

# **A Wait-free Queue with Poly-logarithmic Worst-case Step Complexity**

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## Abstract

In this work, we introduce a novel linearizable wait-free queue implementation. Linearizability and lock-freedom are standard requirements for designing shared data structures. To the best of our knowledge, all of the existing linearizable lock-free queues in the literature have a common problem in their worst case, called the CAS Retry Problem. We show that our algorithm avoids this problem with the helping mechanism which we use and has a worst-case running time better than prior lock-free queues. The amortized number of steps for an **Enqueue** or **Dequeue** in our algorithm is  $O(\log^2 p + \log q)$ , where  $p$  is the number of processes and  $q$  is the size of the queue when the operation is linearized.

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I dedicate this paper to my people, who are fighting for their freedom.

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# 1 Introduction

Shared data structures have become an essential field in distributed algorithms research. We are reaching the physical limits of how many transistors we can place on a CPU core. The industry solution to provide more computational power is to increase the number of cores of the CPU. It is not hard to see why multiple processes cannot share a sequential data structure. For example, consider two processes trying to append to a sequential linked list simultaneously. Processes  $p, q$  read the same tail node,  $p$  changes the next pointer of the tail node to its new node, and then  $q$  does the same. In this run,  $p$ 's update is overwritten. One solution is to use locks; whenever a process wants to update or query a data structure, the process locks the data structure, and others have to wait until the lock is released to read or update the data structure. Using locks has some disadvantages; for example, one process might be slow, and holding a lock for a long time prevents other processes from progressing. Moreover, locks do not allow complete parallelism since only the process holding the lock can progress when the data structure is locked. For these reasons, we are interested in the design of an efficient shared queue without using locks.

A sequential queue stores a sequence of elements and supports two operations, enqueue and dequeue. **Enqueue**( $e$ ) appends the element  $e$  to the sequence stored. **Dequeue** removes and returns the first element in the queue. If the queue is empty, it returns **null**. In a concurrent version of the queue data structure, operations do not happen one at a time. The question that may arise is “What properties matter for the implementation of a shared data structure?”. Since executions on a shared data structure are different from sequential ones, the correctness conditions also differ. To prove a concurrent object works perfectly, we have to show it satisfies safety and progress conditions. A *safety condition* tells us that the data structure does not return wrong responses, and a *progress property* requires that operations eventually terminate.

A system is called *asynchronous* when processes in the system run at arbitrarily varying speeds, i.e., the scheduling of each process is independent from the scheduling of other processes. Our model is an asynchronous shared-memory distributed system of  $p$  processes. Herlihy [12] showed that one

cannot implement concurrent versions of all data structures with multi-reader multi-writer registers alone and introduced a hierarchy based on how powerful objects are to solve the consensus problem among processes. Objects like LL/SC or CAS can be used to reach a consensus among any number of processors. A CAS object provides an atomic Compare&Swap(**new**, **old**) operation: if the value stored in the object is **old**, it updates the value to **new** and returns **true**; otherwise it returns **false**. In this work, we use Compare&Swap (CAS) objects to synchronize among processes.

The standard safety condition is called *linearizability* [13], which ensures that for any concurrent execution on a linearizable object, each operation should appear to take effect instantaneously at some moment between its invocation and response. Figure 1 depicts an example of an execution on a linearizable queue that is initially empty. The arrow shows time, and each rectangle shows the time between the invocation and the termination of an operation. Since **Enqueue(A)** and **Enqueue(B)** are concurrent, **Enqueue(B)** may or may not take effect before **Enqueue(A)**. The execution in Figure 2 is not linearizable since **A** has been enqueued before **B**, so it has to be dequeued first.

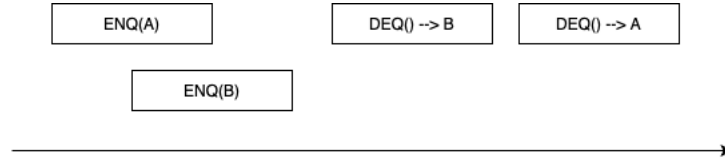


Figure 1: An example of a linearizable execution. Either **Enqueue(A)** or **Enqueue(B)** could take effect first since they are concurrent.

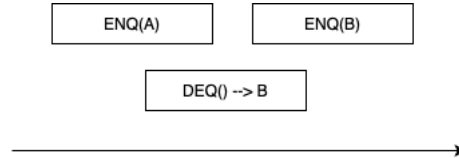


Figure 2: An example of an execution on an empty queue that is not linearizable. **Enqueue(A)** has terminated before **Enqueue(B)** is invoked and the **Dequeue** can occur before **Enqueue(A)** and return **null** or can occur after **Enqueue(A)** and return **A**.

There are various progress properties; the strongest is wait-freedom, and the more common is



lock-freedom. An algorithm is *wait-free* if each operation terminates after a finite number of its own steps. We call an algorithm *lock-free* if, after a sufficient number of steps, one operation terminates. A wait-free algorithm is also lock-free but not vice versa; in an infinite run of a lock-free algorithm, there might be an operation that takes infinitely many steps but never terminates.

We will present a wait-free linearizable queue, which to the best of our knowledge is the first wait-free queue whose operations run in a poly-logarithmic number of steps. We design a tournament tree in which each process tries to propagate its operations along a path from the process's leaf up to the root to be linearized. Processes can do queries to get information about the linearization stored in the root. We create a helping mechanism for when a process is propagating its operation to a node, such that new operations are ensured to be propagated one step further up the tree after at most 2 CAS invocations. The amortized number of steps for an **Enqueue** or **Dequeue**, is  $O(\log^2 p + \log q)$ , where  $q$  is the size of the queue when the operation is linearized. CAS instructions cost more than other instructions for the processor. Each operation in our queue does  $\Theta(\log p)$  CAS operations compared to  $\Omega(p)$  CAS steps for previous queues in the worst case. We use unbounded memory space in our queue and present a way to reduce the memory size to the total number of operations. Furthermore, we do not address garbage collection but we describe a way we think it can be handled.

**Thesis outline** The rest of the thesis is organized as follows. Chapter 2 gives an outline of the related work done in the area and the motivation for our implementation. Previous lock-free queues and their common problem is presented in Section 2.1. In Section 2.2, we mention some restricted lock-free queues. Section 2.3 talks about poly-logarithmic constructions of shared objects. Section 2.4 ends with a lower bound on the amortized time complexity of shared queues.

In Chapter 3 we introduce a poly-logarithmic step wait-free queue. In Section 3.1 we give a high-level description of our implementation. We also discuss the motivation and requirements in our design to achieve poly-log time. Section 3.2 contains the algorithm itself and there is an example of a concurrent execution on the algorithm described in Section 3.3.

We prove the correctness of our queue in Chapter 4 by showing it is linearizable.

Chapter 5 is devoted to complexity analysis for our implementation. We compute the number of CAS instructions our algorithm invokes and the worst-case and amortized running time of the queue. In the end, we prove our queue is wait-free.

Finally, we give some concluding remarks in Chapter 6. It contains how we can improve our algorithm's memory usage and how to use its idea in designing other wait-free data structures.

## 2 Related Work

In this section, we look at previous lock-free queues. The worst-case step complexity of all the lock-free queues includes an  $\Omega(p)$  term that comes from the case where all  $p$  processes try to do an enqueue or a dequeue simultaneously. Morrison and Afek call this the *CAS retry problem* [24].

### 2.1 List-based Queues

Michael and Scott [21] introduced a lock-free queue which we refer to as the MS-queue. A version of it is included in the standard Java Concurrency Package. Their idea is to store the queue elements in a singly-linked list (see Figure 3). A shared variable *Head* points to the first node in the linked list that has not been dequeued, and *Tail* points to the last element in the queue. To insert a node into the linked list, they use atomic primitive operations like LL/SC or CAS. If  $p$  processes try to enqueue simultaneously, only one can succeed, and the others have to retry. This makes the amortized number of steps  $\Omega(p)$  per enqueue. Similarly, dequeue can take  $\Omega(p)$  steps.

Moir, Nussbaum, and Shalev [23] presented a more sophisticated queue by using the *elimination* technique. The elimination mechanism has the dual purpose of allowing operations to complete in parallel and reducing contention for the queue. An Elimination Queue consists of an MS-queue augmented with an elimination array. Elimination works by allowing opposing pairs of concurrent operations such as an enqueue and a dequeue to exchange values when the queue is empty or when concurrent operations can be linearized to empty the queue. Their algorithm makes it possible for long-running operations to eliminate an opposing operation. The empirical evaluation showed the throughput of their work is better than the MS-queue, but the worst case is still the same; in case there are  $p$  concurrent enqueues, their algorithm is not better than MS-queue.

Hoffman, Shalev, and Shavit [14] tried to make the MS-queue more parallel by introducing the Baskets Queue. Their idea is to allow more parallelism by treating the simultaneous enqueue operations as a basket. Each basket has a time interval in which all its nodes' enqueue operations overlap. Since the operations in a basket are concurrent, we can order them in any way. Enqueues

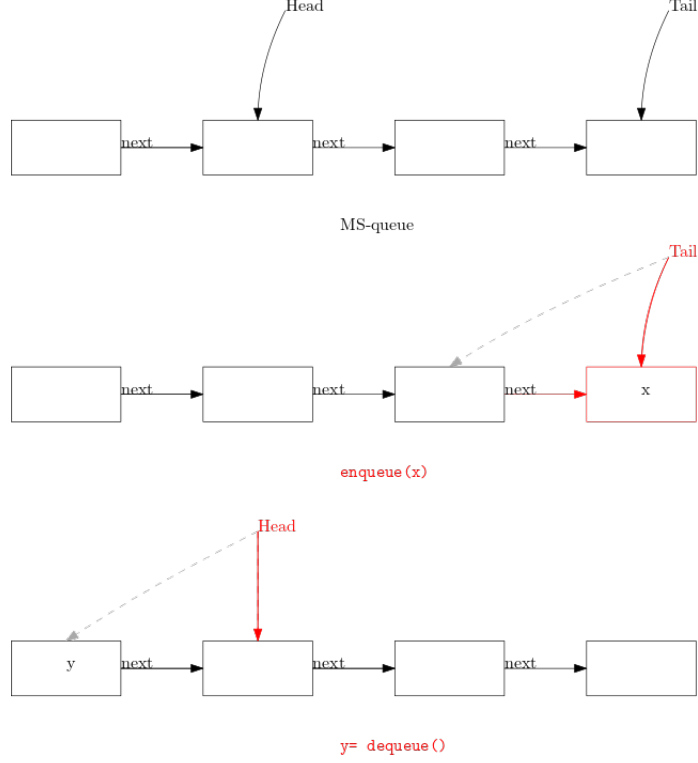


Figure 3: MS-queue structure, enqueue and dequeue operations. In the first diagram, the first element has been dequeued. Red arrows show new pointers and gray dashed arrows show the old pointers.

in a basket try to find their order in the basket one by one using **CAS** operations. However, like the previous algorithms, if there are  $p$  concurrent enqueue operations in a basket, the worst-case step complexity remains  $\Omega(p)$  per operation.

Ladan-Mozes and Shavit [20] presented an optimistic approach to implement a queue. MS-queue uses two **CASes** to do an enqueue: one to change the tail to the new node and another one to change the next pointer of the previous node to the new node. They use a doubly-linked list to do fewer **CAS** operations in an **Enqueue** than MS-queue. As in previous algorithms, the worst case happens in the case where the contention is high: when  $p$  concurrent enqueues happen, their nodes have to be appended to the linked list one by one. The worst-case complexity is still  $\Omega(p)$  **CASes**.

Hendler et al. [11] proposed a new paradigm called flat combining. The key idea behind flat combining is to allow a combiner who has acquired the global lock on the data structure to learn

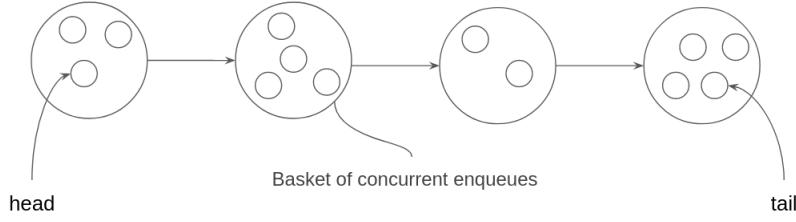


Figure 4: At a time, all operations in a basket ran concurrently, but only one succeeded to do **CAS**. To order the operations in a basket, processes perform another **CAS**. The successful process will be the next one in the basket, and so on.

about the requests of threads on the queue, combine them and apply the combined results on the data structure. Their queue is linearizable but not lock-free and they present experiments that show their algorithm performs well in some situations.

Gidenstam, Sundell, and Tsigas [9] introduced a new algorithm using a linked list of arrays. The queue is stored in a shared array where head and tail pointers point to the current elements in the queue. When the array is full, an empty array is linked to the array and tail pointers are updated. A global head points to the array containing the first element in the queue, and each process has a local head index that points to the first element in that array. Global tail and local tail pointers are similar (see Figure 5). A process updates the position of the pointers after it does an operation. One process might go to sleep before setting the pointers, so the pointers might be behind their real places. They mention how to scan the arrays to update pointers while doing an operation. A process writes an element in the location head by a **CAS** instruction, so if  $p$  processes try to enqueue simultaneously, the step (and **CAS**) complexity remains  $\Omega(p)$ . Their design is lock-free but not wait-free.

Kogan and Petrank [18] introduced wait-free queues based on the MS-queue and use Herlihy's helping technique [12] to achieve wait-freedom. Their worst-case step complexity is  $\Omega(p)$  because of the helping mechanism.

Milman et al. [22] designed a lock-free queue supporting futures. In their queue, operations

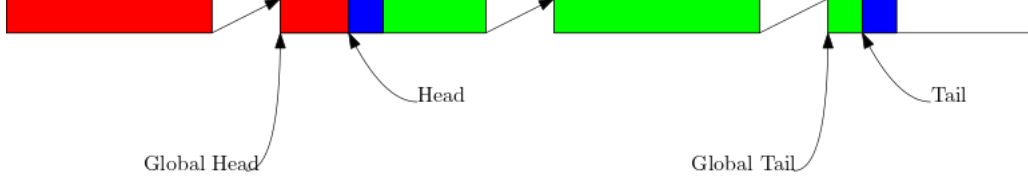


Figure 5: Global pointers point to arrays. Head and Tail elements are blue, dequeued elements are red and current elements of the queue are green.

return future objects instead of responses. Later when the response is needed, it can be evaluated from the future object. They also define a weaker linearizability condition such that each operation can be linearized between its invocation and when its future is evaluated. Their idea of batching allows a sequence of operations to be submitted as a batch for later execution on the MS-queue. They use some properties of the queue size before and after a batch, similar to a part of our work. Their queue is not wait-free: in fact, if the batch sizes are 1, then the queue is like MS-queue.

