

Solving SU(3) Yang-Mills theory on the lattice: a calculation of selected gauge observables with gradient flow

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Introduction

Structure

- Quantum Chromodynamics(QCD).

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- **GLAC.** Will briefly present the code which we developed as well as some benchmarks. We will also present the Metropolis algorithm.

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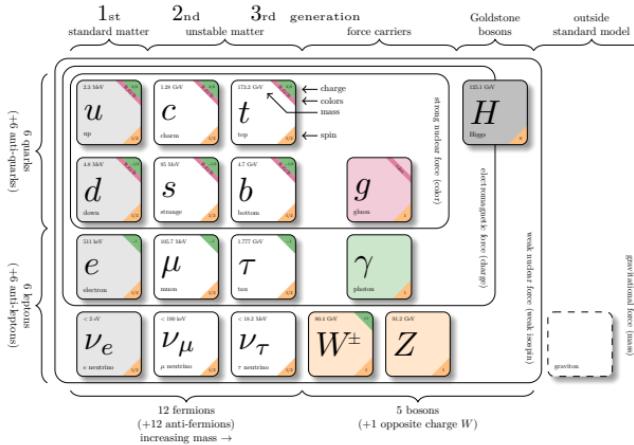
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- **Results.** We will present the results obtained from pure gauge calculations.

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Quantum Chromodynamics(QCD)

The Standard Model



Consists of the innermost square of the **six quarks** and the **eight gluons**.

The non-linearity of QCD

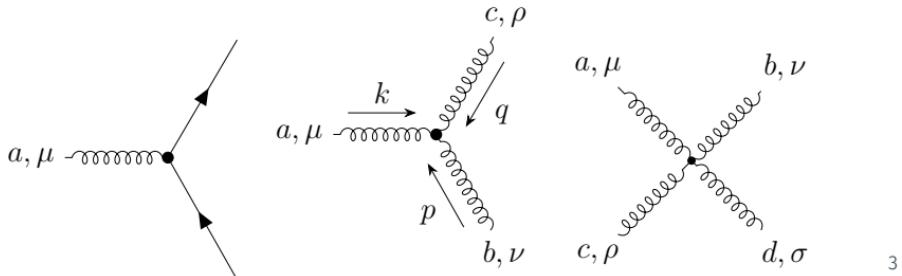
The QCD Lagrangian

$$\mathcal{L}_{\text{QCD}} = \sum_{f=1}^{N_f} \bar{\psi}^{(f)} \left(i \not{D} - m^{(f)} \right) \psi^{(f)} - \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu},$$

with action

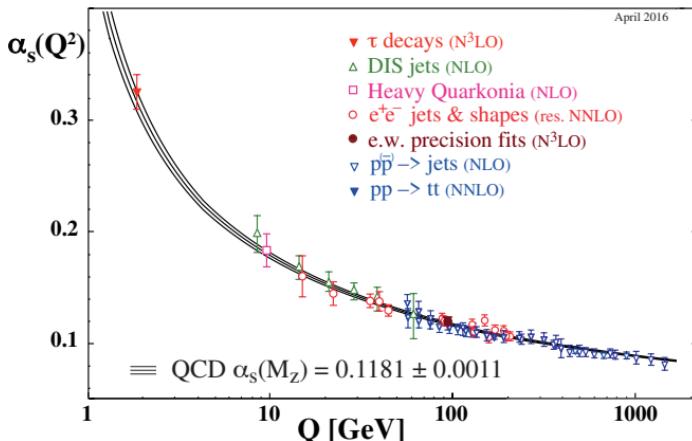
$$S = \int d^4x \mathcal{L}_{\text{QCD}}, \quad (1)$$

is invariant under a SU(3) symmetry.



- *Gluon self-interaction.*
- This central aspect is mostly covered in the pure-gauge/Yang-Mills section of the theory.

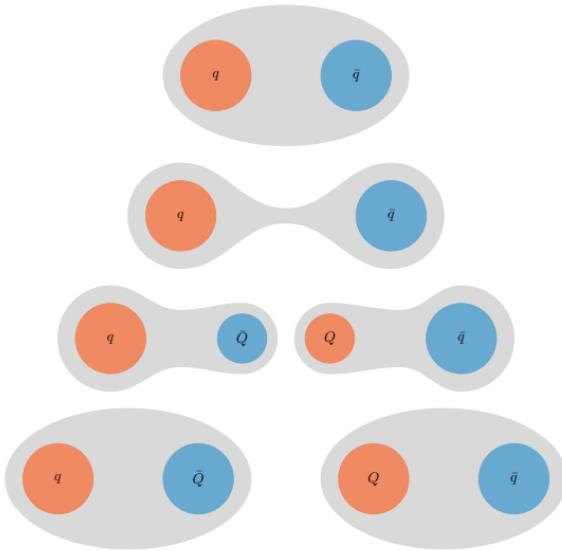
Asymptotic freedom



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- The coupling constant **decreases** as we **increase** the energy.
- Also serves as an *experimental proof* of QCD.
- Other lines of *evidence*: triple γ decay and muon cross section ratio R .
 - Triple γ decay: the number of colors is included in the cross section, which can be measured experimentally.
 - Muon cross section ratio R : the ratio is dependent on having three colors.

Confinement



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If we try to pull apart **two quarks in a meson**, more and more energy is required until we have enough energy to spontaneously create a **quark-antiquark pair**, forming thus **two new mesons**.

Lattice Quantum Chromodynamics(LQCD)

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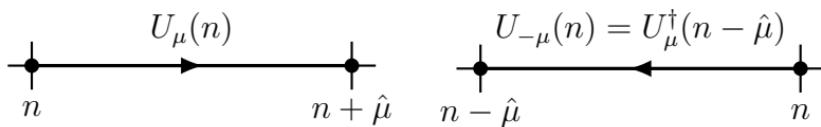
Goal: *maintain gauge invariance*.

Links

A link

$$U_\mu(n) = \exp [iaA_\mu(n)],$$

connects one lattice site to another and is a $SU(3)$ matrix.



where $U_{-\mu}(n) = U_\mu(n - \hat{\mu})^\dagger$.

- Defined from the gauge transporter.
- A link in the positive $\hat{\mu}$ direction is shown in the figure to the left.
- A link in the negative $\hat{\mu}$ direction is shown in the figure to the right.

Gauge invariance on the lattice

Links gauge transform as

$$U_\mu(n) \rightarrow U'_\mu(n) = \Omega(n) U_\mu(n) \Omega(n + \hat{\mu})^\dagger,$$
$$U_{-\mu}(n) \rightarrow U'_{-\mu}(n) = \Omega(n) U_\mu(n - \hat{\mu})^\dagger \Omega(n - \hat{\mu})^\dagger.$$

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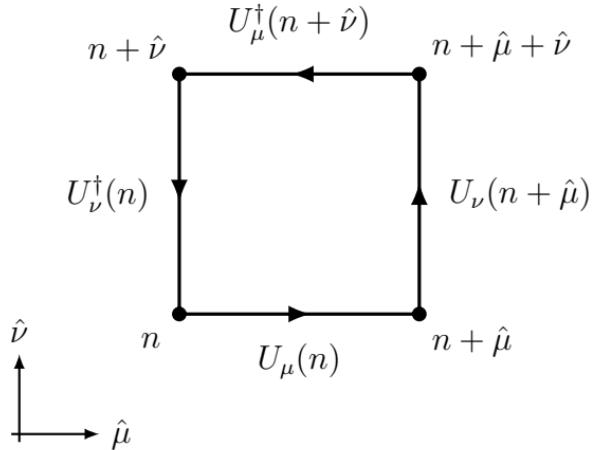
Two main types of gauge invariant objects,

- Fully connected gauge invariant objects.
- Objects with fermions $\psi, \bar{\psi}$ as end points.

The plaquette

The simplest gauge invariant object,

$$\begin{aligned} P_{\mu\nu}(n) &= U_\mu(n) U_\nu(n + \hat{\mu}) U_{-\mu}(n + \hat{\mu} + \hat{\nu}) U_{-\nu}(n + \hat{\nu}) \\ &= U_\mu(n) U_\nu(n + \hat{\mu}) U_\mu(n + \hat{\nu})^\dagger U_\nu(n)^\dagger, \end{aligned}$$



The Wilson gauge action

The Wilson gauge action is given as

$$S_G[U] = \frac{\beta}{3} \sum_{n \in \Lambda} \sum_{\mu < \nu} \text{Re} \operatorname{tr} [1 - P_{\mu\nu}(n)], \quad (2)$$

with $\beta = 6/g_S^2$.

Continuum action recovered when $a \rightarrow 0$!

- Using the definition of the link we saw earlier, we can reproduce the continuum action up to a discretization error of $\mathcal{O}(a^2)$.

Developing a code for solving SU(3) Yang-Mills theory on the lattice

The numerical challenge in lattice QCD

A lattice configuration consists of 3×3 SU(3) matrices,

- The SU(3) matrices are 3×3 matrices of nine complex numbers or 18 real numbers.
- This leads to an absolute **requirement of efficiency**, both in **calculations** and in **input/output**.
- When returning to what ensembles of configurations we generated this will be evident.

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$$\underbrace{N^3}_{\text{Spatial}} \times \underbrace{N_T}_{\text{Temporal}} \times \underbrace{4}_{\text{Links}} \times \underbrace{9}_{\text{SU(3) matrix}} \times \underbrace{2}_{\mathbb{C}\text{-numbers}} = 72N^3N_T,$$

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$\rightarrow 8 \times 72N^3N_T$ bytes.

