SCD and SSF for NCP

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Chain Decon

Complex Reflec Groups Noncrossing Partitions

SCD of

The Group G(d,d,t)A First
Decomposition

A Second
Decomposition

SSP of $\mathcal{NC}_{\mathcal{W}}$

Symmetric Chain Decompositions and the Strong Sperner Property for Noncrossing Partition Lattices

Henri Mühle

LIAFA (Université Paris Diderot)

September 22, 2015

Journées du GT Combinatoire Algébrique du GDR IM

Sperner's Theorem

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Noncrossing Partitions

 $\mathcal{NC}_{G(d,d,n)}$ The Group G(d,d,n)

A First
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SSP of $\mathcal{NC}_{\mathsf{W}}$

• $[n] = \{1, 2, ..., n\}$ for $n \in \mathbb{N}$

ullet antichain: set of pairwise incomparable subsets of [n]

Theorem (E. Sperner, 1928)

The maximal size of an antichain of [n] is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Sperner's Theorem

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SSP of \mathcal{NC}_W

• *k*-family: family of subsets of [*n*] that can be written as a union of at most *k* antichains

Theorem (P. Erdős, 1945)

The maximal size of a k-family of [n] is the sum of the k largest binomial coefficients.

A Generalization

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Decomposition SSP of NC_W

poset perspective:

- antichain of $[n] \longleftrightarrow$ antichain in the Boolean lattice \mathcal{B}_n
- binomial coefficients \longleftrightarrow rank numbers of \mathcal{B}_n

- \mathcal{P} .. graded poset of rank n
- *k*-Sperner: size of a *k*-family does not exceed sum of *k* largest rank numbers
- **strongly Sperner**: k-Sperner for all $k \le r$

A Generalization

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poset perspective:

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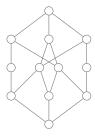
Complex Refle Groups Noncrossing

SCD of

NCG(d,d,n)The Group G(d,d,n)A First
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SSP of NC_W

• a strongly Sperner poset



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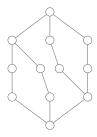
Complex Refle Groups

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SSP of NCW

• a Sperner poset that is not 2-Sperner



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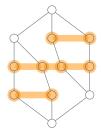
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• a Sperner poset that is not 2-Sperner



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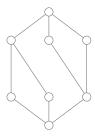
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• a 2-Sperner poset that is not Sperner



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SSP of $\mathcal{NC}_{\mathsf{IA}}$

• strongly Sperner posets:

- Boolean lattices
- divisor lattices
- lattices of noncrossing set partitions
- Bruhat posets of finite Coxeter groups
- weak order lattice of H_3
- non-Sperner posets:
 - lattices of set partitions
 - geometric lattices

strongly Sperner posets:

- Boolean lattices
- divisor lattices

(symmetric chain decompositions)

- lattices of noncrossing set partitions
- Bruhat posets of finite Coxeter groups
- weak order lattice of H_3
- non-Sperner posets:
 - lattices of set partitions
 - geometric lattices

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SSP of $\mathcal{NC}_{\mathcal{V}}$

• strongly Sperner posets:

- Boolean lattices
- divisor lattices

(symmetric chain decompositions)

- lattices of noncrossing set partitions
- Bruhat posets of finite Coxeter groups
- \bullet weak order lattice of H_3 (no symmetric chain decomposition)
- non-Sperner posets:
 - lattices of set partitions
 - geometric lattices

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SSP of NC_{W}

strongly Sperner posets:

- Boolean lattices
- divisor lattices

(symmetric chain decompositions)

- lattices of noncrossing set partitions
- Bruhat posets of finite Coxeter groups
- \bullet weak order lattice of H_3 (no symmetric chain decomposition)
- non-Sperner posets:
 - lattices of set partitions

- accompany a lattices

(of very large sets...)

• geometric lattices

strongly Sperner posets:

- Boolean lattices
- divisor lattices

(symmetric chain decompositions)

- lattices of noncrossing set partitions
- Bruhat posets of finite Coxeter groups
- weak order lattice of H_3 (no symmetric chain decomposition)
- non-Sperner posets:
 - lattices of set partitions

(of very large sets...)

geometric lattices

(certain bond lattices of graphs)

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SSP of NC_V

- \mathcal{P} .. graded poset of rank n
- **decomposition**: partition of \mathcal{P} into connected subposets



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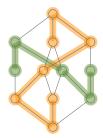
Complex Reflecti Groups Noncrossing

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SSP of \mathcal{NC}_1

• \mathcal{P} .. graded poset of rank n



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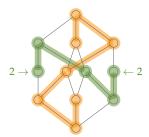
Complex Reflecti Groups Noncrossing

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• \mathcal{P} .. graded poset of rank n



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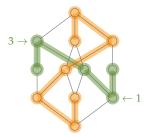
Noncrossing Partitions
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SSP of $\mathcal{NC}_{\mathfrak{p}}$

• \mathcal{P} .. graded poset of rank n



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• \mathcal{P} .. graded poset of rank n



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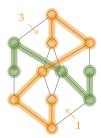
Complex Reflecti Groups Noncrossing

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• \mathcal{P} .. graded poset of rank n



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Symmetric Chain Decom positions

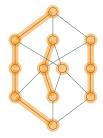
NCP Complex Reflect Groups

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SSP of \mathcal{NC}_{1}

- \mathcal{P} .. graded poset of rank n
- symmetric chain decomposition: symmetric decomposition where parts are chains



SCD and SSF for NCP

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SSP of A

• \mathcal{P} .. graded poset of rank n

Theorem

If $\mathcal P$ admits a symmetric chain decomposition, then $\mathcal P$ is strongly Sperner.

