Parçalı Sürekli Fonksiyonların Laplace Dönüşümü

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Süreksiz ve Periyodik...

SÜREKSİZ VE PERİYODİK FONKSİYONLARIN LAPLACE DÖNÜŞÜMÜ

TRANSFORMS OF DISCONTINUOUS AND PERIODIC FUNCTIONS

In this section we study special functions that often arise when the method of Laplace transforms is applied to physical problems. Of particular interest are methods for handling functions with jump discontinuities. Jump discontinuities occur naturally in physical problems such as electric circuits with on/off switches. To handle such behavior, Oliver Heaviside introduced the following step function.

Unit Step Function

Definition 5. The **unit step function** $u(t)$ is defined by

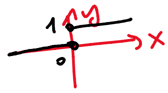
$$(1) \qquad u(t) := \begin{cases} 0\,, & t < 0\,, \\ 1\,, & 0 < t\,. \end{cases}$$

By shifting the argument of $u(t)$, the jump can be moved to a different location. That is,

$$(2) \qquad u(t-a) = \begin{cases} 0\,, & t-a < 0\,, \\ 1\,, & 0 < t-a \end{cases} \quad = \begin{cases} 0\,, & t < a \\ 1\,, & a < t \end{cases}$$

has its jump at $t = a$. By multiplying by a constant M, the height of the jump can also be modified:

$$Mu(t-a) = \begin{cases} 0\,, & t < a\,, \\ M\,, & a < t\,. \end{cases}$$



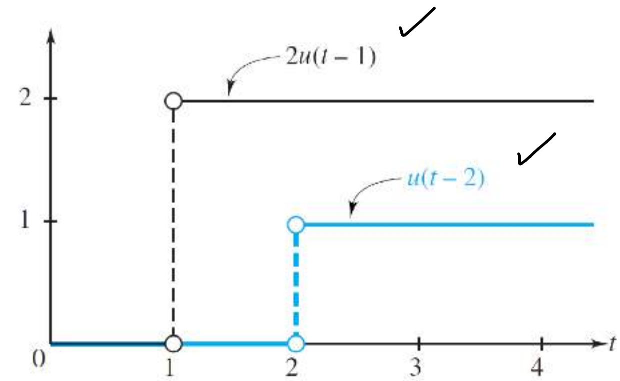


Figure 7.8 Two-step functions expressed using the unit step function

To express piecewise continuous functions, we employ the rectangular window, which turns the step function on and then turns it back off.

Rectangular Window Function

Definition 6. The **rectangular window function** $\Pi_{a,b}(t)$ is defined by†

(3) $$\Pi_{a,b}(t) := u(t-a) - u(t-b) = \begin{cases} 0, & t < a, \\ 1, & a < t < b, \\ 0, & b < t. \end{cases}$$ ✓

The function $\Pi_{a,b}(t)$ is displayed in Figure 7.9, and Figure 7.10, illustrating multiplication of a function by $\Pi_{a,b}(t)$, justifies its name.

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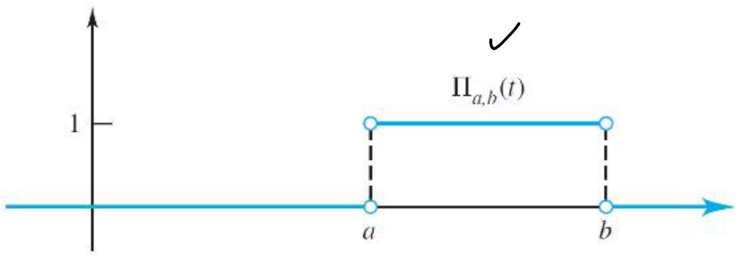

Figure 7.9 The rectangular window

$0 < \infty$

$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

Example 1 Write the function

(4) $$f(t) = \begin{cases} 3, & 0 < t < 2 \\ 1, & 2 < t < 5, \\ t, & 5 < t < 8, \\ t^2/10, & 8 < t \end{cases}$$

1, $t > 8$
0, $t < 8$

(see Figure 7.11 on page 386) in terms of window and step functions.

Clearly, from the figure we want to window the function in the intervals (0, 2), (2, 5), and (5, 8), and to introduce a step for $t > 8$. From (5) we read off the desired representation as

(5) $$f(t) = 3\Pi_{0,2}(t) + 1\Pi_{2,5}(t) + t\Pi_{5,8}(t) + (t^2/10)u(t-8). \quad \blacklozenge$$

The Laplace transform of $u(t-a)$ with $a \geq 0$ is

(6) $$\boxed{\mathcal{L}\{u(t-a)\}(s) = \frac{e^{-as}}{s},}$$ ***

$\mathcal{L}\{u(t)\} = \frac{1}{s}$

$t > a$, 1

since, for $s > 0$,

$$\mathcal{L}\{u(t-a)\}(s) = \int_0^\infty e^{-st}u(t-a)\,dt = \int_a^\infty e^{-st}\,dt$$

$$= \lim \frac{-e^{-st}}{\ } \Big|^N = \frac{e^{-as}}{\ }$$

$$= \lim_{N\to\infty} \frac{-e^{-st}}{s}\Big|_a^N = \frac{e^{-as}}{s} \,. \quad //$$

