Math 239 - Introduction to Combinatorics **Spring 2017**

Lecture 7: May 15th, 2017

Lecturer: Alan Arroyo Guevara *Notes By: Harsh Mistry*

7.1 Product and Sum Lemmas

Recall : Given a set S and a weight function w

$$\phi_s(x) = \sum^{w(\sigma)} = \sum_{n \geq 0} a_n x^n$$

Lemma 7.1 *Sum Lemma*
Let S be a set with a weight function w. Suppose $S_0, S_1, \ldots, S_k$ is a partition of S. Then,

$$\phi_S(x) = \phi_{S_0}(x) + \phi_{S_1}(x) + \ldots + \phi_{S_k}(x)$$

Proof:

$$\begin{aligned} \phi_S(x) = \sum_{\sigma \in S} X^{w(\sigma)} &= \sum_{\sigma \in S_0} X^{w(\sigma)} + \sum_{\sigma \in S_1} X^{w(\sigma)} + \ldots + \sum_{\sigma \in S_k} X^{w(\sigma)} \\ &= \phi_{S_0}(x) + \phi_{S_1}(x) + \ldots + \phi_{S_k}(x) \end{aligned}$$

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Example 7.2 *Let $S = \mathbb{N}_0 = \langle 0, 1, 2, 3, \ldots \rangle$*

$$w(\sigma) = \begin{cases} \frac{\sigma}{2} & \sigma \text{ is even} \\ \sigma & \text{is odd} \end{cases}$$

$w(2) = 1, w(3) = 3$, *now find* $\phi_S(x)$

Solution : $S = E \cup O$

$$E = \langle 0, 2, 4, \ldots \rangle \quad O = \langle 1, 3, 5, 7, \rangle$$

$$\begin{aligned} \phi_E(x) &= x^0 + x^1 + x^2 + \ldots = \frac{1}{1-x} \\ \phi_O(x) &= x^1 + x^3 + x^5 + x^7 + \ldots \\ &= x(x^0 + x^2 + x + x^4 + \ldots) \\ &= x \cdot \left(\frac{1}{1-x^2} \right) \end{aligned}$$

By Sum Lemma

$$\phi_S(x) = \phi_E(x) + \phi_O(x) = \frac{1}{1-x} + \frac{x}{1-x^2}$$

Lemma 7.3 *Product Lemma*
Let A, B be sets.
Suppose A has weight function α
Suppose B has weight function β
Let $S = A \times B$ and suppose that for every $(a, b) \in S$,

$$\phi((a,b)) = \alpha(a) + \beta(b) \implies \phi_S(x) = \phi_A(x) \cdot \phi)_B(x)$$

Proof:

$$\begin{aligned}
\phi_A(x) \cdot \phi_B(x) &= \left(\sum_{a \in A} X^{\alpha(a)}\right)\left(\sum_{b \in B} X^{\beta(b)}\right) \\
&= \sum_{a \in A, b \in B} X^{\alpha(a)} \cdot X^{\beta(b)} \\
&= \sum_{(a,b) \in A \times B} X^{\alpha(a)+\beta(b)} \\
&= \sum_{(a,b) \in S} X^{w((a,b))} \\
&= \phi_S(x)
\end{aligned}$$

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Problem 7.4 *We throw a die (with values 1, 2, 3, 4, 5, 6) and a 4-sided die (With values 1 , 2, 3, 4). Given $k \in \langle 2, 3, \ldots, 10 \rangle$. In how many ways can we get k, as the subset of the dice.*

Equivalent Problem : *Let $A = \langle 1, \ldots, 6 \rangle$, $S = \langle 1, 2, 3, 4 \rangle$, $S = A \times B$, and w((a,b)) = a + b. Find $[x^k]\phi_s(x)$.*

Brute Force Solution :

6 sided / 4 sided	*1*	*2*	*3*	*4*
1	x^2	x^3	x^4	x^5
2	x^3	x^4	x^5	x^6
3	x^4	x^5	x^6	x^7
4	x^5	x^6	x^7	x^8
5	x^6	x^7	x^8	x^9
6	x^7	x^8	x^9	x^{10}

$$\phi_S(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + 4x^6 + 4x^7 + 3x^9 + x^10$$

Using the Product Lemma :
$A = \langle 1, 2, \ldots, 6 \rangle$ and $B = \langle 1, 2, 3, 4 \rangle$

$$\begin{aligned}
\phi_S(x) &= \phi_A(x) \cdot \phi_B(x) \\
&= (x^1 + x^2 + \ldots + x^6)(x^1 + x^2 + x^3 + x^4)
\end{aligned}$$