• \mathcal{P} .. graded poset of rank n

Theorem

If P and Q admit a symmetric chain decomposition, then so does $\mathcal{P} \times \mathcal{Q}$.

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Symmetric Chain Decom positions

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SCD of $NC_{G(d,d,n)}$ The Group G(d,d,n)A First

A Second Decomposition SP of NC_W • \mathcal{P} .. graded poset of rank n; N_i .. size of i^{th} rank

• rank-symmetric: $N_i = N_{n-i}$

• rank-unimodal: $N_0 \leq \cdots \leq N_j \geq \cdots \geq N_n$

 Peck: strongly Sperner, rank-symmetric, rank-unimodal

Theorem

If $\mathcal P$ admits a symmetric chain decomposition, then $\mathcal P$ is Peck.

SCD and SSF for NCP

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SSP of NC_W

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• rank-symmetric: $N_i = N_{n-i}$

• rank-unimodal: $N_0 \leq \cdots \leq N_j \geq \cdots \geq N_n$

 Peck: strongly Sperner, rank-symmetric, rank-unimodal

Theorem

If \mathcal{P} *and* \mathcal{Q} *are Peck, then so is* $\mathcal{P} \times \mathcal{Q}$.

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- - The Group G(d,d,n)
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Complex Reflection Groups

SCD and SSF for NCP

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SSP of \mathcal{NC}_1

- *V* .. *n*-dimensional unitary vector space
- (complex) reflection: unitary transformation of finite order that fixes a hyperplane
- reflecting hyperplane: fixed space of a reflection
- (complex) reflection group: finite subgroup of U(V) generated by reflections
- irreducible: does not preserve a proper subspace of *V*
- rank: codimension of fixed space
- well-generated: irreducible, rank equals minimal number of generators
- **parabolic subgroup**: maximal subgroup that fixes a proper subspace of *V*

Classification of Irreducible Complex Reflection Groups

SCD and SSF for NCP

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Symmetric Chain Decor

NCP Complex Reflection Groups

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 $NC_{G(d,d,n)}$

Decompositi
A Second

SSP of NCm

- one infinite family G(de, e, n):
 - monomial $(n \times n)$ -matrices
 - non-zero entries are (de)th roots of unity
 - \bullet product of non-zero entries is d^{th} root of unity
- 34 exceptional groups G_4, G_5, \ldots, G_{37}

Classification of Irreducible Complex Reflection Groups

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SSP of \mathcal{NC}_W

- one infinite family G(de, e, n):
 - monomial $(n \times n)$ -matrices
 - non-zero entries are $(de)^{th}$ roots of unity
 - product of non-zero entries is *d*th root of unity
- 34 exceptional groups G_4, G_5, \ldots, G_{37}
- well-generated complex reflection groups:
 - $G(1,1,n), n \ge 1$
 - $G(d, 1, n), d \ge 2, n \ge 1$
 - $G(d,d,n),d,n \ge 2$
 - 26 exceptional groups

Classification of Irreducible Complex Reflection Groups

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SSP of NC_N

• one infinite family G(de, e, n):

- monomial $(n \times n)$ -matrices
- non-zero entries are $(de)^{th}$ roots of unity
- product of non-zero entries is *d*th root of unity
- 34 exceptional groups G_4, G_5, \ldots, G_{37}

• finite Coxeter groups:

- $G(1,1,n) \cong A_{n-1}$
- $G(2,1,n) \cong B_n$
- $G(2,2,n) \cong D_n$
- $G(d,d,2) \cong I_2(d)$
- $G_{24} = H_3$, $G_{28} = F_4$, $G_{30} = H_4$, $G_{35} = E_6$, $G_{36} = E_7$, $G_{37} = E_8$

A Distinctive Property

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- SSP of \mathcal{NC}_1

- degrees: degrees of a homogeneous choice of generators of the invariant algebra
- usually denoted by $d_1 \leq d_2 \leq \cdots \leq d_n$
- Coxeter number: largest degree $h = d_n$

Theorem (G. C. Shephard & J. A. Todd, 1954; C. Chevalley, 1955)

A finite group G is a complex reflection group if and only if its algebra of invariant complex polynomials is a polynomial algebra.

A Distinctive Property

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Regular Elements

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SSP of \mathcal{NC}_{1}

- regular vector: vector that does not lie in a reflecting hyperplane
- ζ -regular element: element with eigenvalue ζ so that the corresponding eigenspace contains a regular vector
- regular number: multiplicative order of ζ
- Coxeter element: ζ -regular element of order h, where ζ is a h^{th} root of unity

Regular Elements

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ICP Complex Reflection Groups

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SSP of \mathcal{NC}_1

- regular vector: vector that does not lie in a reflecting hyperplane
- ζ -regular element: element with eigenvalue ζ so that the corresponding eigenspace contains a regular vector
- ullet regular number: multiplicative order of ζ
- Coxeter element: ζ -regular element of order h, where ζ is a h^{th} root of unity

Theorem (G. Lehrer & T. A. Springer, 1999)

If W is a well-generated complex reflection group, then h is a regular number.