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The path integral

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}U O[\psi, \bar{\psi}, U] e^{-S_G[U] - S_F[\psi, \bar{\psi}, U]}.$$

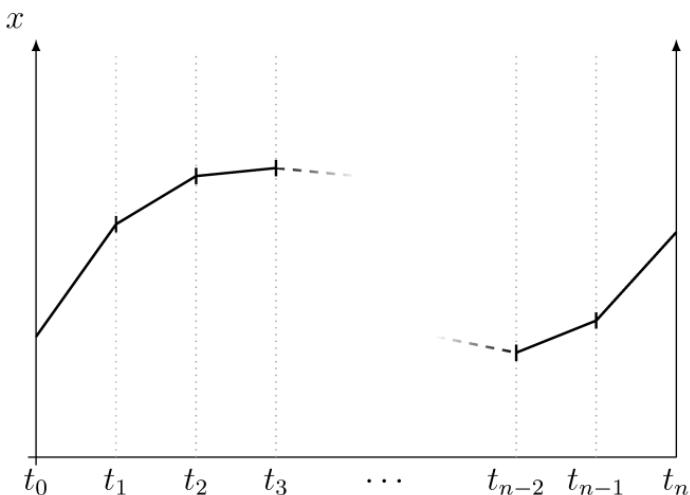
$$Z = \int \mathcal{D}U \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_G[U] - S_F[\psi, \bar{\psi}, U]}.$$

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The path integral



- An example of the discretized path integral, going from time t_0 to t_n , where the end points is taken to be equal, $x_0 = x_{N_T}$. We integrate over all of space at each time t_i finding the most likely position at a given time.

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Measurements on the lattice

The observable becomes an average over the N_{MC} gauge configurations.

$$\langle O \rangle = \lim_{N_{\text{MC}} \rightarrow \infty} \frac{1}{N_{\text{MC}}} \sum_i^{N_{\text{MC}}} O[U_i]$$

We now need to generate configurations...

- We perform an average of the created configurations.

Parallelization

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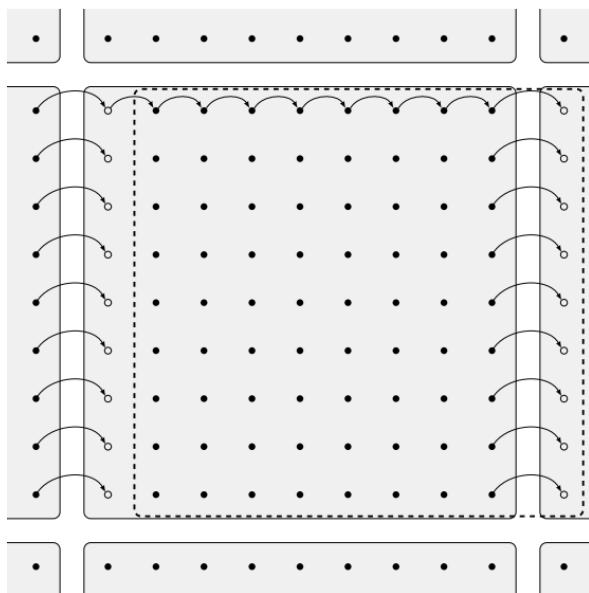
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- Single link sharing used in the Metropolis algorithm.
- *shifts* used in gradient flow and observable sampling

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Shifts



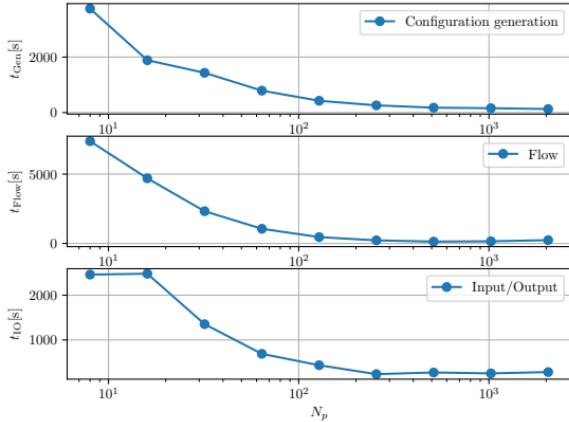
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- An illustration of the lattice shift.
- The links U_ν of the lattice are copied over to a temporary lattice shifted in direction $\hat{\mu}$.
- The face that is shifted over to an adjacent sub-lattice is shared through a non-blocking MPI call, while we copy the links to the temporary lattice.
- Allows for a simplified syntax close to that of the equations we are working with.
- Don't have to write out any loops over the lattice positions.

Scaling

We checked three types of scaling,

- Strong scaling: *fixed problem* and a *variable N_p cores*



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- Strong scaling

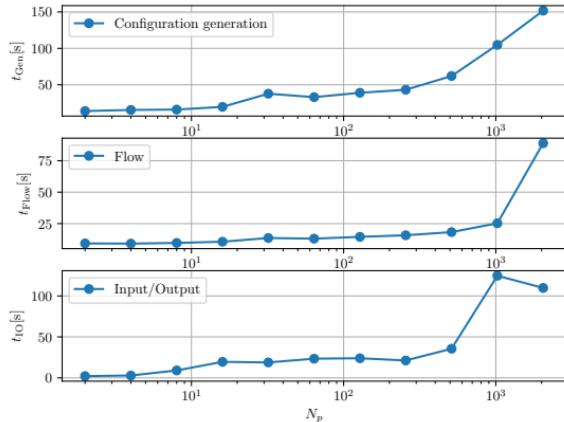
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HERE! FINAL <3-> is redundant!!

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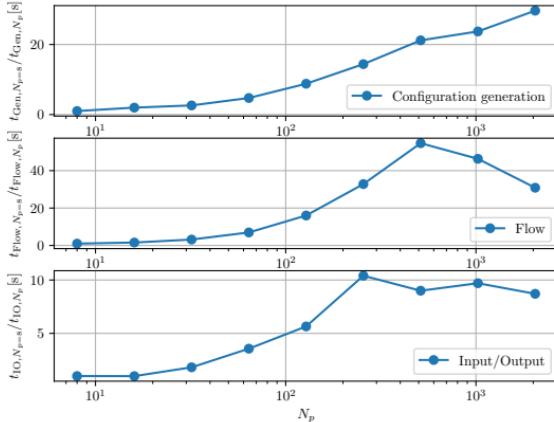
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- **Speedup:** defined as $S(p) = \frac{t_{N_p=0}}{t_{N_p}}$.



We appear to have a plateau around 512 cores. **TODO: FIX THE SLIDES HERE/FINAL** \rightarrow Weak scaling is redundant!!

- The speedup of the configuration generation, flowing, and IO. The speedup is calculated by dividing the run time of each N_p run, with the run time of the run with the least number of processors, $N_p = 8$.

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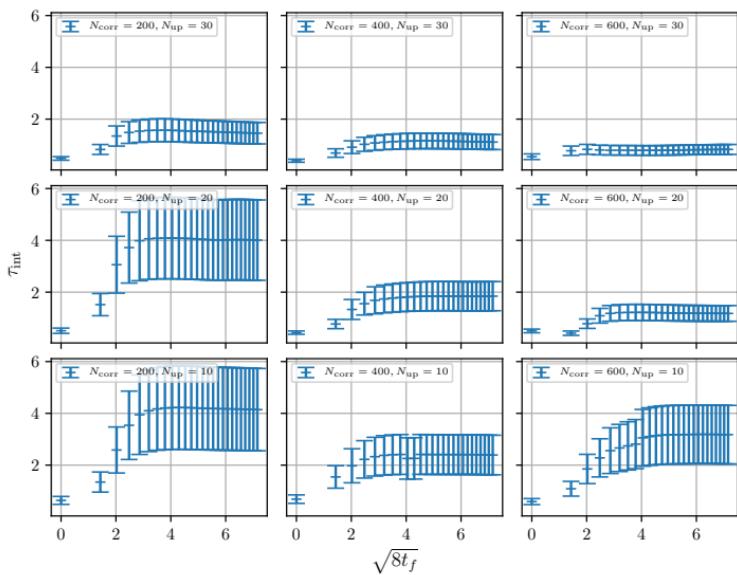
Optimizing the gauge configuration generation

Generated 200 configurations for a lattice of size $N^3 \times N_T = 16^3 \times 32$ and $\beta = 6.0$, for combinations of $N_{\text{corr}} \in [200, 400, 600]$ and $N_{\text{up}} \in [10, 20, 30]$.

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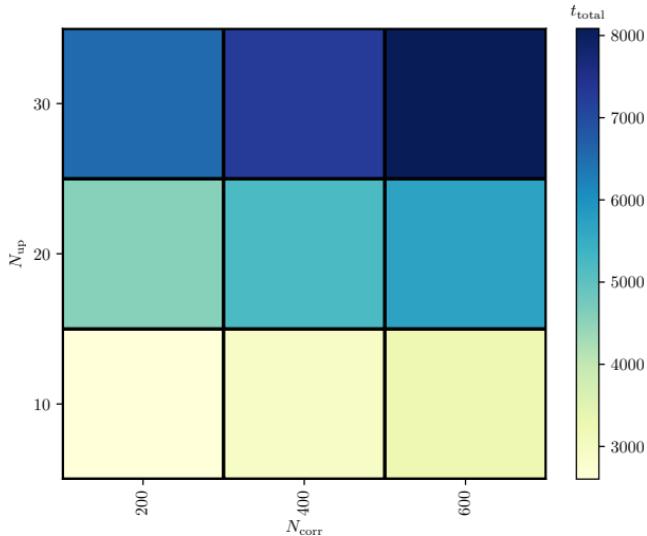
- We run for different values for N_{up} and N_{corr} to see what gives optimizes **computational cost** and **autocorrelation**.
- The integrated autocorrelation time for topological charge $\langle Q \rangle$ for a lattice of size $N = 16$ and $N_T = 32$ with $\beta = 6.0$ for combinations of $N_{\text{corr}} \in [200, 400, 600]$ and $N_{\text{up}} \in [10, 20, 30]$, plotted against flow time $\sqrt{8t_f}$.

