CS 245 - Logic and Computation **Fall 2016**

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4.1 Classifying Formulas

- A formula α is a tautology if any only if for every truth valuation t, $\alpha^t = T$
  - $(p \vee (\neg p))$ is a tautology
- A formula α is a contradiction if and only if for every truth valuation t, $\alpha^t = T$
  - $(p \wedge (\neg P)$ is a contradiction
- A formula α is satisfiable if and only if there is some truth valuation t such that $\alpha^t = T$
  - $(p \implies q)$ is satisfiable if you set both variables to T.

Note : A formula is satisfiable if and only if it is not a contradiction

4.2 "Short-Cutting" a Truth Table

Instead of filling an entire truth table, we can observer what happens if we set a variable to T or F in order to simply the formula. We can use this to evaluate formulas, be creating a valuation tree

4.3 Equivalence of Formulas

Two formulas α**And**β are said to be equivalent if they share the same final column in their respective truth tables. To indicate this we use teh following notion :

$$\alpha \equiv \beta$$

Lemma 4.1 *Suppose that $\alpha \equiv \beta$. Then for any formula γ and any connective *, the formulas $(\alpha * \gamma)$ and $(\beta * \gamma)$ are equivalent : $(\alpha * \gamma) \equiv (\beta * \gamma)$*

4.3.1 Algebra of Formulas

Many equivalences of formulas look much like rules of ordinary arithmetic.

- Commutativity
  - $(\alpha \wedge \beta) \equiv (\beta \wedge \alpha)$

  - $(\alpha \vee \beta) \equiv (\beta \vee \alpha)$

- Associativity
  - $(\alpha \wedge (\beta \wedge \gamma)) \equiv ((\alpha \wedge \beta) \wedge \gamma)$
  - $(\alpha \vee (\beta \vee \gamma)) \equiv ((\alpha \vee \beta) \vee \gamma)$
- Distributivity
  - $(\alpha \vee (\beta \wedge \gamma) \equiv ((\alpha \vee \beta) \wedge (\alpha \vee \gamma)$
  - $(\alpha \wedge (\beta \vee \gamma) \equiv ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$
- Idempotence
  - $(\alpha \vee \alpha) \equiv \alpha$
  - $(\alpha \wedge \alpha) \equiv \alpha$
- Double Negation
  - $(\neg(\neg\alpha)) \equiv \alpha$
- De Morgan's Laws
  - $\neg(\alpha \wedge \beta) \equiv (\neg\alpha \vee \neg\beta)$
  - $\neg(\alpha \vee \beta) \equiv (\neg\alpha \wedge \neg\beta)$
- Simplification I
  - $(\alpha \wedge T) \equiv \alpha$
  - $(\alpha \vee T) \equiv T$
  - $(\alpha \wedge F) \equiv F$
  - $(\alpha \vee F) \equiv \alpha$
- Simplification II
  - $(\alpha \vee (\alpha \wedge \beta)) \equiv \alpha$
  - $(\alpha \wedge (\alpha \vee \beta)) \equiv \alpha$
- Implication
  - $(\alpha \implies \beta) \equiv ((\neg\alpha) \vee \beta)$
- Contrapositive
  - $(\alpha \implies \beta) \equiv ((\neg\beta) \implies (\neg\alpha))$
- Equivalence
  - $(\alpha \iff \beta) \equiv ((\alpha \implies \beta) \wedge (\beta \implies \alpha))$
- Excluded Middle
  - $(\alpha \vee (\neg\alpha)) \equiv T$
- Contradiction
  - $(\alpha \wedge (\neg\alpha)) \equiv F$

4.4 Satisfiability of Sets of Formulas

Let $\sum$ denote a set of formulas and t a valuation define : $\sum^t = \begin{cases} T \text{ if for each } \beta \in \sum, \beta^t = T \\ F \text{ otherwise} \end{cases}$

When $\sum^t = T$, we say that t **satisfies** *sum*

A Set $\sum$ is **Satisfiable** iff there is some valuation t such that $\sum^t = T$

4.5 Logical Consequence (Entailment)

Let *sum* be a set of formulas and let α be a formula. We say that

- α is a logical consequence of *sum*, or
- $\sum$ entails α, or
- $sum \models \alpha$

if and only if for any truth valuation t, if $\sum^t = T$ then also $\alpha^t = T$

4.5.1 Equivalence and Entailment

Equivalence can be expressed using the notion of entailment.

Lemma 4.2 $\alpha \equiv \beta$ *if and only if both*$\{\alpha\} \models \beta$ *and* $\{\beta\} \models \alpha$