THE EXPRESSION OF A TENSOR OR A POLYADIC AS A SUM OF PRODUCTS

By Frank L. Hitchcock

1. Addition and Multiplication.

Tensors are *added* by adding corresponding components. The *product* of a covariant tensor $A_{i_1 \cdot\cdot i_p}$ of order p *into* a covariant tensor $B_{i_{p+1} \cdot\cdot i_{p+q}}$ of order q is defined by writing

$$A_{i_1 \cdot\cdot i_p} B_{i_{p+1} \cdot\cdot i_{p+q}} = C_{i_1 \cdot\cdot i_{p+q}} \tag{1}$$

where the product $C_{i_1 \cdot\cdot i_{p+q}}$ is a covariant tensor of order $p+q$. When no confusion results indices may be omitted giving

$$\mathbf{AB} = \mathbf{C} \tag{1$_a$}$$

equivalent to the n^{p+q} equations (1). Boldface type is convenient for indicating that the letters do not denote merely numbers or scalars. Products of contravariant and of mixed tensors may be similarly defined.

A partial statement of the problem to be considered is as follows: to find under what conditions a given tensor can be expressed as a sum of products of assigned form. A more general statement of the problem will be given below.

2. Polyadic form of a tensor.

Any covariant tensor $A_{i_1 \cdot\cdot i_p}$ can be expressed as the sum of a finite number of tensors each of which is the product of p covariant vectors,

$$A_{i_1 \cdot\cdot i_p} = \sum_{j=1}^{j=h} a_{1j,\, i_1} a_{2j,\, i_2} \cdot\cdot a_{pj,\, i_p} \tag{2}$$

where $a_{1j,\, i_1}$, etc., are a set of hp covariant vectors. When the indices $i_1 \cdot\cdot i_p$ can be omitted this may be written

$$\mathbf{A} = \sum_{j=1}^{j=h} \mathbf{a}_{1j} \mathbf{a}_{2j} \cdot\cdot \mathbf{a}_{pj}. \tag{2$_a$}$$

The right member is now identical in appearance with a Gibbs polyadic, defined as a sum of polyads. A covariant tensor developed as in (2) is *in polyadic form*. Contravariant and mixed tensors may be similarly expanded.

A case of the present problem is to find the minimum value of h for assigned values of n and of p, with or without further restrictions.

3. **Development in terms of basis vectors.**

Let $e_{1,\,i_m}, e_{2,\,i_m}, \cdots, e_{n,\,i_m}$ denote a set of n covariant vectors each of which, at the point considered and with the variables employed, has components *1* or *0* according as its two subscripts are equal or not. We then have identically

$$A_{i_1 \cdot\cdot\, i_p} = A_{\alpha_1 \cdot\cdot\, \alpha_p}\, e_{\alpha_1,\, i_1}\, e_{\alpha_2,\, i_2} \cdot\cdot\, e_{\alpha_p,\, i_p} \tag{3}$$

where the occurrence of Greek indices twice on the right indicates summation from *1* to n. When Roman indices can be omitted this becomes

$$\mathbf{A} = A_{\alpha_1 \cdot\cdot\, \alpha_p}\, \mathbf{e}_{\alpha_1}\, \mathbf{e}_{\alpha_2} \cdot\cdot\, \mathbf{e}_{\alpha_p}. \tag{3$_a$}$$

The set of n vectors $e_{\alpha_m,\, i_m}$ defined above are *basis vectors of the* m^{th} *index*. The n^p products $\mathbf{e}_{\alpha_1} \mathbf{e}_{\alpha_2} \cdot\cdot\, \mathbf{e}_{\alpha_p}$ obtained by assigning all possible sets of values to the subscripts constitute a set of n^p tensors in terms of which any covariant tensor $A_{i_1 \cdot\cdot\, i_p}$ is linearly expressible. Any contravariant or any mixed tensor may be similarly developed.

4. **Differences between tensors and polyadics.**

A polyadic whose components are $A_{i_1 \cdots\, i_p}$ may likewise be developed in the form (3). Like tensors, polyadics are added by adding corresponding components. Their (direct indeterminate) product obeys the rule (1). If therefore a given tensor can be expressed as a sum of products in a particular manner the same is true of that polyadic which possesses components numerically equal to the components of the tensor at the point considered and with the variables employed, and conversely. But a polyadic[1]

[1] I refer to the (affine) polyadics which, with their laws of multiplication, are defined in the author's paper *A Theory of Ordered Determinants with Application to Polyadics*, this Journal, Vol. IV, No. 4, July 1925, pp. 212, 222. These polyadics differ from those of Gibbs only in the fact that the vectors employed have no metric, *i.e.*, there is no dot product and no unit vector.

differs from a tensor in the following essential respect: a polyadic is subject to an arbitrary linear transformation on any assigned index thus,

$$A'_{i_1 \cdot\cdot i_{m-1} i_{(m)} i_{m+1} \cdot\cdot i_p} = A_{i_1 \cdot\cdot i_{m-1} \alpha_m i_{m+1} \cdot\cdot i_p} T_{\alpha_m i_m} \qquad (4)$$

a system of n^p equations in which the occurrence of the Greek index twice on the right indicates summation from *1* to n; the $T_{\alpha_m i_m}$ are a set of n^2 numbers whose determinant is not zero. I shall then say that **A′** is the *transform of* **A** *on the* m^{th} *index by the transformation* T. A polyadic may be thus transformed on any or all indices. The transformation on one index need not be the same as that on any other index. A tensor on the other hand is transformed on every index by the particular transformations known as covariant or contravariant.

5. **The invariant character of a particular formulation.**

The possibility of formulating a tensor in a particular manner is interesting at least in so far as such possibility is an invariant property of the tensor. One way to establish the invariance is to show that the property in question depends on a system of equations which, by their very form, are invariant.[2] In any case, however, the possibility of the formulation will depend on the vanishing, or it may be on the non-vanishing, of certain functions, or upon the vanishing of some and the non-vanishing of others, — excepting, obviously, such formulations as are possible for all tensors. To say that the formulation is possible is the same as saying that a certain set of equations has a solution, *i.e.*, the given tensor is equal to a required tensor of assigned form. The possibility of solution will be a property *invariant in the sense of tensor calculus* if, and only if, the determinants or other functions on which the solution depends are invariant in a quite different sense, namely *in that of the ordinary theory of quantics.*

Under the transformation (4), if a function of the A′ is equal to the same function of the untransformed A multiplied by a power of the determinant of T, I shall say that this function is an *invariant of the polyadic on the* m^{th} *index.* If it is an invariant

[2] See, for example, J. W. Alexander *On the Decomposition of Tensors*, Annals of Math., Vol. 27, No. 4, Sept. 1926, p. 421.

on every index (separately) I shall say that it is a *multiple invariant of the polyadic.*[3] Any property which persists when the polyadic is subject to arbitrary transformation on any or every index as in (4) will be called a *multiply invariant property* of the polyadic.

It is evident that, if a property of a polyadic is *multiply* invariant, it will be an invariant property (in the sense of tensor calculus) of any tensor whose components at the point in question and with the variables employed are numerically equal to those of the polyadic, whether the tensor be covariant, contravariant, or mixed. For if the property persists after an arbitrary linear transformation on each index it will persist when the transformation has on every index one of the particular forms known as covariant or contravariant.

For example, a dyadic A_{rs} can always be written as a sum of dyads thus,

$$A_{rs} = \sum_{j=1}^{j=h} a_{j,r}\, b_{j,s} \tag{5}$$

where the number of dyads h need not exceed n. The same is true of tensors whether the indices r and s are upper or lower. It is well known that h can be less than n if, and only if, the determinant of the A_{rs} vanishes. But this determinant is a multiple invariant of the dyadic. If, therefore, h can be less than n for a given tensor (covariant, contravariant, or mixed) this property is invariant in the sense of tensor calculus.

If, on the other hand, a given property depends on a function of the polyadic which is merely invariant, without being multiply invariant, *i.e.*, if the function is invariant when transformed on all indices simultaneously by the same T but is not invariant when transformed on an arbitrarily chosen single index, the property will be invariant for covariant tensors and for contravariant tensors, but not in general for mixed tensors.

