Ters Laplace Dönüşümü

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ters laplace dönüşümü

$$f(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\rightleftarrows}} F(s)$$

TERS LAPLACE DÖNÜŞÜMÜ

INVERSE LAPLACE TRANSFORM

Inverse Laplace Transform

Definition 4. Given a function $F(s)$, if there is a function $f(t)$ that is continuous on $[0, \infty)$ and satisfies

$$\textbf{(2)} \qquad \mathscr{L}\{f\} = F\ ,$$

then we say that $f(t)$ is the **inverse Laplace transform** of $F(s)$ and employ the notation $\boxed{f = \mathscr{L}^{-1}\{F\}.}$

In case every function $f(t)$ satisfying (2) is discontinuous (and hence not a solution of a differential equation), one could choose any one of them to be the inverse transform; the distinction among them has no physical significance. [Indeed, two *piecewise* continuous functions satisfying (2) can only differ at their points of discontinuity.]

Naturally the Laplace transform tables will be a great help in determining the inverse Laplace transform of a given function $F(s)$.

Example 1 Determine $\mathscr{L}^{-1}\{F\}$, where

(a) $F(s) = \dfrac{2}{s^3}$. **(b)** $F(s) = \dfrac{3}{s^2+9}$. **(c)** $F(s) = \dfrac{s-1}{s^2-2s+5}$.

(a) $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2$

(b) $\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \sin 3t$

(c) $\mathcal{L}^{-1}\left\{\frac{s-1}{s^2-2s+5}\right\}$

$\mathcal{L}\{1\} = \frac{1}{s}$

$\mathcal{L}\{t\} = \frac{1}{s^2}$

$\mathcal{L}\{t^2\} = \frac{2}{s^3}$

$\mathcal{L}\{t^3\} = \frac{6}{s^4}$

$\vdots$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

$= \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\}$

$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$

$= \mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} = e^t \cos 2t$

In practice, we do not always encounter a transform $F(s)$ that exactly corresponds to an entry in the second column of the Laplace transform table. To handle more complicated functions $F(s)$, we use properties of $\mathcal{L}^{-1}$, just as we used properties of $\mathcal{L}$. One such tool is the linearity of the inverse Laplace transform, a property that is inherited from the linearity of the operator $\mathcal{L}$.

Linearity of the Inverse Transform

Theorem 7. Assume that $\mathcal{L}^{-1}\{F\}$, $\mathcal{L}^{-1}\{F_1\}$, and $\mathcal{L}^{-1}\{F_2\}$ exist and are continuous on $[0, \infty)$ and let c be any constant. Then

$$\left.\begin{aligned} &(3) \quad \mathcal{L}^{-1}\{F_1 + F_2\} = \mathcal{L}^{-1}\{F_1\} + \mathcal{L}^{-1}\{F_2\}\,, \checkmark \\ &(4) \quad \mathcal{L}^{-1}\{cF\} = c\mathcal{L}^{-1}\{F\}\,. \checkmark \end{aligned}\right\} \text{Lineer}$$

Example 2 Determine $\mathcal{L}^{-1}\left\{\frac{5}{s-6} - \frac{6s}{s^2+9} + \frac{3}{2s^2+8s+10}\right\}$.

$$= \mathcal{L}^{-1}\left\{\frac{5}{s-6}\right\} - \mathcal{L}^{-1}\left\{\frac{6s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{2s^2+8s+10}\right\}$$

$$= 5\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\} - 6\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$\cos 3t$

$(s+2)^2 + 1$

$e^{-2t}\sin t$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\} = e^{6t}$$

$$= \underline{\underline{5e^{6t} - 6\cos 3t + \frac{3}{2}e^{-2t}\sin t}}$$

Ters Laplace Dönüşümünün Bazı Temel Özellikleri

Özellik 1: Lineerlik

$$L^{-1}\{C_1F_1(s)+C_2F_2(s)\}=C_1L^{-1}\{F_1(s)\}+C_2L^{-1}\{F_2(s)\}$$

Örnek

$$L^{-1}\left\{\frac{2}{s}-\frac{3}{s+5}\right\}=2L^{-1}\left\{\frac{1}{s}\right\}-3L^{-1}\left\{\frac{1}{s+5}\right\}=2-3e^{-5t}$$

