

# Solving SU(3) Yang-Mills theory on the lattice: a calculation of selected gauge observables with gradient flow

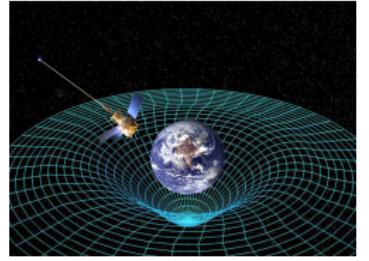
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Hans Mathias Mamen Vege  
04.07.19

Supervisor: *Andrea Shindler*  
Co-supervisor: *Morten Hjorth-Jensen*

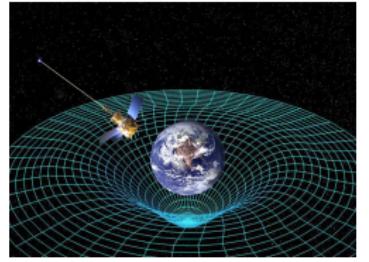
University of Oslo

# The four forces of nature



Gravity

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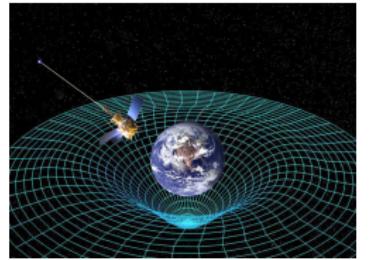


Gravity



Electromagnetism

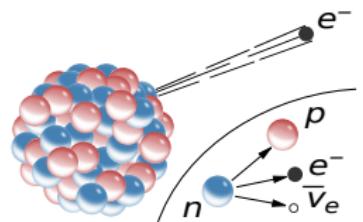
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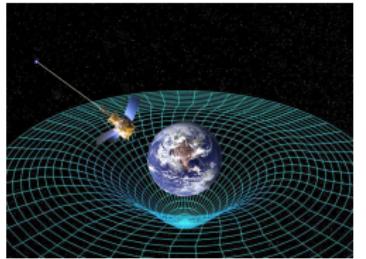


Electromagnetism



Weak nuclear force

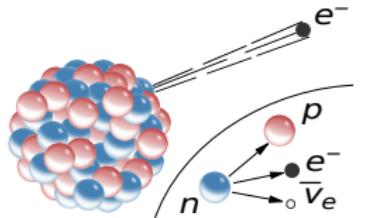
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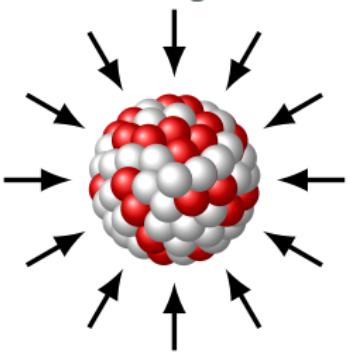
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Strong nuclear force

## What is the strong force?

The mass discrepancy is due to the interaction energy in which gluons are mediators.

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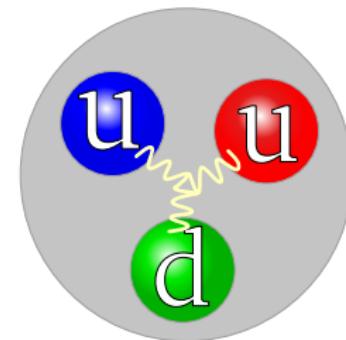
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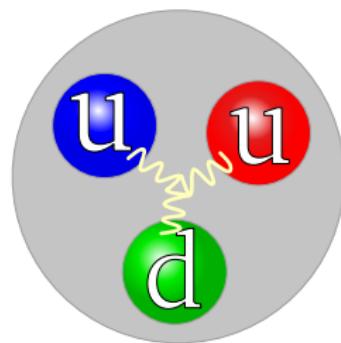
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A **proton** consists of: up-, up- and down-quarks

Mass discrepancy:

$$m_p \neq m_u + m_u + m_d,$$

$$936 \text{ MeV} \neq 3 \text{ MeV} + 3 \text{ MeV} + 6 \text{ MeV}.$$



## Comparing the strong force and QED

Electromagnetism or Quantum Electrodynamics(QED), a U(1) symmetry theory:

$$\mathcal{L}_{\text{QED}} = \sum_{f=\text{fermions}} \bar{\psi}_f (i\gamma^\mu (\partial_\mu + ieQ_f A_\mu) - m_f) \psi_f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Field strength tensor:

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- QED is U(1) theory, and the simply means that it contains a simple rotational symmetry.
- $e$  is the coupling constant and  $Q_f$  is the charge.
- $\bar{\psi}_f$  and  $\psi_f$  are is the fermion fields.
- $A_\mu$  is the gauge fields.
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The **strong nuclear force** or Quantum Chromo Dynamics(QCD), a SU(3) symmetry theory:

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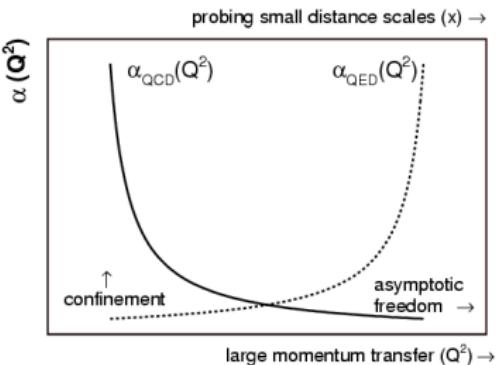
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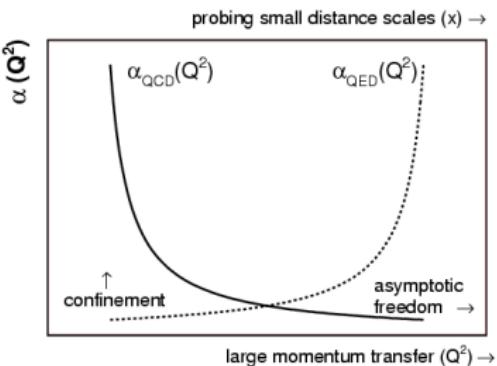
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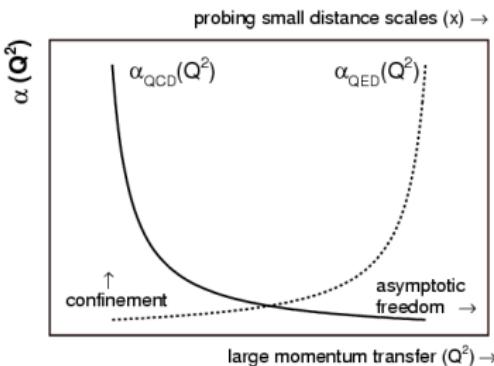


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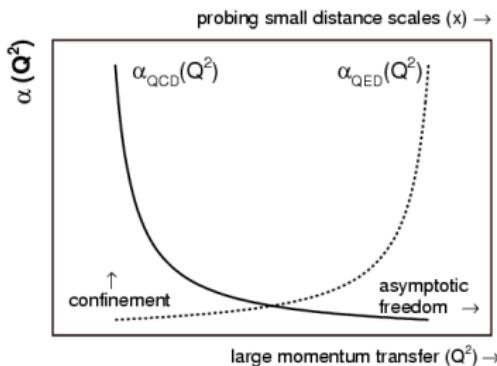


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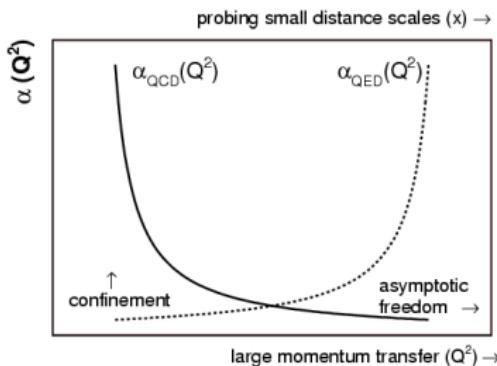


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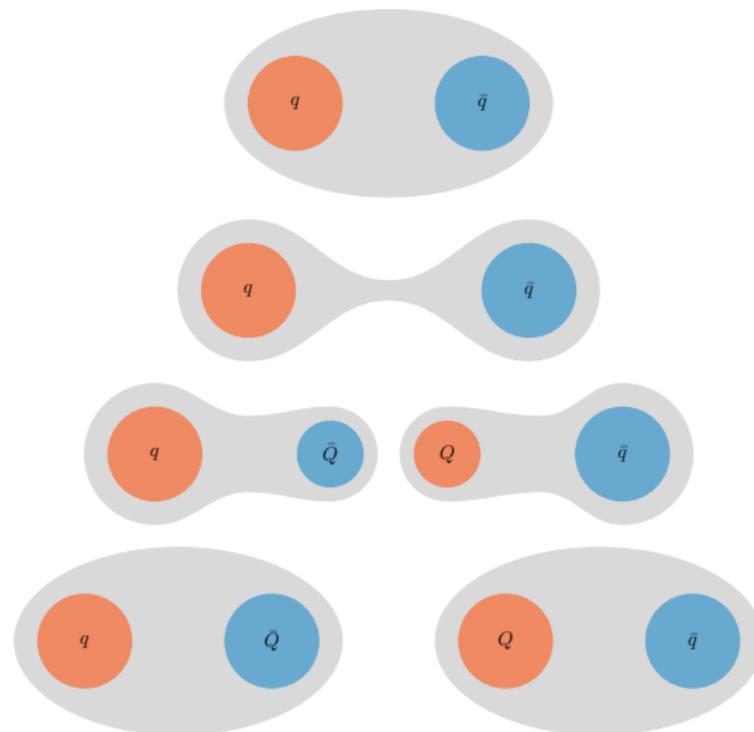
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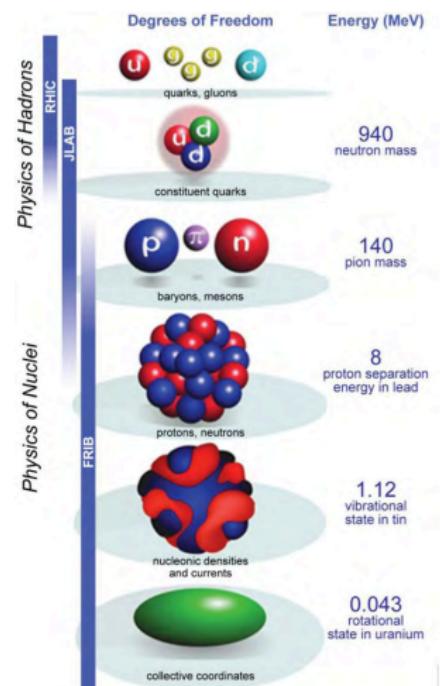
## Confinement: a low-energy phenomena

No free quarks in nature!



If we try to pull apart **two quarks in a meson**, more and more energy is required until we have enough energy to spontaneously create a **quark-antiquark pair**, forming thus **two new mesons**.

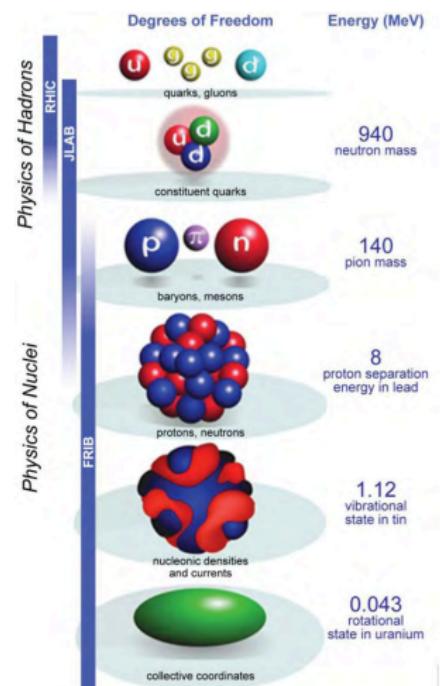
# QCD and nuclear physics



- Need to understand the low-energy regime in order to better understand nuclear physics!

The most fundamental theory we currently have of nuclear physics is QCD. Understanding QCD will help us understand nuclear physics and more *emergent* theories. But to bridge the gaps between these theories is difficult, as QCD contains a large number of degrees of freedom. Thus, a numerical approach is needed.