For example a *symmetric* dyadic can be written

$$A_{rs} = \sum_{j=1}^{j=h} a_{j,r}\, a_{j,s} \tag{6}$$

[3] A fuller discussion of invariants and other concomitants of polyadics is contained in the author's paper *A New Method in the Theory of Quantics,* this Journal, Vol. IV, No. 4, July 1925, p. 238.

and this is an invariant property of the dyadic, for symmetry evidently persists if the $a_{j,r}$ and the $a_{j,s}$ are simultaneously transformed by the same T. It is not a *multiply* invariant property, for symmetry is destroyed if the $a_{j,r}$ are transformed while the $a_{j,s}$ are not. Accordingly symmetry is an invariant property of a covariant tensor A_{rs} or of a contravariant tensor A^{rs}, but not of a mixed tensor A_r^s as is well known.

To illustrate further, take two dimensions, $n=2$. The criterion that the dyadic (5) be symmetrical becomes

$$A_{12}-A_{21}=0 \tag{7}$$

that is

$$\sum_{j=1}^{j=h}(a_{j,1}\,b_{j,2}-a_{j,2}\,b_{j,1})=0 \tag{8}$$

or more briefly

$$\Sigma(\mathbf{ab})=0 \tag{9}$$

where $(\mathbf{ab})$ denotes the space complement of $\mathbf{a}$ and $\mathbf{b}$ which, when $n=2$, is the determinant of their components. The left member is an invariant of the dyadic for it is merely multiplied by the determinant of T when all vectors are transformed by T. This is not true when prefactors only, or postfactors only, are transformed. Thus $\Sigma(\mathbf{ab})$ is *not* a *multiple* invariant of the dyadic.

6. **Plan of the investigation and terminology employed.**

As above shown, any multiply invariant property of a polyadic corresponds to a property of tensors which is invariant in the sense of tensor calculus. (It is not true that, conversely, an invariant property of a tensor must yield an invariant property of a polyadic, because the transformations of a tensor are of a special sort.) The most interesting decompositions (with exception of those which presume symmetry or antisymmetry) appear to be such as are multiply invariant. The problem of decomposition of tensors into sums of products is thus (from a purely algebraic point of view) akin to the following problem: the determination of those properties of polyadics which are multiply invariant, and the discovery of the invariants, or other concomitants, on whose vanishing or non-vanishing the property depends.

It is furthermore evident that the concepts natural to the problem derive more or less from four sources, — tensor calculus, multiple algebra, quantics, and p-way determinants. The main concept is that of a matrix $A_{i_1 \dots i_p}$. The number of indices is called in this paper the *order* of the polyadic and related tensors. This is not free from objection, since it corresponds to the *class* of a p-way determinant.

In keeping with the notion of multiple invariance, any set of n basis vectors $\mathbf{e}_{\alpha_m}$ or e_{α_m, i_m} defined in Art. 3, *i.e.*, the basis vectors of the m^{th} index, is regarded as a *distinct* set, distinct that is from the basis vectors of any other index. When necessary this fact may be emphasized by an upper index, as $\mathbf{e}_{\alpha_m}^{(m)}$.

In the polyadic development (2) any vector $\mathbf{a}_{mj}$ is a *vector factor of the m^{th} index*, and is linearly expressible in terms of the basis vectors of the same index as $a_{mj,\alpha_m}\, \mathbf{e}_{\alpha_m}$. The totality of all possible vectors of this latter form constitutes the *vector complex of the m^{th} index.*

Finally, the word *vector* will in general be employed to denote a factor of a polyad. If the vector is covariant or contravariant in the sense of tensor calculus, the fact, if essential to the argument, will be so stated. Thus the basis vectors of a particular index become merely a set of n quantities, not commutative in multiplication, in terms of which all vectors of the mth index are linearly expressible.

7. **Rank of a p-way matrix.**

Consider the p-way matrix $A_{i_1 \dots i_p}$ of a polyadic or of a tensor. Let all indices be given fixed numerical values except a chosen index i_m. Varying i_m from *1* to n yields n components called a *file of the m^{th} index*. The n^{p-1} files of the m^{th} index may be arranged as a two-way matrix having n rows and n^{p-1} columns. The columns are to stand in a definite order, called *natural order*, defined as follows: let i_k be the first index of the series $i_1, i_2, \dots i_p$ which has a different (fixed) value in two chosen columns; that column *precedes* in which i_k has the smaller value. Any (two-way) determinant of order q from this matrix will be called a *minor* of $A_{i_1 \dots i_p}$ *on the m^{th} index.*

For example, let $n=2$ and $p=3$. The four files of the first index form a matrix

$$\begin{matrix} A_{111} & A_{112} & A_{121} & A_{122} \\ A_{211} & A_{212} & A_{221} & A_{222} \end{matrix} \tag{10}$$

whence there are six second order minors of the first index. In a similar manner there are six second order minors of each of the other indices. Only twelve of these eighteen minors are distinct.

If all minors of order $k+1$ on the m^{th} index vanish, while at least one minor of order k on this index does not vanish, the matrix $A_{i_1 \cdots i_p}$ is of *rank k on the m^{th} index.*

8. Development according to rank on a chosen index.

Consider the polyadic form (2_a) of a polyadic or of a tensor. The vector factors $\mathbf{a}_{mj}$ of the m^{th} index in general lie in an n-space, that is, any $n+1$ of them are linearly related. If any $k+1$ of them are linearly related these vectors may be said to lie in a k-space (although this phraseology is not wholly free from objection).

Theorem I. *If the vector factors of the m^{th} index lie in a k-space, the rank of $A_{i_1 \cdots i_p}$ on the m^{th} index is not greater than k.*

Proof. The h vectors $\mathbf{a}_{mj}$ are linearly expressible in terms of k properly chosen vectors of the m^{th} complex. Let these be b_{r,i_m} where $r=1, 2, \cdots, k$. Substituting for the a_{mj,i_m} in (2) their values in terms of the b_{r,i_m} and collecting terms, the coefficient of each b_{r,i_m} will be a polyadic of order $p-1$ which may be called $\mathbf{B}_{r,i_1 \cdots i_{m-1} i_{m+1} \cdots i_p}$. Thus (2) takes the form

$$A_{i_1 \cdots i_p} = \sum_{r=1}^{r=k} b_{r,i_m} B_{r,i_1 \cdots i_{m-1} i_{m+1} \cdots i_p} \tag{11}$$

which shows that, as a matrix of n rows and n^{p-1} columns, $A_{i_1 \cdots i_p}$ is the product of two matrices which may be written so that the first has k columns and the second k rows. Hence the rank of $A_{i_1 \cdots i_p}$ on the m^{th} index cannot exceed k.

Theorem II. *If the matrix $A_{i_1 \cdots i_p}$ of a polyadic or a tensor is of rank k_1 on the index i_1, rank k_2 on the index i_2, etc., up to k_p on the index i_p,* **A** *can be developed in polyadic form in such a manner that,* *simultaneously, the vector factors* $\mathbf{a}_{mj}$ *of the* m^{th} *index lie in a* k_{m}*-space for* $m=1, 2, \cdots, p$.

Proof. The right member of the identical relation (3_a) may be regarded as the sum of n^{p-1} vectors $A_{\alpha_1 \cdots \alpha_p}\mathbf{e}_{\alpha_1}$ each multiplied into a polyad $\mathbf{e}_{\alpha_2}\mathbf{e}_{\alpha_3}\cdots\mathbf{e}_{\alpha_p}$. If the matrix $A_{i_1 \cdots i_p}$ is of rank k_1 on the first index these vectors (*file vectors of the first index*) may be expressed in terms of a properly chosen set of k_1 linearly independent vectors which may be called $\mathbf{a}_{1j_1}$ where $j_1=1, 2, \cdots, k_1$. Substituting for the vectors $A_{\alpha_1 \cdots \alpha_p}\mathbf{e}_{\alpha_1}$ their values in terms of the $\mathbf{a}_{1j_1}$ and collecting terms, the coefficient into which each $\mathbf{a}_{1j_1}$ is multiplied will be a polyadic of order $p-1$ which may be called $\mathbf{B}_{j_1}$. Thus $\mathbf{A}$, if of rank k_1 on i_1, can be developed as

$$\mathbf{A}=\sum_{j_1=1}^{j_1=k_1} \mathbf{a}_{1j_1}\mathbf{B}_{j_1} \tag{12}$$

which is the same as

$$A_{i_1 \cdots i_p}=\sum_{j_1=1}^{j_1=k_1} a_{1j_1, i_1} B_{j_1, i_2 i_3 \cdots i_p}\,. \tag{12a}$$

Next let $A_{i_1 \cdots i_p}$ be also of rank k_2 on the second index. I shall show that no $\mathbf{B}_{j_1}$ can be of rank greater than k_2 on the index i_2. To say that $\mathbf{A}$ is of rank k_2 on i_2 is the same as saying that $n-k_2$ linearly independent vectors X_{q, i_2} exist such that

$$A_{i_1\alpha_2 i_3 \cdots i_p} X_{q, \alpha_2}=0 \quad (q=1, 2, \cdots, n-k_2) \tag{13}$$

which by (12a) is the same as

$$\sum_{j_1=1}^{j_1=k_1} a_{1j_1, i_1}B_{j_1, \alpha_2 i_3 \cdots i_p}X_{q, \alpha_2}=0 \; (q=1, 2, \cdots, n-k_2). \tag{14}$$

But since the k_1 vectors a_{1j_1, i_1} are linearly independent we must have

$$B_{j_1, \alpha_2 i_3 \cdots i_p}X_{q, \alpha_2}=0 \quad (q=1, 2, \cdots, n-k_2) \tag{15}$$

for each value of j_1. That is, each $\mathbf{B}_{j_1}$ is of rank not greater than k_2 on i_2 as was to be shown.