Conversely, for $a > 0$, we say that the piecewise continuous function $u(t - a)$ is an inverse Laplace transform for e^{-as}/s and we write[†]

$$\boxed{\mathscr{L}^{-1}\left\{\frac{e^{-as}}{s}\right\}(t) = u(t - a) \,.} \quad \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$$

For the rectangular window function, we deduce from (6) that

(7) $\mathscr{L}\{\Pi_{a,b}(t)\}(s) = \mathscr{L}\{u(t - a) - u(t - b)\}(s) = [e^{-sa} - e^{-sb}]/s \,, 0 < a < b. \quad //$

The translation property of $F(s)$ discussed in Section 7.3 described the effect on the Laplace transform of multiplying a function by e^{at}. The next theorem illustrates an analogous effect of multiplying the Laplace transform of a function by e^{-as}.

$\mathscr{L}\{f(t)\} = F(s)$

Translation in t

Theorem 8. Let $F(s) = \mathscr{L}\{f\}(s)$ exist for $s > \alpha \geq 0$. If a is a positive constant, then

(8) $\mathscr{L}\{f(t - a)u(t - a)\}(s) = e^{-as}F(s) \,;$

and, conversely, an inverse Laplace transform[††] of $e^{-as}F(s)$ is given by

(9) $\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = f(t - a)u(t - a) \,.$

$\mathscr{L}^{-1}\{F(s)\} = f(t)$

Proof. By the definition of the Laplace transform, we have

$$\mathscr{L}\{f(t - a)u(t - a)\}(s) = \int_0^\infty e^{-st}f(t - a)u(t - a)\,dt$$
$$= \int_a^\infty e^{-st}f(t - a)\,dt \,,$$

where, in the last equation, we used the fact that $u(t - a)$ is zero for $t < a$ and equals 1 for $t > a$. Now let $v = t - a$. Then we have $dv = dt$, and equation (10) becomes

$t > a$. Now let $v = t - a$. Then we have $dv = dt$, and equation (10) becomes

$$\mathcal{L}\{f(t-a)u(t-a)\}(s) = \int_0^\infty e^{-as}e^{-sv}f(v)dv$$

$$= e^{-as}\int_0^\infty e^{-sv}f(v)dv = e^{-as}F(s) \ . \ \blacklozenge$$

Notice that formula (8) includes as a special case the formula for $\mathcal{L}\{u(t-a)\}$; indeed, if we take $f(t) \equiv 1$, then $F(s) = 1/s$ and (8) becomes $\mathcal{L}\{u(t-a)\}(s) = e^{-as}/s$.

In practice it is more common to be faced with the problem of computing the transform of a function expressed as $g(t)u(t-a)$ rather than $f(t-a)u(t-a)$. To compute $\mathcal{L}\{g(t)u(t-a)\}$, we simply identify $g(t)$ with $f(t-a)$ so that $f(t) = g(t+a)$. Equation (8) then gives

(11) $$\boxed{\mathcal{L}\{g(t)u(t-a)\}(s) = e^{-as}\mathcal{L}\{g(t+a)\}(s) \ .}$$

$e^{-a}(F(s))$

Example 2 Determine the Laplace transform of $t^2u(t-1)$.

$\mathcal{L}\{t^2 u(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2\}$

$\mathcal{L}\{t^2+2t+1\}$

$\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}$

Solution To apply equation (11), we take $g(t) = t^2$ and $a = 1$. Then

$$g(t+a) = g(t+1) = (t+1)^2 = t^2 + 2t + 1 \ .$$

Now the Laplace transform of $g(t+a)$ is

$$\mathcal{L}\{g(t+a)\}(s) = \mathcal{L}\{t^2 + 2t + 1\}(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \ .$$

So, by formula (11), we have

$$\mathcal{L}\{t^2u(t-1)\}(s) = e^{-s}\left\{\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right\} \ . \ \blacklozenge$$

Example 3 Determine $\mathscr{L}\{(\cos t)u(t-\pi)\}$. $= e^{-\pi s}\ \mathcal{L}\{\cos(t+\pi)\}$

$\cos t \cdot \cancel{\cos\pi} - \sin t\ \cancel{\sin\pi}$

$-1 \qquad\qquad\qquad 0$

Solution Here $g(t) = \cos t$ and $a = \pi$. Hence,

$$g(t+a) = g(t+\pi) = \cos(t+\pi) = -\cos t\ ,$$

and so the Laplace transform of $g(t+a)$ is

$\mathcal{L}\{-\cos t\} = -\mathcal{L}\{\cos t\}$

$$\mathscr{L}\{g(t+a)\}(s) = -\mathscr{L}\{\cos t\}(s) = -\frac{s}{s^2+1}\ .$$

Thus, from formula (11), we get

$$\mathscr{L}\{(\cos t)u(t-\pi)\}(s) = -e^{-\pi s}\frac{s}{s^2+1}\ . \ \blacklozenge$$

Example 4 Determine $\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$ and sketch its graph.

