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CS 245 - Logic and Computation **Fall 2016**

Lecture 15 - 18: November 1 - 10, 2016

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15.1 Axioms

Definition 15.1 *A **Axiom** is a formula that is assumed as a premise in any proof. An **Axiom Schema** is a set of axioms defined by a pattern or rule. Axioms Often behave like additional inference rules*

15.1.1 Peano Axioms

Fix the domain as $\mathbb{N}$, the natural numbers. Interpret the constant symbol 0 as zerp and the unary function symbol s as success. $s(x) \to x+1$

Thus each number in $\mathbb{N}$ has a term : $0, s(0), s(s(0)), \ldots$

Zero and successor satisfy the following axioms

- PA1 : $\forall x s(x) \neq 0$, "zero is not a successor"
- PA2 : $\forall x \forall y((s(x) = s(y) \to x = y)$., "nothing has two predecessors

Addition and Multiplication Axioms :

- PA3 : $\forall x(x + 0 = x)$, Adding zero to any number yields the same number
- PA4 : $\forall x \forall y(x + s(y) = s(x + y))$, Adding a successor yields the successor of adding the number
- PA5 : $\forall (x \times 0) = 0$, multiplying by zero yields zero
- PA6 : $\forall x \forall y(x \times s(y) = x \times y + x)$

Induction Axiom :

- PA7 : For each formula φ and variable x

  $\varphi[0/x] \to (\forall x(\varphi \to \varphi[s(x)/x]) \to \forall x \varphi)$

15.1.2 Properties of Peano Axioms

The Peano Axioms imply all of the familiar properties of the natural numbers

Theorem 15.2 $\vdash_{PA} \forall x \forall y(x + y = y + x)$

Lemma 15.3 $\forall y(x + y = y + x) \vdash_{PA} \forall y(s(x) + y = y + s(x))$

15.1.3 Definability

Let the formula φ have free variables $x_1 \dots x_k$.

Given an interpretation I, a formula φ defined the k-ary relation of tuples that make φ true - that is, the relation

$$\{\langle a_1 \dots a_k \rangle \in dom(I) \mid \varphi^{(I, \theta[x_1 \to a_1] \dots [x_k \to a_k]))} = T$$

A relation R is definable (In I) if and only if $R = R_\varphi$ for some formula φ

15.1.3.1 Properties of Defined Relations

The PA axioms allow one to show that the defined relation $\leq$ has the usual properties

- $x \leq y$ and $y \leq z$ imply $x \leq z$ (Transitivity)
- If $x \leq y$ and $y \leq x$ then $x = y$

15.2 Lists

Vocabulary for Lists :

- a constant symbol **e**, represents empty list
- a binary function symbol **cons**, connects two lists

Short-hand Notation :

- $\langle\rangle$ denotes the empty list e
- $\langle a \rangle$ denotes a list with a single item a
- If $\langle \gamma \rangle$ denotes a non-empty list. Then $\langle a, \gamma \rangle$ denotes a list with a and the rest of the items in γ.

15.2.1 Axioms of Basic Lists

- List 1 : $\forall x \forall y cons(x, y) \neq e$
- List 2 : $\forall x \forall y \forall z \forall w (cons(x, y) = cons(z, w) \to (x = z \land y = w))$
- List 3 : For each formula $\varphi(x)$ and each variable y not free in φ

  $$\varphi[e/x] \to (\forall x(\varphi \to \forall y \varphi[cons(y, x)/x]) \to \forall x \varphi)$$