Math 239 - Introduction to Combinatorics **Spring 2017**

Lecture 9: May 19th, 2017

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Problem 9.1 *How many compositions of n are there?*
Solution :

1. $\mathbb{N} = \{1, 2, \ldots\}$
   For $k = 0, 1, 2, \ldots$, *Let* $S_k = \mathbb{N}^k$ *and* $S = \cup_{k \geq 0} S_k$

$$w(C_1, \ldots, C_k) = C_1 + \ldots + c_k, \quad \forall (C_1, \ldots, C_k) \in S$$
$$w(c) = c, \forall C \in \mathbb{N}]$$

2.

$$\begin{aligned}
\phi_{\mathbb{R}}(x) &= x^1 + x^2 + \ldots \\
&= x(x^0 + x^1 + \ldots) \\
&= \frac{x}{1-x} \\
\phi_{S_k}(x) &= \phi_{N^k}(x) \\
&= (\phi_N(x))^k \\
&= \left(\frac{x}{1-x}\right)^k \\
\phi_S(x) &= \phi_{\cup s_k}(x) = \sum \phi_{S_k}(x) = \sum (\frac{x}{1-x})^k = \frac{1-x}{1-2x}
\end{aligned}$$

3.

$$\begin{aligned}
\phi_S(x) &= [x^n] \frac{1-x}{1-2x} \\
&= [x^n] \frac{1-x-x+x}{1-2x} \left[\frac{1-2x}{1-2x} + \frac{x}{1-2x}\right] \\
&= [x^n] \left(1 + \frac{x}{1-2x}\right) \\
&= [x^n] \left(1 + x \sum_{l \geq 0} (2x)^l\right) \\
&= [x^n] \left(1 + \sum_{l \geq 0} 2^l x^{l+1}\right) \qquad = \begin{cases} 1 \ , \ n = 0 \\ 2^{n-1} \ , \ n \geq 1 \end{cases}
\end{aligned}$$

Problem 9.2 *How many compositions of n are there with k parts, each part being at most 5.*
Solution:

1. $A = \{1, 2, 3, 4, 5\}$, $S = A^k$

$$w(C_1, \ldots, C_k) = C_1 + \ldots + C_k$$

$$w(c) = c \forall c \in A$$

2.

$$\begin{aligned}\phi_A(x) &= x^1 + x^2 + x^3 + x^4 + x^5 \\ &= x(1 + x^2 + x^3 + x^4) \\ &= x\left(\frac{1 - x^5}{1 - x}\right) \\ \phi_S(x) &= \phi_{A^k}(x) = (\phi_A(x))^k = x^k(1 - x^5)^k(1 - x)^{-k}\end{aligned}$$

3.

$$\begin{aligned}x^k(1-5)^k(1-x)^{-k} &= [x^{n-k}](1-x^5)^k - (1-x)^{-k} \\ &= [x^{n-k}]\left(\sum_{l=0}\binom{k}{l}(-x^5)^l\right)\left(\sum_{m\geq 0}\binom{m+k-1}{k-1}x^m\right)\end{aligned}$$

$$[x^{n-k}]\sum_{t\geq 0}\left(\sum_{S=0}^{t} a_S b_{t-S}\right)x^t = \sum_{s=c}^{n-k} a_S b_{n-k-S}$$