$\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

Solution To use the translation property (9), we first express e^{-2s}/s^2 as the product $e^{-as}F(s)$. For this purpose, we put $e^{-as} = e^{-2s}$ and $F(s) = 1/s^2$. Thus, $a = 2$ and

$$f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}(t) = t \ .$$

$(t-2).u(t-2)$

It now follows from the translation property that

$$\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}(t) = f(t-2)u(t-2) = (t-2)u(t-2) \ .$$

See Figure 7.12. ◆

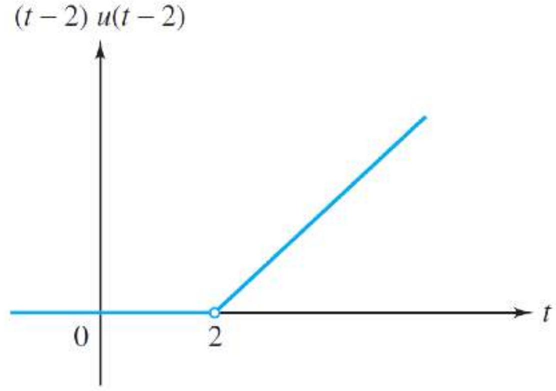

Figure 7.12 Graph of solution to Example 4

Example 5 The current I in an LC series circuit is governed by the initial value problem

$$\textbf{(12)} \qquad I''(t) + 4I(t) = g(t) \;; \qquad I(0) = 0 \;, \qquad I'(0) = 0 \;,$$

where $1 \cdot \Pi_{0,1}(t) - \Pi_{1,2}(t) + \cancel{0 \cdot u(t-2)}$ $\quad = u(t-0) - u(t-1)$

$u(t-1) - u(t-2)$

$$g(t) := \begin{cases} 1 \;, & 0 < t < 1 \;, \\ -1 \;, & 1 < t < 2 \;, \\ 0 \;, & 2 < t \;. \end{cases}$$

$\mathcal{L}\{I'' + 4I\} = \mathcal{L}\{g(t)\}$

$\mathcal{L}\{I''\} = s^2 J(s) - s\underbrace{I(0)}_{0} - \underbrace{I'(0)}_{0}$

$= s^2 J(s)$

Determine the current as a function of time t.

Solution Let $J(s) := \mathcal{L}\{I\}(s)$. Then we have $\mathcal{L}\{I''\}(s) = s^2 J(s)$.

Writing $g(t)$ in terms of the rectangular window function $\Pi_{a,b}(t) = u(t-a) - u(t-b)$, we get

$$g(t) = \Pi_{0,1}(t) + (-1)\Pi_{1,2}(t) = u(t) - u(t-1) - [u(t-1) - u(t-2)]$$
$$= u(t) - 2u(t-1) + u(t-2),$$

$\mathcal{L}\{g(t)\} =$

and so

$$\boxed{\mathcal{L}\{g\}(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}\;.}$$

Thus, when we take the Laplace transform of both sides of (12), we obtain

$$\mathscr{L}\{I''\}(s) + 4\mathscr{L}\{I\}(s) = \mathscr{L}\{g\}(s)$$

$(s^2+4)J(s)$

$$s^2J(s) + 4J(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}$$

$$J(s) = \frac{1}{s(s^2+4)} - \frac{2e^{-s}}{s(s^2+4)} + \frac{e^{-2s}}{s(s^2+4)} .$$

To find $I = \mathscr{L}^{-1}\{J\}$, we first observe that

$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+4)} - \frac{2e^{-s}}{s(s^2+4)} + \frac{e^{-2s}}{s(s^2+4)}\right\}$$

$$J(s) = F(s) - 2e^{-s}F(s) + e^{-2s}F(s) ,$$

where

$F(s)$

$$F(s) := \frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{4}\left(\frac{s}{s^2+4}\right) .$$

$$\frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$

(s^2+4) (s)

Computing the inverse transform of $F(s)$ gives

$$f(t) := \mathscr{L}^{-1}\{F\}(t) = \frac{1}{4} - \frac{1}{4}\cos 2t .$$

$1 = A(s^2+4) + (Bs+C).s$

$1 = As^2 + 4A + Bs^2 + Cs$

$A+B = 0$, $B = -\frac{1}{4}$

$C = 0$

$4A = 1 \Rightarrow A = \frac{1}{4}$

Hence, via the translation property (9), we find

$$\begin{aligned} I(t) &= \mathscr{L}^{-1}\{F(s) - 2e^{-s}F(s) + e^{-2s}F(s)\}(t) \\ &= f(t) - 2f(t-1)u(t-1) + f(t-2)u(t-2) \\ &= \left(\frac{1}{4} - \frac{1}{4}\cos 2t\right) - \left[\frac{1}{2} - \frac{1}{2}\cos 2(t-1)\right]u(t-1) \\ &\quad + \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-2)\right]u(t-2) . \end{aligned}$$

$\mathscr{L}^{-1} = \frac{1}{4}\mathscr{L}^{-1}\{\frac{1}{s}\} - \frac{1}{4}\mathscr{L}^{-1}\{\frac{s}{s^2+4}\}$