Regular Elements

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Symmetric

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Complex Refle froups Vancrossing

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SSP of \mathcal{NC}_1

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Noncrossing Partitions

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SSP of $\mathcal{NC}_{\mathcal{V}}$

- *W* .. complex reflection group; *T* .. reflections of *W*; *c* .. Coxeter element
- absolute length: $\ell_T(w) = \min\{k \mid w = t_1t_2\cdots t_k, t_i \in T\}$
- **absolute order**: $u \leq_T v$ if and only if

$$\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$$

• W-noncrossing partitions:

$$NC_W(c) = \{ w \in W \mid w \leq_T c \}$$

• write $\mathcal{NC}_W(c) = (\mathcal{NC}_W(c), \leq_T)$

Noncrossing Partitions

for NCP
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Theorem (V. Reiner, V. Ripoll & C. Stump, 2015)

For any well-generated complex reflection group W, and any two Coxeter elements $c, c' \in W$ we have $\mathcal{NC}_W(c) \cong \mathcal{NC}_W(c')$.

Theorem (D. Bessis, 2003; D. Bessis, 2015; D. Bessis & R. Corran, 2006; T. Brady, 2001; T. Brady & C. Watt, 2002; T. Brady & C. Watt, 2008; G. Kreweras, 1972; V. Reiner, 1997)

The poset NC_W is a lattice for any well-generated complex reflection group W.

Catalan Numbers

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• W-Catalan number:

$$Cat_W = \prod_{i=1}^n \frac{d_i + h}{d_i}$$

Theorem (C. A. Athanasiadis & V. Reiner, 2004; D. Bessis, 2003; D. Bessis, 2015; D. Bessis & R. Corran, 2006; G. Kreweras, 1972; V. Reiner, 1997)

We have $|NC_W| = Cat_W$ for any well-generated complex reflection group W.

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Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$

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SSP of ACm

- $W = G(1,1,n) \cong \mathfrak{S}_n$; T .. transpositions; $c = (1 \ 2 \ \dots \ n)$
- $\mathcal{NC}_{G(1,1,n)}(c)$ is isomorphic to the lattice of noncrossing set partitions of [n]
- $R_k = \{ w \in NC_{G(1,1,n)}(c) \mid w(1) = k \}, R_k = (R_k, \leq_T)$

Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$

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SSP of \mathcal{NC}

• $W = G(1,1,n) \cong \mathfrak{S}_n$; T .. transpositions; $c = (1 \ 2 \ \dots \ n)$

- $\mathcal{NC}_{G(1,1,n)}(c)$ is isomorphic to the lattice of noncrossing set partitions of [n]
- $R_k = \{ w \in NC_{G(1,1,n)}(c) \mid w(1) = k \}, \mathcal{R}_k = (R_k, \leq_T)$
- ⊎ .. disjoint set union; 2 .. 2-chain

Lemma (R. Simion & D. Ullmann, 1991)

We have $\mathcal{R}_1 \uplus \mathcal{R}_2 \cong \mathbf{2} \times \mathcal{NC}_{G(1,1,n-1)}$, and $\mathcal{R}_i \cong \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i+1)}$ whenever $3 \leq i \leq n$. Moreover, this decomposition is symmetric.

Symmetric Chain Decompositions of $\mathcal{NC}_{G(1,1,n)}$

SCD and SSP for NCP

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Symmetric Chain Decompositions

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A First
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SSP of NC_W

• $W = G(1,1,n) \cong \mathfrak{S}_n$; T .. transpositions; $c = (1 \ 2 \ \dots \ n)$

• $\mathcal{NC}_{G(1,1,n)}(c)$ is isomorphic to the lattice of noncrossing set partitions of [n]

• $R_k = \{ w \in NC_{G(1,1,n)}(c) \mid w(1) = k \}, \mathcal{R}_k = (R_k, \leq_T)$

ullet .. disjoint set union; ${f 2}$.. 2-chain

Theorem (R. Simion & D. Ullmann, 1991)

The lattice $NC_{G(1,1,n)}$ admits a symmetric chain decomposition for each $n \geq 1$.

Example: $\mathcal{NC}_{\mathfrak{S}_4}((1\ 2\ 3\ 4))$

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Chain Decon positions

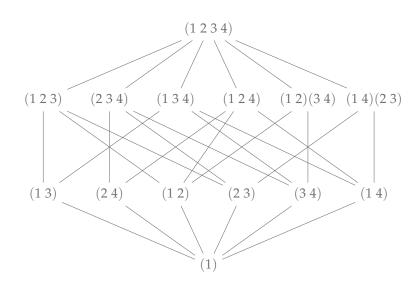
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Example: $\mathcal{NC}_{\mathfrak{S}_4}((1\ 2\ 3\ 4))$

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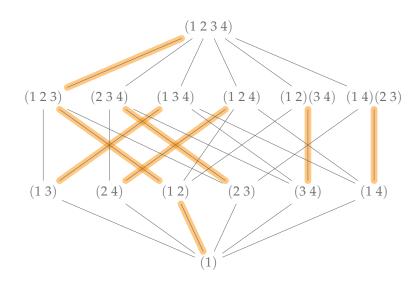
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Example: $\mathcal{NC}_{\mathfrak{S}_4}((1\ 2\ 3\ 4))$