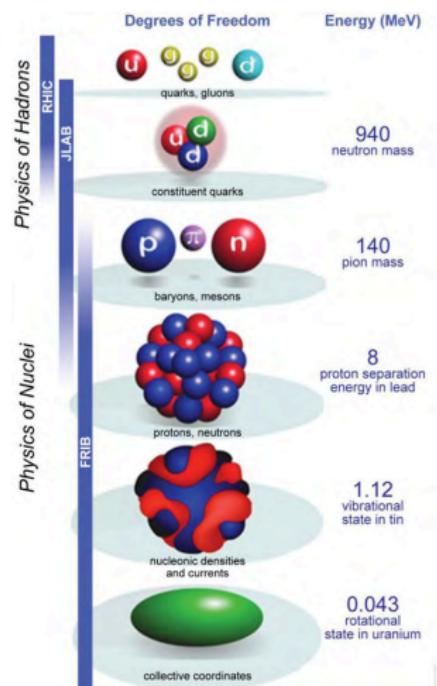
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- → numerical methods(e.g. lattice QCD)

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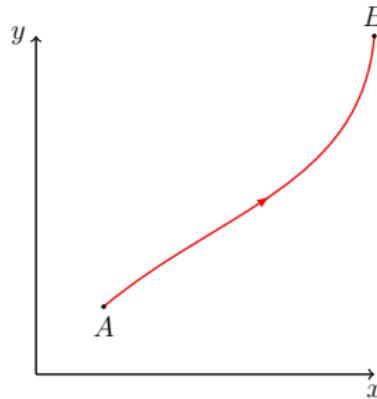
But first, we need to know *what* a path integral is.

## How we measure: path integrals

Going from  $t_0$  at  $A$  to a time  $t_1$  at  $B$  can be given in terms of a path integral.

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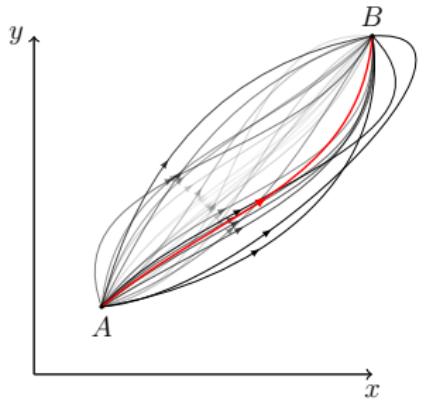


Classically, only one possible path obtained from the principle of least action.

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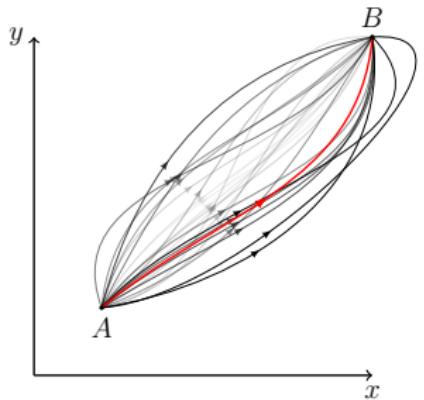


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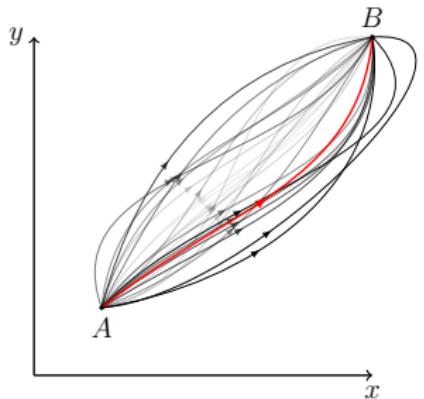
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Sum over all possible paths → the most likely path.

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## Path integrals

Given a field  $\phi_M$  in Minkowski space, the *partition function*  $Z$  is given by

$$Z = \int \mathcal{D}\phi_M e^{\frac{i}{\hbar} S_M[\phi_M]}$$

$\downarrow \quad \hbar = 1, \quad \tau \rightarrow -it$       imaginary time( $\rightarrow$  Euclidean space)!

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where  $\mathcal{D}$  is an integration of all possible paths in space.

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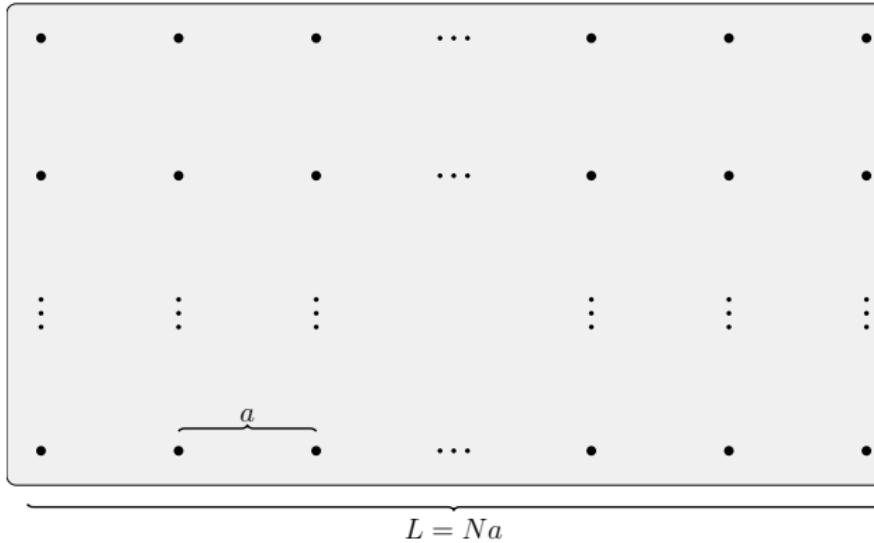
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## Discretizing the path integral

$N$  is the number of points on the lattice.



- We now have a integral over the field at every spacetime point.

Path integral integration measure becomes

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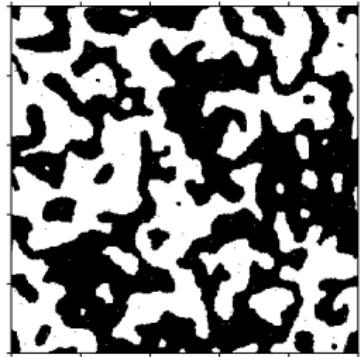
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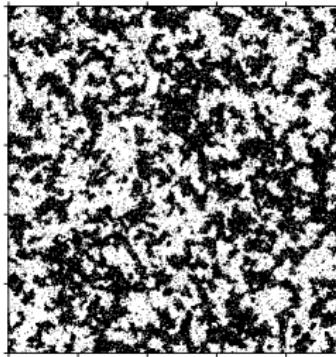
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Looking at a spin lattice of the Ising model,

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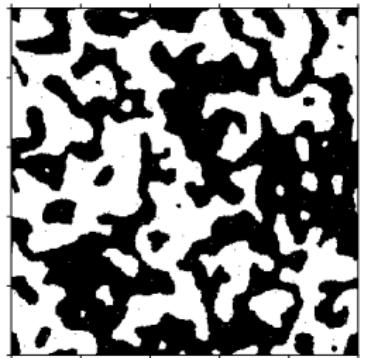


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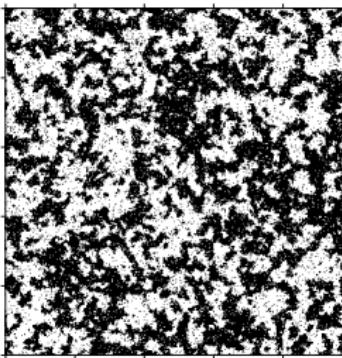
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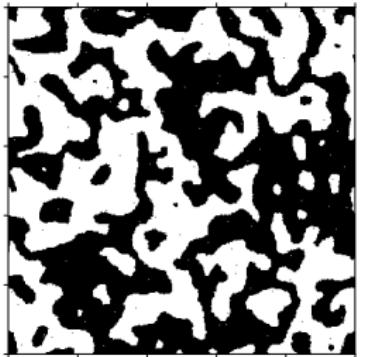
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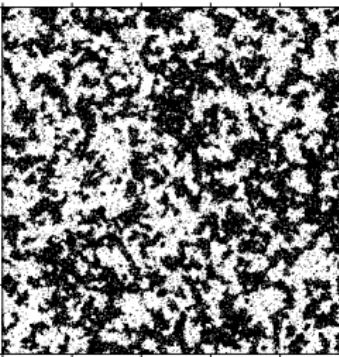
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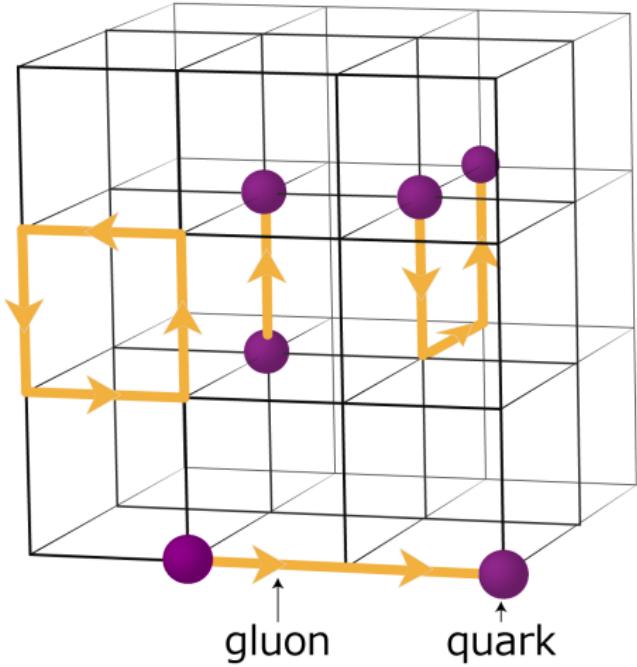
- A **configuration** in the Ising model is a given *arrangement of the spins*.
- A **configuration** in LQCD is a given *arrangement of the gauge field*.

An expectation value becomes

$$\langle O \rangle = \frac{1}{N_{\text{cfg}}} \sum_{i=1}^{N_{\text{cfg}}} O[\phi_i] + \mathcal{O}\left(\frac{1}{\sqrt{N_{\text{cfg}}}}\right)$$

where  $\phi_i$  is a generated gauge configuration(or just a general configuration).

# QCD on the lattice



- The lattice is a cube in 4D.
- Quarks at lattice, gluons in-between (**links**).
- Maintains the SU(3) symmetry by introducing links.
- Closed loops are gauge invariant.
- Smallest possible object: the plaquette.
- Paths of links with fermions as end points are gauge invariant.
- However, from now on we will ignore any fermions.

[http://www.jicfus.jp/en/wp-content/uploads/2012/12/  
LatticeQCD.png](http://www.jicfus.jp/en/wp-content/uploads/2012/12/LatticeQCD.png)