A set of k_2 vectors $\mathbf{a}_{2j_2}$ where $j_2=1, 2, \cdots, k_2$ therefore exists in terms of which every vector $B_{j_1, \alpha_2 i_3 \cdots i_p} \mathbf{e}_{\alpha_2}$ is linearly expressible. Substituting and collecting terms each $\mathbf{a}_{2j_2}$, for a particular value of j_1, will be multiplied into a polyadic of order $p-2$ which may be called $\mathbf{C}_{j_1 j_2}$. Thus each $\mathbf{B}_{j_1}$ develops as

$$\mathbf{B}_{j_1} = \sum_{j_2=1}^{j_2=k_2} \mathbf{a}_{2j_2} \mathbf{C}_{j_1 j_2}. \tag{16}$$

Then by (12) $\mathbf{A}$, if of rank k_1 on i_1 and rank k_2 on i_2 can be developed as

$$\mathbf{A} = \sum_{j_1=1}^{j_1=k_1} \sum_{j_2=1}^{j_2=k_2} \mathbf{a}_{1j_1} \mathbf{a}_{2j_2} \mathbf{C}_{j_1 j_2}. \tag{17}$$

Proceeding in a similar manner, assuming in succession that $\mathbf{A}$ is of rank k_3 on i_3, etc., up to k_{p-1} on i_{p-1}, we arrive at

$$\mathbf{A} = \sum_{j_1=1}^{j_1=k_1} \sum_{j_2=1}^{j_2=k_2} \cdots \sum_{j_{p-1}=1}^{j_{p-1}=k_{p-1}} \mathbf{a}_{1j_1} \mathbf{a}_{2j_2} \cdots \mathbf{a}_{(p-1)j_{p-1}} \mathbf{g}_{j_1 j_2 \cdots j_{p-1}} \tag{18}$$

where the $\mathbf{g}$'s are a set of vectors in the complex of the p^{th} index, in number equal to the product $k_1 k_2 \cdots k_{p-1}$. The vectors $\mathbf{a}_{mj_m}$ are, for a chosen value of m, in number k_m, that is they lie in a k_m-space.

Finally, by similar reasoning, assuming $\mathbf{A}$ of rank k_p on i_p, the $\mathbf{g}$'s lie in a k_p-space. The theorem is thus proved.

Corollary I. If in (18) each $\mathbf{g}$ be expressed in terms of k_p linearly independent vectors $\mathbf{a}_{pj_p}$ we have

$$\mathbf{A} = \sum_{j_1=1}^{j_1=k_1} \cdots \sum_{j_p=1}^{j_p=k_p} \mathbf{a}_{1j_1} \cdots \mathbf{a}_{pj_p}\, g_{j_1 \cdots j_p} \tag{19}$$

where the $g_{j_1 \cdots j_p}$ are a set of scalar coefficients in number equal to the product of the ranks $k_1 \cdots k_p$.

Corollary II. By a properly chosen transformation $T_{\alpha_m i_m}$ as in (4) the vectors $\mathbf{a}_{mj_m}$ can be identified with the first k_m basis vectors of the m^{th} complex $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{k_m}$. Thus without altering any multiply invariant property, **A** can be assumed of the form

$$\mathbf{A} = \sum_{j_1=1}^{j_1=k_1} \cdots \sum_{j_p=1}^{j_p=k_p} \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_p} g_{j_1 \cdots j_p} \tag{20}$$

provided the ranks on $i_1 \cdots i_p$ are $k_1 \cdots k_p$ respectively.

Corollary III. If none of the ranks $k_1 \cdots k_p$ exceeds a single number k, it may be assumed, without altering any multiply invariant property, that **A** has the form

$$\mathbf{A} = \sum_{j_1=1}^{j_1=k} \cdots \sum_{j_p=1}^{j_p=k} \mathbf{e}_{j_1} \cdots \mathbf{e}_{j_p} g_{j_1 \cdots j_p} \tag{21}$$

which is the form of a polyadic in k dimensions instead of in n dimensions.

It may be remarked that (20) assumes every component of **A** to be zero for which any index i_m is greater than k_m, and similarly for (21). The problem of finding the minimum h in (2) is the same as choosing the **a**'s in (19) so as to annul as many *more* components as possible. In other words, a knowledge of the ranks on the indices enables us to reduce the number of terms in (2) from n^p to $k_1 k_2 \cdots k_{p-1}$ by an expansion (19) but in general fails to give the minimum h.

In the case of a dyadic, however, the rank on either index is equal to the rank in the usual sense, and (19) merely confirms known results.

Also, that a polyadic can be expressed as a single polyad it is necessary and sufficient that rank on every index be unity.

9. Multiplex rank.

Consider once more the p-way matrix of the $A_{i_1 \cdots i_p}$. Suppose two chosen indices i_r and i_s to be varied, all other indices retaining a chosen set of fixed values. We thus obtain a set of n^2 components, a *duplex file on* $i_r i_s$, to be taken in natural order (Art. 7). The n^{p-2} duplex files on $i_r i_s$ may be arranged as a two-way matrix having n^2 rows and n^{p-2} columns. The columns also are to stand in natural order. Any (two-way) determinant of order q from this matrix is a *duplex minor* of order q on the indices $i_r i_s$ of $A_{i_1 \cdots i_p}$. (When confusion might arise minor and rank on a single index will be distinguished as *simple rank* and *simple minor* on the index in question.)

Similarly, ordering n^t components selected by varying t chosen indices yields a *multiplex* (t-plex) *file* on those indices. The n^{p-t} multiplex files on a chosen set of indices can be arranged as a two-way matrix of n^t rows and n^{p-t} columns, which are to be in natural order. A determinant of order q from this matrix is a *t-plex minor of order* q on the designated indices of $A_{i_1} \ldots {}_{i_p}$. The rank of this matrix is the *t-plex rank* of **A** on the t chosen indices.

Lemma. It is evident that the t-plex rank of **A** on t chosen indices is equal to the $(p-t)$-plex rank on the remaining indices.

10. Development on multipartite indices.

Let the p indices of $A_{i_1} \ldots {}_{i_p}$ be divided into π groups of $t_1, t_2, \cdots, t_\pi$ indices each. The selection of indices for each group is arbitrary, *i.e.*, the indices of a group need not be consecutive indices in the series $i_1 \cdots i_p$. Within each group, however, the indices may stand in the order in which they occur in $i_1 \ldots i_p$. Any array or summation in which the t_m indices of a group all run from *1* to n may be taken in natural order (Art. 7). This is the same as replacing the t_m indices of a group by a single multipartite index which runs from *1* to n^{t_m}. If $u_1, u_2, \cdots, u_\pi$ are the new indices we have identically

$$A_{i_1} \ldots {}_{i_p} = A_{u_1} \ldots {}_{u_\pi}. \tag{22}$$

This rearrangement of indices is equivalent to treating a polyadic as a *multiple polyadic*, *i.e.*, a sum of multiple polyads, wherein factors are polyadics instead of vectors. By mere grouping of terms in (3) we can expand as

$$A_{i_1} \ldots {}_{i_p} = \sum_{j=1}^{j=h} B_{1j,u_1} B_{2j,u_2} \cdots B_{\pi j,u_\pi} \tag{23}$$

where, for a chosen value of m, the B_{mj,u_m} are a set of h polyadics of order t_m, the *polyadic factors on the* m^{th} *multipartite index.*

The problem of finding the minimum value of h in (23) is more general than the problem of expressing **A** as the sum of a minimum number of products as defined in Art. 1, because now, on the right of (23), the order of indices $i_1 \ldots i_p$ is no longer preserved.

