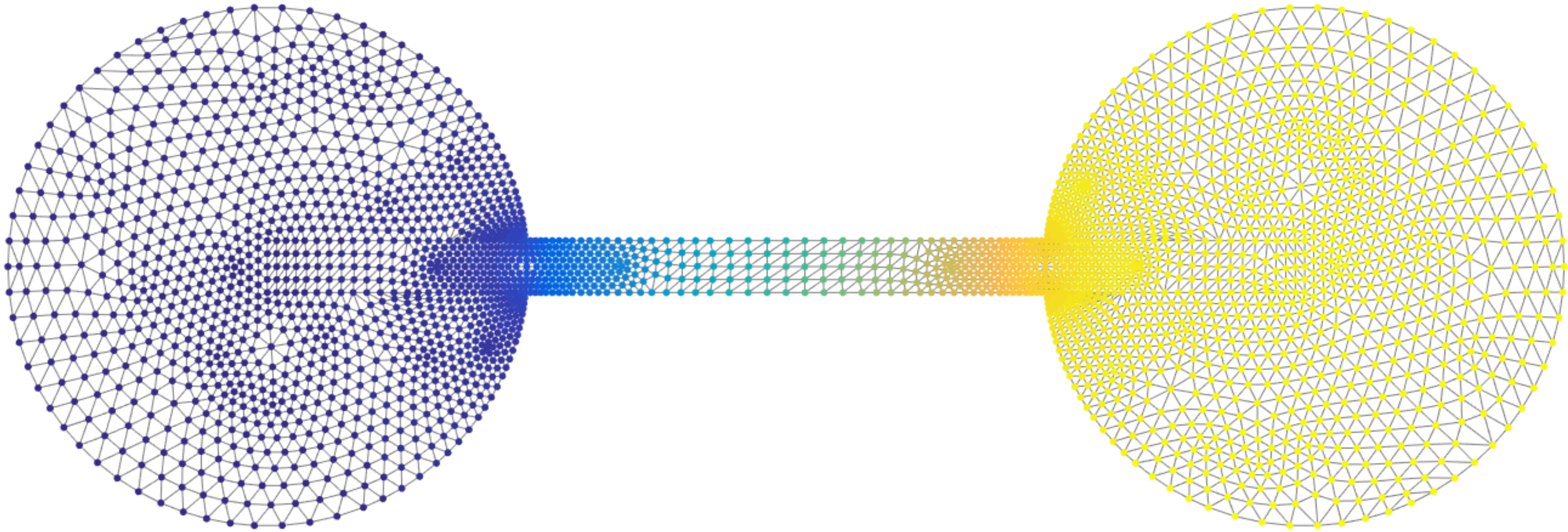
- Symmetric
- Positive definite

https://en.wikipedia.org/wiki/Laplacian_matrix

Q:

What is the
smallest eigenvalue
of the graph Laplacian?

Second-Smallest Eigenvector



$$Lx = \lambda x$$

Used for graph partitioning

Fiedler vector ("algebraic connectivity")

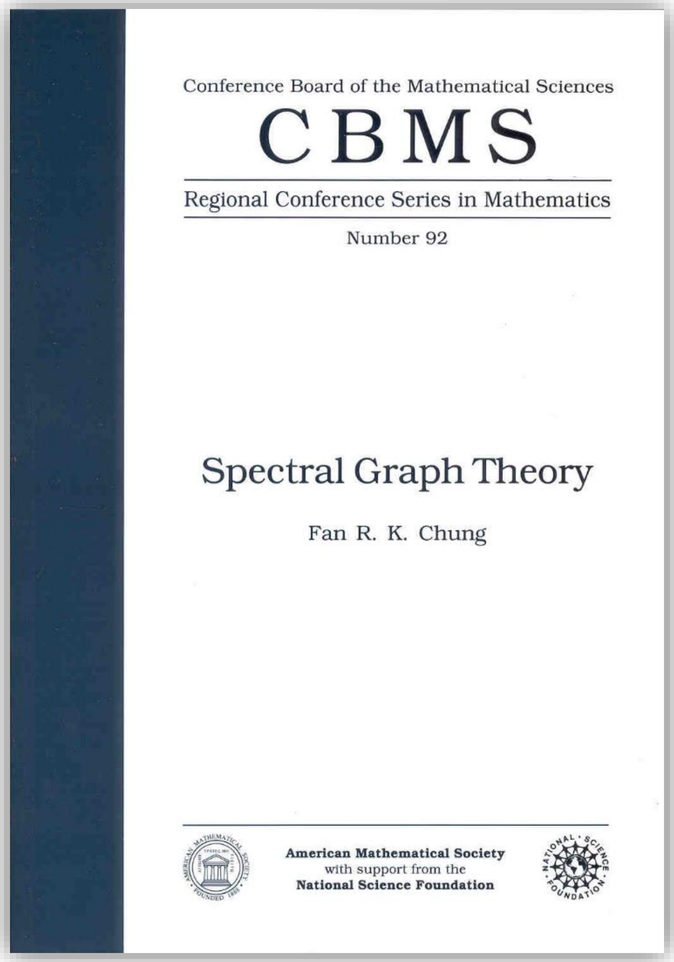
Mean Value Property

$$L_{vw} = A - D = \begin{cases} 1 & \text{if } v \sim w \\ -\text{degree}(v) & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$(Lx)_v = 0$$

