

# An introduction to Lattice Quantum Chromodynamics

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## Abstract

## 1 Introduction

The aim of this paper is to give a hands-on introduction to the how one can go from a rudimentary understanding of quantum mechanics and quantum field theory, to simulating quantum chromodynamics on the lattice. In order to kick this is of, let us begin by discussing our end-goal, Quantum Chromodynamics(QCD).

QCD is the theory for interacting quarks and gluons. It has since its inception gotten a reputation notoriously difficult to work with, much because it is not linear due to its three- and four-vertex gluon self interactions. This makes solving it rather difficult, and in our search for understanding this theory we come to embrace lattice QCD.

The Lattice QCD machinery is at its core a application of the Metropolis algorithm on solving path integrals. LQCD may at first glance seem familiar to the much simpler Ising model, but in contrast to the Ising model there are many more tools which need to be mastered and understood. Questions such as how to create a discretized field theory, how to analyze sparse datasets or parallelize code requiring communication between nodes will all be addressed in due time. The first step will be to understand the path integral formalism. Since this is a quantum theory, we will refresh ourselves on some basic quantum mechanics.

If the reader feels so inclined, feel free to skip to chapter about the Path Integral Formalism.

## 2 Refreshing quantum mechanics

### 2.1 The postulates of quantum mechanics

Let us begin by recapitulating the postulates of quantum mechanics. Depending on what literature you are looking at, a different number of postulates will be listed [R Shankar, Griffiths, Laurent Nottale]. However, let us start by looking at those which count as the most fundamental ones, and expand upon them<sup>1</sup>.

- I *Complex state function.* The state of a quantum mechanical system can be described by a complex valued vector,  $|\psi\rangle$  that lives in Hilbert space  $\mathcal{H}$ . Vectors in Hilbert space is often referred to as states.
- II *The correspondence principle.* A physical observable has an Hermitian operator. For every dynamical variable in classical mechanics, there is a corresponding quantum mechanical one. Further, the principle states we reproduce the classical results in the limit of large quantum numbers. Further, on order to convert to from classical to quantum mechanics, we replace the Poisson brackets with the the commutator ones divided by  $i\hbar$ .

$$\{\cdot, \cdot\} \rightarrow \frac{1}{i\hbar}[\cdot, \cdot] \quad (1)$$

Further, any observable  $\hat{\Omega}$  measured will result in a eigenvalue  $\omega$  associated with the observable. These are related through the equation eigenvalue equation,  $\hat{\Omega}|\psi\rangle = \omega|\psi\rangle$

- III *Von Neumann's postulate(collapse of state).* If we measure a system  $|\psi\rangle$  with an observable  $\Lambda$  and get the eigenvalue of  $\omega_i$ , then the system will be in a state  $|\psi_i\rangle$  after the measurement<sup>2</sup>. We say that the system has *collapsed* into state  $|\psi_i\rangle$ .
- IV *The Schrödinger equation.* We require a state vector  $|\psi(t)\rangle$  to abide by the Schrödinger equation,

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle \quad (2)$$

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<sup>1</sup>If one wants to, there are actually three subsets of postulates in QM[see laurent nottale], the fundamental ones, the secondary ones which can be derived from the fundamental ones, and the principles which are more consequences of the fundamental ones.

<sup>2</sup>A measurement is here an ideal experiment which we minimally disturb the system and it is in compliance with theory. In classical mechanics, an ideal measurement is a measurement in which the system remains unaffected.

where the Hamiltonian  $\hat{H}$  is a linear Hermitian operator constructed accordingly to the correspondence principle.

V *Born's postulate.* The wave function squared  $|\psi|^2$  is interpreted as a the probability of the system at a given configuration. For instance, the Probability of getting the eigenvalue  $\omega_i$  in a state  $|\psi\rangle$  is given by  $|\langle\omega_i|\psi\rangle|^2$ . This implies that the wave function must be normalized,

$$\langle\psi|\psi\rangle = 1 \quad (3)$$

From these postulates follow several other properties and consequences. The first being expectation values.

