Math 239 - Introduction to Combinatorics **Spring 2017**

Lecture 26: June 28th, 2017

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26.1 Euler's Formula

$$| U | - | E | + | F |$$

Theorem 26.1 *Let f be the set of faces of a planar embedding of a connected graph $G = (V, E)$ Then* $| U | - | E | + | F |= 2$

Proof: Induction on $M =| E |$

- Base : m = 1 : $2 - 1 + 1 = 2$
- IH : Suppose all planar embedding of graphs with m-2 edges satisfy Euler's Formula.
- IS : Consider a planar embedding of a connected graph $G = (V, E)$ and with m edges
  - Case 1 : G is a tree $| V | - | E | + | F |= 2$
  - Case 2 : G is not a tree. Then G has a cycle C. Pick $e \in E(C)$.
    Consider $G - e$ and Let f_1 and f_e be the faces incident to e in G.

    After e is removed f_1 and f_e merge into one face f and the rest stay in the same place. Let F' be the set of faces in the embedding of $G - e$. Since $G - e$ is planar embedding connected and has $m - 1$ edges, By IH

    $| V | - | E | + | F |=| V(G-e) | -(| E(G-e) | +1)+(| F' | +1) =| V(G-e) | - | E(G-e) | + | F' |= 2$

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Remark

If G is not connected, $| V | - | E | + | F |= 1 + C$ where C is the number of components

26.2 Non-Planar Graphs

Theorem 26.2 *Let ℓ be the smallest degree of a face in a planar embedding of a graph $G = (V, E)$. Then* $| E |\leq \frac{\ell}{\ell-2}(| V | -2)$

Proof: Let F be the faces.

- Faceshaking : $2 \mid E \mid = \sum_{f \in F} deg(f) \geq \sum_{f \in F} \ell = \mid F \mid \cdot \ell$
- Euler's Formula : $\mid V \mid - \mid E \mid + \mid F \mid = 2 \implies \mid F \mid = 2 + \mid E \mid - \mid V \mid$

So, $2 \mid E \mid \geq \mid F \mid \cdot \ell = \ell \cdot (2 + \mid E \mid - \mid V \mid) \implies \ell(\mid V \mid -2) \geq (\ell - 2) \mid E \mid \implies \mid E \mid \leq \frac{\ell}{\ell - 2}(\mid V \mid -2)$ ■