We may say, however, that if $A_{i_1 \cdots i_p}$ is developed as in (23) we have thereby expressed an *isomer* of **A** as a sum of products in the sense of Art. 1.[4]

Lemma. *For an assigned value of h, the possibility of* (23) *is a multiply invariant property of* **A**.

For by a transformation (4) on any index, any polyadic factor goes into a polyadic of the same order.

It follows that any relationships on which the possibility of (23) depends have multiple invariant character.

It also follows that if (23) exists for a polyadic, it exists for any tensor whose components at the point in question and with the variables employed are respectively equal to those of the polyadic.

11. Development according to multiplex rank.

The polyadic factors B_{mj_m, u_m} of (23) are, with respect to all questions of linear dependence, precisely the same as vectors in a space of n^{t_m} dimensions; for a fixed value of m, any $n^{t_m}+1$ of them are linearly related. If, in a particular case, it holds true that any k_m+1 of them are linearly related, these polyadics may be said to lie in a k_m-space (although this is not free from objection).

Theorem III. *If the polyadic factors on the m^{th} multipartite index lie in a k_m space, the t_m-plex rank of $A_{i_1 \cdots i_p}$ on the indices of the m^{th} group is not greater than k_m.*

Proof. The h polyadics B_{mj, u_m} are linearly expressible in terms of k_m properly chosen polyadics bearing the index u_m. Let these be C_{r, u_m} where $r=1, 2, \cdots, k_m$. Substituting for the B_{mj, u_m} their values and collecting terms, the coefficient of each C_{r, u_m} will be a polyadic of order $p-t_m$ which may be called

$$D_{r, u_1 \cdots u_{m-1} u_{m+1} \cdots u_\pi}$$

so that we may write

$$A_{i_1 \cdots i_p} = \sum_{r=1}^{r=k_m} C_{r, u_m} D_{r, u_1 \cdots u_{m-1} u_{m+1} \cdots u_\pi} \tag{24}$$

[4] An isomer (conjugate) is a new polyadic or tensor obtained from $A_{i_1 \cdots i_p}$ by a permutation of indices.

Therefore by the laws of matrix multiplication **A** is of rank not greater than k_m on the multipartite index u_m as was to be shown.

Theorem IV. *With indices in* π *groups, if* $A_{i_1 \cdot\cdot i_p}$ *is of multiplex rank* k_1 *on the indices of the first group,* k_2 *on those of the second group, and so on up to* k_π *on those of the* π^{th} *group, the polyadic or tensor can be developed in the multiple polyadic form* (23) *in such a manner that, simultaneously, the polyadic factors of the* m^{th} *multipartite index lie in a* k_m*-space for* $m=1, 2, \cdot\cdot, \pi$.

Proof. Let $\beta_1, \beta_2, \cdot\cdot, \beta_\pi$ be dummy indices corresponding to the multipartite indices $u_1, u_2, \cdot\cdot, u_\pi$. Furthermore let the (ordinary) indices of the first group have the dummies $\rho_1, \cdot\cdot, \rho_{t_1}$. Let the polyads $\mathbf{e}_{\rho_1}\mathbf{e}_{\rho_2}\cdot\cdot\mathbf{e}_{\rho_{t_1}}$, in natural order, be $\mathbf{E}_{\beta_1}$, so that as β_1 runs from 1 to n^{t_1} these polyads all occur once and once only. Treating the other groups similarly, an isomer **A**′ of **A** can be written

$$\mathbf{A}' = A_{\beta_1 \cdot\cdot \beta_\pi}\mathbf{E}_{\beta_1}\cdot\cdot\mathbf{E}_{\beta_\pi} \tag{25}$$

identically. Thus **A**′ takes the form of a polyadic of order π. The proof now proceeds as in Theorem II, multiplex rank replacing simple rank, and multipartite indices behaving like ordinary indices of that proof. We arrive at a development of **A**′ which has precisely the form of the right member of (18) but wherein the factors, instead of being vectors, are polyadics of orders $t_1, t_2, \cdot\cdot t_\pi$. Restoring the original order of indices we have a development of the form (23) in which any set of polyadic factors $B_{mj,\,u_m}$ lies in a k_m-space. The theorem is therefore proved.

Corollary I. With the hypotheses of Theorem IV we may write

$$A_{i_1 \cdot\cdot i_p} = A_{u_1 \cdot\cdot u_\pi} = \sum_{j_1=1}^{j_1=k_1} \cdots \sum_{j_\pi=1}^{j_\pi=k_\pi} B_{ij_1,\,u_1} \cdots B_{\pi j_\pi,\,u_\pi} g_{j_1 \cdot\cdot j_\pi} \tag{26}$$

where the B_{mj_m,u_m} are polyadics or tensors of order t_m and the $g_{j_1 \cdot\cdot j_\pi}$ are a set of scalar coefficients in number equal to the product of the multiplex ranks $k_1 \cdot\cdot k_\pi$.

It may be remarked that we have here no corollary analogous to Corollary II of Theorem II. We cannot in general by linear transformations (4) identify a set of polyadic factors $B_{mj_m,\,u_m}$ with the first k_m polyads $\mathbf{E}_1 \cdot\cdot \mathbf{E}_{k_m}$.

Somewhat analogous with Theorem II, Corollary III we have

Corollary II. If none of the multiplex ranks $k_1 \ldots k_\pi$ exceeds a single number k, a suitably chosen isomer $\mathbf{A}'$ of $\mathbf{A}$ can be written

$$\mathbf{A}' = \sum_{j_1=1}^{j_1=k} \cdots \sum_{j_\pi=1}^{j_\pi=k} \mathbf{B}_{1j_1} \mathbf{B}_{2j_2} \cdots \mathbf{B}_{\pi j_\pi} g_{j_1 \ldots j_\pi} \tag{27}$$

which is the *form of* a polyadic in k dimensions instead of n dimensions. Any set of k polyadic factors $\mathbf{B}_{mj_m}$ can be so chosen as to be linearly independent, hence are like basis vectors in a k-space (this of course in a purely algebraic sense). The g's are scalar coefficients, analogous to the components of a tensor in the k-space. The utility of this analogy between polyadic factors and vectors will appear shortly.

Corollary III. The simple or multiplex rank of a polyadic on one or several indices is a multiple invariant.

For by Theorem IV the possibility of a particular development of form (23) depends on such rank, and this form is multiply invariant.

When $\pi = 2$ we have a case of (23) treated by Dr. J. W. Alexander.[5]

When $\pi = 2$ and p is even, with $t_1 = t_2 = k$, the indices of the first group being $i_1 \ldots i_k$, in order, we have "double polyadics".[6] In this case $g_{j_1 j_2}$ behaves like a dyadic and (23) becomes analogous to (5).

Again, that a polyadic or one of its isomers may be reduced to a product of polyadics of lower order it is necessary and sufficient that the rank on each group of indices shall be unity for at least one manner of grouping.

12. Character of the problem when π exceeds 2.

When π exceeds 2 a knowledge of the ranks on various groups of indices, while quite essential, does not in general solve the problem of the minimum value of h in (23). Reduction to fewer

[5] Loc. cit. note 2.

[6] Proc. Amer. Acad. Arts and Sci., Vol.58, No. 10, May 1923. The multiple dot products of this paper may be regarded as mere summations on multipartite indices u_1 and u_2.

than $k_1 \, . \, . \, k_\pi$ terms depends on invariants of another character, the quantities elsewhere denoted by $\mathbf{F}_k((\mathbf{P}^q))$ where $\mathbf{P}$ is the polyadic.[7] They are always expressible as p-way determinants, and, from their very form, are multiply invariant. These statements I shall illustrate in the next article by a study of the simplest possible case where π exceeds 2, namely the problem of expressing a triadic in two-space as a sum of two triads.

13. **The case** $\pi = p = 3$, $n = 2$, $h = 2$.