Nikolaev and Ravindran [25] present a wait-free queue that uses the fast-path slow-path methodology introduced by Kogan and Petrank [19]. Their work is based on a circular queue using bounded memory. When a process wishes to do an enqueue or a dequeue, it starts two paths. The fast path ensures good performance while the slow path ensures termination. They show that these two paths do not affect each other and the queue remains consistent. If a process makes no progress, other processes help its slow path to finish. The helping phase suffers from the CAS Retry Problem, because processes compete in a CAS loop to decide which succeeds to help. Because of this, the worst-case complexity cannot be better than  $\Omega(p)$ .

The CAS Retry Problem is not limited to list-based queues; array-based queues also share it as well [5, 27, 28]. Our motivation is to overcome this problem and present a wait-free sublinear queue.

## 2.2 Restricted Queues

David introduced the first sublinear concurrent queue [6]. Even though his algorithm does  $O(1)$  steps for each operation, it is a single-enqueuer single-dequeuer queue and uses infinite memory.

The author states that to reduce memory usage to be bounded, the time per operation increases linearly.

Jayanti and Petrovic introduced a wait-free poly-logarithmic multi-enqueuer single-dequeuer queue [16]. We use their idea of having a tournament tree among processes to agree on the linearization of operations to design a poly-logarithmic multi-enqueuer multi-dequeuer queue. Unlike their work, our algorithm does not put a limit on the number of concurrent dequeuers.

### 2.3 Universal Constructions and Other Poly-log Time Data Structures

A *universal construction* is an algorithm that can implement a shared version of any given sequential object. The first universal construction was introduced by Herlihy [12]. We can implement a concurrent queue using a universal construction. Jayanti proved an  $\Omega(\log p)$  lower bound on the worst-case shared-access time complexity of  $p$ -process universal constructions [15]. He also mentions that the universal construction by Afek, Dauber, and Touitou [1] can be modified to  $O(\log p)$  worst-case step complexity, using atomic access to  $\Omega(p \log p)$ -bit words. Chandra, Jayanti and Tan introduced a semi-universal construction that achieves  $O(\log^2 p)$  shared accesses [4]. However their algorithm cannot be used to create a queue. We mention a non-practical universal construction with a poly-log number of CAS instructions in the last paragraph of the Page 13.

Ellen and Woelfel introduced an implementation of a Fetch&Inc object with step complexity of  $O(\log p)$  using  $O(\log n)$ -bit LL/SC objects, where  $n$  is the number of operations [7]. Their idea to achieve logarithmic complexity is to use a tree storing the Fetch&Inc operations invoked by processes. When a process wants to do a Fetch&Inc it adds its Fetch&Inc to the tree and returns the number of elements in the tree. There are some similarities between designing a queue and a Fetch&Inc object. A Fetch&Inc object can be constructed from a queue. The algorithm by Ellen and Woelfel is interesting because of the similarities between Fetch&Inc objects and queues. Also, it is one of the few wait-free data structures achieving poly-log complexity.

## 2.4 Attiya–Fouren Lower Bound

Because of the CAS retry problem in previous list-based queues one might guess the  $\Omega(p)$  term is inherent in the time complexity of concurrent queues. Attiya and Fouren gave a lower bound on amortized complexity of lock-free queues with regard to  $c$ , the number of concurrent processes. Their result says if  $c$  is  $O(\log \log p)$ , any implementation of queues using reads, writes and conditional operations like CAS has  $\Omega(c)$  amortized complexity [2]. The surprising point is that since their result is about when contention is low, their lower bound does not contradict with our algorithm's complexity and we manage to reach poly-log time complexity.



### 3 Queue Implementation

In our model there are  $p$  processes doing **Enqueue** and **Dequeue** operations on a queue concurrently. We design a linearizable wait-free queue with  $O(\log^2 p + \log q)$  steps per operation, where  $q$  is the number of elements in the queue at the time of linearization. We avoid the  $\Omega(p)$  worst-case step complexity of existing shared queues based on linked lists or arrays, which suffer from the CAS Retry Problem.

There is a shared binary tree among the processes to agree on one total ordering of the operations invoked by processes. The tree is called a *tournament tree* (see Figure 6). Each process has a leaf in which the operations invoked by the process are stored in order. When a process wishes to do an operation it appends the operation to its leaf and tries to propagate its new operation up to the tree's root. Each node of the tree keeps an ordering of operations propagated up to it. All processes agree on the sequence of operations in the root and this ordering is used as the linearization ordering.

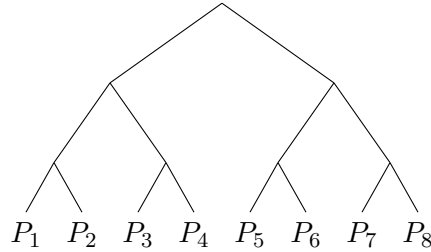


Figure 6: Each of the processes  $P_1, P_2, \dots, P_p$  has a leaf and in each node there is an ordering of operations stored. Each process tries to propagate its operations up to the root, which stores a total ordering of all operations.

To propagate operations to a node  $n$  in the tree, a process observes the operations in both of  $n$ 's children that are not already in  $n$ , merges them to create an ordering and then tries to append the ordering to the sequence stored in  $n$ . We call this procedure  $n$ .**Refresh** (see Figure 7). A **Refresh** on  $n$  with a successful append helps to propagate their operations up to  $n$ . We shall prove that if a process invokes **Refresh** on the node  $n$  two times and fails to append the new operations to  $n$

both times, the operations that were in  $n$ 's children before the first **Refresh** are guaranteed to be in  $n$  after the second failed **Refresh**. We sketch the argument here.

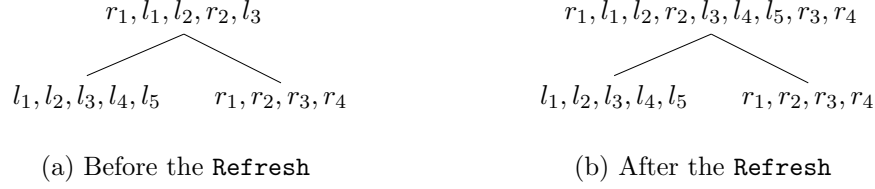


Figure 7: Before and after a  $n$ .**Refresh** with a successful append. Operations propagating from the left child are labelled with  $l$  and from the right child with  $r$ .

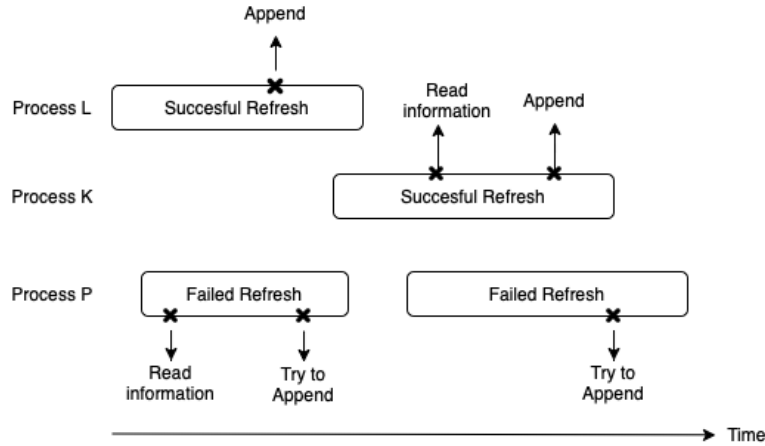


Figure 8: Time relations between the concurrent successful **Refreshes** and the two consecutive failed **Refreshes**.

We use **CAS** (Compare&Swap) instructions to implement the **Refresh**'s attempt to append described in the previous paragraph. The second failed **Refresh** of  $P$  is assuredly concurrent with a successful **Refresh** that has read its information after the invocation of the first failed **Refresh** (see Figure 8). This is because some process  $L$  does a successful append during  $P$ 's first failed attempt, and some process  $K$  performs a **Refresh** that reads its information after  $L$ 's append and then performs a successful append during  $P$ 's second failed **Refresh**. Process  $K$ 's **Refresh** helps to append the new operations that were in  $n$ 's children before  $P$ 's first failed **Refresh**, in case they were not already appended. After a process appends its operation into its leaf it can call **Refresh**

on the path up to the root at most two times on each node. So, with  $O(\log p)$  CASes an operation can ensure it appears in the linearization. This cooperative solution allows us to overcome the CAS Retry Problem.

It is not efficient to explicitly store the sequence of operations in each node because each operation would have to be copied all the way up to the root; doing this would not be possible in poly-logarithmic time. Instead we use an implicit representation of the operations propagated together. Furthermore, we do not need to maintain an ordering on operations propagated together in a node until they have reached the root. It is sufficient to only keep track of sets of operations propagated together in each **Refresh** and then define the linearization ordering only in the root (see Figure 9). Achieving a constant-sized implicit representation of operations in a **Refresh** allows us to do CAS instructions on fixed-size objects in each **Refresh**. To do that, we introduce *blocks*. A block stores information about the operations propagated by a **Refresh**. It contains the number of operations from the left and the right child propagated to the node by the **Refresh** procedure. See Figure 10 for an example. A node stores an array of blocks of operations propagated up to it. A propagate step aggregates the new blocks in the children into a new block and stores the new block in the parent. We call the aggregated blocks *subblocks* of the new block and the new block the *superblock* of them. In each **Refresh** there is at most one operation from each process trying to be propagated, because one process cannot invoke two operations concurrently. Thus, there are at most  $p$  operations in a block. Furthermore, since the operations in a **Refresh** step are concurrent we can linearize them among themselves in any order we wish, because if two operations are read in one successful **Refresh** step in a node they are going to be propagated up to the root together. Our choice is to put the operations propagated from the left child before the operations propagated from the right child. In this way, if we know the number of operations from the left child and the number of operations from the right child in a block, we have a complete ordering of the operations.

So far, we have a shared tree that processes use to agree on the implicit ordering stored in its root. With this agreement on the linearization ordering, we can design a universal construction; for a given object, we can perform an operation  $op$  by applying all the operations up until  $op$  in

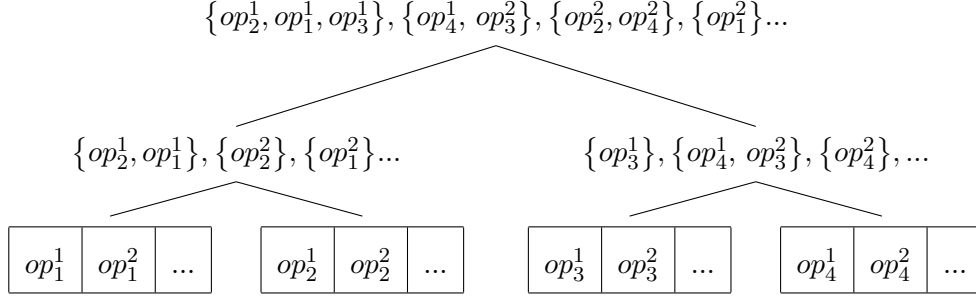


Figure 9: Leaves are for processes  $P_1$  to  $P_4$  from left to right. In each internal node one can arbitrarily linearize the sets of concurrent operations propagated together in a **Refresh**. For example  $op_4^1$  and  $op_3^2$  have propagated together in one **Propagate** step and they will be propagated up to the root together. Since their execution time intervals overlap, they can be linearized in any order.

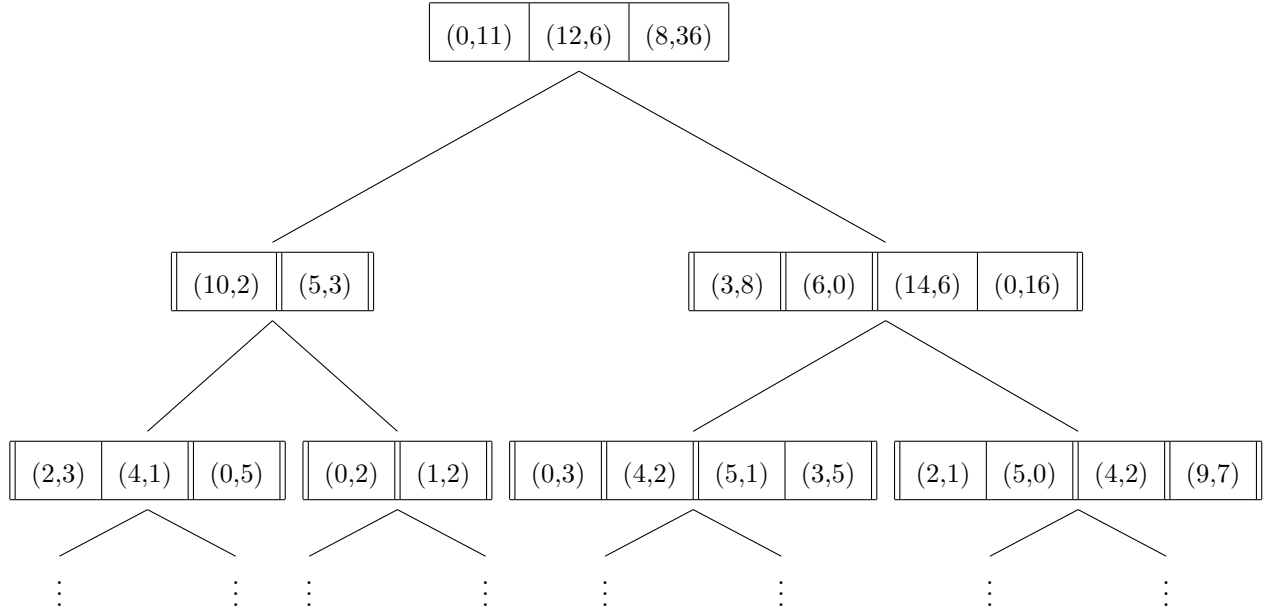


Figure 10: Using blocks to represent operations. Blocks between two lines  $||$  are propagated together to the parent. Each block consists of a pair (left, right) indicating the number of operations from the left and the right child, respectively. For example, (12,6) in the root contains (10,2) from the left and the right child, respectively. The third block in the root (8,36) is created by merging (5,3) from the left child and (14,6) and (0,16) from the right child. (5,3) is the superblock of (0,5) and (1,2) and (5,1), (3,5) and (4,2) are subblocks of (14,6).

the root on a local copy of the object and then returning response for  $op$ . However, this approach is not enough for an efficient queue. We show that we can build an efficient queue if we can compute two things about the ordering in the root: (1) the  $i$ th propagated operation and (2) the rank of a propagated operation in the linearization. We explain how to implement (1) and (2) in poly-logarithmic steps.

