and 14

CS 245 - Logic and Computation **Fall 2016**

Lecture 13 and 14: October 25 - 27, 2016

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13.1 Proofs in First-Order Logic using Natural Deduction

13.1.1 New Natural Deduction Rules

- $\forall - Elimination$: if $\sum \vdash \forall x\alpha$ then $\sum \vdash \alpha[t/x]$

$$\frac{\forall x\alpha}{\alpha[t/x]}$$

- $\exists - Introduction$: if $\sum \vdash \alpha[t/x]$ then $\sum \vdash \exists x\alpha$

$$\frac{\alpha[t/x]}{\exists x\alpha}$$

- $\forall - Introduction$: if $\sum \vdash \alpha[y/x]$ and y not free in $\sum$ or α, then $\sum \vdash \forall x\alpha$

$$\begin{array}{c} y\text{fresh} \\ \vdots \\ \alpha[y/x] \\ \hline \forall x\alpha \end{array}$$

  - A variable is fresh in a sub proof if it occurs nowhere outside the box of the subproof. (i.e not a free variable outside the subproof.

- $\exists$ - Elimination : if $\sum, \alpha[u/x] \vdash \beta$ with u fresh, then $\sum, \exists x\alpha \vdash \beta$

$$\begin{array}{cc} & \alpha[u/x], \text{u fresh} \\ \exists x\alpha & \vdots \\ & \beta \\ \hline \multicolumn{2}{c}{\beta} \end{array}$$

13.1.2 Soundness of $\forall$ Elimination and $\exists$ Introduction

- For any formula φ, variable x and term t

$$\forall x\varphi \models \varphi[t/x] \text{ and } \varphi[t/x] \models \exists x\varphi$$

- For every formula φ, variable x and t

$$\varphi[t/x]^{(I,E)} = \varphi^{(I,E[x \to t^{(I,E)}])}$$

13.1.3 Defining Substitution

For a variable x and term t

1. If α is $(Qx\beta)$, then $\alpha[t/x]$ is α
2. If α is $(Qy\beta)$ for some other variable y, then
   - if y does not occur in t, then $\alpha[t/x]$ is $(Qy\beta[t/x])$
   - Otherwise, let z be a variable that occurs in neither α not t; then $\alpha[t/x]$ is $(Qz(\beta[z/y])[t/x])$

With this definition, we always get that, as required

$$(Qy\alpha)^{(I,E[x \to t^{(I,E)}])} = ((Qy\alpha)[t/x])^{(I,E)}$$

13.1.4 FOL with Equality

First order logic with equality is first order logic with the restriction that the symbol "=" must be interpreted as equality on the domain

$$(=)^I = \{(d,d) \mid d \in dom(I)\}$$

There are two way to account for this restriction in our proofs.

1. Add deduction rules for symbol =
   - Introduction
   $$\frac{}{t = t} = i$$
   - Elimination
   $$\frac{t_1 = t_2 \qquad \alpha[t_1/x]}{\alpha[t_2/x]}$$
2. Use axioms rather than deduction rules
   (a) $\forall\alpha\alpha = \alpha$ is an axiom
   (b) For Each formula α and variable z : $\forall\alpha\forall y(\alpha = y \to (\alpha[\alpha/z] \to \alpha[y/z]))$ is an axiom

   These axioms imply
   - Symmetry of =: $\vdash \forall x\forall y(x = y \to y = x)$
   - Transitivity of =: $\vdash \forall w\forall x\forall y(x = y \to (y = w \to x = w))$

13.1.5 Derived rules for Equality

- EQtrans(k) :
$$\frac{t_1 = t_2 \qquad \dots \qquad t_k = t_{k+1}}{t_1 = t_{k+1}}$$
- EQsubs(r) :
$$\frac{t_1 = t_2}{r[t_1/z] = r[t_2/z]}$$