Solving $\mathrm{SU}(3)$ Yang-Mills theory on the lattice: a calculation of selected gauge observables with gradient flow

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Introduction

Structure

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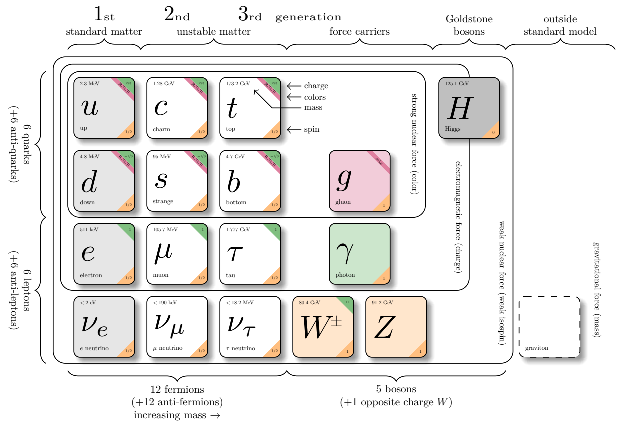
The Standard Model



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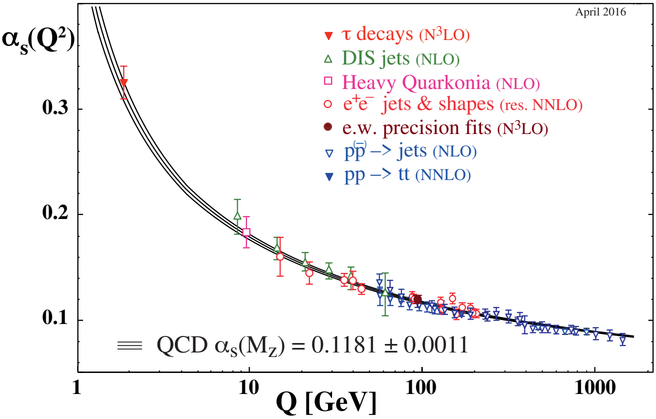
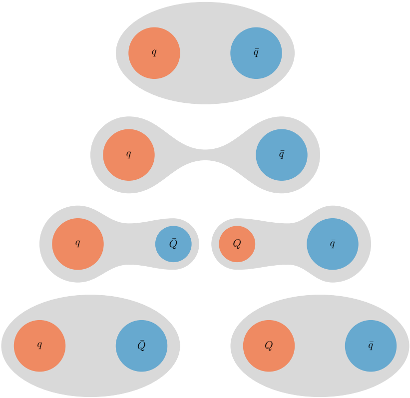
Consists of the innermost square of the six quarks and the gluons.

Asymptotic freedom



- The coupling constant **decreases** as we **increase** the energy
- One of the experimental proofs of QCD along with triple γ decay and muon cross section ration R.



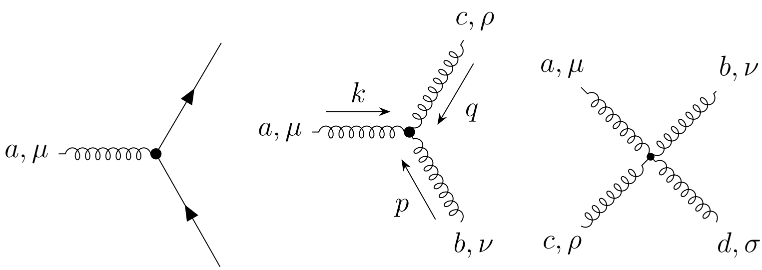
If we try to pull apart **two mesons**, more and more energy is required until we have enough energy to spontaneously create a **quark-antiquark** pair, forming thus **two new mesons.**

The QCD Lagrangian

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f=1}^{N_f} \bar{\psi}^{(f)} \left(i\not{D} - m^{(f)} \right) \psi^{(f)} - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu},$$

with action

$$S = \int \mathrm{d}^4 x \mathcal{L}_{\mathrm{QCD}}. \tag{1}$$



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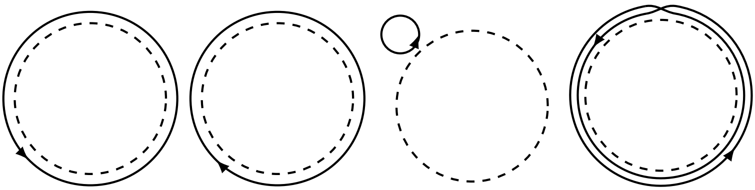

(a) $\nu = +1$ (b) $\nu = -1$ (c) $\nu = 0$ (d) $\nu = +2$

Figure 1: The figure is taken from Forkel [1, p. 32].

