Biçimsel Diller ve Otomata Teorisi

Sunu IX
Düzenli Diller

İZZET FATİH ŞENTÜRK

Closure Properties

- A language that can be defined by a RE is called a **regular language**
- Are all languages regular?
  - The answer is no
- Before proving this fact, let us discuss some of the properties of regular languages

Properties of Regular Languages

- If L_1 and L_2 are regular languages then
  - $L_1 + L_2$ (The language of all words either in L_1 or L_2)
  - L_1L_2 (The language of all words formed by concatenating a word from L_1 with a word from L_2)
  - $L_1{}^*$ are also regular languages (Strings that are concatenation of arbitrarily many factors from L_1)
- The set of regular languages is *closed* under union, concatenation, and Kleene closure

Proof 1 – By Regular Expressions

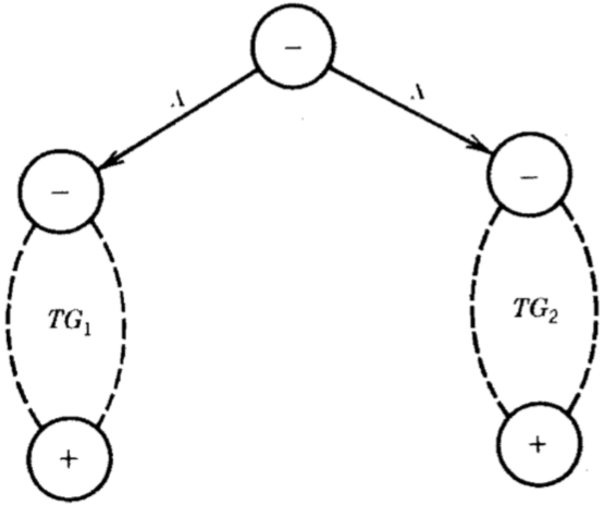
- If L_1 and L_2 are regular languages, there are regular expressions r_1 and r_2 that define these languages. Then $(r_1 + r_2)$ is a RE that defines the language $L_1 + L_2$. The language L_1L_2 can be defined by the RE r_1r_2. The language L_1^* can be defined by the RE $(r_1)^*$
  - All three of these sets of words are definable by regular expressions and so are themselves regular languages

- Regular languages can also be defined in terms of machines

Proof 2 – By Machines

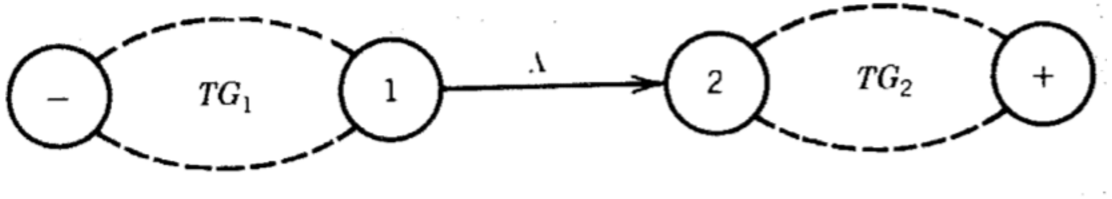
- L_1 and L_2 are regular languages
  - There must be TGs that accept them
  - Let TG_1 accept L_1 and TG_2 accept L_2
  - TG_1 and TG_2 each have a unique start state and a unique separate final state (If this is not the case originally, we can modify TGs so that it becomes true)

The TG accepts the language $L_1 + L_2$
This machines prove that $L_1 + L_2$ is regular

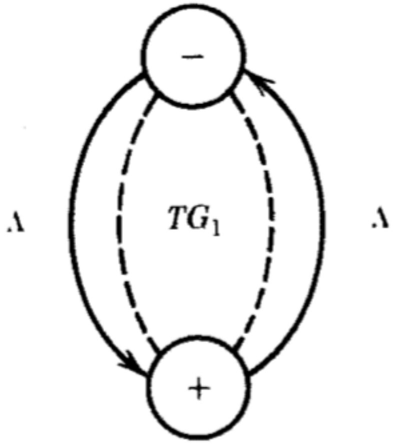


Proof 2 – By Machines

The TG accepts the language L_1L_2
1 is the former + of TG_1 and 2 is the former – of TG_2

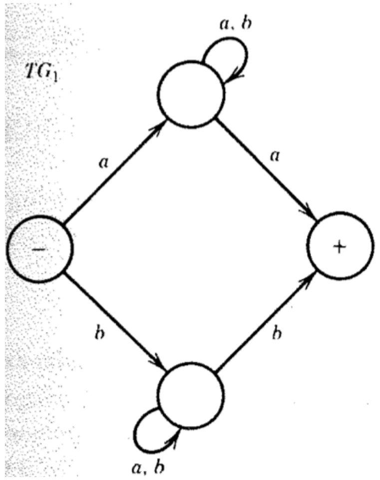


The TG accepts the language L_1^*
We begin at the – of TG_1 and trace a path to the + of TG_1



Example

- Let the alphabet be $\Sigma = \{a\ b\}$ and
  - L_1 = all words of two or more letters that begin and end with the same letter
  - L_2 = all words that contain the substring *aba*
- We will use the following TGs and REs

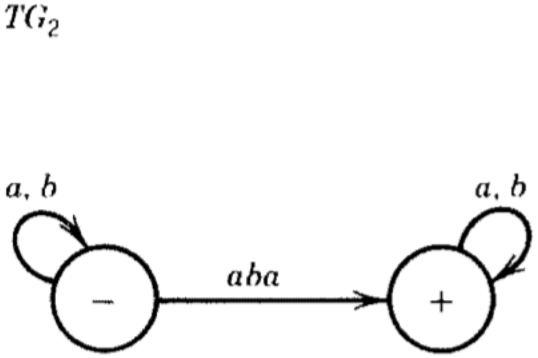


$\mathbf{r}_1$

$\mathbf{a(a + b)^*a + b(a + b)^*b}$

$\mathbf{r}_2$

$\mathbf{(a + b)^*aba(a + b)^*}$



Example

- The language $L_1 + L_2$ is regular because it can be defined by the RE

$$[a(a + b)^*a + b(a + b)^*b] + [(a + b)^*aba(a + b)^*]$$

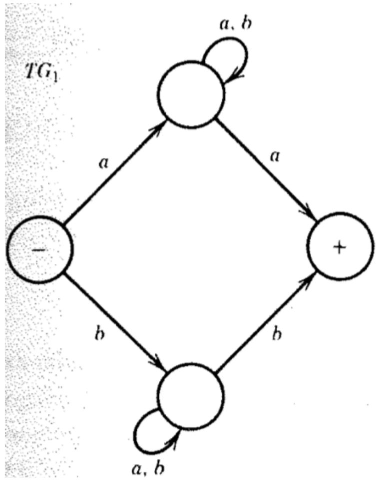
For the purpose of clarity, we have employed brackets instead of nested parantheses