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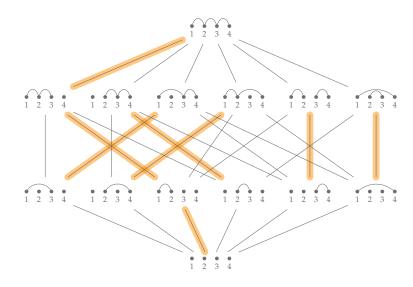
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The Groups G(d, d, n), $d, n \ge 2$

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SSP of $\mathcal{NC}_{\mathcal{V}}$

• subgroups of \mathfrak{S}_{dn} , permuting elements of

$$\left\{1^{(0)}, \dots, n^{(0)}, 1^{(1)}, \dots, n^{(1)}, \dots, 1^{(d-1)}, \dots, n^{(d-1)}\right\}$$

- $w \in G(d, d, n)$ satisfies $w(k^{(s)}) = \pi(k)^{(s+t_k)}$
 - $\bullet \ \sum_{k=1}^n t_k \equiv 0 \ (\bmod \ d)$
 - $\pi \in \mathfrak{S}_n$, and t_k depends on w and k

The Groups G(d, d, n), $d, n \ge 2$

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A Second Decomposition

SSP of \mathcal{NC}_{W}

• elements can be decomposed into "cycles":

$$\left(\left(k_1^{(t_1)} \dots k_r^{(t_r)} \right) \right) = \left(k_1^{(t_1)} \dots k_r^{(t_r)} \right) \left(k_1^{(t_1+1)} \dots k_r^{(t_r+1)} \right) \\
 \dots \left(k_1^{(t_1+d-1)} \dots k_r^{(t_r+d-1)} \right),$$

and

$$\begin{bmatrix} k_1^{(t_1)} & \dots & k_r^{(t_r)} \end{bmatrix}_s = \begin{pmatrix} k_1^{(t_1)} & \dots & k_r^{(t_r)} & k_1^{(t_1+s)} & \dots \\ k_r^{(t_r+s)} & \dots & k_1^{(t_1(d-1)s)} & \dots & k_r^{(t_r+(d-1)s)} \end{pmatrix}.$$

The Lattices $\mathcal{NC}_{G(d,d,n)}$, $d,n \geq 2$

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SSP of NC_V

• Coxeter element
$$c = [1^{(0)} \dots (n-1)^{(0)}]_1 [n^{(0)}]_{d-1}$$

• matrix representation:

$$c = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \zeta_d & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \zeta_d^{d-1} \end{pmatrix},$$

where
$$\zeta = e^{2\pi\sqrt{-1}/d}$$

The Lattices $\mathcal{NC}_{G(d,d,n)}$, $d,n \geq 2$

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SSP of NC_V

Proposition (*, 2015)

For $d, n \ge 2$, the atoms in $NC_{G(d,d,n)}(c)$ are of one of the following forms:

- $((a^{(0)} b^{(s)}))$ for $1 \le a < b < n$ and $s \in \{0, d-1\}$, or
- $((a^{(0)} n^{(s)}))$ for $1 \le a < n$ and $0 \le s < d$.

The Lattices $\mathcal{NC}_{G(d,d,n)}$, $d, n \geq 2$

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Proposition (%, 2015)

For $d, n \geq 2$, the coatoms in $NC_{G(d,d,n)}(c)$ are of one of the following forms:

- $\left[1^{(0)} \dots a^{(0)} (b+1)^{(0)} \dots (n-1)^{(0)}\right]_1 \left[n^{(0)}\right]_{d-1}$ $\left(\left((a+1)^{(0)} \dots b^{(0)}\right)\right) \text{ for } 1 \leq a < b < n,$
- $\left(\left(1^{(0)} \dots a^{(0)} n^{(s-1)} (a+1)^{(d-1)} \dots (n-1)^{(d-1)} \right) \right)$ for $1 \le a < n$ and $0 \le s < d$.

Example: *d*= 5,*n*= 3

SCD and SSP for NCP

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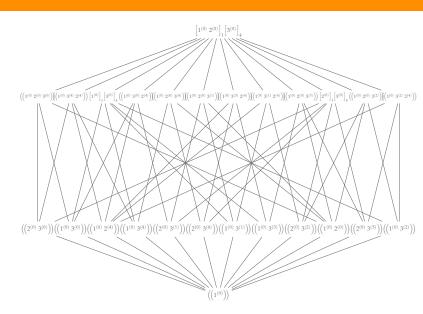
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SSP of NC_W

•
$$R_k^{(s)} = \left\{ w \in NC_{G(d,d,n)}(c) \mid w(1^{(0)}) = k^{(s)} \right\}$$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

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$$\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\!\left(1^{(0)}\right) = k^{(s)} \right\}$$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

Lemma (**, 2015)

The sets $R_1^{(s)}$ and $R_k^{(s')}$ are empty for $2 \le s < d$ as well as $2 \le k < n$ and $1 \le s' < d - 1$.

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$$\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\!\left(1^{(0)}\right) = k^{(s)} \right\}$$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

Lemma (**, 2015)

The poset $\mathcal{R}_1^{(0)} \uplus \mathcal{R}_2^{(0)}$ is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(d,d,n-1)}$. Moreover, its least element has length 0, and its greatest element has length n.

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$$\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\left(1^{(0)}\right) = k^{(s)} \right\}$$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

Lemma (**, 2015)

The poset $\mathcal{R}_n^{(s)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-1)}$ for $0 \leq s < d$. Moreover, its least element has length 1, and its greatest element has length n-1.