We exclude fermions to only look at the gauge fields,

$$S_G = \frac{1}{2} \int d^4x \text{tr} (G_{\mu\nu})^2$$

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with  $\beta = 6/g_S^2$

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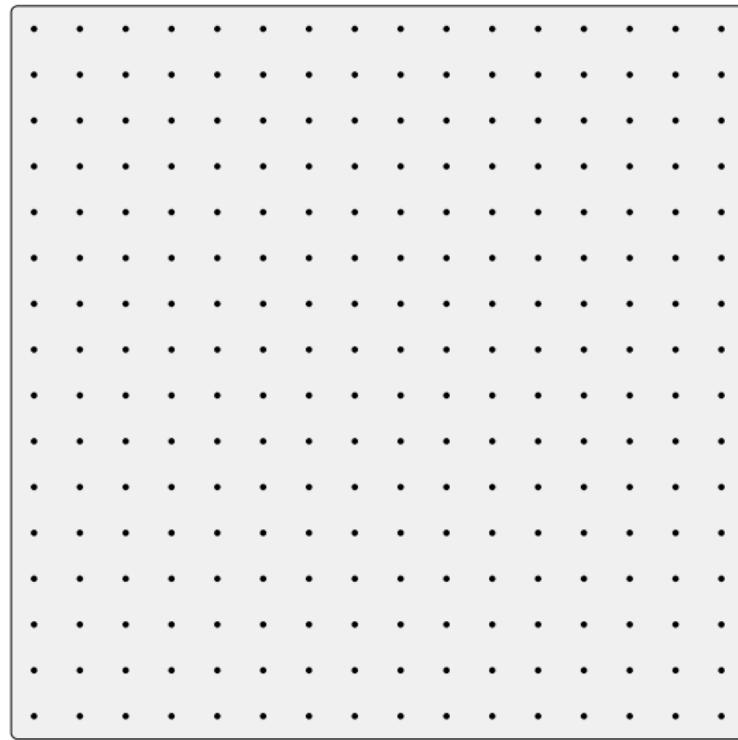
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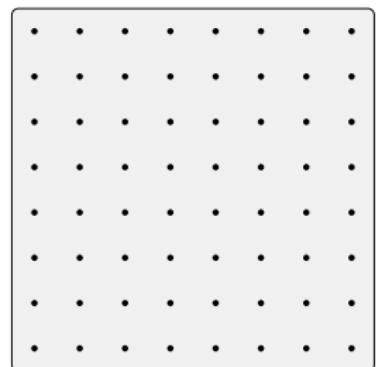
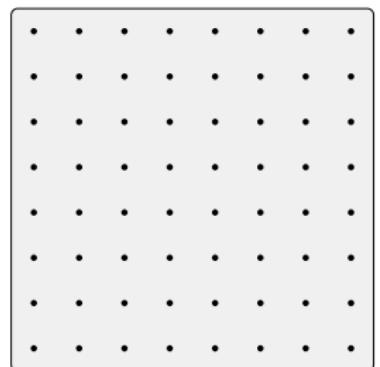
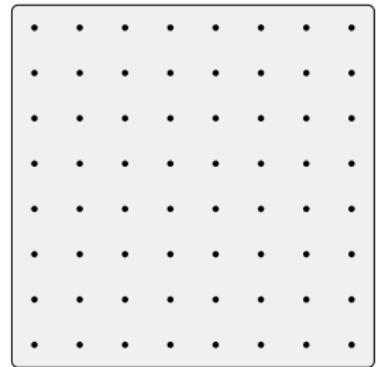
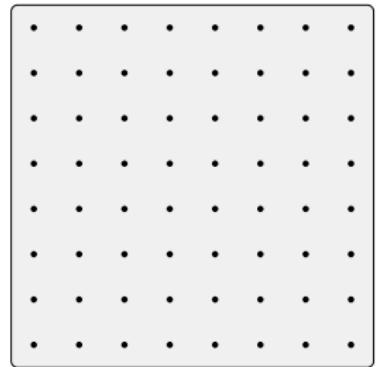
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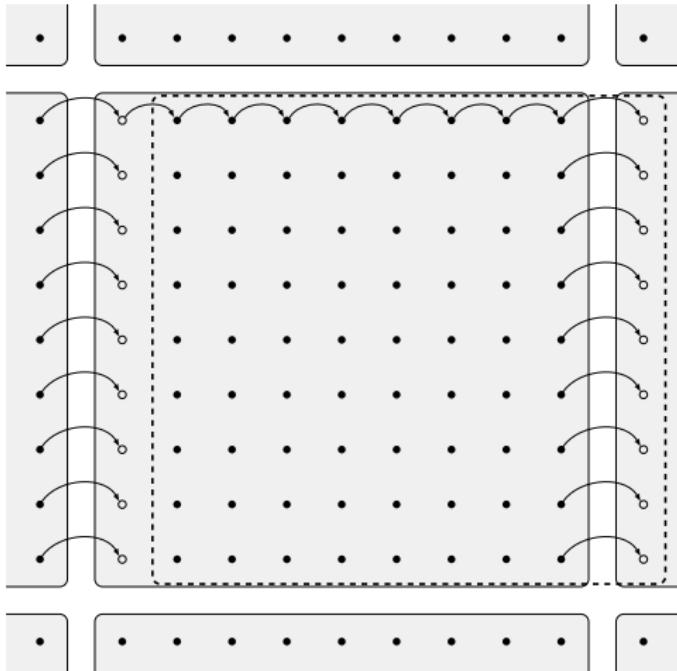
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An analogy: the diffusion equation:

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## Results

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## Ensembles

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$D_2$	6.45	48	96	250	0.0478(3)	5.695

- I implemented the methods discussed under a code I call GLAC, and will now present some of the results I generated using this code.
- The main ensembles made for this thesis.
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- Notice that the size of a lattice is 16 times larger when doubling the dimensions.
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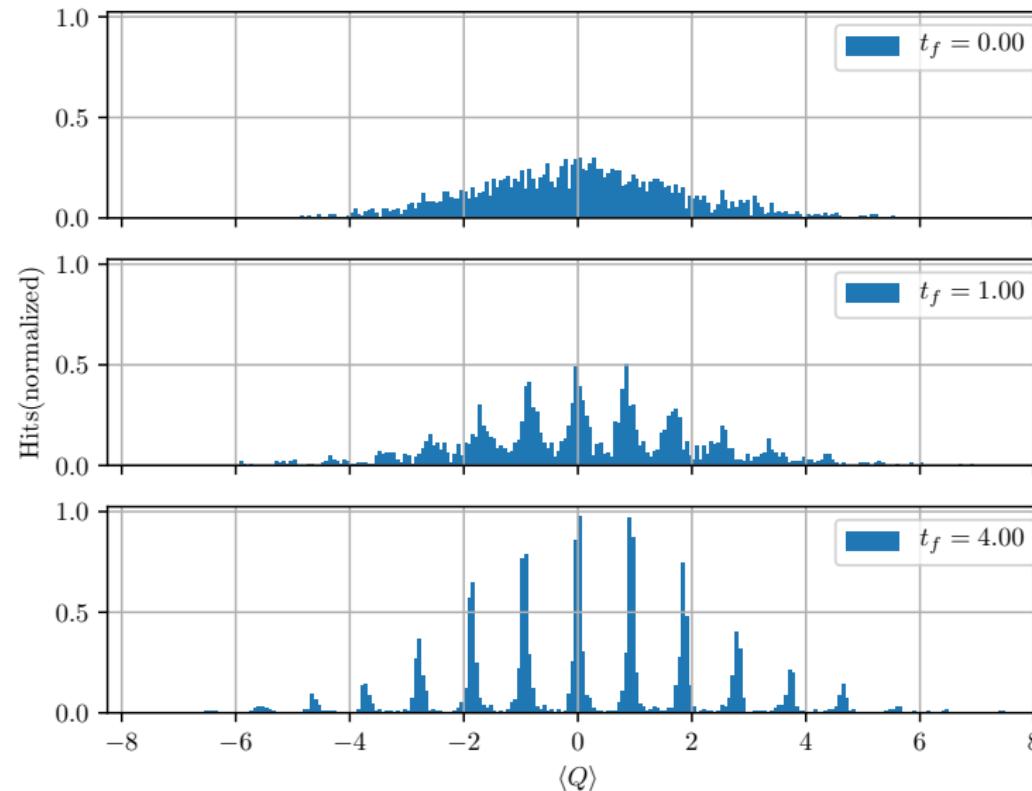
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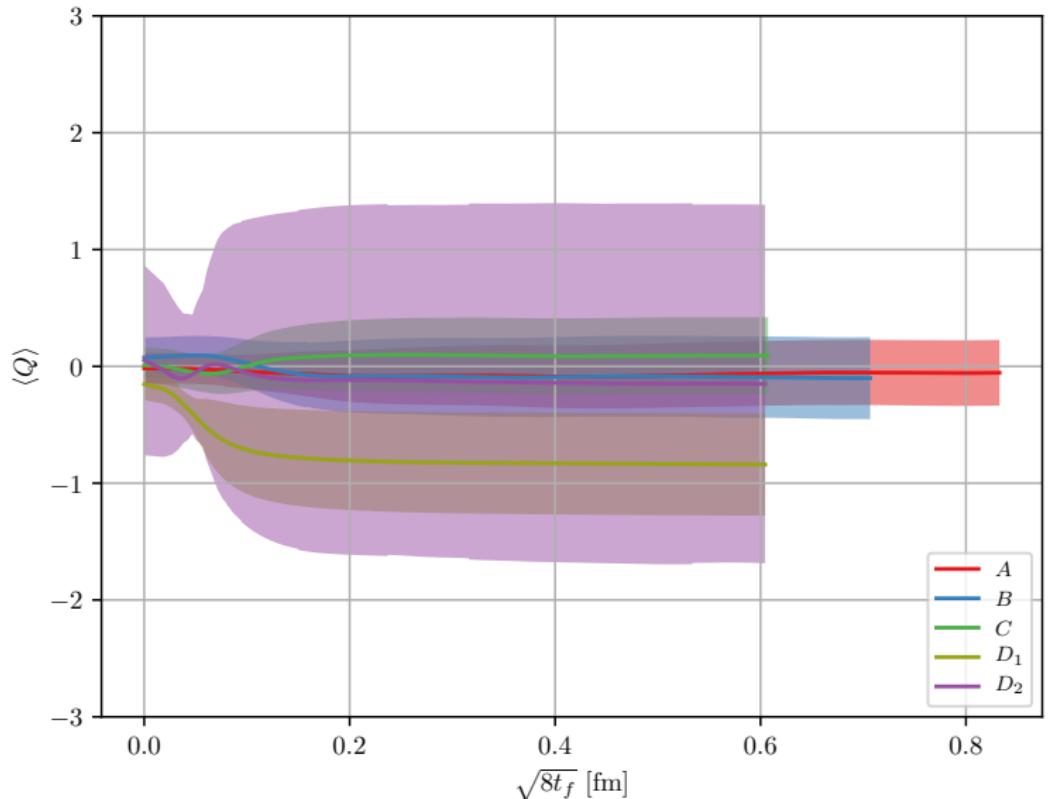
## Topological charge distribution

Histograms for the  $Q$  for ensemble  $G$  with a lattice of size  $N^3 \times N_T = 16^3 \times 32$  with  $\beta = 6.1$ , taken at different flow times  $t_f/a^2 = 0.0, 1.0, 4.0$  fm.



## Topological charge

## Topological charge for our main ensembles



## Autocorrelations

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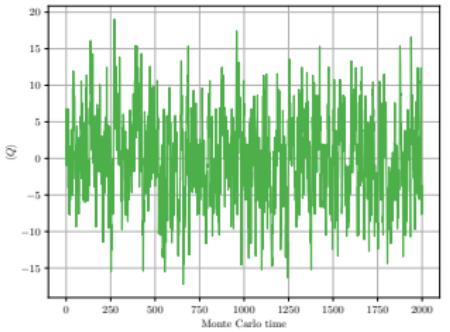
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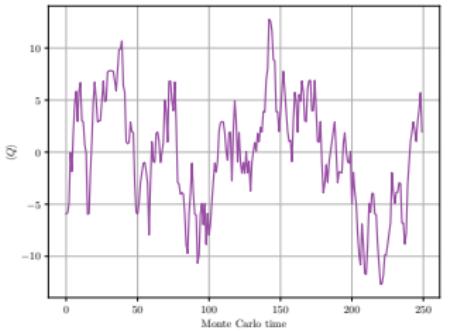
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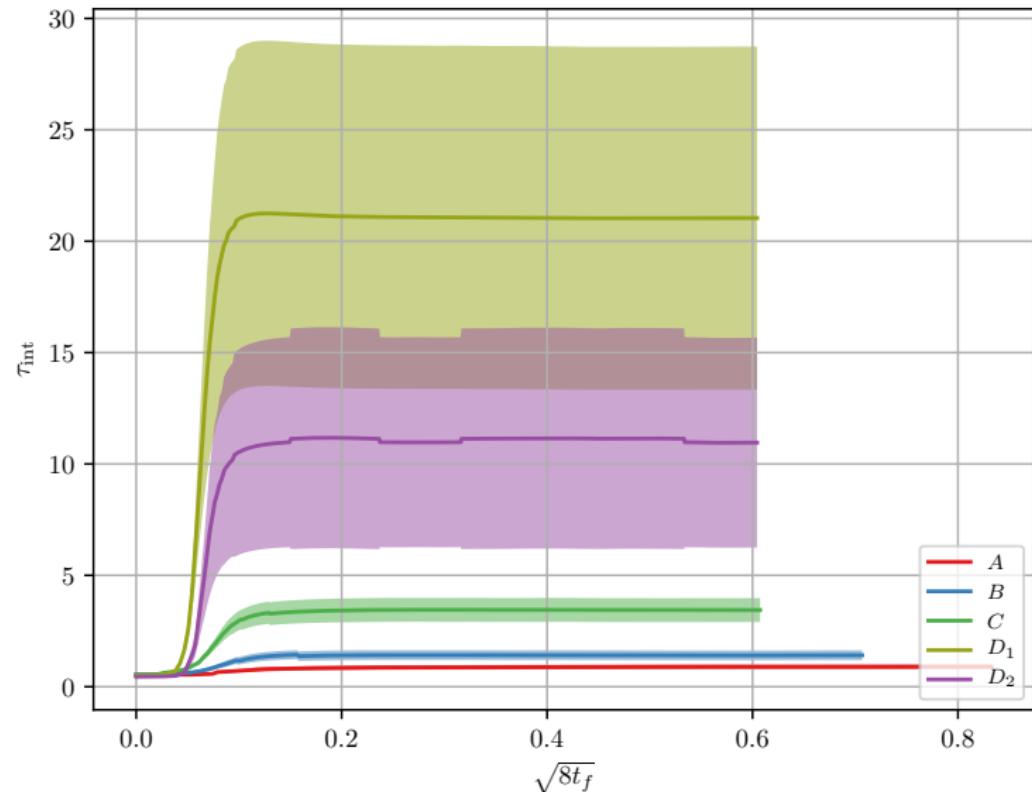


Ensemble  $D_2$ ,  $48^3 \times 96$ ,  $\beta = 6.45$

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## Topological charge autocorrelation

- The integrated autocorrelation  $\tau_{\text{int}}$  for topological charge for the five main ensembles.