Value at *v* is average of neighboring values

For More Information…



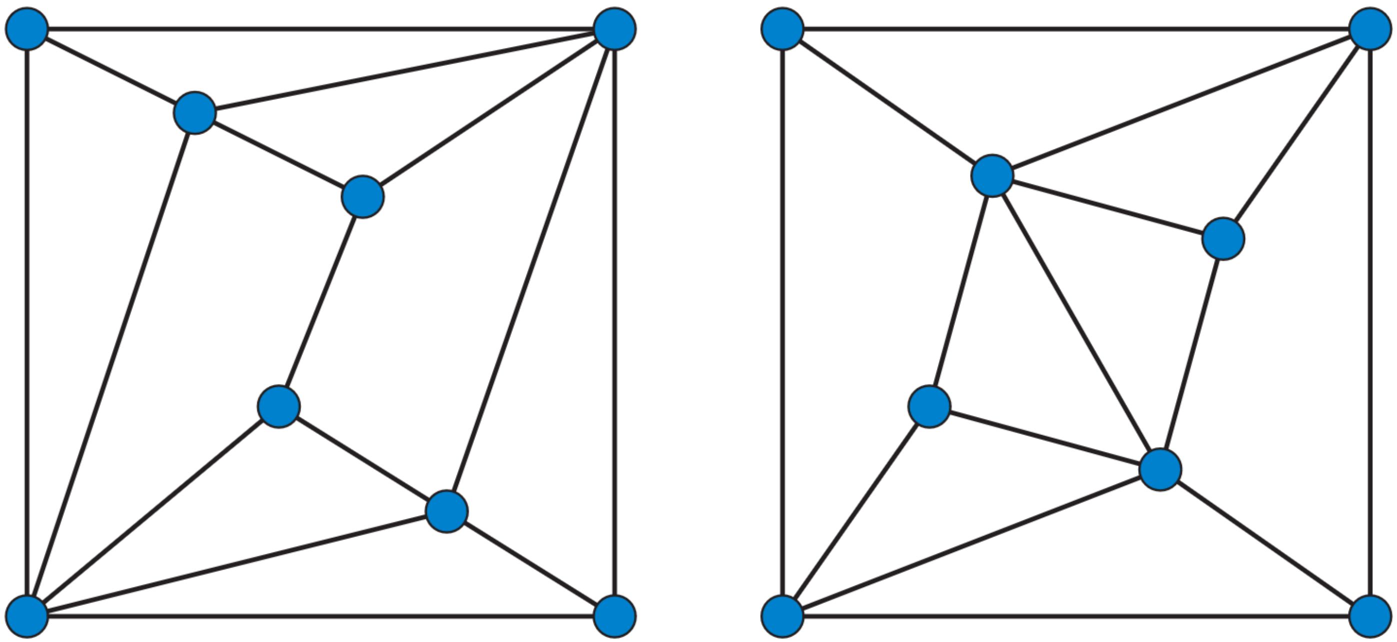
Graph Laplacian encodes lots of information!

Example: Kirchoff's Theorem

Number of spanning trees equals

$$\frac{1}{n}\lambda_2\lambda_3\cdots\lambda_n$$

Hear the Shape of a Graph?



“Enneahedra”

No!

Our Progression

- Line segments
- Regions in $\mathbb{R}^2$
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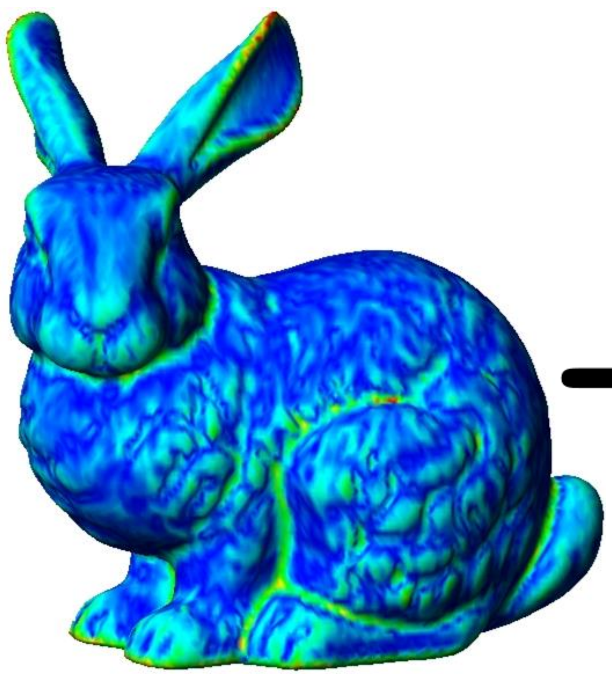
- Surfaces/manifolds

Recall:

Scalar Functions

$f:$  $\longrightarrow \mathbb{R}$

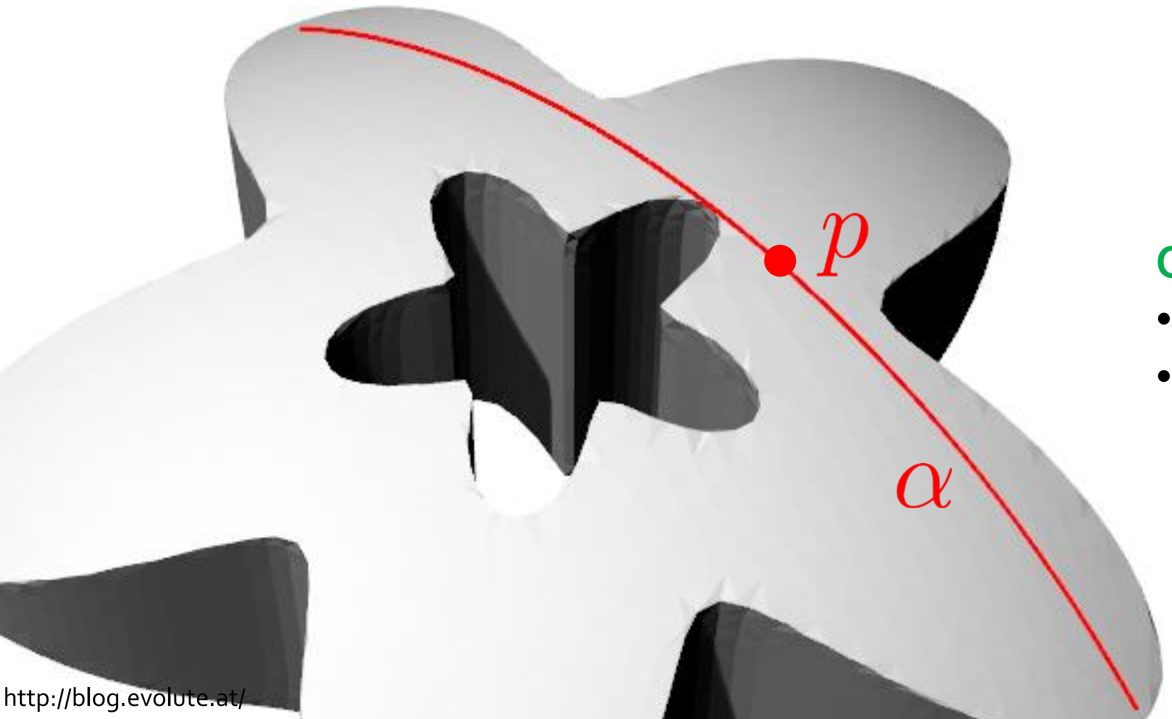
http://www.ieeta.pt/polymeco/Screenshots/PolyMeCo_OneView.jpg

Map points to real numbers

Differential of a Map

Suppose $f: S \to \mathbb{R}$ and take $p \in S$. For $v \in T_pS$, choose a curve $\alpha: (-\varepsilon, \varepsilon) \to S$ with $\alpha(0) = p$ and $\alpha'(0) = v$. Then the differential of f is $df: T_pS \to \mathbb{R}$ with

$$(df)_p(v) := \left.\frac{d}{dt}\right|_{t=0} (f \circ \alpha)(t) = (f \circ \alpha)'(0).$$



On the board (time-permitting):