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 $\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\!\left(1^{(0)}\right) = k^{(s)} \right\}$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

Lemma (**%**, 2015)

The poset $\mathcal{R}_i^{(0)}$ is isomorphic to $\mathcal{NC}_{G(d,d,n-i+1)} \times \mathcal{NC}_{G(1,1,i-2)}$ whenever $3 \leq i < n$. Moreover, its least element has length 1, and its greatest element has length n-1.

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 $\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\!\left(1^{(0)}\right) = k^{(s)} \right\}$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

Lemma (%, 2015)

The poset $\mathcal{R}_i^{(d-1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-i)} \times \mathcal{NC}_{G(d,d,i-1)}$ whenever $3 \leq i < n$. Moreover, its least element has length 1, and its greatest element has length n-1.

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$$\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\left(1^{(0)}\right) = k^{(s)} \right\}$$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

Lemma (**, 2015)

The poset $\mathcal{R}_1^{(1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 2, and its greatest element has length n-1.

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$$\bullet \ R_k^{(s)} = \left\{ w \in N\!C_{G(d,d,n)}(c) \mid w\!\left(1^{(0)}\right) = k^{(s)} \right\}$$

$$\bullet \ \mathcal{R}_k^{(s)} = \left(R_k^{(s)}, \leq_T\right)$$

Lemma (**, 2015)

The poset $\mathcal{R}_2^{(d-1)}$ is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length n-2.

Example: *d*= 5,*n*= 3

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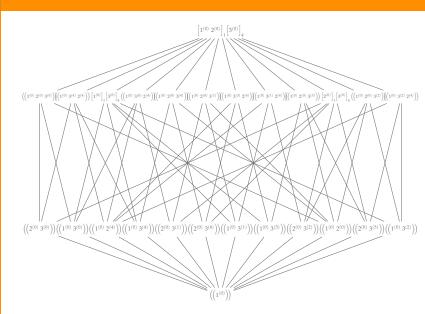
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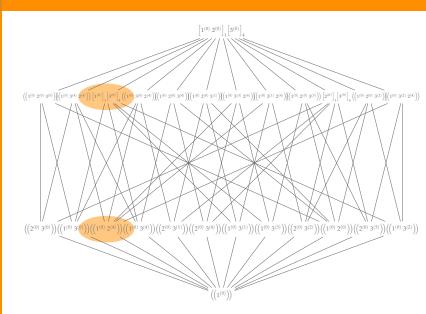
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ullet bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

A Second Decomposition

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to NC_{G(d,d,n)}(c), \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)}\right)\right) x$$

A Second Decomposition

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- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

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- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

- this map is an injective involution
- its image consists of permutations $w \in R_n^{(d-1)}$ with $w\left(n^{(d-1)}\right) = 1^{(0)}$

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- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

- this map is an injective involution
- its image is the interval

$$\left[\left(\left(1^{(0)} n^{(d-1)}\right)\right), \left(\left(1^{(0)} n^{(d-1)}\right)\right) \left(\left(2^{(0)} \dots (n-1)^{(0)}\right)\right)\right]_{T}$$

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SSP of \mathcal{NC}_W

• bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

• consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

Lemma (¥, 2015)

The interval
$$(f_1(R_1^{(1)}), \leq_T)$$
 is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$.

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- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

• define $D_1 = R_1^{(1)} \uplus f_1\left(R_1^{(1)}\right)$, and $\mathcal{D}_1 = (D_1, \leq_T)$

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• bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

• consider the map

$$f_1: R_1^{(1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(1^{(0)} \ n^{(d-2)} \right) \right) x$$

• define $D_1 = R_1^{(1)} \uplus f_1(R_1^{(1)})$, and $\mathcal{D}_1 = (D_1, \leq_T)$

Lemma (¥, 2015)

The poset \mathcal{D}_1 is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length n-1.

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• bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

• consider the map

$$f_2: R_2^{(d-1)} \to NC_{G(d,d,n)}(c), \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)}\right)\right) x$$

- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$$

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- bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$
- consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(\hat{d}-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$$

- this map is an injective involution
- its image consists of permutations $w \in R_n^{(d-1)}$ with $w\left(n^{(d-1)}\right) = 2^{(d-1)}$

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• bad parts:
$$R_1^{(1)}$$
 and $R_2^{(d-1)}$

• consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(\hat{d}-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$$

- this map is an injective involution
- its image is the interval

$$\left[\left(\left(1^{(0)} \ n^{(d-1)} \ 2^{(d-1)} \right) \right), \left(\left(1^{(0)} \ n^{(d-1)} \ 2^{(d-1)} \ \dots \ (n-1)^{(d-1)} \right) \right) \right]_T$$

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• bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

• consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$$

Lemma (¥, 2015)

The interval
$$(f_2(R_2^{(d-1)}), \leq_T)$$
 is isomorphic to $\mathcal{NC}_{G(1,1,n-2)}$.