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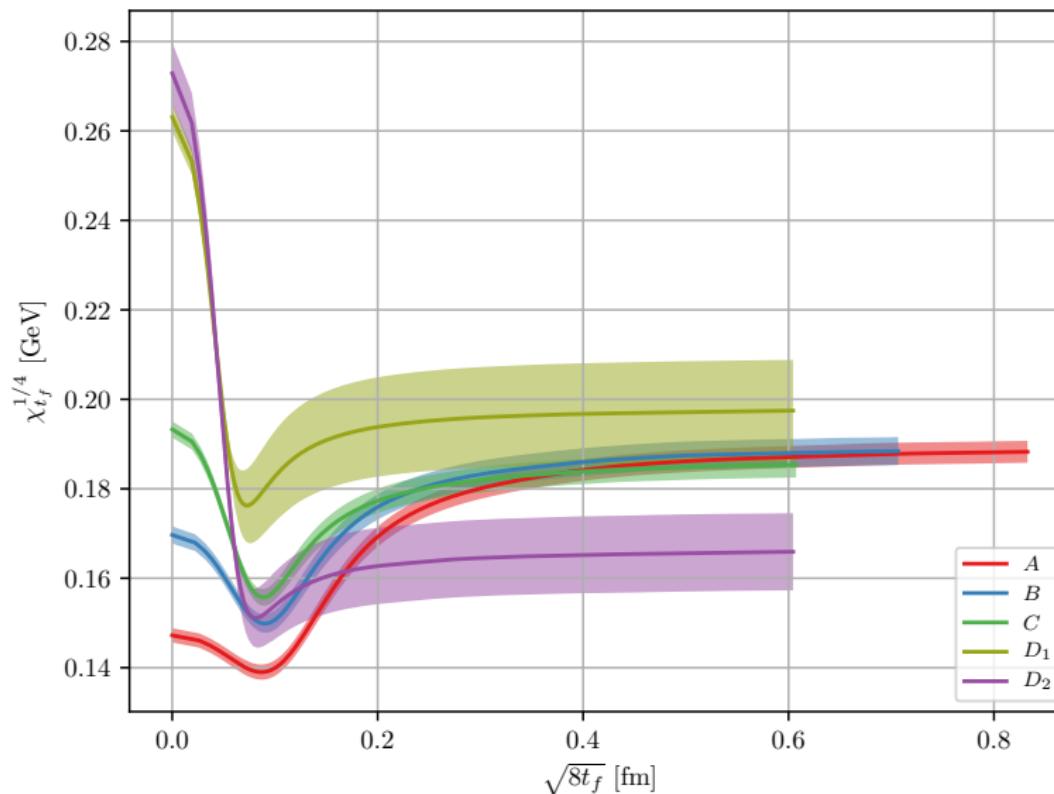
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## Topological susceptibility



- The topological susceptibility  $\chi_{tf}^{1/4}$  of the **main ensembles**.
- We have a **UV divergence at zeroth flow time**, hence to need for gradient flow which renormalizes this quantity.
- **Bootstrapped**  $N_{\text{bs}} = 500$  times.
- Corrected for autocorrelations with  $\sigma = \sqrt{2\tau_{\text{int}}}\sigma_0$ .

Ensembles	$\chi_{tf}^{1/4} (\langle Q^2 \rangle) [\text{GeV}]$	$N_f$	$\chi^2/\text{d.o.f}$
$A, B, C, D_2$	0.179(10)	3.75(29)	2.38
$A, B, C, D_1$	0.186(6)	3.21(25)	0.83
$A, B, C$	0.184(6)	3.37(26)	0.33

## The fourth cumulant

- Highly unstable, as we shall see.
- Will provide insight into the goodness of our ensembles.
- An  $R$ -value away from 1 will indicate that QCD cannot be described by the dilute instanton gas model.

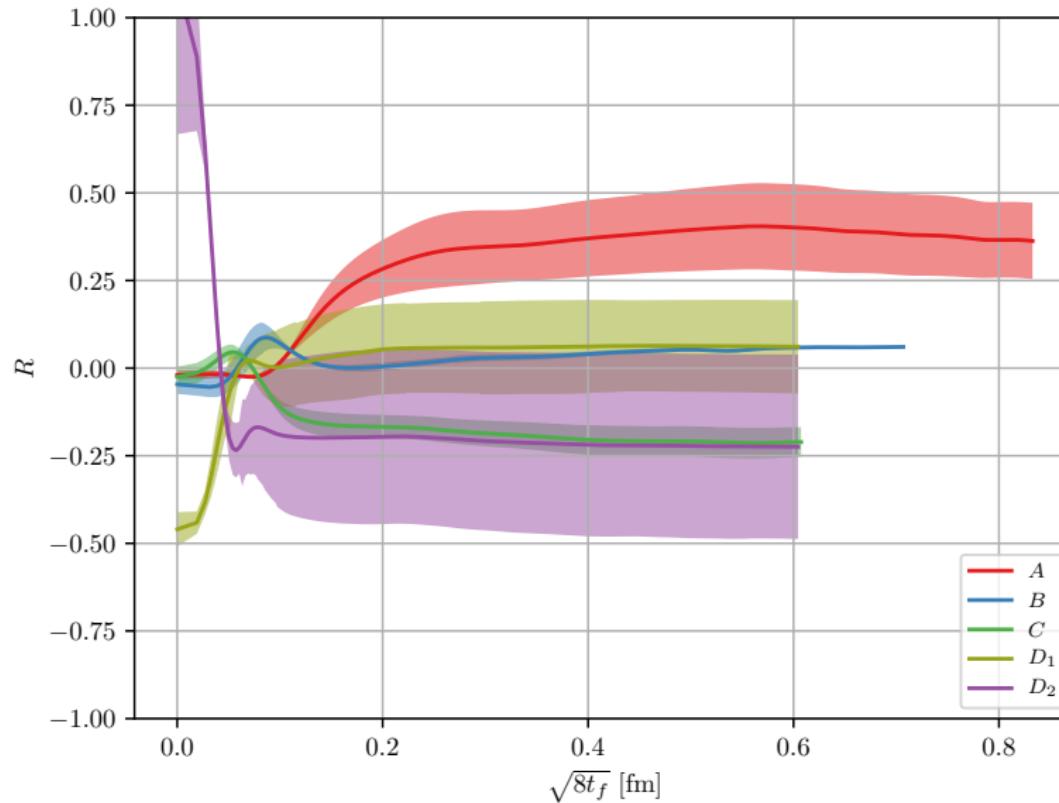
$$\langle Q^4 \rangle_c = \frac{1}{V^2} \left( \langle Q^4 \rangle - 3 \langle Q^2 \rangle^2 \right).$$

From this, we can also measure the ratio  $R$ ,

$$R = \frac{\langle Q^4 \rangle_c}{\frac{1}{V} \langle Q^2 \rangle} = \frac{1}{V} \frac{\langle Q^4 \rangle - 3 \langle Q^2 \rangle^2}{\langle Q^2 \rangle},$$

## The fourth cumulant

- The fourth cumulant ratio  $R = \langle Q^4 \rangle_C / \langle Q^2 \rangle$ .
- The results was analyzed using  $N_{\text{bs}} = 500$  bootstrap samples, with the error corrected for by  $\sqrt{2\tau_{\text{int}}}$ .



## The fourth cumulant at reference flow times

The fourth cumulant is taken at their individual reference scales seen in the third column. The data were analyzed with using a bootstrap analysis of  $N_{bs} = 500$  samples, with error corrected by the integrated autocorrelation,  $\sqrt{2\tau_{\text{int}}}$ .

Ensemble	$L/a$	$t_0/a^2$	$\langle Q^2 \rangle$	$\langle Q^4 \rangle$	$\langle Q^4 \rangle_C$	$R$
$A$	2.24	3.20(3)	0.78(4)	2.13(27)	0.282(67)	0.359(65)
$B$	2.21	4.43(4)	0.81(5)	1.98(23)	0.036(11)	0.044(11)
$C$	2.17	6.01(6)	0.77(4)	1.6(2)	-0.174(40)	-0.226(64)
$D_1$	1.53	12.2(1)	1.00(20)	3.01(1.07)	0.03(12)	0.03(12)
$D_2$	2.29	12.2(1)	0.497(100)	0.64(20)	-0.103(95)	-0.21(23)

## Comparing fourth cumulant

We can compare with article by Cè et al. [2015]

## Comparing fourth cumulant

- Parameters of the ensembles presented by Cè et al. [2015]. The first column is the ensemble name from the article. The letter indicates the volume, while the subindex indicates the  $\beta$  value. Ensembles of similar letters keep approximately the same length  $L$ .

Ensemble	$\beta$	$L/a$	$L$ [fm]	$a$ [fm]	$t_0/a^2$	$t_0/r_0^2$	$N_{\text{cfg}}$
$F_1$	5.96	16	1.632	0.102	2.7887(2)	0.1113(9)	1 440 000
$B_2$	6.05	14	1.218	0.087	3.7960(12)	0.1114(9)	144 000
$\tilde{D}_2$		17	1.479		3.7825(8)	0.1110(9)	
$B_3$	6.13	16	1.232	0.077	4.8855(15)	0.1113(10)	144 000
$\tilde{D}_3$		19	1.463		4.8722(11)	0.1110(10)	
$B_4$	6.21	18	1.224	0.068	6.2191(20)	0.1115(11)	144 000
$\tilde{D}_4$		21	1.428		6.1957(14)	0.1111(11)	

## Comparing fourth cumulant

Article	Thesis	Ratio( $\langle Q^2 \rangle$ )	Ratio( $\langle Q^4 \rangle$ )	Ratio( $\langle Q^4 \rangle_C$ )	Ratio( $R$ )
$F_1$	$A$	1.08(6)	1.34(18)	19.03(5.81)	17.64(4.48)
$B_2$	$A$	1.02(5)	1.15(15)	3.60(1.09)	3.54(90)
	$B$	1.04(6)	1.06(11)	0.480(74)	0.46(4)
$\tilde{D}_2$	$A$	1.02(5)	1.19(15)	8.31(1.99)	8.15(1.56)
	$B$	1.05(6)	1.10(12)	1.1(1)	1.06(3)
$B_3$	$B$	1.06(6)	1.10(12)	0.550(86)	0.52(5)
$\tilde{D}_3$	$B$	1.05(6)	1.11(12)	1.51(23)	1.4(1)
$B_4$	$C$	0.99(5)	0.86(8)	-2.32(46)	-2.35(59)
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- A comparison between the results obtained in this thesis on the fourth cumulant, and by those similar in volume form Cè et al. [2015]. *Ratio* indicates that we are dividing our results by the ones in previous table. **1 is perfect overlap.**

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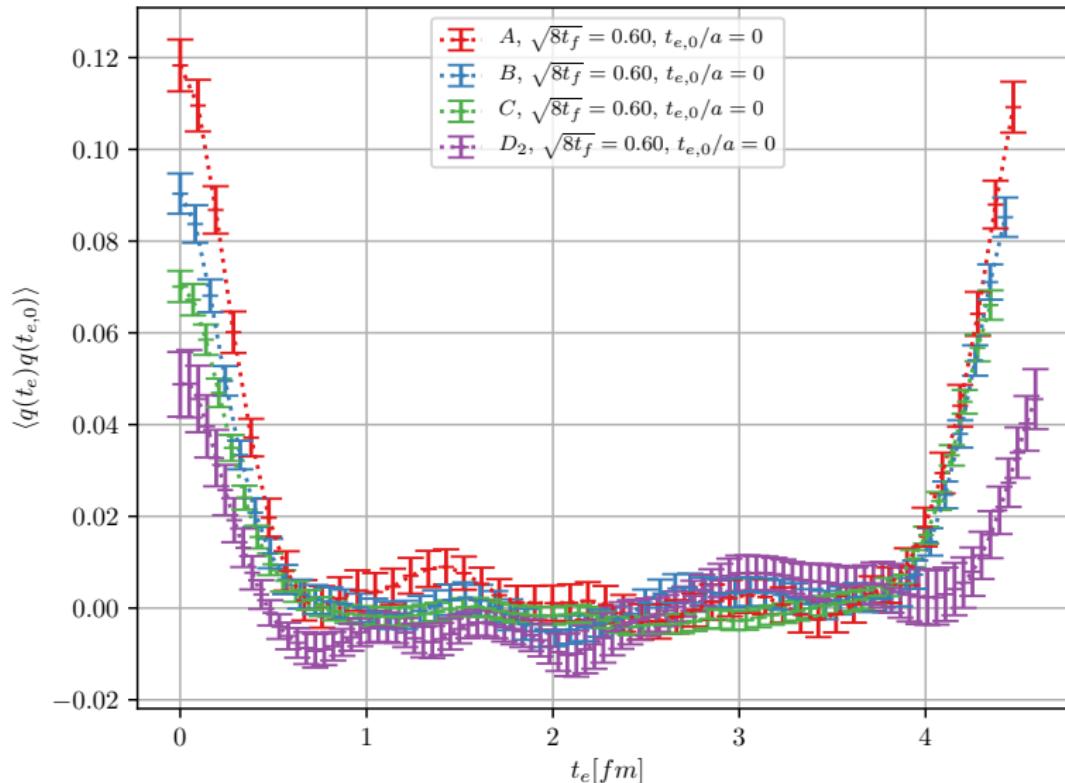
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from which the **effective glueball mass** can be extracted as