### 2.1.1 Expectation values

Given some observable  $\hat{\Omega}$  and a system described by a normalized wave function  $\psi$ , we have that the expectation value of that is given as the mean value from statistics.

$$\begin{aligned} \langle\Omega\rangle &= \sum_i P(\omega_i)\omega_i = \sum_i |\langle\omega_i|\psi\rangle|^2\omega_i \\ &= \sum_i \langle\psi|\omega_i\rangle\langle\omega_i|\psi\rangle\omega_i \\ &= \sum_i \langle\psi|\omega_i|\omega_i\rangle\langle\omega_i|\psi\rangle \\ &= \sum_i \langle\psi|\hat{\Omega}|\omega_i\rangle\langle\omega_i|\psi\rangle \\ &= \langle\psi|\hat{\Omega} \cdot I|\psi\rangle \end{aligned}$$

We used that  $I = \sum_i |\omega_i\rangle\langle\omega_i|$  and that the  $\omega_i|\omega_i\rangle = \hat{\Omega}|\omega_i\rangle$ . This gives us the expression for the expectation value,

$$\langle\Omega\rangle = \langle\psi|\hat{\Omega}|\psi\rangle \quad (4)$$

### 2.1.2 Superposition principle

Applying the superposition principle in quantum mechanics, we get from the linearity of the Hamiltonian  $\hat{H}$  in the Schrödinger equation(2) that a

quantum mechanical state is built up from a set of linear independent states. E.g.,

$$|w\rangle = a|u\rangle + b|v\rangle$$

### 2.1.3 Expansion in eigenfunctions

A state  $|\psi\rangle$  can be expanded in a basis of eigenfunctions  $|\psi_n\rangle \in \mathcal{H}$  as

$$\psi = \sum_n c_n |\psi_n\rangle \quad (5)$$

We demand orthogonality from this set, such that

$$\langle \psi_n | \psi_m \rangle = \delta_{nm} \quad (6)$$

### 2.1.4 Properties of Hilbert space vectors

As both the superposition principle and expansion in

### 2.1.5 Probability interpretation

The probability for finding a particle at a position  $x$  (assume we are in position space) is given by a function  $|\psi(x)|^2$ , probability per unit length. This is called the *probability density*. The *probability amplitude* is given by  $\psi(x)$ . The probability for finding the particle between two positions  $x_1$  and  $x_2$  is given by  $\int_{x_1}^{x_2} |\psi(x)|^2 dx$ . The probability of finding a particle at position  $x$  over a series of experiments is given by  $\langle \psi | x | \psi \rangle = \int^{\mathcal{R}} \psi^*(x) x \psi(x) dx$ .

### 2.1.6 Formalism

As have been alluded to earlier, *IS NEXT SECTION NEEDED?* Let us blast off with recapping some of the formalism of quantum mechanics, in particular what that is needed to perform calculations we are interested in.

## 3 The path integral formalism

The path integral formalism saw its complete form in 1948 with the help of Richard Feynman [1] (but was however conceptualized somewhat earlier). The concept is both intuitive and elegant, and forms the basis of modern quantum field theory.

### **3.1 Correlators**

### **3.2 Path integral formalism in quantum mechanics**

Consider a particle

### **3.3 The Euclidean picture**

### **3.4 Observables for the path integral**

### **3.5 Switching between the Heisenberg and Schrödinger picture.**

### **3.6 Wick rotations**

## **4 The Metropolis algorithm**

## **5 Statistical analyses**

What is autocorrelation? Autocorrelation versus correlation?

### **5.1 Bootstrapping**

### **5.2 Jackknife method**

### **5.3 Blocking**

## **6 Quantum Field Theory and its fundamentals**

random citation from peskin [2]

## 6.1 Observables

## 6.2 Action

# 7 Lattice Quantum Chromodynamics

## 7.1 Making a theory gauge invariant

## 7.2 The Plaquette

## 7.3 The Wilson gauge action

## 7.4 Notes on a Lattice QCD simulation

### 7.4.1 Updating matrices

### 7.4.2 Generating random $SU(3)$ matrices

### 7.4.3 Generating random $SU(2)$ matrices

## References

- [1] R. P. Feynman. Space-time approach to non-relativistic quantum mechanics. *Rev. Mod. Phys.*, 20:367–387, Apr 1948.
- [2] D.V. Schroeder M.E. Peskin. *An Introduction to Quantum Field Theory*. Westview Press, 1995.