$$\overset{r_1}{a(a + b)^*a + b(a + b)^*b}$$

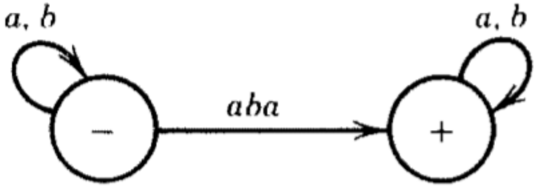
$$\overset{r_2}{(a + b)^*aba(a + b)^*}$$

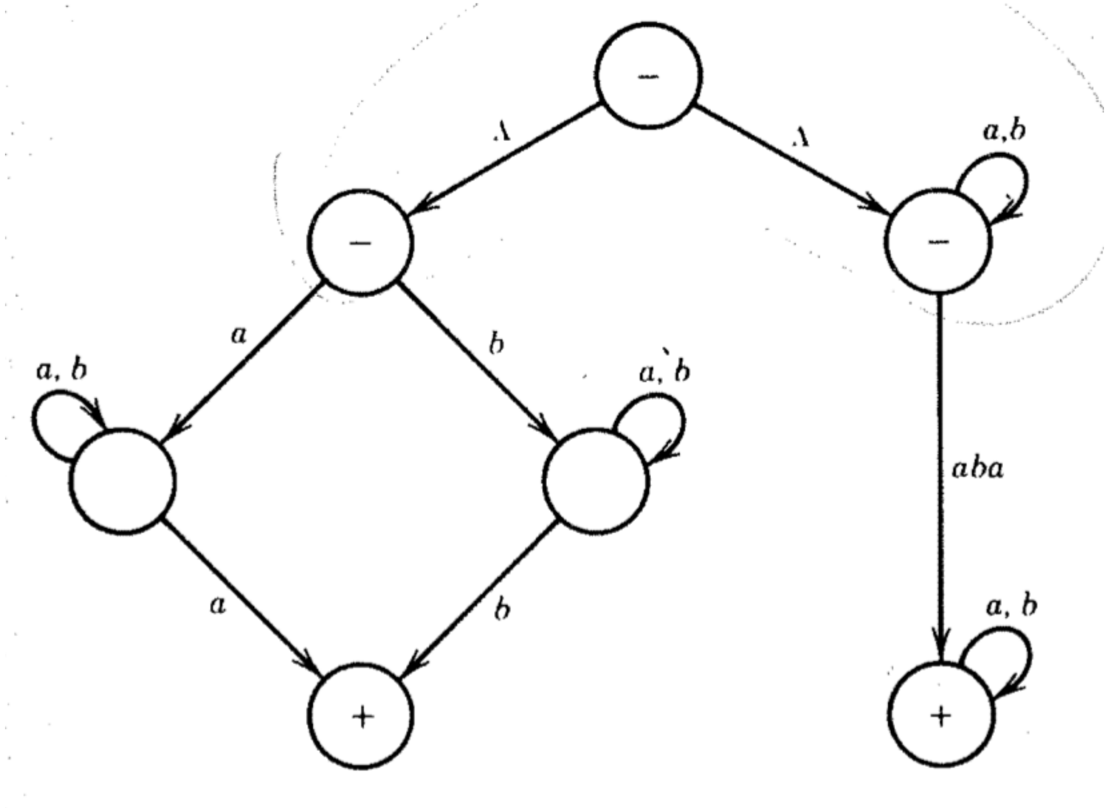
Example

$L_1 + L_2$ is accepted by the following TG



TG_2





Example

- The language L_1L_2 is regular because it can be defined by the RE

$$[\mathbf{a}(\mathbf{a}+\mathbf{b})^*\mathbf{a}+\mathbf{b}(\mathbf{a}+\mathbf{b})^*\mathbf{b}]\quad[(\mathbf{a}+\mathbf{b})^*\mathbf{aba}(\mathbf{a}+\mathbf{b})^*]$$

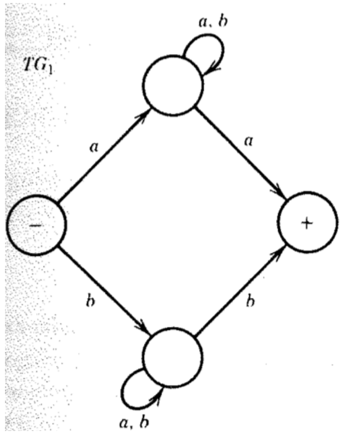
$\mathbf{r}_1$

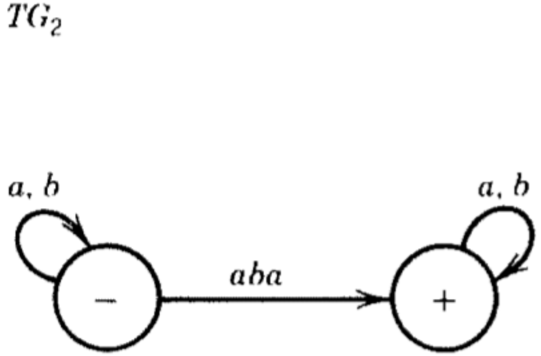
$$\mathbf{a}(\mathbf{a}+\mathbf{b})^*\mathbf{a}+\mathbf{b}(\mathbf{a}+\mathbf{b})^*\mathbf{b}$$