We have to solve the polyadic equation

$$A_{\alpha_1\alpha_2\alpha_3}\mathbf{e}_{\alpha_1}\mathbf{e}_{\alpha_2}\mathbf{e}_{\alpha_3} = \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 + \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3 \qquad (28)$$

where the components $A_{\alpha_1\alpha_2\alpha_3}$ are 8 given numbers and the 6 vectors on the right are required. This is equivalent to 8 ordinary equations

$$A_{i_1i_2i_3} = a_{1,i_1}a_{2,i_2}a_{3,i_3} + b_{1,i_1}b_{2,i_2}b_{3,i_3} \qquad (28_a)$$

where the 12 numbers on the right are unknown.

To solve, equate values of the covariant $\mathbf{F}_1((\mathbf{A}^2))$ obtained from the two members of (28). The operation $\mathbf{F}_1$ implies two steps: first, multiply each member into itself, preserving the natural order of indices (multiple indeterminate product); second, replace each group of vectors of like index by its space complement (folding), excepting the third index.

Thus from the right of (28) we first form the product

$$\mathbf{a}_1\mathbf{a}_1\mathbf{a}_2\mathbf{a}_2\mathbf{a}_3\mathbf{a}_3 + \mathbf{a}_1\mathbf{b}_1\mathbf{a}_2\mathbf{b}_2\mathbf{a}_3\mathbf{b}_3 + \mathbf{b}_1\mathbf{a}_1\mathbf{b}_2\mathbf{a}_2\mathbf{b}_3\mathbf{a}_3 + \mathbf{b}_1\mathbf{b}_1\mathbf{b}_2\mathbf{b}_2\mathbf{b}_3\mathbf{b}_3 \qquad (29)$$

We then take the space complements of the 8 dyads $\mathbf{a}_1\mathbf{a}_1$, $\mathbf{a}_2\mathbf{a}_2$, $\mathbf{a}_1\mathbf{b}_1$, $\mathbf{a}_2\mathbf{b}_2$, $\mathbf{b}_1\mathbf{a}_1$, $\mathbf{b}_2\mathbf{a}_2$, $\mathbf{b}_1\mathbf{b}_1$, and $\mathbf{b}_2\mathbf{b}_2$, which, because $n=2$, are the respective determinants of the components of the vectors concerned. The space complements of the first, second, seventh, and eighth of these dyads therefore vanish and the result is

$$(\mathbf{a}_1\mathbf{b}_1)(\mathbf{a}_2\mathbf{b}_2)\mathbf{a}_3\mathbf{b}_3 + (\mathbf{b}_1\mathbf{a}_1)(\mathbf{b}_2\mathbf{a}_2)\mathbf{b}_3\mathbf{a}_3 \qquad (30)$$

where $(\mathbf{a}_1\mathbf{b}_1)$, etc., are determinants. This may be written

$$(\mathbf{a}_1\mathbf{b}_1)(\mathbf{a}_2\mathbf{b}_2)[\mathbf{a}_3\mathbf{b}_3 + \mathbf{b}_3\mathbf{a}_3] \qquad (31)$$

where the quantity in brackets is a symmetrical dyadic.

[7] Loc. cit. note 1, p. 233.

The same process applied to the left member of (28) yields a cubic determinant with vector elements which develops[8] into the symmetrical dyadic

$$\mathbf{e}_1\mathbf{e}_1 m_{11} + (\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) m_{12} + \mathbf{e}_2\mathbf{e}_2 m_{22} \tag{32}$$

where

$$m_{11} = 2(\mathrm{A}_{111}\mathrm{A}_{221} - \mathrm{A}_{121}\mathrm{A}_{211}), \quad m_{22} = 2(\mathrm{A}_{112}\mathrm{A}_{222} - \mathrm{A}_{122}\mathrm{A}_{212})$$
$$m_{12} = \mathrm{A}_{111}\mathrm{A}_{222} - \mathrm{A}_{211}\mathrm{A}_{122} - \mathrm{A}_{121}\mathrm{A}_{212} + \mathrm{A}_{221}\mathrm{A}_{112} \tag{33}$$

so that by equating the two values of $\mathbf{F}_1((\mathbf{A}^2))$ from (31) and (32)

$$\mathbf{e}_1\mathbf{e}_1 m_{11} + (\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1) m_{12} + \mathbf{e}_2\mathbf{e}_2 m_{22} = (\mathbf{a}_1\mathbf{b}_1)(\mathbf{a}_2\mathbf{b}_2)[\mathbf{a}_3\mathbf{b}_3 + \mathbf{b}_3\mathbf{a}_3] \tag{34}$$

We now consider three cases.

Case I. The binary quadric

$$x_1{}^2 m_{11} + 2x_1x_2 m_{12} + x_2{}^2 m_{22} \tag{35}$$

does not vanish identically and has linearly independent factors.

Let these factors be

$$x_1 r_3 - x_2 r_3' \quad \text{and} \quad x_1 s_3 - x_2 s_3' \tag{36}$$

and assume

$$\mathbf{a}_3 = \mathbf{e}_1 r_3 - \mathbf{e}_2 r_3' \quad \text{and} \quad \mathbf{b}_3 = \mathbf{e}_1 s_3 - \mathbf{e}_2 s_3' \tag{37}$$

These two vectors not being parallel we may certainly write

$$\mathbf{A} = \mathbf{X}\mathbf{a}_3 + \mathbf{Y}\mathbf{b}_3 \tag{38}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are dyadics. I shall now show that $\mathbf{X}$ is the product of two vectors, hence may be written $\mathbf{a}_1\mathbf{a}_2$, and similarly for $\mathbf{Y}$; so that (38) takes the required form (28).

Proof. Let the result of multiplying $\mathbf{A}$ into $\mathbf{b}_3$ and taking space complements of $\mathbf{b}_3$ with the postfactors of $\mathbf{A}$ be denoted[9] by $\mathbf{A}\times\mathbf{b}_3$. Thus from (38)

$$\mathbf{A}\times\mathbf{b}_3 = \mathbf{X}(\mathbf{a}_3\mathbf{b}_3) \tag{39}$$

both members of which are dyadics. The operation $\mathbf{A}\times\mathbf{b}_3$ is the same as replacing $\mathbf{e}_1$ and $\mathbf{e}_2$, basis vectors of the third index in $\mathbf{A}$, by $-s_3'$ and $-s_3$ respectively; for $(\mathbf{e}_1\mathbf{b}_3) = -s_3'$ and $(\mathbf{e}_2\mathbf{b}_3) = -s_3$, by (37). Multiply each side of (39) into itself and fold on both factors, *i.e.*, perform the operation F_0. The right side gives

[8] Loc. cit. note 1, p. 235.

[9] Using the notation of Dr. Naess, which is consistent because the space complement is a generalization of Gibbs' "cross product."

$(\mathbf{a}_3\mathbf{b}_3)^2F_0((\mathbf{X}^2))$, that is $(\mathbf{a}_3\mathbf{b}_3)^2\mathbf{X}{\times\atop\times}\mathbf{X}$, which is $2(\mathbf{a}_3\mathbf{b}_3)^2|X|$ where $|X|$ is the determinant of $\mathbf{X}$. The left side gives the same result as making the above mentioned replacement in (32), *i.e.*, $-s_3'$ for $\mathbf{e}_1$ and $-s_3$ for $\mathbf{e}_2$, because it is immaterial whether we make this replacement before or after folding $\mathbf{A}$ on its first two indices; but this result is zero, by definition of $\mathbf{b}_3$ in (37). Hence $|X|=0$ and $\mathbf{X}$ can be written as a dyad. Similarly $|Y|=0$ and the required form (28) is established.

It has thus been shown that (28), or (28_a), is a possible development for $A_{i_1i_2i_3}$ (with $n=2$) provided (35) has linearly independent factors; *i.e.*, if we write

$$I_a=2(m_{11}m_{22}-m^2_{12}) \tag{40}$$

this sufficient condition is that I_a be not zero. If we denote (32) by $\mathbf{M}$ we have

$$F_0((\mathbf{M}^2))=I_a \tag{41}$$

that is $I_a=\mathbf{M}{\times\atop\times}\mathbf{M}$, a multiple invariant of $\mathbf{A}$. Comparing with (34) we have

$$I_a=2(\mathbf{a}_1\mathbf{b}_1)^2(\mathbf{a}_2\mathbf{b}_2)^2(\mathbf{a}_3\mathbf{b}_3)^2. \tag{42}$$

Case II. $I_a=0$ and $\mathbf{A}$ is of rank unity on at least one index.