The current is graphed in Figure 7.13. Note that $I(t)$ is smoother than $g(t)$; the former has discontinuities in its second derivative at the points where the latter has jumps. ◆

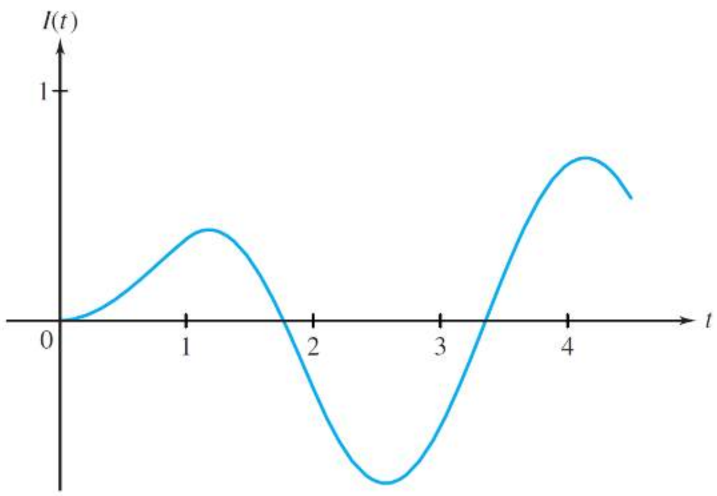

Figure 7.13 Solution to Example 5

Periodic Function

Definition 7. A function $f(t)$ is said to be **periodic of period T** ($\neq 0$) if

$$f(t+T) = f(t)$$

for all t in the domain of f.

$\cos(x+2\pi) = \cos x$ $\quad$ $\tan(x+\pi) = \tan x$

$\sin(x+2\pi) = \sin x$ $\quad$ $\cot(x+\pi) = \cot x$

As we know, the sine and cosine functions are periodic with period 2π and the tangent function is periodic with period π.† To specify a periodic function, it is sufficient to give its values over one period. For example, the square wave function in Figure 7.14 can be expressed as

$$f(t) := \begin{cases} 1\,, & 0 < t < 1\,, \\ -1\,, & 1 < t < 2\,, \end{cases} \quad \text{and } f(t) \text{ has period } 2. \tag{13}$$

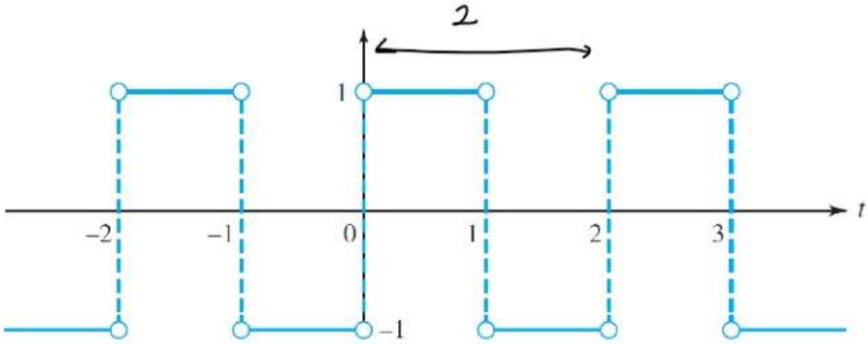

Figure 7.14 Graph of square wave function $f(t)$

$\int_0^\infty$, $\dfrac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$ **

Transform of Periodic Function

Theorem 9. If f has period T and is piecewise continuous on $[0, T]$, then the Laplace transforms $F(s) = \int_0^\infty e^{-st}f(t)\,dt$ and $F_T(s) = \int_0^T e^{-st}f(t)\,dt$ are related by

(15) $F_T(s) = F(s)[1 - e^{-sT}]$ or $\boxed{F(s) = \dfrac{F_T(s)}{1-e^{-sT}}}$.

Proof. From (14) and the periodicity of f, we have

(16) $f_T(t) = f(t)u(t) - f(t)u(t-T) = f(t)u(t) - f(t-T)u(t-T)$,

so taking transforms and applying (8) yields $F_T(s) = F(s) - e^{-sT}F(s)$, which is equivalent to (15). ◆

Example 6 Determine $\mathscr{L}\{f\}$, where f is the square wave function in Figure 7.14.

Solution Here $T = 2$. Windowing the function results in $f_T(t) = \Pi_{0,1}(t) - \Pi_{1,2}(t)$, so by (7) we get $F_T(s) = (1 - e^{-s})/s - (e^{-s} - e^{-2s})/s = (1 - e^{-s})^2/s$. Therefore (15) implies

$$\mathscr{L}\{f\} = \frac{(1 - e^{-s})^2/s}{1 - e^{-2s}} = \frac{1 - e^{-s}}{(1 + e^{-s})s} \ . \ ◆$$

For functions with power series expansions we can find their transforms by using the formula $\mathscr{L}\{t^n\}(s) = n!/s^{n+1}$, $n = 0, 1, 2, \ldots$.

