Math 239 - Introduction to Combinatorics **Spring 2017**

Lecture 3: May 5th, 2017

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In general, given a set S, a partition of S is a collection $S_0, S_1, \ldots, S_n$ of subsets of S such that :

1. $S = S_0 \cup S_1 \cup \ldots \cup S_n$
2. $S_i \cap S_j = \emptyset$ for $i \neq j$

Tip

In proofs of combinatorial identities, partitions are used to represent sums.

Proposition 3.1

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, 1 \leq k \leq n$$

Proof: Let S be the set of k-subsets of $\langle 1, \ldots, n \rangle$
Clearly $\mid S \mid = \binom{n}{k}$, so we want a partition such that

$$\mid S_0 \mid = \binom{n-1}{k-1} \text{ and } \mid S_1 \mid = \binom{n-1}{k}$$

Define

- S_0 = all k-subsets of $\langle 1, \ldots, n \rangle$, that contains n.
- S_1 = all k subsets of $\langle 1, \ldots, n \rangle$, that do not contain n

Clearly, S_0, S_1 is a partition.

So, Let T be the set of (k - 1) subsets$\langle 1, \ldots, n \rangle$.

Using this we can define $f : S \rightarrow T$, and $f^{-1} : T \rightarrow S$ such that :

- $f(A) = A \setminus \{n\}, \forall A \in S_0$
- $f^{-1}(B) = B \cup \{n\}, \forall B \in T$

Exercise

Prove f^{-1} is the inverse of f

Therefore , $\mid S_n \mid = \mid T \mid = \binom{n-1}{k-1}$ and $\mid S_1 \mid = \binom{n-1}{k}$, since S_1 is the set of k-subsets of $\langle 1, \dots, n-1 \rangle$.

Thus,

$$\binom{n}{k} = \mid S \mid = \mid S_0 \mid \; + \mid S_1 \mid = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$\blacksquare$

Proposition 3.2

$$\binom{n+k}{n} = \sum_{i=0}^{k} \binom{n+i-1}{n-1}$$

Proof: Let S be the set of n-subsets of $\langle 1, \dots, n, \dots, n+k \rangle$.

$$\mid S_0 \mid = \binom{n+k}{n}$$

Consider a partition $S_0 \cup S_1 \cup \dots \cup S_k$ of S. For $i = 0, 1, \dots, k$ we let

$$S_i = \langle A \cup \langle n_i \rangle : A \subset \langle 1, \dots, n+i-1 \rangle, \mid A \mid = n-1 \rangle$$

$$\mid S_i \mid = \binom{n+i-1}{n-1}$$

The elements in S_i text are in correspondence with the elements in $\{A : A \subset \langle 1, \dots, n+i-1 \rangle, \mid A \mid = n-1\}$

S_i can also be described as the set of n-sets of $\langle 1, \dots, n+k \rangle$, that have $n+i$ as its maximum element.

So, $S_i \cap S_j = \emptyset$ for $i \neq j$. As a result

$$S = S_0 \cup S_1 \cup \dots \cup S_k$$

because for every $A \subset \langle 1, \dots, n+1 \rangle$, $\mid A \mid$ and A have $n+i$ as the largest element $\blacksquare$