Math 128 : Calculus 2 for the Sciences **Winter 2016**

Lecture 4: January 11, 2016

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4.1 Half Angle Identities

- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Example 4.1

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} \cos^2 x dx &= \int_0^{\frac{\pi}{3}} \frac{1}{2}(1 + \cos 2x) dx \\
&= \frac{1}{2}(x + \frac{\sin 2x}{2}) \mid_0^{\frac{\pi}{3}} \\
&= \frac{1}{2}(\frac{\pi}{3} + \frac{\frac{\sqrt{3}}{2}}{2}) \\
&= \frac{\pi}{6} + \frac{\sqrt{3}}{8}
\end{aligned}$$

Example 4.2

$$\begin{aligned}
\int \sin^2 x \cos^2 x dx \ &= \frac{1}{4}\int (1 - \cos 2x)(1 + \cos 2x) dx \\
&= \frac{1}{4}\int (1 - \cos^2 2x) \\
&= \frac{1}{4}\int (1 - \frac{1}{2}(1 + cos^4 x)) dx \\
&= \frac{1}{4}\int (\frac{1}{2} - \frac{1}{2}\cos 4x) \\
&= \frac{1}{8}(x - \frac{\sin 4x}{4}) + C
\end{aligned}$$

4.2 Integrals Involving Tan and Sec

$$\int \tan^m x \sec^n x dx, m, n \in \mathbb{Z}, m, n \geq 0$$

Known Derivatives

$$\frac{d}{dx}\tan x = \sec^2 x \quad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\tan^m x = m\tan^{m-1} x \sec^2 x \qquad \frac{d}{dx}\sec^m = n\sec^{n-1} x \sec x \tan x = n\sec^n x \tan x$$

- Try to manipulate expression, so we have $\tan x$ or $\sec^2 x$
- Use identity $1 + \tan^2 x = \sec^2 x$

Example 4.3 -

$$\begin{aligned}
\int \sec^9 x \tan^5 x dx &= \int \sec^9 x \tan^4 x \tan x dx \\
&= \int \sec^9 x (\sec^2 x - 1)^2 \tan x dx \\
&= \int \sec^9 x (\sec 3x - 1)^2 \tan x dx \\
&= \int u^8 (u^2 - 1)^2 du && u = \sec x \implies du = \sec x \tan x \\
&= \int u^8 (u^4 - 2u^2 + 1) du \\
&= \frac{u^1 3}{13} - \frac{2u^1 1}{11} + \frac{u^9}{8} + C \\
&= \frac{\sec^1 3x}{13} - \frac{2\sec^1 1x}{11} + \frac{\sec^9 x}{9} + C
\end{aligned}$$

Example 4.4 -

$$\begin{aligned}
\int \sec^4 x \tan^2 x dx &= \int \sec 3x \tan^2 x \sec^2 x dx \\
&= \int (1 + tan^2 x) \tan 3x \sec^2 x dx \\
&= \int (1 + u^2) u^2 du && u = \tan x \implies du = \sec^2 x \\
&= \frac{u^3}{3} + \frac{u^5}{5} + C \\
&= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C
\end{aligned}$$

Example 4.5 -

$$\begin{aligned}
\int \sec^3 x dx &= \sec x \tan x - \int \tan^2 x \sec x dx \\
&= \sec x \tan x - \int (\sec^2 x - 1) \sec x dx && u = \sec x \implies du = \sec x \tan x \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx && dv = \sec^2 x dx \implies v = \tan x \\
\int \sec^3 x dx &= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln \mid \sec x + \tan x \mid + C
\end{aligned}$$

Note : Integrals involving Cot and Csc share a similar strategy

4.3 Trig Substitution

Interested in integrals containing $\sqrt{a^2 - b^2x^2}$, $\sqrt{a^2 + b^2x^2}$ or $\sqrt{b^2x^2 - a^2}$

Example 4.6 -

$$\int \frac{1}{\sqrt{1-x^2}}dx (= \arcsin x + C)$$

Another Way to Solve :

Inverse Subsitution $\begin{cases} x = \sin\beta \\ dx = \cos\beta d\beta \end{cases}$

$$\int \frac{\cos\beta}{\sqrt{1 - sin^2\beta}} = \int \frac{\cos}{\sqrt{\cos 3x}}$$

End of Lecture Notes

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