After propagating an operation  $op$  to the root, processes can find out information about the linearization ordering using (1) and (2). To get the  $i$ th operation in the root, we find the block  $B$  containing the  $i$ th operation in the root, and then recursively find the subblock of  $B$  in the descendent of the root that contains that  $i$ th operation. When we reach a block in a leaf, the operation is explicitly stored there. To make this search faster, instead of iterating over all blocks in the node, we store the prefix sum of the number of elements in the blocks sequence to permit a binary search for the required block. We also store pointers to determine the range of subblocks of a block to make the binary search faster. In each block, we store the prefix sum of operations from the left child and from the right child. Moreover, for each block, we store two attributes  $\mathbf{end}_{\text{left}}$  and  $\mathbf{end}_{\text{right}}$ , the indices of the last left and right subblock (see Figure 11). We know a block size is at most  $p$ , so binary search takes at most  $O(\log p)$  time, since the  $\mathbf{end}_{\text{left}}$  and  $\mathbf{end}_{\text{right}}$  indices of a block and its previous block reduce the search range size to  $O(p)$ .

To compute the rank in the root of an operation in a leaf, we need to find the superblock of the block that the operation is in. After a block is installed in a node we store the approximate index of its superblock in it to make this faster.

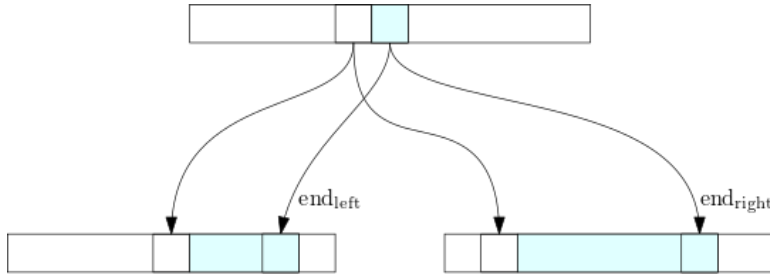


Figure 11: Each block stores the index of its last subblock in each child.

In an execution on a queue where no dequeue operation returns `null`, the  $k$ th dequeue returns the argument of the  $k$ th enqueue. In the general case a dequeue returns `null` if and only if the queue size after the previous operation is 0. We refer to such a dequeue as a *null dequeue*. If the dequeue is the  $k$ th non-null dequeue, it returns the argument of the  $k$ th enqueue. Having the size of the queue after an operation we can compute the number of non-null dequeues from the number of enqueues before the operation. So, if we store the size of the queue after each block of operations in the root, we can compute the index of the enqueue whose argument is the response to a given dequeue in constant time.

In our case of implementing a queue, a process only needs to compute the rank of a **Dequeue** and get an **Enqueue** with a specific position. We know we can linearize operations in a block in any order; here, we choose to put **Enqueue** operations in a block before **Dequeue** operations. Consider the following operations, where operations in a cell are concurrent.

Deq	Enq(5), Enq(2), Enq(1), Deq	Enq(3), Deq	Enq(4), Deq, Deq, Deq, Deq
-----	-----------------------------	-------------	----------------------------

Table 1: An Execution on a queue separated by the operations in the root blocks

The **Dequeue** operations return `null`, 5, 2, 1, 3, 4, `null`, respectively. Now, we claim that by knowing the size of the queue, we can compute the rank of the required **Enqueue** for any non-null **Dequeue**. We apply this approach to blocks; if we store the size of the queue after each block of operations, we can compute the index of each **Dequeue**'s result in  $O(1)$  steps.

	Deq	Enq(5), Enq(2), Enq(1), Deq	Enq(3), Deq	Enq(4), Deq, Deq, Deq, Deq
#Enqs	0	3	1	1
#Deqs	1	1	1	4
Size at end	0	2	2	0

Table 2: Augmented history of operation blocks on the queue.

The size of the queue after the  $b$ th block in the root could be computed as

$$\max\left(\text{size after } (b-1)\text{th block} + \#\text{Enqueues in } b\text{th block} - \#\text{Dequeues in } b\text{th block}, 0\right).$$

Moreover, the total number of non-null dequeues in blocks  $1, 2, \dots, b$  in the root is

$$\sum_{i=1}^b \#\text{Enqueues in } i\text{th block} - \text{size after } b\text{th block}.$$

Given a **Dequeue** is in block  $B$ , its response is the argument of the **Enqueue** whose rank is

$$\#\text{non-null Dequeues in blocks } 1, 2, \dots, b-1 + \text{index of the Dequeue among } B\text{'s Dequeue}$$

if 
$$\left(\text{size of the queue after } b-1\text{th block} + \#\text{Enqueues in } b\text{th block} - \text{index of Dequeue in } B\text{'s Dequeues}\right) \geq 0.$$

Otherwise, the response would be **null**.

### 3.1 Details of the Implementation

Section 3.2 gives the pseudocode for the queue implementation. It uses the following two types of objects.

**Node** In each **Node** we store pointers to its parent and its left and right child, an array of **Blocks** called **blocks** and the index **head** of the first empty entry in **blocks**.

**Block** The information stored in a **Block** depends on whether the **Block** is in an internal node or a leaf. If it is in a leaf, we use a **LeafBlock** which stores one operation. If a block  $B$  is in an internal node  $n$ , then it contains subblocks in the left and right children of  $n$ . The left subblocks of  $B$  are some consecutive blocks in the left child of  $n$  starting from where the left subblocks of the block prior to  $B$  ended. The right subblocks of  $B$  are defined similarly. In each **block** we store four essential fields that implicitly summarize which operations are in the block **sum<sub>enq-left</sub>**, **sum<sub>deq-left</sub>**, **sum<sub>enq-right</sub>**, **sum<sub>deq-right</sub>**. The **sum<sub>enq-left</sub>** field is the total number of **Enqueue** operations in the blocks before the last subblock of  $B$  in the left child. The other fields' semantics are similar. The

`endleft` and `endright` field store the index of the last subblock of a block in the left and the right child, respectively. The approximate index of the superblock of non-root blocks is stored in their `super` field. The `size` field in a block in the root node stores the size of the queue after the operations in the block have been performed.

We now describe the routines used in the implementation.

**Enqueue( $e$ )** An **Enqueue** operation does not return a response, so it is sufficient to propagate the **Enqueue** operation to the root and then use its position in the linearization for future **Dequeue** operations. **Enqueue( $e$ )** creates a **LeafBlock** with `element` =  $e$ , sets its `sumenq` and `sumdeq` fields and then appends it to the tree.

**Dequeue()** **Dequeue** creates a **LeafBlock**, sets its `sumenq` and `sumdeq` fields and appends it to the tree. Then, it computes the position of the appended **Dequeue** operation in the root using **IndexDequeue** and after that finds the response of the **Dequeue** by calling **FindResponse**.

**Append( $B$ )** The `head` field is the index of the first empty slot in `blocks` in a **LeafBlock**. There are no multiple write accesses on `head` and `blocks` in a leaf because only the process that the leaf belongs to appends to it. **Append( $B$ )** adds  $B$  to the end of the `blocks` field in the leaf, increments `head` and then calls **Propagate** on the leaf's `parent`. When **Propagate** terminates, it is guaranteed that the appended block is a subblock of a block in the `root`.

**Propagate()** **Propagate** on node  $n$  uses the double refresh idea described earlier and invokes two **Refreshes** on  $n$  in Lines 52 and 53. Then, it invokes **Propagate** on  $n.parent$  recursively until it reaches the root.

**Refresh() and Advance()** The goal of a **Refresh** on node  $n$  is to create a block of  $n$ 's children's new blocks and append it to  $n.blocks$ . The variable `h` is read from  $n.head$  at Line 60. The new block created by **Refresh** will be inserted into  $n.blocks[h]$ . Lines 61–66 of  $n.Refresh$  help to **Advance**  $n$ 's children. **Advance** increments the children's `head` if necessary and sets the `super` field



of their most recently appended blocks. The reason behind this helping is explained later when we discuss **IndexDequeue**. After helping to **Advance** the children, a new block called **new** is created in Line 67. Then, if **new** is empty, **Refresh** returns **true** because there are no new operations to propagate, and it is unnecessary to add an empty block to the tree. Later we will use the fact that all blocks contain at least one operation. Line 70 tries to install **new**. If it was successful, all is good. If not, it means someone else has already put a block in  $n.\text{blocks}[h]$ . In this case, **Refresh** helps advance  $n.\text{head}$  to  $h+1$  and update the **super** field of  $n.\text{blocks}[h]$  at Line 71.

**CreateBlock()**  $n.\text{CreateBlock}(h)$  is used by **Refresh** to construct a block containing new operations of  $n$ 's children. The block **new** is created in Line 80 and its fields are filled similarly for both left and right directions. The variable  $\text{index}_{\text{prev}}$  is the index of the block preceding the first subblock in the child in direction **dir** that is aggregated into **new**. Field  $\text{new}.\text{end}_{\text{dir}}$  stores the index of the rightmost subblock of **new** in the child. Then  $\text{sum}_{\text{enq-dir}}$  is computed from the sum of the number of **Enqueue** operations in the **new** block from direction **dir** and the value stored in  $n.\text{blocks}[h-1].\text{sum}_{\text{enq-dir}}$ . The field  $\text{sum}_{\text{deq-dir}}$  is computed similarly. Then, if **new** is going to be installed in the **root**, the **size** field is also computed.

**IndexDequeue**( $b, i$ ) A call to  $n.\text{IndexDequeue}(b, i)$  computes the block and the rank within the block in the root of the  $i$ th **Dequeue** of the  $b$ th block of  $n$ . Let  $R_n$  be the successful **Refresh** on node  $n$  that did a successful  $\text{CAS}(\text{null}, B)$  into  $n.\text{blocks}[b]$ . Let  $par$  be  $n.\text{parent}$ . Without loss of generality, assume for the rest of this section that  $n$  is the left child of  $par$ . Let  $R_{par}$  be the first successful  $par.\text{Refresh}$  that reads some value greater than  $b$  for **left.head** and therefore contains  $B$  as a subblock of its created block in Line 67. Let  $j$  be the index of the block that  $R_{par}$  puts in  $par.\text{blocks}$ .

Since the index of the superblock of  $B$  is not known until  $B$  is propagated,  $R_n$  cannot set the **super** field of  $B$  while creating  $B$ . One approach for  $R_{par}$  is to set the **super** field of  $B$  after propagating  $B$  to  $par$ . This solution would not be efficient because there might be up to  $p$

subblocks that  $R_{par}$  propagated, which need to update their **super** field. However, intuitively, once  $B$  is installed, its superblock is going to be close to  $n.parent.head$  at the time of installation. If we know the approximate position of the superblock of  $B$  then we can search for the real superblock when it is needed. Thus,  $B.super$  does not have to be the exact location of the superblock of  $B$ , but we want it to be close to  $j$ . We can set  $B.super$  to  $par.head$  while creating  $B$ , but the problem is that there might be many **Refreshes** on  $par$  that could happen after  $R_n$  reads  $par.head$  and before propagating  $B$  to  $par$ . If  $R_n$  sets  $B.super$  to  $par.head$  after appending  $B$  to  $n.blocks$  (Line 76),  $R_n$  might go to sleep at some time after installing  $B$  and before setting  $B.super$ . In this case, the next **Refreshes** on  $n$  and  $par$  help fill in the value of  $B.super$ .

Block  $B$  is appended to  $n.blocks[b]$  on Line 70. After appending  $B$ ,  $B.super$  is set on Line 76 of a call to **Advance** from  $n.Refresh$  by the same or another process or by Line 64 of a  $n.parent.Refresh$ . We shall show that this is sufficient to ensure that  $B.super$  differs from the index of  $B$ 's superblock by at most 1.

**FindResponse**( $b, i$ ) To compute the response of the  $i$ th **Dequeue** in the  $b$ th block of the root Line 19 computes whether the queue is empty or not. If there are more **Dequeues** than **Enqueues** the queue would become empty before the requested **Dequeue**. If the queue is not empty, Line 22 computes the rank  $e$  of the **Enqueue** whose argument is the response to the **Dequeue**. Knowing the response is the  $e$ th **Enqueue** in the root (which is before the  $b$ th block), we find the block and position containing the **Enqueue** operation using **DoublingSearch** and after that **GetEnqueue** finds its **element**.

**GetEnqueue**( $b, i$ ) and **DoublingSearch**( $e, end$ ) We can describe an operation in a node in two ways: the rank of the operation among all the operations in the node or the index of the block containing the operation in the node and the rank of the operation within that block. If we know the block and rank within the block of an operation, we can find the subblock containing the operation and the operation's rank within that subblock in poly-log time. To find the response of

a `Dequeue`, we know about the rank of the response `Enqueue` in the root (`e` in Line 22). We also know the `eth Enqueue` is in `root.blocks[1..end]`. `DoublingSearch` uses doubling to find the range that contains the answer block (Lines 38–41) and then tries to find the required indices with a binary search (Line 42). A call to `n.GetEnqueue(b, i)` returns the `element` of the  $i$ th enqueue in the  $b$ th block of  $n$ . The range of subblocks of a block is determined using the `endleft` and `endright` fields of the block and its previous block. Then, the subblock is found using binary search on the `sumenq` field (Lines 99 and 103).

## 3.2 Pseudocode

We present our algorithm in pseudocode. Page 23 contains the description of the fields in the tree nodes and the blocks. Value of any uninitialized field is `null`. Page 24 contains major routines and the rest of this section consists of the auxiliary routines. The abbreviations below are used in the pseudocode and the proof of correctness.

- `blocks[b].sumx = blocks[b].sumx-left + blocks[b].sumx-right` (for internal blocks where  $b \geq 0$  and  $x \in \{\text{enq}, \text{deq}\}$ )
- `blocks[b].numx = blocks[b].sumx - blocks[b-1].sumx` (for all blocks where  $b > 0$  and  $x \in \{\text{enq}, \text{deq}, \text{enq-left}, \text{enq-right}, \text{deq-left}, \text{deq-right}\}$ )

---

**Algorithm** Tree Fields Description

---

◊ *Shared*

- A binary tree of *Nodes* with one leaf for each process. *root* is the root node.

◊ *Local*

- *Node leaf*: process's leaf in the tree.

► *Node*

- *\*Node left, right, parent* : Initialized when creating the tree.
- *Block[] blocks* : Initially *blocks[0]* contains an empty block with all fields equal to 0.
- *int head= 1*: #blocks in *blocks*. *blocks[0]* is a block with all integer fields equal to zero.

