Math 239 - Introduction to Combinatorics **Spring 2017**

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11.1 Sum, Product, and Star Lemma for Binary Strings

In the context of strings, the weight function is always defined to be the length of each string.

$$w(\epsilon) = 0, w(0) = 1, w(0101) = 4$$

Lemma 11.1 ***Sum Lemma*** *(For strings) If A, B are sets of strings and $A \cup B$ is unambiguous then,*

$$\phi_{A\cup B}(x) = \phi_A(x) + \phi_B(x)$$

Lemma 11.2 ***Product Lemma*** *(For strings) If A, B are sets of strings and AB is unambiguous, then*

$$\phi_{AB}(x) = \phi_A(x)\phi_B(x)$$

Proof: Refer to course notes for the full proof, but the key idea is :
If AB is unambiguous then there is a bijection between $AB = A \times B$ ■

Lemma 11.3 ***Star Lemma*** - *If A is a set of strings and A^* is unambiguous , then*

$$\phi_{A^*}(x) = \frac{1}{1 - \phi_A(x)}$$

Proof:

$$\begin{aligned}
\phi_A(x) &= \phi_{\cup_{k\geq 0} A^k}(x) \\
&= \sum_{k\geq 0} \phi_{A^k}(x) \\
&= \sum_{k\geq 0} (\phi_A(x))^k \\
&= \frac{1}{1-\phi_A(x)} \text{ By product lemma}
\end{aligned}$$

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Problem 11.4 *Let $S = \{1\}^*(\{0\}\{11\}\{11\}^*)^*$, now find $\phi_S(x)$*
Solution :

$$\begin{aligned}
\phi_S(X) &= \phi_{\{1\}^*}(x) \cdot \phi_{(\{0\}\{11\}\{11\}^*)^*}(x) \\
&= \frac{1}{1-\phi_{\{1\}}(x)} \frac{1}{1-\phi_{(\{0\}\{11\}\{11\}^*)}(x)} \\
&= \frac{1}{1-x} \frac{1}{1-\frac{x^3}{1-x^2}} \\
&= \dots \\
&= \frac{1+x}{1-x^3-x^3}
\end{aligned}$$

11.2 Decomposition Rules

The goal of decomposition is to describe sets of strings in unambiguous ways.
Consider : s_1 = all binary strings and s_2 = binary strings where each 0 is a preceded by an odd number of 1's

- **Decompose after each occurrence of 0's** : Describe your strings as "concatenations" of strings 111 10.
- **Decompose after each block of 0's** : Describe your strings as "concatenation" of strings of the form 111 100 0