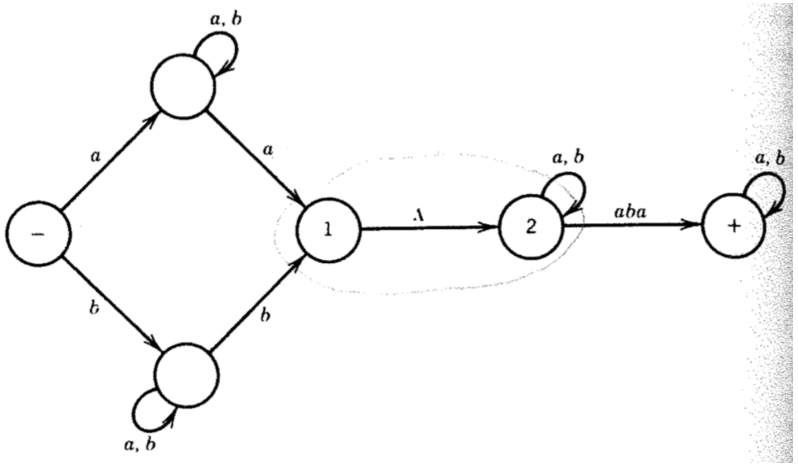
$\mathbf{r}_2$

$$(\mathbf{a}+\mathbf{b})^*\mathbf{aba}(\mathbf{a}+\mathbf{b})^*$$

- The language is accepted by the TG





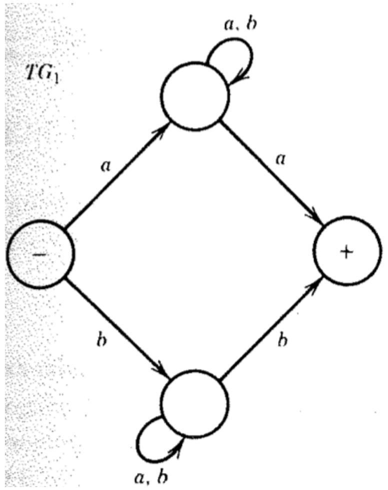


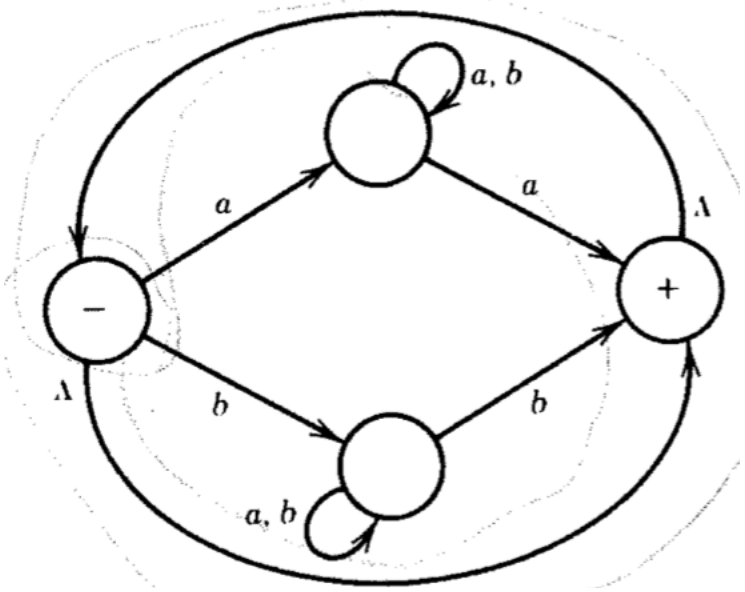
Example

- The language L_1^* is regular because it can be defined by the RE

$$[a(a + b)^*a + b(a + b)^*b]^*$$

- The language is accepted by the TG





Complements and Intersections

- If L is a language over the alphabet Σ, we define its **complement**, L', to be the language of all strings of letters from Σ that are not words in L
- Example: If L is the language over the alphabet Σ = {*a b*} of all words that have a double *a* in them, then L' is the language of all words that do not have a double *a*
- The complement of the language L' is the language L
    - (L')' = L
- If L is a regular language, then L' is also a regular language
    - The set of regular languages is closed under complementation

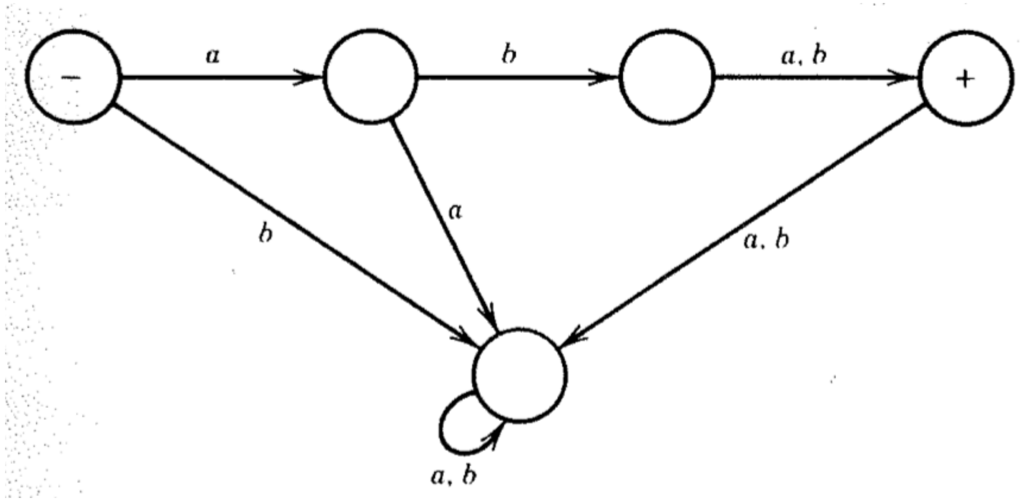
Complements

- If L is a regular language, then L’ is also a regular language
- Proof: If L is a regular language, there is some FA that accepts the language L
  - Some of the states of this FA are final states and, most likely, some are not
  - Let us reverse the final status of each state; that is, if it was a final state, make it a nonfinal state, and if it was a nonfinal state, make it a final state