► *Block*

- *int super* : approximate index of the superblock, read from *parent.head* when appending the block to the node

► *InternalBlock* extends *Block*

- *int end<sub>left</sub>, end<sub>right</sub>* : indices of the last subblock of the block in the left and right child
- *int sum<sub>enq-left</sub>*: #enqueues in *left.blocks[1..end<sub>left</sub>]*
- *int sum<sub>deq-left</sub>*: #dequeues in *left.blocks[1..end<sub>left</sub>]*
- *int sum<sub>enq-right</sub>*: #enqueues in *right.blocks[1..end<sub>right</sub>]*
- *int sum<sub>deq-right</sub>*: #dequeues in *right.blocks[1..end<sub>right</sub>]*

► *LeafBlock* extends *Block*

- *Object element* : Each block in a leaf represents a single operation. If the operation is *enqueue(x)* then *element=x*, otherwise *element=null*.
- *int sum<sub>enq</sub>, sum<sub>deq</sub>* : # enqueue, dequeue operations in this block and its previous blocks in the leaf

► *RootBlock* extends *InternalBlock*

- *int size* : size of the queue after performing all operations in this block and its previous blocks in the root
-

---

**Algorithm Queue**

---

```
1: void Enqueue(Object e)                                ▷ Creates a block with element e and adds it to the tree.
2:   block newBlock= new(LeafBlock)
3:   newBlock.element= e
4:   newBlock.sumenq= leaf.blocks[leaf.head].sumenq+1
5:   newBlock.sumdeq= leaf.blocks[leaf.head].sumdeq
6:   leaf.Append(newBlock)
7: end Enqueue

8: Object Dequeue()  ▷ Creates a block with null value element, appends it to the tree and returns its response.
9:   block newBlock= new(LeafBlock)
10:  newBlock.element= null
11:  newBlock.sumenq= leaf.blocks[leaf.head].sumenq
12:  newBlock.sumdeq= leaf.blocks[leaf.head].sumdeq+1
13:  leaf.Append(newBlock)
14:  <b, i>= IndexDequeue(leaf.head, 1)
15:  output= FindResponse(b, i)
16:  return output
17: end Dequeue

18: element FindResponse(int b, int i)                    ▷ Returns the response to  $D_i(root, b)$ , the  $i$ th Dequeue in
   root.blocks[b].
19:   if root.blocks[b-1].size + root.blocks[b].numenq - i < 0 then    ▷ Check if the queue is empty.
20:     return null
21:   else
22:     e= i + (root.blocks[b-1].sumenq-root.blocks[b-1].size)        ▷ The response is  $E_e(root)$ , the  $e$ th
   Enqueue in the root.
23:     return root.GetEnqueue(root.DoublingSearch(e, b))
24:   end if
25: end FindResponse
```

---

---

**Algorithm** *Node*

---

$\rightsquigarrow$  Precondition: `blocks[start..end]` contains a block with `sumenq` greater than or equal to `x`

▷ Does a binary search for the value `x` of `sumenq` field and returns the index of the leftmost block in `blocks[start..end]` whose `sumenq` is  $\geq x$ .

```
26: int BinarySearch(int x, int start, int end)
27:   while start<=end do
28:     int mid= (start+end)/2
29:     if blocks[mid].sumenq<x then
30:       start= mid+1
31:     else
32:       end= mid
33:     end if
34:   end while
35:   return start
36: end BinarySearch
```

---

---

**Algorithm** *Root*

---

$\rightsquigarrow$  Precondition: `root.blocks[end].sumenq  $\geq e$`

▷ Returns `<b,i>` such that  $E_e(\text{root}) = E_i(\text{root}, b)$ , i.e., the `e`th Enqueue in the `root` is the `i`th Enqueue within the `b`th block in the `root`.

```
37: <int, int> DoublingSearch(int e, int end)
38:   start= end-1
39:   while root.blocks[start].sumenq>=e do
40:     start= max(start-(end-start), 0)
41:   end while
42:   b= root.BinarySearch(e, start, end)
43:   i= e- root.blocks[b-1].sumenq
44:   return <b,i>
45: end DoublingSearch
```

---

---

**Algorithm** *Leaf*

---

```
46: void Append(block B) ▷ Only called by the owner of the leaf.
47:   blocks[head]= B
48:   head= head+1
49:   parent.Propagate()
50: end Append
```

---

---

**Algorithm** *Node*

---

```
▷ n.Propagate propagates operations in this.children up to this when it terminates.
51: void Propagate()
52:   if not Refresh() then
53:     Refresh()
54:   end if
55:   if this is not root then
56:     parent.Propagate()
57:   end if
58: end Propagate

▷ Creates a block containing new operations of this.children, and then tries to append it to this.
59: boolean Refresh()
60:   h= head
61:   for each dir in {left, right} do
62:     hdir= dir.head
63:     if dir.blocks[hdir]!=null then
64:       dir.Advance(hdir)
65:     end if
66:   end for
67:   new= CreateBlock(h)
68:   if new.num==0 then return true
69:   end if
70:   result= blocks[h].CAS(null, new)
71:   this.Advance(h)
72:   return result
73: end Refresh
```

---



---

**Algorithm Node**

---

```
74: void Advance(int h)                                ▷ Sets blocks[h].super and increments head from h to h+1.
75:     hp = parent.head
76:     blocks[h].super.CAS(null, hp)
77:     head.CAS(h, h+1)
78: end Advance

79: Block CreateBlock(int i)                            ▷ Creates and returns the block to be installed in blocks[i].
80:     block new = new(InternalBlock)
81:     for each dir in {left, right} do
82:         indexprev = blocks[i-1].enddir
83:         new.enddir = dir.head-1                      ▷ new contains dir.blocks[blocks[i-1].enddir..dir.head-1].
84:         blockprev = dir.blocks[indexprev]
85:         blocklast = dir.blocks[new.enddir]
86:         new.sumenq-dir = blocks[i-1].sumenq-dir + blocklast.sumenq - blockprev.sumenq
87:         new.sumdeq-dir = blocks[i-1].sumdeq-dir + blocklast.sumdeq - blockprev.sumdeq
88:     end for
89:     if this is root then
90:         new.type = InternalBlock-->RootBlock
91:         new.size = max(root.blocks[i-1].size + new.numenq - new.numdeq, 0)
92:     end if
93:     return new
94: end CreateBlock
```

---

```

95: element GetEnqueue(int b, int i)                                ▷ Returns the element of  $E_i(\text{this}, b)$ .
96:     if this is leaf then
97:         return blocks[b].element
98:     else if i <= blocks[b].numenq-left then                    ▷  $E_i(\text{this}, b)$  is in the left child of this node.
99:         subblockIndex= left.BinarySearch(i+blocks[b-1].sumenq-left, blocks[b-1].endleft+1,
                                     blocks[b].endleft)
100:        return left.GetEnqueue(subblockIndex, i)
101:     else
102:         i= i-blocks[b].numenq-left
103:         subblockIndex= right.BinarySearch(i+blocks[b-1].sumenq-right, blocks[b-1].endright+1,
                                     blocks[b].endright)
104:        return right.GetEnqueue(subblockIndex, i)
105:     end if
106: end GetEnqueue

```

```

107: <int, int> IndexDequeue(int b, int i)                                ▷ Returns <x, y> if  $D_i(\text{this}, b) = D_y(\text{root}, x)$ .
108:     if this is root then
109:         return <b, i>
110:     else
111:         dir= (parent.left==n ? left: right)
112:         superblockIndex= parent.blocks[blocks[b].super].sumdeq-dir > blocks[b].sumdeq ?
                                blocks[b].super: blocks[b].super+1
113:         if dir is left then
114:             i+= blocks[b-1].sumdeq-parent.blocks[superblockIndex-1].sumdeq-left
115:         else
116:             i+= blocks[b-1].sumdeq-parent.blocks[superblockIndex-1].sumdeq-right
117:             i+= parent.blocks[superblockIndex].numdeq-left
118:         end if
119:         return this.parent.IndexDequeue(superblockIndex, i)
120:     end if
121: end IndexDequeue

```

### 3.3 Example Execution

Here we want to demonstrate how four processes help each other to propagate their operations up to the root and then compute their responses.

Initially, each node in the tree contains one empty block. In Figure 12, processes  $P1, P2, P4$  have appended the blocks containing  $ENQ_1, DEQ_2, DEQ_4$ . Line 47 appends a block to the process's leaf.

Then as in Figure 13,  $P2$  does a successful **Refresh** on the node  $n5$  that propagates both the operations of  $P1$  and  $P2$ .  $P4$  does also a successful **Refresh** on the node  $n6$  and propagates  $DEQ_4$ .  $P1$ 's **Refresh** on  $n5$  is slower than  $P2$ 's and creates its new block after  $P2$ 's successful **CAS** that propagated  $ENQ_1$  and  $ENQ_2$ . So the block created by  $P1$  is empty as there is nothing to propagate ( $P1$ 's Line 60 is after  $P2$ 's Line 70).

Since the block that  $P1$  creates is empty, it returns **true** in Line 68 as shown in Figure 14.  $P3$  propagates  $ENQ_3$  to  $n6$  and  $P4$  is very fast and it propagates  $DEQ_4$  to the root even before  $P2$  puts its block into  $n5$ .

Then as shown in Figure 15,  $P4$  goes to sleep, and  $P2$  helps to increment **root.head** after its failed **CAS** on the root. After incrementing **root.head**,  $P3$  succeeds to do **root.Refresh** with a single **CAS** on the root and puts operations from  $P1, P2$  and  $P3$  into the root.  $P1$  fails to do **Refresh** on the root two times, but it is guaranteed that its operation is propagated before its second failed **CAS** on **root.blocks**.

All the fields of selected blocks from Figure 15 are shown in Figure 16.  $P1$  and  $P3$ 's operations terminated after that they appended the **Enqueues** to the root.  $P2$  needs to compute the index of the block that its **Dequeue** is in. It computes the superblock of its leaf block, which is the second block in  $n5$ . Then,  $P2$  computes the superblock of the block containing  $ENQ_1, DEQ_2$ . As we can see the **super** field of  $ENQ_1, DEQ_2$  is 1 but the index of its superblock is 2. The queue size after the previous block is 0 so  $DEQ_2$ 's response is  $ENQ_1$ 's element. Then  $P2$  tries to get  $ENQ_1$ 's element.  $ENQ_1$  is in the left subblock of block  $ENQ_1, ENQ_3, DEQ_2$  because 1 is less than the

.`sumenq-left` field of the block containing  $ENQ_1, ENQ_3, DEQ_2$ . Similarly ,  $P_2$  determines that  $ENQ_1$  came from the left child of  $n_5$ .  $P_2$  therefore reaches the leaf  $n_1$  and returns  $ENQ_1$ 's element.  $P_4$  similarly finds the block in the `root` that contains  $DEQ_4$ . Since it is the first operation in the block and the size of the queue after the previous (empty) block is 0,  $DEQ_4$  returns `null`.

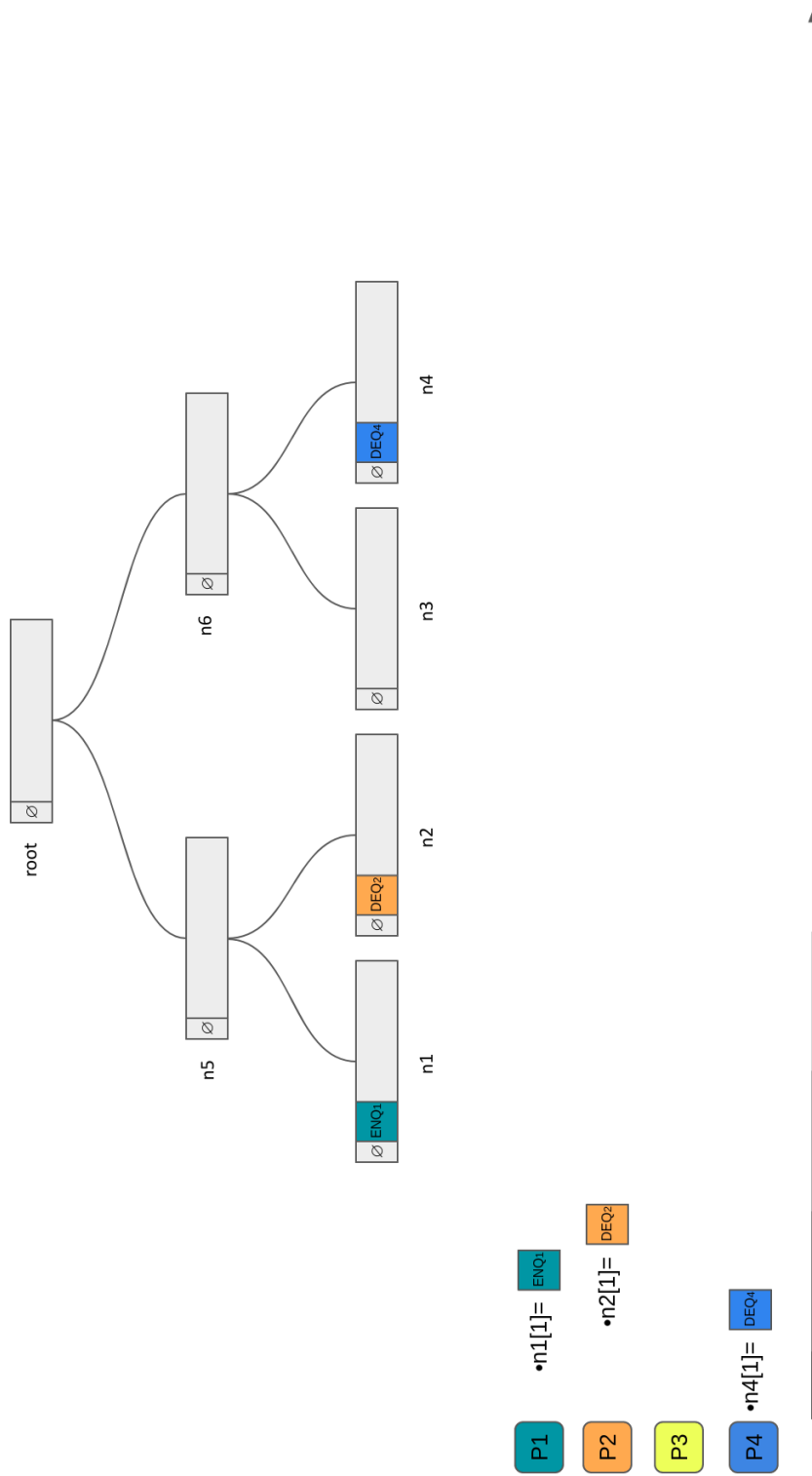


Figure 12:  $P1, P2, P4$  append operations to their leaves.