The form (28) is at once established by Theorem IV. For suppose $\mathbf{A}$ of rank *1* on i_3. We may write $\mathbf{A}=\mathbf{Ba}$ where $\mathbf{B}$ is a dyadic and $\mathbf{a}$ is a vector. $\mathbf{B}$ may certainly be written as the sum of two dyads, hence (28) exists, with $\mathbf{a}_3=\mathbf{b}_3=\mathbf{a}$. Similarly (28) exists if $\mathbf{A}$ is of rank *1* on i_1 or on i_2.

Case III. $I_a=0$ and $\mathbf{A}$ is of rank *2* on all indices.

The form (28) is now impossible, for if (28) exists (42) also exists. If $I_a=0$ one of the three determinants on the right must vanish. But then, by (28) $\mathbf{A}$ must be of rank *1* on some index, contrary to hypothesis.

For example, if the $A_{i_1i_2i_3}$, taken in natural order (Art. 7) are the first 8 integers 1, 2, $\cdot\cdot$, 8, it will be found that $I_a=0$ while ranks on i_1, i_2, and i_3 are all *2*, and (28) is impossible.

Remarks on the above cases. It is evident that every triadic in two dimensions which does not vanish identically must belong to one of the above three types, the classification being with reference to rank on the indices and to I_a, all of which are multiple invariants. Thus the classification applies without change to any tensors of the third order when $n=2$. This relatively simple example illustrates very well the statement of Art. 12 that classification of tensors and polyadics (with respect at least to their expression as sums of products) depends on quantities $\mathbf{F}_k((\mathbf{P}^q))$.

14. Case $\pi=p=3$, $h=2$, n arbitrary. Reduction of any triadic or third order tensor to a sum of two triads.

The results of Art. 13 may be at once extended to any triadic or third order tensor by virtue of the following:

Theorem V. *If the polyadic development* (2) *exists for any assigned value of h, the simple rank of* $\mathbf{A}$ *on any index cannot exceed h.*

For there are only h vector factors of any one index. Hence these factors lie in an h-space, and, by Theorem I, the rank on this index is not greater than h.

If, therefore, we assume $h=2$ for any triadic, the rank on any index can be at most 2. By Corollary III of Theorem II we may reduce the triadic to the form of a triadic in 2-space without altering any multiply invariant property. The entire argument of Art. 13 may then be applied.

What I_a now becomes deserves mention. This quantity was formed by first taking $\mathbf{M}=\mathbf{F}_1((\mathbf{A}^2))$, which may be indicated by the briefer notation $(\mathbf{A}^2)_1$. Then $\mathrm{I}_a=\mathrm{F}_0((\mathbf{M}^2))=\mathbf{M}{\times\atop\times}\mathbf{M}$. Thus we may briefly write

$$\mathrm{I}_a=(\mathbf{A}^2)_1{\times\atop\times}(\mathbf{A}^2)_1. \tag{43}$$

The operations F_0 and $\mathbf{F}_1$ are both mutiply invariant. Applying the results of Art. 13 we have the result: *any triadic or third order tensor whose rank on any index does not exceed 2 may be expressed as the sum of two triads except when the quantity* $(\mathbf{A}^2)_1{\times\atop\times}(\mathbf{A}^2)_1$ *vanishes and none of these ranks is unity.* It should be noticed that this quantity is no longer a scalar.

15. Case $\pi=3$, $h=2$, n and p arbitrary. Reduction of any polyadic or tensor to a sum of two terms each of which is the product of three polyadics or tensors of lower order.

In a similar manner, the argument of Art. 13 applies to any polyadic or tensor when we divide the indices into three groups and assign $h=2$ in the development (23). This is by:

Theorem VI. *If the multiple polyadic development* (23) *exists for an assigned value of h, the multiplex rank of* **A** *on any multipartite index cannot exceed h.*

For there are only h polyadic factors of any one multipartite index. Hence by Theorem III the multiplex rank on this index is not greater than h.

If therefore, with three multipartite indices, we assume $h=2$ in (23), the multiplex rank on none of these indices exceeds *2*. By Corollary II of Theorem IV we may reduce **A** (or an isomer) to the form of a triadic in 2-space. The argument of Art. 13 may then be applied, the scalars $g_{j_1 j_2 j_3}$ (*cf.* (27)) replace the $A_{i_1 i_2 i_3}$ of Art. 13 and the **B**'s replace basis vectors.

Interesting questions arise when we consider a generalization of I_a for this case, to be obtained from the original **A** instead of using the g's. The form would be that of the right of (43), but the folding operations apply to polyadic factors instead of to vectors. It can be shown that the results of these operations are still invariants.[10] Applying Art. 13 we have the result: any polyadic or tensor may be expressed as the sum of two products of the form (23) if indices may be taken in three groups in such a manner that no multiplex rank on these groups exceeds 2, with the following exceptional case: when none of these ranks is unity and the quantity $(\mathbf{A}^2)_1 \overset{\times}{\times} (\mathbf{A}^2)_1$, in which the folding operations refer to the multipartite indices, vanishes.

16. Case $\pi=p=4$, $n=h=2$. Reduction of a tetradic or fourth order tensor in 2-space to the sum of two tetrads.

We now consider the next simplest possibility, a tetradic in two-space $n=2$ and desire to express it as the sum of two tetrads,

$$\mathbf{A}=\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3\mathbf{a}_4+\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3\mathbf{b}_4 \tag{44}$$

or

$$A_{i_1 i_2 i_3 i_4}=a_{1,i_1}a_{2,i_2}a_{3,i_3}a_{4,i_4}+b_{1,i_1}b_{2,i_2}b_{3,i_3}b_{4,i_4} \tag{44$_a$}$$

a system of 16 equations, the 16 numbers on the right unknown. As a first step in the solution, note the following:

Theorem VII. *If the polyadic development* (2) *exists for an*

[10] The demonstration is contained in a forthcoming paper.

assigned value of h, the multiplex rank of $\mathbf{A}$ *on any group of indices cannot exceed* h.

For grouping the vector factors into polyads to correspond with the grouping of indices, (2) takes the form (23) with the same value of h. There are now h polyad factors on any one multipartite index. Hence by Theorem III the multiplex rank on this index is not greater than h.

In the present case it follows that, if (44) exists, the respective duplex ranks on i_1i_2, i_1i_3, and i_1i_4 do not exceed *2*.

Next, equate values of $F_0((\mathbf{A}^2))$ from the two members of (44), *i.e.*, fold each side into itself on both factors, giving

$$|A_{i_1i_2i_3i_4}| = (\mathbf{a}_1\mathbf{b}_1)(\mathbf{a}_2\mathbf{b}_2)(\mathbf{a}_3\mathbf{b}_3)(\mathbf{a}_4\mathbf{b}_4) \tag{45}$$

where $|A_{i_1i_2i_3i_4}|$ is the four-way full-sign determinant[11] of the components of $\mathbf{A}$, that is

$$\begin{aligned}|A_{i_1i_2i_3i_4}| = {} & A_{1111}A_{2222} - A_{1112}A_{2221} - A_{1121}A_{2212} + A_{1122}A_{2211} \\ & - A_{1211}A_{2122} + A_{1212}A_{2121} + A_{1221}A_{2112} - A_{1222}A_{2111}.\end{aligned} \tag{46}$$

We now consider three cases.

Case I. The determinant (46) is not zero.

Since the duplex rank on i_1i_2 must not exceed *2* we may write by Theorem IV

$$\mathbf{A} = \mathbf{B}_1\mathbf{C}_1 + \mathbf{B}_2\mathbf{C}_2 \tag{47}$$

where the factors on the right are dyadics. I shall now show that $\mathbf{B}_1$ and $\mathbf{B}_2$ may be so chosen as to be dyads, and that when they are so chosen $\mathbf{C}_1$ and $\mathbf{C}_2$ are also dyads, thus establishing (44).

Proof. At least one of the dyadics $\mathbf{B}_1$, $\mathbf{B}_2$ can be taken as a dyad. For if $\mathbf{B}_1$ is not a dyad at least one dyadic of the pencil $\mathbf{B}_1 + x\mathbf{B}_2$ is a dyad, since equating the determinant of the pencil to zero gives a quadratic in x having at least one root. Take this dyad instead of $\mathbf{B}_1$. Thus assuming $\mathbf{B}_1$ a dyad, by a suitable transformation (4) on i_1 and another on i_2 we can identify $\mathbf{B}_1$ with $\mathbf{e}_1\mathbf{e}_1$; and (46), being a multiple invariant,[12] will not be made to vanish. Assume $\mathbf{B}_2 = b_{12}\mathbf{e}_1\mathbf{e}_2 + b_{21}\mathbf{e}_2\mathbf{e}_1 + b_{22}\mathbf{e}_2\mathbf{e}_2$. If b_{22} is not zero a dyadic

[11] Loc. cit. note 1, p. 234, Theorem IX.
[12] Loc cit. note 3, p. 246, Ex. 1.

can be chosen from the pencil $\mathbf{B}_1+x\mathbf{B}_2$ which shall be a dyad. Take this instead of $\mathbf{B}_2$.