Optimizing the gauge configuration generation



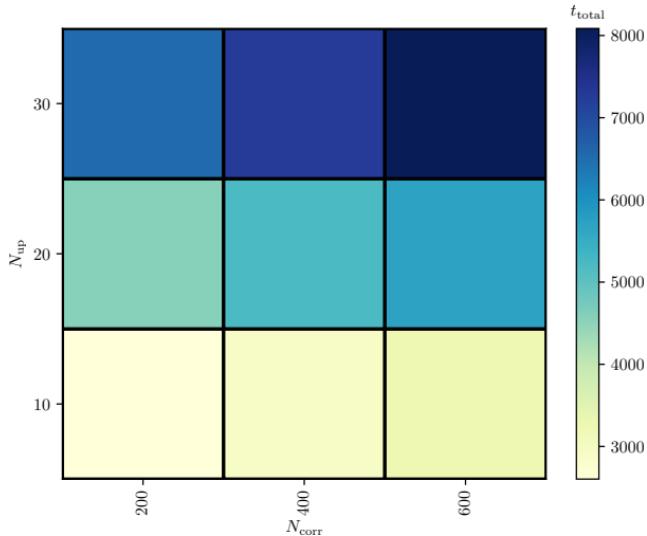
- The time taking to generate 200 configurations and flowing them $N_{\text{flow}} = 250$ flow steps for a lattice of size $N = 16$ and $N_T = 32$, with $\beta = 6.0$ for combinations of $N_{\text{corr}} \in [200, 400, 600]$ and $N_{\text{up}} \in [10, 20, 30]$.

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- What we see is that increasing N_{up} is a cheaper alternative compared to using N_{corr}

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- **Integration testing.** Random matrix generation, lattice objects, parallelization, ect.
- **Validation testing.** Cross checking results with a configuration from Chroma.

Gradient flow

The flow equation

The flow of the SU(3) gauge fields are denoted by $B_\mu(x, t_f)$ which are Lie algebra valued gauge fields,

$$\frac{d}{dt_f} B_\mu(x, t_f) = D_\nu G_{\nu \mu}(x, t_f), \quad (3)$$

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with the initial conditions being the fundamental gauge field,

$$B_\mu(x, t_f)|_{t_f=0} = A_\mu(x).$$

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A bad approximation: *the diffusion equation*,

$$\frac{\partial}{\partial t_f} B_\mu(x, t_f) \approx \partial^2 B_\mu(x, t_f)$$

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The smearing radius increases as $\sqrt{8t_f}$.

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- Topological charge preserved and is more pronounced.
- Renormalizes the topological charge at non-zero flow time.

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with the initial conditions being the fundamental gauge field,

$$B_\mu(x, t_f)|_{t_f=0} = A_\mu(x).$$

A bad approximation: *the diffusion equation*,

$$\frac{\partial}{\partial t_f} B_\mu(x, t_f) \approx \partial^2 B_\mu(x, t_f)$$

The smearing radius increases as $\sqrt{8t_f}$.

20

- The flow equation in the continuum is defined by this differential equation.
- With the covariant derivative given by following, with the \cdot being the derivative with respect to flow time.
- The field strength tensor of the flown fields is given in the regular format.
- The initial condition is the un-flowed gauge field, A_μ .
- Bad approx.: diffusion equation.
- Topological charge preserved and is more pronounced.
- Renormalizes the topological charge at non-zero flow time.

Gradient flow on the lattice

$$\dot{V}_{tf}(x, \mu) = -g_S^2 \left\{ \partial_{x,\mu} S_G[V_{tf}] \right\} V_{tf}(x, \mu),$$

- On the lattice, the flow equation takes the shape in terms of the link variables.

Gradient flow on the lattice

$$\dot{V}_{tf}(x, \mu) = -g_S^2 \left\{ \partial_{x,\mu} S_G[V_{tf}] \right\} V_{tf}(x, \mu),$$

with initial condition,

$$V_{tf}(x, \mu)|_{t_f=0} = U(x, \mu)$$

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Solving gradient flow with Runge-Kutta 3

With

$$\dot{V}_{tf} = -g_S^2 \left\{ \partial_{x,\mu} S_G[V_{tf}] \right\} V_{tf} = Z(V_{tf}) V_{tf},$$

- We rewrite the equations slightly,

Solving gradient flow with Runge-Kutta 3

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$$\dot{V}_{t_f} = -g_S^2 \left\{ \partial_{x,\mu} S_G[V_{t_f}] \right\} V_{t_f} = Z(V_{t_f}) V_{t_f},$$

we get

$$W_0 = V_{t_f},$$

$$W_1 = \exp \left[\frac{1}{4} Z_0 \right] W_0,$$

$$W_2 = \exp \left[\frac{8}{9} Z_1 - \frac{17}{36} Z_0 \right] W_1,$$

$$V_{t_f + \epsilon_f} = \exp \left[\frac{3}{4} Z_2 - \frac{8}{9} Z_1 + \frac{17}{36} Z_0 \right] W_2,$$

with coefficients from ?].

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- We rewrite the equations slightly,
- and use a structure preserving integrator with coefficients from ?].

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Verifying the integration

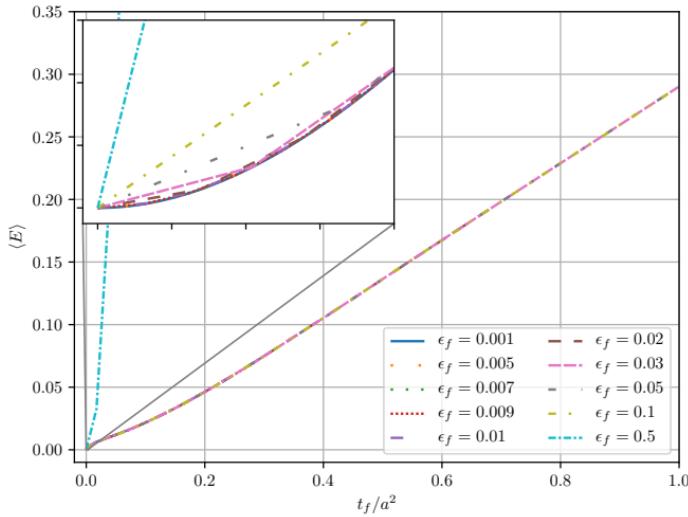
Testing the integrator for different integration steps ϵ_f .

ϵ_f	0.001	0.005	0.007	0.009	0.01	0.02	0.03	0.05	0.1	0.5
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- The values we will test the integrator against.

Verifying the integration

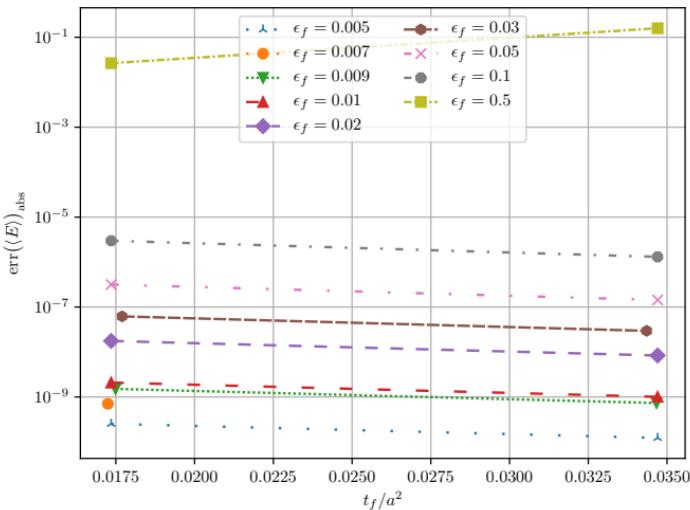
Lattice size $N^3 \times N_T = 24^3 \times 48$ with $\beta = 6.0$.



- The values we will test the integrator against.
- The energy flowed for different the different ϵ_f values.

Verifying the integration

The absolute difference between the smallest flow time $\epsilon_f = 0.001$ and those shown previously.