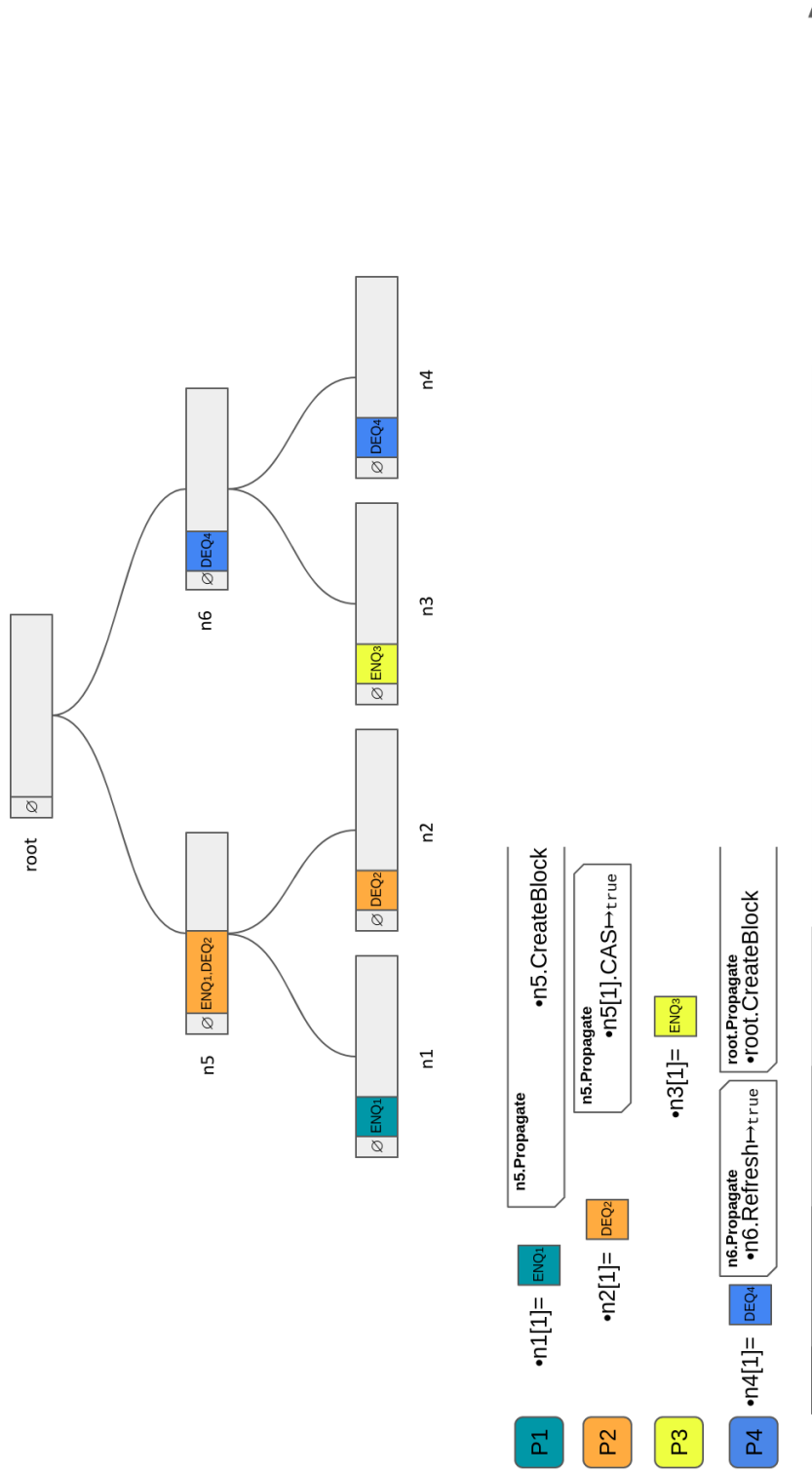


Figure 13: In the diagram, we use  $n[i]$  as an abbreviation for  $n.blocks[i]$ .  $P2$  helps  $P1$ .  $P4$  is faster than  $P3$ .

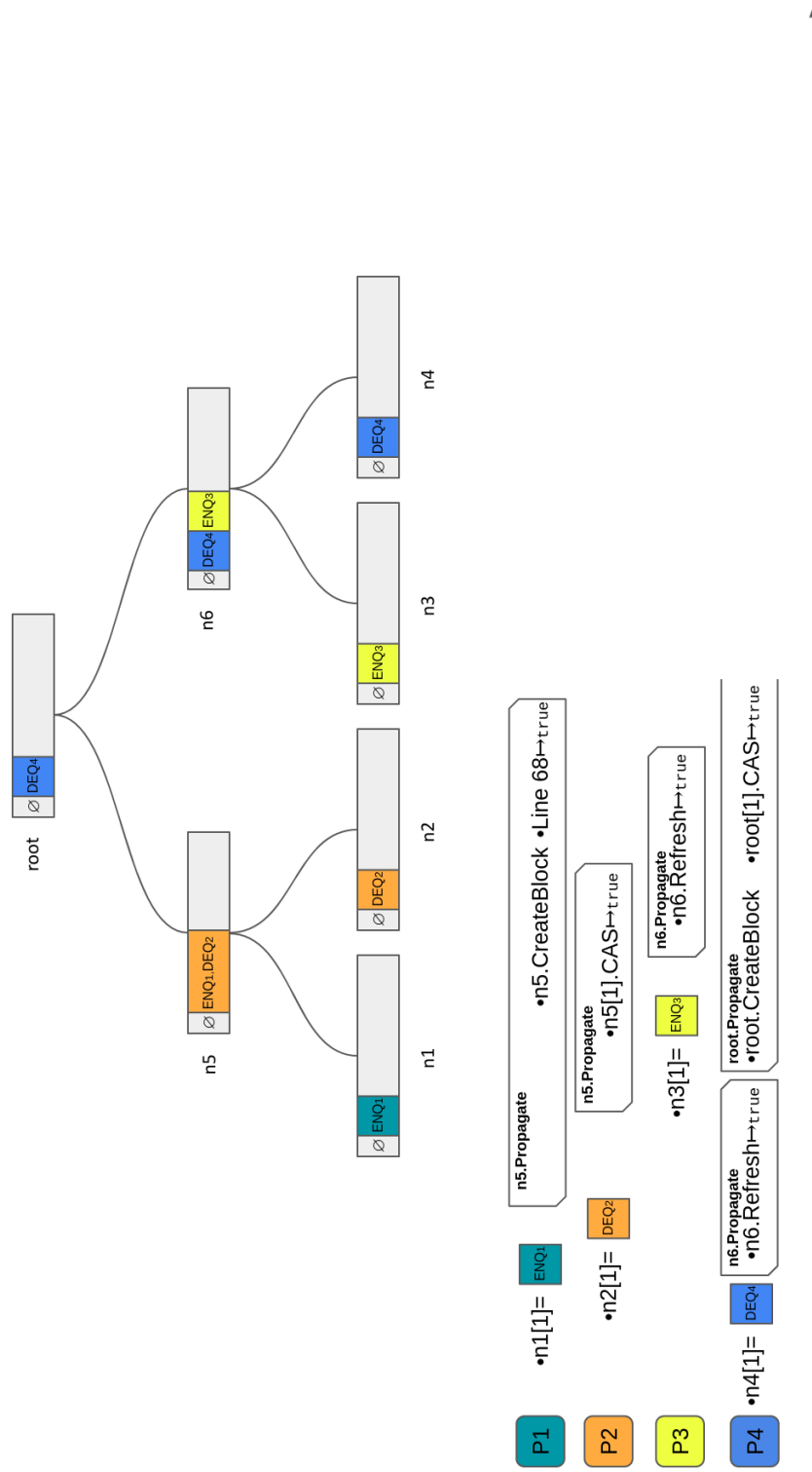


Figure 14:  $P_4$  puts its block into the root before  $P_2$  appends its block to  $n5$ .

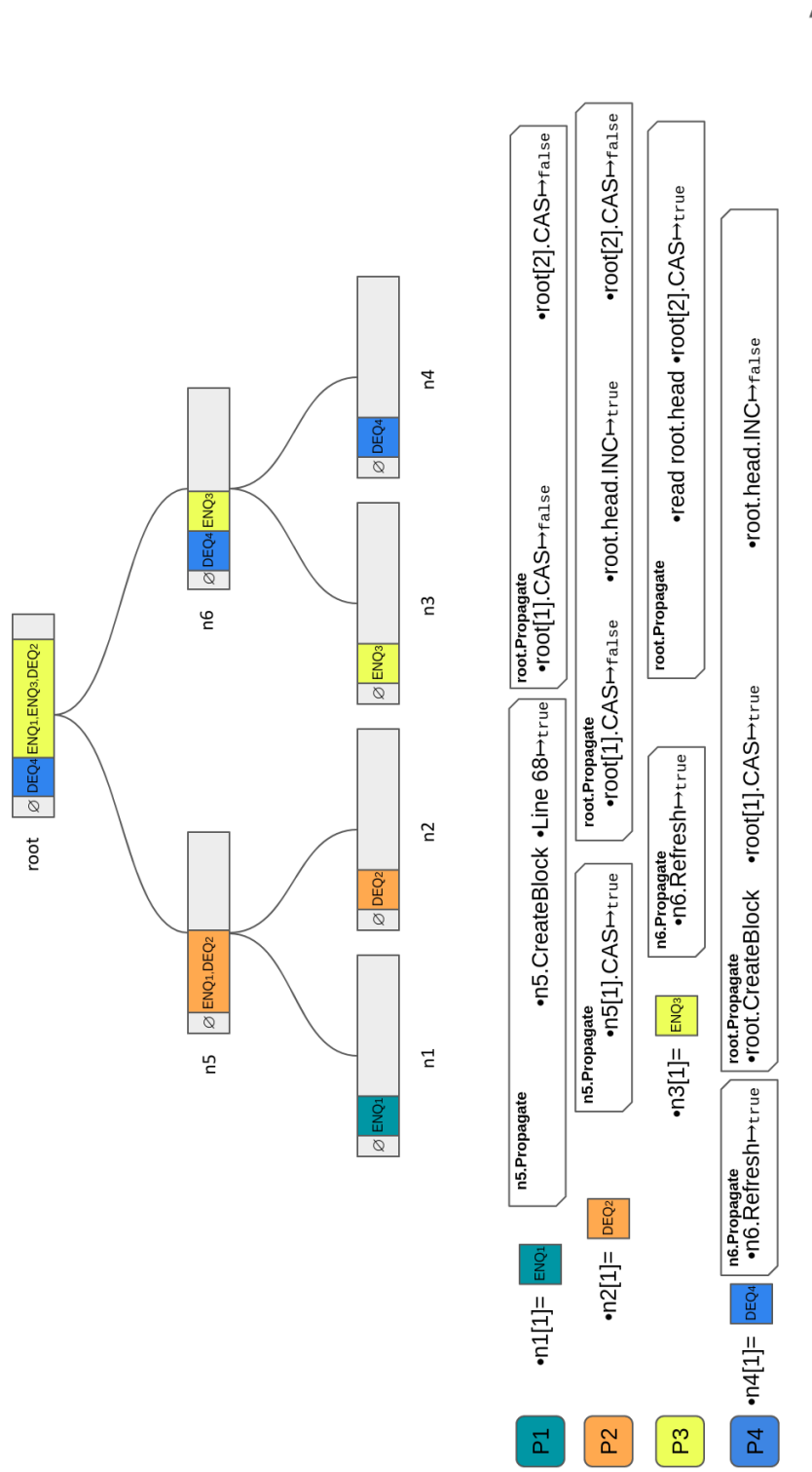


Figure 15:  $P_3$ 's CAS on `root.blocks` succeeds.  $P_1$  and  $P_2$ 's operations are done, each after two failed CAS attempts on `root.blocks`.





## 4 Proof of Correctness

We adopt linearizability as our definition of correctness. In our case, where we create the linearization ordering in the root, we need to prove (1) the ordering is legal, i.e, for every execution on our queue if operation  $op_1$  terminates before operation  $op_2$  then  $op_1$  is linearized before operation  $op_2$  and (2) if we do operations sequentially in their linearization order, operations get the same results as in our queue. The proof is structured like this. First, we define and prove some facts about blocks and the node's `head` field. Then, we introduce the linearization ordering formally. Next, we prove double `Refresh` on a node is enough to propagate its children's new operations up to the node, which is used to prove (1). After this, we prove some claims about the size and operations of each block, which we use to prove the correctness of `DoublingSearch()`, `GetEnqueue()` and `IndexDequeue()`. Finally, we prove the correctness of the way we compute the response of a dequeue, which establishes (2).

### 4.1 Basic Properties

In this subsection, we talk about some properties of blocks and fields of the tree nodes.

A block is an object storing some statistics, as described in Algorithm Queue. A block in a node implicitly represents a set of operations.

**Definition 1** (Ordering of a block in a node). Let  $B$  be  $n.\text{blocks}[b]$  and  $B'$  be  $n.\text{blocks}[j]$ . We call  $i$  the *index* of block  $B$ . Block  $B$  is *before* block  $B'$  in node  $n$  if and only if  $i < j$ .

Next, we show that the value of `head` in a node can only be increased. By the termination of a `Refresh`, `head` has been incremented by the process doing the `Refresh` or by another process.

**Observation 2.** *For each node  $n$ ,  $n.\text{head}$  is non-decreasing over time.*

*Proof.* The claim follows trivially from the code since `head` is only changed by incrementing in Line 77 of `Advance`. □

**Lemma 3.** *Let  $R$  be an instance of **Refresh** on a node  $n$  that creates a non-empty block  $\text{new}$  ( $\text{new.sum} \neq 0$ ). After  $R$  terminates,  $n.\text{head}$  is greater than the value read in Line 60 of  $R$ .*

*Proof.* If the CAS in Line 77 is successful, then the claim holds. Otherwise,  $n.\text{head}$  has changed from the value read in Line 60. By Observation 2 this means another process has incremented  $n.\text{head}$ . □

Now we show  $n.\text{blocks}[n.\text{head}]$  is either the last block written into node  $n$  or the first empty block in  $n$ .

**Invariant 4** (headPosition). If the value of  $n.\text{head}$  is  $h$  then  $n.\text{blocks}[i] = \text{null}$  for  $i > h$  and  $n.\text{blocks}[i] \neq \text{null}$  for  $0 \leq i < h$ .

*Proof.* Initially the invariant is true since  $n.\text{head} = 1$ ,  $n.\text{blocks}[0] \neq \text{null}$  and  $n.\text{blocks}[x] = \text{null}$  for every  $x > 0$ . The truth of the invariant may be affected by writing into  $n.\text{blocks}$  or incrementing  $n.\text{head}$ . We show that if the invariant holds before such a change, then it still holds after the change.

In the algorithm,  $n.\text{blocks}$  is modified only on Line 70, which updates  $n.\text{blocks}[h]$  where  $h$  is the value read from  $n.\text{head}$  in Line 60. Since the CAS in Line 70 is successful, it means  $n.\text{head}$  has not changed from  $h$  before doing the CAS: if  $n.\text{head}$  had changed before the CAS then it would be greater than  $h$  by Observation 2 and hence  $n.\text{blocks}[h] \neq \text{null}$  and by the induction hypothesis, so the CAS would fail. Writing into  $n.\text{blocks}[h]$  when  $h = n.\text{head}$  preserves the invariant, since the claim does not talk about the content of  $n.\text{blocks}[n.\text{head}]$ .