- Does not depend on choice of α
- Linear map

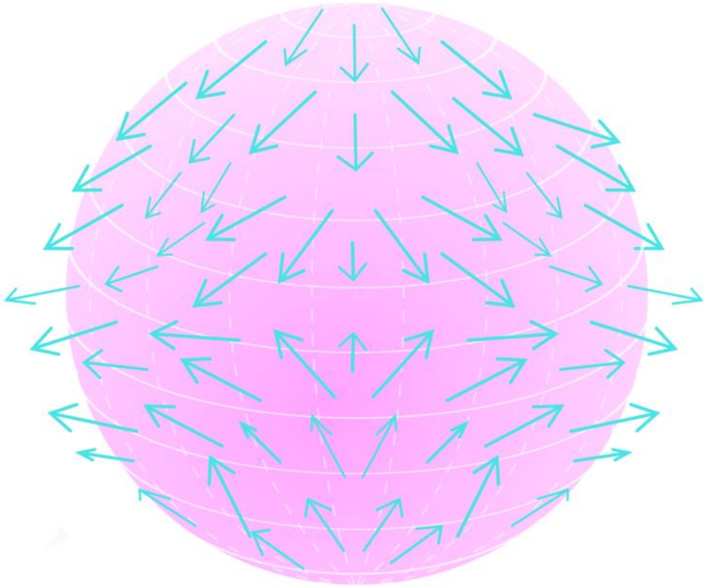
http://blog.evolute.at/

Following *Curves and Surfaces*, Montiel & Ros

Gradient Vector Field

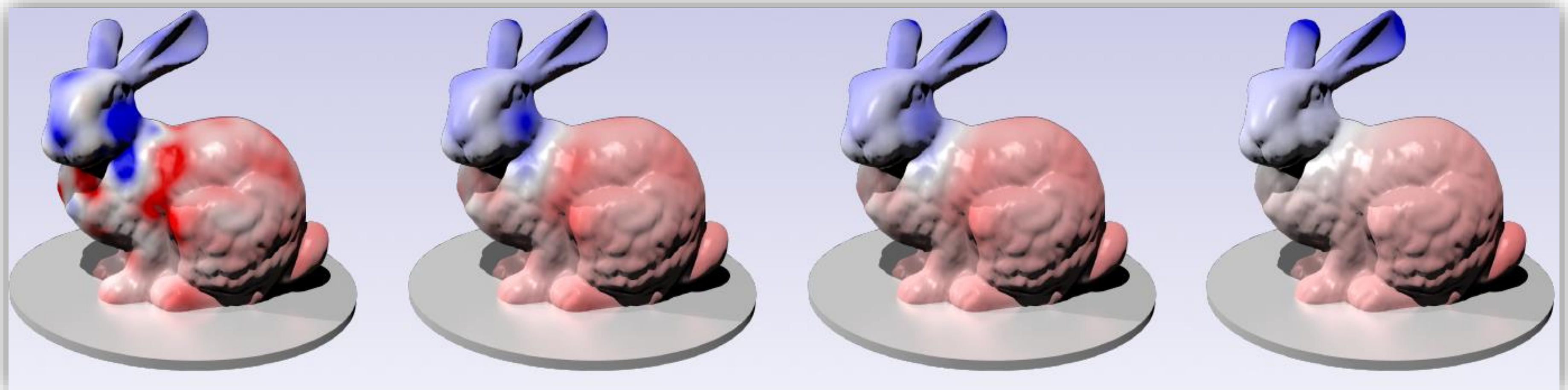
$$\nabla f : S \to \mathbb{R}^3 \text{ with}$$

$$\begin{cases} \langle (\nabla f)(p), v \rangle = (df)_p(v), v \in T_pS \\ \langle (\nabla f)(p), N(p) \rangle = 0 \end{cases}$$



Following *Curves and Surfaces*, Montiel & Ros

Dirichlet Energy



Decreasing *E*

$$E[f] := \int_S \|\nabla f\|_2^2 \, dA$$

Images made by E. Vouga

From Inner Product to Operator

$$\langle f, g \rangle_\Delta := \int_S \nabla f(x) \cdot \nabla g(x)\, dA$$
$$:= \langle f, \Delta g \rangle$$

Implies
$\langle f, f \rangle \geq 0$

On the board:
"Motivation" from finite-dimensional linear algebra.

Laplace-Beltrami operator

What is Divergence?

$$V : S \to \mathbb{R}^3 \text{ where } V(p) \in T_pS$$
$$dV_p : T_pS \to \mathbb{R}^3$$
$$\{e_1, e_2\} \subset T_pS \text{ orthonormal basis}$$

$$(\nabla \cdot V)_p := \sum_{i=1}^{2} \langle e_i, dV(e_i) \rangle_p$$

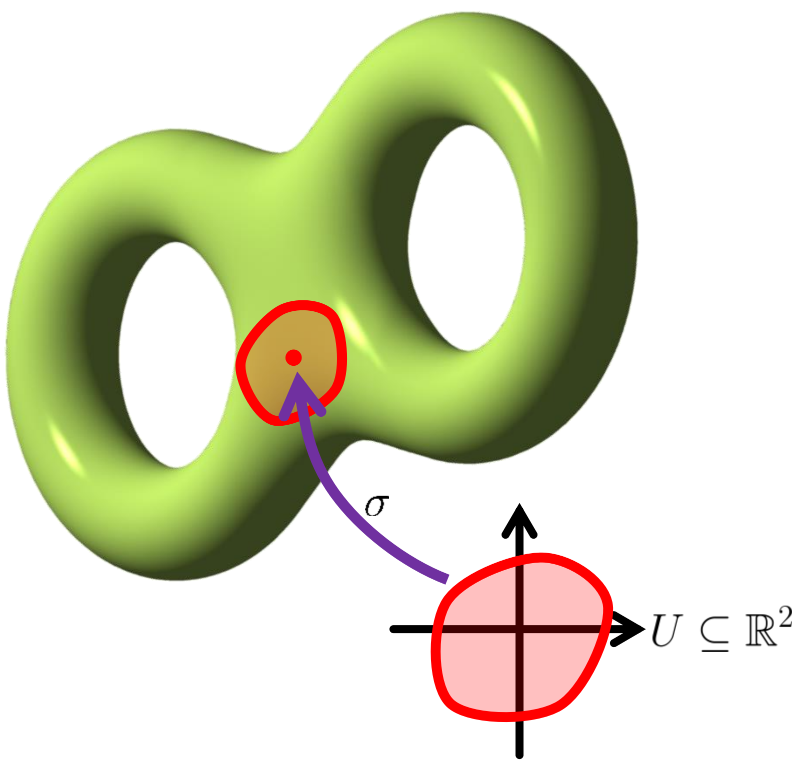
Things we should check (but probably won't):

- **Independent of choice of basis**
- $\boldsymbol{\Delta = \nabla \cdot \nabla}$

