Lecture 6: May 12th, 2017

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$$(1-x)^{-k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n$$

Notation : Let A(x) $= \sum_{n\geq 0} a_n x^n$

$$\left[x^i\right] A(x) = a_i$$

Problem 6.1 *Find* $\left[x^4\right](1-2x)^{-2}(1-x^2)^{-6}$
Solution :

$$(1-2x)^{-2} = (1-(2x))^{-2} = \sum_{n\geq 0} \binom{n+2-1}{2-1}(2x)^n = \sum_{n\geq 0} \binom{n+1}{1} 2^n x^n$$

$$(1-x^2)^{-6} = \sum_{n\geq 0} \binom{n+6-1}{6-1}(x^2)^n = \sum_{n\geq 0} \binom{n+5}{5} x^{2n}$$

$$(1-2x)^{-2}(1-x^2)^{-6} = \left(\sum_{n\geq 0}(n+1)2^n x^n\right)\left(\sum_{n\geq 0}\binom{n+5}{5}x^{2n}\right)$$

Tip

Redefine coefficients in order to apply the definition of multiplication.

- *For* $i \geq 0$, $a_i = (i+1)2^i$
- *For* $j \geq 0$,

$$b_j = \begin{cases} 0 \text{ , } j \text{ is odd} \\ \binom{\frac{j}{2}+5}{5} \text{ , } j \text{ is even} \end{cases}$$

$$\begin{aligned}(1-2x)^{-2}(1-x^2)^{-6} &= \left(\sum_{i\geq 0} a_i x^i\right)\left(\sum_{j\geq 0} b_j x^j\right) \\ &= \sum_{n\geq 0}\left(\sum_{l=0}^{n} a_l b_{n-l}\right) x^n\end{aligned}$$

$$\begin{aligned}
\left[x^4\right](1-2X)^{-2}(1-x^2)^{-6} &= \sum_{l=0} a_l b_{4-l} \\
&= a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b \\
&= (0+1)2^0 \binom{4/2+5}{5} + (2+1)2^2 \binom{2/2+5}{5} + (4+1)2^4 \binom{0/2+5}{5}
\end{aligned}$$

Proposition 6.2 $1 + x + \ldots + x^k = \frac{1 = x^{k+1}}{1-x}$

Proof:

$$\begin{aligned}
(1-x^{k+1})(1-x)^{-1} &= (1-x^{k+1})(1+x+x^2+\ldots) \\
&= 1 + x + x^2 + x^3 + \ldots + x^k + x^k + x^{k+1} + \ldots - x^{k-1} - x^{k+2} - \ldots \\
&= 1 + x + x^2 + \ldots + x^k
\end{aligned}$$

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Compositions of Formal Power Series

Fact

A(B(x)) is only defined when B(x) has 0 as a constant coefficient.
For more information, read 1.7.10 in the course notes