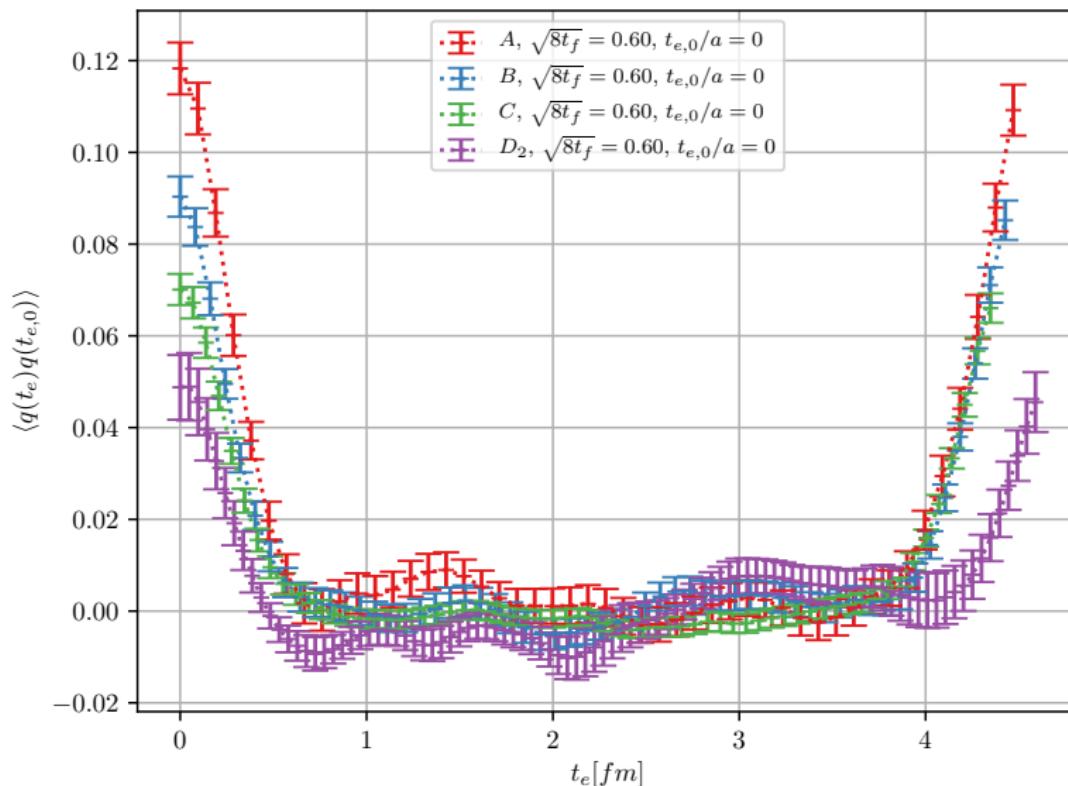
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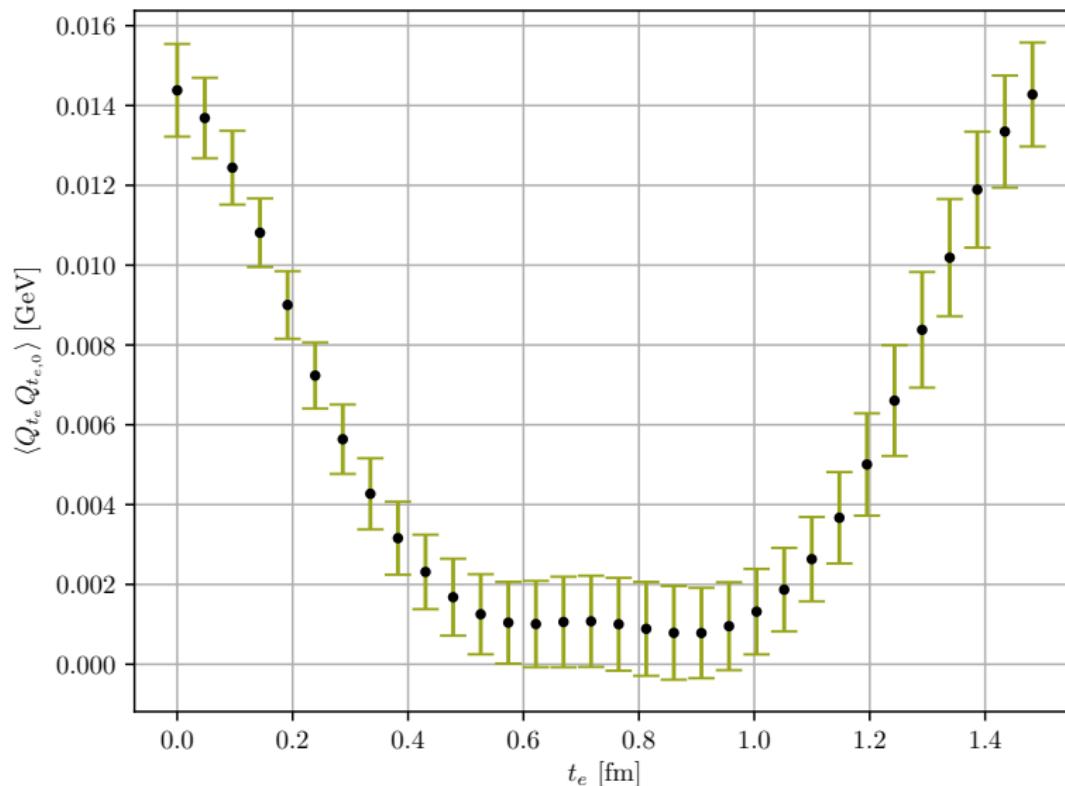


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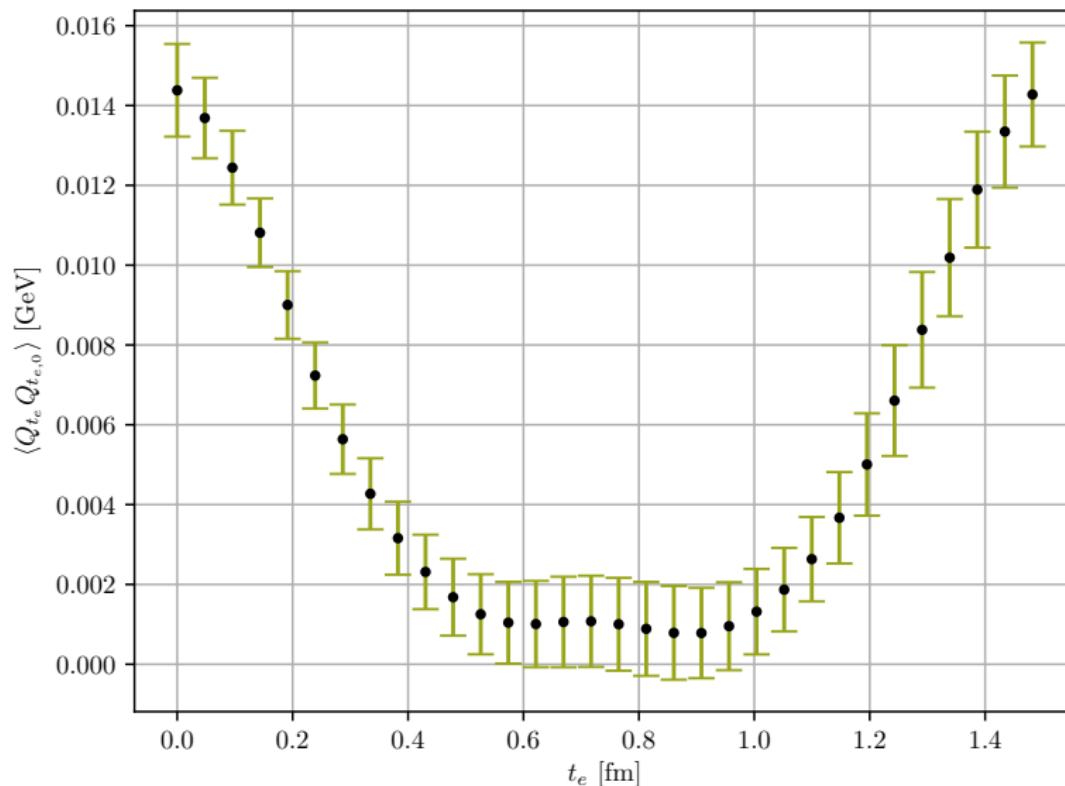
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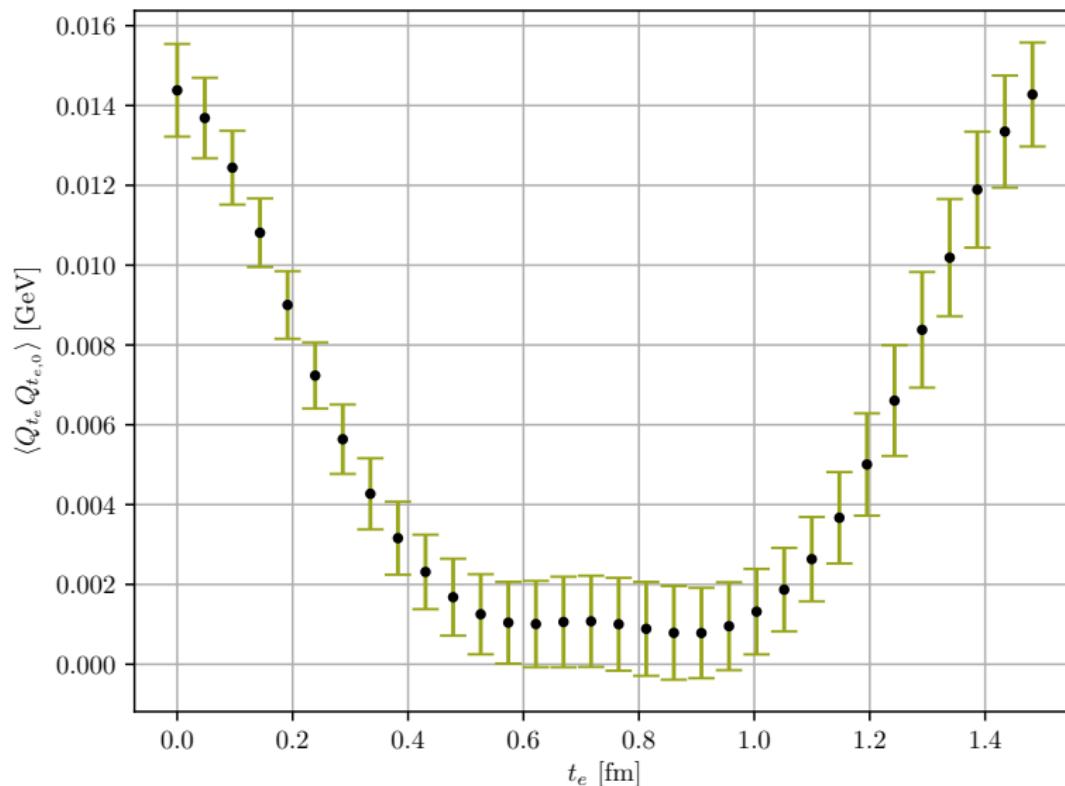
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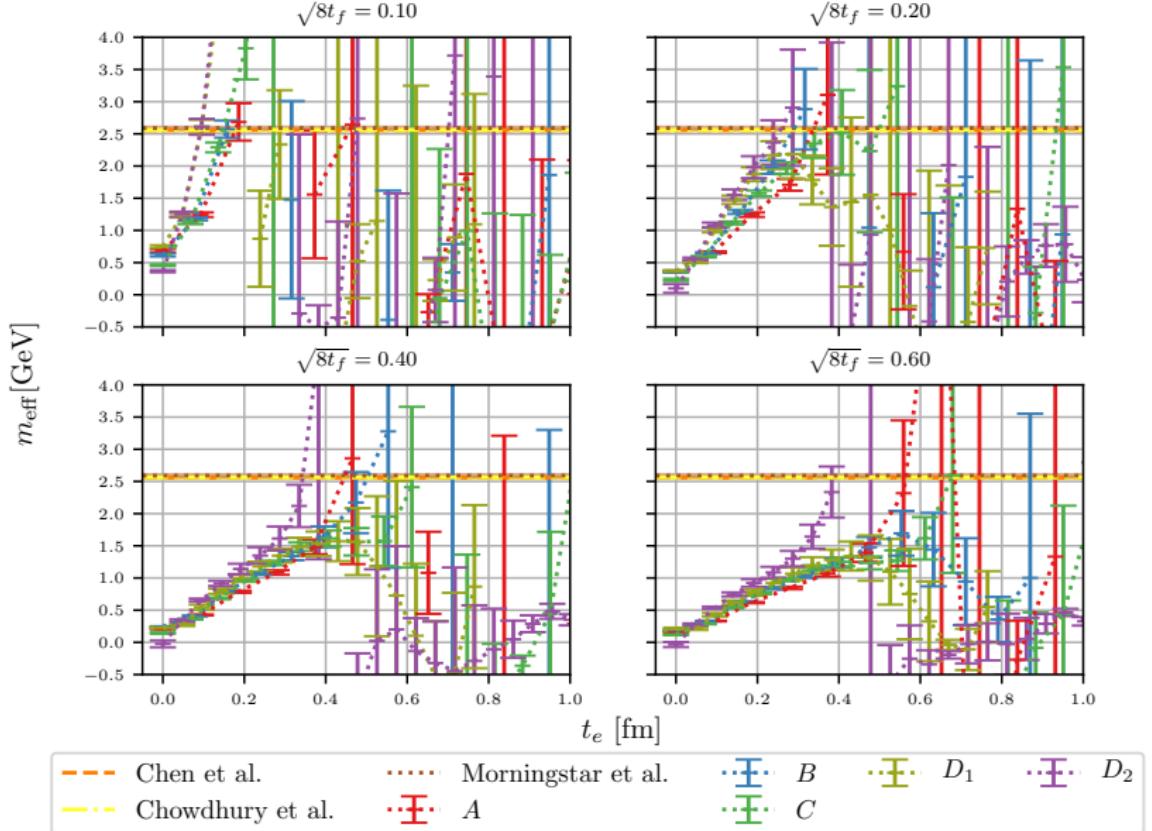
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Conclusion, future developments  
and final thoughts