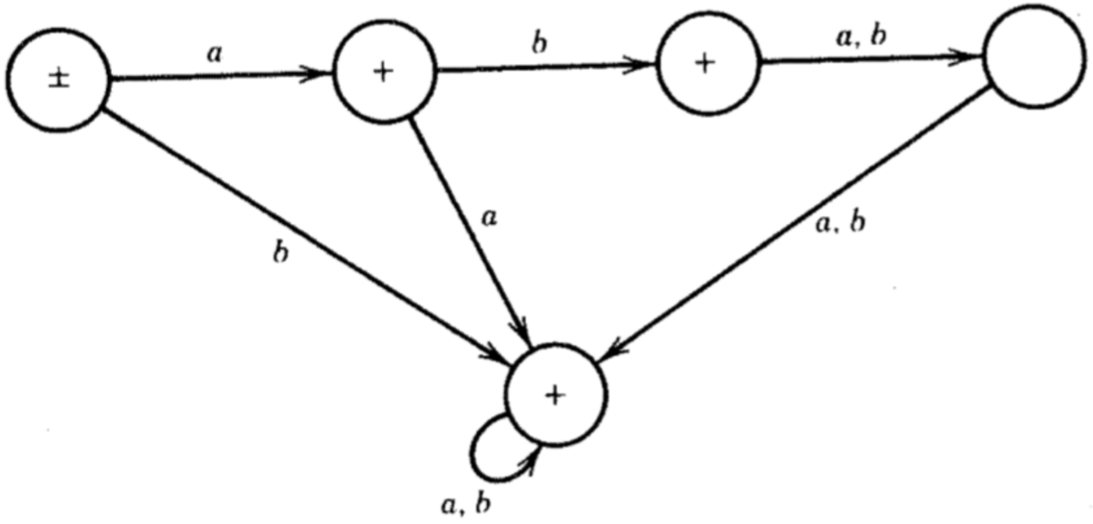
    $- \leftrightarrow \pm$
  - If an input string formerly ended in a nonfinal state, it now ends in a final state and vice versa
  - This new machine we have built accepts all input strings that were not accepted by the original FA and rejects all the input strings that the FA used to accept

Example

- An FA that accepts only the strings *aba* and *abb*



- An FA that accepts all strings other than *aba* and *abb*

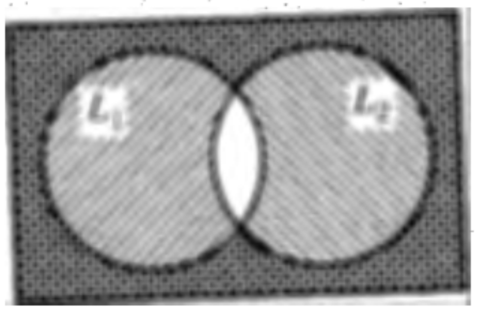


Intersections

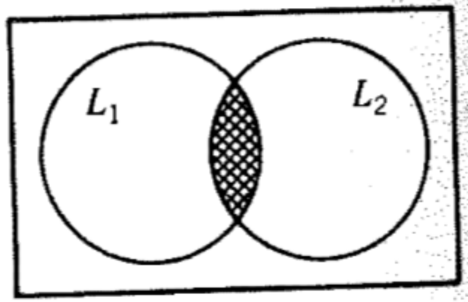
- If L_1 and L_2 are regular languages, then $L_1 \cap L_2$ is also a regular language
  - The set of regular languages is closed under intersection
- Proof: By DeMorgan's law for sets of any kind (regular languages or not)
  - $L_1 \cap L_2 = (L_1' + L_2')'$

L_1' and L_2' are regular
$L_1' + L_2'$ is regular

$(L_1' + L_2') =$



$(L_1' + L_2')' =$

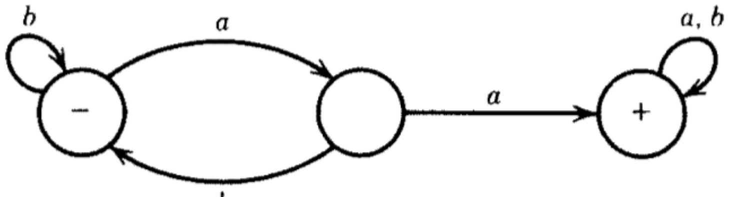


$= L_1 \cap L_2$

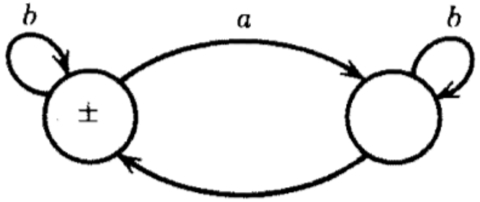
Example

- Two languages, Σ = {a b}
  - L_1 = all strings with a double *a*
  - L_2 = all strings with an even number of *a*'s
- These languages are not the same
  - *aaa* is in L_1 but not in L_2
  - *aba* is in L_2 but not in L_1
- They are both regular languages
  - $\mathbf{r_1} = (\mathbf{a} + \mathbf{b})^*\mathbf{aa}(\mathbf{a} + \mathbf{b})^*$
  - $\mathbf{r_2} = \mathbf{b}^*(\mathbf{ab}^*\mathbf{ab}^*)^*$

FA_1



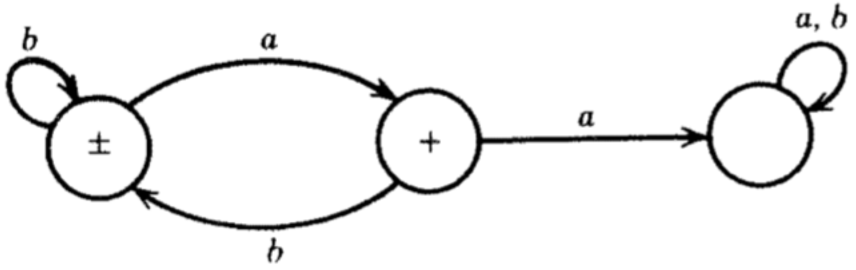
FA_2



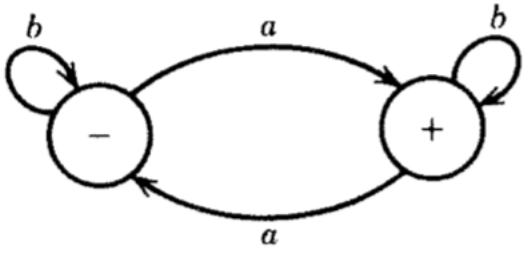
Example

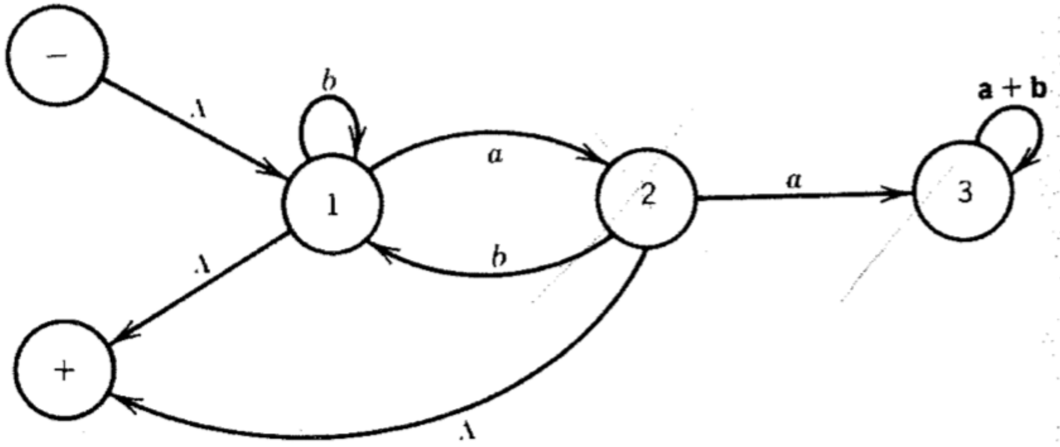
- The first step for $L_1 \cap L_2$ is to find L_1' and L_2'

