Math 239 - Introduction to Combinatorics **Spring 2017**

Lecture 30: July 10th, 2017

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Goal : Prove 5-Colour Theorem

Observation 1 : If we consider a k-colouring of G and permute the colours, then such new colouring is a k-colouring of G.

Definition 30.1 *Let G be a k-coloured graph using colours in 6. For $t, j \in 6, (i \neq j)$, the subgraph $G(i,j)$ consists of all vertices in G that are coloured i or j, and all the edges that have ends coloured i and j*

Observation 2 : Let G be a k-coloured graph using colours in ζ. Suppose u, v are vertices coloured i and j, respectively $i \neq j$. If u and v are on different components of $G(i,j)$, then there exists a k-colouring of G for which u and v have coloured t.

1. Consider a k-colouring of G
2. Look at $G(i,j)$
3. Swap colours in component containing v
4. Restore other colours / vertices of G

Theorem 30.2 ***5-colour Theorem :*** *Every planar graph is 5-colourable*

Proof:

Base : 0 vertices

I.H. : Every planar graph with at most n-1 vertices has a 5-colouring using $\zeta = \langle 1, \ldots, 6 \rangle$

I.C. : Let G be a planar graph with n vertices and let v be a vertex of degree of at most 5. Consider a 5-colouring of $G - v$. We may assume that $deg(v) = 5$ and that the 5 colours are used to colour the neighbours of v (otherwise, we are down).
By permuting colours (if necessary), we may assume that colours 1, 2,3,4,5 occurs in a cyclic order at v_j in this planar embedding of G. Label neighbours of v as u_1, u_2, u_3, u_4, u_4 where u_i is coloured i

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