The value of  $n.\text{head}$  is modified only in Line 77 of **Advance**. If  $n.\text{head}$  is incremented to  $h + 1$  it is sufficient to show  $n.\text{blocks}[h] \neq \text{null}$ . **Advance** is called in Lines 64 and 71. For Line 64,  $n.\text{blocks}[h] \neq \text{null}$  because of the if condition in Line 63. For Line 71, Line 70 was finished before doing Line 71. Whether Line 70 is successful or not,  $n.\text{blocks}[h] \neq \text{null}$  after the  $n.\text{blocks}[h].\text{CAS}$ . □

We define the subblocks of a block recursively.

**Definition 5** (Subblock). A block is a *direct subblock* of the  $i$ th block in node  $n$  if it is in

$$n.\text{left.blocks}[n.\text{blocks}[i-1].\text{end}_{\text{left}}+1 \cdots n.\text{blocks}[i].\text{end}_{\text{left}}]$$

or in

$$n.\text{right.blocks}[n.\text{blocks}[i-1].\text{end}_{\text{right}}+1 \cdots n.\text{blocks}[i].\text{end}_{\text{right}}].$$

Block  $B$  is a *subblock* of block  $C$  if  $B$  is a direct subblock of  $C$  or a subblock of a direct subblock of  $C$ . We say block  $B$  has been *propagated* to node  $n$  if  $B$  is in  $n.\text{blocks}$  or is a subblock of a block in  $n.\text{blocks}$ .

The following lemma is used to prove that subblocks of two blocks in a node are disjoint.

**Lemma 6.** *If  $n.\text{blocks}[b] \neq \text{null}$  ( $b > 0$ ) then  $n.\text{blocks}[i].\text{end}_{\text{left}} \geq n.\text{blocks}[i-1].\text{end}_{\text{left}}$  and  $n.\text{blocks}[i].\text{end}_{\text{right}} \geq n.\text{blocks}[i-1].\text{end}_{\text{right}}$ .*

*Proof.* Consider the block  $B$  written into  $n.\text{blocks}[b]$  by CAS at Line 70. Block  $B$  is created by the `CreateBlock( $b$ )` called at Line 67. Prior to this call to `CreateBlock( $b$ )`,  $n.\text{head} = b$  at Line 60, so  $n.\text{blocks}[b-1]$  is already a non-null value  $B'$  by Invariant 4. Thus, the `CreateBlock( $b-1$ )` that created  $B'$  has terminated before the invocation of `CreateBlock( $b$ )` that created  $B$ . The value written into  $B.\text{end}_{\text{left}}$  at Line 83 of `CreateBlock( $b$ )` was one less than the value of  $n.\text{left.head}$  read at Line 83 of `CreateBlock( $b$ )`. Similarly, the value in  $n.\text{blocks}[b-1].\text{end}_{\text{left}}$  was one less than the value read from  $n.\text{left.head}$  during the call to `CreateBlock( $b-1$ )`. By Observation 2,  $n.\text{left.head}$  is non-decreasing, so  $B'.\text{end}_{\text{left}} \leq B.\text{end}_{\text{left}}$ . The proof for  $\text{end}_{\text{right}}$  is similar.  $\square$

**Lemma 7.** *Set of the subblocks of any two blocks in a node are distinct.*

*Proof.* We are going to prove the lemma by contradiction. Consider the lowest node  $n$  in the tree that violates the claim. Then, subblocks of  $n.\text{blocks}[i]$  and  $n.\text{blocks}[j]$  overlap for some  $i < j$ . Since  $n$  is the lowest node in the tree violating the claim, direct subblocks of blocks of  $n.\text{blocks}[i]$  and  $n.\text{blocks}[j]$  have to overlap. Without loss of generality, assume left child subblocks of  $n.\text{blocks}[i]$  overlap with the left child subblocks of  $n.\text{blocks}[j]$ . By Lemma 6 we

have  $n.\text{blocks}[i].\text{end}_{\text{left}} \leq n.\text{blocks}[j-1].\text{end}_{\text{left}}$ , so the range  $[n.\text{blocks}[i-1].\text{end}_{\text{left}} + 1 \dots n.\text{blocks}[i].\text{end}_{\text{left}}]$  cannot have overlap with the range  $[n.\text{blocks}[j-1].\text{end}_{\text{left}} + 1 \dots n.\text{blocks}[j].\text{end}_{\text{left}}]$ . Therefore, direct subblocks of  $n.\text{blocks}[i]$  and  $n.\text{blocks}[j]$  cannot overlap, which is in contradiction with the assumption.  $\square$

**Definition 8** (Superblock). Block  $B$  is *superblock* of block  $C$  if  $C$  is a direct subblock of  $B$ .

**Corollary 9.** *Every block has at most one superblock.*

*Proof.* A block having more than one superblock contradicts Lemma 7.  $\square$

Now we can define the operations of a block using the definition of subblocks.

**Definition 10** (Operations of a block). A block  $B$  in a leaf represents an **Enqueue** if  $B.\text{element} \neq \text{null}$ . Otherwise, if  $B.\text{element} = \text{null}$ ,  $B$  represents a **Dequeue**. The set of operations of block  $B$  is the union of the operations in leaf subblocks of  $B$ . We denote the set of operations of block  $B$  by  $\text{ops}(B)$  and the union of operations of a set of blocks  $\mathcal{B}$  by  $\text{ops}(\mathcal{B})$ . We also say  $B$  contains  $op$  if  $op \in \text{ops}(B)$ .

The next lemma proves that each operation appears at most once in the blocks of a node.

**Lemma 11.** *For any node  $n$ , if  $op$  is in  $n.\text{blocks}[i]$  then there is no  $j \neq i$  such that  $op$  is in  $n.\text{blocks}[j]$ .*

*Proof.* We prove this claim by contradiction using Lemma 7. Assume  $op$  is in the subblocks of both  $n.\text{blocks}[i]$  and  $n.\text{blocks}[j]$ . From Lemma 7 we know that the subblocks of these blocks are different, so there are two leaf blocks containing  $op$ . Since each process puts each operation in only one block of its leaf,  $op$  cannot be in two leaf blocks. This is a contradiction.  $\square$

**Definition 12.**  $n.\text{blocks}[i]$  is *established* if  $n.\text{head} > i$ . An operation is *established* in node  $n$  if it is in an established block of  $n$ .  $EST_n^t$  is the set of established operations in node  $n$  at time  $t$ .

Now we want to say that blocks of a node grow over time.

**Observation 13.** *If time  $t < \text{time } t'$  ( $t$  is before  $t'$ ), then  $\text{ops}(n.\text{blocks})$  at time  $t$  is a subset of  $\text{ops}(n.\text{blocks})$  at time  $t'$ .*

*Proof.* Blocks are only appended (not modified) with CAS to  $n.\text{blocks}[n.\text{head}]$ , so the set of the blocks of a node after the CAS contains the set of the blocks before the CAS.  $\square$

## 4.2 Ordering Operations

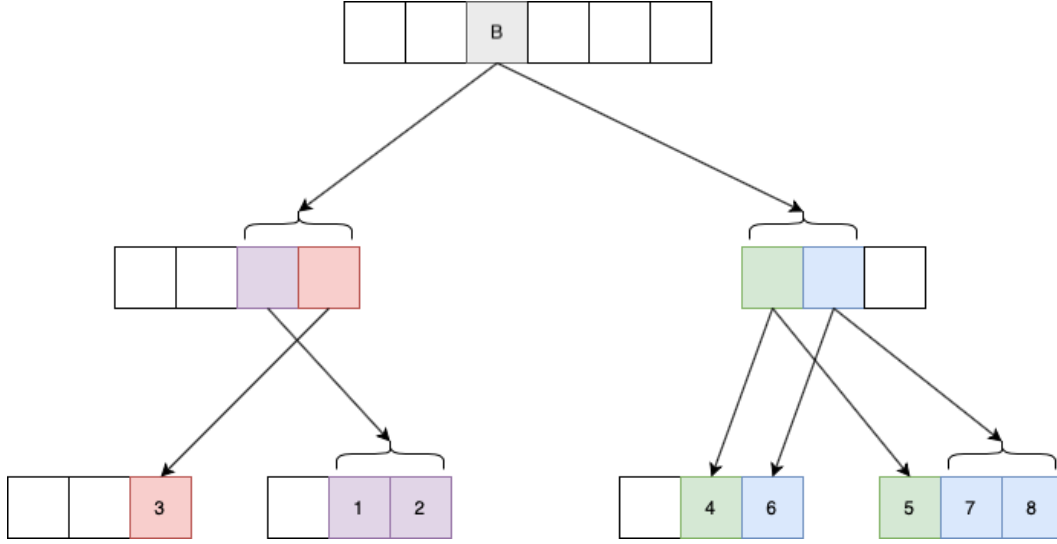


Figure 17: Order of operations in the block B. Operations in the leaves are ordered in the numerical order shown in the drawing.

Now we define the ordering of operations stored in each node. In the non-root nodes, we only need to order operations of a type among themselves that is, we order the **Enqueues** in the node and order the **Dequeues** in the node separately. Processes are numbered from 1 to  $p$ , and leaves of the tree are assigned from left to right. We will show in Lemma 27 that there is at most one operation from each process in a given block.

**Definition 14** (Ordering of operations inside the nodes).

- $E(n, b)$  is the sequence of enqueue operations in  $\text{ops}(n.\text{blocks}[b])$  defined recursively as follows.  $E(\text{leaf}, b)$  is the single enqueue operation in  $\text{ops}(\text{leaf}.\text{blocks}[b])$  or an empty

sequence if  $leaf.blocks[b]$  represents a dequeue operation. If  $n$  is an internal node, then

$$E(n, b) = E(n.left, n.blocks[b-1].end_{left} + 1) \cdots E(n.left, n.blocks[b].end_{left}).$$

$$E(n.right, n.blocks[b-1].end_{right} + 1) \cdots E(n.right, n.blocks[b].end_{right}).$$

- $E_i(n, b)$  is the  $i$ th enqueue in  $E(n, b)$ .
- The order of the enqueue operations in the node  $n$  is  $E(n) = E(n, 1) \cdot E(n, 2) \cdot E(n, 3) \cdots$
- $E_i(n)$  is the  $i$ th enqueue in  $E(n)$ .
- $D(n, b)$  is the sequence of dequeue operations in  $ops(n.blocks[b])$  defined recursively as follows.  $D(leaf, b)$  is the single dequeue operation in  $ops(leaf.blocks[b])$  or an empty sequence if  $leaf.blocks[b]$  represents an enqueue operation. If  $n$  is an internal node, then

$$D(n, b) = D(n.left, n.blocks[b-1].end_{left} + 1) \cdots D(n.left, n.blocks[b].end_{left}).$$

$$D(n.right, n.blocks[b-1].end_{right} + 1) \cdots D(n.right, n.blocks[b].end_{right}).$$

- $D_i(n, b)$  is the  $i$ th dequeue in  $D(n, b)$ .
- The order of the dequeue operations in the node  $n$  is  $D(n) = D(n, 1) \cdot D(n, 2) \cdot D(n, 3) \cdots$
- $D_i(n)$  is the  $i$ th dequeue in  $D(n)$ .

The linearization ordering is given by the order in which operations appear in the blocks in the root.

**Definition 15** (Linearization).

$$L = E(root, 1) \cdot D(root, 1) \cdot E(root, 2) \cdot D(root, 2) \cdot E(root, 3) \cdot D(root, 3) \cdots$$

The following observation follows from the Definition of  $num_x$  on Page 22.

**Observation 16.** For any node  $n$  and indices  $i < j$  of blocks in  $n$ , we have

$$n.blocks[j].sum_x - n.blocks[i].sum_x = \sum_{k=i+1}^j n.blocks[k].num_x$$

where  $x$  in  $\{enq, deq, enq-left, enq-right, deq-left, deq-right\}$ .

The next claim is also valid if we replace `enq` with `deq` and  $E$  with  $D$ .

**Lemma 17.** *Let  $B$  and  $B'$  be  $n.\text{blocks}[b]$  and  $n.\text{blocks}[b-1]$ , respectively.*

*If  $n$  is an internal node, then*

$$(1) \ B.\text{num}_{\text{enq-left}} = \left| E(n.\text{left}, B'.\text{end}_{\text{left}} + 1) \cdots E(n.\text{left}, B.\text{end}_{\text{left}}) \right|.$$

$$(2) \ B.\text{num}_{\text{enq-right}} = \left| E(n.\text{right}, B'.\text{end}_{\text{right}} + 1) \cdots E(n.\text{right}, B.\text{end}_{\text{right}}) \right|.$$

*And for every node  $n$ , we have*

$$(3) \ B.\text{num}_{\text{enq}} = \left| E(n, b) \right|.$$

*Proof.* We prove the claim by induction on the height of node  $n$ . For the base case when  $n$  is a leaf, statement (3) is trivial, and (1) and (2) are vacuously true. Supposing the claim is true for  $n$ 's children, we prove the claim for  $n$ .

$$\begin{aligned} B.\text{num}_{\text{enq-left}} &= B.\text{sum}_{\text{enq-left}} - B'.\text{sum}_{\text{enq-left}} \\ &= B'.\text{sum}_{\text{enq-left}} + n.\text{left}.\text{blocks}[B.\text{end}_{\text{left}}].\text{sum}_{\text{enq}} \\ &\quad - n.\text{left}.\text{blocks}[B'.\text{end}_{\text{left}}].\text{sum}_{\text{enq}} - B'.\text{sum}_{\text{enq-left}} \\ &= n.\text{left}.\text{blocks}[B.\text{end}_{\text{left}}].\text{sum}_{\text{enq}} - n.\text{left}.\text{blocks}[B'.\text{end}_{\text{left}}].\text{sum}_{\text{enq}} \\ &= \sum_{i=B'.\text{end}_{\text{left}}+1}^{B.\text{end}_{\text{left}}} n.\text{left}.\text{blocks}[i].\text{num}_{\text{enq}} \\ &= \left| E(n.\text{left}, B'.\text{end}_{\text{left}} + 1) \cdots E(n.\text{left}, B.\text{end}_{\text{left}}) \right| \end{aligned}$$

First line follows from the Definition of `numenq`. Second line is similar to the way `sumenq-left` is computed in the `CreateBlock` routine. Observation 16 results the third line and the last line holds because of the induction hypothesis (3). (2) is similar to (1). Now we prove (3) starting from the Definition of  $E(n, b)$ .

$$E(n, b) = E(n.\text{left}, n.\text{blocks}[b-1].\text{end}_{\text{left}} + 1) \cdots E(n.\text{left}, n.\text{blocks}[b].\text{end}_{\text{left}}).$$

$$E(n.\text{right}, n.\text{blocks}[b-1].\text{end}_{\text{right}} + 1) \cdots E(n.\text{right}, n.\text{blocks}[b].\text{end}_{\text{right}}).$$

By (1) and (2) we have  $\left| E(n, b) \right| = B.\text{num}_{\text{enq-left}} + B.\text{num}_{\text{enq-right}} = B.\text{num}_{\text{enq}}$ . □



The next claim is also true if we replace **enq** with **deq** and  $E$  with  $D$ .