Sanity Check: Local Version

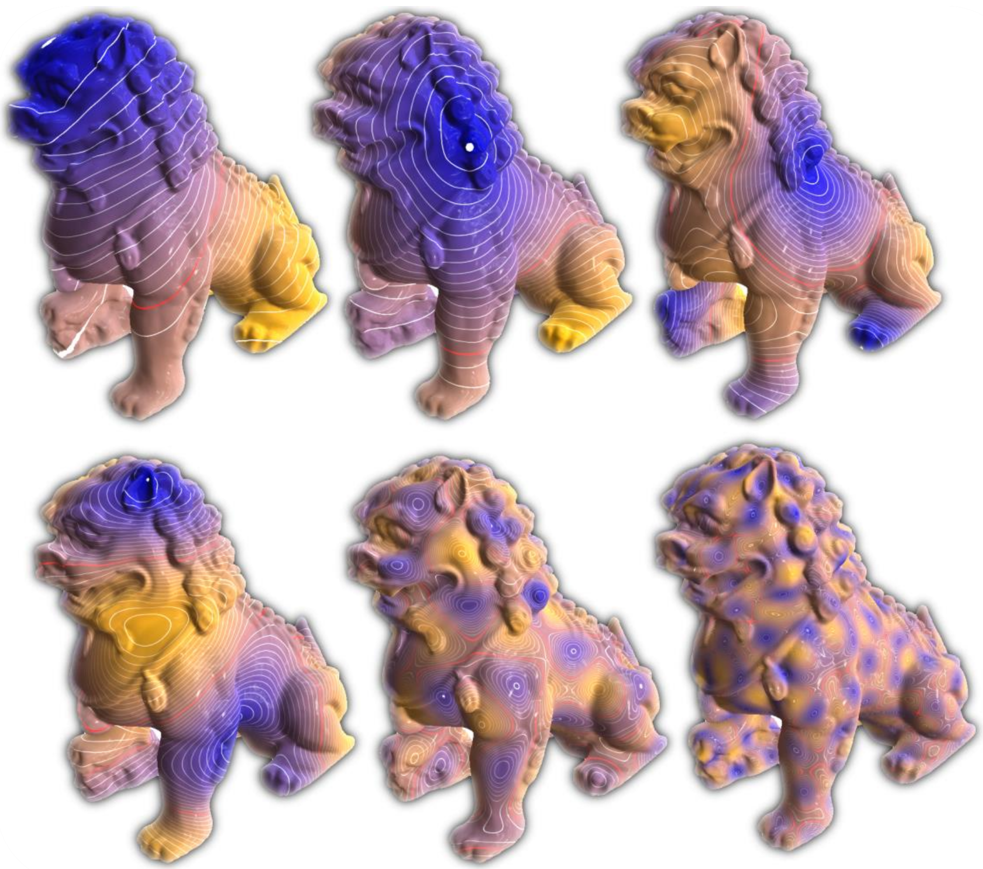
$$f : S \to \mathbb{R}$$

$$\text{Pullback: } \sigma^* f := f \circ \sigma : U \to \mathbb{R}$$



Laplace-Beltrami coincides with Laplacian on $\mathbb{R}^2$ when σ takes x, y axes to orthonormal vectors.

Eigenfunctions



$$\Delta\psi_i = \lambda_i\psi_i$$

Vibration modes of surface (not volume!)

http://alice.loria.fr/publications/papers/2008/ManifoldHarmonics//photo/dragon_mhb.png

