Discrete Mathematics

Counting

Ayşegül Gençata Yayımlı H. Turgut Uyar

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Permutations are for lists (order matters) and combinations are for groups (order doesn't matter).

License

Topics

çiğköfte yapma

eldeki malzemelerle pasta yapma

Combinatorics

farklı nesne düzenleme (seçme) işidir ve bu süreçte sıranın bu düzenleme de önemi yoktur.

- combinatorics: study of arrangement of objects
- enumeration: counting of objects with certain properties

- to solve a complicated problem:
- break it down into smaller problems
- piece together solutions to these smaller problems

Sum Rule

- $task_1$ can be performed in n_1 distinct ways
- $task_2$ can be performed in n_2 distinct ways
- $task_1$ and $task_2$ cannot be performed simultaneously
- performing either $task_1$ or $task_2$ can be accomplished in $n_1 + n_2$ ways

Sum Rule Example

- a college library has 40 textbooks on sociology, and 50 textbooks on anthropology
- to learn about sociology or anthropology a student can choose from $40 + 50 = 90$ textbooks

seçmeli ders: 10 bilg, 5 elektronikten var. 1 dersi 15 farklı yolla

Sum Rule Example

- a computer science instructor has two colleagues
- one colleague has 3 textbooks on "Introduction to Programming"
- the other colleague has 5 textbooks on the same subject
- n: maximum number of different books that can be borrowed
- $5 \leq n \leq 8$
- both colleagues may own copies of the same book

Product Rule

- a procedure can be broken down into $stage_1$ and $stage_2$
- n_1 possible outcomes for $stage_1$
- for each of these, n_2 possible outcomes for $stage_2$
- procedure can be carried out in $n_1 \cdot n_2$ ways

Product Rule Example

- drama club is holding tryouts for a play
- 6 men and 8 women auditioning for the leading roles
- director can cast leading couple in $6 \cdot 8 = 48$ ways

Product Rule Example

- license plates with 2 letters, followed by 4 digits
- how many possible plates?
- no letter or digit can be repeated:
  $26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 \cdot 7 = 3,276,000$
- repetitions allowed:
  $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6,760,000$
- repetitions allowed, only vowels and even digits:
  $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 15,625$

Product Rule Example

- a byte consists of 8 bits
- a bit has two possible values: 0 or 1
- number of possible values for a byte:
  $2 \cdot 2 \cdots 2 = 2^8 = 256$

Counting Example

- pastry shop menu:
  6 kinds of muffins, 8 kinds of sandwiches
  hot coffee, hot tea, icea tea, cola, orange juice
- buy either a muffin and a hot beverage,
  or a sandwich and a cold beverage
- how many possible purchases?
- muffin and hot beverage: $6 \cdot 2 = 12$
- sandwich and cold beverage: $8 \cdot 3 = 24$
- total: $12 + 24 = 36$

Permutation

- permutation: a linear arrangement of distinct objects
- order important

Permutation Example

- a class has 10 students: $A, B, C, \ldots, I, J$
- 4 students are to be seated in a row for a picture: $BCEF, CEFI, ABCF, \ldots$
- how many such arrangements?
- filling of a position: a stage of the counting procedure $10 \cdot 9 \cdot 8 \cdot 7 = 5,040$

Permutation Example

$$\begin{aligned} 10 \cdot 9 \cdot 8 \cdot 7 &= 10 \cdot 9 \cdot 8 \cdot 7 \cdot \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\ &= \frac{10!}{6!} \end{aligned}$$

Permutations

- n distinct objects
- number of permutations of size r (where $1 \leq r \leq n$):

$$\begin{aligned} P(n, r) &= n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

- if repetitions are allowed: n^r

Permutations Example

- if size equals number of objects: $r = n$

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

example

- number of permutations of the letters in "COMPUTER": 8!

Arrangements Example

- number of arrangements of the letters in "BALL"
- two L's are indistinguishable

A	B	L	L	L	A	B	L
A	L	B	L	L	A	L	B
A	L	L	B	L	B	A	L
B	A	L	L	L	B	L	A
B	L	A	L	L	L	A	B
B	L	L	A	L	L	B	A

- number of arrangements: $\frac{4!}{2} = 12$

Arrangements Example

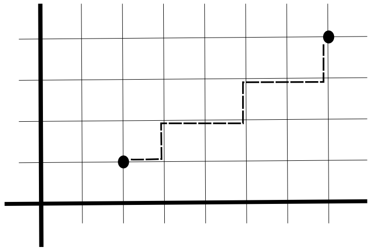
- arrangements of all letters in "DATABASES"
- for each arrangement where A's are **indistinguishable**, $3! = 6$ arrangements where A's are **distinguishable**:
  $DA_1TA_2BA_3SES$, $DA_1TA_3BA_2SES$, $DA_2TA_1BA_3SES$, $DA_2TA_3BA_1SES$, $DA_3TA_1BA_2SES$, $DA_3TA_2BA_1SES$
- for each of these, 2 arrangements where S's are distinguishable:
  $DA_1TA_2BA_3S_1ES_2$, $DA_1TA_2BA_3S_2ES_1$
- number of arrangements: $\frac{9!}{2! \cdot 3!} = 30,240$

Generalized Rule

- n objects
- n_1 indistinguishable objects of $type_1$
  n_2 indistinguishable objects of $type_2$
  ...
  n_r indistinguishable objects of $type_r$
- $n_1 + n_2 + \ldots + n_r = n$
- number of linear arrangements:

$$\frac{n!}{n_1! \cdot n_2! \cdots n_r!}$$

Arrangements Example



- go from $(2, 1)$ to $(7, 4)$
- each step one unit to the right (R) or one unit upwards (U)
- *RURRURRU*, *URRRUURR*
- how many such paths?