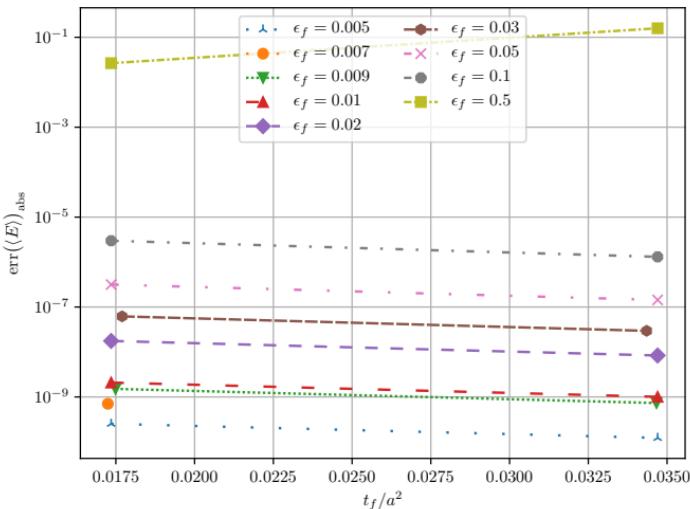


23

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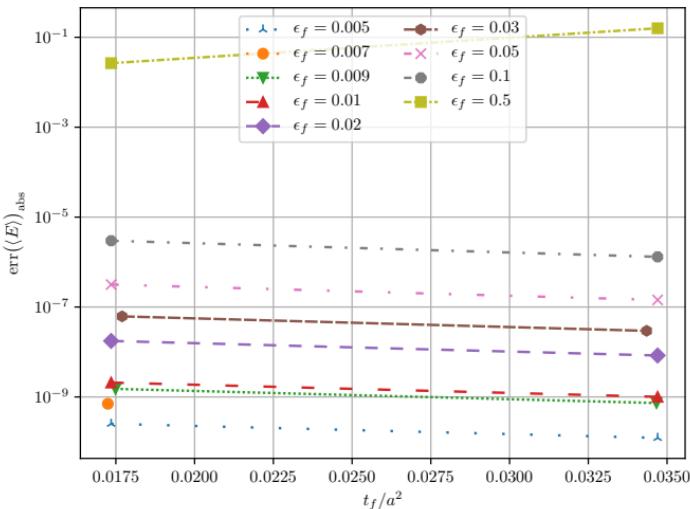


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- An **example** of the flowing, can be seen by observing the **energy evolving over flow time**.

Results

Ensembles

Ensemble	β	N	N_T	N_{cfg}	ϵ_{flow}	Config. size[GB]
A	6.0	24	48	1000	0.01	0.356
B	6.1	28	56	1000	0.01	0.659
C	6.2	32	64	2000	0.01	1.125
D_1	6.45	32	32	1000	0.02	0.563
D_2	6.45	48	96	250	0.02	5.695

- We use $N_{\text{corr}} = 1600$ for $\beta = 6.45$ ensembles, $N_{\text{corr}} = 600$ for the rest.

- The main ensembles made for this thesis.
- Every configuration was flown with $N_{\text{flow}} = 1000$ flow steps.
- We should also mention that we generated a few additional ensembles for investigating other aspects of the topological charge.

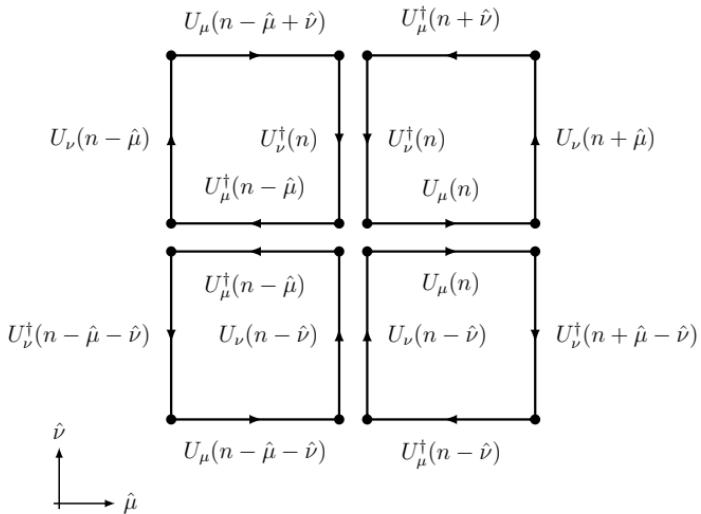
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- $N_{\text{up}} = 30$.

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The clover field strength definition



- We will use the clover field strength definition in gauge observables.

Energy definition

$$E = \frac{a^4}{2|\Lambda|} \sum_{n \in \Lambda} \sum_{\mu, \nu} (F_{\mu\nu}^{\text{clov}}(n))^2$$

- We can use this definition to set a scale.

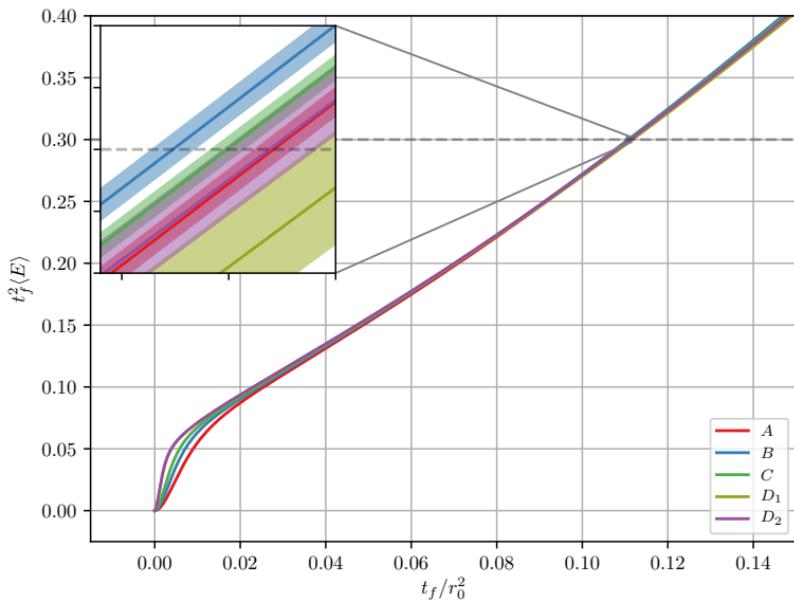
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We can use this definition to set a scale t_0 ,

$$\left\{ t_f^2 \langle E(t) \rangle \right\}_{t_f=t_0} = 0.3.$$

- We can use this definition to set a scale.



Scale setting t_0

Ensemble	L/a	L [fm]	a [fm]
A	24	2.235(9)	0.0931(4)
B	28	2.214(10)	0.0791(3)
C	32	2.17(1)	0.0679(3)
D_1	32	1.530(9)	0.0478(3)
D_2	48	2.29(1)	0.0478(3)

Scale setting t_0

Ensemble	$t_0[\text{fm}^2]$	t_0/a^2	t_0/r_0^2
A	0.02780(2)	3.20(3)	0.11121(9)
B	0.02769(2)	4.43(4)	0.11075(10)
C	0.02775(2)	6.01(6)	0.11099(8)
D_1	0.02779(5)	12.2(1)	0.1112(2)
D_2	0.02794(9)	12.2(1)	0.1117(3)

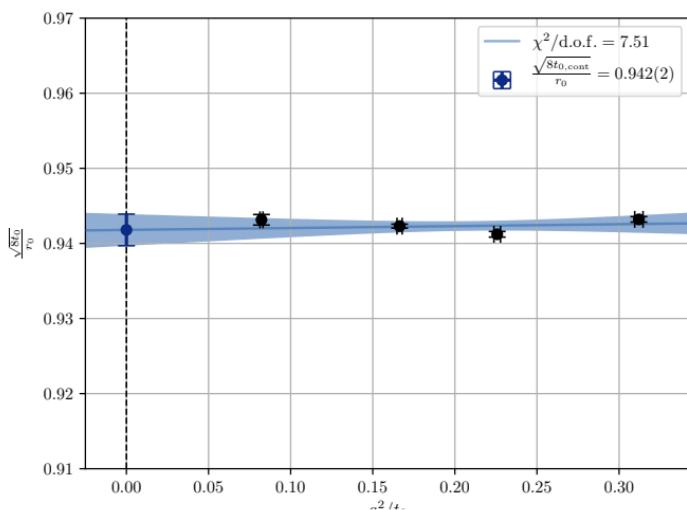
- Extrapolation results for t_0 , where we retrieved the exact point of intersection between $t_f^2 \langle E \rangle$ and 0.3 using $N_{\text{bs}} = 500$ bootstrap fits. Extrapolating to the continuum gives us $t_{0,\text{cont}}/r_0^2 = 0.11087(50)$.

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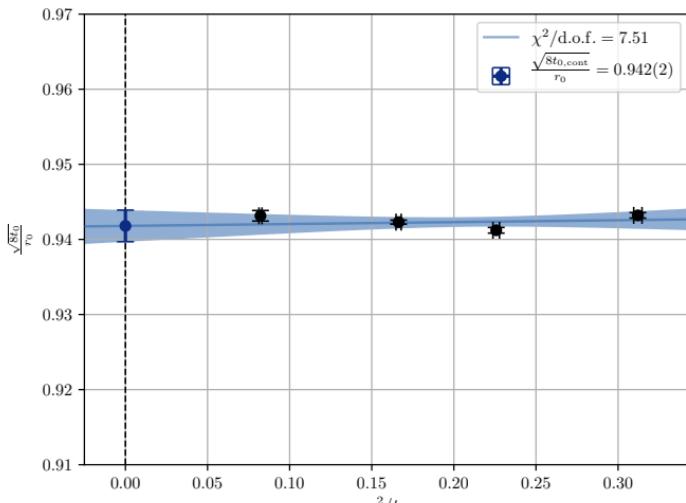
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Scale setting t_0



- The continuum extrapolation $a \rightarrow 0$ for t_0 of the four ensembles A , B , C , and D_2 .

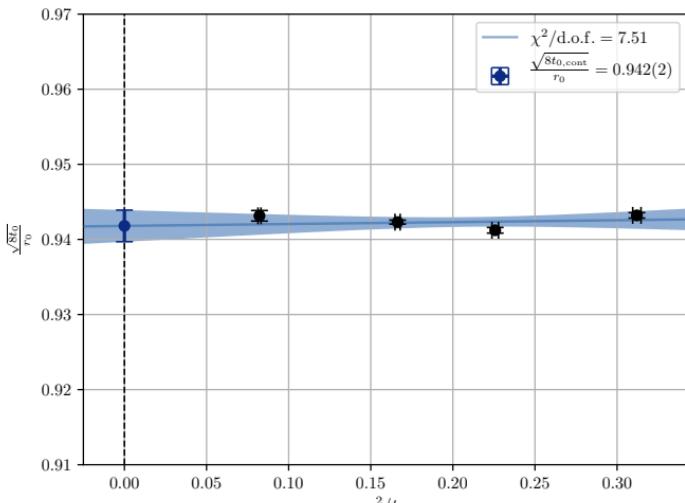
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Continuum extrapolation using ensembles A , B , C , and D_2 gives $t_{0,\text{cont}}/r_0^2 = 0.11087(50)$.

