

# Solving SU(3) Yang-Mills theory on the lattice: a calculation of selected gauge observables with gradient flow

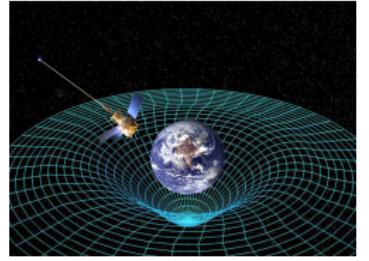
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04.07.19

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Co-supervisor: *Morten Hjorth-Jensen*

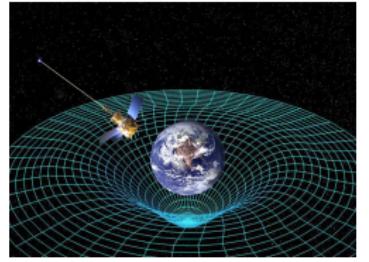
University of Oslo

# The four forces of nature



Gravity

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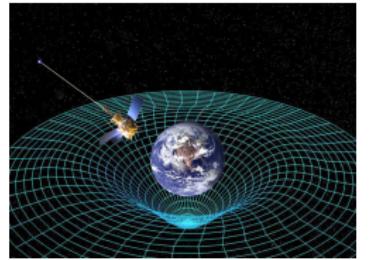


Gravity



Electromagnetism

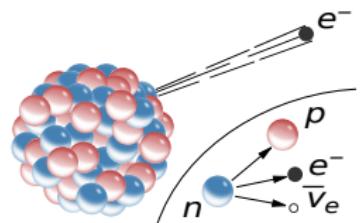
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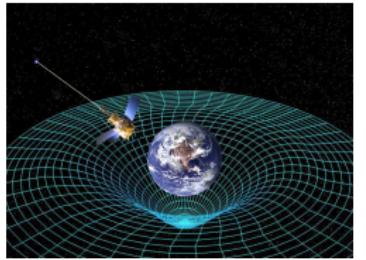


Electromagnetism



Weak nuclear force

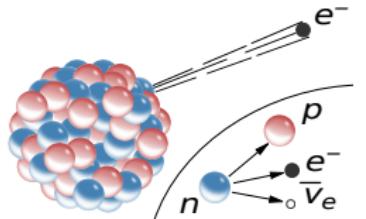
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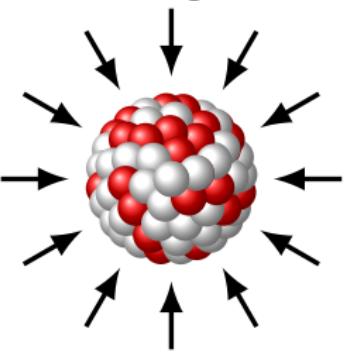
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Strong nuclear force

## What is the strong force?

The mass discrepancy is due to the interaction energy in which gluons are mediators.

Consists of:

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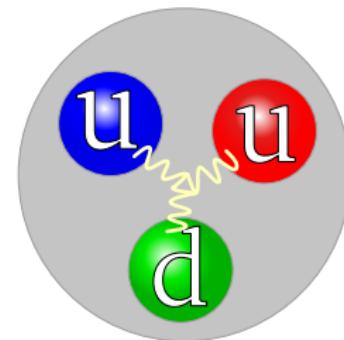
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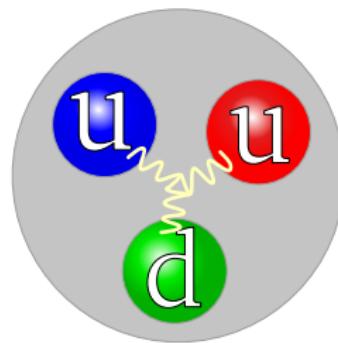
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A **proton** consists of: up-, up- and down-quarks

Mass discrepancy:

$$m_p \neq m_u + m_u + m_d,$$

$$936 \text{ MeV} \neq 3 \text{ MeV} + 3 \text{ MeV} + 6 \text{ MeV}.$$



## Comparing the strong force and QED

$e$  is the coupling constant and  $Q_f$  is the charge.

QED Quantum Electrodynamics(Electromagnetism), a U(1) theory:

$$\mathcal{L}_{\text{QED}} = \sum_{f=\text{fermions}} \bar{\psi}_f (i\gamma^\mu (\partial_\mu + ieQ_f A_\mu) - m_f) \psi_f - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Field strength tensor:

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The strong nuclear force, QCD, a SU(3) theory:

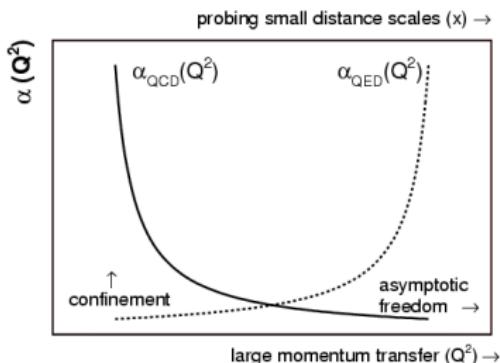
$$\mathcal{L}_{\text{QCD}} = \sum_{f=u,d,\dots} \bar{\psi}_f (i\gamma^\mu (\partial_\mu + ig_S A_\mu^a T^a) - m_f) \psi_f - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$$

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$$F_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_S f^{abc} A_\mu^b A_\nu^c$$

# Why is the strong force strong?

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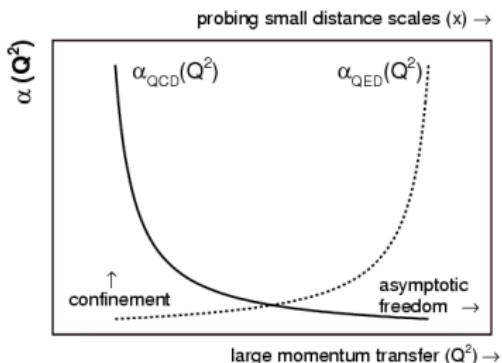


Coupling constant  $\alpha$  strength of the force in an interaction.

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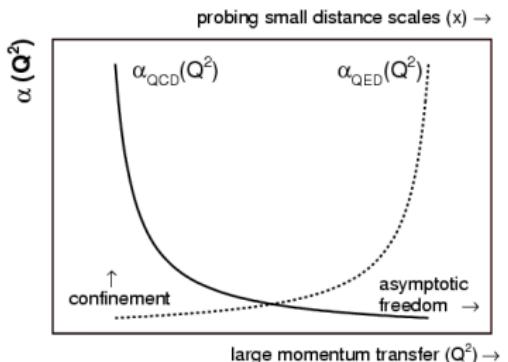


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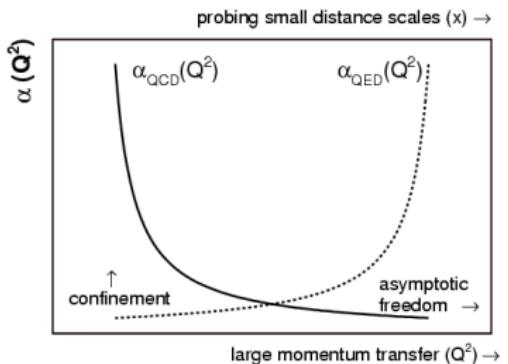
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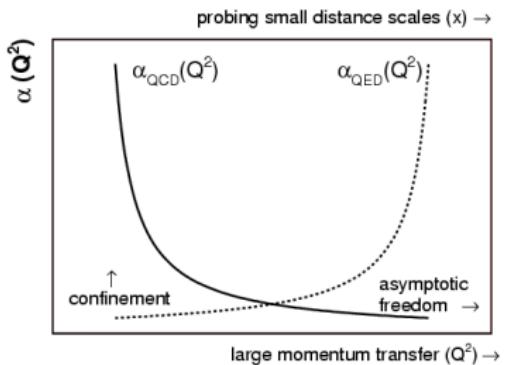
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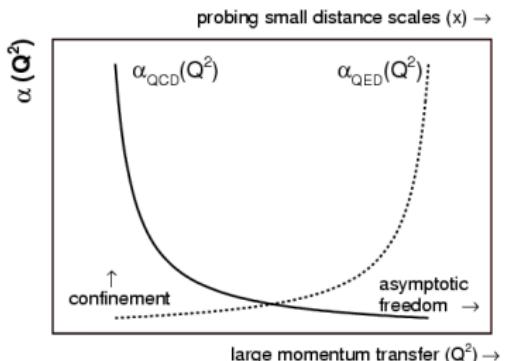
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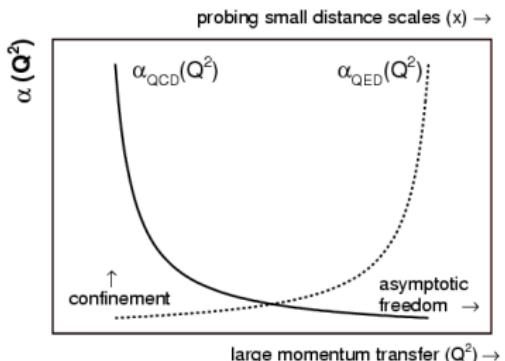
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- Which is really a shame, since many interesting phenomena such as **confinement** is a low-energy phenomena.

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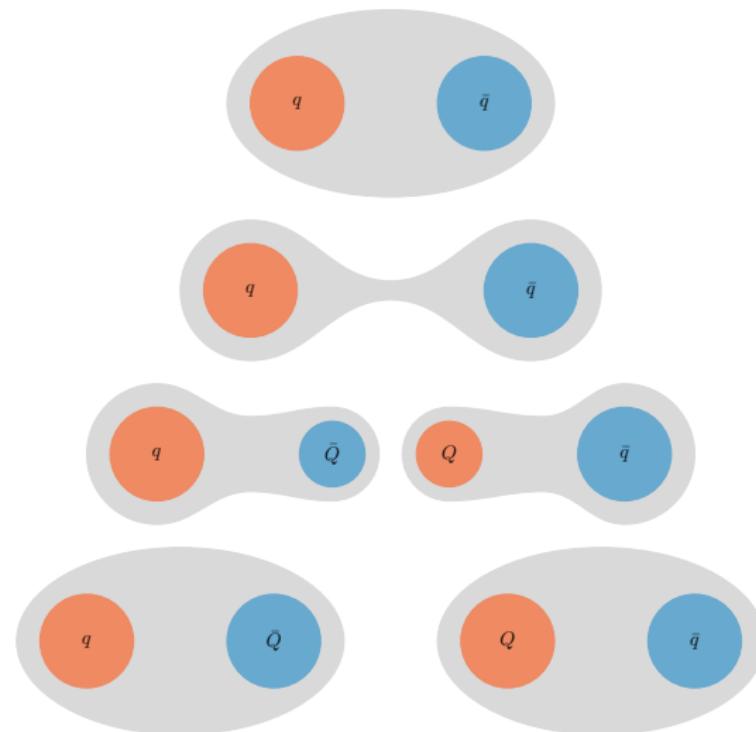
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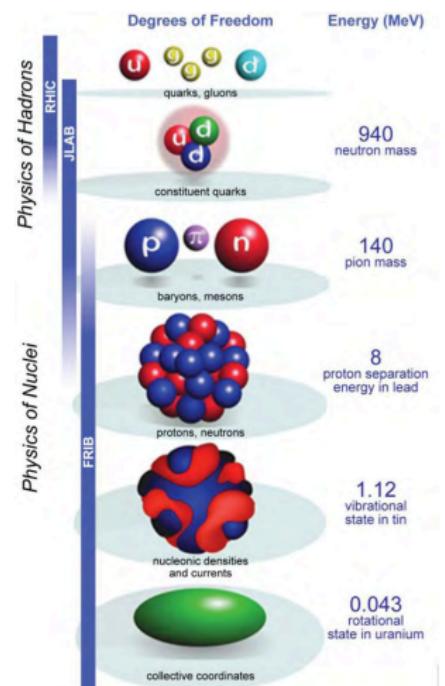
## Confinement: a low-energy phenomena

No free color charges in nature!



If we try to pull apart **two quarks in a meson**, more and more energy is required until we have enough energy to spontaneously create a **quark-antiquark pair**, forming thus **two new mesons**.