Özellik 2: Öteleme

$$L^{-1}\{F(s-k)\}=e^{kt}L^{-1}\{F(s)\}$$

Örnek

$$L^{-1}\left\{\frac{s+3}{(s+3)^2-\omega^2}\right\}=e^{-3t}L^{-1}\left\{\frac{s}{s^2-\omega^2}\right\}=e^{-3t}\cosh(\omega t)$$

$$\boxed{\mathcal{L}\{y'\}=sY(s)-y(0)}$$

Özellik 3:

$$\boxed{L^{-1}\{sF(s)\}=\frac{d}{dt}L^{-1}\{F(s)\}}$$ **

*

$$\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+4}\right\}=$$

$$=\mathcal{L}^{-1}\left\{s\left(\frac{s}{s^2+4}\right)\right\}=\frac{d}{dt}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$\cos 2t$

$$=-2\sin 2t$$

Örnek

$\cos 3t$

$$L^{-1}\left\{\frac{s}{s^2+9}\right\}=L^{-1}\left\{s\left(\underbrace{\frac{1}{s^2+9}}_{F(s)}\right)\right\}=\frac{d}{dt}\frac{1}{3}L^{-1}\left\{\frac{3}{s^2+9}\right\}=\frac{d}{dt}\left(\frac{\sin 3t}{3}\right)=\cos 3t$$

$\frac{1}{3}\sin 3t$ $\cancel{3}\cos 3t$

Özellik 4:

$$\boxed{L^{-1}\left\{\frac{F(s)}{s}\right\}=\int_0^t L^{-1}\{F(s)\}dt}$$

$f(t)$

$$3(e^{2t}-e^0)$$

Örnek

$$L^{-1}\left\{\frac{6}{s(s-2)}\right\}=L^{-1}\left\{\frac{1}{s}\left(\frac{6}{(s-2)}\right)\right\}=\int_0^t L^{-1}\left\{\frac{6}{s-2}\right\}dt=6\int_0^t e^{2t}dt=6\left.\frac{e^{2t}}{2}\right|_0^t=3e^{2t}-3$$

12) $\mathcal{L}\{te^{3t}\}$

Ötleme

$\mathcal{L}\{e^{3t}\, t\} = \dfrac{1}{(s-3)^2}$ ✓

Türev alma

$\mathcal{L}\{t\, e^{3t}\} = -\dfrac{d}{ds}\mathcal{L}\{e^{3t}\} = -\dfrac{d}{ds}\left(\dfrac{1}{s-3}\right)$

$= -\left(-\dfrac{1}{(s-3)^2}\right)$

$= \dfrac{1}{(s-3)^2}$ ✓

Özellik 5:

$$\boxed{L^{-1}\left\{\frac{d^n F(s)}{ds^n}\right\} = (-t)^n L^{-1}\{F(s)\}}$$

$\left(\dfrac{1}{s}\right)' = -\dfrac{1}{s^2}$

$\left(-\dfrac{1}{s^2}\right)' = \dfrac{2}{s^3}$

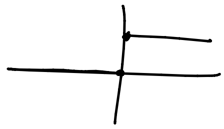
Örnek

t^2

$$L^{-1}\left\{\frac{2}{s^3}\right\} = L^{-1}\left\{\frac{d^2}{ds^2}\left(\frac{1}{s}\right)\right\} = (-t)^2 L^{-1}\left\{\frac{1}{s}\right\} = t^2 \times 1 = t^2$$

1

Özellik 6:



$$\boxed{L^{-1}\left\{e^{-t_0 s}F(s)\right\} = u(t-t_0)\,f(t-t_0)}$$ ✓

Bozonak fonk

Örnek

Basamak fonk.