FA_1'



FA_2'

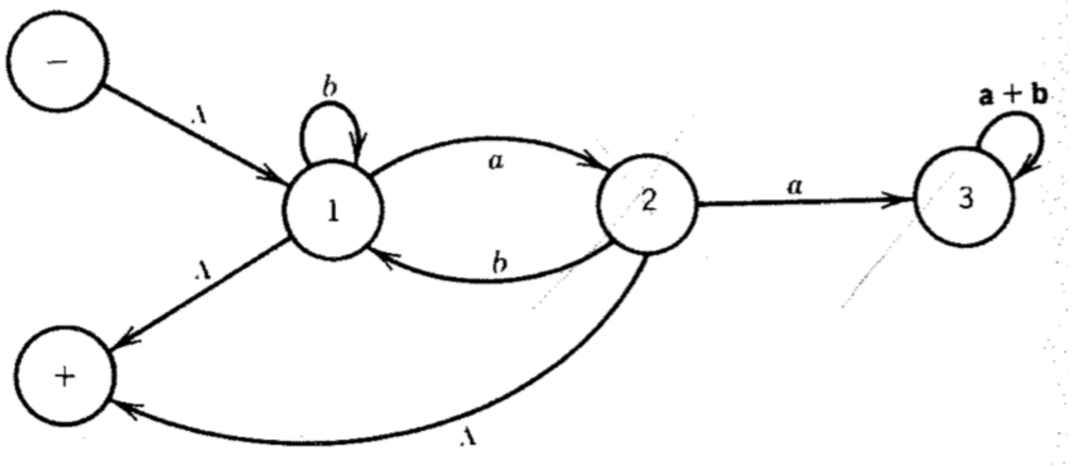


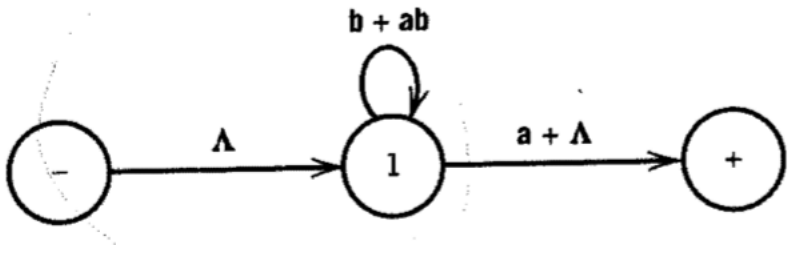


State 3 is part of no path from – to +, it can be dropped

Example

- To bypass state 2, we need to join the incoming a-edge with both outgoing edges (b-edge to 2 and Λ-edge to +)
  - When we add the two loops, we get **b** + **ab** and the sum of the two edges from 1 to + is a + Λ



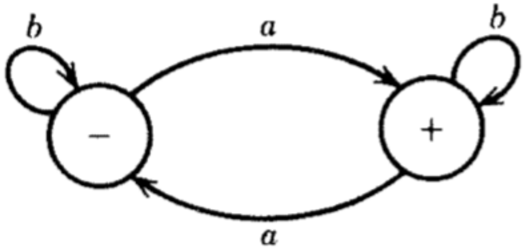


r_1’ = (**b** + **ab**)*(**a** + **Λ**)

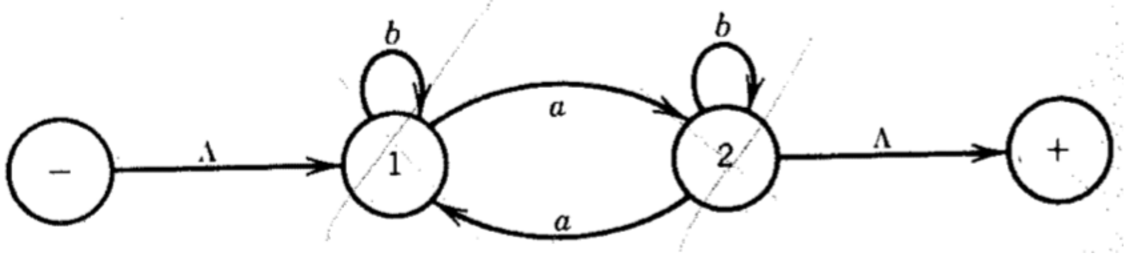
Example

- Let us do the same thing for L_2'

FA_2'

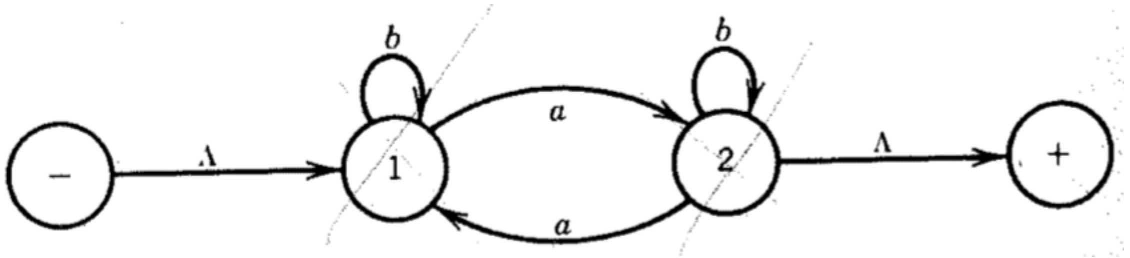


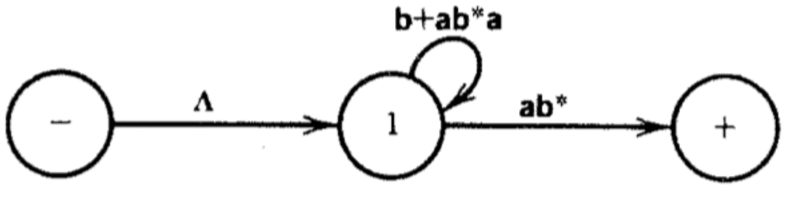
- FA_2' becomes



Example

- Start simplification by eliminating state 2
  - There is one incoming edge, a loop, and two outgoing edges
  - We need to replace them with only two edges:
    - The path 1-2-2-1 becomes a loop at 1
    - The path 1-2-2-+ becomes an edge from 1 to +
  - After bypassing state 2 and adding the two loop labels, we have