Example 47: Find the laplace transform of the triangular wave of period 2a given by

$$f(t) = \begin{cases} t, & 0 < t < a \\ 2a - t, & a < t < 2a \end{cases}$$

$t\Pi_{0,a}(t) + (2a-t)\Pi_{a,2a}(t)$

$T = 2a$

$$\mathcal{L}\{f(t)\} = \frac{\int_0^{2a} e^{-st} f(t)\,dt}{1 - e^{-2as}} = \frac{1}{1 - e^{-2as}} \underbrace{\int_0^a e^{-st} \cdot t\,dt + \int_a^{2a} e^{-st}(2a - t)\,dt}_{\text{Kısmi integrasyon}}$$

$\mathcal{L}\{ u(t) - \cancel{u(t-a)} + \cancel{u(t-a)} - \mathcal{L}(u(t-2a)\}$

Example 49: Find the laplace transform of the square wave (or *meander*) function of a period a defined as: $f(t) = 1$ when $0 < t < a/2$

$= -1$, when $a/2 < t < a$ $F_T(s)$ [UP Tech, 2004]

$T = a$

$1 \cdot \Pi_{0,\frac{a}{2}}(t) + (-1)\,\Pi_{\frac{a}{2},a}(t)$

$$\mathcal{L}\{f(t)\} = \frac{\int_0^a e^{-st} f(t)\,dt}{1-e^{-as}} = \frac{\overbrace{\int_0^{\frac{a}{2}} e^{-st}\cdot 1\,dt + \int_{\frac{a}{2}}^{a} e^{-st}\cdot(-1)\,dt}^{F_T(s)}}{1-e^{-as}}$$

$$F_T(s) = -\frac{1}{s}e^{-st}\Big|_0^{a/2} - \left(-\frac{1}{s}e^{-st}\Big|_{\frac{a}{2}}^{a}\right)$$

$$= -\frac{1}{s}\left(e^{-s\frac{a}{2}} - \underbrace{e^{0}}_{1}\right) + \frac{1}{s}\left(e^{-sa} - e^{-s\frac{a}{2}}\right)$$

$$= -\frac{1}{s}e^{-s\frac{a}{2}} + \frac{1}{s} + \frac{1}{s}e^{-sa} - \frac{1}{s}e^{-s\frac{a}{2}} = \frac{1+e^{-sa}-2e^{-s\frac{a}{2}}}{s} \quad **$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}} = \frac{e^x-e^{-x}}{e^x+e^{-x}} = \frac{1-e^{-2x}}{1+e^{-2x}}$$

$$\mathcal{L}\{f(t)\} = \frac{F_T(s)}{1-e^{-as}} = \frac{e^{-sa}-2e^{-s\frac{a}{2}}+1}{s(1-e^{-as})} = \frac{\left(1-e^{-s\frac{a}{2}}\right)^{\cancel{2}}}{s\left(\cancel{1-e^{-s\frac{a}{2}}}\right)\left(1+e^{-s\frac{a}{2}}\right)} = \frac{\left(1-e^{-\frac{as}{2}}\right)}{s\left(1+e^{-\frac{as}{2}}\right)} \;//$$

$$= \frac{1}{s}\tanh\left(\frac{as}{4}\right) \;//$$

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$F_T(s) = ?$ $\quad f(t) = (1)\,\Pi_{0,\frac{a}{2}}(t) + (-1)\,\Pi_{\frac{a}{2},a}(t)$

$$= \left[u(t-0) - u\left(t-\tfrac{a}{2}\right)\right] - \left[u\left(t-\tfrac{a}{2}\right) - u(t-a)\right]$$

$$= u(t) - 2u\left(t-\tfrac{a}{2}\right) + u(t-a)$$

$$F_T(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\left\{u(t) - 2u\left(t-\tfrac{a}{2}\right) + u(t-a)\right\}$$

$$= \frac{1}{s} - \frac{2e^{-\frac{as}{2}}}{s} + \frac{e^{-as}}{s} = \frac{1-2e^{-\frac{as}{2}}+e^{-as}}{s} \quad *** \;//$$

Dirac Delta Function

Definition 10. The **Dirac delta function** $\delta(t)$ is characterized by the following two properties:

$\delta(t)$

$$\textbf{(1)} \qquad \delta(t) = \begin{cases} 0 \ , & t \neq 0 \ , \\ \text{"infinite,"} & t = 0 \ , \end{cases}$$

and

$$\textbf{(2)} \qquad \int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$$

for any function $f(t)$ that is continuous on an open interval containing $t = 0$.

By shifting the argument of $\delta(t)$, we have $\delta(t-a)=0$, $t \neq a$, and

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = f(a) \tag{3}$$

for any function $f(t)$ that is continuous on an interval containing $t = a$.

It is obvious that $\delta(t-a)$ is not a function in the usual sense; instead it is an example of what is called a **generalized function** or a **distribution.** Despite this shortcoming, the Dirac delta function was successfully used for several years to solve various physics and engineering problems before Laurent Schwartz mathematically justified its use!