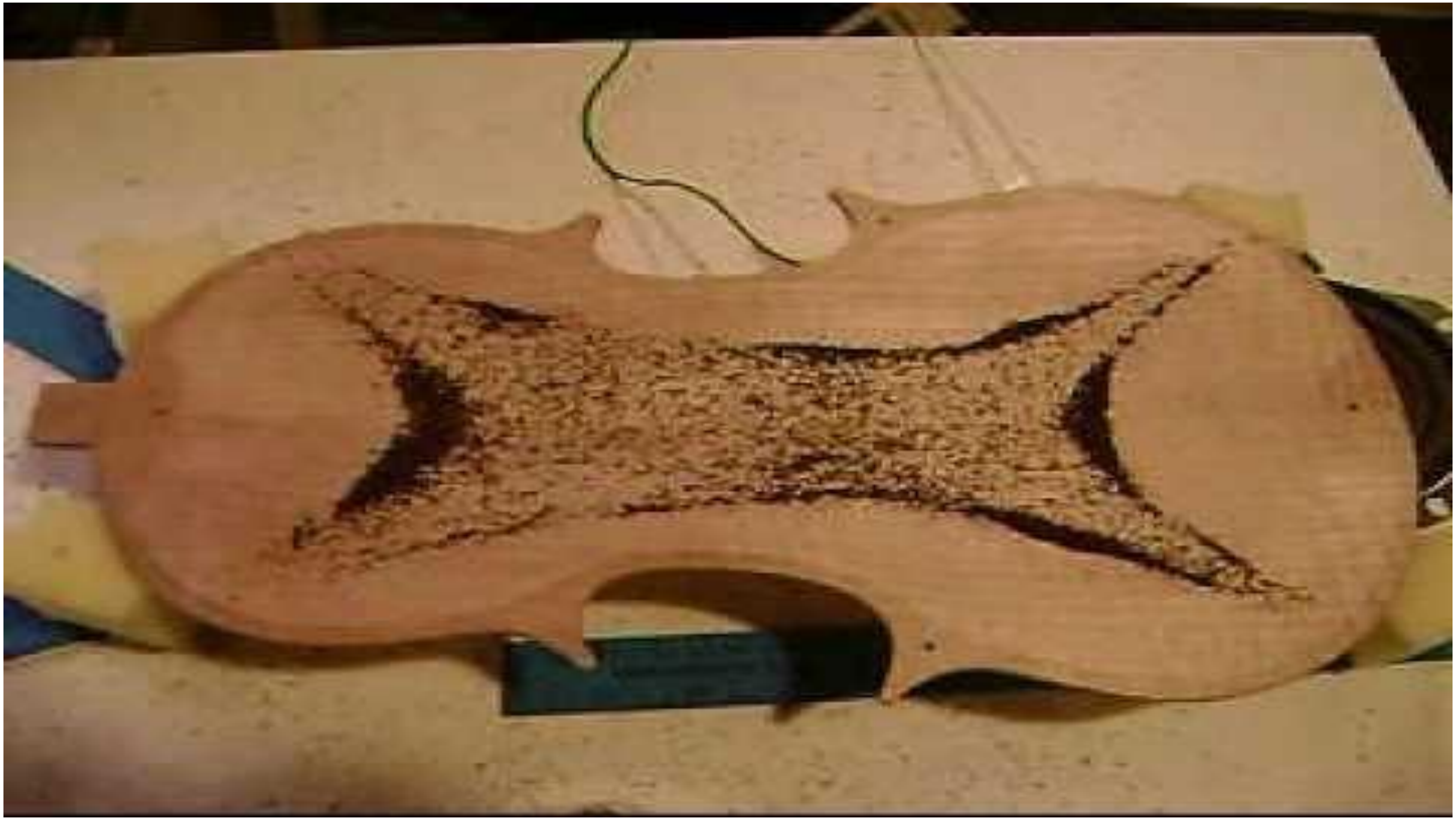
Chladni Plates

https://www.youtube.com/watch?v=CGiiSlMFFlI

Performance Art?