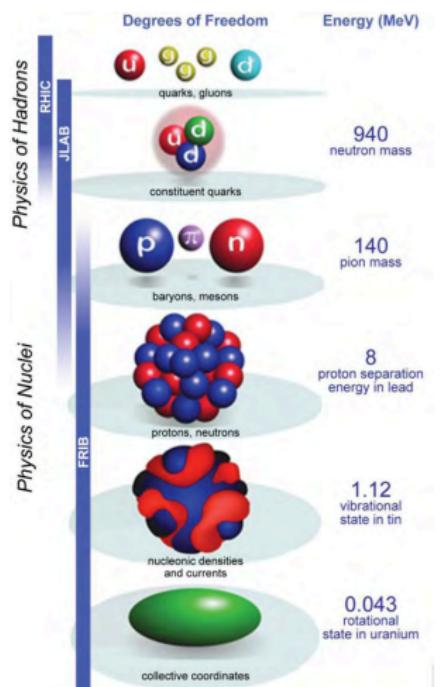
# QCD and nuclear physics



Need to understand the low-energy regime in order to better understand nuclear physics!

The most fundamental theory we currently have of nuclear physics is QCD. Understanding QCD will help us understand nuclear physics and more *emergent* theories. But to bridge the gaps between these theories is difficult, as QCD contains a large number of degrees of freedom. Thus, a numerical approach is needed.

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→ numerical methods(e.g. lattice QCD)

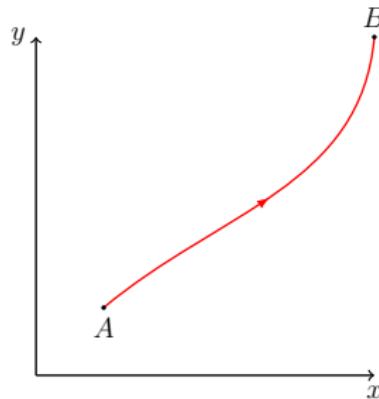
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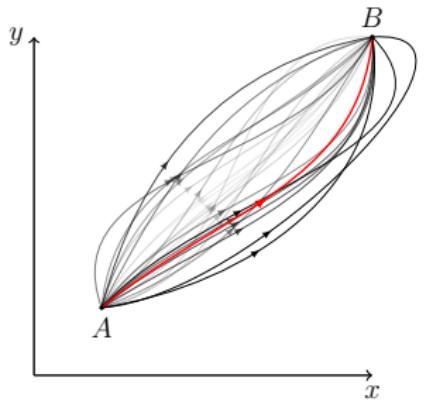


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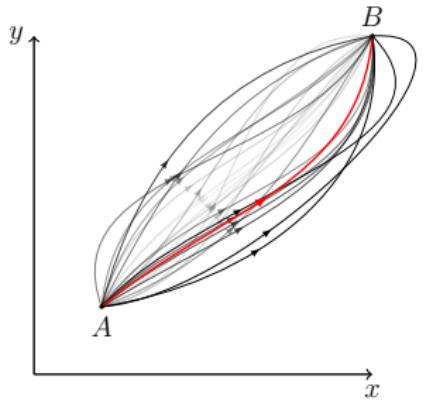


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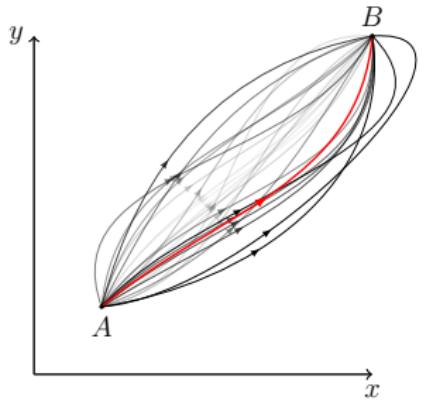
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Sum over all possible paths → the most likely path.

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## Path integrals

Given a field  $\phi^M$  in Minkowski space, the *partition function*  $Z$  is given by

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$\downarrow \quad \hbar = 1, \quad \tau \rightarrow -it \quad \text{imaginary time(euclidean space)!}$

$$= \int \mathcal{D}\phi e^{-S[\phi]}$$

where  $\mathcal{D}$  is an integration of all possible paths in space.

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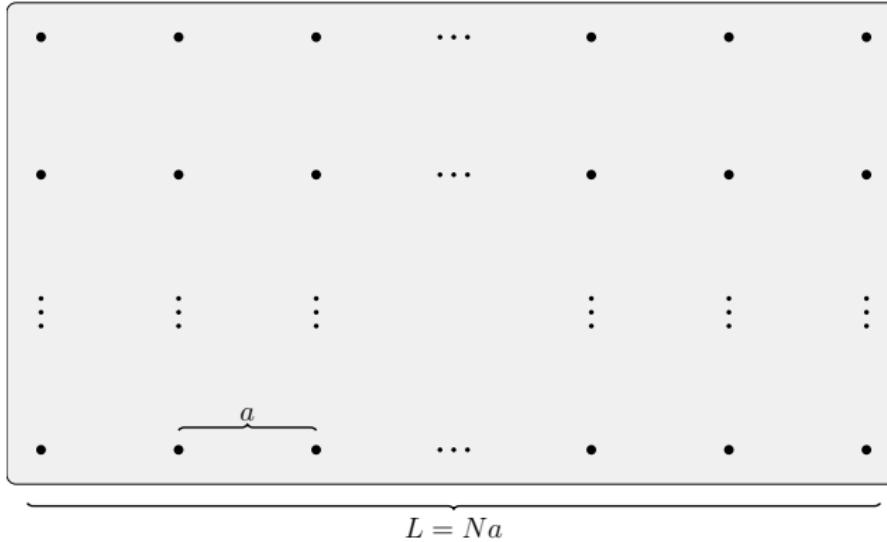
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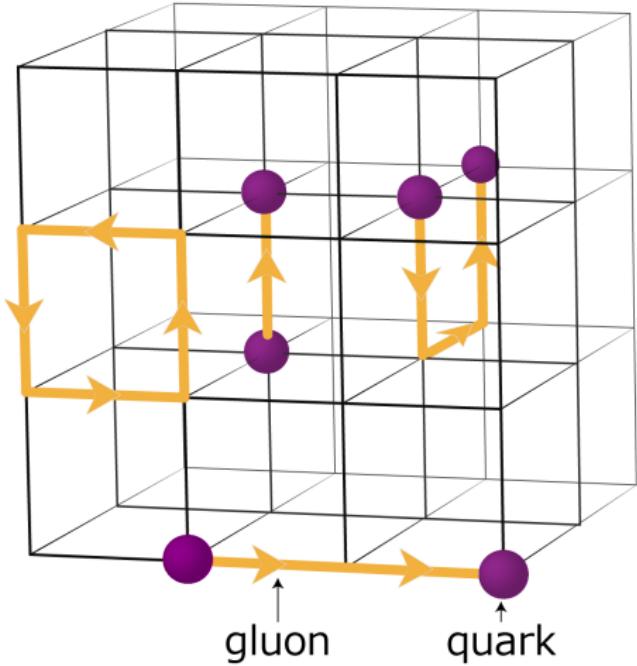
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# QCD on the lattice



- The lattice is a cube in 4D.
- Quarks at lattice, gluons in-between (**links**).
- Maintains the SU(3) symmetry by introducing links.
- Closed loops are gauge invariant.
- Smallest possible object: the plaquette.
- Paths of links with fermions as end points are gauge invariant.
- However, from now on we will ignore any fermions.

[http://www.jicfus.jp/en/wp-content/uploads/2012/12/  
LatticeQCD.png](http://www.jicfus.jp/en/wp-content/uploads/2012/12/LatticeQCD.png)

We exclude fermions to only look at the gauge fields,

$$S_G = \frac{1}{2} \int d^4x \text{tr} (F_{\mu\nu})^2$$

## Links

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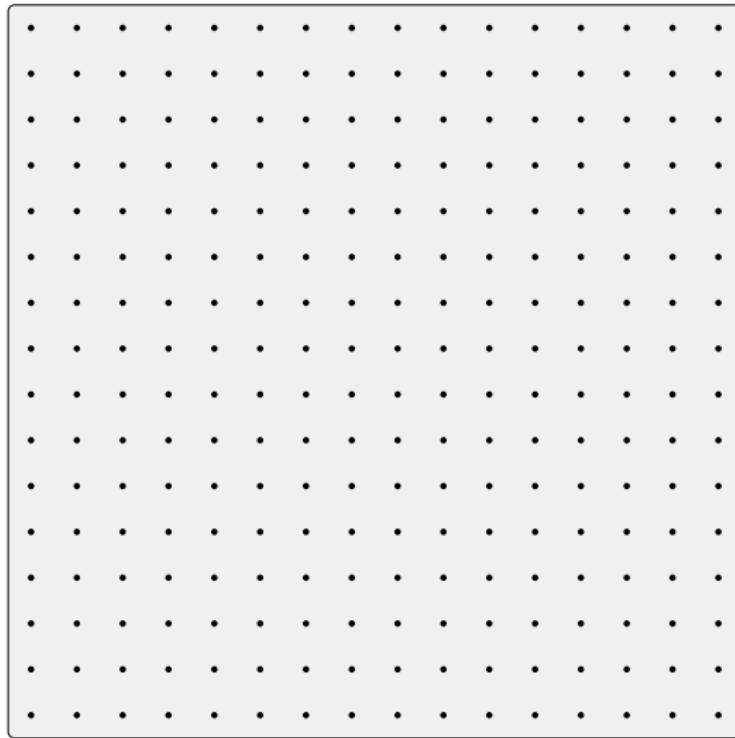
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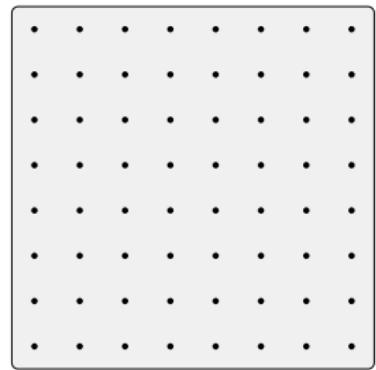
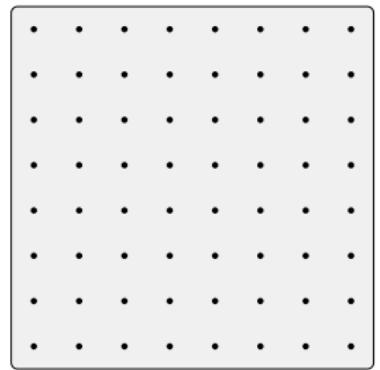
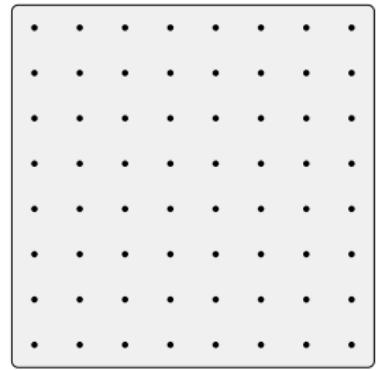
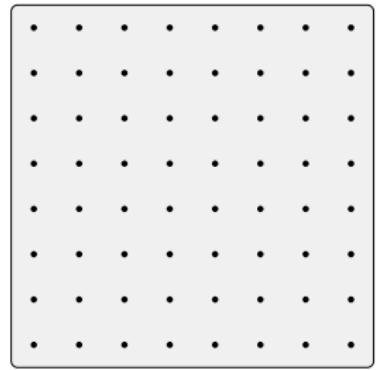
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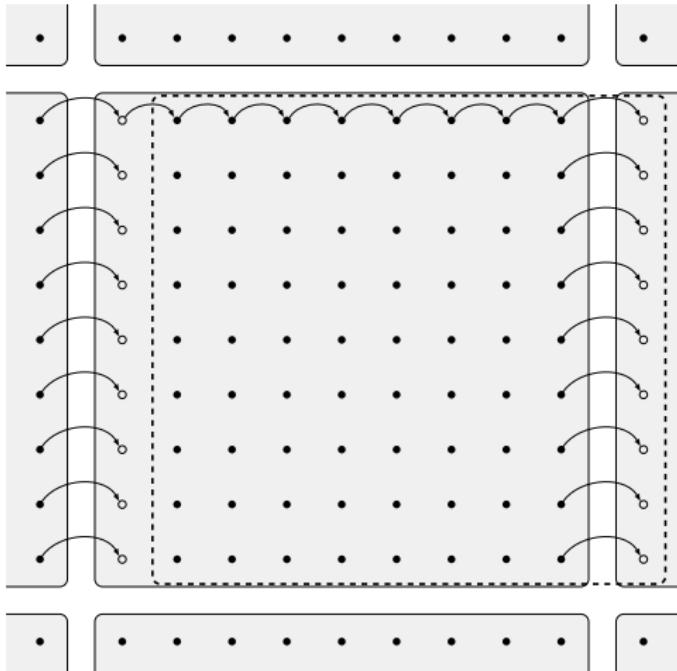
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An analogy: the diffusion equation:

$$\frac{\partial}{\partial t_f} B_\mu(x, t_f) \approx \partial_x^2 B_\mu(x, t_f)$$

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Solve on the lattice using the symplectic, structure preserving Runge-Kutta 3 solver<sup>2</sup>

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## Results

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## Ensembles

Points in lattice given by  $N^3 \times N_T$ .