$$L^{-1}\left\{\frac{2e^{-3s}}{s^3}\right\} = u(t-3)L^{-1}\left\{\frac{2}{s^3}\right\} = u(t-3)(t^2)_{t\to(t-t_0)} = u(t-3)(t-3)^2$$

t^2

Özellik 7:

$$L^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right)$$

Örnek

$\mathcal{L}^{-1}\left\{ \frac{s}{s^2+3^2} \right\} = \cos 3t$

$$L^{-1}\left\{\frac{5s}{25s^2+9}\right\} = L^{-1}\left\{\frac{(5s)}{(5s)^2+3^2}\right\} = \frac{1}{5}\cos 3\frac{t}{5}$$

Example 3 Determine $\mathscr{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{5}{s^4}\right\} = 5\,\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{5}{6}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\}$$

$$= \frac{5}{6}t^3$$

$$\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^4}\right\} = \frac{5}{6}e^{-2t}t^3 \;//$$

$s = s+2$

Example 4 Determine $\mathscr{L}^{-1}\left\{\dfrac{3s+2}{\underbrace{s^2+2s+10}_{(s+1)^2+9}}\right\}$.

$\mathcal{L}^{-1}\left\{\dfrac{P(s)}{Q(s)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{3s}{(s+1)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+9}\right\}$$

$$= 3\,\mathcal{L}^{-1}\left\{\frac{s+1-1}{(s+1)^2+9}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+9}\right\}$$

$$= 3\,\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+9}\right\} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+9}\right\}$$

$$= 3\,\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+9}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s+1)^2+9}\right\}$$

$$= 3e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t \quad //$$

Thus, if we are faced with the problem of computing $\mathscr{L}^{-1}$ of a rational function such as $F_1(s)$, we will first express it, as we did $F_2(s)$, as a sum of simple rational functions. This is accomplished by the **method of partial fractions.**

We briefly review this method. Recall from calculus that a rational function of the form $P(s)/Q(s)$, where $P(s)$ and $Q(s)$ are polynomials with the degree of P less than the degree of Q, has a partial fraction expansion whose form is based on the linear and quadratic factors of $Q(s)$. (We assume the coefficients of the polynomials to be real numbers.) There are three cases to consider:

1. Nonrepeated linear factors.
2. Repeated linear factors.
3. Quadratic factors.

1. Nonrepeated Linear Factors

If $Q(s)$ can be factored into a product of distinct linear factors,

$$Q(s) = (s - r_1)(s - r_2) \cdots (s - r_n) ,$$

where the r_i's are all distinct real numbers, then the partial fraction expansion has the form

$$\frac{P(s)}{Q(s)} = \frac{A_1}{s - r_1} + \frac{A_2}{s - r_2} + \cdots + \frac{A_n}{s - r_n} ,$$

where the A_i's are real numbers. There are various ways of determining the constants $A_1, \ldots, A_n$. In the next example, we demonstrate two such methods.

Example 5 Determine $\mathscr{L}^{-1}\{F\}$, where

$$F(s) = \frac{7s-1}{(s+1)(s+2)(s-3)} .$$

$$\mathscr{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\} = ?$$

$$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}$$

$$(s+2)(s-3) \quad (s+1)(s-3) \quad (s+1)(s+2)$$

$$7s-1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$$

$$s=-1 \Rightarrow -8 = A\cdot 1\cdot(-4) \Rightarrow A = 2$$

$$s=-2 \Rightarrow -15 = B\cdot(-1)(-5) \Rightarrow B = -3$$

$$s=3 \Rightarrow 20 = C\cdot(4)\cdot(5) \Rightarrow C = 1$$

$$\mathscr{L}^{-1}\left\{\frac{7s-1}{(s+1)(s+2)(s-3)}\right\} = 2\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - 3\mathscr{L}\left\{\frac{1}{s+2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= 2e^{-t} - 3e^{-2t} + e^{3t}$$

2. Repeated Linear Factors

(Kökler katlı ise)

Let $s - r$ be a factor of $Q(s)$ and suppose $\boxed{(s - r)^m}$ is the highest power of $s - r$ that divides $Q(s)$. Then the portion of the partial fraction expansion of $P(s)/Q(s)$ that corresponds to the term $(s - r)^m$ is

$$\frac{A_1}{s - r} + \frac{A_2}{(s - r)^2} + \cdots + \frac{A_m}{(s - r)^m},$$

where the A_i's are real numbers.