$$r'_2 = (b + ab^*a)^*ab^*$$

Example

- We now have regular expressions for L_1' and L_2' and we can write L_1' + L_2'

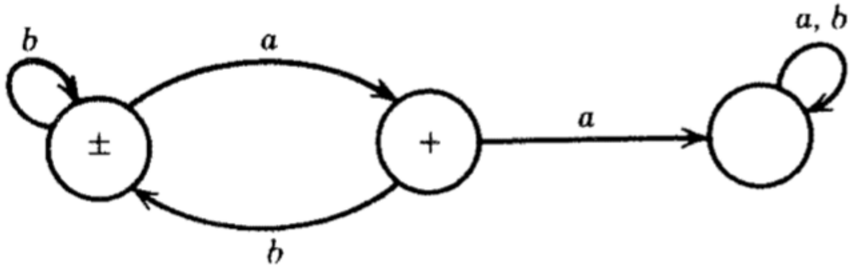
$$\mathbf{r}_1' + \mathbf{r}_2' = (\mathbf{b} + \mathbf{ab})^*(\Lambda + \mathbf{a}) + (\mathbf{b} + \mathbf{ab^*a})^*\mathbf{ab^*}$$

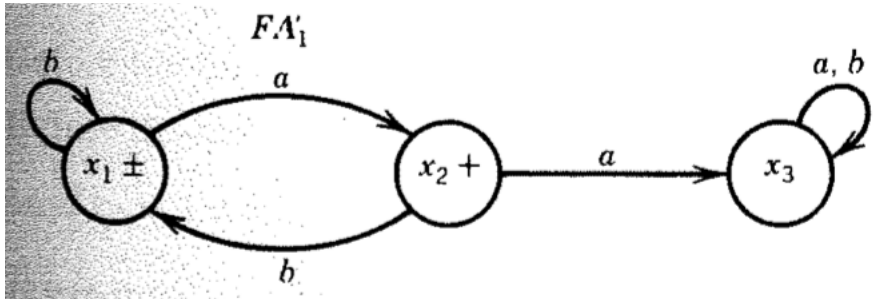
- We must now go in the other direction and make this RE into an FA so that we can take its complement to get the FA that defines $L_1 \cap L_2$
- To build the FA that corresponds to a complicated RE is mo picnic, but it can be done. However, we can find a better way

Example

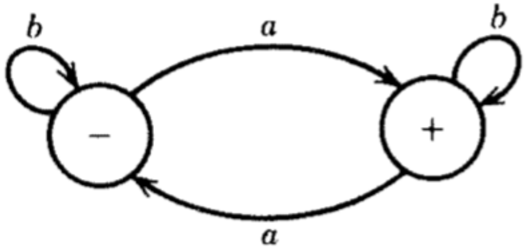
- An alternative approach is to make the machine for L_1' + L_2' directly from the machines for L_1' and L_2' without resorting to regular expressions

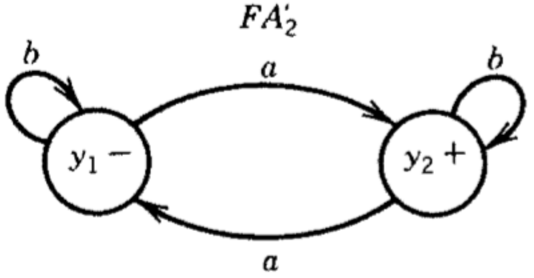
FA'_1





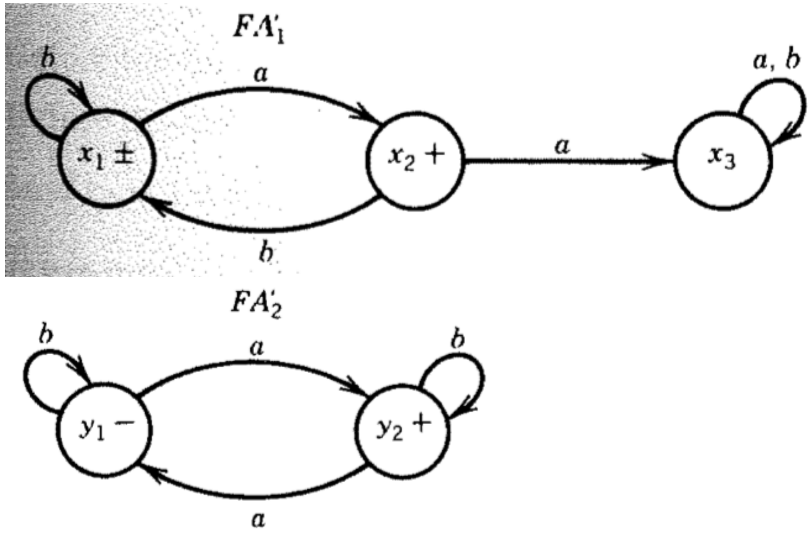
FA'_2





Example

- The start states are x_1 and y_1 and the final states are x_1, x_2, and y_2. The six possible combination states are



$z_1 = x_1$ or y_1 start, final (words ending here are accepted in FA'_1)

$z_2 = x_1$ or y_2 final (words ending here are accepted on FA'_1 and FA'_2)

$z_3 = x_2$ or y_1 final (words ending here are accepted on FA'_1)

$z_4 = x_2$ or y_2 final (words ending here are accepted on FA'_1 and FA'_2)

$z_5 = x_3$ or y_1 not final on either machine

$z_6 = x_3$ or y_2 final (words ending here are accepted on FA'_2)