Ensemble	$\beta = 6/g_S^2$	$N$	$N_T$	$N_{\text{cfg}}$	$a$ [fm]	Config. size[GB]
$A$	6.0	24	48	1000	0.0931(4)	0.356
$B$	6.1	28	56	1000	0.0791(3)	0.659
$C$	6.2	32	64	2000	0.0679(3)	1.125
$D_1$	6.45	32	32	1000	0.0478(3)	0.563
$D_2$	6.45	48	96	250	0.0478(3)	5.695

- I implemented the methods discussed under a code I call GLAC, and will now present some of the results I generated using this code.
- The main ensembles made for this thesis.
- Notice that the size of a lattice is 16 times larger when doubling the dimensions.
- Every configuration was flown with  $N_{\text{flow}} = 1000$  flow steps.
- Since there are so few data configurations, resampling techniques such as bootstrapping were implemented.

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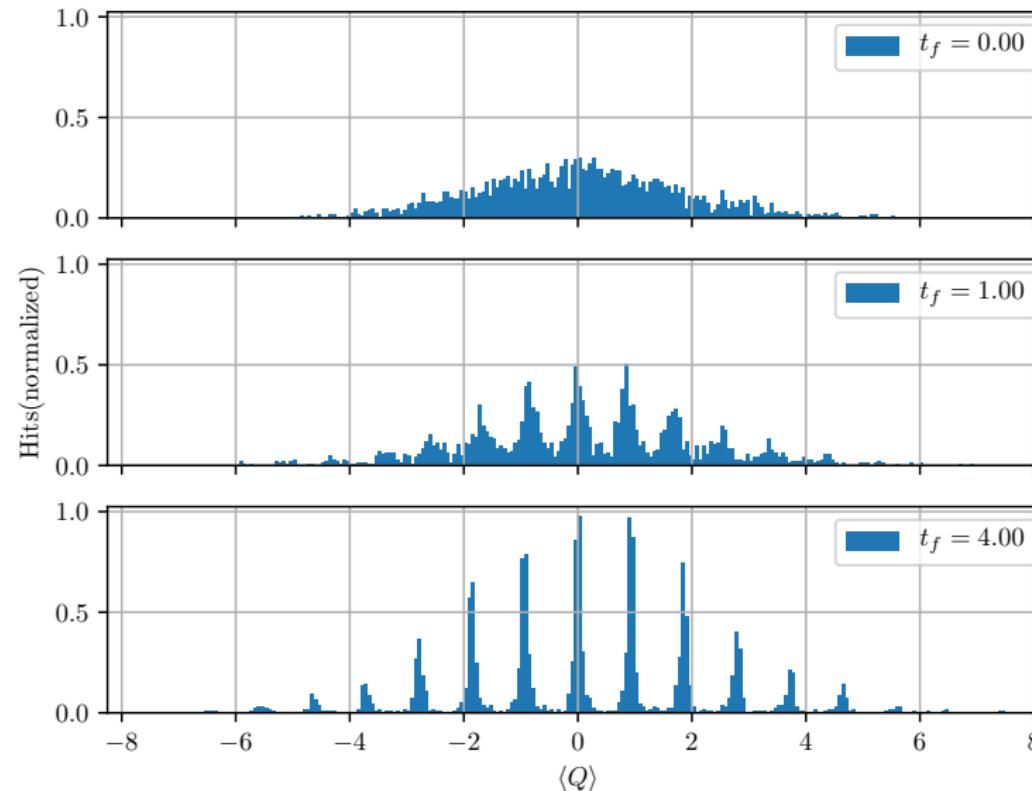
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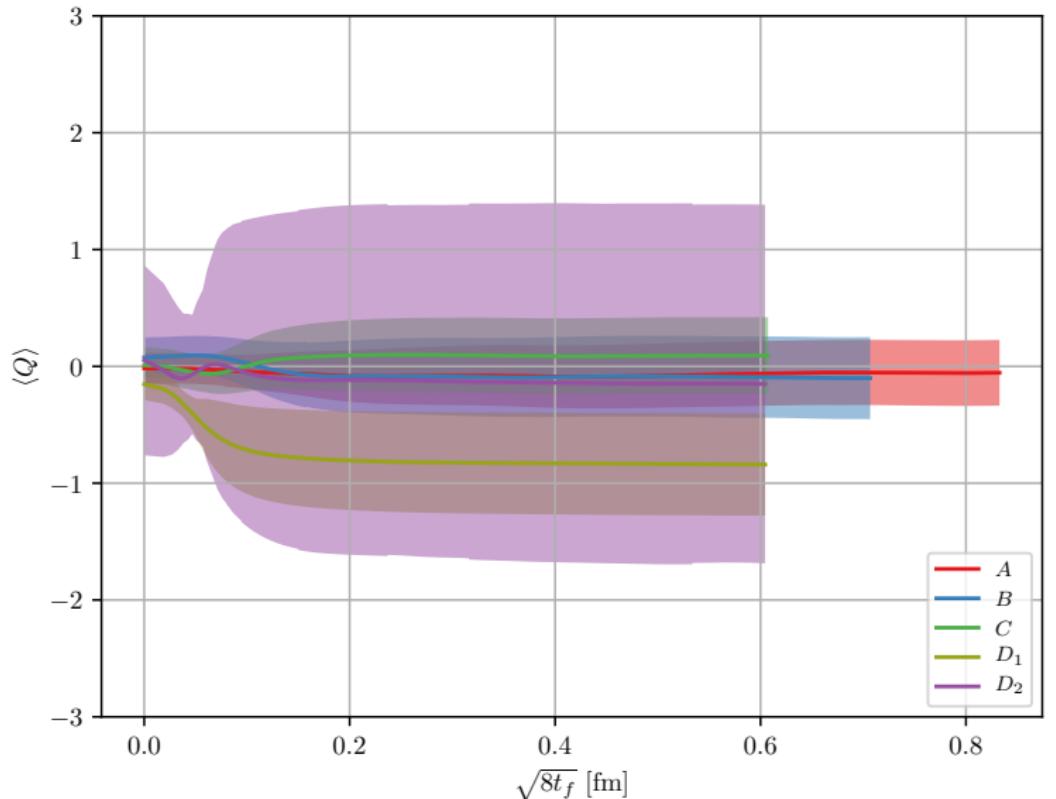
## Topological charge distribution

Histograms for the  $Q$  for ensemble  $G$  with a lattice of size  $N^3 \times N_T = 16^3 \times 32$  with  $\beta = 6.1$ , taken at different flow times  $t_f/a^2 = 0.0, 1.0, 4.0$  fm.



## Topological charge

## Topological charge for our main ensembles



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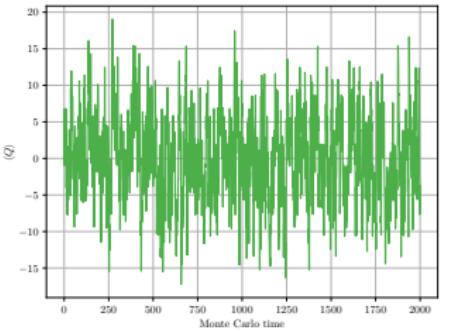
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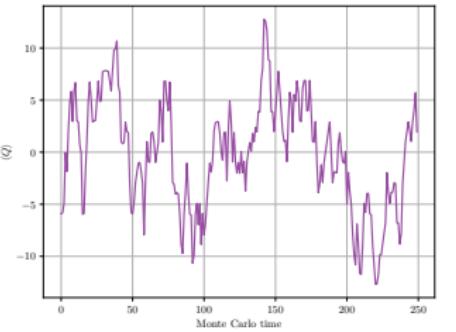
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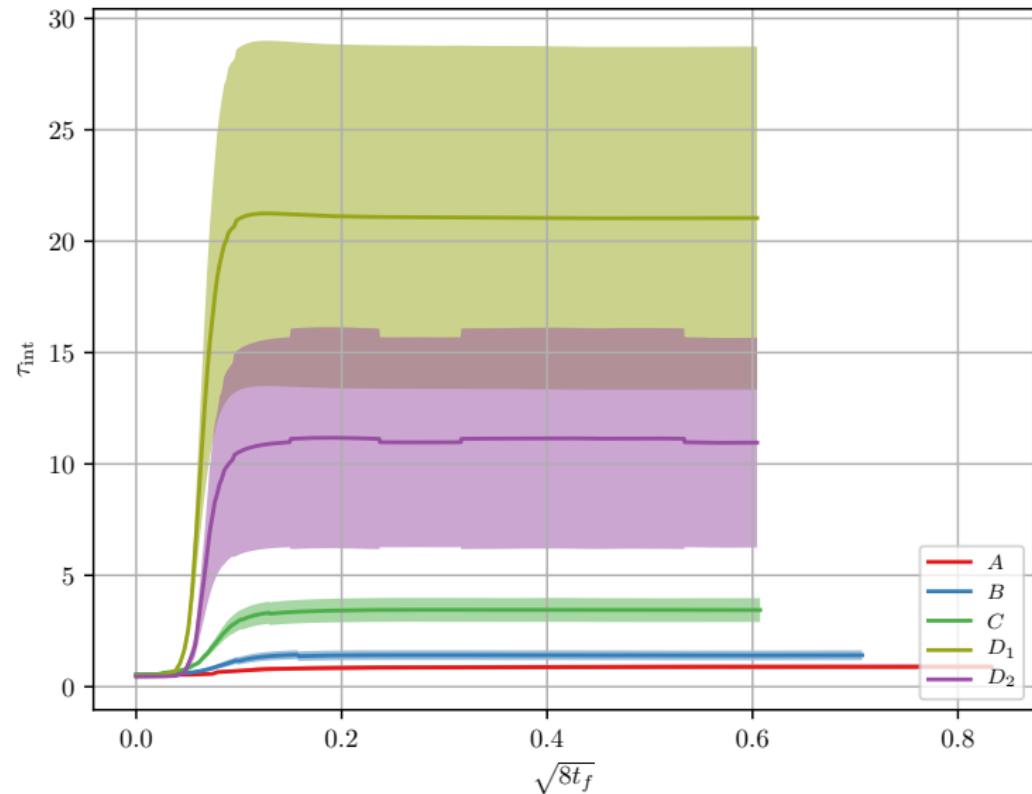
Ensemble  $C$ ,  $32^3 \times 64$ ,  $\beta = 6.2$



Ensemble  $D_2$ ,  $48^3 \times 96$ ,  $\beta = 6.45$

## Topological charge autocorrelation

- The integrated autocorrelation  $\tau_{\text{int}}$  for topological charge for the five main ensembles.