- **Instantons** are local minimums to the Yang-Mills action in Euclidean space.
- Measuring **topological charge** is a measure of the *Winding number* of the gauge field.
- An illustration of how one can view the winding number given a function f that parametrizes a path around a circle S^1. Given that it starts and ends at the same point, we have that the number of times it wraps around the circle gives us the winding number.

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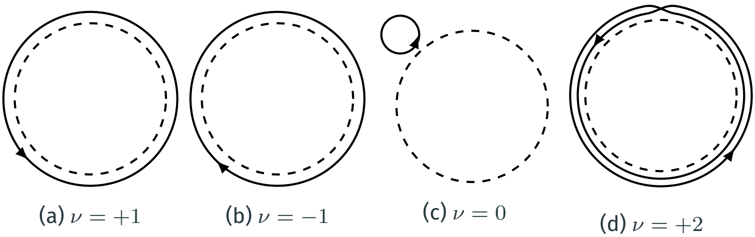

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The Witten-Veneziano relation

A formula connecting pure gauge theory and full QCD.

$$m_{\eta'}^2 = \frac{2N_f}{f_\pi^2}\chi_{\text{top}} \tag{2}$$

- Pion decay constant $f_\pi = 0.130(5)/\sqrt{2}$ GeV.
- η' meson mass $m_{\eta'} = 0.95778(6)$ GeV.
- χ_{top} is the *topological susceptibility*.

- We use the experimental values for the pion decay constant and the η' mass.
- Allows us to estimate the number of flavors in our theory N_f.
- χ_{top} is the topological susceptibility, calculated from the expectation value of Q.

Lattice Quantum Chromodynamics(LQCD)

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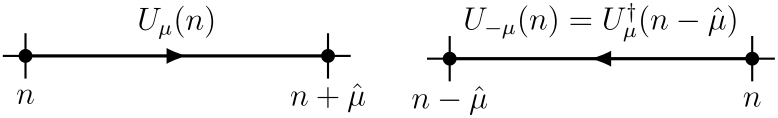
2. Fermions live on the each *point* in the cube.
3. The gauge fields live on the sites *in between* the points, and is called links.

Links

A link

$$U_\mu(n) = \exp\left[iaA_\mu(n)\right],$$

connects one lattice site to another and is a $\mathrm{SU}(3)$ matrix.



where $U_{-\mu}(n) = U_\mu(n-\hat{\mu})^\dagger$.

- Defined from the gauge transporter.
- A link in the positive $\hat{\mu}$ direction is shown in the figure to the left.
- A link in the negative $\hat{\mu}$ direction is shown in the figure to the right.

Gauge invariance on the lattice

Links gauge transform as

$$
\begin{aligned}
U_\mu(n) \to U'_\mu(n) &= \Omega(n)\, U_\mu(n)\Omega(n+\hat{\mu})^\dagger, \\
U_{-\mu}(n) \to U'_{-\mu}(n) &= \Omega(n)\, U_\mu(n-\hat{\mu})^\dagger\Omega(n-\hat{\mu})^\dagger.
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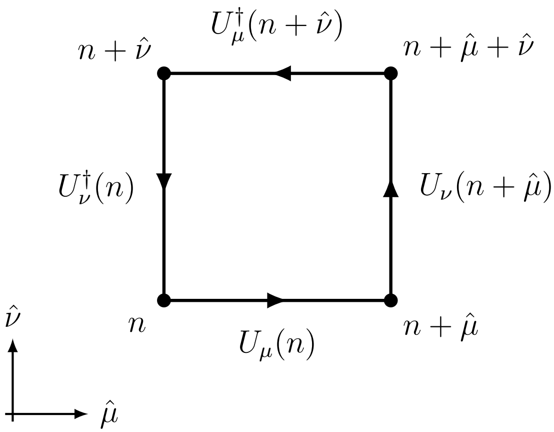
Two main types of gauge invariant objects,

- Fully connected gauge invariant objects.
- Objects with fermions ψ, $\bar{\psi}$ as end points.