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• bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

• consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$$

• define $D_2 = R_2^{(d-1)} \uplus f_2(R_2^{(d-1)})$, and $\mathcal{D}_2 = (D_2, \leq_T)$

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• bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

• consider the map

$$f_2: R_2^{(d-1)} \to R_n^{(d-1)}, \quad x \mapsto \left(\left(2^{(0)} \ n^{(0)} \right) \right) x$$

• define
$$D_2 = R_2^{(d-1)} \uplus f_2(R_2^{(d-1)})$$
, and $\mathcal{D}_2 = (D_2, \leq_T)$

Lemma (¥, 2015)

The poset \mathcal{D}_2 is isomorphic to $\mathbf{2} \times \mathcal{NC}_{G(1,1,n-2)}$. Moreover, its least element has length 1, and its greatest element has length n-1.

SCD and SSF for NCP

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Symmetric Chain Decon positions

Complex Reflecti Groups Noncrossing Partitions

CD of $VC_{G(d,d,n)}$ The Group G

A First Decomposition A Second Decomposition

Decomposition SSP of NC_{W}

• bad parts: $R_1^{(1)}$ and $R_2^{(d-1)}$

• define
$$D = R_n^{(d-1)} \setminus \left(f_1\left(R_1^{(1)}\right) \uplus f_2\left(R_2^{(d-1)}\right) \right)$$
, and $\mathcal{D} = (D, \leq_T)$

Lemma (¥, 2015)

The poset \mathcal{D} is isomorphic to $\biguplus_{i=3}^{n-1} \mathcal{NC}_{G(1,1,i-2)} \times \mathcal{NC}_{G(1,1,n-i)}$. Morever, its minimal elements have length 2, and its maximal elements have length n-2.

The Main Result

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SSP of NC

Theorem (¥, 2015)

For $d, n \geq 2$ the lattice $NC_{G(d,d,n)}$ admits a symmetric chain decomposition. Consequently, it is Peck.

Example: *d*= 5,*n*= 3

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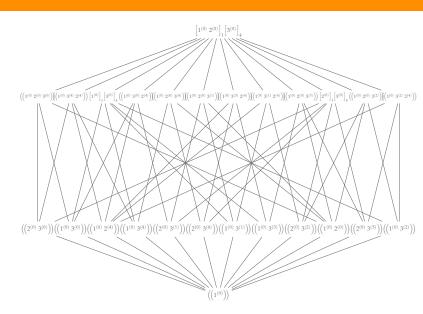
 $NC_{G(d,d,n)}$ The Group G(d,d,n)

Decomposition

A Second

Decomposition

SSP of \mathcal{NC}_{W}



Example: *d*= 5,*n*= 3

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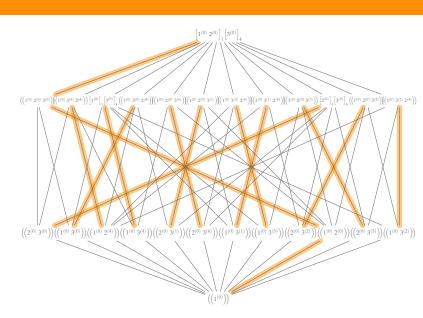
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Outline

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 - Symmetric Chain Decompositions of $\mathcal{NC}_{G(d,d,n)}$
 - The Group G(d, d, n)
 - A First Decomposition
 - A Second Decomposition
 - **5** Strong Sperner Property of \mathcal{NC}_W

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 $\mathcal{NC}_{G(d,d,n)}$ The Group G(d,d,r)A First Decomposition

SSP of ACM

- so far: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$ admit symmetric chain decompositions
- what about the other well-generated complex reflection groups?

SCD and SSF for NCP

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SSP of \mathcal{NC}_{W}

• so far: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$ admit symmetric chain decompositions

what about the other well-generated complex reflection groups?

Theorem (V. Reiner, 1997)

The lattice $NC_{G(2,1,n)}$ admits a symmetric chain decomposition for any $n \ge 1$.

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Complex Reflection Groups Noncrossing Partitions

SCD of $NC_{G(d,d,n)}$ The Group G(d,d,n)A First

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SSP of NC_W

• so far: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$ admit symmetric chain decompositions

- what about the other well-generated complex reflection groups?
- we have $\mathcal{NC}_{G(2,1,n)} \cong \mathcal{NC}_{G(d,1,n)}$ for $d \geq 2$ and $n \geq 1$

Theorem (V. Reiner, 1997)

The lattice $NC_{G(2,1,n)}$ admits a symmetric chain decomposition for any $n \ge 1$.

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- SSP of \mathcal{NC}_W

- ullet so far: $\mathcal{NC}_{G(1,1,n)}$ and $\mathcal{NC}_{G(d,d,n)}$ admit symmetric chain decompositions
- what about the other well-generated complex reflection groups?
- we have $\mathcal{NC}_{G(2,1,n)} \cong \mathcal{NC}_{G(d,1,n)}$ for $d \geq 2$ and $n \geq 1$
- only the 26 exceptional groups remain

Theorem (V. Reiner, 1997)

The lattice $NC_{G(2,1,n)}$ admits a symmetric chain decomposition for any $n \ge 1$.