A heuristic argument for the existence of the Dirac delta function can be made by considering the impulse of a force over a short interval. If a force $\mathcal{F}(t)$ is applied from time t_0 to time t_1, then the **impulse** due to $\mathcal{F}$ is the integral

$$\text{Impulse} := \int_{t_0}^{t_1} \mathcal{F}(t)\,dt \ .$$

By Newton's second law, we see that

$$\int_{t_0}^{t_1} \mathcal{F}(t)dt = \int_{t_0}^{t_1} m\frac{dv}{dt}dt = mv(t_1) - mv(t_0) \ , \tag{4}$$

where m denotes mass and v denotes velocity. Since mv represents the momentum, we can interpret equation (4) as saying: **The impulse equals the change in momentum.**

When a hammer strikes an object, it transfers momentum to the object. This change in momentum takes place over a very short period of time, say, $[t_0, t_1]$. If we let $\mathscr{F}_1(t)$ represent the force due to the hammer, then the *area* under the curve $\mathscr{F}_1(t)$ is the impulse or change in momentum (see Figure 7.26 on page 406). If, as is illustrated in Figure 7.27 on page 406, the same change in momentum takes place over shorter and shorter time intervals—say, $[t_0, t_2]$ or $[t_0, t_3]$—then the average force must get greater and greater in order for the impulses (the areas under the curves $\mathscr{F}_n$) to remain the same. In fact, if the forces $\mathscr{F}_n$ having the same impulse act, respectively, over the intervals $[t_0, t_n]$, where $t_n \to t_0$ as $n \to \infty$, then $\mathscr{F}_n$ approaches a function that is zero for $t \neq t_0$ but has an infinite value for $t = t_0$. Moreover, the areas under the $\mathscr{F}_n$'s have a common value. Normalizing this value to be 1 gives

$$\int_{-\infty}^{\infty} \mathscr{F}_n(t)\,dt = 1$$

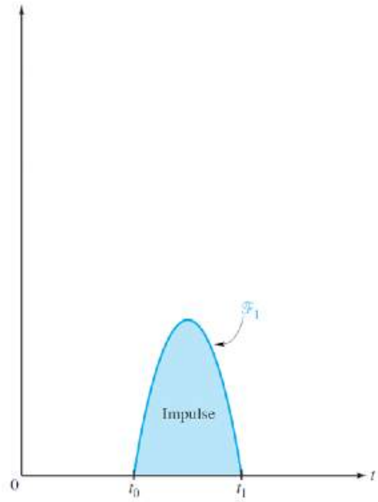

Figure 7.26 Force due to a blow from a hammer

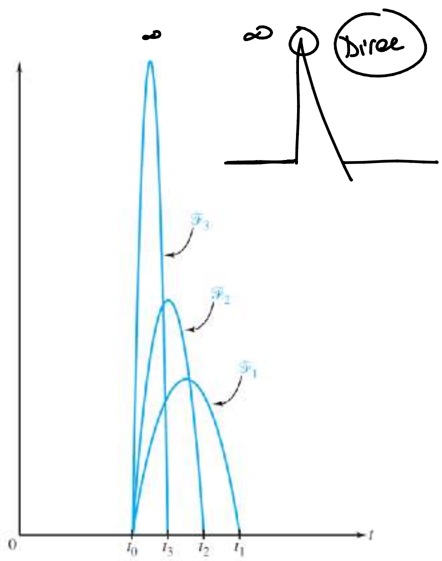

Figure 7.27 Forces with the same impulse

When $t_0 = 0$, we derive from the limiting properties of the $\mathscr{F}_n$'s a "function" δ that satisfies property (1) and the integral condition

(5) $$\int_{-\infty}^{\infty} \delta(t)\,dt = 1 \ .$$

Notice that (5) is a special case of property (2) that is obtained by taking $f(t) \equiv 1$. It is interesting to note that (1) and (5) actually imply the general property (2) (see Problem 33).

The Laplace transform of the Dirac delta function can be quickly derived from property (3). Since $\delta(t-a) = 0$ for $t \neq a$, then setting $f(t) = e^{-st}$ in (3), we find for $a \geq 0$

$$\int_0^{\infty} e^{-st}\delta(t-a)\,dt = \int_{-\infty}^{\infty} e^{-st}\delta(t-a)\,dt = e^{-as} \ .$$

Thus, for $a \geq 0$,

(6) $$\boxed{\mathscr{L}\{\delta(t-a)\}(s) = e^{-as} \ .}$$

$$\boxed{\mathcal{L}\{\delta(t)\} = 1}$$

An interesting connection exists between the unit step function and the Dirac delta function. Observe that as a consequence of equation (5) and the fact that $\delta(x-a)$ is zero for $x < a$ and for $x > a$, we have

$$\textbf{(7)} \qquad \int_{-\infty}^{t} \delta(x-a)\,dx = \begin{cases} 0\,, & t < a\,, \\ 1\,, & t > a \end{cases}$$
$$= u(t-a)\ .$$

If we formally differentiate both sides of (7) with respect to t (in the spirit of the fundamental theorem of calculus), we find

$$\boxed{\delta(t-a) = u'(t-a)}\,.$$

Thus it appears that the Dirac delta function is the derivative of the unit step function. That is, in fact, the case if we consider "differentiation" in a more general sense.†

Example 1 A mass attached to a spring is released from rest 1 m below the equilibrium position for the mass–spring system and begins to vibrate. After π seconds, the mass is struck by a hammer exerting an impulse on the mass. The system is governed by the symbolic initial value problem

$$\boxed{\frac{d^2x}{dt^2} + 9x = 3\delta(t - \pi) \; ; \qquad x(0) = 1 \; , \qquad \frac{dx}{dt}(0) = 0 \; ,} \tag{11}$$

where $x(t)$ denotes the displacement from equilibrium at time t. Determine $x(t)$.

