**Math 239 - Introduction to Combinatorics** **Spring 2017**

# Lecture 5: May 10th, 2017

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**Definition 5.1** *Given* $A(x) = \sum_{n\geq 0} a_n x^n$ *and* $B(x) = \sum_{n\geq 0} b_n x^n$

$$A(x) + B(x) = \sum_{n\geq 0} (a_n + b_n)x^n$$

$$A(x) \cdot B(x) = \sum_{n\geq 0} \left( \sum_{i=0}^{n} a_i \cdot b_{n-1} \right) x^n$$

$$A(x) - B(x) = \sum_{n\geq 0} (a_n - b_n)x^n$$

$$cA(x) = \sum_{n/geq0} ca_n x^n, \ c \in Q$$

**Definition 5.2** *B(x) is the inverse of* $A(x)$ *if*

$$A(x) \cdot B(x) = 1$$

***Note :*** *B(x) is usually denoted as* $A^{-1}(x)$ *or* $\frac{1}{A(x)}$

**Proposition 5.3** *Geometric Series*

$$(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

**Proof:**

$$\begin{aligned}(1-x)(1+x+x^2+x^3+\ldots) &= 1x^0 + (1\cdot 1 - 1\cdot 1)x^1 + (1\cdot 1 - 1\cdot 1)x^2 + \ldots \\ &= 1\end{aligned}$$

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**Tip**

Use the $(1-x)^{-1} = 1 + x + x^2 + \ldots$ to find other inverses.

**Example 5.4** *Inverse Examples*

$$(1+x)^{-1} = (1-(-x)) = 1 + (-x) + (-x)^2 + \ldots + \ldots = 1 - x + x^2 + \ldots$$

$$(1-x+2x^2)^{-1} = (1-(x-2x^2))^{-1} = 1 + (x-2x^2) + (x-2x^2)^2 + \ldots$$

**Note :** Not every A(x) has an inverse

**Problem 5.5** *Show $A(x) = x$ has no inverse*
***Solution :*** *Suppose $B(x) = \sum_{n\geq 0} b_n x^n$ is the inverse of $x$.*

$$\begin{aligned} 1 &= x \cdot B(x) \\ &= x \cdot (b_0 + b_1 x + b_2 x^2 + \ldots) \\ &= 0 + b_0 x + b_1 x^2 + b_2 x^3 + \ldots \end{aligned}$$

*The resulting expression has 0 has a constant coefficient. Therefore, contraction. Thus, $A(x) = x$ has no inverse.*

**Theorem 5.6**

$$Q(x) = \sum_{n\geq 0} q_n x^n \text{ has an inverse } \iff q_0 \neq 0$$

**Proposition 5.7** *(Inverse binomial series)*

$$(1-x)^{-k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n$$

**Proof:** Induction on k

<u>Base</u>

$$\sum_{n\geq 0} \binom{n+1-1}{1-1} x^n \sum_{n\geq 0} \binom{n}{0} x^n = \sum_{n\geq 0} x^n = (1-x)^{-1}$$

<u>I.H.</u> For k = m we assume

$$(1-x)^{-m} = \sum_{n\geq 0} \binom{n+m-1}{m-1} x^n$$

<u>I.S.</u> We must prove for k = m + 1

$$\begin{aligned} (1-x)^{-(m+1)} &= (1-x)^{-m} \cdot (1-x)^{-1} \\ &= \left( \sum_{n\geq 0} \binom{n+m-1}{m-1} x^n \right) \left( \sum_{n\geq 0} x^n \right) \\ &= \sum_{n\geq 0} \left( \sum_{i=0}^{n} a_i \cdot b_{n-i} \right) x^n \\ &= \sum_{n\geq 0} \left( \sum_{i=0}^{n} \binom{i+m-1}{m-1} \right) x^n \\ &= \sum_{n\geq 0} \binom{n+m}{m} x^n \\ &= \sum_{n\geq 0} \binom{n+(m+1)-1}{(m+1)-1} x^n \end{aligned}$$

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