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## Conclusion

- Created a code (GLAC) capable of generating and flowing gauge configurations.
  - Verified to match other code bases down to machine precision.
- Created a code (LatViz) for visualizing gauge fields.
- $\langle Q \rangle \neq 0$  for some ensembles.
- The topological susceptibility  $\langle \chi_f^{1/4} \rangle$  and  $N_f$
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- Implement better actions with operators that have smaller error contributions.
- Fermions and HMC(Hybrid Monte Carlo).

Thank you for listening.

Questions?

## References

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S. Borsanyi, S. Durr, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, T. Kurth, L. Lellouch, T. Lippert, C. McNeile, and K. K. Szabo. High-precision scale setting in lattice QCD. *Journal of High Energy Physics*, 2012(9), September 2012. ISSN 1029-8479. doi: [10.1007/JHEP09\(2012\)010](https://doi.org/10.1007/JHEP09(2012)010). URL <http://arxiv.org/abs/1203.4469>. arXiv: 1203.4469.

Marco Cè, Cristian Consonni, Georg P. Engel, and Leonardo Giusti. Non-Gaussianities in the topological charge distribution of the SU(3) Yang–Mills theory. *Physical Review D*, 92(7), October 2015. ISSN 1550-7998, 1550-2368. doi: [10.1103/PhysRevD.92.074502](https://doi.org/10.1103/PhysRevD.92.074502). URL <http://arxiv.org/abs/1506.06052>. arXiv: 1506.06052.

Martin Lüscher. Properties and uses of the Wilson flow in lattice QCD. *Journal of High Energy Physics*, 2010(8), August 2010. ISSN 1029-8479. doi: [10.1007/JHEP08\(2010\)071](https://doi.org/10.1007/JHEP08(2010)071). URL <http://arxiv.org/abs/1006.4518>. arXiv: 1006.4518.

Hans Munthe-Kaas. Runge–Kutta methods on Lie groups. *BIT Numerical Mathematics*, 38(1):92–111, March 1998. ISSN 0006-3835, 1572-9125. doi: [10.1007/BF02510919](https://doi.org/10.1007/BF02510919). URL <http://link.springer.com/10.1007/BF02510919>.

## Extra slides

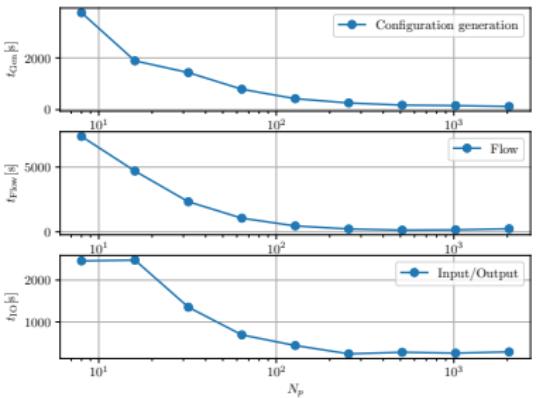
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# Scaling

- Strong scaling

We checked three types of scaling,

- Strong scaling: *fixed problem* and a variable  $N_p$  cores

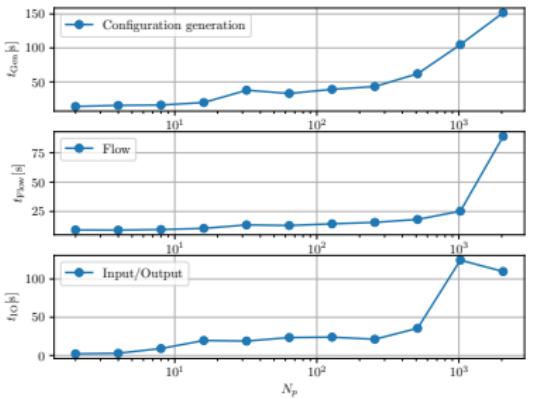


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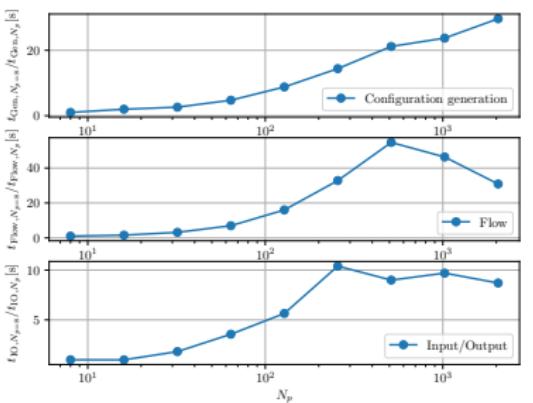
- **Strong scaling:** *fixed problem and a variable  $N_p$  cores*
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We appear to have a plateau around 512 cores.

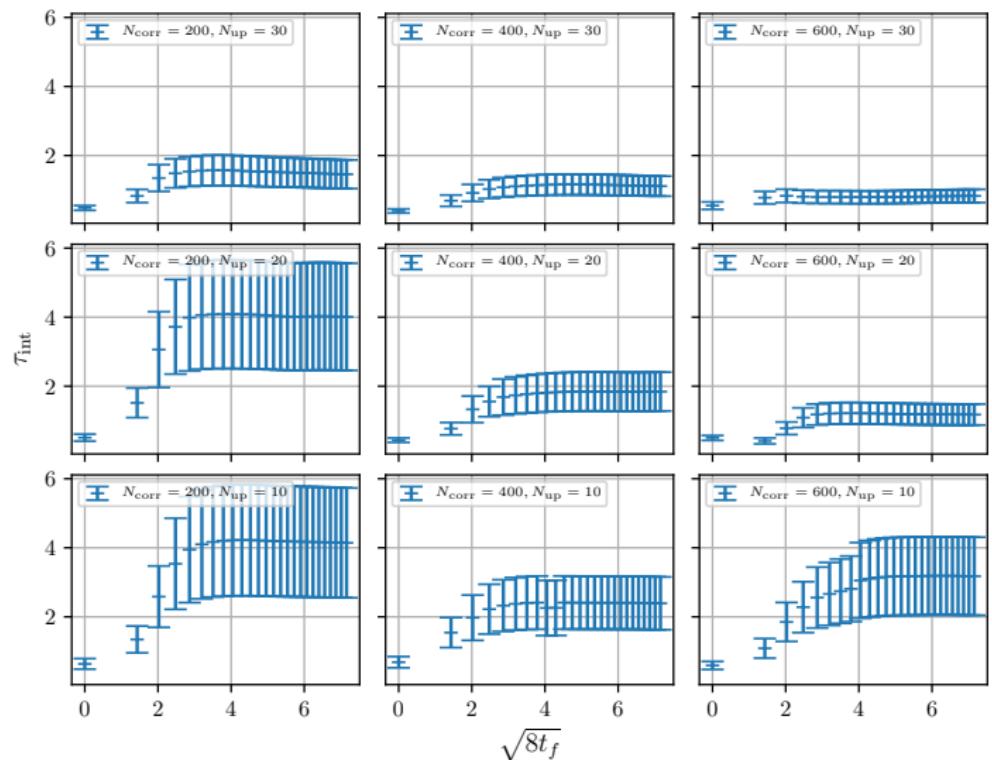
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## Optimizing the gauge configuration generation

- We run for different values for  $N_{\text{up}}$  and  $N_{\text{corr}}$  to see what gives optimizes **computational cost** and **autocorrelation**.
- The integrated autocorrelation time for topological charge  $\langle Q \rangle$  for a lattice of size  $N = 16$  and  $N_T = 32$  with  $\beta = 6.0$  for combinations of  $N_{\text{corr}} \in [200, 400, 600]$  and  $N_{\text{up}} \in [10, 20, 30]$ , plotted against flow time  $\sqrt{8t_f}$ .

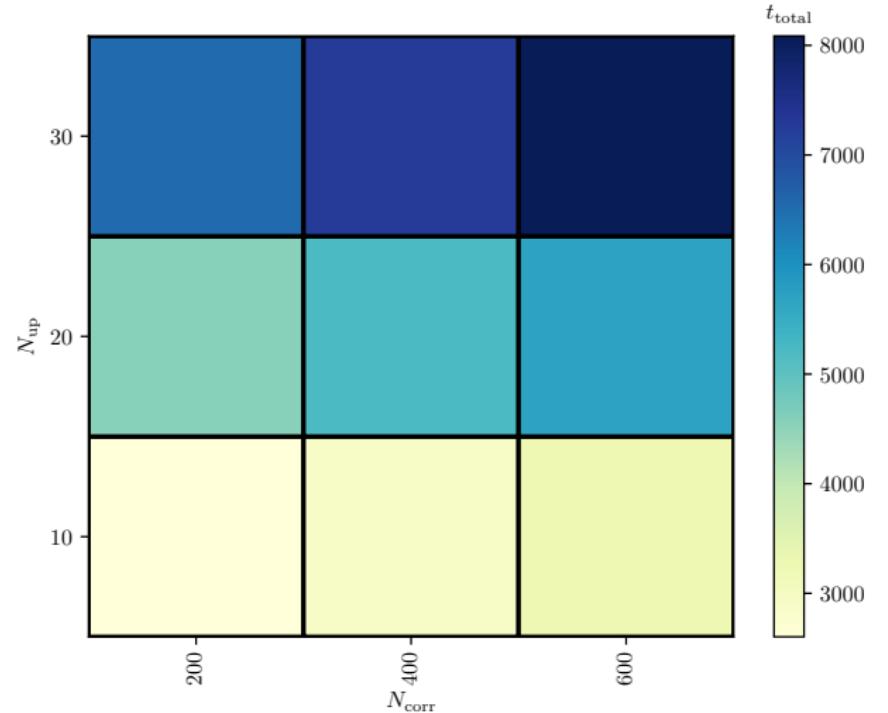
Generated 200 configurations for a lattice of size  $N^3 \times N_T = 16^3 \times 32$  and  $\beta = 6.0$ , for combinations of  $N_{\text{corr}} \in [200, 400, 600]$  and  $N_{\text{up}} \in [10, 20, 30]$ .