- each path consists of 5 R's and 3 U's
- number of paths: $\frac{8!}{5! \cdot 3!} = 56$

Circular Arrangements Example

- 6 people seated around a round table: A, B, C, D, E, F
- arrangements considered to be the same when one can be obtained from the other by rotation: *ABEFCD*, *DABEFC*, *CDABEF*, *FCDABE*, *EFCDAB*, *BEFCDA*
- how many different circular arrangements?

- each circular arrangement corresponds to 6 linear arrangements
- number of circular arrangements: $\frac{6!}{6} = 120$

(n-1)!

Combination

- combination: choosing from distinct objects
- order not important

Combination Example

- a deck of 52 playing cards
- 4 suits: clubs, diamonds, hearts, spades
- 13 ranks in each suit: Ace, 2, 3, ..., 10, Jack, Queen, King
- draw 3 cards in succession, without replacement
- how many possible draws?

$$52 \cdot 51 \cdot 50 = \frac{52!}{49!} = P(52, 3) = 132,600$$

P(52,3)

Combination Example

- one such draw:
  AH (ace of hearts), $9C$ (9 of clubs), KD (king of diamonds)
- if order doesn't matter
- 6 permutations of $(AH, 9C, KD)$ correspond to just one selection

$$\frac{52!}{3! \cdot 49!} = 22,100$$

C(52,3)

Number of Combinations

- n distinct objects
- each combination of r objects: $r!$ permutations of size r
- number of combinations of size r (where $0 \leq r \leq n$):

$$C(n,r) = \binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r! \cdot (n-r)!}$$

Number of Combinations

- number of combinations:

$$C(n,r) = \frac{n!}{r! \cdot (n-r)!}$$

- note that:

$$\begin{aligned} C(n,0) &= 1 = & C(n,n) \\ C(n,1) &= n = & C(n,n-1) \end{aligned}$$

Number of Combinations Example

- Lynn and Patti buy a powerball ticket
- match five numbers selected from 1 to 49
- and then match powerball, 1 to 42
- how many possible tickets?
- Lynn selects five numbers from 1 to 49: $C(49,5)$
- Patti selects the powerball from 1 to 42: $C(42,1)$
- possible tickets: $\binom{49}{5}\binom{42}{1} = 80,089,128$

Number of Combinations Examples

- for a volleyball team, gym teacher must select nine girls from junior and senior classes
- 28 junior and 25 senior candidates
- how many different ways?
- if no restrictions: $\binom{53}{9} = 4,431,613,550$
- if two juniors and one senior are best spikers and must be on the team: $\binom{50}{6} = 15,890,700$
- if there has to be four juniors and five seniors: $\binom{28}{4}\binom{25}{5} = 1,087,836,750$

(50*49*48*47*46*45) / 6!

Binomial Theorem

Theorem

if x and y are variables and n is a positive integer, then:

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots \\ &\quad + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 \\ &= \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}\end{aligned}$$

(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3

- $\binom{n}{k}$: binomial coefficient

Binomial Theorem Examples

- in the expansion of $(x+y)^7$, coefficient of x^5y^2: $\binom{7}{5} = \binom{7}{2} = 21$

Multinomial Theorem

Theorem

For positive integers n, t, the coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}\cdots x_t^{n_t}$ in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots n_t!}$$

where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$, and $n_1 + n_2 + n_3 + \ldots + n_t = n$.

Multinomial Theorem Examples

- in the expansion of $(x + y + z)^7$, coefficient of $x^2y^2z^3$:

$$\binom{7}{2,2,3} = \frac{7!}{2! \cdot 2! \cdot 3!} = 210$$

Combinations with Repetition Example

- 7 students visit a restaurant
- each of them orders one of the following:
  cheeseburger (c), hot dog (h), taco (t), fish sandwich (f)
- how many different purchases are possible?

Combinations with Repetition Example

n=10, r=7 r arrangement'ı yapılacak nesne

c c h h t t f	x x | x x | x x | x
c c c c h t f	x x x x | x | x | x
c c c c c c f	x x x x x x | | | x
h t t f f f f	| x | x x | x x x x
t t t t t t t	| | x x x x x x x |
f f f f f f f	| | | x x x x x x x

- enumerate all arrangements of 10 symbols
  consisting of seven x's and three $|$'s
- number of different purchases: $\frac{10!}{7! \cdot 3!} = \binom{10}{7} = 120$

Number of Combinations with Repetition

- select, with repetition, r of n distinct objects
- considering all arrangements of r x's and $n - 1$ $|$'s

$$\frac{(n + r - 1)!}{r! \cdot (n - 1)!} = \binom{n + r - 1}{r}$$

Number of Combinations with Repetition Example

- distribute 7 bananas and 6 oranges among 4 children
- each child receives at least one banana
- how many ways?

4 çocuk

- step 1: give each child a banana
- step 2: distribute 3 bananas to 4 children

1 1 1 0	b | b | b |
1 0 2 0	b | | b b |
0 0 1 2	| | b | b b
0 0 0 3	| | | b b b

- $C(6,3) = 20$ ways

n=7, r=3

Number of Combinations with Repetition Example

- step 3: distribute 6 oranges to 4 children

1 2 2 1	o | o o | o o | o
1 2 0 3	o | o o | | o o o
0 3 3 0	| o o o | o o o |
0 0 0 6	| | | o o o o o o

- $C(9,6) = 84$ ways
- step 4: by the rule of product: $20 \cdot 84 = 1,680$ ways

n=10,r=6

References

Required Reading: Grimaldi

- Chapter 1: Fundamental Principles of Counting
  - 1.1. The Rules of Sum and Product
  - 1.2. Permutations
  - 1.3. Combinations
  - 1.4. Combinations with Repetition