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- $r_0 = 0.5$ fm.

Scale setting t_0



Continuum extrapolation using ensembles A , B , C , and D_2 gives $t_{0,\text{cont}}/r_0^2 = 0.11087(50)$. This matches the values retrieved by [?].

- The continuum extrapolation $a \rightarrow 0$ for t_0 of the four ensembles A , B , C , and D_2 .
- $r_0 = 0.5$ fm.

Scale setting t_0

Extrapolations for different ensemble-combinations

Ensembles	$\frac{t_0, \text{cont}}{r_0^2}$	$\chi^2/\text{d.o.f}$
A, B, C, D_2	0.11087(50)	7.51
B, C, D_2	0.1115(3)	0.41
A, B, C, D_1	0.1119(6)	0.88

Scale setting w_0

Can also set a scale using the derivative which offers more granularity for small flow times,

$$W(t)|_{t=w_0^2} = 0.3,$$
$$W(t) \equiv t_f \frac{d}{dt_f} \{ t_f^2 \langle E \rangle \}.$$

Scale setting w_0

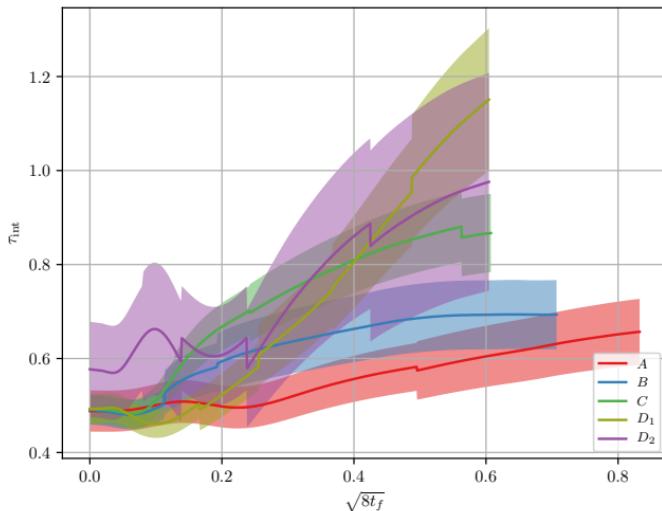
Ensembles	$w_{0\text{cont}}[\text{fm}]$	$\chi^2/\text{d.o.f}$
A, B, C, D_2	0.1695(5)	7.12
B, C, D_2	0.1702(3)	0.53
A, B, C, D_1	0.1706(6)	0.86

Scale setting w_0

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Comparable to [?] which included dynamical fermions, with
 $w_{0,\text{cont}} = 0.1755(18)(04)$ fm.

Autocorrelation in the energy



The autocorrelation of the energy. A value of $\tau_{\text{int}} = 0.5$ indicates that we have zero autocorrelation.

Topology in QCD

TODO: Flytte den delen til der jeg presenterer topologisk ladning.

- Instantons

- **Instantons** are local minimums to the Yang-Mills action in Euclidean space.

Topology in QCD

TODO: Flytte den delen til der jeg presenterer topologisk ladning.

- Instantons
- Topological charge, Q

33

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- Measuring **topological charge** is a measure of the *Winding number* of the gauge field.

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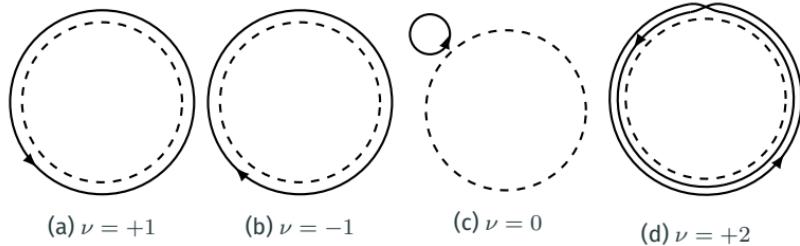


Figure 2: The figure is taken from ? , p. 32].

- **Instantons** are local minimums to the Yang-Mills action in Euclidean space.
- Measuring **topological charge** is a measure of the *Winding number* of the gauge field.
- An illustration of how one can view the winding number given a function f that parametrizes a path around a circle S^1 . Given that it starts and ends at the same point, we have that the number of times it wraps around the circle gives us the winding number.

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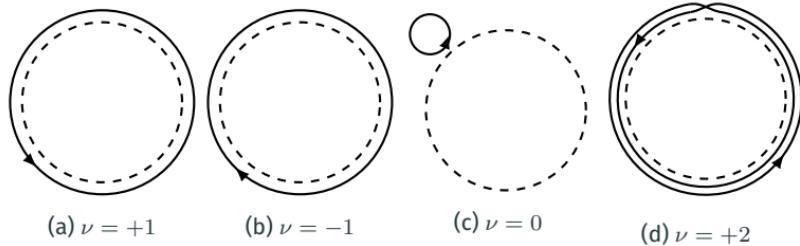


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Topological charge definition

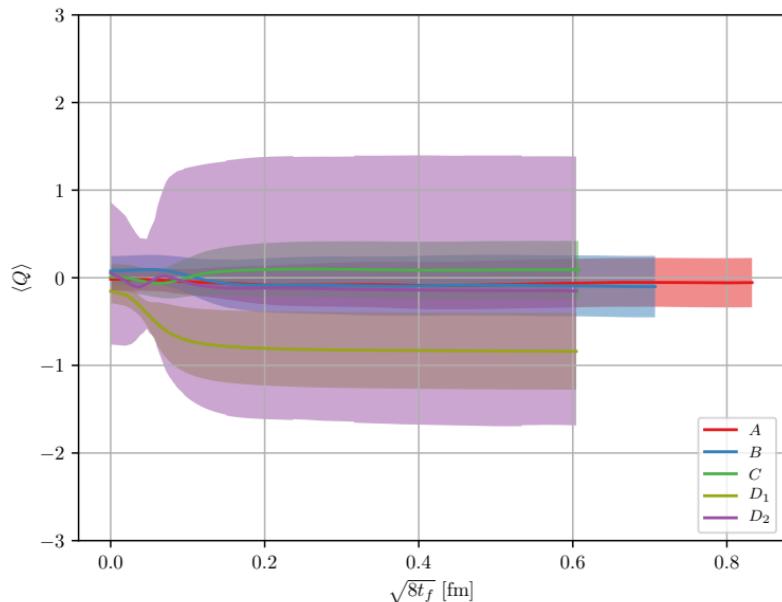
$$Q = a^4 \sum_{n \in \Lambda} q(n),$$

with the charge density given by

$$q(n) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr} [F_{\mu\nu}(n) F_{\rho\sigma}(n)].$$

- We will use the clover field strength definition.
- Symmetries will allow us to reduce the effective number of clovers need to calculate from 24 to 6.

Topological charge



35

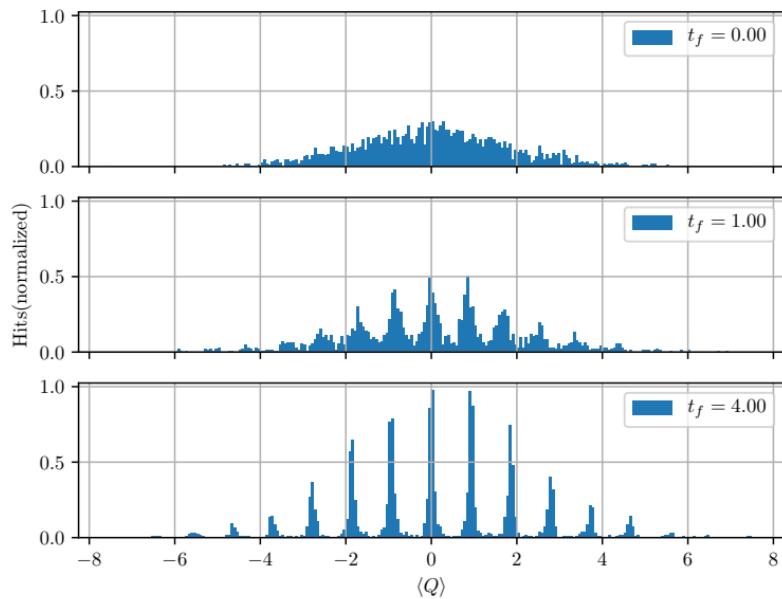
- Topological charge Q as evolved in flow time for the five main ensembles.
- Bootstrapped data with $N_{\text{bs}} = 500$ bootstrap samples.
- Corrected for autocorrelations with $\sigma = \sqrt{2\tau_{\text{int}}}\sigma_0$.

Additional ensembles

Ensemble	N	N_T	N_{cfg}	N_{corr}	N_{up}	a [fm]	L [fm]
E	8	16	8135	600	30	0.0931(4)	0.745(3)
F	12	24	1341	200	20	0.0931(4)	1.118(5)
G	16	32	2000	400	20	0.0790(3)	1.265(6)

- Additional ensembles made in order to illuminate additional aspects of the topological charge.
- Supporting ensembles made on Smaug. All ensembles were flown $N_{\text{flow}} = 1000$ steps with $\epsilon_{\text{flow}} = 0.01$.