https://www.youtube.com/watch?v=Fyzqd2_T09Q

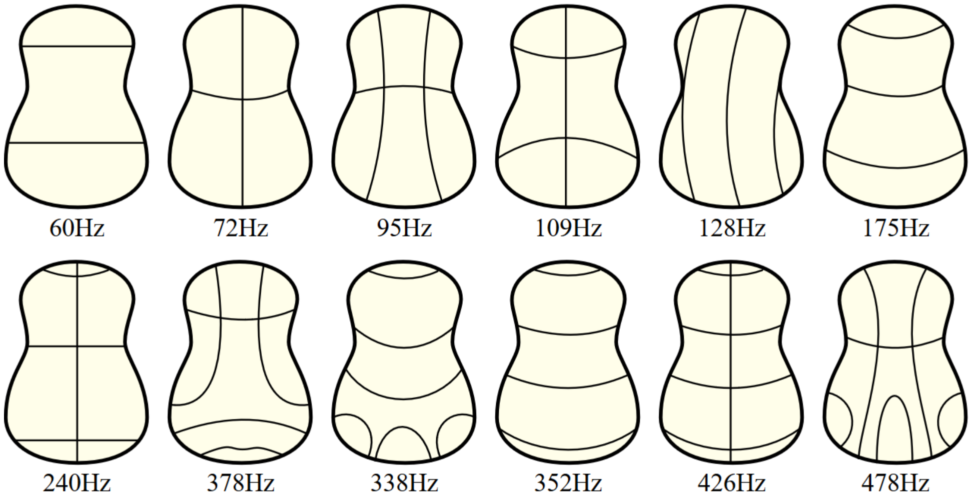
Practical Application



https://www.youtube.com/watch?v=3uMZzVvnSiU

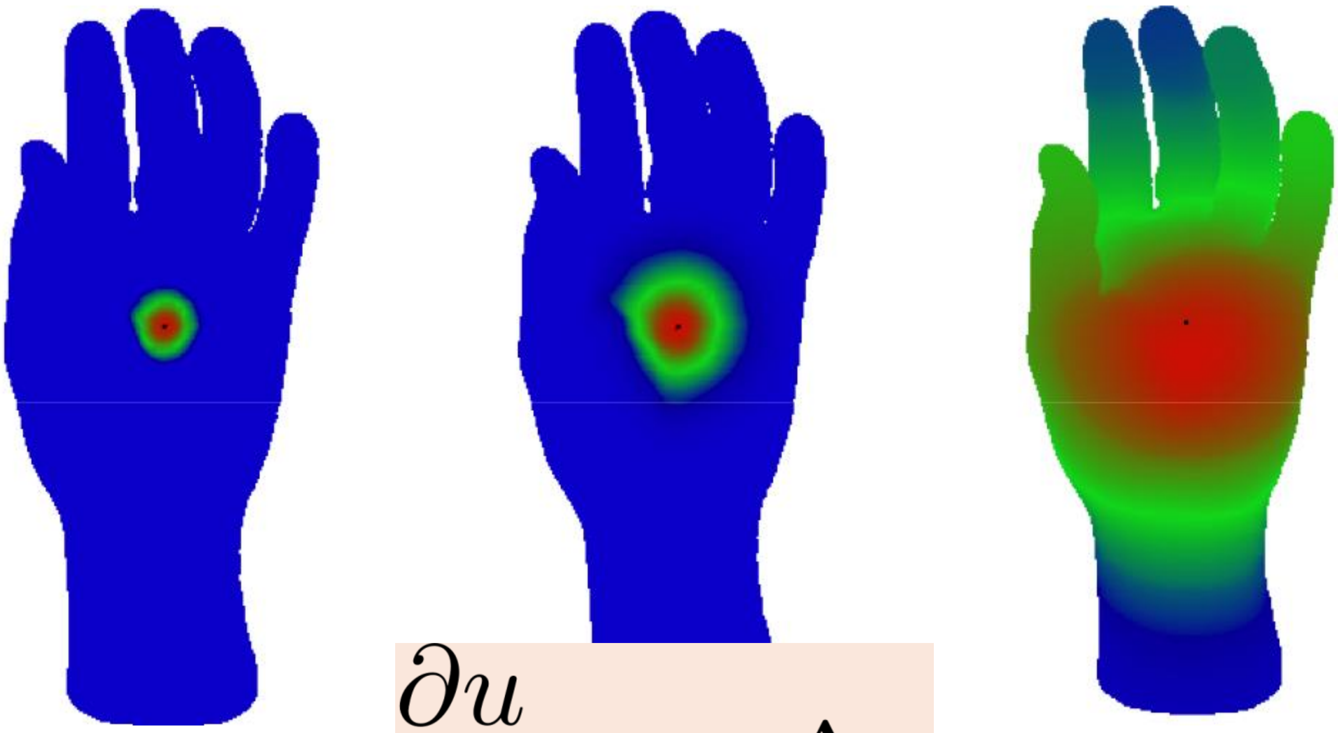
Nodal Domains

Theorem (Courant). **The *n*-th eigenfunction of the Dirichlet boundary value problem has at most *n* nodal domains.**