**Corollary 18.** *Let  $B$  be  $n.\text{blocks}[b]$ .*

(1) *If  $n$  is an internal node then  $B.\text{sum}_{\text{enq-left}} = \left| E(n.\text{left}, 1) \cdots E(n.\text{left}, B.\text{end}_{\text{left}}) \right|$ .*

(2) *If  $n$  is an internal node then  $B.\text{sum}_{\text{enq-right}} = \left| E(n.\text{right}, 1) \cdots E(n.\text{right}, B.\text{end}_{\text{right}}) \right|$ .*

(3)  $B.\text{sum}_{\text{enq}} = \left| E(n, 1) \cdot E(n, 2) \cdots E(n, b) \right|$ .

*Proof.* Result (1) can be proved using the previous lemma.

$$\begin{aligned}
B.\text{sum}_{\text{enq-left}} &= n.\text{blocks}[1].\text{num}_{\text{enq-left}} + \cdots + n.\text{blocks}[b].\text{num}_{\text{enq-left}} \\
&= \left| E(n.\text{left}, 1) \cdots E(n.\text{left}, n.\text{blocks}[1].\text{end}_{\text{left}}) \right| + \\
&\quad \vdots \\
&+ \left| E(n.\text{left}, n.\text{blocks}[b-1].\text{end}_{\text{left}}) \cdots E(n.\text{left}, n.\text{blocks}[b].\text{end}_{\text{left}}) \right| \\
&= \left| E(n.\text{left}, 1) \cdots E(n.\text{left}, B.\text{end}_{\text{left}}) \right|
\end{aligned}$$

We can prove (2) and (3) the same as (1). □

### 4.3 Propagating Operations to the Root

This section explains why two **Refreshes** are enough to propagate a node's operations to its parent.

**Definition 19.** Let  $t^{op}$  be the time  $op$  is invoked,  $^{op}t$  be the time  $op$  terminates,  $t_l^{op}$  be the time immediately before running Line  $l$  of operation  $op$  and  $^{op}t_l$  be the time immediately after running Line  $l$  of operation  $op$ . We sometimes suppress  $op$  and write  $t_l$  or  ${}_l t$  if  $op$  is clear from the context. In the text,  $v_l$  is the value of variable  $v$  immediately after line  $l$  for the process we are talking about and  $v_t$  is the value of variable  $v$  at time  $t$ .

**Definition 20** (Successful Refresh). An instance of **Refresh** is *successful* if its **CAS** in Line 70 returns **true**.

In the next two results, we show that for every successful **Refresh**, all the operations established in the children before the **Refresh** are in the parent after the **Refresh**'s successful **CAS** at Line 70.

**Lemma 21.** *If  $R$  is a successful instance of  $n.\text{Refresh}$ , then we have  $EST_{n.\text{left}}^{t^R} \cup EST_{n.\text{right}}^{t^R} \subseteq ops(n.\text{blocks}_{70})$ .*

*Proof.* We show

$$\begin{aligned} EST_{n.\text{left}}^{t^R} &= ops(n.\text{left.blocks}[0..n.\text{left.head}_{t^R} - 1]) \\ &\subseteq ops(n.\text{blocks}_{70}) = ops(n.\text{blocks}[0..n.\text{head}_{70}]). \end{aligned}$$

In every node,  $\text{blocks}[0]$  is an empty block without any operations. Line 70 stores a block **new** in  $n$  that has  $\text{end}_{\text{left}} = n.\text{left.head}_{83} - 1$ . Therefore, by Definition 5, after the successful CAS in Line 70 we know all blocks in  $n.\text{left.blocks}[1 \dots n.\text{left.head}_{83} - 1]$  are sub-blocks of  $n.\text{blocks}[1 \dots n.\text{head}_{60}]$ . Because of Observation 2 we have  $n.\text{left.head}_{t^R} - 1 \leq n.\text{left.head}_{83} - 1$  and  $n.\text{head}_{60} \leq n.\text{head}_{70}$ . From Observation 13 the claim follows. The proof for the right child is the same.  $\square$

**Corollary 22.** *If  $R$  is a successful instance of  $n.\text{Refresh}$  that terminates, then we have*

$$EST_{n.\text{left}}^{t^R} \cup EST_{n.\text{right}}^{t^R} \subseteq EST_n^{Rt}.$$

*Proof.* The left-hand side is the same as Lemma 21, so it is sufficient to show when  $R$  terminates the established blocks in  $n$  are a superset of  $n.\text{blocks}_{70}$ . Line 70 writes the block **new** in  $n.\text{blocks}[h]$  where  $h$  is value of  $n.\text{head}$  read at Line 60. Because of Lemma 3 we are sure that  $n.\text{head} > h$  when  $R$  terminates. So the block **new** appended to  $n$  at Line 70 is established at  $Rt$ .  $\square$

In the next lemma, we show that if two consecutive instances of **Refresh** by the same process on node  $n$  fail, then the blocks established in the children of  $n$  before the first **Refresh** are guaranteed to be in  $n$  after the second **Refresh**.

**Lemma 23.** *Consider two consecutive terminating instances  $R_1, R_2$  of **Refresh** on internal node  $n$  by process  $p$ . If neither  $R_1$  nor  $R_2$  is a successful **Refresh**, then we have  $EST_{n.\text{left}}^{t^{R_1}} \cup EST_{n.\text{right}}^{t^{R_1}} \subseteq EST_n^{R_2t}$ .*

*Proof.* Let  $R_1$  read  $i$  from  $n.\text{head}$  at Line 60. By Lemma 3,  $R_1$  and  $R_2$  cannot both read the same value  $i$ . By Observation 2,  $R_2$  reads a larger value of  $n.\text{head}$  than  $R_1$ .

Consider the case where  $R_1$  reads  $i$  and  $R_2$  reads  $i + 1$  from Line 60. As  $R_2$ 's CAS in Line 70 returns **false**, there is another successful instance  $R'_2$  of  $n.\text{Refresh}$  that has done a CAS successfully into  $n.\text{blocks}[i+1]$  before  $R_2$  tries to CAS.  $R'_2$  creates its block **new** after reading the value  $i+1$  from  $n.\text{head}$  (Line 60) and  $R_1$  reads the value  $i$  from  $n.\text{head}$ . By Observation 2 we have  $R_1 t < t_{60}^{R_1} < t_{60}^{R'_2}$  (see Figure 18). By Lemma 21 we have  $EST_{60t}^{n,\text{left}} \cup EST_{60t}^{n,\text{right}} \subseteq ops(n.\text{blocks}_{t_{70}^{R'_2}})$ . Also by Lemma 3 on  $R_2$ , the value of  $n.\text{head}$  is more than  $i+1$  after  $R_2$  terminates, so the block appended by  $R'_2$  into  $n.\text{blocks}[i]$  is established by the time  $R_2$  terminates. To summarize,  $R_1 t$  is before  $R'_2$ 's read of  $n.\text{head}$  ( $t_{60}^{R'_2}$ ), so we have  $EST_{n.\text{left}}^{t_{R_1}} \cup EST_{n.\text{right}}^{t_{R_1}} \subseteq ops(n.\text{blocks}_{t_{70}^{R'_2}})$ .  $R'_2$ 's successful CAS ( $t_{70}^{R'_2}$ ) is before  $R_2$ 's termination ( $t^{R_2}$ ), so by Lemma 3  $n.\text{head}$  has been incremented when  $R_2$  terminates and the block  $R'_2$  put into  $n$  is established by then. So we have  $ops(n.\text{blocks}_{t_{70}^{R'_2}}) \subseteq EST_n^{R_2 t}$ .

If  $R_2$  reads some value greater than  $i + 1$  in Line 60 it means  $n.\text{head}$  has been incremented at least two times since  $\frac{R_1}{60} t$ . By Invariant 4, when  $n.\text{head}$  is incremented from  $i + 1$  to  $i + 2$ ,  $n.\text{blocks}[i + 1]$  is non-null. Let  $R_3$  be the Refresh on  $n$  that has put the block in  $n.\text{blocks}[i+1]$ .  $R_3$  read  $n.\text{head} = i + 1$  at Line 60 and has put its block in  $n.\text{blocks}[i + 1]$  before  $R_2$ 's read of  $n.\text{head}$  at Line 60. So we have  $t^{R_1} < \frac{R_3}{60} t < \frac{R_3}{70} t < t_{60}^{R_2} < t^{R_2}$ . From Observation 13 on the operations before and after  $R_3$ 's CAS and Lemmas 21 and 3 on  $R_3$  the claim holds.  $\square$

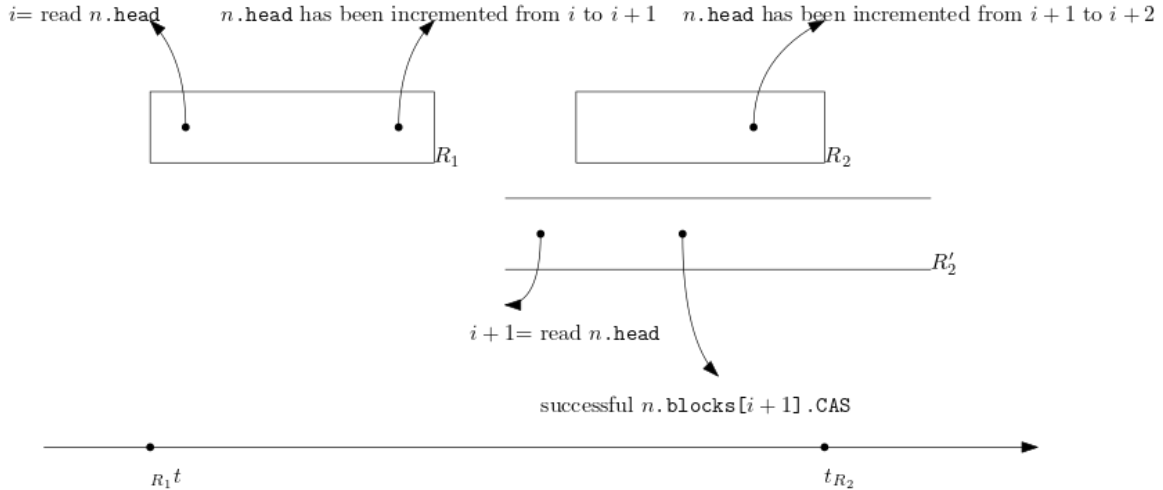


Figure 18:  $R_1 t < t_{60}^{R_1} < \text{incrementing } n.\text{head from } i \text{ to } i + 1 < t_{60}^{R'_2} < t_{70}^{R'_2} < \text{incrementing } n.\text{head from } i + 1 \text{ to } i + 2 < t_{R_2}$

**Corollary 24.**  $EST_{n.\text{left}}^{52t} \cup EST_{n.\text{right}}^{52t} \subseteq EST_n^{t53}$

*Proof.* If the first **Refresh** in line 52 returns **true**, then by Corollary 22 the claim holds. If the first **Refresh** failed and the second **Refresh** succeeded, the claim still holds by Corollary 22. Otherwise, both failed, and the claim is implied by Lemma 23.  $\square$

Now we show that after **Append**( $B$ ) on a leaf finishes, the operation contained in  $B$  will be established in **root**.

**Corollary 25.** *For  $A = l.\text{Append}(B)$  we have  $ops(b) \subseteq EST_n^{tA}$  for each node  $n$  in the path from  $l$  to **root**.*

*Proof.*  $A$  adds  $B$  to the assigned leaf of the process, establishes it at Line 48 and then calls **Propagate** on the parent of the leaf where it appended  $B$ . For every node  $n$ ,  $n.\text{Propagate}$  appends  $B$  to  $n$ , establishes it in  $n$  by Corollary 24 and then calls  $n.\text{parent}.\text{Propagate}$  until  $n$  is **root**.  $\square$

**Corollary 26.** *After  $l.\text{Append}(B)$  finishes,  $B$  is a subblock of exactly one block in each node along the path from  $l$  to the **root**.*

*Proof.* By the previous corollary and Lemma 11 there is exactly one block in each node containing  $B$ .  $\square$

#### 4.4 Correctness of **GetEnqueue**

First, we prove some claims about the size and operations of a block. These lemmas will be used later for the correctness and analysis of **GetEnqueue**( ).

**Lemma 27.** *Each block contains at most one operation of each process and, therefore, at most  $p$  operations in total.*

*Proof.* To derive a contradiction, assume there are two operations  $op_1$  and  $op_2$  of process  $p$  in block  $B$  in node  $n$ . Without loss of generality  $op_1$  is invoked earlier than  $op_2$ . Process  $p$  cannot invoke

more than one operation concurrently, so  $op_1$  has to be finished before  $op_2$  begins. By Corollary 26, before  $op_2$  calls **Append**,  $op_1$  exists in every node of the tree on the path from  $p$ 's leaf to the root. Since  $b$  contains  $op_2$ , it must be created after  $op_2$  is invoked. The fact that  $op_2$ .**Append** is invoked after  $op_1$ .**Append** terminated means that there is some block  $B'$  in  $n$  before  $B$  that contains  $op_1$ . The existence of  $op_1$  in  $B$  and  $B'$  contradicts Lemma 11.  $\square$

**Lemma 28.** *Each block has at most  $p$  direct subblocks.*

*Proof.* From the definition of operations in a block in Definition 10 we know the operations in a block are the union of the operations in the direct subblocks of the block. The claim follows from the equivalence of the number of operations in a block and the number of operations in the subblocks of the block and the observation that each block appended to an internal node contains at least one operation due to the test on Line 68. Also, blocks in the leaves contain only one **Enqueue** or **Dequeue** operation.  $\square$

`DoublingSearch`( $e$ ,  $end$ ) returns a pair  $\langle b, i \rangle$  such that the  $i$ th **Enqueue** in the  $b$ th block of the root is the  $e$ th **Enqueue** in the sequence stored in the root.