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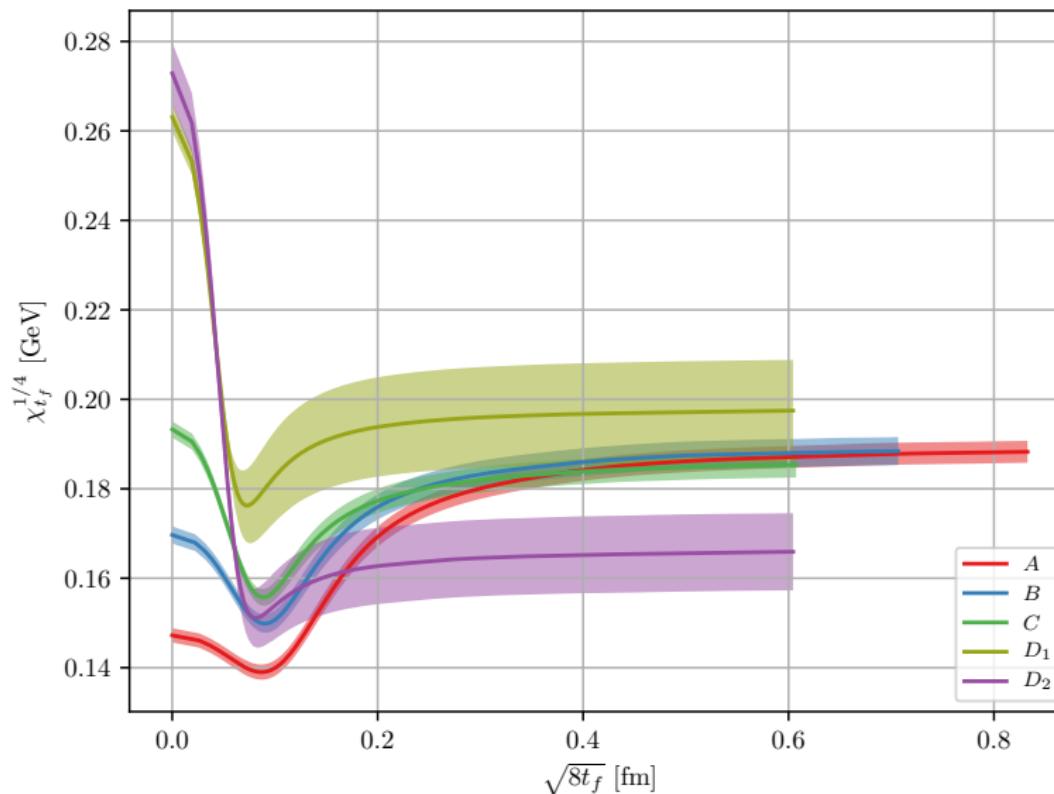
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## Topological susceptibility



- The topological susceptibility  $\chi_{tf}^{1/4}$  of the **main ensembles**.
- We have a **UV divergence at zeroth flow time**, hence to need for gradient flow which renormalizes this quantity.
- **Bootstrapped**  $N_{\text{bs}} = 500$  times.
- Corrected for autocorrelations with  $\sigma = \sqrt{2\tau_{\text{int}}}\sigma_0$ .

Ensembles	$\chi_{tf}^{1/4} (\langle Q^2 \rangle) [\text{GeV}]$	$N_f$	$\chi^2/\text{d.o.f}$
$A, B, C, D_2$	0.179(10)	3.75(29)	2.38
$A, B, C, D_1$	0.186(6)	3.21(25)	0.83
$A, B, C$	0.184(6)	3.37(26)	0.33

## The fourth cumulant

- Highly unstable, as we shall see.
- Will provide insight into the goodness of our ensembles.
- An  $R$ -value away from 1 will indicate that QCD cannot be described by the dilute instanton gas model.

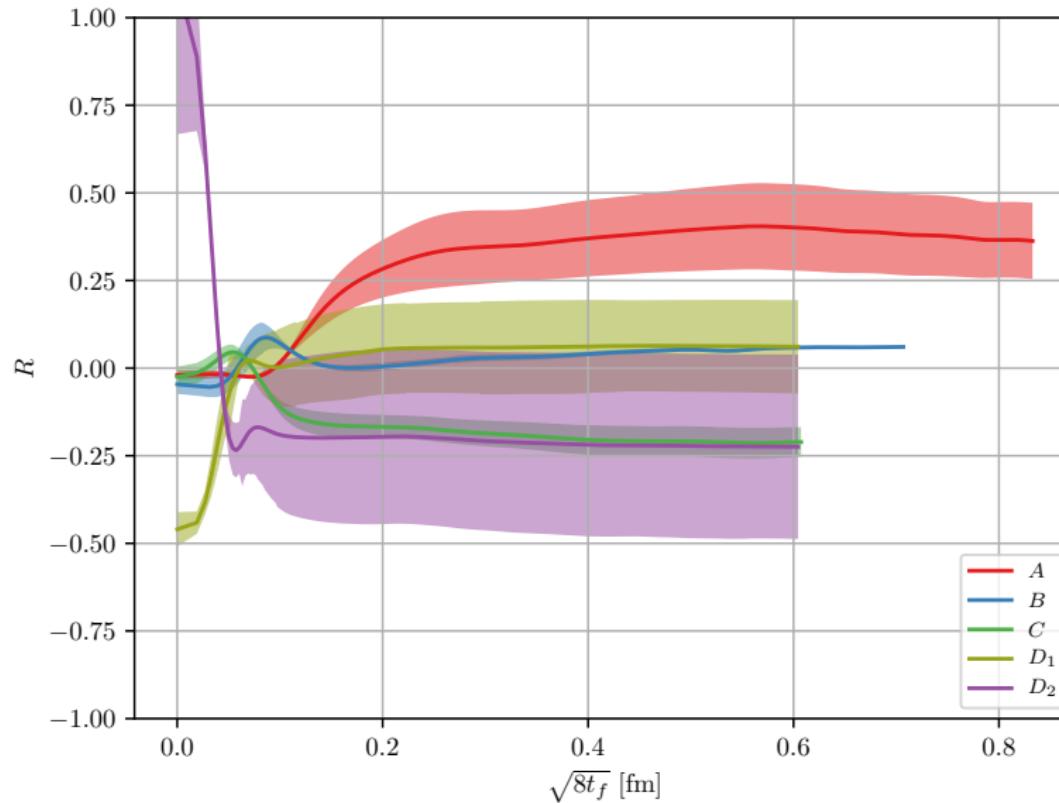
$$\langle Q^4 \rangle_c = \frac{1}{V^2} \left( \langle Q^4 \rangle - 3 \langle Q^2 \rangle^2 \right).$$

From this, we can also measure the ratio  $R$ ,

$$R = \frac{\langle Q^4 \rangle_c}{\frac{1}{V} \langle Q^2 \rangle} = \frac{1}{V} \frac{\langle Q^4 \rangle - 3 \langle Q^2 \rangle^2}{\langle Q^2 \rangle},$$

## The fourth cumulant

- The fourth cumulant ratio  $R = \langle Q^4 \rangle_C / \langle Q^2 \rangle$ .
- The results was analyzed using  $N_{\text{bs}} = 500$  bootstrap samples, with the error corrected for by  $\sqrt{2\tau_{\text{int}}}$ .



## The fourth cumulant at reference flow times

The fourth cumulant is taken at their individual reference scales seen in the third column. The data were analyzed with using a bootstrap analysis of  $N_{bs} = 500$  samples, with error corrected by the integrated autocorrelation,  $\sqrt{2\tau_{\text{int}}}$ .

Ensemble	$L/a$	$t_0/a^2$	$\langle Q^2 \rangle$	$\langle Q^4 \rangle$	$\langle Q^4 \rangle_C$	$R$
$A$	2.24	3.20(3)	0.78(4)	2.13(27)	0.282(67)	0.359(65)
$B$	2.21	4.43(4)	0.81(5)	1.98(23)	0.036(11)	0.044(11)
$C$	2.17	6.01(6)	0.77(4)	1.6(2)	-0.174(40)	-0.226(64)
$D_1$	1.53	12.2(1)	1.00(20)	3.01(1.07)	0.03(12)	0.03(12)
$D_2$	2.29	12.2(1)	0.497(100)	0.64(20)	-0.103(95)	-0.21(23)

## Comparing fourth cumulant

We can compare with article by Cè et al. [2015]

## Comparing fourth cumulant

- Parameters of the ensembles presented by Cè et al. [2015]. The first column is the ensemble name from the article. The letter indicates the volume, while the subindex indicates the  $\beta$  value. Ensembles of similar letters keep approximately the same length  $L$ .

Ensemble	$\beta$	$L/a$	$L$ [fm]	$a$ [fm]	$t_0/a^2$	$t_0/r_0^2$	$N_{\text{cfg}}$
$F_1$	5.96	16	1.632	0.102	2.7887(2)	0.1113(9)	1 440 000
$B_2$	6.05	14	1.218	0.087	3.7960(12)	0.1114(9)	144 000
$\tilde{D}_2$		17	1.479		3.7825(8)	0.1110(9)	
$B_3$	6.13	16	1.232	0.077	4.8855(15)	0.1113(10)	144 000
$\tilde{D}_3$		19	1.463		4.8722(11)	0.1110(10)	
$B_4$	6.21	18	1.224	0.068	6.2191(20)	0.1115(11)	144 000
$\tilde{D}_4$		21	1.428		6.1957(14)	0.1111(11)	

## Comparing fourth cumulant

Article	Thesis	Ratio( $\langle Q^2 \rangle$ )	Ratio( $\langle Q^4 \rangle$ )	Ratio( $\langle Q^4 \rangle_C$ )	Ratio( $R$ )
$F_1$	$A$	1.08(6)	1.34(18)	19.03(5.81)	17.64(4.48)
$B_2$	$A$	1.02(5)	1.15(15)	3.60(1.09)	3.54(90)
	$B$	1.04(6)	1.06(11)	0.480(74)	0.46(4)
$\tilde{D}_2$	$A$	1.02(5)	1.19(15)	8.31(1.99)	8.15(1.56)
	$B$	1.05(6)	1.10(12)	1.1(1)	1.06(3)
$B_3$	$B$	1.06(6)	1.10(12)	0.550(86)	0.52(5)
$\tilde{D}_3$	$B$	1.05(6)	1.11(12)	1.51(23)	1.4(1)
$B_4$	$C$	0.99(5)	0.86(8)	-2.32(46)	-2.35(59)
$\tilde{D}_4$	$C$	0.98(5)	0.85(8)	-3.95(96)	-4.05(1.19)

- A comparison between the results obtained in this thesis on the fourth cumulant, and by those similar in volume form Cè et al. [2015]. *Ratio* indicates that we are dividing our results by the ones in previous table.  
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- $q(0)$  is not required to be at  $n_t = 0$ .