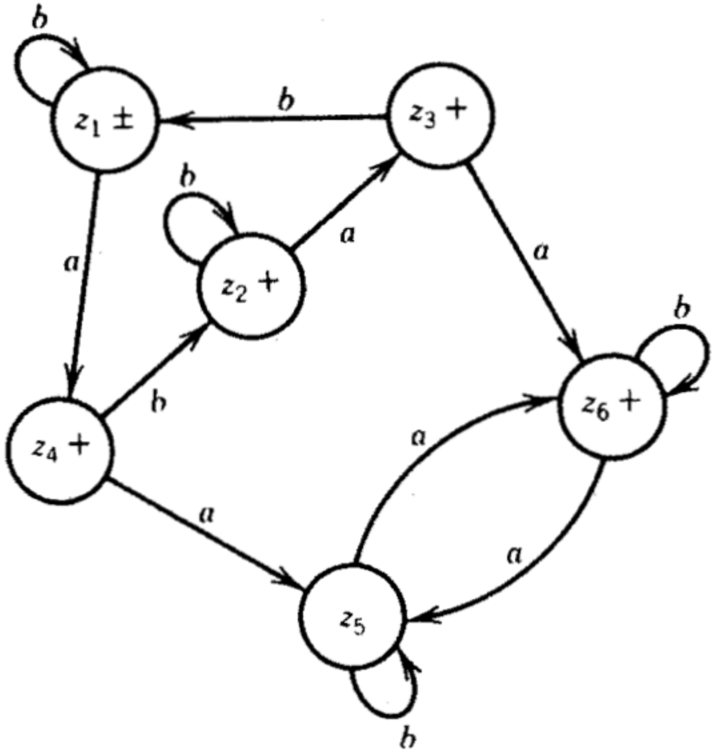
- The transition table for this machine is

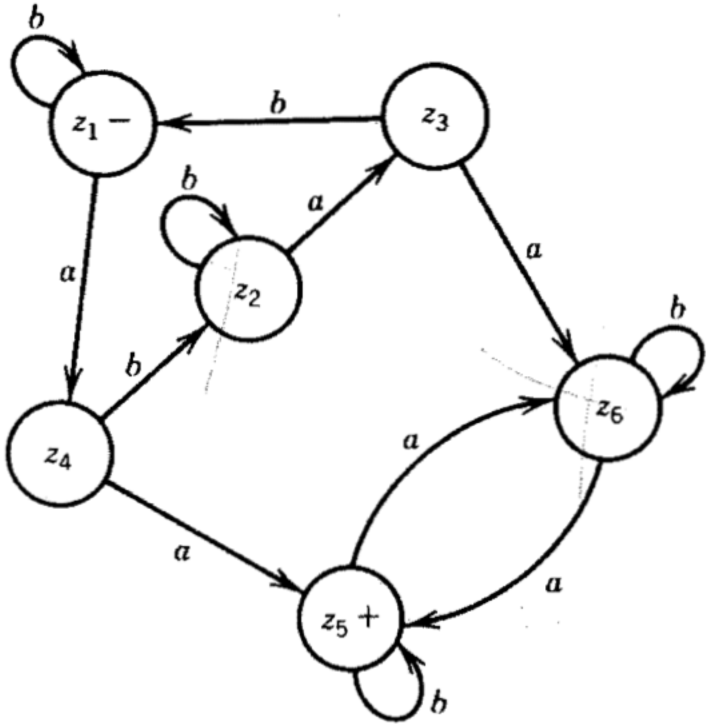
	a	b
$\pm z_1$	z_4	z_1
$+ z_2$	z_3	z_2
$+ z_3$	z_6	z_1
$+ z_4$	z_5	z_2
z_5	z_6	z_5
$+ z_6$	z_5	z_6

Example

- The union machine can be pictured as

	a	b
$\pm z_1$	z_4	z_1
$+ z_2$	z_3	z_2
$+ z_3$	z_6	z_1
$+ z_4$	z_5	z_2
z_5	z_6	z_5
$+ z_6$	z_5	z_6

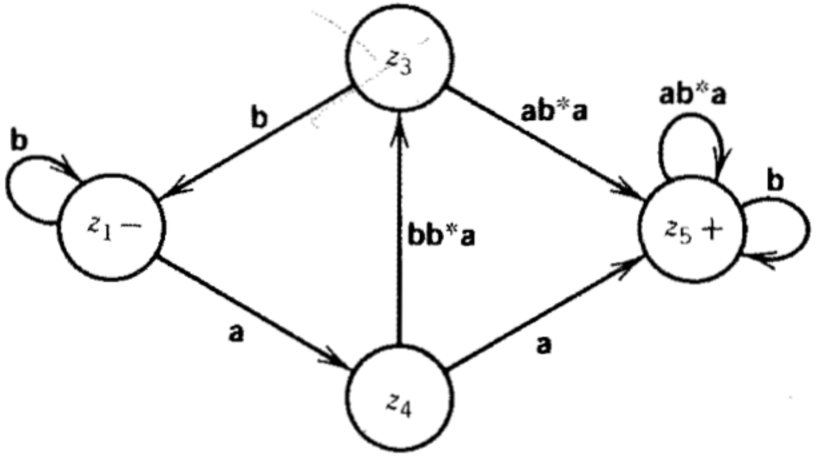




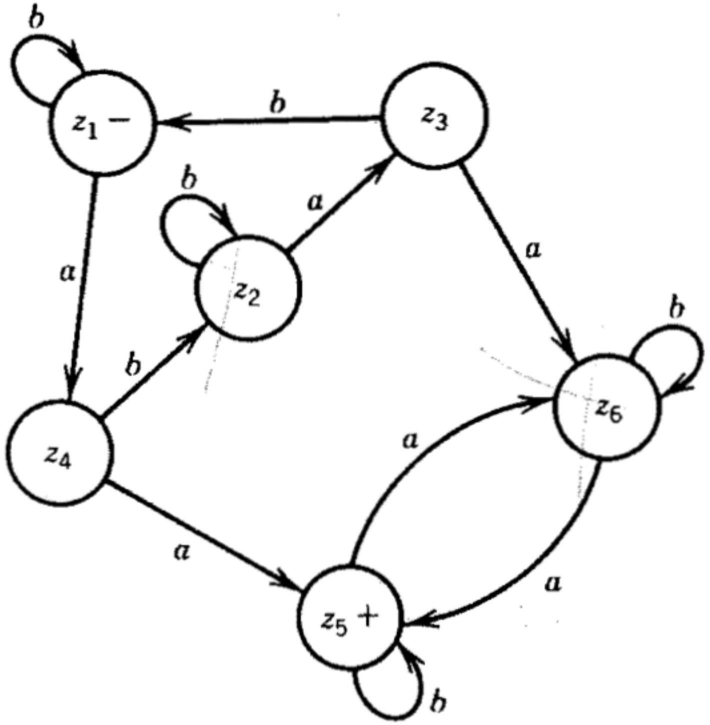
- This is an FA that accepts the language $L_1' + L_2'$. We produce an FA for the language $L_1' + L_2'$ when we reverse the status of each state from final to nonfinal and vice versa