The simplest gauge invariant object,

$$\begin{aligned} P_{\mu\nu}(n) &= U_\mu(n)\,U_\nu(n+\hat{\mu})\,U_{-\mu}(n+\hat{\mu}+\hat{\nu})\,U_{-\nu}(n+\hat{\nu}) \\ &= U_\mu(n)\,U_\nu(n+\hat{\mu})\,U_\mu(n+\hat{\nu})^\dagger\,U_\nu(n)^\dagger, \end{aligned}$$



The Wilson gauge action

The Wilson gauge action is given as

$$S_G[U] = \frac{\beta}{3} \sum_{n\in\Lambda} \sum_{\mu<\nu} \mathrm{Re}\,\mathrm{tr}\left[1 - P_{\mu\nu}(n)\right], \tag{3}$$

with $\beta = 6/g_S$.

- Using the definition of the link we saw earlier, we can reproduce the continuum action up to an discretization error of $\mathcal{O}(a^2)$.

Developing a code for solving $\mathrm{SU}(3)$ Yang-Mills theory on the lattice

The numerical challenge in lattice QCD

A lattice configuration consists of $\mathrm{SU}(3)$ matrices,

- The $\mathrm{SU}(3)$ matrices are 3×3 matrices of nine complex numbers or 18 real numbers.
- This leads to an absolute **requirement of efficiency**, both in **calculations** and in **input/output**.
- When returning to what ensembles of configurations we generated this will be evident.

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$$\underbrace{N^3}_{\text{Spatial}} \times \underbrace{N_T}_{\text{Temporal}} \times \underbrace{4}_{\text{Links}} \times \underbrace{9}_{\mathrm{SU}(3)\text{ matrix}} \times \underbrace{2}_{\mathbb{C}\text{-numbers}} = 72N^3N_T,$$

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Which is $\rightarrow 8 \times 72N^3N_T$ bytes.

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The Metropolis algorithm

Algorithm of choice when sampling gauge configurations.

The path integral I

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \, O[\psi, \bar{\psi}, A] \mathrm{e}^{-S_G[A] - S_F[\psi, \bar{\psi}, A]}.$$

with

$$Z = \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \, e^{-S_G[A] - S_F[\psi, \bar{\psi}, A]}.$$

The path integral I

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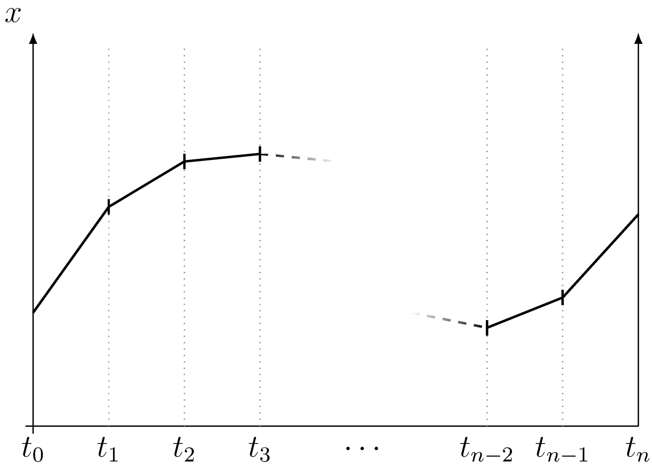
The path integral II



An example of the discretized path integral, going from time t_0 to t_{N_T}, where the end points is taken to be equal, $x_0 = x_{N_T}$. We integrate over all of space at each time t_i finding the most likely position at a given time.

Link sharing

Scaling

Measuring observables on the lattice

How to measure

Topological charge

Energy

Gradient flow

The flow equation

Solving gradient flow on the lattice

Smearing the lattice

Results

Ensembles

Topological charge

Topological susceptibility

The fourth cumulant

The topological charge correlator

The effective glueball mass

Conclusion

Questions?

References

[1] Hilmar Forkel. A Primer on Instantons in QCD. *arXiv:hep-ph/0009136*, September 2000. URL http://arxiv.org/abs/hep-ph/0009136. arXiv: hep-ph/0009136.