Math 136 - Linear Algebra **Winter 2016**

Lecture 17: February 10, 2016

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17.1 Special Subspaces Continued

Example 17.1 *For $\vec{a} \neq 0 \in \mathbb{R}^n$ Find a basis for the range of $Proj_{\vec{n}}$*

$$Range(Proj_{\vec{a}}) = \{Proj_{\vec{a}}(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$$

$$\forall \vec{x} \in \mathbb{R}^n Proj_{\vec{n}}(\vec{x}) = \frac{\vec{x} \cdot \vec{n}}{\|\vec{a}\|^2}\vec{a} \in Span\{\vec{a}\} \implies Range(Proj_{\vec{a}}) \subset Span\{\vec{a}\}n$$

On the other hand, $Proj_{\vec{a}}(t\vec{a}) = \frac{t\vec{a} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} = t\vec{a} \ \forall t \in \mathbb{R}$

$$\implies Span\{\vec{a}\} \subset Range(Proj_{\vec{a}}) \implies Range(proj_{\vec{a}}) = Span\{\vec{a}\}$$

Theorem 17.2 *Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear with the standard matrix $\begin{bmatrix} L \end{bmatrix} = (L(\vec{e_1}, \ldots, L(\vec{e_n}))$ Then,*

$$Range(L) = Span\{L(\vec{e_1}, \ldots, L(\vec{e_n})\}$$

Proof:

$$L(\vec{x}) = [\mathrm{L}]\vec{x} = \begin{bmatrix} L(\vec{e_1}, \ldots, L(\vec{e_n}) \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 L(\vec{e_1} + \ldots + x_n L(\vec{e_n})$$

$$\therefore L(\vec{x}) \in Range(L) \iff L(\vec{x}) \in Span\{L(\vec{e_1}, \ldots, L(\vec{e_n})\}$$

∎

Definition 17.3 *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping, we define its kernel as*

$$Ker(L) := \{\vec{x} \in \mathbb{R}^n \mid L(\vec{x}) = 0\}$$

For L : $\mathbb{R}^n \to \mathbb{R}^m$ and Standard matrix [L]

$L(\vec{x}) = \vec{0}$ iff $\begin{bmatrix} L \end{bmatrix} \vec{x} = \vec{0}$

$\vec{x} \in Ker(L)$ iff $\vec{x}$ is a soution of $(\begin{bmatrix} L \end{bmatrix} \mid \vec{0})$

Theorem 17.4 *If $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear then $Ker(L)$ is a subspace of $\mathbb{R}^n$*

Example 17.5 *Fix* $\vec{a} \neq 0, \vec{a} \in \mathbb{R}$ *, What is the* $Ker(Proj_{\vec{a}})$*? For* $\vec{a} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$ *write down a basis for* $Ker(Proj_{\vec{a}})$

$$\vec{x} \in Ker(Proj_{\vec{a}}) \iff Proj_{\vec{a}}(\vec{x}) = 0 \iff \frac{x \cdot a}{\|\vec{a}\|^2}\vec{a} = \vec{0} \iff \vec{x} \cdot \vec{a} = 0$$

$$\implies Ker(Proj_{\vec{a}}) = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{n} = 0\}$$

Since $\vec{a} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

$$\implies Ker(Proj_{\vec{a}}) = \{\vec{x} \in \mathbb{R}^3 \mid 2x_1 - x_2 + 5x_3 = 0\}$$

$$\therefore \text{ The basis is } \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Definition 17.6 *Given* $A \in M_{m \times n}(\mathbb{R})$ *we define the nullspace of* A *as,*

$$Null(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0\}$$

and if $A = (\vec{a_1}, \ldots, \vec{a_n})$ *we define its column space as*

$$Col(A) = Span\{\vec{a_1}, \ldots, \vec{a_n}\} \subset \mathbb{R}^m$$

for given A $\in M_{m \times n}(\mathbb{R})$ we define L : $\mathbb{R}^n \to \mathbb{R}^m$ as,

$$L(\vec{x}) = A\vec{x} = \big[L\big]\, \vec{x}$$

then, [L] = A

$$Ker(L) = \{\vec{x} \in \mathbb{R}^n \mid L(\vec{x}) = 0\} = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = 0\} = Null(A)$$
$$Range(L) = Span\{\vec{a_1}, \ldots, \vec{a_n}\} = Col(A)$$

End of Lecture Notes
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