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SSP of MCw

- \mathcal{P} .. graded poset of rank n
- ullet $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed

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Chain Decompositions

NCP Complex I

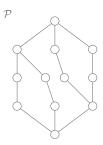
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A Second Decomposition

- \mathcal{P} .. graded poset of rank n
- ullet $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed



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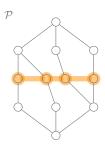
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- \mathcal{P} .. graded poset of rank n
- ullet $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed



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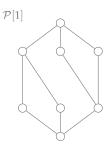
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SSP of \mathcal{NC}_W

- \mathcal{P} .. graded poset of rank n
- ullet $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed



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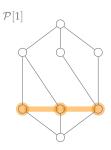
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- \mathcal{P} .. graded poset of rank n
- ullet $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed



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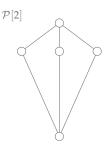
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- \mathcal{P} .. graded poset of rank n
- $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed



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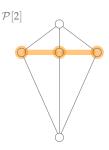
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A Second Decomposition

- \mathcal{P} .. graded poset of rank n
- ullet $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed



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- \mathcal{P} .. graded poset of rank n
- ullet $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed

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SSP of NC_{10}

- \mathcal{P} .. graded poset of rank n
- ullet $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed

 $\mathcal{P}[3]$



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SSP of NCw

• \mathcal{P} .. graded poset of rank n

ullet $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed

 $\mathcal{P}[4]$

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- SSP of \mathcal{NC}_W

- \mathcal{P} .. graded poset of rank n
- ullet $\mathcal{P}[i]$.. subposet of \mathcal{P} with i largest ranks removed

Proposition (%, 2015)

A graded poset P of rank n is strongly Sperner if and only if P[i] is Sperner for all $i \in \{0, 1, ..., n\}$.

• antichains in $\mathcal{P}[i]$ are antichains in $\mathcal{P}[s]$ for s < i

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SSP of \mathcal{NC}_W

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 SAGE has a fast implementation to compute the size of the largest antichain of a poset

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 SAGE has a fast implementation to compute the width of a poset

Theorem (**, 2015)

The lattice NC_W is Peck for any well-generated exceptional complex reflection group W.

A Decomposition Argument

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SSP of \mathcal{NC}_{W}

 SAGE has a fast implementation to compute the width of a poset

Theorem (**, 2015)

The lattice NC_W is Peck for any well-generated complex reflection group W.

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SSP of NCm

W .. well-generated complex reflection group; c ..
 Coxeter element of W

• *m*-divisible noncrossing partition: *m*-multichain of noncrossing partitions $\rightsquigarrow NC_W^{(m)}(c)$

$$(w)_m = (w_1, w_2, \dots, w_m)$$
 with $w_1 \leq_T w_2 \leq_T \dots \leq_T w_m \leq_T c$

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SCD of $NC_{G(d,d,n)}$ The Group G(d,d,n)

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SSP of \mathcal{NC}_{W}

- W .. well-generated complex reflection group; c ..
 Coxeter element of W
- *m*-divisible noncrossing partition: *m*-multichain of noncrossing partitions $\rightsquigarrow NC_W^{(m)}(c)$
- *m*-delta sequence: sequence of "differences" of elements in a multichain

$$(w)_m = (w_1, w_2, \dots, w_m) \text{ with } w_1 \leq_T w_2 \leq_T \dots \leq_T w_m \leq_T c$$

 $\partial(w)_m = [w_1; w_1^{-1} w_2, w_2^{-1} w_3, \dots, w_{m-1}^{-1} w_m, w_m^{-1} c]$

SCD and SSF for NCP

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- W .. well-generated complex reflection group; c ..
 Coxeter element of W
- *m*-divisible noncrossing partition: *m*-multichain of noncrossing partitions $\rightsquigarrow NC_W^{(m)}(c)$
- *m*-delta sequence: sequence of "differences" of elements in a multichain
- partial order: $(u)_m \leq (v)_m$ if and only if $\partial(u)_m \leq_T \partial(v)_m \qquad \rightsquigarrow \mathcal{NC}_W^{(m)}(c)$

Question (D. Armstrong, 2009)

Are the posets $NC_W^{(m)}$ strongly Sperner for any W and any m > 1?

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SCD of $\mathcal{NC}_{G(d,d,n)}$

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SSP of \mathcal{NC}_{W}

• affirmative answer for m = 1

Question (D. Armstrong, 2009)

Are the posets $NC_W^{(m)}$ strongly Sperner for any W and any m > 1?

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SSP of \mathcal{NC}_{W}

• affirmative answer for m = 1

• what about m > 1?

• $\mathcal{NC}_W^{(m)}$ is antiisomorphic to an order ideal in $\left(\mathcal{NC}_W\right)^m$

• $(\mathcal{NC}_W)^m$ is Peck

• $\mathcal{NC}_W^{(m)}$ is not rank-symmetric \leadsto no symmetric chain decomposition

Question (D. Armstrong, 2009)

Are the posets $NC_W^{(m)}$ strongly Sperner for any W and any m > 1?

Example: $\mathcal{NC}_{\mathfrak{S}_4}^{(2)}$

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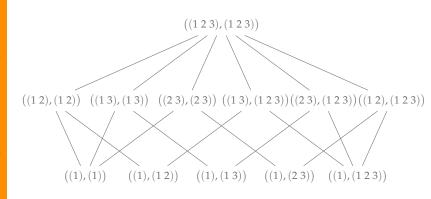
Noncrossing

Noncrossing Partitions

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SSP of NCu



Example: $\mathcal{NC}_{\mathfrak{S}_4}^{(2)}$

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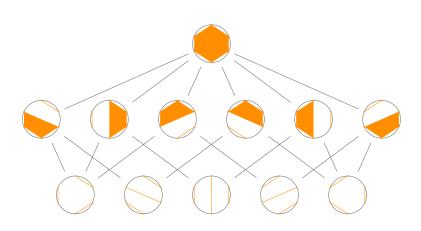
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SSP of $\mathcal{NC}_{\mathsf{IA}}$

Thank You.

