CS 370 - Numerical Computation **Fall 2018**

Parametric Curves And Intro to ODEs

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- Our interpolants so far only handled functions $y = p(x)$ or one coordinate is a function of the other. This prevents us from modelling more complicated curves such as ones that fold back over itself
- Parametric curves enable us to model more general curves

3.1 Parametric Curves

- Let x and y each be seperate functions of a new parameter, t. Then a points position is given by the vector $\vec{P(t)} = (x(t), y(t))$
- Parameter t increases monotonically along the curve, but x and y may increase and decrease as needed to describe any shape.
- An example would be parameter t might represent time, so an object's coordinates, (x, y) change as time passes

Note

The notion of parametric curves is not specific to any type of curve or interpolating function (like piecewise linear, Hermite, cubic splines, or even fancier Bezier/B-spline curves, etc.)

It's a more powerful/flexible way of describing curves in **general**

- We say that the curve is "parametrized" by t. For example, the (x, y) position on the curve is dictated by parameter t

Example 3.1 *Line Example*
The simple line $y = 3x + 2$ can equivalently be described by the two coordinate functions

$$x(t) = t$$

$$y(t) = 3t + 2$$

Example 3.2 *Semi-Circle Example*
Consider a curve along a semi-circle in the upper half plane, oriented from $(1, 0)$ to $(-1, 0)$.
The usual implicit equation for a unit circle is $x^2 + y^2 = 1$

The Parametric for is

$$x(t) = \cos(\pi t) \quad y(t) = \sin(\pi t) \textit{ for } 0 \le t \le 1$$

3.1.1 Non-Uniqueness

A given curve can be "parametrized" in different ways, while yielding the exact same shape.

Example 3.3 -

1. $x(t) = \cos(\pi t)$, $y(t) = \sin(\pi t)$ *for* $0 \leq t \leq 1$
2. $x(t) = \cos(\pi(1-t)), y(t) = \sin(\pi(1-t))$ *for* $0 \leq t \leq 1$

Parametrization traverses the curve in the opposite direction (left to right) as t *goes from 0 to 1*

3.1.2 Speeds

2 parametrizations can also traverse the curve in the same direction, but at different speeds/rates./

Example 3.4 -

1. $x(t) = \cos(\pi t)$, $y(t) = \sin(\pi t)$ *for* $0 \leq t \leq 1$
2. $x(t) = \cos(\pi t^2)$, $y(t) = \sin(\pi t^2)$ *for* $0 \leq t \leq 1$

Both Curves cover the same semi-circle curve, in the same direction, but return different poitns for any given value of t

3.1.3 ODE

- ODE (Ordinary Differential Equations) are equations that provide the relationship between variable and its derivative given by the known function $y'(t) = f(t, y(t))$. Normally, $y(t)$ is not given explicitly in closed form
- The general form us a differential equation is

$$y'(t) = f(t, y(t))$$

  where f is specified and the initial values are $y(t_0) = y_0$

3.1.3.1 Complications

1. Involving more than one unknown variable, or a system of differential equations, the problem can get complicated
2. Involving higher order differential equations can also cause complications

3.1.3.2 Systems of Differential Equations

If given multiple dynamic functions and multiple initial conditions, we can the equations as vectors

The system…

$$x'(t) = f_x\big(t, x(t), y(t)\big), \text{with } x(t_0) = x_0$$
$$y'(t) = f_y\big(t, x(t), y(t)\big), \text{with } y(t_0) = y_0$$

can be written in *vector form* by stacking:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}' = \begin{bmatrix} f_x\big(t, x(t), y(t)\big) \\ f_y\big(t, x(t), y(t)\big) \end{bmatrix} \text{ with } \begin{bmatrix} x(t_0) \\ y(t_0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

or in vector notation:

$$\vec{\boldsymbol{x}}'(t) = \vec{\boldsymbol{f}}(t, \vec{\boldsymbol{x}}(t)) \text{ with } \vec{\boldsymbol{x}}(t_0) = \vec{\boldsymbol{x}}_0.$$

3.1.3.3 Time-Stepping

Given initial conditions, we repeatedly step sequentially forward to the next time instant, using the derivative info, y' and a timestep, h

Set $n = 0, t = t_0, y = y_0$

1. Compute y_{n+1}
2. Increment time, $t_{n+1} = t_n + h$
3. Advance, $n = n + 1$
4. Repeat

- Single Step Time Stepping involves using information from current time
- Multi Step Time Stepping involves using information from previous time steps
- Explicit time stepping is when y_{n+1} is given as an explicit function to evaluate
- Implicit time stepping is when y_{n+1} is not given as an explicit function to evaluate
- Timestep Size h can be constant or Variable