If b_{22} *could* be zero, by transformations (4) we could make $\mathbf{B}_2=\mathbf{e}_1\mathbf{e}_2+\mathbf{e}_2\mathbf{e}_1$ and then write by (47)

$$\begin{aligned}\mathbf{A}=\mathbf{e}_1\mathbf{e}_1(&\mathbf{e}_1\mathbf{e}_1A_{1111}+\mathbf{e}_1\mathbf{e}_2A_{1112}+\mathbf{e}_2\mathbf{e}_1A_{1121}+\mathbf{e}_2\mathbf{e}_2A_{1122})\\ &+(\mathbf{e}_1\mathbf{e}_2+\mathbf{e}_2\mathbf{e}_1)(\mathbf{e}_1\mathbf{e}_1A_{1211}+\mathbf{e}_1\mathbf{e}_2A_{1212}+\mathbf{e}_2\mathbf{e}_1A_{1221}+\mathbf{e}_2\mathbf{e}_2A_{1222}).\end{aligned}\tag{48}$$

The two-way matrix of the duplex files on i_1i_3 is now

$$\begin{array}{cccc} A_{1111} & A_{1112} & A_{1211} & A_{1212}\\ A_{1121} & A_{1122} & A_{1221} & A_{1222}\\ A_{1211} & A_{1212} & 0 & 0\\ A_{1221} & A_{1222} & 0 & 0\end{array}\tag{49}$$

If this is of rank not exceeding *2* we must have

$$A_{1211}A_{1222}-A_{1212}A_{1221}=0.$$

But this reduces $|A_{i_1i_2i_3i_4}|$ to zero for this determinant is now $-2(A_{1211}A_{1222}-A_{1212}A_{1221})$. Hence (48) is impossible and our assumption that $b_{22}=0$ is contradictory. Thus b_{22} is not zero and $\mathbf{B}_2$ can be taken as a dyad.

Taking $\mathbf{B}_1$ and $\mathbf{B}_2$ as dyads, by suitable transformations (4) we can identify these with $\mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_2\mathbf{e}_1$. By (47) we may write

$$\begin{aligned}\mathbf{A}=\mathbf{e}_1\mathbf{e}_2(&\mathbf{e}_1\mathbf{e}_1A_{1211}+\mathbf{e}_1\mathbf{e}_2A_{1212}+\mathbf{e}_2\mathbf{e}_1A_{1221}+\mathbf{e}_2\mathbf{e}_2A_{1222})\\ &+\mathbf{e}_2\mathbf{e}_1(\mathbf{e}_1\mathbf{e}_1A_{2111}+\mathbf{e}_1\mathbf{e}_2A_{2112}+\mathbf{e}_2\mathbf{e}_1A_{2121}+\mathbf{e}_2\mathbf{e}_2A_{2122}).\end{aligned}\tag{50}$$

The two-way matrix of duplex files on i_1i_3 is now

$$\begin{array}{cccc} 0 & 0 & A_{1211} & A_{1212}\\ 0 & 0 & A_{1221} & A_{1222}\\ A_{2111} & A_{2112} & 0 & 0\\ A_{2121} & A_{2122} & 0 & 0\end{array}\tag{51}$$

whence by inspection the determinants of $\mathbf{C}_1$ and $\mathbf{C}_2$ vanish if this is of rank *2*. $\mathbf{C}_1$ and $\mathbf{C}_2$ may be taken as dyads, and (47) takes the form (44). Thus it has been shown that (44) exists if $|A_{i_1i_2i_3i_4}|$ is not zero and the duplex ranks are *2*. (It may be noted that if none of these ranks exceeds *2* while some one of them is unity, the determinant (46) vanishes, as is easily proved.)

Case II. The four-way determinant (46) vanishes and one of the simple ranks is unity.

By Theorem IV we may now write

$$A_{i_1 i_2 i_3 i_4} = a_r B_{stu} \tag{52}$$

where a_r is a vector whose index is that one of the four indices $i_1 i_2 i_3 i_4$ on which the rank is unity, and B_{stu} is a triadic whose indices correspond to the remaining three. The solution may then be completed by Art. 13.

Case III. The determinant (46) vanishes while all simple ranks are *2*. By (45) the required form (44) is then impossible.

Remarks on the above cases. The solution of Case I was arrived at without assuming the duplex rank on $i_1 i_4$ to be *2*, but this is a necessary condition. Hence it follows that if neither of the ranks of $i_1 i_2$ nor $i_1 i_3$ exceeds *2*, the same is true of that on $i_1 i_4$. It also follows from the method of proof that, with these ranks not exceeding *2*, the condition that the pencil $x\mathbf{B}_1 + y\mathbf{B}_2$, in the notation of (47), shall contain only one dyad is the vanishing of (46).

The four-way determinant (46) in this example plays the same part as did I_a in Art. 13, an interesting analogy. The difference in form of these two invariants illustrates the contrast between polyadics of odd and of even order.

17. The case $\pi = 4$, $h = 2$, n and p arbitrary.

As in articles 14 and 15 we may at once generalize to any tetradic and further to any polyadic with indices in four groups. The necessary conditions being imposed upon the ranks, and development made by (21) or (27), the reasoning of Art. 16 applies without formal change.

18. Case $\pi = p = n = h = 3$. Reduction of a triadic or third order tensor in three-space to a sum of three triads.

Taking $n = 3$ the simplest case which is of interest arises when we assign $h = 3$. For, as shown in Art. 14, if we assign h = 2 the necessary conditions on the ranks practically reduce the problem to a binary one. Let

$$\mathbf{A} = \mathbf{a}_1\mathbf{a}_2\mathbf{a}_3 + \mathbf{b}_1\mathbf{b}_2\mathbf{b}_3 + \mathbf{c}_1\mathbf{c}_2\mathbf{c}_3 \tag{53}$$

equivalent to 27 equations. As in Art. 13 equate values of $\mathbf{F}_1((\mathbf{A}^2))$ from both sides. The right side gives

$$(\mathbf{a}_1\mathbf{b}_1)(\mathbf{a}_2\mathbf{b}_2)[\mathbf{a}_3\mathbf{b}_3+\mathbf{b}_3\mathbf{a}_3]+(\mathbf{b}_1\mathbf{c}_1)(\mathbf{b}_2\mathbf{c}_2)[\mathbf{b}_3\mathbf{c}_3+\mathbf{c}_3\mathbf{b}_3] \\ +(\mathbf{c}_1\mathbf{a}_1)(\mathbf{c}_2\mathbf{a}_2)[\mathbf{c}_3\mathbf{a}_3+\mathbf{a}_3\mathbf{c}_3] \tag{54}$$

where the space complements $(\mathbf{a}_1\mathbf{b}_1)$, etc., are now vectors because $n=3$, and may be denoted by $\mathbf{a}_1\times\mathbf{b}_1$, etc., whenever convenient. The quantities in brackets are symmetrical dyadics. Hence (54) is a tetradic. The left of (53) gives a cubic determinant[13] which develops as

$$\mathbf{e}_\alpha\mathbf{e}_\beta\mathbf{C}_{\alpha\beta} \tag{55}$$

where the $\mathbf{C}_{\alpha\beta}$ are a set of 9 symmetrical dyadics; their scalar components resemble m_{11}, m_{12}, and m_{22} of Art. 13. We may write

$$\mathbf{C}_{ab}=\mathbf{e}_r\mathbf{e}_s\mathrm{m}_{abrs} \tag{56}$$

where

$$\mathrm{m}_{11rs}=\mathrm{A}_{22r}\mathrm{A}_{33s}-\mathrm{A}_{23r}\mathrm{A}_{32s}-\mathrm{A}_{32r}\mathrm{A}_{23s}+\mathrm{A}_{33r}\mathrm{A}_{22s} \tag{57}$$

$$\mathrm{m}_{12rs}=-\mathrm{A}_{21r}\mathrm{A}_{33s}+\mathrm{A}_{23r}\mathrm{A}_{31s}+\mathrm{A}_{31r}\mathrm{A}_{23s}-\mathrm{A}_{33r}\mathrm{A}_{21s} \tag{58}$$

and so on. Thus there are 54 of these m's, cubic determinants of the second order, becoming twice a flat determinant when $r=s$. And $\mathrm{m}_{abrs}=\mathrm{m}_{absr}$.

