Math 136 - Linear Algebra **Winter 2016**

Lecture 5: January 13, 2016

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5.1 Subspace Basis Examples

Find the basis for a given subspace

$$S_2 = \left\{ \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix} \mid a,b,c \in \mathbb{R}^3 \right\}$$

$$\forall \vec{x} \in S_2 \ \vec{x} = \begin{bmatrix} a-b \\ b-c \\ c-a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{x} \in Span \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$S_2 = Span \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} = Span \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ which is linearly independant}$$

$$\therefore \text{ this is a basis for } S_2$$

5.2 Dot Product

Recall :

- In $\mathbb{R}^2, \forall \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2$
- $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2} = \sqrt{\vec{x} \cdot \vec{x}}$
- $\vec{x} \cdot \vec{y} = \|x\| \|y\| \cos\theta$

Proof:

$$\vec{x} - \vec{y} = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix} \cos\theta = \frac{\|\vec{x}\|^2 + \|\vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2}{2\|\vec{x}\|\|\vec{y}\|}$$

$$= \frac{x_1^2 + x_2^2 + y_2^2 + y_2^2 - ((x_1 - y_1)^2 + (x_2 - y_2)^2)}{2\|\vec{x}\|\|\vec{y}\|}$$

$$\cos\theta = \frac{2x_1y_1 + 2x_2y_2}{2\|\vec{x}\|\|\vec{y}\|} = \frac{x_1\theta_1 + x_2\theta_2}{\|\vec{x}\|\|\vec{y}\|}$$

$$= \vec{x} \cdot \vec{y} = x_1 y_2 + x_2 y_2$$

$$= \|\vec{x}\|\|\vec{y}\| \cos\theta$$

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Definition 5.1 $\forall \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n, \vec{x} \cdot \vec{y} = x_1 y_1 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i y_i$

Proposition 5.2

$\forall \vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n, \forall s, t \in \mathbb{R}$

1. $\vec{x} \cdot \vec{x} = 0$ *iff* $\vec{x} = \vec{0}$
2. $\vec{x} \cdot \vec{y} = \vec{y} = \vec{x}$
3. $\vec{x}(t\vec{y} + s\vec{z}) = t(\vec{x} \cdot \vec{y}) + s(\vec{x} \cdot \vec{z})$

Definition 5.3 $\forall \vec{x} \in \mathbb{R}$ *The length (or norm) of* $\vec{x}$ *is* $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$

Also , $\vec{x}$ is called the unit vector if $\|\vec{x}\| = 1$

Theorem 5.4 $\forall \vec{x}\vec{y} \in \mathbb{R}^n, t \in \mathbb{R}$ *we have*

1. $\|\vec{x}\| \geq 0$ *iff* $\vec{x} = \vec{0}$
2. $\|t\vec{x}\| = \mid t \mid \|\vec{x}\|$
3. *Cauchy Schwarz inequality :* $\|\vec{x} \cdot \vec{y}\| \leq \|\vec{x}\| \|\vec{y}\|$
4. *Triangle inequality :* $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Proof: Cauchy Schwarz inequality

If $\vec{x} = \vec{0}$ it is clear, suppose $\vec{x} \neq \vec{0}$

$0 \leq \|t\vec{x} + t\vec{y}\|^2 = (t\vec{x} + y) \cdot (t\vec{x} + y) = t^2 \vec{x} \cdot \vec{x} + 2t\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y}$

$0 \leq \|x\|^2 t^2 + 2(\vec{x} \cdot \vec{y})t + \|\vec{y}\|, \forall t \in \mathbb{R}$

$4(\vec{x} \cdot \vec{y})^2 \leq 4\|\vec{x}\|^2 \|\vec{y}\|^2$ Using the discriminant

$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

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Remarks : If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

$\vec{x} - \vec{y} = x_1 y_1 + \ldots + x_n y_n$

$\|\vec{x}\| = x_1^2 + \ldots x_n^2 \quad \|\vec{y}\| = y_1^2 + \ldots y_n^2\|$

Coordinate Form of C S

$\|x_1 y_1 + \ldots + x_n y_n\| \leq \sqrt{x_1^2 + \ldots x_n^2} \sqrt{y_1^2 + \ldots y_n^2}$

Try letting n = 3

End of Lecture Notes
Notes By : Harsh Mistry