Solution Let $X(s) = \mathscr{L}\{x\}(s)$. Since

$$\mathscr{L}\{x''\}(s) = s^2X(s) - s \quad \text{and} \quad \mathscr{L}\{\delta(t-\pi)\}(s) = e^{-\pi s} \; ,$$

taking the Laplace transform of both sides of (11) and solving for $X(s)$ yields

$$s^2X(s) - s + 9X(s) = 3e^{-\pi s} \quad +s$$

$X(s)(s^2+9)$

$x(t) = \mathcal{L}^{-1}\{X(s)\}$

$$X(s) = \frac{s}{s^2+9} + e^{-\pi s}\frac{3}{s^2+9}$$

$$= \mathscr{L}\{\cos 3t\}(s) + e^{-\pi s}\mathscr{L}\{\sin 3t\}(s) \; .$$

$\mathcal{L}^{-1}\{e^{-\pi s} \cdot \frac{3}{s^2+9}\} = ?$

$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)\,f(t-a)$

$\sin 3t$

$u(t-\pi)\sin(3t-3\pi)$ //

Using the translation property (cf. page 386) to determine the inverse Laplace transform of $X(s)$, we find

$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t>a \end{cases}$$

$$\begin{aligned} x(t) &= \cos 3t + [\sin 3(t-\pi)]u(t-\pi) \\ &= \begin{cases} \cos 3t\ , & t<\pi\ , \\ \cos 3t - \sin 3t\ , & \pi < t \end{cases} \\ &= \begin{cases} \cos 3t\ , & t<\pi\ , \\ \sqrt{2}\cos\left(3t+\dfrac{\pi}{4}\right)\ , & \pi<t\ . \end{cases} \end{aligned}$$

The graph of $x(t)$ is given in color in Figure 7.28. For comparison, the dashed curve depicts the displacement of an undisturbed vibrating spring. Note that the impulse effectively adds 3 units to the momentum at time $t=\pi$. ◆

SORU

$$y'' - 2y' - 3y = 2\delta(t-1) - \delta(t-3)$$
$$y(0) = 2\ , \qquad y'(0) = 2$$

$$\mathcal{L}\{y''-2y'-3y\} = \mathcal{L}\{2\delta(t-1) - \delta(t-3)\}$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2s - 2\ ,$$

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 2\ ,$$

$$\mathcal{L}\{2\delta(t-1)\} = 2e^{-s}$$

$$\mathcal{L}\{\delta(t-3)\} = e^{-3s}$$

$$s^2Y(s) - 2s - 2 - 2sY(s) + 4 - 3Y(s) = 2e^{-s} - e^{-3s}$$

$$Y(s)(s^2-2s-3) = 2e^{-s} - e^{-3s} + 2s - 2$$

$$Y(s) = \frac{2e^{-s}}{s^2-2s-3} - \frac{e^{-3s}}{s^2-2s-3} + \frac{2s-2}{s^2-2s-3}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{2e^{-s}}{s^2-2s-3} - \frac{e^{-3s}}{s^2-2s-3} + \frac{2s-2}{s^2-2s-3} \right\}$$

$$\mathcal{L}^{-1}\left\{ \frac{2s-2}{\ } \right\} = \mathcal{L}^{-1}\left\{ \frac{2(s-1)}{\ } \right\}$$

$$\left(\frac{}{s^2-2s-3} \quad \frac{}{s^2-2s-3} \quad \frac{}{s^2-2s-3} \right)$$

$$F(s) = \frac{1}{s^2-2s-3}, \quad f(t) = ?$$

$$s^2-2s-3 = (s-1)^2-4$$

$$\mathcal{L}^{-1}\left\{ \frac{2s-2}{s^2-2s-3} \right\} = \mathcal{L}^{-1}\left\{ \frac{2(s-1)}{(s-1)^2-4} \right\}$$

$$= 2\cosh(2t)e^t$$

$$f(t) = \frac{1}{2}\sinh 2t \cdot e^t$$

$$y(t) = \cancel{2} \cdot \frac{1}{\cancel{2}} \cdot \left[\sinh(2t-2)\, e^{t-1} \cdot u(t-1) \right] - \left[\frac{1}{2}\sinh(2t-6)\, e^{t-3} \cdot u(t-3) \right]$$

$$+ 2\cosh(2t)\, e^t$$

$$\boxed{y(t) = \sinh(2t-2)\, e^{t-1} u(t-1) - \frac{1}{2}\sinh(2t-6)\, e^{t-3} u(t-3) + 2\cosh(2t)e^t}$$