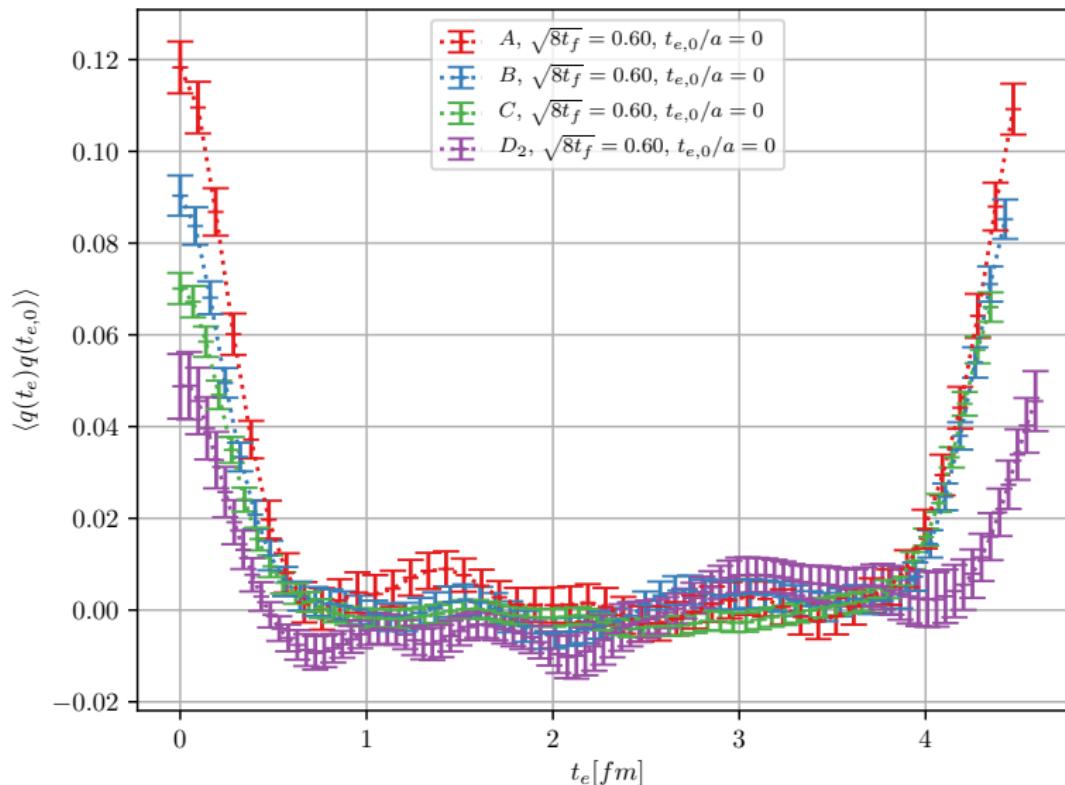
The **topological charge correlator**

$$C(n_t) = \langle q(n_t)q(0) \rangle,$$

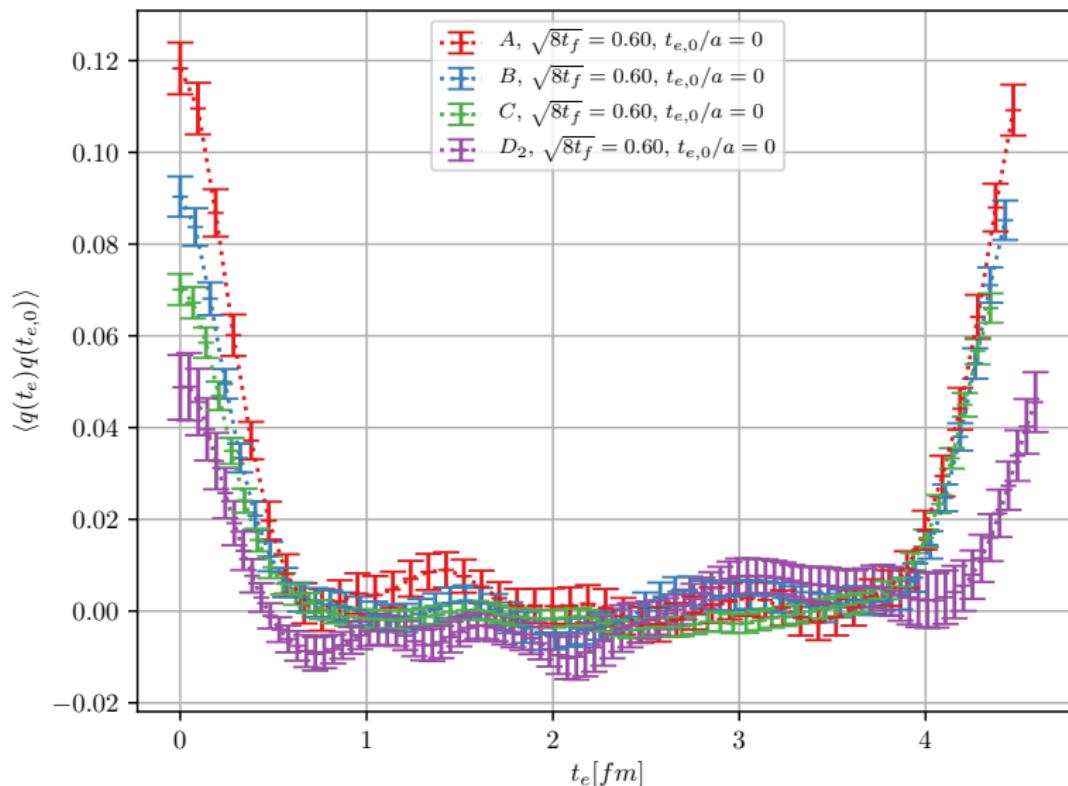
$q(0)$  is the *source* placed at a fixed Euclidean time, and  $q(n_t)$  is the *sink* which at some Euclidean times  $t$ .

## The topological charge correlator

- The topological charge correlator for all of the ensembles except  $D_1$ . The  $x$ -axis contains the sink-source separation, as the source  $q(0)$  is placed at  $t_e = 0$  fm, and the sink  $q(t_e)$  is taken at  $t_e$ .

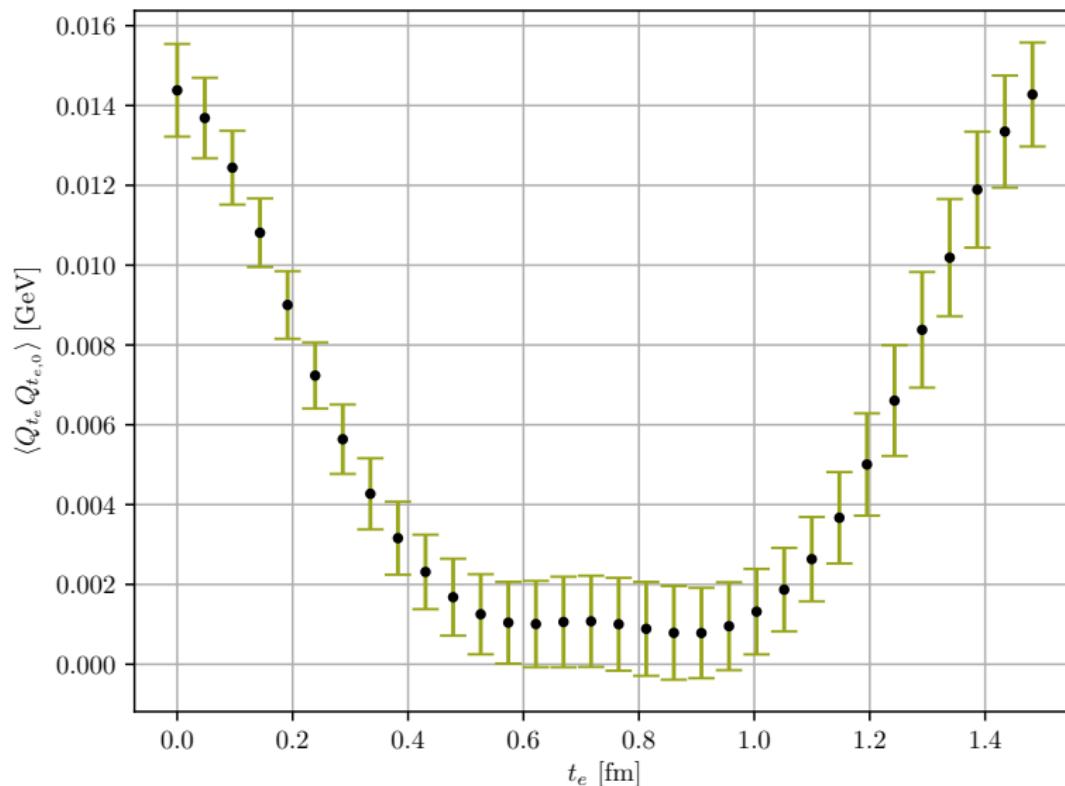


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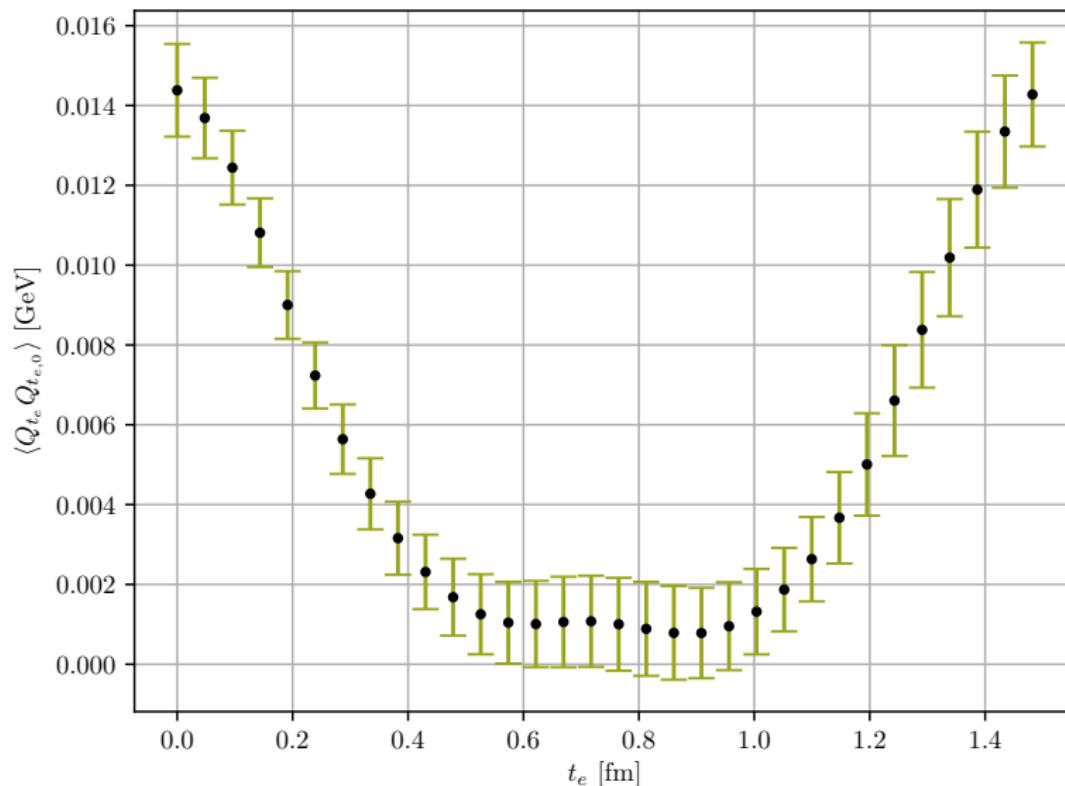
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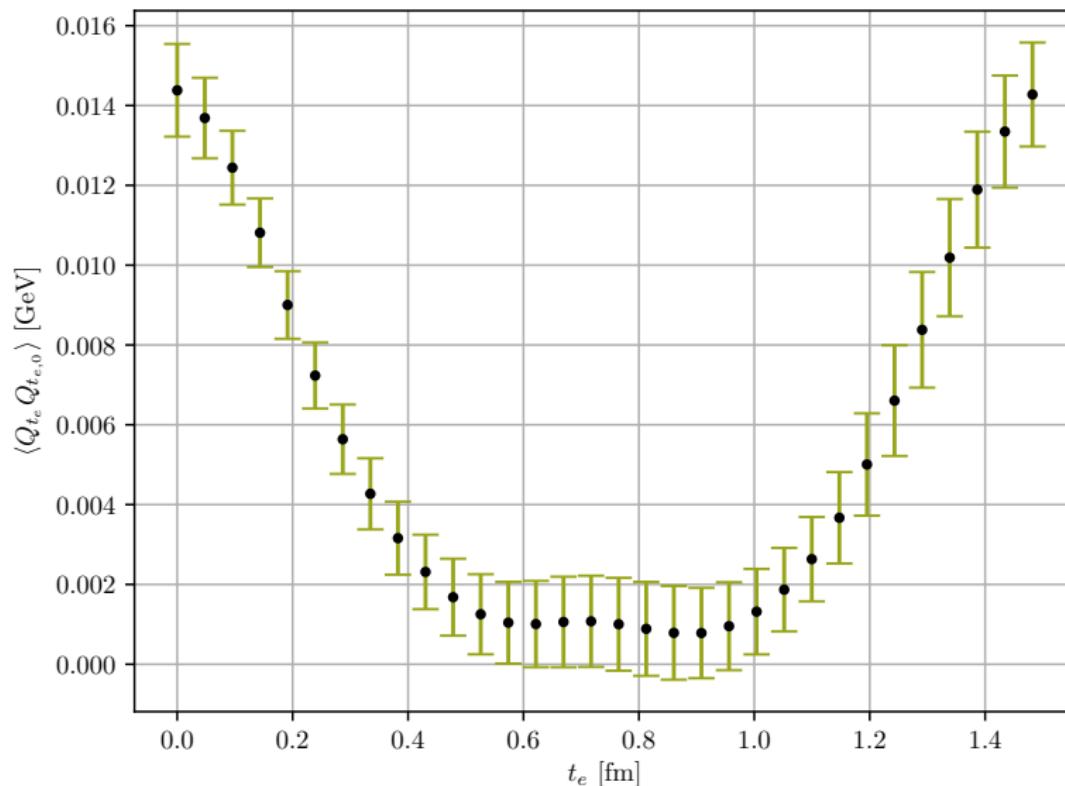
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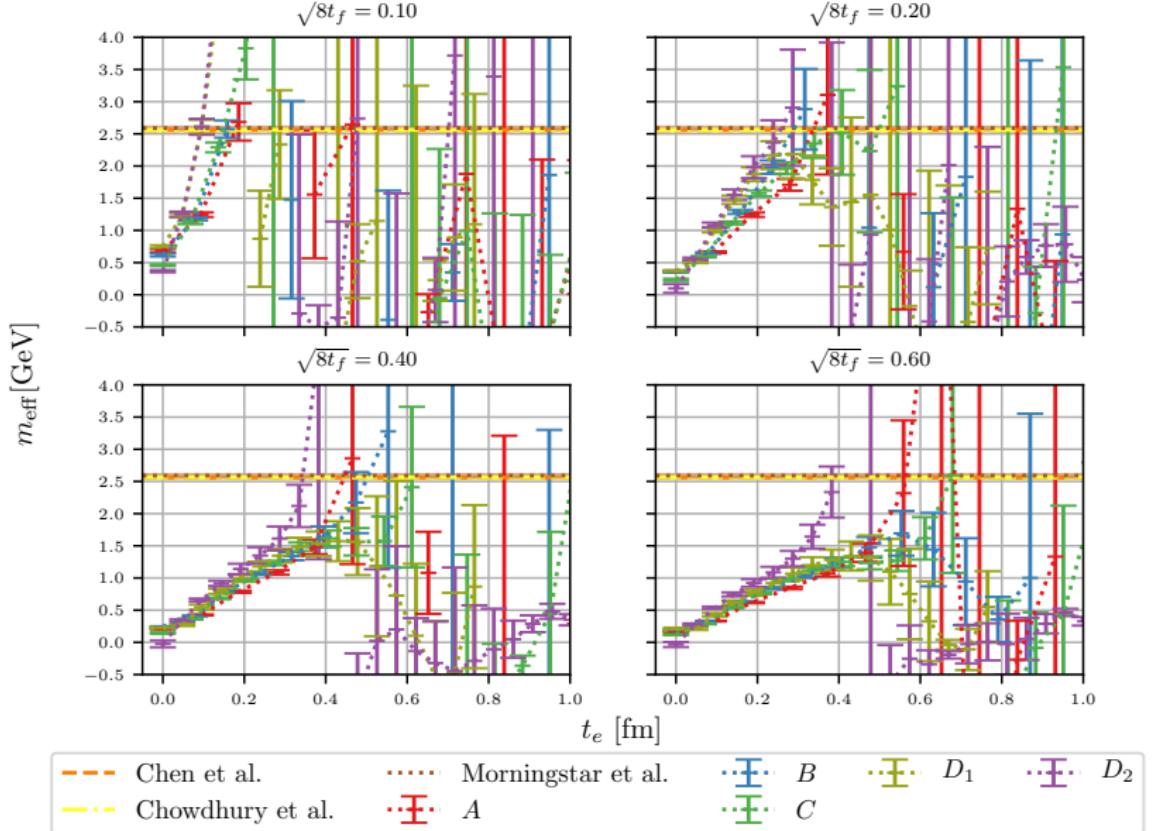
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Conclusion, future developments  
and final thoughts