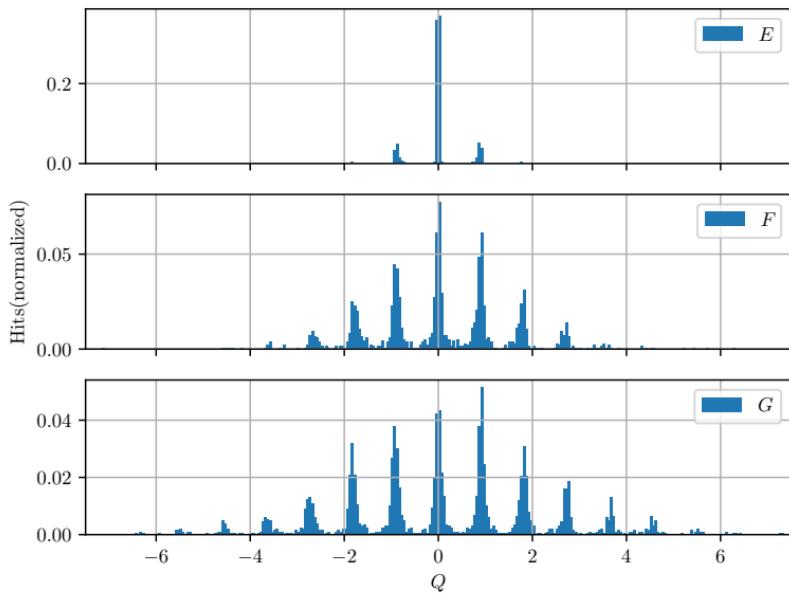
Topological charge distribution



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Histograms for the topological charge for ensemble G with a lattice of size $N^3 \times N_T = 16^3 \times 32$ with $\beta = 6.1$, taken at different flow times $t_f/a^2 = 0.0, 1.0, 4.0$ fm.

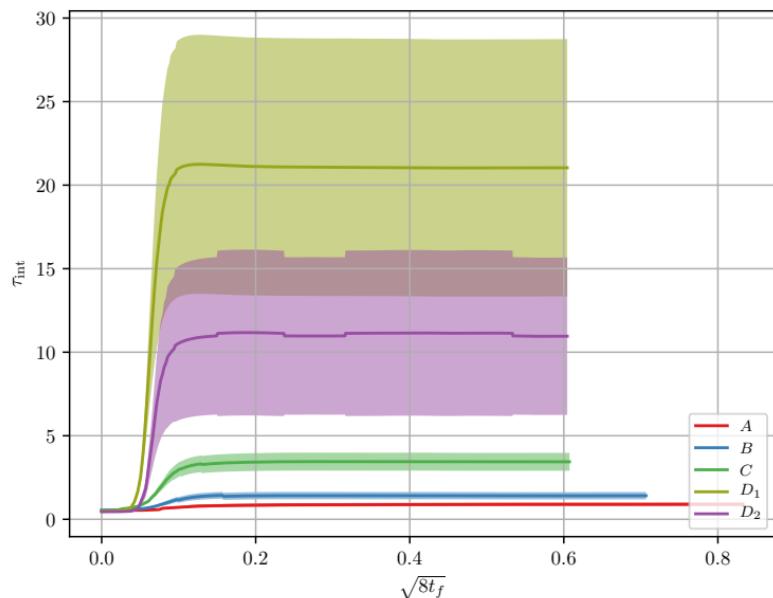
Topological charge distribution in flow time



38

Histograms of topological charge for the supporting ensembles seen at $t_f/a^2 = 0.25$ fm.

Topological charge autocorrelation



The integrated autocorrelation τ_{int} for topological charge for the five main ensembles.

Topological susceptibility

$$\chi_{\text{top}}^{1/4} = \frac{1}{V^{1/4}} \langle Q^2 \rangle^{1/4}$$

40

- We can use the Witten-Veneziano formula in order to extract an estimate for N_f using the topological susceptibility

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- This can help us understand the "quality" of the ensemble, as we would expect to be around $N_f = 3$.

Topological susceptibility

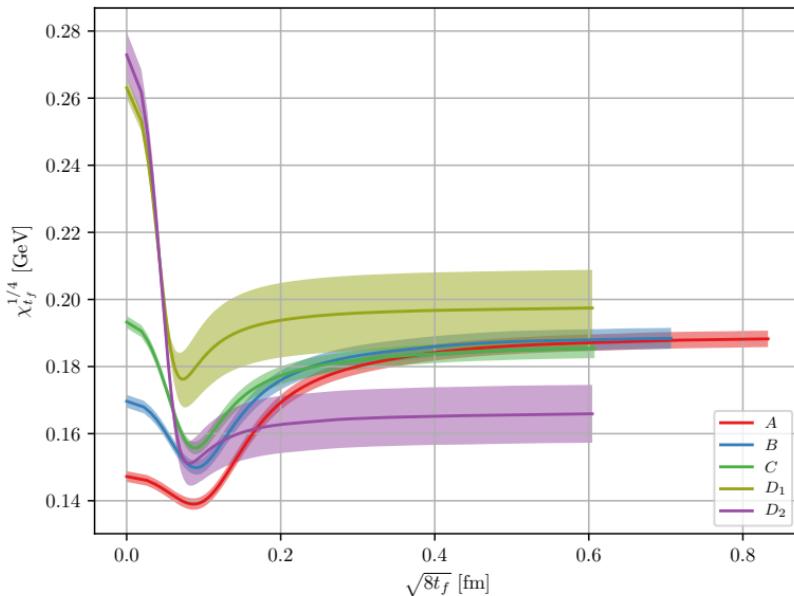
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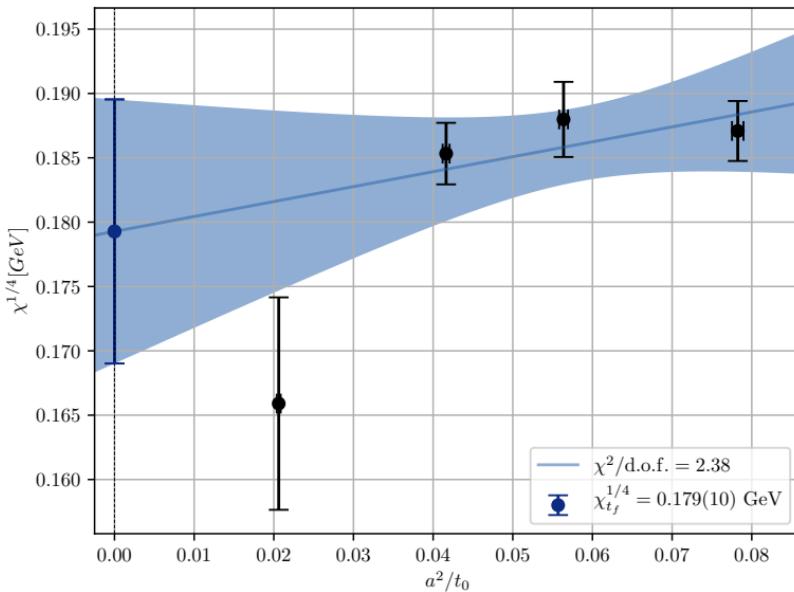
- The topological susceptibility $\chi_{tf}^{1/4}$ of the main ensembles.
- Bootstrapped $N_{\text{bs}} = 500$ times.
- Corrected for autocorrelations with $\sigma = \sqrt{2\tau_{\text{int}}}\sigma_0$.

Ensemble	$\chi_{tf}^{1/4}$ [GeV]	$\chi_{tf}^{1/4}$ [GeV], corrected	$\sqrt{2\tau_{int}}$
<i>A</i>	0.1877(23)	0.1877(24)	1.028(46)
<i>B</i>	0.1880(21)	0.1880(29)	1.346(81)
<i>C</i>	0.1853(14)	0.1853(24)	1.762(104)
<i>D</i> ₁	0.1971(22)	0.1971(101)	4.523(675)
<i>D</i> ₂	0.1656(33)	0.1656(86)	2.624(441)

Error corrected for autocorrelations with $\sigma = \sqrt{2\tau_{int}}\sigma_0$.

The topological susceptibility for the main ensembles together with the correction factor from the integrated autocorrelation time. The second column have not had its results corrected by $\sqrt{2\tau_{int}}$. None of the results have been analyzed with bootstrapping.

Topological susceptibility continuum extrapolation



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- A continuum extrapolation of the topological susceptibility $\chi_{tf}^{1/4}$ for the main ensembles excluding the D_1 ensemble.
- The points for $\chi_{tf}^{1/4}$ is taken at $\sqrt{8t_{f,0}} = 0.6 \text{ fm}$.

The Witten-Veneziano relation

TODO: Flytte den delen til der jeg presenterer

Witten-Veneziano-formelen. A formula connecting pure gauge theory and full QCD.

$$m_{\eta'}^2 = \frac{2N_f}{f_\pi^2} \chi_{\text{top}} \quad (6)$$

- Pion decay constant $f_\pi = 0.130(5)/\sqrt{2}$ GeV.
- η' meson mass $m_{\eta'} = 0.95778(6)$ GeV.
- $\chi_{\text{top}} = \langle Q^2 \rangle$ is the *topological susceptibility*.

44

TODO: MORE ON WV FORMULA. CLARIFY WHAT IT IS AND WHAT IT EXPLAINS.

- We use the experimental values for the pion decay constant and the η' mass.
- R.h.s. is full QCD and l.h.s is from pure gauge theory.
- Allows us to estimate the number of flavors in our theory N_f .
- χ_{top} is the topological susceptibility, calculated from the expectation value of Q .

