Math 239 - Introduction to Combinatorics **Spring 2017**

Lecture 8: May 17th, 2017

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Remark

The sum lemma generalizes when $S_0, S_1, S_2, \ldots$ is a partition of S.

$$\phi_S = \sum_{n \geq 0} \phi_{S_n}(x)$$

8.1 Compositions of an Integer

Definition 8.1 *For integer $n \geq 0, k \geq 0$, a composition of n is a k-tuple of positive integers $c_1, c_2, \ldots, c_k$ such that $c_1 + c_2 + \ldots + c_k = n$*

Remark

- The c_i's are called the parts of the composition
- There is one composition of 0, The empty composition

Tackling a Composition Problem

Notation : $\mathbb{N} = \langle 1, 2, 3, \ldots \rangle, \mathbb{N}_0 = \langle 0, 1, 2, 3, \ldots \rangle$

Goal : Craft a set S and weight function w, such that the number of compositions of n we are counting is $[x^n]\phi_S(x)$

Steps :

1. Find a set S that fits your problem
   - If your compositions have k parts, let S be a product of k sets.
   - w usually is the sum of the parts
   - If the number of parts is unrestricted then

     $$S = S_0 \cup S_1 \cup S_2 \cup \ldots$$

     each S_k is the number of compositions with k parts that we are considering.
2. Find $\phi_S(x)$ by applying Sum and Product Lemmas
3. Find $[x^n]\phi_S(x)$, Using the tricks provided by the Inverse Bin. Series)

Problem 8.2 *Let K, $n \in \mathbb{N}, n \geq k$. How many compositions of n with k parts are there?*

Solution :

1. $S = \frac{\mathbb{N}\times\ldots\times\mathbb{N}}{k} = \mathbb{N}^k$
   For $(C_1, C_2, \ldots, C_k) \in S$, *let* $w(C_1, \ldots, C_k) = C_1 + C_2 + \ldots + C_k$

2. *By Product Lemma,*

$$
\begin{aligned}
\phi_S(x) &= \phi_{\mathbb{N}\times\ldots\times\mathbb{N}}(x) \\
&= \phi_{\mathbb{N}}(x) \cdot \phi_{\mathbb{N}}(x) \cdot \ldots \cdot \phi_{\mathbb{N}}(x) \\
&= (\phi_{\mathbb{N}}(x))^k \\
&= (x^1 + x^2 + x^3 1 \ldots)^k \\
&= (x(x^0 + x^1 + x^2 + \ldots))^k \quad = \left(\frac{x}{1-x}\right)^k
\end{aligned}
$$

3.

$$
\begin{aligned}
\left(\frac{1}{1-x}\right)^k &= [x^n](x^k)(1-k)^{-k} \\
&= [x^{n-k}](1-x)^{-k} \\
&= [x^{n-k}]\sum_{j=0} \binom{j+k-1}{k-1} x^j \\
&= \binom{n-k+k-1}{k-1} \\
&= \binom{n-1}{k-1}
\end{aligned}
$$

Problem 8.3 *How many compositions of n are there?*
Solution : *Next lecture*