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Thank you for listening.

Questions?

## References

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## Extra slides

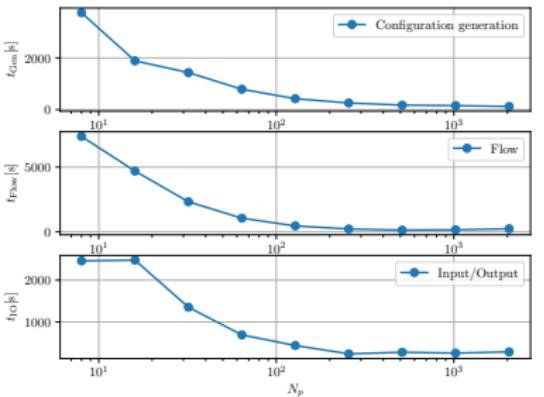
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# Scaling

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We checked three types of scaling,

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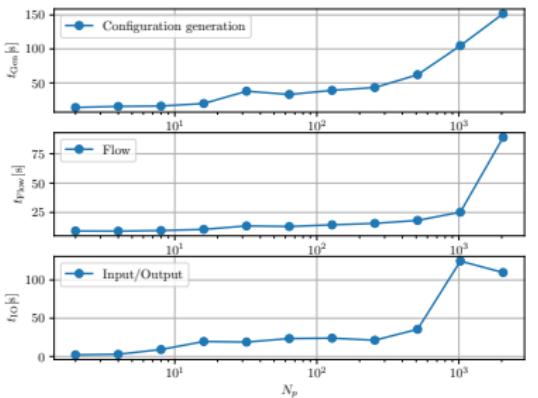


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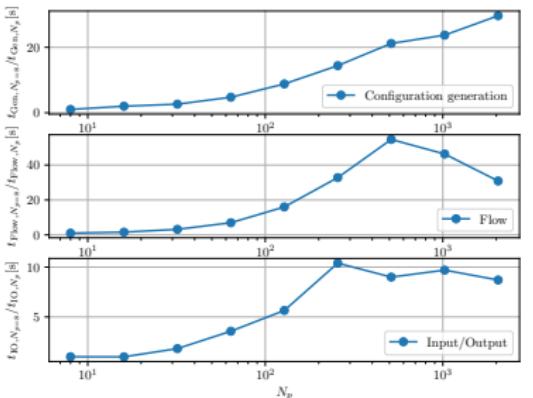
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We appear to have a plateau around 512 cores.

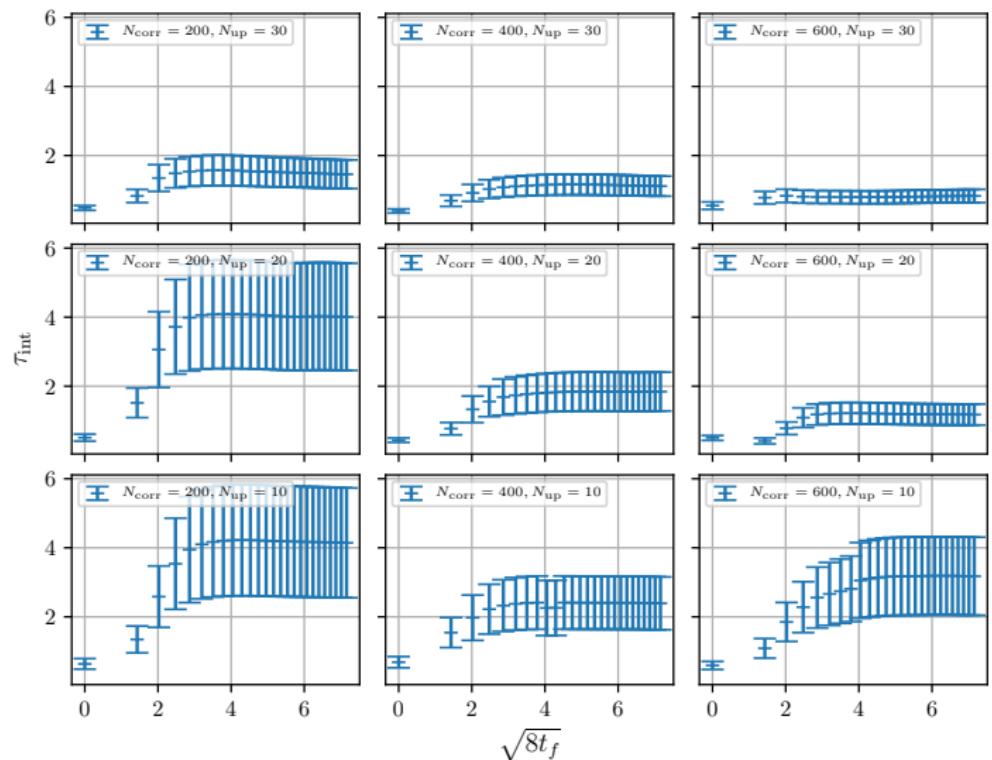
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## Optimizing the gauge configuration generation

- We run for different values for  $N_{\text{up}}$  and  $N_{\text{corr}}$  to see what gives optimizes **computational cost** and **autocorrelation**.
- The integrated autocorrelation time for topological charge  $\langle Q \rangle$  for a lattice of size  $N = 16$  and  $N_T = 32$  with  $\beta = 6.0$  for combinations of  $N_{\text{corr}} \in [200, 400, 600]$  and  $N_{\text{up}} \in [10, 20, 30]$ , plotted against flow time  $\sqrt{8t_f}$ .

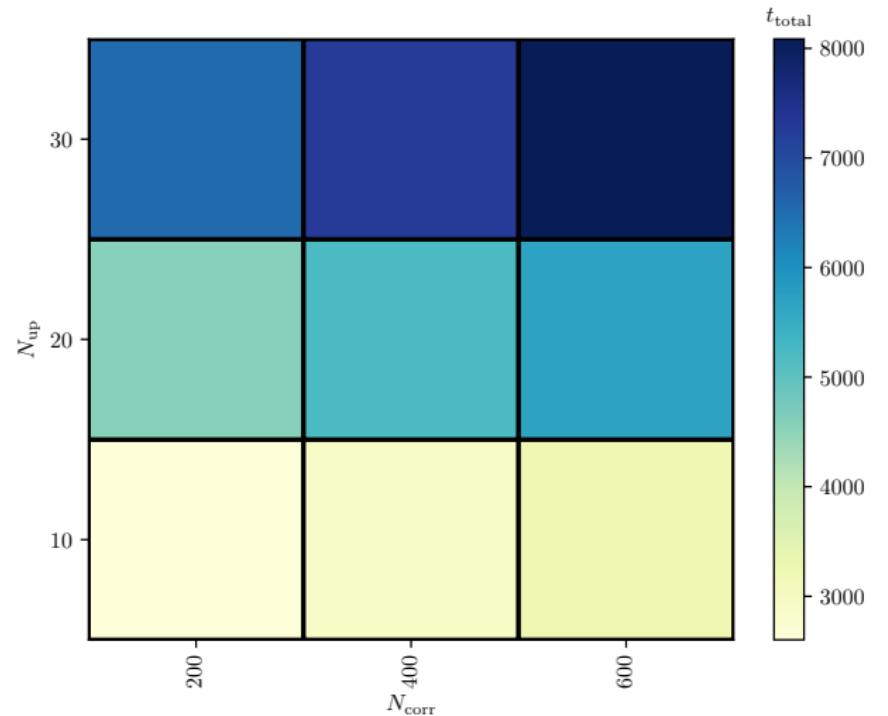
Generated 200 configurations for a lattice of size  $N^3 \times N_T = 16^3 \times 32$  and  $\beta = 6.0$ , for combinations of  $N_{\text{corr}} \in [200, 400, 600]$  and  $N_{\text{up}} \in [10, 20, 30]$ .

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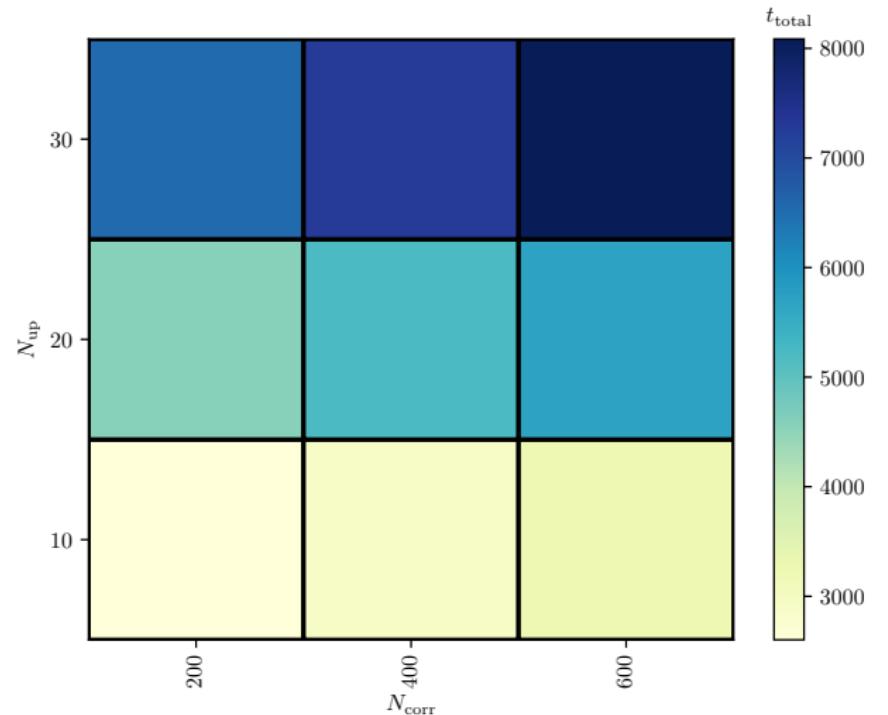
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- **Integration testing.** Random matrix generation, lattice objects, parallelization, ect.
- **Validation testing.** Cross checking results with a configuration from Chroma.