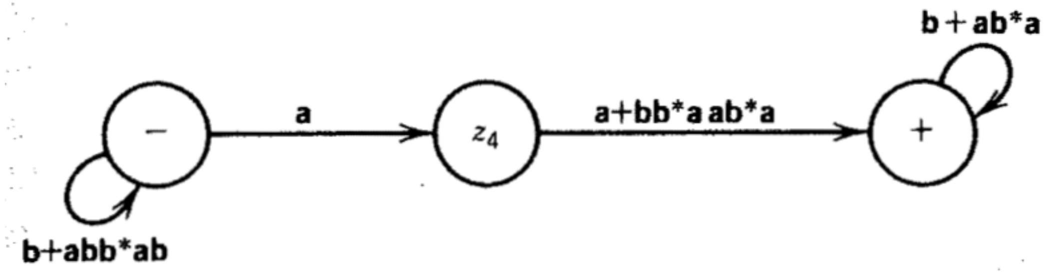
Example

- Bypassing z_2 and z_6 gives



- Then bypassing z_3 gives





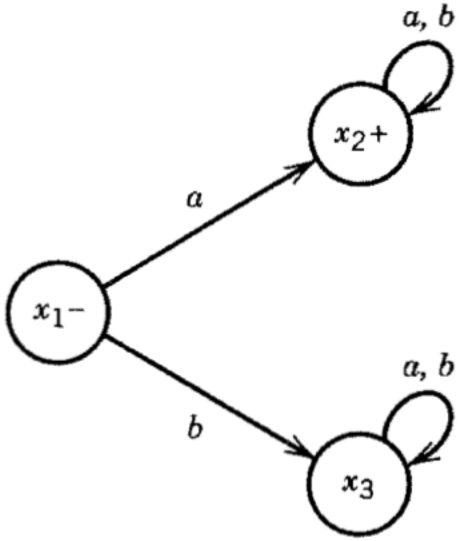
$$\boxed{(b + abb^*ab)^*a(a + bb^*aab^*a)(b + ab^*a)^*}$$

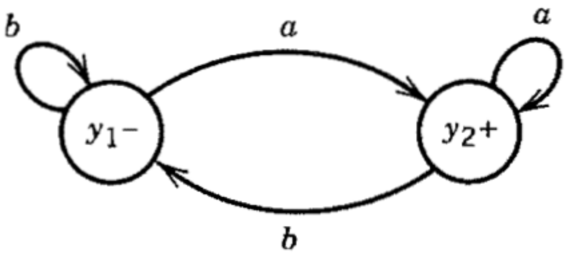
Example

- L_1 = all words that begin with an a
- L_2 = all words that end with an a
- $r_1 = \mathbf{a}(\mathbf{a} + \mathbf{b})^*$
- $r_2 = (\mathbf{a} + \mathbf{b})^*\mathbf{a}$
- The intersection language will be
  - $L_1 \cap L_2$ = all words that begin and end with the letter *a*
  - Obviously regular, it can be defined by the RE: $\mathbf{a}(\mathbf{a} + \mathbf{b})^*\mathbf{a} + \mathbf{a}$

Example

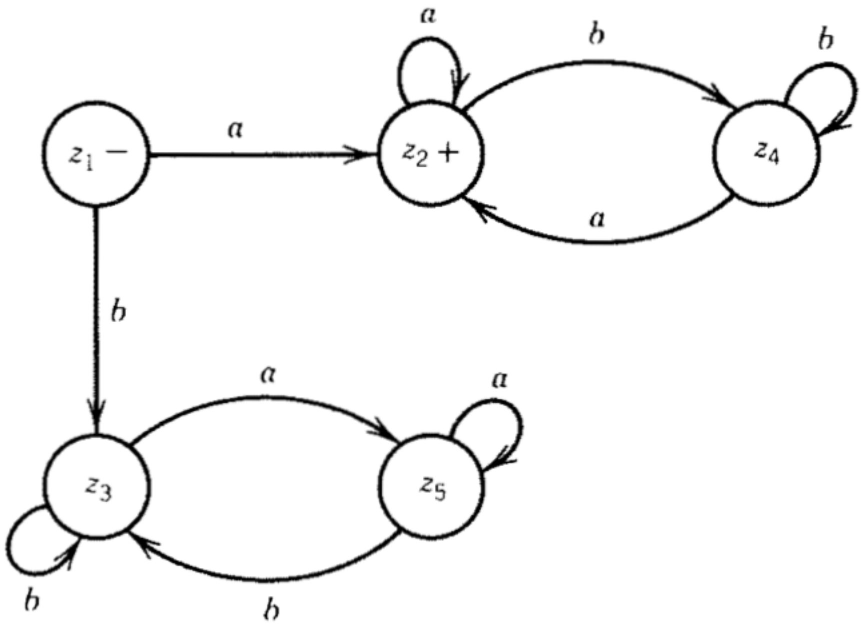
- $r_1 = \mathbf{a}(\mathbf{a} + \mathbf{b})^*$
- $r_2 = (\mathbf{a} + \mathbf{b})^*\mathbf{a}$





We now build the transition table of the machine that runs its input strings on FA_1 and FA_2 simultaneously

	State	Read *a*	Read *b*
$-\ z_1$	x_1 or y_1	x_2 or y_2	x_3 or y_1
$+\ z_2$	x_2 or y_2	x_2 or y_2	x_2 or y_1
z_3	x_3 or y_1	x_3 or y_2	x_3 or y_1
z_4	x_2 or y_1	x_2 or y_2	x_2 or y_1
z_5	x_3 or y_2	x_3 or y_2	x_3 or y_1



We construct the machine for $L_1 \cap L_2$ = all words in both L_1 and L_2 we put + only in the state that represents acceptance by both machines at once