SCD and SSP for NCP Henri Müble

$$NC_{G(d,d,n)}(c) = R_1^{(0)} \uplus R_1^{(1)} \uplus \bigcup_{i=2}^{n-1} \left(R_i^{(0)} \uplus R_i^{(d-1)} \right) \uplus \bigcup_{s=0}^{d-1} R_n^{(s)}$$

SCD and SSP for NCP

Henri Mühle

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$NC_{G(d,d,n)}(c) = R_1^{(0)} \uplus R_1^{(1)} \uplus \biguplus_{i=2}^{n-1} \left(R_i^{(0)} \uplus R_i^{(d-1)} \right) \uplus \biguplus_{s=0}^{d-1} R_n^{(s)}$$

SCD and SSI for NCP

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\begin{aligned} \mathsf{Cat}_{G(d,d,n+2)} &= 2 \cdot \mathsf{Cat}_{G(d,d,n+1)} + 2 \cdot \mathsf{Cat}_{G(1,1,n)} + d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \sum_{i=3}^{n+1} \mathsf{Cat}_{G(d,d,n-i+3)} \mathsf{Cat}_{G(1,1,i-2)} \end{aligned}$$

SCD and SSP for NCP

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\begin{aligned} \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \cdot \sum_{i=2}^{n+2} \mathsf{Cat}_{G(d,d,n-i+3)} \cdot \mathsf{Cat}_{G(1,1,i-2)} \end{aligned}$$

SCD and SSP for NCP

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\begin{split} \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \cdot \sum_{i=0}^n \mathsf{Cat}_{G(d,d,n-i+1)} \cdot \mathsf{Cat}_{G(1,1,i)} \end{split}$$

SCD and SSP for NCP

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

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SCD and SSF for NCP

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\begin{aligned} \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \cdot \sum_{i=0}^{n} \mathsf{Cat}_{G(d,d,i+1)} \cdot \mathsf{Cat}_{G(1,1,n-i)} \\ \mathsf{Cat}_{G(d,d,n+2)} &= \left(\prod_{i=1}^{n+1} \frac{di + (n-1)d}{di} \right) \frac{n + (n-1)d}{n} \end{aligned}$$

SCD and SSF for NCP

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\begin{split} \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \cdot \sum_{i=0}^{n} \mathsf{Cat}_{G(d,d,i+1)} \cdot \mathsf{Cat}_{G(1,1,n-i)} \\ \mathsf{Cat}_{G(d,d,n+2)} &= \Big((n+1)d + n + 2 \Big) \cdot \mathsf{Cat}_{G(1,1,n+1)} \end{split}$$

SCD and SSF for NCP

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\begin{aligned} \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ 2 \cdot \sum_{i=0}^{n} \mathsf{Cat}_{G(d,d,i+1)} \cdot \mathsf{Cat}_{G(1,1,n-i)} \\ \mathsf{Cat}_{G(d,d,n+2)} &= d \cdot \mathsf{Cat}_{G(1,1,n+1)} \\ &+ \left(nd + n + 2 \right) \cdot \mathsf{Cat}_{G(1,1,n+1)} \end{aligned}$$

SCD and SSF for NCP

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$nd \cdot \operatorname{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} = 2 \cdot \sum_{i=0}^{n} (id+i+1) \cdot \operatorname{Cat}_{G(1,1,i)} \cdot \operatorname{Cat}_{G(1,1,n-i)}$$

SCD and SSF for NCP

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\begin{split} nd \cdot \mathsf{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} &= \\ 2 \cdot \sum_{i=0}^n \left(id \cdot \mathsf{Cat}_{G(1,1,i)} \cdot \mathsf{Cat}_{G(1,1,n-i)} \right) + 2 \cdot \sum_{i=0}^n \binom{2i}{i} \cdot \mathsf{Cat}_{G(1,1,n-i)} \end{split}$$

SCD and SSF for NCP

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$nd \cdot \operatorname{Cat}_{G(1,1,n+1)} + \binom{2(n+1)}{n+1} = \\ 2d \cdot \sum_{i=0}^{n} \left(i \cdot \operatorname{Cat}_{G(1,1,i)} \cdot \operatorname{Cat}_{G(1,1,n-i)} \right) + \binom{2(n+1)}{n+1}$$

SCD and SSF for NCP

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For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\frac{n}{2} \cdot \text{Cat}_{G(1,1,n+1)} = \sum_{i=0}^{n} i \cdot \text{Cat}_{G(1,1,i)} \cdot \text{Cat}_{G(1,1,n-i)}$$

SCD and SSP for NCP

Henri Mühle

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n} i \cdot Cat_{G(1,1,i)} \cdot Cat_{G(1,1,n-i)} = \binom{2n+1}{n-1}$.

$$\binom{2n+1}{n-1} = \sum_{i=0}^{n} i \cdot \operatorname{Cat}_{G(1,1,i)} \cdot \operatorname{Cat}_{G(1,1,n-i)}$$

SCD and SSP for NCP

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Proposition (Y. Kong, 2000)

For
$$n \ge 0$$
 we have $\sum_{i=0}^{n-1} Cat_{G(1,1,i)} \cdot \binom{2(n-i)}{n-i-1} = \binom{2n+1}{n-1}$.