## Verifying the integration

- The values we will test the integrator against.

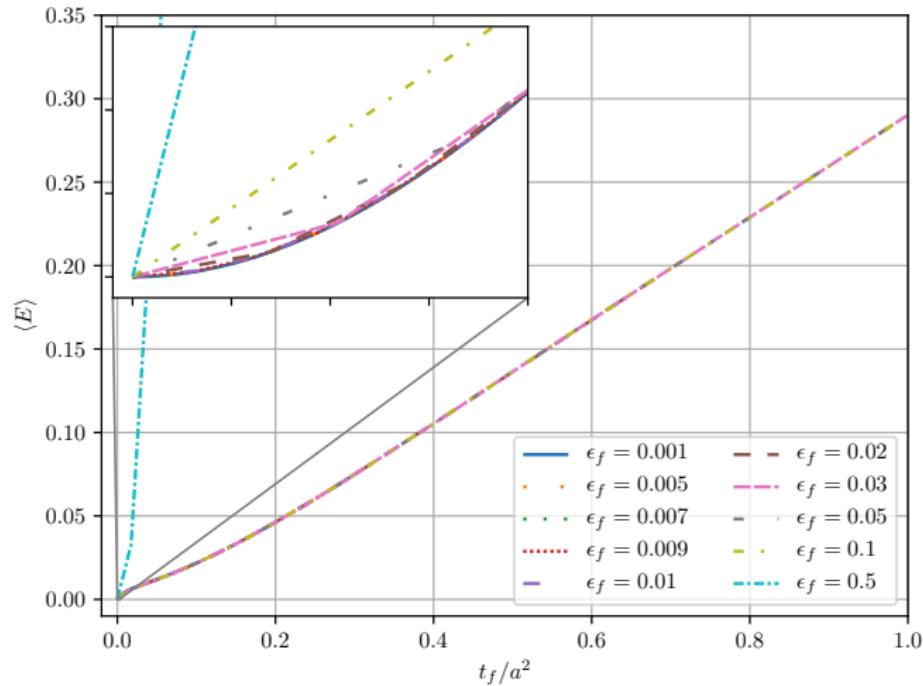
Testing the integrator for different integration steps  $\epsilon_f$ .

$\epsilon_f$	0.001	0.005	0.007	0.009	0.01	0.02	0.03	0.05	0.1	0.5
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## Verifying the integration

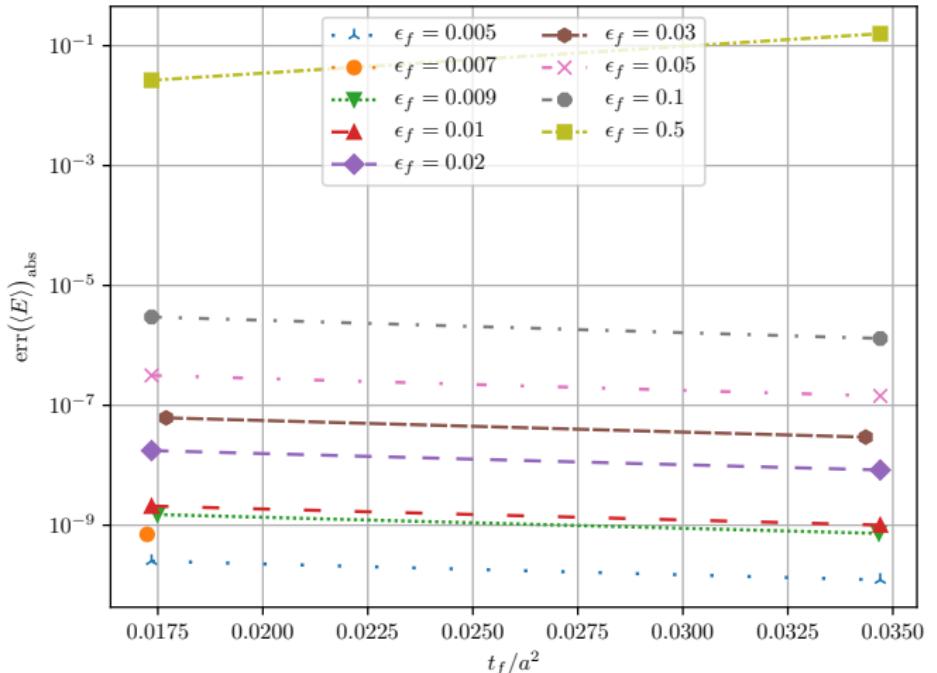
Lattice size  $N^3 \times N_T = 24^3 \times 48$  with  $\beta = 6.0$ .

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## Verifying the integration

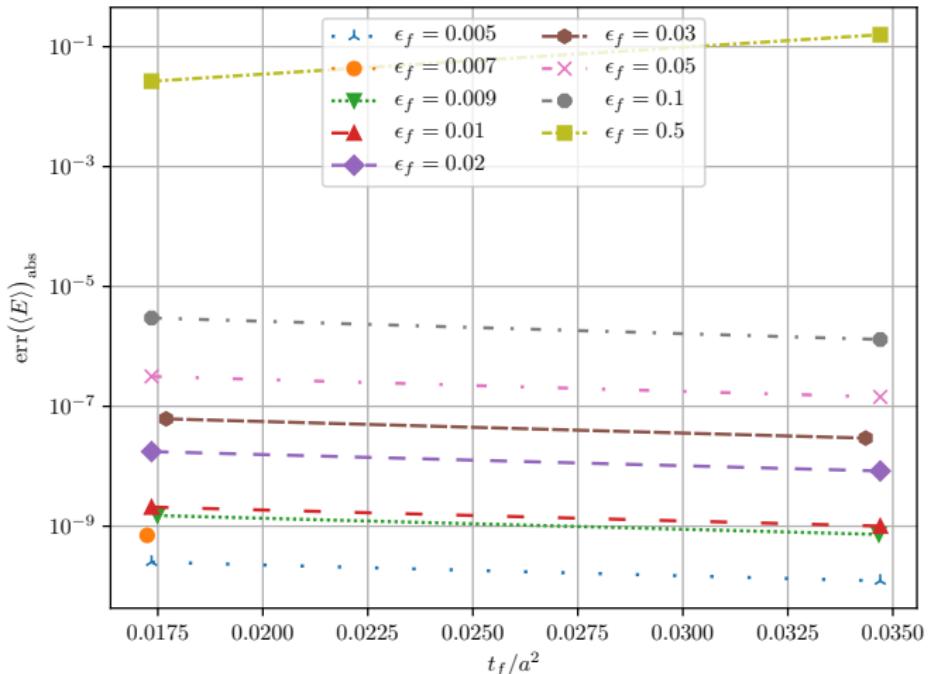
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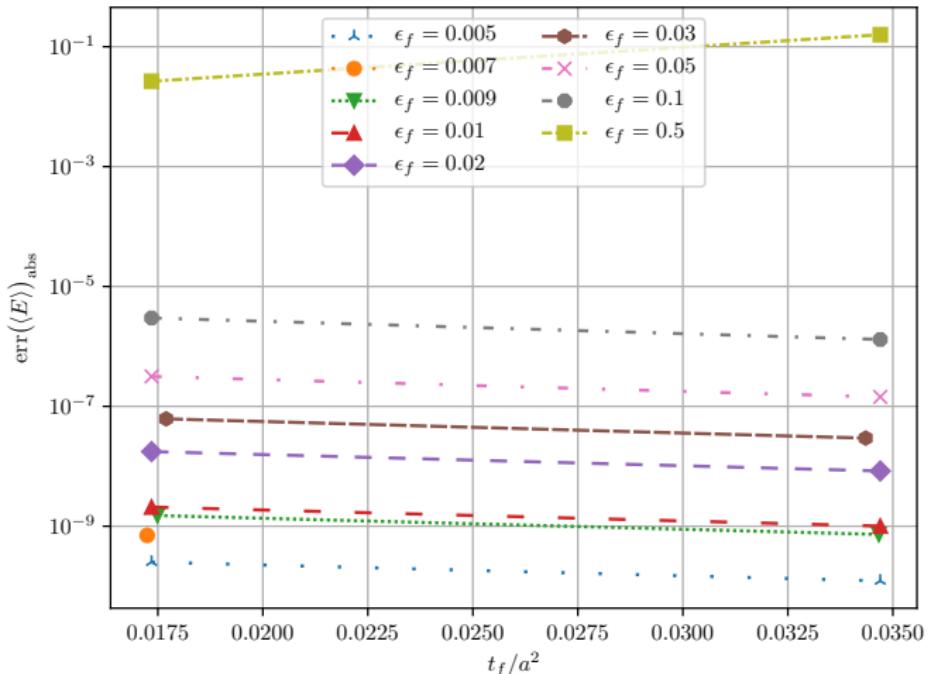
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# The non-linearity of QCD

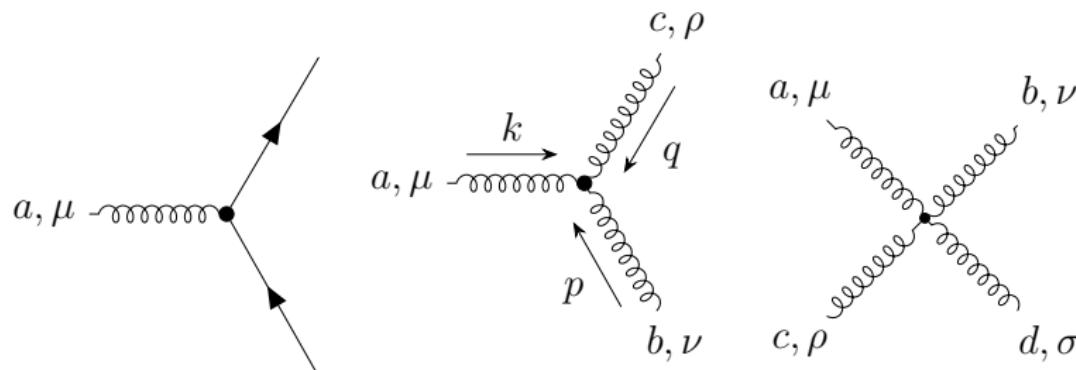
The QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \sum_{f=1}^{N_f} \bar{\psi}^{(f)} \left( i \not{D} - m^{(f)} \right) \psi^{(f)} - \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu},$$

with action

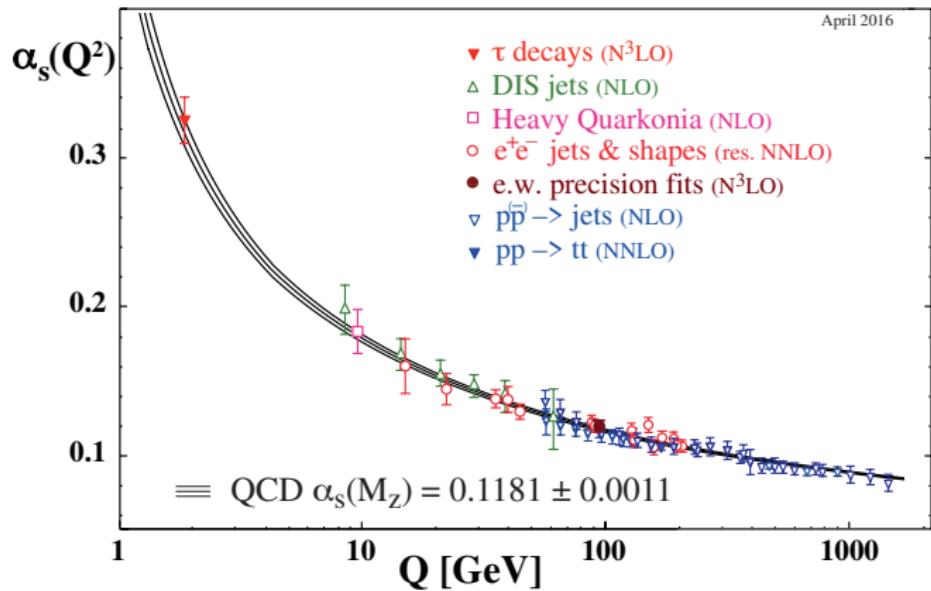
$$S = \int d^4x \mathcal{L}_{\text{QCD}}, \quad (1)$$

is invariant under a SU(3) symmetry.



