Math 239 - Introduction to Combinatorics **Spring 2017**

Lecture 14: May 31st, 2017

Lecturer: Alan Arroyo Guevara *Notes By: Harsh Mistry*

14.1 Homogeneous Linear Recurrences

Definition 14.1 *A sequence $\{a_n\}_{n\geq 0}$ is defined by a linear homogeneous recurrence, if for some integer $n \geq k$,*

$$a_n + q_1 a_{n-1} + \ldots + q_k a_{n-k} = 0$$

and initial conditions $a_0, a_1, a_2, \ldots, a_{k-1}$ are given. The characteristic polynomial of this recurrence is :

$$C(x) = x^k + q_1 x^{k+1} + q_2 x^{k+2} + \ldots + q_k$$

Theorem 14.2 *Let $\beta_1, \ldots, \beta_j$ be distinct roots of C(x) and suppose $m_1, m_2, \ldots, m_j$ are such that*

$$C(x) = (x-\beta_1)^m (x-\beta_2)^m \ldots (x-\beta_j)^m$$

Then,

$$a_n = P_1(n)\beta_1^n + P_2(n)\beta_2^n + \ldots + P_j(n)\beta_j^n$$

Where for each i, $P_i(n)$ is a polynomial on n of degree less than m_1, whose coefficients are determined by the initial values $a_0, a_1, \ldots, a_{k-1}$

Problem 14.3 *Find a_n explicitly where*

$$a_n - 4a_{n-1} - 5a_{n-2} + 2a_{n-3} \quad (n \geq 3)$$

$$a_0 = 4, a_1 = 9, a_2 = 17$$

Solution:

1. *Find C(x)*

$$a_n - 4a_{n-1} + 5a_{n-2} - 2a_{n-3} = 0$$

$$C(x) = x^3 - 4x^2 + 5x - 2$$

2. *Factor C(x)*

$C(1) = 0,$ *so 1 is a root*

$$\begin{aligned} C(x) &= (x-1)(x^2 - 3x + 2) \\ &= (x-1)(x-2)(x-1) \\ &= (x-1)^2(x-2) \end{aligned}$$

Roots Are : $\beta_1 = 1, \beta_2 = 2, m_1 = 2, m_2 = 1$

3. *Apply the Theorem that solves L.H.R*

$$a_n = A + Bn(1)^m + C(2)^n = A + Bn + C(2)^n$$

4. *Obtain unknowns by using initial conditions*
   - $4 = a_0 = A + B(0) + C \cdot 2^0 = A + C$
   - $9 = a_1 = A + B(1) + C \cdot 2^1 = A + B + 2C$
   - $17 = a_2 = A + B(2) + C \cdot 2^2 = A + 2B + 4C$

   Solve the linear system A = 1, B = 2, C = 3. So,

$$a_n = 1 + 2n + 3(2^n)$$

14.2 From Recurrences to Coefficients

Problem 14.4 *Find the formal power series* $\sum_{n\geq j} a_n x^n$ *so that* $\langle a_n \rangle_{n\geq 0}$ *satisfy*

$$a_n - a_{n-1} - a_{n-2} + a_{n-3} = 0 \qquad (n \geq 4)$$

$$a_0 = 8, a_1 = 2, a_2 = 10, a_3 = 6$$

Solution :

1. $q(x) = 1 - x - x^2 + x^3$
   Complete q(x) that has same coefficients as C(x) but in reverse direction.
2. *Let* $p(x) = q(x) \sum_{n\geq 0} a_n x^n$

$$\begin{aligned} P(x) &= (1 - x - x^2 + x^3)(a_0 + a_1 x + a_2 x^2 + \ldots) \\ &= a_0 + (a_1 - a_0)x(a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1 + a_0)x^3 + (a_2 - a_3 - a_2 + a_1)x^4 + x^5 \end{aligned}$$