Topological susceptibility continuum extrapolation

Ensembles	$\chi_{tf}^{1/4} (\langle Q^2 \rangle) [\text{GeV}]$	N_f	$\chi^2/\text{d.o.f}$
A, B, C, D_2	0.179(10)	3.75(29)	2.38
A, B, C, D_1	0.186(6)	3.21(25)	0.83
B, C, D_1	0.187(24)	3.18(24)	1.63
B, C, D_2	0.166(24)	5.06(39)	2.05
A, B, C	0.184(6)	3.37(26)	0.33

The fourth cumulant

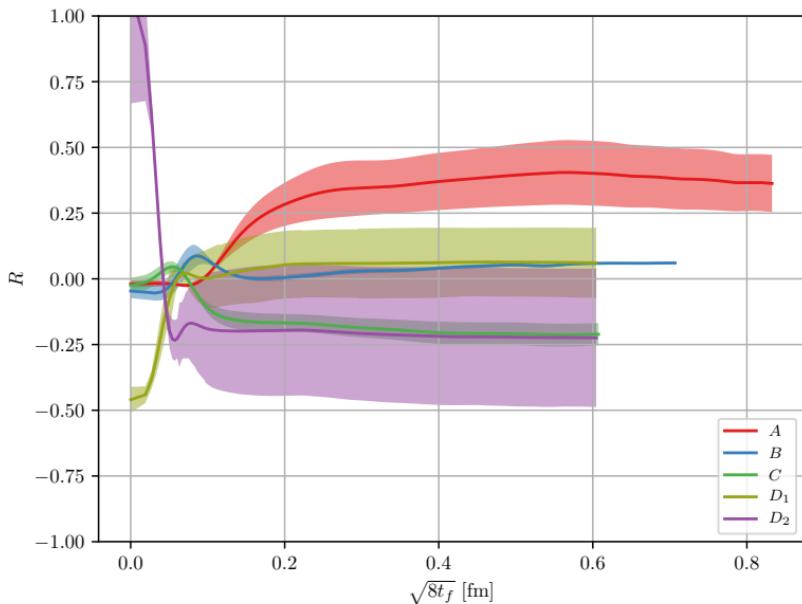
TODO: WHAT DO THE R RATIO INDICATE WHEN AWAY FROM ZERO?!

$$\langle Q^4 \rangle_c = \frac{1}{V^2} \left(\langle Q^4 \rangle - 3 \langle Q^2 \rangle^2 \right).$$

From this, we can also measure the ratio R ,

$$R = \frac{\langle Q^4 \rangle_c}{\frac{1}{V} \langle Q^2 \rangle} = \frac{1}{V} \frac{\langle Q^4 \rangle - 3 \langle Q^2 \rangle^2}{\langle Q^2 \rangle},$$

The fourth cumulant



- The fourth cumulant ratio $R = \langle Q^4 \rangle_C / \langle Q^2 \rangle$.
- The results were analyzed using $N_{\text{bs}} = 500$ bootstrap samples, with the error corrected for by $\sqrt{2\tau_{\text{int}}}$.

The fourth cumulant at reference flow times

TODO: SCALE THE TABLES

Ensemble	L/a	t_0/a^2	$\langle Q^2 \rangle$	$\langle Q^4 \rangle$	$\langle Q^4 \rangle_C$	
A	2.24	3.20(3)	0.78(4)	2.13(27)	0.282(67)	0.359
B	2.21	4.43(4)	0.81(5)	1.98(23)	0.036(11)	0.044
C	2.17	6.01(6)	0.77(4)	1.6(2)	-0.174(40)	-0.226
D_1	1.53	12.2(1)	1.00(20)	3.01(1.07)	0.03(12)	0.03
D_2	2.29	12.2(1)	0.497(100)	0.64(20)	-0.103(95)	-0.21

The fourth cumulant is taken at their individual reference scales seen in the third column. The data were analyzed with using a bootstrap analysis of $N_{bs} = 500$ samples, with error corrected by the integrated autocorrelation, $\sqrt{2\tau_{int}}$.

Comparing fourth cumulant

TODO: WHAT ARTICLE?! SCALE TABLE

Ensemble	β	L/a	L [fm]	a [fm]	t_0/a^2	t_0/r_0^2	
F_1	5.96	16	1.632	0.102	2.7887(2)	0.1113(9)	1 440
B_2	6.05	14	1.218	0.087	3.7960(12)	0.1114(9)	144
\tilde{D}_2		17	1.479		3.7825(8)	0.1110(9)	
B_3	6.13	16	1.232	0.077	4.8855(15)	0.1113(10)	144
\tilde{D}_3		19	1.463		4.8722(11)	0.1110(10)	
B_4	6.21	18	1.224	0.068	6.2191(20)	0.1115(11)	144
\tilde{D}_4		21	1.428		6.1957(14)	0.1111(11)	

- Parameters of the ensembles presented by ?]. The first column is the ensemble name from the article. The letter indicates the volume, while the subindex indicates the β value. Ensembles of similar letters keep approximately the same length L .

Comparing fourth cumulant

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Ensemble	$\langle Q^2 \rangle_{\text{normed}}$	$\langle Q^4 \rangle_{\text{normed}}$	$\langle Q^4 \rangle_{C,\text{normed}}$	R_{normed}
F_1	0.728(1)	1.608(4)	0.016(1)	0.022(1)
B_2	0.772(3)	1.873(19)	0.085(4)	0.110(5)
\tilde{D}_2	0.770(3)	1.817(17)	0.037(4)	0.048(5)
B_3	0.760(3)	1.805(17)	0.074(3)	0.097(4)
\tilde{D}_3	0.769(3)	1.801(14)	0.027(1)	0.035(1)
B_4	0.776(3)	1.874(18)	0.069(3)	0.089(4)
\tilde{D}_4	0.785(3)	1.891(17)	0.040(4)	0.052(5)

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- Results as presented by [?], normalized by the lattice volume.

Comparing fourth cumulant

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Article	Thesis	Ratio($\langle Q^2 \rangle$)	Ratio($\langle Q^4 \rangle$)	Ratio($\langle Q^4 \rangle_C$)	Ratio(R)
F_1	A	1.08(6)	1.34(18)	19.03(5.81)	17.64(4.48)
B_2	A	1.02(5)	1.15(15)	3.60(1.09)	3.54(90)
	B	1.04(6)	1.06(11)	0.480(74)	0.46(4)
\tilde{D}_2	A	1.02(5)	1.19(15)	8.31(1.99)	8.15(1.56)
	B	1.05(6)	1.10(12)	1.1(1)	1.06(3)
B_3	B	1.06(6)	1.10(12)	0.550(86)	0.52(5)
\tilde{D}_3	B	1.05(6)	1.11(12)	1.51(23)	1.4(1)
B_4	C	0.99(5)	0.86(8)	-2.32(46)	-2.35(59)
\tilde{D}_4	C	0.98(5)	0.85(8)	-3.95(96)	-4.05(1.19)

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TODO: WHAT ARTICLE?! SCALE TABLE

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The topological charge correlator

TODO: CLEAN UP! REMOVE GENERAL EXPRESSION(ITS JUST MORE CONFUSING!)TODO: EXPLAIN HERE WHAT SOURCE/SINK IS
A general correlator is given as,

$$C(n_t) = \left\langle \hat{O}_2(\mathbf{0}, n_t) \hat{O}_1(\mathbf{0}, 0) \right\rangle = \sum_k \langle 0 | \hat{O}_2 | k \rangle \langle k | \hat{O}_1 | 0 \rangle e^{-n_t E_k}$$

where n_t is the Euclidean time in which the correlator is taken and E_k is states of energy.

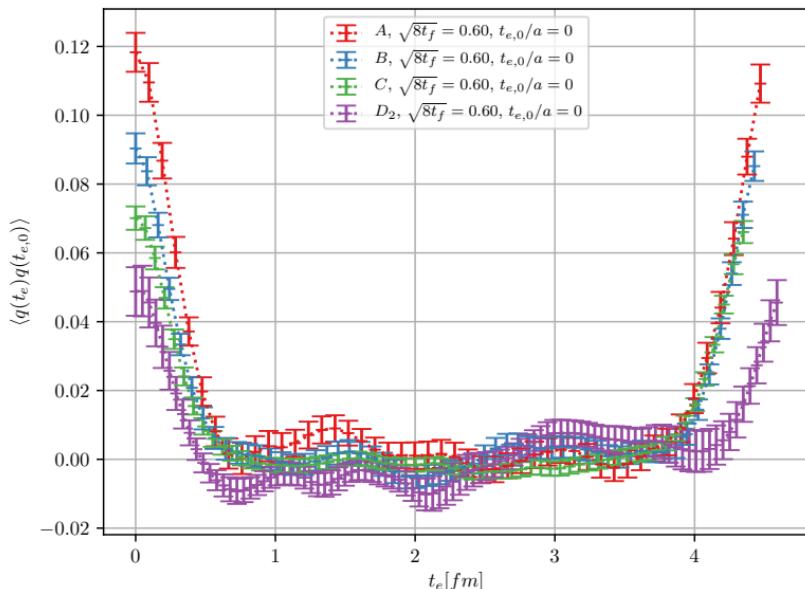
The **topological charge correlator**

$$C(n_t) = \langle q(n_t) q(0) \rangle,$$

$q(0)$ is the *source* and $q(n_t)$ is the *sink*.

- $q(0)$ is not required to be at $n_t = 0$.