- *Gluon self-interaction.*
- This central aspect is mostly covered in the pure-gauge/Yang-Mills section of the theory.
- **Two important features:** confinement and asymptotic freedom.

# Asymptotic freedom



- The coupling constant **decreases** as we **increase** the energy.
- Also serves as an *experimental proof* of QCD.
- Other lines of evidence: triple  $\gamma$  decay and muon cross section ratio  $R$ .
  - Triple  $\gamma$  decay: the number of colors is included in the cross section, which can be measured experimentally.
  - Muon cross section ratio  $R$ : the ratio is dependent on having three colors.

- We rewrite the equations slightly,

With

$$\dot{V}_{t_f} = -g_S^2 \{ \partial_{x,\mu} S_G[V_{t_f}] \} V_{t_f} = Z(V_{t_f}) V_{t_f},$$

## Solving gradient flow with Runge-Kutta 3

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and  $Z_i = \epsilon_f Z(W_i)$  we get

$$W_0 = V_{t_f},$$

$$W_1 = \exp \left[ \frac{1}{4} Z_0 \right] W_0,$$

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$$V_{t_f + \epsilon_f} = \exp \left[ \frac{3}{4} Z_2 - \frac{8}{9} Z_1 + \frac{17}{36} Z_0 \right] W_2,$$

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## Additional ensembles

- Additional ensembles made in order to illuminate additional aspects of the topological charge.
- Supporting ensembles made on Smaug. All ensembles were flown  $N_{\text{flow}} = 1000$  steps with  $\epsilon_{\text{flow}} = 0.01$ .

Ensemble	$N$	$N_T$	$N_{\text{cfg}}$	$N_{\text{corr}}$	$N_{\text{up}}$	$a$ [fm]	$L$ [fm]
$E$	8	16	8135	600	30	0.0931(4)	0.745(3)
$F$	12	24	1341	200	20	0.0931(4)	1.118(5)
$G$	16	32	2000	400	20	0.0790(3)	1.265(6)

## Energy definition

We use  $t_0$

$$E = \frac{a^4}{2|\Lambda|} \sum_{n \in \Lambda} \sum_{\mu, \nu} (F_{\mu\nu}^{\text{clov}}(n))^2$$

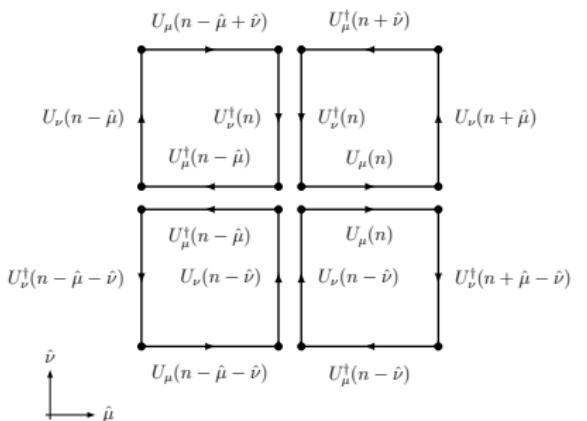
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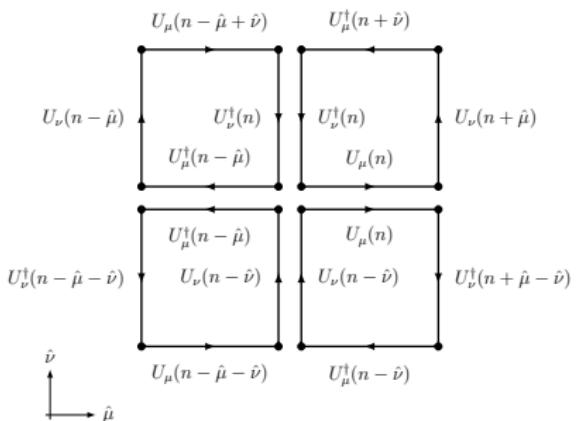
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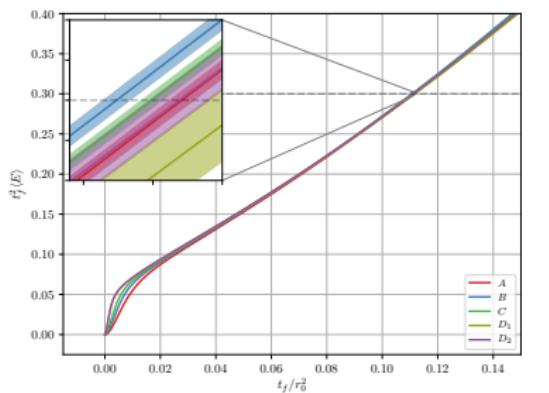
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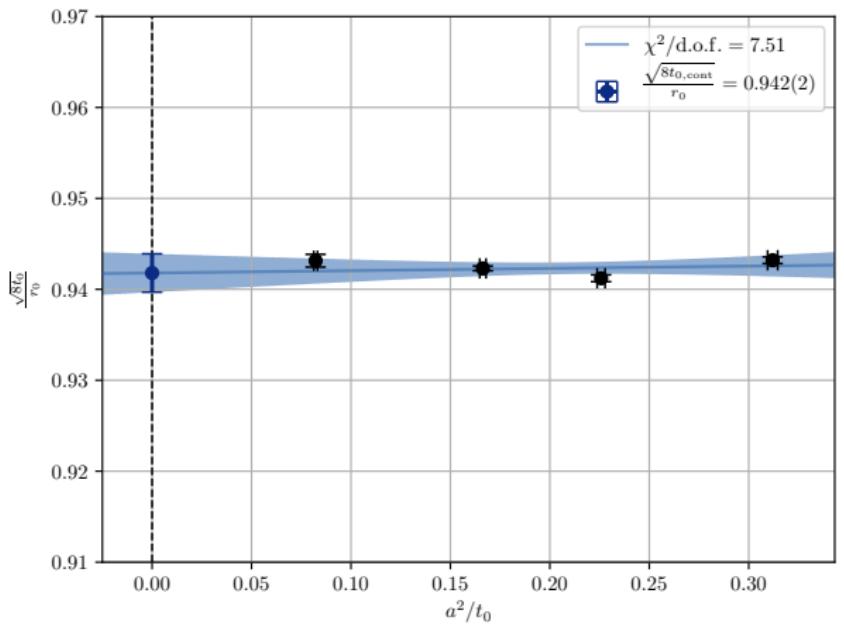
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Using scale definition  $t_0$  from Lüscher [2010],

$$\{t_f^2 \langle E(t) \rangle\}_{t_f=t_0} = 0.3.$$



## Scale setting $t_0$



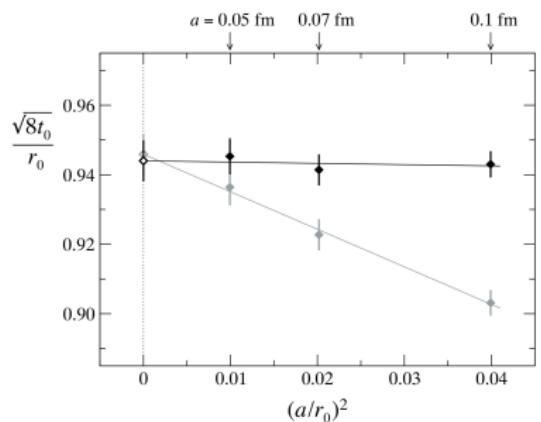
Continuum extrapolation using ensembles  $A$ ,  $B$ ,  $C$ , and  $D_2$  gives  
 $t_0,\text{cont}/r_0^2 = 0.11087(50)$ .

- The continuum extrapolation  $a \rightarrow 0$  for  $t_0$  of the four ensembles  $A$ ,  $B$ ,  $C$ , and  $D_2$ .
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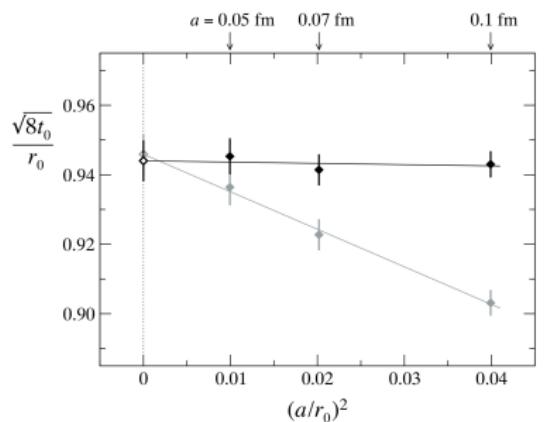
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## Scale setting $t_0$

Ensemble	$L/a$	$L$ [fm]	$a$ [fm]
$A$	24	2.235(9)	0.0931(4)
$B$	28	2.214(10)	0.0791(3)
$C$	32	2.17(1)	0.0679(3)
$D_1$	32	1.530(9)	0.0478(3)
$D_2$	48	2.29(1)	0.0478(3)

## Scale setting $t_0$

- Extrapolation results for  $t_0$ , where we retrieved the exact point of intersection between  $t_f^2 \langle E \rangle$  and 0.3 using  $N_{\text{bs}} = 500$  bootstrap fits. Extrapolating to the continuum gives us  $t_{0,\text{cont}}/r_0^2 = 0.11087(50)$ .

Ensemble	$t_0[\text{fm}^2]$	$t_0/a^2$	$t_0/r_0^2$
$A$	0.02780(2)	3.20(3)	0.11121(9)
$B$	0.02769(2)	4.43(4)	0.11075(10)
$C$	0.02775(2)	6.01(6)	0.11099(8)
$D_1$	0.02779(5)	12.2(1)	0.1112(2)
$D_2$	0.02794(9)	12.2(1)	0.1117(3)

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## Scale setting $t_0$

- Notice the  $\chi^2/\text{d.o.f.}$  of the extrapolation versus the two other extrapolations.

Extrapolations for different ensemble-combinations

Ensembles	$t_{0,\text{cont}}/r_0^2$	$\chi^2/\text{d.o.f.}$
$A, B, C, D_2$	0.11087(50)	7.51
$B, C, D_2$	0.1115(3)	0.41
$A, B, C, D_1$	0.1119(6)	0.88

## Scale setting $w_0$

Can also set a scale using the derivative which offers more granularity for small flow times,

$$W(t)|_{t=w_0^2} = 0.3,$$

$$W(t) \equiv t_f \frac{d}{dt_f} \{ t_f^2 \langle E \rangle \}.$$

First presented by Borsanyi et al. [2012].

## Scale setting $w_0$

Ensembles	$w_{0,\text{cont}}[\text{fm}]$	$\chi^2/\text{d.o.f}$
$A, B, C, D_2$	0.1695(5)	7.12
$B, C, D_2$	0.1702(3)	0.53
$A, B, C, D_1$	0.1706(6)	0.86

Ensembles	$w_{0,\text{cont}}[\text{fm}]$	$\chi^2/\text{d.o.f}$
$A, B, C, D_2$	0.1695(5)	7.12
$B, C, D_2$	0.1702(3)	0.53
$A, B, C, D_1$	0.1706(6)	0.86

Comparable to Borsanyi et al. [2012] which included dynamical fermions, with  $w_{0,\text{cont}} = 0.1755(18)(04)$  fm.

## Autocorrelation in the energy

The autocorrelation of the energy. A value of  $\tau_{\text{int}} = 0.5$  indicates that we have zero autocorrelation.