Example 6 Determine $\mathscr{L}^{-1}\left\{\dfrac{s^2+9s+2}{(s-1)^2(s+3)}\right\}$.

$$\frac{s^2+9s+2}{(s-1)^2\cdot(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$$

$(s-1)(s-1)$ $\quad (s+3)(s-1) \quad (s+3) \quad (s-1)^2$

$$s^2+9s+2 = A(s+3)(s-1) + B(s+3) + C(s-1)^2$$

$s=1 \Rightarrow \quad 12 = B.4 \quad \Rightarrow \quad B=3$

$s=-3 \Rightarrow \quad -16 = C.16 \Rightarrow \quad C=-1$

$s=0 \Rightarrow \quad 2 = -3A + 3B + C \Rightarrow \quad A=2$

$$\mathcal{L}^{-1}\left\{\frac{s^2+9s+2}{(s-1)^2(s+3)}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= 2e^t + 3te^t - e^{-3t}$$

3. Quadratic Factors

Let $(s-\alpha)^2+\beta^2$ be a quadratic factor of $Q(s)$ that cannot be reduced to linear factors with real coefficients. Suppose m is the highest power of $(s-\alpha)^2+\beta^2$ that divides $Q(s)$. Then the portion of the partial fraction expansion that corresponds to $(s-\alpha)^2+\beta^2$ is

$$\frac{C_1 s+D_1}{(s-\alpha)^2+\beta^2}+\frac{C_2 s+D_2}{\left[(s-\alpha)^2+\beta^2\right]^2}+\cdots+\frac{C_m s+D_m}{\left[(s-\alpha)^2+\beta^2\right]^m}.$$

As we saw in Example 4, it is more convenient to express $C_i s+D_i$ in the form $A_i(s-\alpha)+\beta B_i$ when we look up the Laplace transforms. So let's agree to write this portion of the partial fraction expansion in the equivalent form

$$\frac{A_1(s-\alpha)+\beta B_1}{(s-\alpha)^2+\beta^2}+\frac{A_2(s-\alpha)+\beta B_2}{\left[(s-\alpha)^2+\beta^2\right]^2}+\cdots+\frac{A_m(s-\alpha)+\beta B_m}{\left[(s-\alpha)^2+\beta^2\right]^m}.$$

Example 7 Determine $\mathscr{L}^{-1}\left\{\dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

$\Delta = b^2 - 4ac$

$= 4 - 4 \cdot 1 \cdot 5 = -16 < 0$

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{As+B}{s^2-2s+5} + \frac{C}{s+1}$$

$(s+1)$ $\quad$ (s^2-2s+5)

$$2s^2+10s = (As+B)(s+1) + C(s^2-2s+5)$$

$$2s^2+10s = As^2 + As + Bs + B + Cs^2 - 2Cs + 5C$$

$-/ \; A + C = 2$

$A + B - 2C = 10$

$B + 5C = 0$

$-/ \; B - 3C = 8$

$B + 5C = 0$

$8C = -8$

$\boxed{C=-1}$ $\quad$ $\boxed{B=5}$ $\quad$ $\boxed{A=3}$

$$\mathscr{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\} = \mathscr{L}^{-1}\left\{\frac{3s+5}{s^2-2s+5}\right\} + \mathscr{L}^{-1}\left\{\frac{-1}{s+1}\right\} \longrightarrow -e^{-t}$$

