**CS 370 - Numerical Computation** **Fall 2018**

# Time-Stepping

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- Time stepping is also known as "Time-Integration" We are integrating over time to approximate $y$ from $y'$

## 4.1 Time-Stepping Schemes

- Forward Euler
$$y_{n+1} = y_n + hf(t_n, y_n)$$
- Trapezoidal
$$y_{n+1} = y_n + \frac{h}{2}(f(t_n, y_n) + f(t_{n+1}, y_{n+1}))$$
- Modified Euler
$$y^*_{n+1} = y_n + hf(t_n, y_n)$$
$$y_{n+1} = y_n + \frac{h}{2}((f(t_n, y_n), f(t_{n+1}, y^*_{n+1}))$$
- Runge-Kutta
$$\begin{aligned} k_1 &= hf(t_n, y_n) \\ k_2 &= hf(t_n + \frac{h}{2}, y_n + \frac{k_1}{2}) \\ k_3 &= hf(t_n + \frac{h}{2}, y_n + \frac{k_2}{2}) \\ k_4 &= hf(t_n + h, y_n + k_3) \\ y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} \end{aligned}$$
- Backwards Euler
$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$$

## 4.2 Explicit v.s Implicit Schemes

- Explicit
  - Simpler and fast to compute per step
  - Less stable - requires smaller timesteps
- Implicit
  - Often more complex and expensive to solve per step.
  - More stable  can safely use larger timesteps.

## 4.3 Global Error

$$\#steps = \frac{t_{final} - t_0}{h} = O(h^{-1})$$

$$\textbf{Global Error} \leq \textbf{Local Error} \cdot O(h^-1)$$

## 4.4 Multi-Step Schemes

One way to derive multi-step schemes is to fit curves to current and earlier points.

- Backwards Differentiation Formulas
  1. Fit an interpolant p(t)with Lagrane polynomials to the unknown point $(t_{n+1}, y_{n+1})$
  2. Determine its derivative, $p'(t)$, by differentiating
  3. Require end-of-step slope to match so $p'(t_{n+1}) = f(t_{n+1}, y_{n+1})$
- Explicit multistep : 2nd order Adams-Bashforth . LTE is $O(h^3)$

$$y_{n+1} = y_n + \frac{3}{2}hf(t_n, y_n) - \frac{1}{2}hf(t_{n-1}, y_{n-1})$$

## 4.5 Summary

| Name | Single/Multi-Step | Explicit/Implicit | Global Truncation Error |
|---|---|---|---|
| Forward Euler | Single | Explicit | $O(h)$ |
| Improved Euler and Midpoint (2nd order Runge Kutta schemes) | Single | Explicit | $O(h^2)$ |
| 4th Order Runge Kutta | Single | Explicit | $O(h^4)$ |
| Trapezoidal | Single | Implicit | $O(h^2)$ |
| Backwards/Implicit Euler (BDF1) | Single | Implicit | $O(h)$ |
| BDF2 | Multi | Implicit | $O(h^2)$ |
| 2-step Adams-Bashforth | Multi | Explicit | $O(h^2)$ |
| 3rd order Adams-Moulton | Multi | Implicit | $O(h^3)$ |
| Etc., ad nauseum! | | | |