https://en.wikipedia.org/wiki/Ernst_Chladni

Additional Connection to Physics

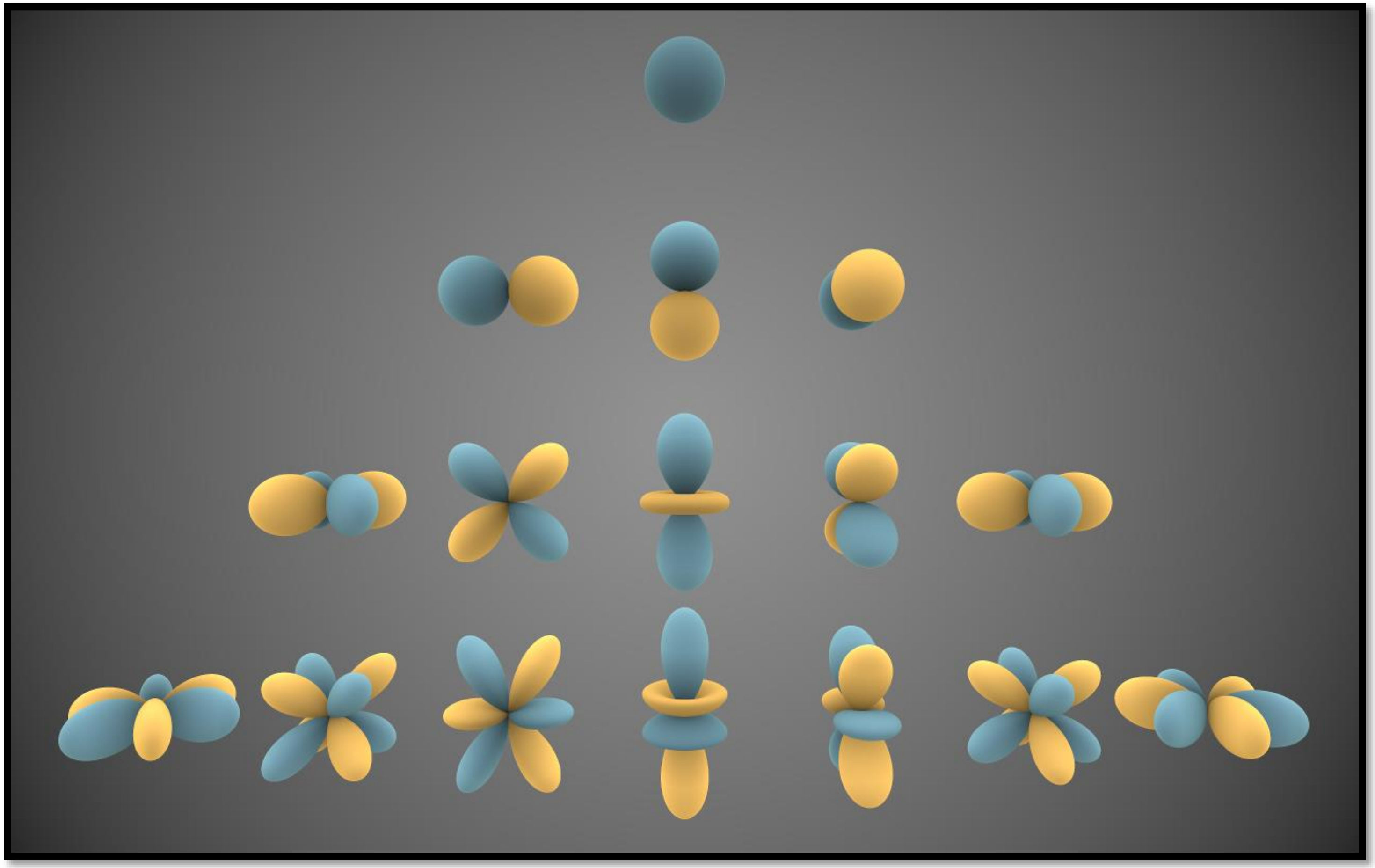


$$\frac{\partial u}{\partial t} = -\Delta u$$

http://graphics.stanford.edu/courses/cs468-10-fall/LectureSlides/11_shape_matching.pdf

Heat equation

Spherical Harmonics



https://en.wikipedia.org/wiki/Spherical_harmonics

Weyl's Law

$$N(\lambda) := \#\ \text{eigenfunctions} \leq \lambda$$

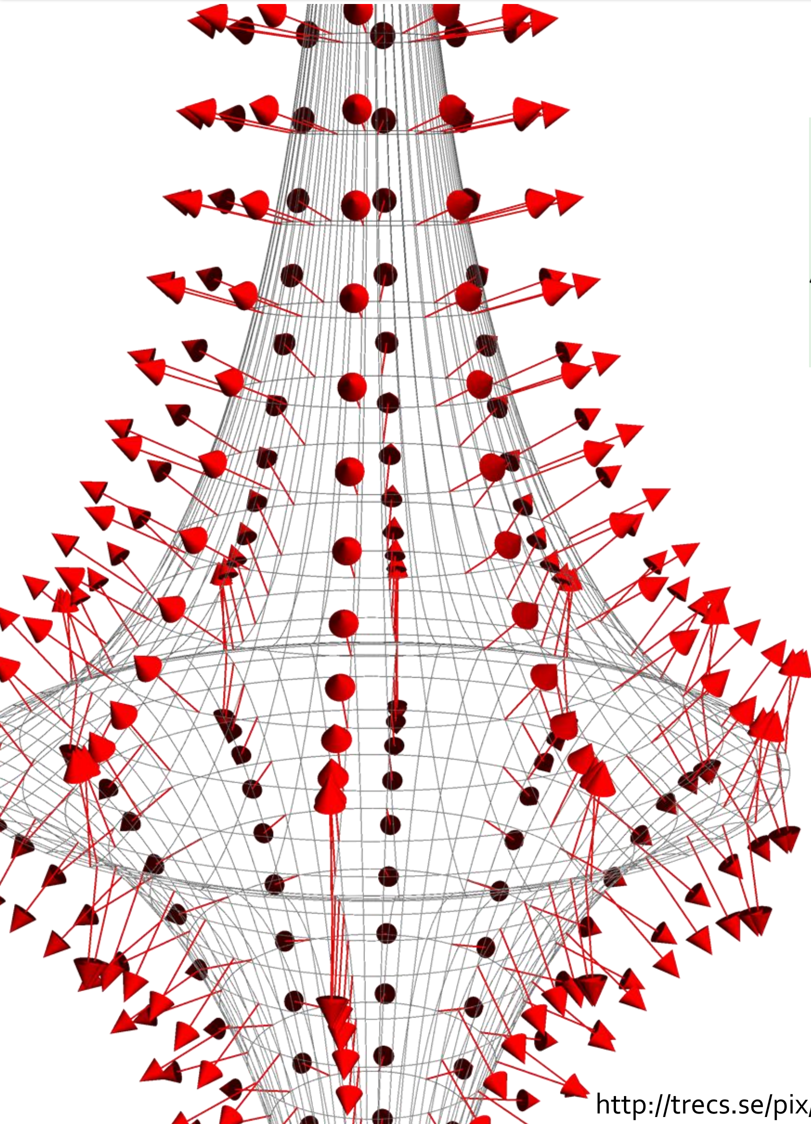
$$\omega_d := \text{volume of unit ball in } \mathbb{R}^d$$

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{d/2}} = (2\pi)^{-d} \omega_d \mathrm{vol}(\Omega)$$

Corollary: $\mathrm{vol}(\Omega) = (2\pi)^d \lim_{R \to \infty} \frac{N(R)}{R^{d/2}}$

For surfaces: $\lambda_n \sim \dfrac{4\pi}{\mathrm{vol}(\Omega)} n$

Laplacian of *xyz* function



http://trecs.se/pix/pseudosphereNormalVectorField4096.png

$$\Delta \vec{x} = \frac{1}{2}(\kappa_1 + \kappa_2)N$$

Intuition:

Laplacian measures difference with neighbors.

Introducing the Laplacian Operator

Justin Solomon
MIT, Spring 2017