The topological charge correlator



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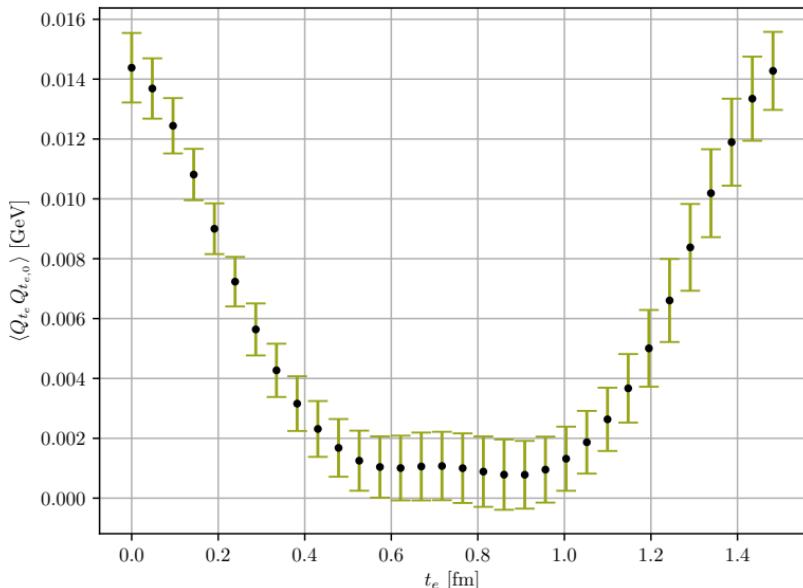
- The topological charge correlator for all of the ensembles except D_1 . The x -axis contains the sink-source separation, as the source $q(0)$ is placed at $t_e = 0$ fm, and the sink $q(t_e)$ is taken at t_e .

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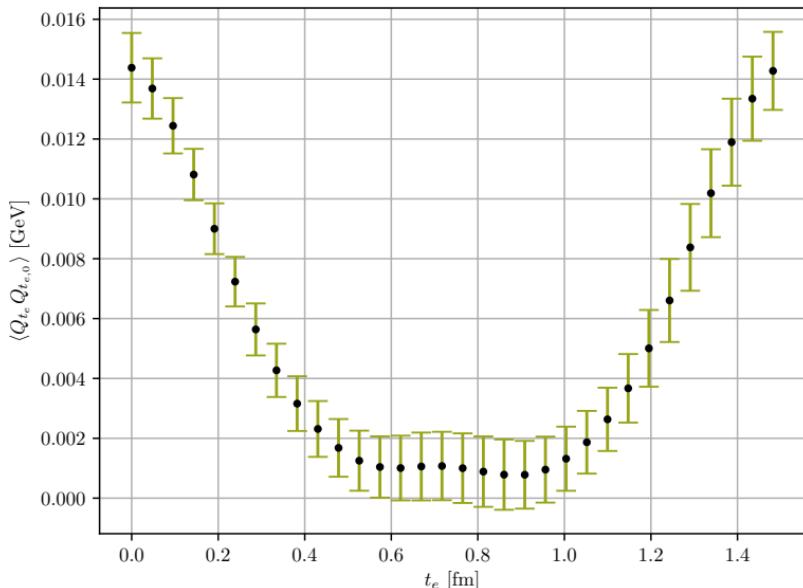
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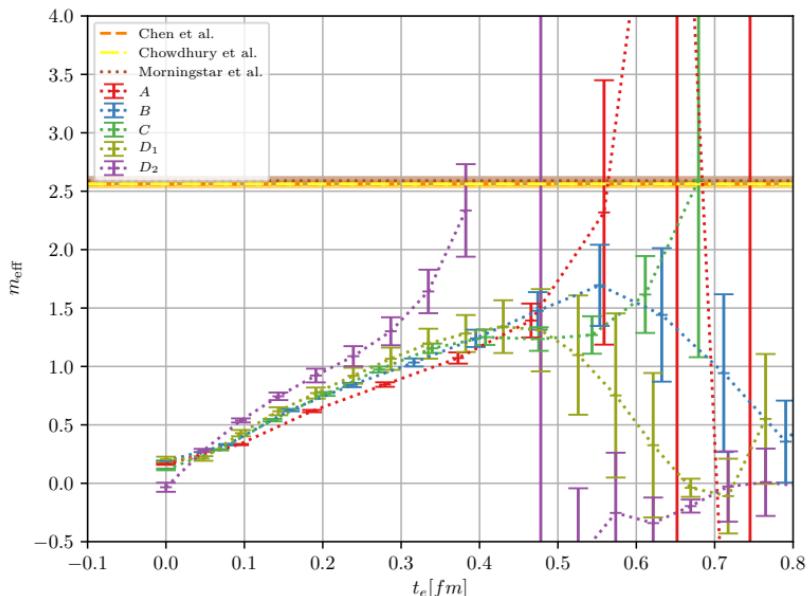
The ground state in the correlator is given as

$$C(n_t) = A_0 e^{-n_t E_0} + A_1 e^{-n_t E_1} + \dots$$

which can be extracted as

$$am_{\text{eff}} = \log \left(\frac{C(n_t)}{C(n_t + 1)} \right),$$

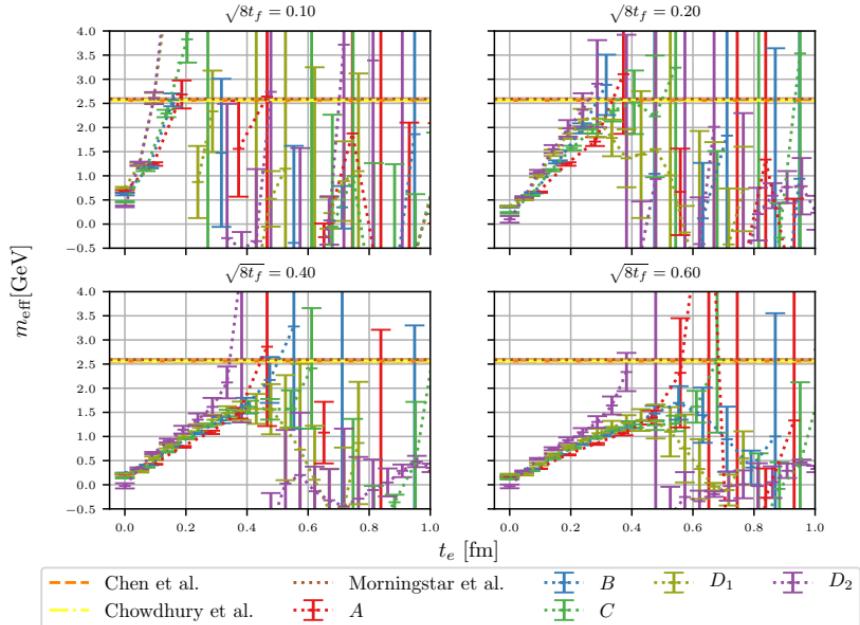
The effective glueball mass



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- The effective mass of the glueball, as extracted from the topological charge correlator in Euclidean time.

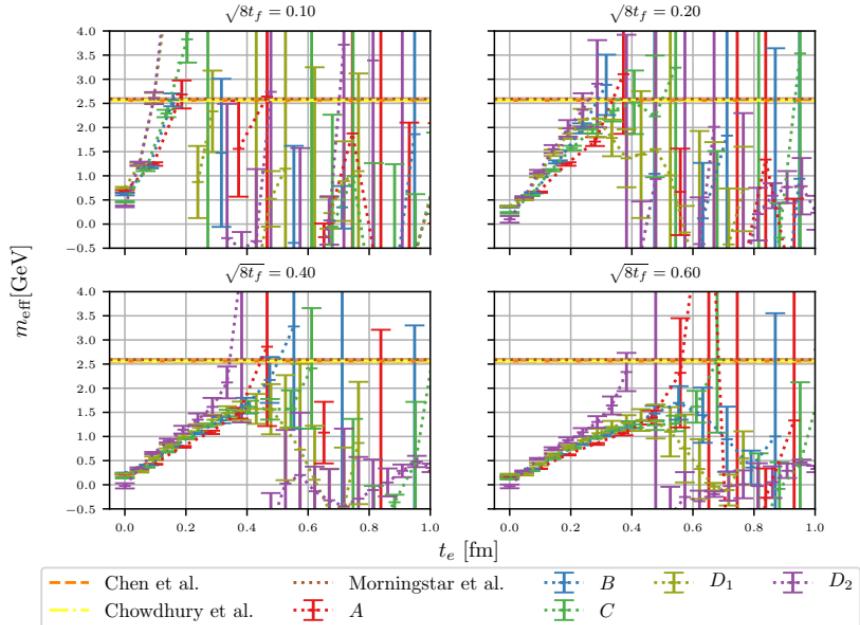
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~~TODO: CLEAN UP COMMENT ON FRAME!~~ The effective mass of the charge correlator in Euclidean time,

$\sqrt{8t_f} \in [0.1, 0.2, 0.3, 0.4, 0.6]$: low statistics and critical slowdown \rightarrow poor signal.

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Conclusion, future developments and final thoughts

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- Fermions and HMC(Hybrid Monte Carlo).

TODO: FIX THIS! 5.0pt

Questions?

- [] S. Borsanyi, S. Durr, Z. Fodor, C. Hoelbling, S. D. Katz, S. Krieg, T. Kurth, L. Lellouch, T. Lippert, C. McNeile, and K. K. Szabo. High-precision scale setting in lattice QCD. *Journal of High Energy Physics*, 2012(9), September 2012. ISSN 1029-8479. doi: 10.1007/JHEP09(2012)010. URL <http://arxiv.org/abs/1203.4469>. arXiv: 1203.4469.
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