Math 136 - Linear Algebra **Winter 2016**

Lecture 11: January 27, 2016

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11.1 Rank Continued

r(A) $\leq r(A \mid \vec{b})$
r(A) $= r(A \mid \vec{0})$

Corollary 11.1 *A constant linear system $(A \mid \vec{b})$ has a unique solution iff $r(A) = n$ (no free variables) In addition, a homogeneous system $(A|\ \vec{0})$ only has a trivial solution iff $r(A) = n$*

Theorem 11.2 *A set of n vectors $\{\vec{v_1} \dots \vec{v_n}\} \subset \mathbb{R}^n$ is linear independent iff it spans $\mathbb{R}^n$*
$\implies \{\vec{v_1} \dots \vec{v_n}\}$ *is a basis iff it is linear independent*

Proof: Given $\{\vec{v_1} \dots \vec{v_n}\} \subset \mathbb{R}^n$, Let A $= \{\vec{v_1} \dots \vec{v_n}\}$
$SpanA = \mathbb{R}^n \iff \forall \vec{x} \in \mathbb{R}^n, \vec{x} = t_1\vec{v_1} + \dots + t_n\vec{v_n} \iff (A \mid \vec{x})$ is constant for any $\vec{x} \in \mathbb{R}^n \iff$ $r(A) = n \iff (A \mid \vec{0}$ only has a trivial solution $\iff t_1\vec{v_1} + \dots + t_n\vec{v_n} = 0$ only has a trivial solution $\iff \{\vec{v_1} \dots \vec{v_n}\}$ is linear independent ∎

Theorem 11.3 *Let $(A \mid \vec{b})$ be a constant linear system, $r(A) = r$ then the solution set has he form*

$$\{\vec{x} \in \mathbb{R}^n \mid \vec{x} = \vec{x_0} + t_1\vec{v_1} + \dots + t_{n-r}\vec{v_{n-r}}, t_i \in \mathbb{R}\}\}$$

where $\{\vec{v_1} \dots \vec{v_{n-r}}\}$ is linear independent

Example 11.4

$$\begin{bmatrix} 1 & 2 & 0 & | & 1 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \to \begin{cases} x_1 + 2x_2 = 1 \\ x_3 = 1 \end{cases} \quad \text{let } t = x_3 \to \vec{x} = \begin{bmatrix} 1 - 2x \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

11.2 Rank and Linear Independence

$$\begin{aligned} \{\vec{v_1} \dots \vec{v_n}\} \subset \mathbb{R}^n &\iff t_1\vec{v_1} + \dots + t_k\vec{v_k} = \vec{0} \text{ has a trivial solution} \\ &\iff (\vec{v_1} \dots \vec{v_k} \mid \vec{0}) \text{ only has a trivial solution} \\ &\iff r(\vec{v_1} \dots \vec{v_k}) = k \end{aligned}$$

Example 11.5 *Show* $\left\{ \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}0\\2\\3\end{bmatrix} \begin{bmatrix}1\\1\\2\end{bmatrix} \right\}$ *is linear independent*

$$\begin{bmatrix}1&0&1\\1&2&1\\0&3&2\end{bmatrix} \to \begin{bmatrix}1&0&1\\0&2&1\\0&3&2\end{bmatrix} \to \begin{bmatrix}1&0&1\\0&1&0\\0&3&2\end{bmatrix} \to \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$$

$$r\begin{bmatrix}1&0&1\\1&2&1\\0&3&2\end{bmatrix} = 3$$

Since there is 3 vectors and the rank is 3, the set is linear independent

Example 11.6 $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix} \begin{bmatrix}-1\\0\\2\end{bmatrix} \right\} \subset \mathbb{R}^3$ *is linear independent*

$$\begin{bmatrix}1&-1\\2&0\\3&2\end{bmatrix} \to \begin{bmatrix}1&-1\\0&2\\0&5\end{bmatrix} \to \begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix} \implies Rank = 2 \implies \text{ the set is linear independent}$$

Example 11.7 *Prove* $\left\{ \begin{bmatrix}2\\-1\\2\end{bmatrix} \begin{bmatrix}-1\\1\\0\end{bmatrix} \begin{bmatrix}2\\0\\5\end{bmatrix} \right\}$ *is a basis of* $\mathbb{R}^3$

$$r\begin{bmatrix}2&-1&2\\-1&1&0\\2&0&5\end{bmatrix} = 3 \implies \text{The set is linear independent} \implies \text{The set spans } \mathbb{R}^3$$

End of Lecture Notes
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