$\downarrow$ $(s-1)^2+4$

$$= 3\,\mathscr{L}^{-1}\left\{\frac{s-1+1}{(s-1)^2+4}\right\} + 5\,\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}$$

$$= 3\,\mathscr{L}^{-1}\left\{\frac{s-1}{(s-1)^2+4}\right\} + \frac{8}{2}\,\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^2+4}\right\}$$

$$= 3e^t\cos 2t + 4e^t\sin 2t \quad = e^t(3\cos 2t + 4\sin 2t)$$

$$\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\} = e^{t}(3\cos 2t+4\sin 2t)-e^{-t}$$

$\mathcal{L}^{-1}\{F(s)\} = f(t)$

$\mathcal{L}^{-1}\{G(s)\} = g(t)$

KONVOLÜSYON TEOREMİ

Diferensiyel denklemleri Laplece yöntemiyle çözerken çoğu zaman doğrudan ters Laplace'ı olmayan, ancak s'ye bağlı ve ters Laplace'ı bilinen iki fonksiyonun çarpımı şeklinde ifade edilebilen $Y(s)$ fonksiyonlarıyla karşılaşırız. Diğer bir ifadeyle $\boxed{Y(s) = F(s)G(s)}$ şeklinde ifade edilebilmekte ve buradaki $F(s)$ ve $G(s)$ fonksiyonlarının ters Laplace dönüşümleri olan $f(t)$ ve $g(t)$ bilinmektedir. Bu tür durumlarda $Y(s)$ fonksiyonunun ters Laplace'ı **konvolüsyon teoremi** ile belirlenebilir. Bu teorem;

$$L^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

olarak ifade edilir (τ geçici değişkendir). Bu integrale $f(t)$ ve $g(t)$ fonksiyonlarının *konvolüsyonu* adı verilir ve bazen $f(t) * g(t)$ şeklinde gösterilir. Alınan integralin üst limiti t olduğundan, sonuçta t'ye bağlı bir ifade çıkacaktır.

ÖRNEK

Konvolüsyon teoremini kullanarak aşağıdaki ifadenin ters Laplace dönüşümünü bulunuz.

$$Y(s) = \frac{s}{(s+1)(s^2+1)}$$

$\mathcal{L}^{-1}\{Y(s)\} = ?$, $y(t) = ?$

$F(s) = \frac{1}{s+1}$, $G(s) = \frac{s}{s^2+1}$

$\mathcal{L}^{-1}\{F(s)\} = e^{-t} = \underline{f(t)}$ $\qquad$ $\mathcal{L}^{-1}\{G(s)\} = \cos t = \underline{g(t)}$

$$\mathcal{L}^{-1}\{F(s).G(s)\} = \int_0^t f(t-x)\,g(x)\,dx \quad \text{veya} \quad \int_0^t f(x)\,g(t-x)\,dx$$

$$= \int_0^t e^{-(t-x)} \cos x\,dx \quad , \quad \int_0^t e^{-x} \cos(t-x)\,dx$$

$$= \int_0^t e^{-t} \cdot e^{x} \cos x\,dx$$

$$= \underline{e^{-t}} \int_0^t e^{x} \cos x\,dx$$

$e^x = u$, $\cos x\,dx = dv$

$e^x dx = du$, $\sin x = v$

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

$e^x = u$, $\sin x\,dx = dv$

$e^x dx = du$, $-\cos x\,dx = v$

$$\int e^x \cos x\,dx = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x\,dx\right]$$

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

$$\int_0^t e^x \cos x\,dx = \left[\frac{e^x}{2}(\sin x + \cos x)\right]_0^t$$

$$= \left[\frac{e^t}{2}(\sin t + \cos t)\right] - \left[\frac{e^0}{2}(\sin 0 + \cos 0)\right]$$

$$= \underline{\frac{e^t}{2}}(\sin t + \cos t) - \frac{1}{2}$$

$$= \frac{e^{t}}{2}(\sin t + \cos t) - \frac{1}{2}$$

$$\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = e^{-t}\left[\frac{e^{t}}{2}(\sin t + \cos t) - \frac{1}{2}\right]$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\cdot G(s)\} = \frac{1}{2}(\sin t + \cos t) - \frac{1}{2}e^{-t}} \quad //$$