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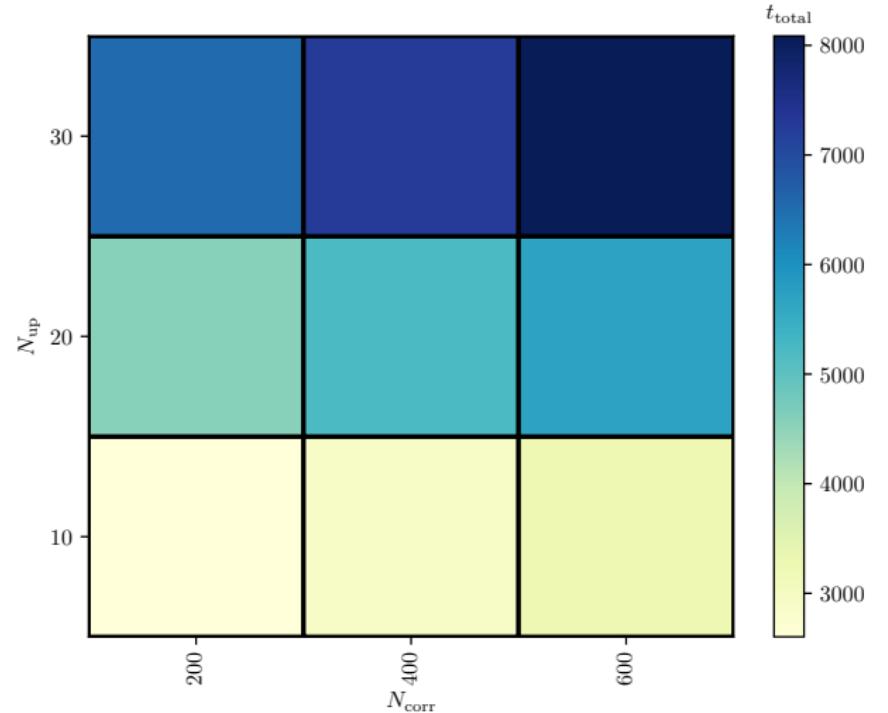
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## Verifying the integration

- The values we will test the integrator against.

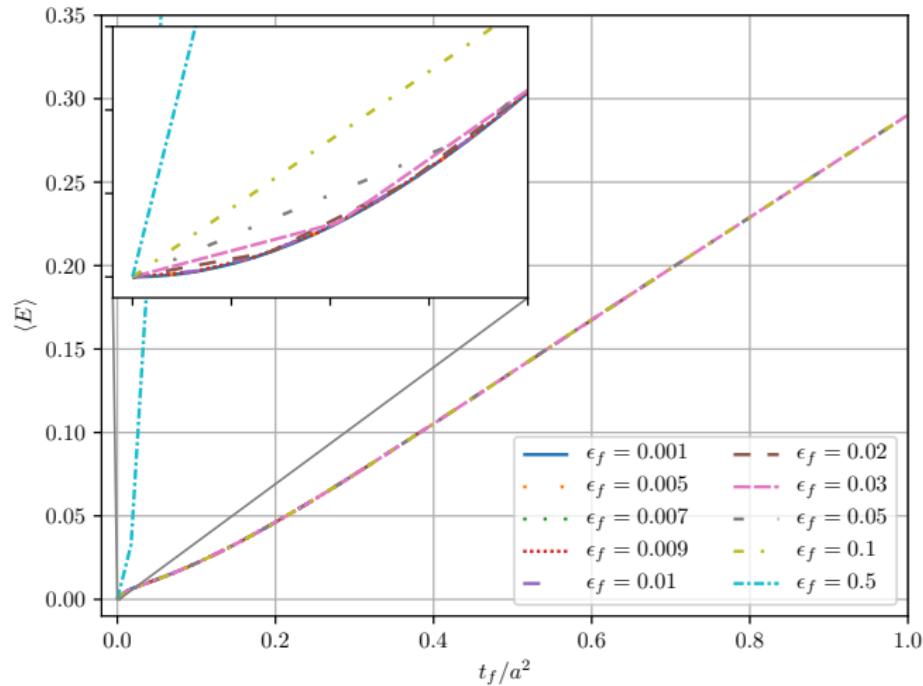
Testing the integrator for different integration steps  $\epsilon_f$ .

$\epsilon_f$	0.001	0.005	0.007	0.009	0.01	0.02	0.03	0.05	0.1	0.5
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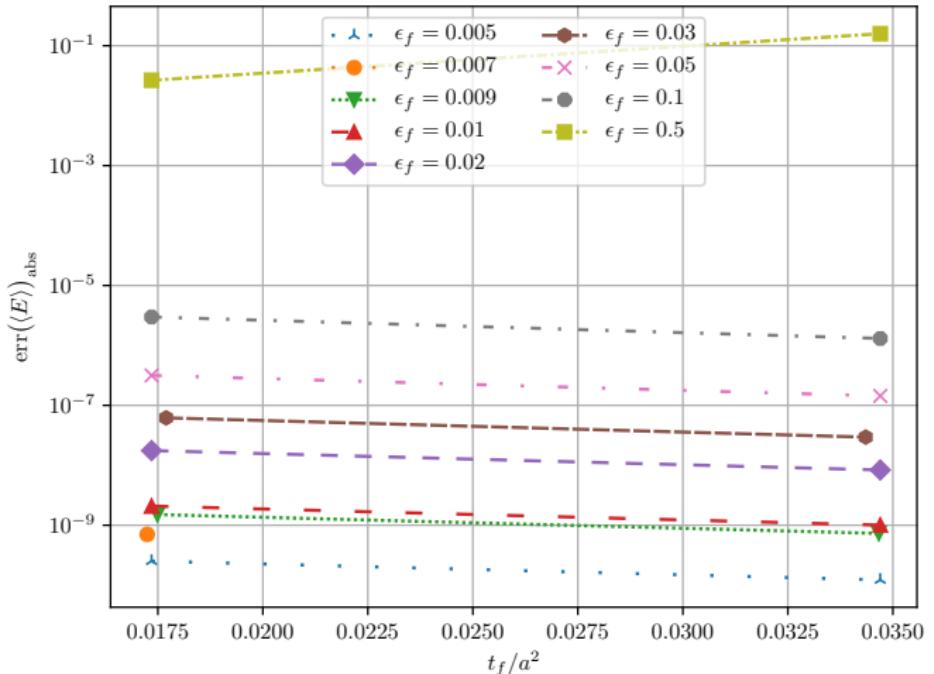
Lattice size  $N^3 \times N_T = 24^3 \times 48$  with  $\beta = 6.0$ .

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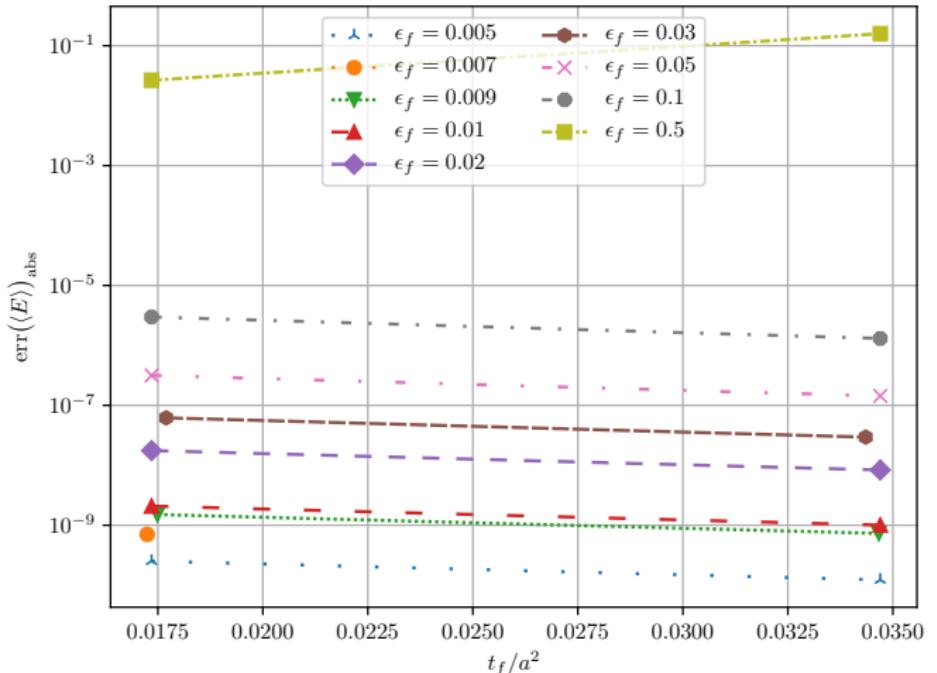
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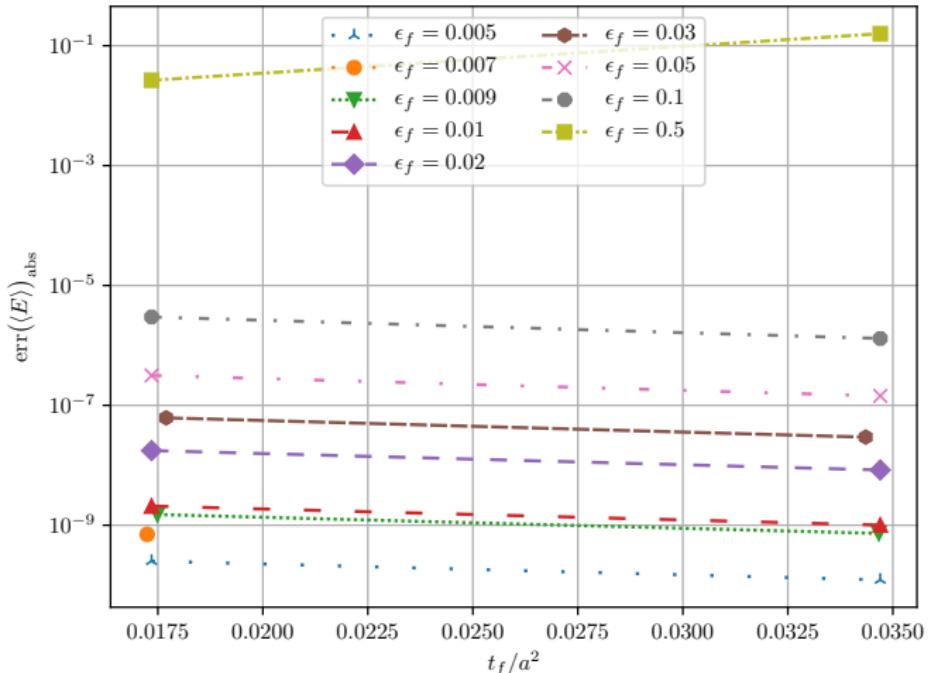
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# The non-linearity of QCD

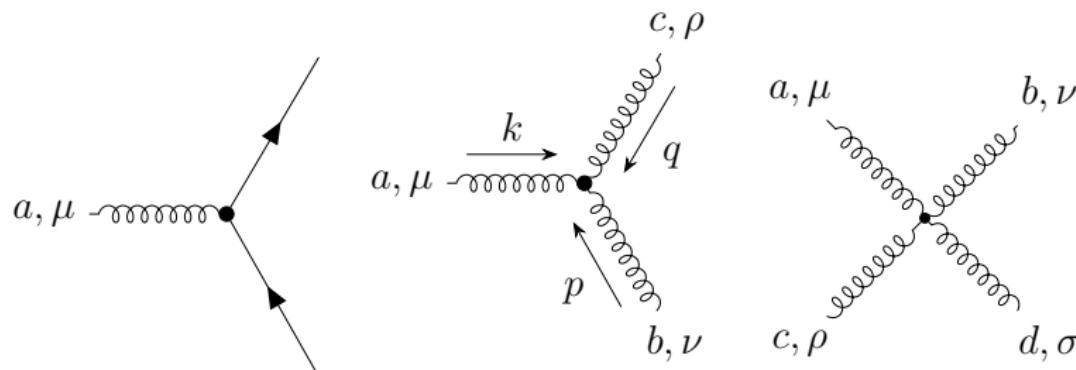
The QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \sum_{f=1}^{N_f} \bar{\psi}^{(f)} \left( i \not{D} - m^{(f)} \right) \psi^{(f)} - \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu},$$

with action

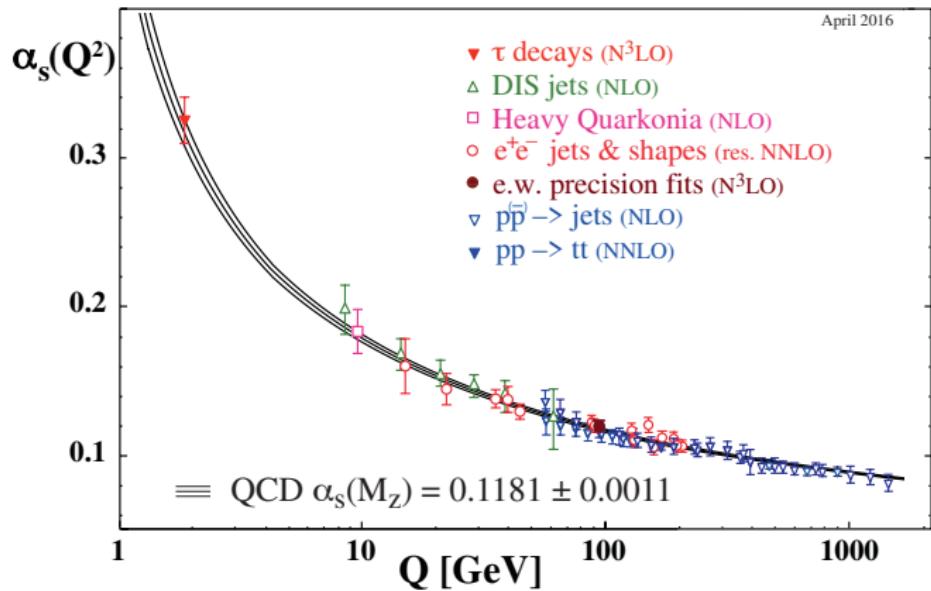
$$S = \int d^4x \mathcal{L}_{\text{QCD}}, \quad (1)$$

is invariant under a SU(3) symmetry.