**

SORU

$$y'' - y = 4\delta(t-2) + t^2$$

$$y(0) = 0, \qquad y'(0) = 2$$

$$\mathcal{L}\{y''-y\} = \mathcal{L}\{4\delta(t-2) + t^2\}$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 2$$

$$\mathcal{L}\{\delta(t-2)\} = e^{-2s}$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3}$$

$$s^2Y(s) - 2 - Y(s) = 4e^{-2s} + \frac{2}{s^3}$$

$$Y(s)(s^2-1) = 4e^{-2s} + \frac{2}{s^3} + 2$$

$$Y(s) = \frac{4e^{-2s}}{s^2-1} + \frac{2}{s^3(s^2-1)} + \frac{2}{s^2-1}$$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{4e^{-2s}}{s^2-1} + \frac{2}{s^3(s^2-1)} + \frac{2}{s^2-1} \right\} \quad 2\sinh t$$

$$4\sinh(t-2)\, u(t-2)$$

$$\frac{2}{s^3(s^2-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-1} + \frac{E}{s+1} \quad ??$$

= Sonuç. ??

SORU

Solve the initial value problem

(13) $\quad y'' + 2ty' - 4y = 1, \qquad y(0) = y'(0) = 0.$

$$\mathcal{L}\{y'' + 2ty' - 4y\} = \mathcal{L}\{1\}$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s)$$

$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s)$

$\mathcal{L}\{ty'\} = -\frac{d}{ds}\mathcal{L}\{y'\} = -\frac{d}{ds}\left(sY(s)\right) = -\left(Y(s) + sY'(s)\right) = -Y(s) - sY'(s)$

$s^2Y(s) - 2Y(s) - 2sY'(s) - 4Y(s) = \frac{1}{s}$

$-2sY'(s) + (s^2-6)\,Y(s) = \frac{1}{s}$

$Y'(s) + \left(\frac{s^2-6}{-2s}\right)Y(s) = -\frac{1}{2s^2}$ (Liner d.d.)

$e^{\int\left(-\frac{s}{2}+\frac{3}{s}\right)ds} \cdot Y(s) = -\int \frac{1}{2s^2}\, e^{\int\left(-\frac{s}{2}+\frac{3}{s}\right)ds}\, ds$

$e^{-\frac{s^2}{4}+3\ln s} \cdot Y(s) = -\int \frac{1}{2s^2}\, e^{-\frac{s^2}{4}+3\ln s}\, ds$

$e^{-\frac{s^2}{4}} \cdot s^3 Y(s) = -\int \frac{1}{2s^2} \cdot e^{-\frac{s^2}{4}} \cdot s^3\, ds$

$e^{-\frac{s^2}{4}} \cdot s^3 Y(s) = -\frac{1}{2}\int s\, e^{-\frac{s^2}{4}}\, ds$

$-\frac{s^2}{4} = t$

$-\frac{s}{2}\,ds = dt$

$\frac{s\,ds}{2} = -dt$

$= \int e^t dt$

$\cancel{e^{-\frac{s^2}{4}}} s^3 Y(s) = \cancel{e^{-\frac{s^2}{4}}}$

$Y(s) = \frac{1}{s^3} \Rightarrow \boxed{y(t) = \frac{t^2}{2}}$ **

SORU (ÖDEV)

$y'' + ty' - y = 0$;

$y(0) = 0$, $y'(0) = 3$

SORU

$y'' + 4y = g(t)$; $y(0) = -1$; $y'(0) = 0$,

where

$g(t) = \begin{cases} t, & t < 2, \\ 5, & t > 2 \end{cases}$

$g(t) = t.\Pi_{0,2}(t) + 5\,u(t-2)$

$= t\left[u(t) - u(t-2)\right] + 5\,u(t-2))$

$= t.u(t) - t\,u(t-2) + 5\,u(t-2)$

$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{g(t)\}$

$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + s$

$\mathcal{L}\{g(t)\} = \mathcal{L}\{t\,u(t) - t\,u(t-2) + 5\,u(t-2)\}$

$= \frac{1}{s^2} - e^{-2s}\mathcal{L}\{t+2\} + \frac{5e^{-2s}}{s}$

$\mathcal{L}\{g(t)\} =$

$$= \frac{1}{s^2} - e^{-2s}\mathcal{L}\{t+2\} + \frac{5e^{-2s}}{s}$$

$$= \frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s} + \frac{5e^{-2s}}{s}$$

$$= \frac{1}{s^2} - \frac{e^{-2s}}{s^2} + \frac{3e^{-2s}}{s} = \frac{1-e^{-2s}+3se^{-2s}}{s^2}$$

$$s^2Y(s) + s + 4Y(s) = \frac{1-e^{-2s}+3se^{-2s}}{s^2}$$

$$Y(s)(s^2+4) = \frac{1-e^{-2s}+3se^{-2s}}{s^2} - s$$

$$Y(s) = \frac{1-e^{-2s}+3se^{-2s}}{s^2(s^2+4)} - \frac{s}{s^2+4} \Rightarrow y(t) = ??$$

ÖDEV

SORU

$$w'' - 2w' + w = 6t - 2 \ ;$$
$$w(-1) = 3 \ ; \qquad w'(-1) = 7$$