Math 239 - Introduction to Combinatorics	**Spring 2017**

Lecture 2: May 3rd, 2017

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2.1 Admin Info

Alan's office hours
Mon, Fri : 12:30 - 13:30
Tue : 11:30 - 12:30

2.2 Recap

$$\binom{n}{k} = \frac{n!}{k(n-k)!} = \# \text{ k-subsets of an n-element set} = \# \text{ of ways we can choose k objects amoung n}$$

2.3

Definition 2.1 *A* <u>*Bijection*</u> *is a function* $f : S \to T$ *such that*

- f *is 1-1 : for every* $x_1, x_2 \in S, x_1 \neq x_2, f(x_1) \neq f(x_2)$
- f *is onto : for every* $y \in T$ *there exists* $x \in S$ *such that* $f(x) = y$

Definition 2.2 *An inverse for f is a function* $f^{-1} : T \to S$ *such that*

- $f^{-1}(f(x)) = x$, $\forall x \in S$
- $f(f^{-1}(y)) = Y$, $\forall y \in T$

Lemma 2.3 *f is a bijection* $\iff$ *f has an inverse*

Problem 2.4 *Show that* $\binom{n}{k} = \binom{n}{n-k}$ *by showing that there is a bijection between*

$S =$ *set of all k-subsets of* $\{1, ..., n\}$

$T =$ *set of all (n-k) subsets of* $\{1, \ldots, n\}$

Solution : *Define* $f : S \to T$ *and* $f^{-1} : T \to S$.
$f(A) = \{1, \ldots, n\} \setminus A, \forall A \in S$
$f^{-1}(B) = \{1, \ldots, n\} \setminus B, \forall B \in T$

Note : Note that f and f^{-1} are well defined because
$\{1,\ldots,n\}\setminus A$ has (n-k) elements $\forall A \in S$
$\{1,\ldots,n\}\setminus b$ has k elements $\forall B \in T$

f^{-1} is the inverse of f because

$$\begin{aligned} f^{-1}(f(A)) &= f^{-1}(\{1,\ldots,n\}\setminus A) \\ &= \{1\ldots,n\}\ (\{1,\ldots,n\})\setminus A \\ &= A, \forall A \in S \end{aligned}$$

2.4 Binomial Theorem

$(1+x)^2 = 1 + 2x + x^2$

Theorem 2.5

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \binom{n}{0} x^0 + \binom{n}{1} x^1 + \ldots + \binom{n}{n} x^n$$

Proof:

$$\begin{aligned} (1+x)^n &= (1+x) + (1+x) + \ldots + (1+x) \\ &= \underline{1}1 + \underline{(}n)x + \underline{\binom{n}{2}} x^2 + \ldots + \underline{\binom{n}{k}} x^k + \ldots + \underline{\binom{n}{n}} x^n \end{aligned}$$

To get x^k among the n copies of (1 + x) we choose k. Thre are n choose k ways to do this ∎

2.5 Combinatorial Proofs

Proposition 2.6 $2^n = \sum_{k=0}^{n} \binom{n}{k}$

Algebraic Proof : Plug x = 1 in to the binomial theorem

Combinatorial Proof :
The idea is to count a set in two differing ways such that one way gives you the <u>left hand side</u> f the identity you want to prove and the another way gives you the the <u>right hand side</u>

Proof: Let S be the subset of all subsets $\{1,\ldots,n\}$

LHS : $\mid S \mid = s^n$, since every element has two possibilities. Its either within a subset or not included within a subsets

RHS : For every $k \in \{0,\ldots,n\}$, let s_k = k-subsets of $\{1,\ldots,n\}$

$$\begin{aligned} S &= S_0 \cup S_1 \cup S_2 \cup \ldots \cup S_n \\ S_i \cap S_j &= \emptyset \text{ for } i \neq j \\ \mid S \mid &= \mid S_0 \mid + \ldots + \mid S_n \mid \\ &= \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} \end{aligned}$$

$\blacksquare$