**CS 370 - Numerical Computation** **Fall 2018**

# Fourier Analysis

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- Basic Idea
  - Transform some function/data into a form that reveals frequency of information in data
  - Process/analyze it in this frequency domain form, which makes some tasks easier
  - Transform back
- Original data/function is in "time domain" if $f$ is a function of time $f(t)$
- Original data/function is in "Spatial domain" if $f$ is a function of space/position $f(x)$

## 5.1 Continuous Fourier Series

- Consider some periodic function f(t) with period T so, $f(t \pm T) = f(t)$
- The goal is to represent any $f(t)$ as an infinite sum of trig functions

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\frac{2\pi kt}{T}) + \sum_{k=1}^{\infty} b_k \sin(\frac{2\pi kt}{T})$$

  $a_k, b_k$ indicate the amplitude for each sinusoid of a specific period $\frac{T}{k}$ or frequency $\frac{k}{T}$
  Higher Integer k indicates shorter period & higher wave frequency

### 5.1.1 Determine the Coefficients

- Assume that the range of t is $t \in [0, 2\pi]$ and period T= 2n

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt)$$

**Orthogonality Relations**

- $\int_0^{2\pi} \cos(kt)\sin(jt)dt = 0$ for any integers k,j
- $\int_0^{2\pi} \cos(kt)\cos(jt)dt = 0$ for $k \neq j$
- $\int_0^{2\pi} \sin(kt)\sin(jt)dt = 0$ for $k \neq j$
- $\int_0^{2\pi} \cos(kt)dt = 0$
- $\int_0^{2\pi} \sin(kt)dt = 0$

Using this we can find the coefficients by solving integrals

$$a_0 = \frac{\int_0^{2\pi} f(t)dt}{2\pi 2}$$

$$a_k = \frac{\int_0^{2\pi} f(t)\cos(kt)dt}{\int_0^{2\pi} \cos^2(kt)dt}$$

$$b_k = \frac{\int_0^{2\pi} f(t)\sin(kt)dt}{\int_0^{2\pi} \sin^2(kt)dt}$$

### 5.1.2 Fourier Series with Complex Exponentials

- the sinusoidal expression can be represented as

$$f(t) = \sum_{k=-\infty}^{+\infty} c_k e^{ikt}$$

  Where $c_k$ are complex numbers

- We can derive a simple conversion

$$a_0 = c_0$$

$$c_k = \frac{a_k}{2} - \frac{ib_k}{2}$$

$$c_{-k} = \frac{a_k}{2} + \frac{ib_k}{2}$$

- Which gives us the relationships

$$|a_0| = |c_0|$$

$$|c_k| = |c_{-k}| = \frac{1}{2}\sqrt{a_k^2 + b_k^2}$$

- The orthogonality property is

$$\int_0^{2\pi} e^{ikt} e^{-ilt} dt = \begin{cases} 0; k \neq 1 \\ 2\pi; k = 1 \end{cases}$$

- Which gives us

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} e^{ikt} f(t) dt$$

## 5.2 Discrete Fourier Transform

- Given N point and period T, assuming N is even the series can be approximated as

$$f(t) \approx \sum_{-\frac{N}{2}+1}^{N/2} c_k e^{\frac{(2\pi i)kt}{T}}$$

  By plugging in N data points will give N equations with unknown
  This leads towards the (inverse) Discrete Fourier Transform

### 5.2.1 Inverse Discrete Fourier Transform

- Discrete data $f_n$ is expressed as sum of coefficients $F_k$ times complex exponentials

$$f_n = \sum_{k=0}^{N-1} F_k e^{i(\frac{2\pi nk)}{N}} = \sum_{k=0}^{N-1} F_k W^{nk}$$

- With $W = e^{\frac{2\pi i}{N}}$ , $W^k = e^{\frac{2\pi ik}{N}}$ which represents the Nth roots of unit
- Using this we find that the inverse is (DFT)

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n W^{-nk}$$

## 5.3 DFT Properties

- The sequence $\{F_k\}$ is doubly infinite and periodic
- Conjugate symmetry : if data $f_n$ is real, $F_k = \bar{F}_{N-k}$
  Hence the $|F_k|$ are symmetric about $k = \frac{N}{2}$

## 5.4 Fast Fourier Transform

If $N \neq 2^m$, pad initial data with 0's

1. Split The full DFT into two DFT's of half the length
2. Repeat recursively
3. Finish at the base case : the DFT of individual pairs of numbers

### 5.4.1 Dividing Up

$$g_n = \frac{1}{2}(f_n + f_{n+\frac{N}{2}})$$
$$h_n = \frac{1}{2}(f_n - f_{n+\frac{N}{2}})W^{-n}$$

Then $F_{even}$ = DFT(g) and $F_{odd}$ = DFT(h)