- *Gluon self-interaction.*
- This central aspect is mostly covered in the pure-gauge/Yang-Mills section of the theory.
- **Two important features:** confinement and asymptotic freedom.

# Asymptotic freedom



- The coupling constant **decreases** as we **increase** the energy.
- Also serves as an *experimental proof* of QCD.
- Other lines of evidence: triple  $\gamma$  decay and muon cross section ratio  $R$ .
  - Triple  $\gamma$  decay: the number of colors is included in the cross section, which can be measured experimentally.
  - Muon cross section ratio  $R$ : the ratio is dependent on having three colors.

- We rewrite the equations slightly,

With

$$\dot{V}_{t_f} = -g_S^2 \{ \partial_{x,\mu} S_G[V_{t_f}] \} V_{t_f} = Z(V_{t_f}) V_{t_f},$$

## Solving gradient flow with Runge-Kutta 3

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and  $Z_i = \epsilon_f Z(W_i)$  we get

$$W_0 = V_{t_f},$$

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$$V_{t_f + \epsilon_f} = \exp \left[ \frac{3}{4} Z_2 - \frac{8}{9} Z_1 + \frac{17}{36} Z_0 \right] W_2,$$

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## Additional ensembles

- Additional ensembles made in order to illuminate additional aspects of the topological charge.
- Supporting ensembles made on Smaug. All ensembles were flown  $N_{\text{flow}} = 1000$  steps with  $\epsilon_{\text{flow}} = 0.01$ .

Ensemble	$N$	$N_T$	$N_{\text{cfg}}$	$N_{\text{corr}}$	$N_{\text{up}}$	$a$ [fm]	$L$ [fm]
$E$	8	16	8135	600	30	0.0931(4)	0.745(3)
$F$	12	24	1341	200	20	0.0931(4)	1.118(5)
$G$	16	32	2000	400	20	0.0790(3)	1.265(6)

## Energy definition

We use  $t_0$

$$E = \frac{a^4}{2|\Lambda|} \sum_{n \in \Lambda} \sum_{\mu, \nu} (F_{\mu\nu}^{\text{clov}}(n))^2$$

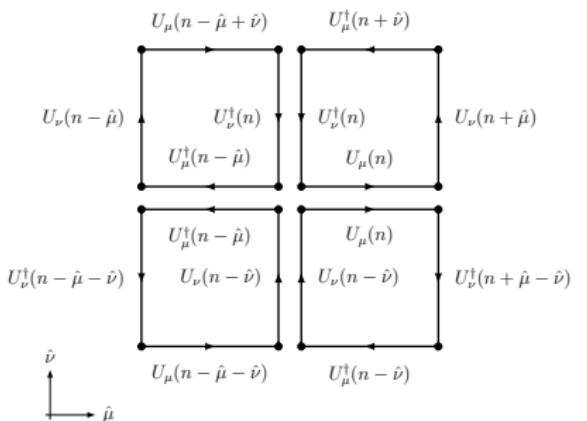
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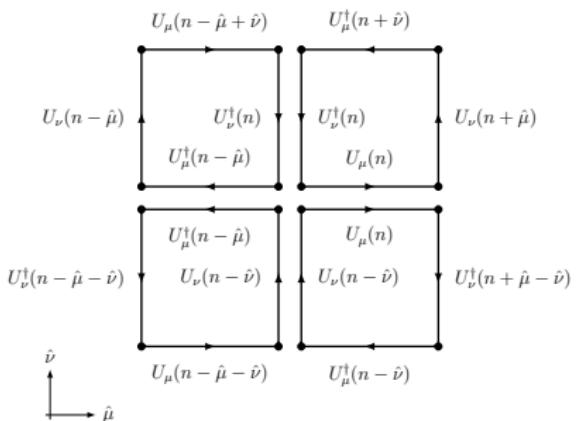
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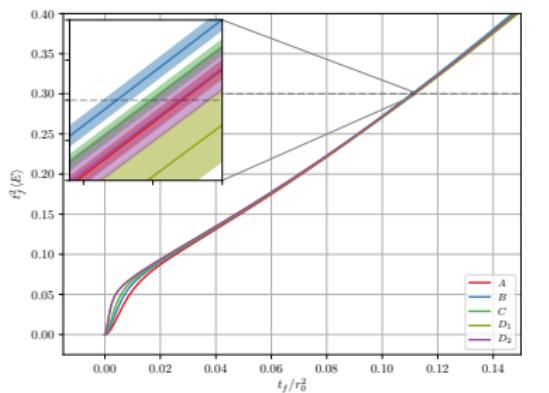
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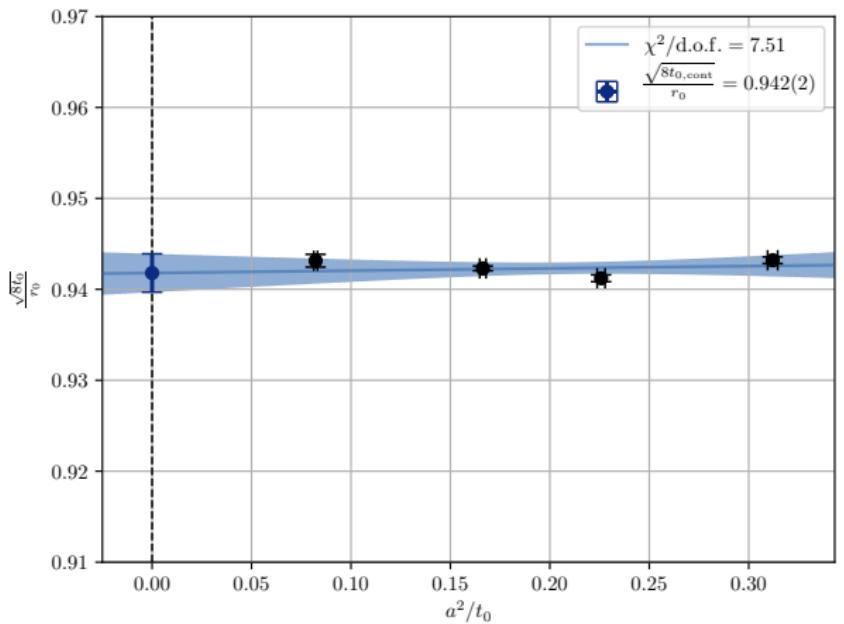
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Using scale definition  $t_0$  from Lüscher [2010],

$$\{t_f^2 \langle E(t) \rangle\}_{t_f=t_0} = 0.3.$$



## Scale setting $t_0$



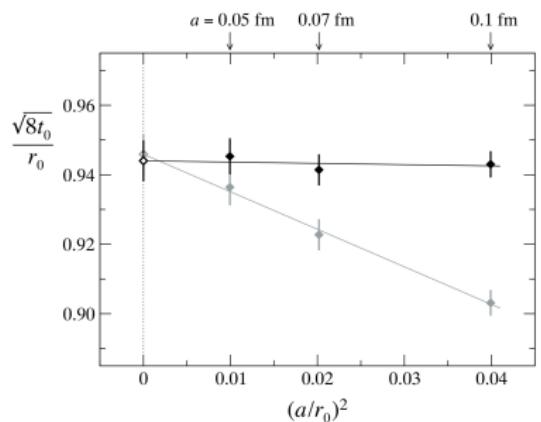
Continuum extrapolation using ensembles  $A$ ,  $B$ ,  $C$ , and  $D_2$  gives  
 $t_0,\text{cont}/r_0^2 = 0.11087(50)$ .

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- $r_0 = 0.5$  fm.

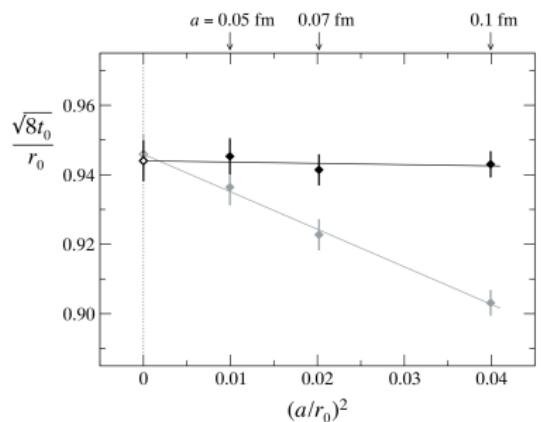
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Ensemble	$L/a$	$L$ [fm]	$a$ [fm]
$A$	24	2.235(9)	0.0931(4)
$B$	28	2.214(10)	0.0791(3)
$C$	32	2.17(1)	0.0679(3)
$D_1$	32	1.530(9)	0.0478(3)
$D_2$	48	2.29(1)	0.0478(3)

## Scale setting $t_0$

- Extrapolation results for  $t_0$ , where we retrieved the exact point of intersection between  $t_f^2 \langle E \rangle$  and 0.3 using  $N_{\text{bs}} = 500$  bootstrap fits. Extrapolating to the continuum gives us  $t_{0,\text{cont}}/r_0^2 = 0.11087(50)$ .

Ensemble	$t_0[\text{fm}^2]$	$t_0/a^2$	$t_0/r_0^2$
$A$	0.02780(2)	3.20(3)	0.11121(9)
$B$	0.02769(2)	4.43(4)	0.11075(10)
$C$	0.02775(2)	6.01(6)	0.11099(8)
$D_1$	0.02779(5)	12.2(1)	0.1112(2)
$D_2$	0.02794(9)	12.2(1)	0.1117(3)

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- Notice the  $\chi^2/\text{d.o.f.}$  of the extrapolation versus the two other extrapolations.

Extrapolations for different ensemble-combinations

Ensembles	$t_{0,\text{cont}}/r_0^2$	$\chi^2/\text{d.o.f.}$
$A, B, C, D_2$	0.11087(50)	7.51
$B, C, D_2$	0.1115(3)	0.41
$A, B, C, D_1$	0.1119(6)	0.88

## Scale setting $w_0$

Can also set a scale using the derivative which offers more granularity for small flow times,

$$W(t)|_{t=w_0^2} = 0.3,$$

$$W(t) \equiv t_f \frac{d}{dt_f} \{ t_f^2 \langle E \rangle \}.$$

First presented by Borsanyi et al. [2012].

## Scale setting $w_0$

Ensembles	$w_{0,\text{cont}}[\text{fm}]$	$\chi^2/\text{d.o.f}$
$A, B, C, D_2$	0.1695(5)	7.12
$B, C, D_2$	0.1702(3)	0.53
$A, B, C, D_1$	0.1706(6)	0.86

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Comparable to Borsanyi et al. [2012] which included dynamical fermions, with  $w_{0,\text{cont}} = 0.1755(18)(04)$  fm.

## Autocorrelation in the energy

The autocorrelation of the energy. A value of  $\tau_{\text{int}} = 0.5$  indicates that we have zero autocorrelation.

