- 28

Stat 230 - Probability **Fall 2016**

Lecture 24 - 28: November 7 - 16, 2016

Lecturer: Nagham Mohammad *Notes By: Harsh Mistry*

24.1 Rectangular Distribution

A continuous random variable, X is said to have a Uniform distribution from a to b on the interval [a.b], U(a,b), if all subintervals of fixed length are equally likely

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$\mu = E(X) = \frac{(a+b)}{2} \qquad \sigma^2 = V(X) = \frac{(b-a)^2}{12}$$

24.1.1 The Cumulative Distribution function

$$F(x) = P[X \leq x] = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & c > b \end{cases}$$

24.2 Exponential Distribution

- Widely used in engineering and science disciplines
- Exponential distribution is used to describe the time or distance until some event happens
- In Poisson process for events in time, let X be the length of time we wait for the first event occurrence
- X is said to have Exponential distribution if p.d.f of C is

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

- Where $\lambda > 0$

24.2.1 The Cumulative Distribution function

$$f(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Where : $\lambda = \frac{1}{\mu}$

24.2.2 Mean and Variance

$$E(X) = \mu = \frac{1}{\lambda}$$
$$\sigma^2 = \frac{1}{\lambda^2}$$
$$\sigma = \frac{1}{\lambda}$$

24.2.3 Alternate Form

It is common to use the parameter $\theta = \frac{1}{\lambda}$ in the Exponential distribution.

$$E(x) = \mu = \theta$$
$$f_X(x) = \begin{cases} \frac{1}{\theta}e^{-x/\theta} & x > 0 \\ 0 & x \leq 0 \end{cases}$$
$$F_X(x) = \begin{cases} 1 - e^{-x/\theta} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Note

- Average rate of occurrence $= \lambda$
- Average waiting time for an occurrence $= \theta$

24.3 Lack of Memory Property

$$P(X > c + b \mid X > b) = P(X > c)$$
$$P(X \leq c + b \mid X \geq b) = P(X \leq c)$$

For a Poisson process, gievn that you have waited b units of time for the next event, the probability you wait an additional c time does not depend on b, but only depends on c.

24.4 Mean and Variance

- Finding μ and σ^2 directly involves integration by parts
- An easier solution uses properties of Gamma functions

24.4.1 Gamma Function

$\Gamma(\alpha)$, is called a gamma function of α where $\alpha > 0$, is defined as :

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx$$

Integration for $\Gamma(\alpha)$ by parts yields

1. $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$
2. $\Gamma(\alpha) = (\alpha - 1)!$, If α is a positive integer
3. $\Gamma(x + 1) = x\Gamma(x)$
4. $\Gamma(1/2) = \sqrt{\pi}$

24.4.2 Gamma Distribution

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

With a shape parameter $\alpha > 0$ and a scale parameter $\beta > 0$

- The gamma distribution is a two-parameter family of continuous probability distributions.
- The common Exponential distribution and Chi-squared distribution are special cases of the Gamma distribution.

24.4.3 Gamma Mean/Variance

$$E(X) = \alpha\beta$$
$$Var(X) = \alpha\beta^2$$

24.5 The Normal Distribution

A random variable, X, is said to have a Normal distribution with mean and variance σ^2, if X is a continuous random variable with probability density function f (x):

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad \begin{array}{c} \sigma > 0 \\ -\infty < x < +\infty \\ -\infty < \mu < +\infty \end{array}$$

The Normal distribution is often denoted by : X ˜ $N(\mu, \sigma^2)$

24.6 The Standard Normal Distribution

- The Normal distribution with parameter values = 0 and =1 is called a standard Normal distribution
- A r.v. has a standard Normal distribution is called a standard Normal random variable and denoted by Z.
- Z - N(0,1) where $\mu = 0$ and $\sigma = 1$ has the probability density function

$$\phi(z) = \frac{1}{\sqrt{2pi}} e^{-\frac{1}{2}z^2} \text{ for } z \in R$$

24.6.1 Standardized Score

- Also known as "standard score" or "z-score"
- The standardized score is a number that represents the number of standard deviations a data point is from the mean

$$\text{z-score } = \frac{\text{observed value - mean}}{\text{standard deviation}}$$

$$z = \frac{x - \mu}{\sigma}$$

24.7 Percentiles and Standardized Scores

To find percentiles for normal curves, you need :

1. Your own value
2. The mean of the population
3. The standard deviation for the population(s.d)
4. Find the standardized score

Then you can use the table to find percentiles.

24.7.1 Useful Results

1. $P(Z \leq -a) = 1 - P(Z \leq a)$
2. $P(Z > -a) = P(Z \leq a)$
3. $P(\mid Z \mid \leq a) = 2P(Z \leq a) - 1$