Math 239 - Introduction to Combinatorics	**Spring 2017**

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4.1 Generating Series

General Counting Problem :

Suppose S is a set, and for each element $\sigma \in S$ has associated a non-negative weight $w(\sigma)$.
Given $k \in \langle 0, 1, 2, \ldots \rangle$, how many elements in s have weight k?

Example 4.1 *General Counting problem with $S =$ all subsets of $\langle 1, 2, 3, \rangle$.*

$w(\sigma) = \mid \sigma \mid$

$S = \langle \emptyset, \{1\}, \{2\}, \{3\}, \{2, 1\}, \{2, 3\}, \{1, 2, 3\} \rangle$

k	*Number of σ with $w(\sigma) = k$*
0	*1*
1	*3*
2	*3*
3	*1*

Instead of using a table we encode this information as a coefficients of a polynomial

Definition 4.2 *Let S be a set, suppose each $\sigma \in S$ has a non-negative weight $w(\sigma)$. The generating series for S with respect to w is defined as :*

$$\phi_s(x) = \sum_{\sigma in S} x^{w(\sigma)}$$

Equivalently, this is equal to

$$\phi_s(x) = \sum_{n \geq 0} a_n x^n$$

where a_n denotes the number of elements in S that have weight k

Problem 4.3 *Find the generating series of the previous example*

σ	$w(\sigma)$	$x^{w(\sigma)}$
$\emptyset$	*0*	x^0
$\{1\}$	*1*	x^1
$\{2\}$	*1*	x^1
$\{3\}$	*1*	x^1
$\{1, 2\}$	*2*	x^2
$\{1, 3\}$	*2*	x^2
$\{2, 3\}$	*2*	x^2
$\{1, 2, 3\}$	*3*	x^3

$$\phi_S(x) = x^0 + 3x^1 + 3x^2 + x^3$$

In general, if S *is the set of all subsets of* $\langle 1, \dots, n\rangle$, $w(\sigma) = |\ \sigma\ |$ *then,*

$$\phi_S(x) = \underline{\binom{n}{0}}x^0 + \underline{\binom{n}{1}}x^1 + \ldots + \underline{\binom{n}{n}}x^n = (1+x)^n$$

Theorem 4.4 *Let* S *be a finite set with a weight function* w*.*

(a) $\phi_S(1) = |\ S\ |$

(b) $\phi'_S(1) =$ *sum of all weights* $= \sum_{\sigma\in S} w(\sigma)$

(c) $\frac{\phi' S(1)}{\phi_S(1)} =$ *average weight* $= \frac{1}{|S|}\sum_{\sigma\in S} w(\sigma)$

Example 4.5 $S =$ *all subsets of* $\langle 1, \dots, n\rangle$

$w(\sigma) = |\ \sigma\ |$
$\phi_S(x) = (1+x)^n$
$\phi_S(1) = (1+1)^n = 2^n$
$\phi'_S = ((1+x)^n) = n(1+x)^{n-1}$

$\phi'_S = n \cdot (1+1)^{n-1} = n \cdot 2^{n-1} =$ *sum of all weights* *average* $= \dfrac{n\cdot 2^{n-1}}{2^n} = \dfrac{n}{2}$

Proof:

(a)

$$\phi_S(1) = \sum_{\sigma\in S}(1)^{w(\sigma)} = \sum 1 = |\ S\ |$$

(b)

$$\phi'_S(x) = (\sum_{\sigma\in S} x^{w(\sigma)})\prime = \sum_{\sigma\in S} w(\sigma)\cdot x^{w(\sigma)-1}$$

$$\phi'_S(1) = \sum_{\sigma\in S} w(\sigma)1^{w(\sigma)-1} = \sum_{\sigma\in S} w(\sigma)$$

(c) Follows from (a) and (b)

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4.2 Formal Power Series

Definition 4.6 *Given a sequence* $a_n, a_1, a_3, \ldots$ *of rational numbers (i.e* $\frac{2}{3}, 4$, *and not* π*)*

$$A(x) = a_0 + a_1x + a_2x^2 + \ldots = \sum_{n\geq 0} a_n x^n = \sum_{n=0} a_n X^n$$

Example 4.7

$$(1, 1, \ldots,) \to P(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

$$(0, 1, 2, 3, \ldots) \to Q(x) = 0 + x + 2x^2 + 3x^3 + \ldots = \sum_{n \geq 0} nx^n$$

$$P(x) + Q(x) = (1 + 0) + (1 + 1)x + (1 + 2)x^2 + \ldots$$