If the basis vectors of the third index i_3 of $\mathrm{A}_{i_1i_2i_3}$ be replaced by a set of scalar variables x_{i_3}, the vectors $\mathbf{a}_3$, $\mathbf{b}_3$, and $\mathbf{c}_3$ become linear forms $\mathrm{a}_{3,\alpha_3}x_{\alpha_3}$, etc., which may be abbreviated a_x, b_x, and c_x. The vectors $(\mathbf{a}_1\mathbf{b}_1)$, etc., may be abbreviated $\mathbf{r}_1$, $\mathbf{p}_1$, $\mathbf{q}_1$ and $\mathbf{r}_2$, $\mathbf{p}_2$, $\mathbf{q}_2$. Then (54) becomes $2\mathbf{p}_1\mathbf{p}_2\mathrm{c}_x\mathrm{a}_x+2\mathbf{q}_1\mathbf{q}_2\mathrm{a}_x\mathrm{b}_x+2\mathbf{r}_1\mathbf{r}_2\mathrm{b}_x\mathrm{c}_x$. The 9 symmetrical dyadics $\mathbf{C}_{\alpha\beta}$ becomes 9 ternary quadrics which may be written $\overset{x}{\mathrm{C}}_{\alpha\beta}$ and regarded as conics. Equating results from (54) and (55) we then have

$$\mathbf{e}_\alpha\mathbf{e}_\beta\overset{x}{\mathrm{C}}_{\alpha\beta}=2\mathbf{p}_1\mathbf{p}_2\mathrm{c}_x\mathrm{a}_x+2\mathbf{q}_1\mathbf{q}_2\mathrm{a}_x\mathrm{b}_x+2\mathbf{r}_1\mathbf{r}_2\mathrm{b}_x\mathrm{c}_x \tag{59}$$

equivalent to 9 equations which must hold if (53) exists. Evidently *any four of the nine conics must be linearly related,* and, when three conics C_1, C_2, and C_3 are chosen in terms of which all

[13] Loc. cit. note 1, p. 227, determinant (28) with the two triadics equal. The method of development is very fully exemplified in that paper.

may be expressed, *their Jacobian must have three linear factors or else vanish identically*. These are necessary conditions which the $A_{i_1 i_2 i_3}$ must satisfy if (53) is possible.

Conversely, assume the nine conics expressible in terms of three conics C_1, C_2, C_3, whose Jacobian has three factors not linearly related, a_x, b_x, and c_x. Let $\mathbf{a}_3$, $\mathbf{b}_3$, $\mathbf{c}_3$ be the vectors obtained by restoring $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ in place of x_1, x_2, x_3. We may write

$$\mathbf{A} = \mathbf{X}\mathbf{a}_3 + \mathbf{Y}\mathbf{b}_3 + \mathbf{Z}\mathbf{c}_3 \tag{60}$$

where $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are dyadics. I shall show that $\mathbf{X}$ is the product of two vectors, hence may be written $\mathbf{a}_1\mathbf{a}_2$, and similarly for $\mathbf{Y}$ and $\mathbf{Z}$; so that (60) takes the required form (53).

Proof. Let the result of multiplying $\mathbf{A}$ into $\mathbf{b}_3\mathbf{c}_3$ and taking space complements with the postfactors of $\mathbf{A}$ be denoted[14] by $\mathbf{A}^1\times{}^2\mathbf{b}_3\mathbf{c}_3$. Thus by (60)

$$\mathbf{A}^1\times{}^2\mathbf{b}_3\mathbf{c}_3 = \mathbf{X}(\mathbf{a}_3\mathbf{b}_3\mathbf{c}_3) \tag{61}$$

both members of which are dyadics. The operation $\mathbf{A}^1\times{}^2\mathbf{b}_3\mathbf{c}_3$ is the same as replacing $\mathbf{e}_1$,$\mathbf{e}_2$, and $\mathbf{e}_3$, basis vectors of the third index in $\mathbf{A}$, by their cofactors in the determinant

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ b_{3,1} & b_{3,2} & b_{3,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{vmatrix}. \tag{62}$$

Multiply each side of (61) into itself and fold on both factors, *i.e.*, perform the operation $\mathbf{F}_0$. The right side gives $(\mathbf{a}_3\mathbf{b}_3\mathbf{c}_3)^2\mathbf{F}_0((\mathbf{X}))^2$, that is $(\mathbf{a}_3\mathbf{b}_3\mathbf{c}_3)^2\mathbf{X}{}^\times_\times\mathbf{X}$, whose vanishing is necessary and sufficient that $\mathbf{X}$ be the product of two vectors. The left side gives the same result as making the above replacement in (56), because it is immaterial whether we make it before or after folding $\mathbf{A}$ on its first two indices; but this result is zero, by definition of $\mathbf{b}_3$ and $\mathbf{c}_3$, (three conics pass through double points of their Jacobian). Hence $\mathbf{X}$ can be written as a dyad, and similarly for $\mathbf{Y}$ and $\mathbf{Z}$; and the required form (53) is established.

Remarks on the necessary conditions for (53). To say that any four of the nine conics $C_{\alpha\beta}$ are linearly related is the same as saying that the four-way matrix of the components of the tetradic $\mathbf{F}_1((\mathbf{A}^2))$ is of duplex rank 3 on its first two indices, the 81 (54 dis-

[14] Again using the notation of Dr. Naess.

tinct) quantities m_{abrs} being arranged in a two-way matrix as described in Art. 9. The m_{abrs} are cubic determinants picked from the original matrix $A_{i_1 i_2 i_3}$, cubic minors, we may say, of the second order.

The Jacobian of any three of the nine conics must vanish identically or have three linear factors. It is well known that a cubic has three linear factors when and only when it is proportional to its own Hessian. Hessians and Jacobians thus arising out of the m_{abrs} are components of polyadics obtainable by further folding of $\mathbf{F}_1((\mathbf{A}^2))$. Special cases arise according to whether the linear factors are linearly dependent, or, perhaps, all Jacobians vanish identically.

19. **Conclusion.** Continuance of the problem requires further extension of the notion of rank, which would too greatly lengthen the present paper. The simple and duplex ranks which have so far arisen have been two-way, in the sense that the p-way matrix of the $A_{i_1 \ldots i_p}$ was arranged in some manner as a flat matrix in defining these ranks. But to continue the present problem it is necessary to consider the vanishing of k-way determinants picked from the p-way matrix without flattening it.

A number of relations exist among the simple or multiplex ranks already defined, some of which were suggested in the course of the work. These appear more clearly if the matrix is *not* flattened, *i.e.*, a multiplex file, instead of being straightened out and treated like a vector, is left in position, as a *couche* in the original matrix.[15] In this paper the analogy of a multiplex file with a vector has been expressly emphasized, bringing out the analogy of Theorem IV with Theorem II; this is undoubtedly somewhat artificial, but it enables us to extend results when $\pi = p$ immediately to arbitrary p, with the same value of π, by virtue of (27).

Finally, a word as to the space complement and notations for expressing it appears necessary. Any space complement, by Dr. Naess' definition, is a determinant. Its *components* can be expressed by summation with the $\epsilon_{r_1 \ldots r_n}$, etc., of Ricci and Levi-Civita. The

[15] My colleague, Mr. L. H. Rice, to whom I am indebted for many helpful suggestions, will treat these relations in a paper shortly to appear.

determinant of Naess having elements of every row the basis vectors $\mathbf{e}_1 \cdots \mathbf{e}_n$ is analogous to the pseudo scalar of Grassmanian analysis. What notation we employ is partly a matter of convenience, but should embody, if possible, those analogies which are of most importance in the problem concerned and in the logic employed. In the algebra of tensors or polyadics, of which the present problem is a part, the most important analogy, in the author's opinion, is with the symbolic method of handling invariants and other concomitants of quantics. Real vector factors behave like the symbolic factors of that method. Folding is analogous with "Faltung," due regard being had for the non-commutative character of the symbols. The quantities $\mathbf{F}_k((\mathbf{A}^q))$ are analogous with transvectants. This is why the notation $(\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3)$, etc., has been employed for space complements.