Lecture 5: September 22, 2016

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5.1 Definability of Connectives

Formulas $\alpha \implies \beta$ and $\neg\alpha \vee \beta$ are equivalent. This, $\implies$ is said to be **Definable** in terms of $\neg$ and $\vee$

5.2 Adequate Sets

A set of connectives is said to be adequate iff any n-ary ($n \geq 1$) connective can be defined in terms of the ones in the set.

5.3 Proof In Propositional Logic : Resolution

We notate there is a proof with assumptions *sum* and conclusion φ by

$$\sum \vdash \varphi$$

We can be read as $\sum$ proves φ

5.3.1 Inference Rules

In general, an inference rule is written as : $\frac{\alpha_1 \alpha_2 \ldots \alpha_i}{\beta}$

The notation means if $\alpha_1\alpha_2\ldots\alpha_i$ already appears in the proof, then one may infer the formula β

5.3.2 Approaches

- Direct Proofs : Establish $\sum \vdash \varphi$ by stating the assumptions and from their derive φ
- Refutation (Contradiction) : Give a direct proof of $\sum \cup \{\neg\varphi\} \models \perp$

5.3.3 The "Resolution" System and Rule

Resolution is a refutation system, with the following inference rule:

$$\frac{(\alpha \vee p)(\neg p \vee \beta)}{a \vee B}$$

for any variable p and formulas α and β.

We consider the following as special cases:

$$\textbf{Unit Resolution (Eliminate P)} \; \frac{(\alpha \vee P)(\neg p)}{\alpha}$$

$$\textbf{Contradiction (Refuation is complete)} \; \frac{p(\neg p)}{\perp}$$

A proof is complete when one derives a contradiction $\perp$. In this case, the original assumptions are refuted.

5.3.4 Connective Normal Form

The Resolution rule can only be used successfully on formulas of a restricted form.

CNF :

- A **Literal** is a variable or the negation of a variable
- A **Clause** is a disjunction of literals
- A formula in **CNF** if it is a conjunction of clauses

In essence, a formula is in CNF if and only if

- its only connectives are $\neg, \vee$, **and/or** $\wedge$,
- $\neg$ applies only to variables, and
- $\vee$ applies only to sub formulas with no occurrence of $\wedge$

5.3.5 Converting to CNF

1. Eliminate Implication and Equivalence
   - Replace $\alpha \implies \beta$ by $\neg\alpha \vee \beta$
   - Replace $\alpha \iff \beta$ by $(\neg\alpha \vee \beta) \wedge (\alpha \vee \neg\beta)$
2. Apply De Morgans and double-negation laws as often as possible.
   - Replace $\neg(\alpha \vee \beta)$ with $\neg\alpha \wedge \neg\beta$
   - Replace $\neg(\alpha \wedge \beta)$ with $\neg\alpha \vee \neg\beta$
   - Replace $\neg\neg\alpha$ with α
3. Transform into a conjunction of clauses using distributivity
   - Replace $(\alpha \vee (\beta \wedge \gamma))$ with $(\alpha \vee \beta) \wedge (\alpha \vee \alpha)$
4. Simplify using idempotence, contradiction, excluded middle and Simplification I & II.

5.3.6 The Resolution Proof Procedure

1. Convert each formula in $\sum$ to CNF
2. Convert $\neg\varphi$ to CNF
3. Split the CNF formulas at the $\wedge$'s, yielding a set of clauses
4. Form the resulting set of clauses, keep applying the resolution until either :
   - The empty clause $\perp$ results. In this case φ is a theorem
   - The rule can no longer be applies to give a new formula. In this case, φ is not a theorem