**Lemma 29** (`DoublingSearch` correctness). *If  $1 \leq e \leq \text{root.blocks}[end].\text{sum}_{\text{enq}}$ , then `DoublingSearch`( $e$ ,  $end$ ) returns  $\langle b, i \rangle$  such that  $E_i(\text{root}, b) = E_e(\text{root})$ .*

*Proof.* From Lines 86 and 87 we know the `sumenq-left`, and `sumenq-right` fields of blocks in each node are sorted in non-decreasing order. Since `sumenq` = `sumenq-left` + `sumenq-right`, the `sumenq` values of `root.blocks[0 .. end]` are also non-decreasing. By Corollary 18 we know that the `sumenq` field in a block is the sum of the number of **Enqueue** operations in that block and the prefix blocks of that block in the node. Furthermore, since `root.blocks[0].sumenq` = 0 and `root.blocks[end].sumenq`  $\geq e$ , there is a  $b$  such that `root.blocks[b-1].sumenq`  $< e \leq$  `root.blocks[b].sumenq`. Block `root.blocks[b]` contains  $E_i(\text{root}, b)$ . Lines 38–41 doubles the search range in Line 40 and will eventually reach `start` such that `root.blocks[start].sumenq`  $\leq e \leq$  `root.blocks[end].sumenq`. Then, in Line 42, the binary search finds the  $b$  such that `root.blocks[b-1].sumenq`  $< e \leq$  `root.blocks[b].sumenq`.

By Corollary 18, `root.blocks[b]` is the block that contains  $E_e(\text{root})$ . Finally,  $i$  is computed using the definition of `sumenq` and Corollary 18.  $\square$

**Lemma 30** (GetEnqueue correctness). *If  $1 \leq i \leq n.\text{blocks}[b].\text{num}_{\text{enq}}$  then  $n.\text{GetEnqueue}(b, i)$  returns  $E_i(n, b).\text{element}$ .*

*Proof.* We will prove this lemma by induction on the node  $n$ 's height. For the base case, suppose  $n$  is a leaf. Leaf blocks each contain exactly one operation,  $n.\text{blocks}[b].\text{sum}_{\text{enq}} \leq 1$ , which means only  $n.\text{GetEnqueue}(b, 1)$  can be called when  $n$  is a leaf and  $n.\text{blocks}[b]$  must contain an **Enqueue** operation. Line 97 of  $n.\text{GetEnqueue}(b, 1)$  returns the `element` of the **Enqueue** operation stored in the  $b$ th block of leaf  $n$ , as required.

For the induction step, we prove if  $n.\text{dir}.\text{GetEnqueue}(b', i)$  returns  $E_i(n.\text{dir}, b')$  then  $n.\text{GetEnqueue}(b, i)$  returns  $E_i(n, b)$ . From Definition 14 of  $E(n, b)$ , we know that operations from the left subblocks come before the operations from the right subblocks in a block (see Figure 19). By Lemma 17, the `numenq-left` field in  $n.\text{blocks}[b]$  is the number of **Enqueue** operations from the blocks' subblocks in the left child of  $n$ . So the  $i$ th **Enqueue** operation in  $n.\text{blocks}[b]$  is propagated from the right child if and only if  $i$  is greater than  $n.\text{blocks}[b].\text{num}_{\text{enq-left}}$ . Line 98 decides whether the  $i$ th enqueue in the  $b$ th block of internal node  $n$  is in the left child or right child subblocks of  $n.\text{blocks}[b]$ . By Definitions 5 and 10, to find an operation in the subblocks of  $n.\text{blocks}[b]$  we need to search in the range

$$\begin{aligned} & n.\text{left}.\text{blocks}[n.\text{blocks}[b-1].\text{end}_{\text{left}}+1..n.\text{blocks}[b].\text{end}_{\text{left}}] \text{ or} \\ & n.\text{right}.\text{blocks}[n.\text{blocks}[b-1].\text{end}_{\text{right}}+1..n.\text{blocks}[b].\text{end}_{\text{right}}]. \end{aligned}$$

First, we consider the case where the **Enqueue** we are looking for is in the left child. There are  $eb = n.\text{blocks}[b-1].\text{sum}_{\text{enq-left}}$  **Enqueues** in the blocks of  $n.\text{left}$  before the left subblocks of  $n.\text{blocks}[b]$ , so  $E_i(n, b)$  is  $E_{i+eb}(n.\text{left})$  which is  $E_{i'}(n.\text{left}, b')$  for some  $b'$  and  $i'$ . We can compute  $b'$  and then search for the  $i'$ th **Enqueue** in  $n.\text{left}.\text{blocks}[b']$ , where  $i'$  is  $i+eb-n.\text{left}.\text{blocks}[b'-1].\text{sum}_{\text{enq}}$ . The parameters in Line 99 are for searching  $E_{i+eb}(n.\text{left})$

in  $n.\text{left}.\text{blocks}$  in the range of left subblocks of  $n.\text{blocks}[b]$ , so this `BinarySearch` returns the index of the subblock containing  $E_i(n, b)$ .

Otherwise, the `Enqueue` we are looking for is in the right child. Because `Enqueues` from the left subblocks are ordered before the ones from the right subblocks, there are  $n.\text{blocks}[b].\text{num}_{\text{enq-left}}$  enqueues ahead of  $E_i(n, b)$  from the left child. So we need to search for  $i - n.\text{blocks}[b].\text{num}_{\text{enq-left}} + n.\text{blocks}[b-1].\text{sum}_{\text{enq-right}}$  in the right child (Line 103). Other parameters for the right child are chosen similarly to the left child.

So, in both cases, the direct subblock containing  $E_i(n, b)$  is computed in Line 99 or 103.  $\text{subblockIndex}$  is the index of the block in  $n.\text{dir}$  containing  $E_i(n, b)$ . Finally,  $n.\text{child}.\text{GetEnqueue}(\text{subblockIndex}, i)$  is invoked and it returns  $E_i(n, b).\text{element}$  by the hypothesis of the induction.  $\square$

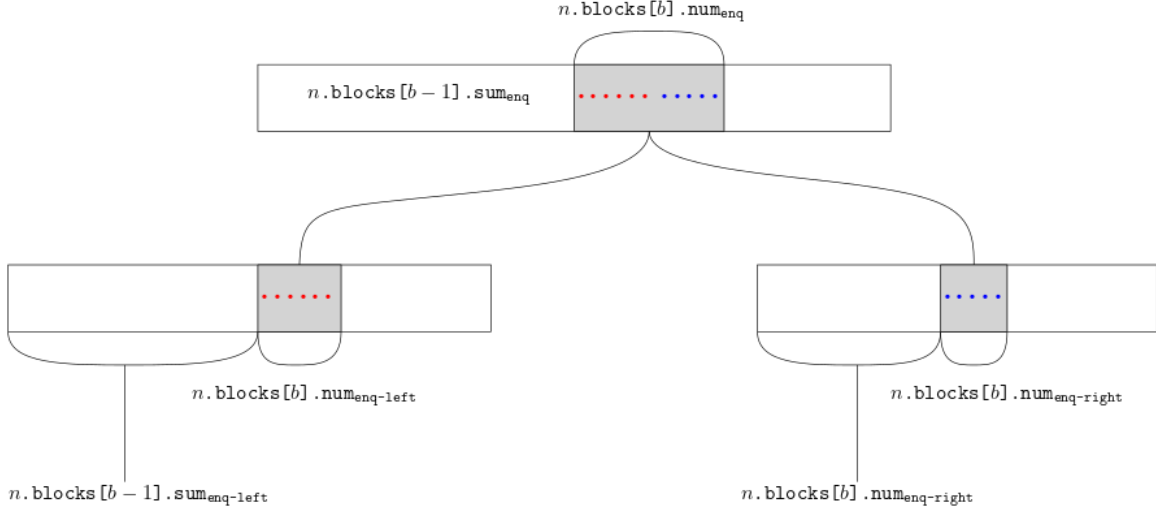


Figure 19: The number and order of the `Enqueue` operations propagated from the left and the right child to  $n.\text{blocks}[b]$ . Both  $n.\text{blocks}[b]$  and its subblocks are shown in grey. `Enqueue` operations from the left child (colored red), are ordered before the `Enqueue` operations from the right child (colored blue).

## 4.5 Correctness of IndexDeque

The next few results show that the `super` field of a block is accurate within one of the actual index of the block's superblock in the parent node. Then we explain how it is used to compute the rank of a given Dequeue in the root.

**Definition 31.** If a `Refresh` instance  $R_1$  does its CAS at Line 70 earlier than `Refresh` instance  $R_2$  we say  $R_1$  has *happened before*  $R_2$ .

**Observation 32.** After  $n.\text{blocks}[i].\text{CAS}(\text{null}, B)$  succeeds,  $n.\text{head}$  cannot increase from  $i$  to  $i + 1$  until  $B.\text{super}$  is set.

*Proof.* From Observation 2 we know that  $n.\text{head}$  changes only by the increment on Line 77. Before an instance of `Advance` increments  $n.\text{head}$  on Line 77, Line 76 ensures that  $n.\text{blocks}[\text{head}].\text{super}$  was set at Line 76. □

**Corollary 33.** If  $n.\text{blocks}[i].\text{super}$  is null, then  $n.\text{head} \leq i$  and  $n.\text{blocks}[i + 1]$  is null.

*Proof.* By Invariant 4 and Observation 32. □

Now let us consider how the `Refreshes` that took place on the parent of node  $n$  after block  $B$  was stored in  $n$  will help to set  $B.\text{super}$  and propagate  $B$  to the parent.

**Observation 34.** If the block created by an instance  $R_p$  of  $n.\text{parent.Refresh}$  contains block  $B = n.\text{blocks}[b]$  then  $R_p$  reads a value greater than  $b$  from  $n.\text{head}$  in Line 83.

**Lemma 35.** If  $B = n.\text{blocks}[b]$  is a direct subblock of  $n.\text{parent.blocks}[\text{superblock}]$  then  $B.\text{super} \leq \text{superblock}$ .

*Proof.* Let  $R_p$  be the instance of  $n.\text{parent.Refresh}$  that does a successful CAS on the superblock of  $B$  which is  $n.\text{parent.blocks}[\text{superblock}]$ . By Observation 34 if  $R_p$  propagates  $B$  it has to read a greater value than  $b$  from  $n.\text{head}$ , which means  $n.\text{head}$  was incremented from  $b$  to  $b + 1$  in Line 77. By Observation 32  $B.\text{super}$  was already set in Line 76. The value written in  $B.\text{super}$ ,



was read in Line 75 before the CAS that sets  $B.\text{super}$  in Line 76. From Observation 2 we know  $n.\text{parent.head}$  is non-decreasing so  $B.\text{super} \leq \text{superblock}$ , since  $n.\text{parent.head}$  is still equal to  $\text{superblock}$  when  $R_p$  executes its CAS at Line 70 by Lemma 6. The reader may wonder when the case  $b.\text{super} = \text{superblock}$  happens. This can happen when  $n.\text{parent.blocks}[B.\text{super}] = \text{null}$  when  $B.\text{super}$  is written and  $R_p$  puts its created block into  $n.\text{parent.blocks}[B.\text{super}]$  afterwards.  $\square$

**Lemma 36.** *Let  $R_n$  be a Refresh that puts  $B$  in  $n.\text{blocks}[b]$  at Line 70. Then, the block created by one of the next two successful  $n.\text{parent.Refreshes}$  according to Definition 31 contains  $B$  and  $B.\text{super}$  is set when the second successful  $n.\text{parent.Refresh}$  reaches Line 67.*

*Proof.* Let  $R_{p1}$  and  $R_{p2}$  be the next two successful  $n.\text{parent.Refreshes}$  after  $R_n$ . To derive a contradiction assume  $B$  was neither propagated to  $n.\text{parent}$  by  $R_{p1}$  nor by  $R_{p2}$ .

Since  $R_{p2}$ 's created block does not contain  $B$ , by Observation 34 the value  $R_{p2}$  reads from  $n.\text{head}$  in Line 83 is at most  $b$ . From Observation 2 the value  $R_{p2}$  reads in Line 62 is also at most  $b$ .

$R_n$  puts  $B$  into  $n.\text{blocks}[b]$  so  $R_n$  reads the value  $b$  from  $n.\text{head}$ . Since  $R_{p2}$ 's CAS into  $n.\text{parent.blocks}$  is successful there should be a Refresh instance  $R'_p$  on  $n.\text{parent}$  that increments  $n.\text{parent.head}$  (Line 77) after  $R_{p1}$ 's Line 70 and before  $R_{p2}$ 's Line 60. We assumed  $t_{70}^{R_n} < t_{70}^{R_{p1}} < t_{70}^{R_{p2}}$  by Definition 31. Finally, Line 62 is after Line 60 and  $R_{p2}$ 's Line 60 is after  $R'_p$ 's Line 77, which is after  $R_n$ 's  $n.\text{blocks.CAS}$ .

$$\left. \begin{array}{l} \frac{R_n}{70} t < \frac{R_{p1}}{70} t \\ \frac{R_{p1}}{70} t < \frac{R_{p'}}{77} t < \frac{R_{p2}}{60} t \\ \frac{R_{p2}}{60} t < \frac{R_{p2}}{62} t \end{array} \right\} \implies \frac{R_n}{70} t < \frac{R_{p2}}{62} t$$

So  $R_{p2}$  reads a value greater than or equal to  $b$  for  $n.\text{head}$  by Observation 2.

Therefore  $R_{p2}$  reads  $n.\text{head} = b$ .  $R_{p2}$  calls  $n.\text{Advance}$  at Line 64, which ensures  $n.\text{head}$  is incremented from  $b$ . So the value  $R_{p2}$  reads in Line 83 of **CreateBlock** is greater than  $b$  and  $R_{p2}$ 's created block contains  $B$ . This is in contradiction with our hypothesis.

Furthermore, if  $B.\text{super}$  was not set earlier, it is set by  $R_{p2}$ 's call to  $n.\text{Advance}$  invoked from Line 64.  $\square$

**Corollary 37.** *If  $B = n.\text{blocks}[b]$  is propagated to  $n.\text{parent}$ , then  $B.\text{super}$  is equal to or one less than the index of the superblock of  $B$ .*

*Proof.* Let  $R_n$  be the  $n.\text{Refresh}$  that put  $B$  in  $n.\text{blocks}$  and let  $R_{p1}$  be the first successful  $n.\text{parent.Refresh}$  after  $R_n$  and  $R_{p2}$  be the second next successful  $n.\text{parent.Refresh}$ . Before  $B$  can be propagated to  $n$ 's  $\text{parent}$ ,  $n.\text{head}$  must be greater than  $b$ , so by Observation 32  $B.\text{super}$  is set. From Lemma 36 we know that  $B$  is propagated by the second next successful  $\text{Refresh}$ 's CAS on  $n.\text{parent.blocks}$ . To summarize, we have  $n.\text{parent.head}_{\frac{R_{p2}}{70}t} = n.\text{parent.head}_{\frac{R_{p1}}{70}t} + 1$  and  $n.\text{parent.head}_{\frac{R_{p1}}{70}t} \leq n.\text{parent.head}_{\frac{R_n}{70}t}$  from Definition 31 and Observation 2. The value that is set in  $B.\text{super}$  is read from  $n.\text{parent.head}$  after  $\frac{R_n}{70}t$ . So  $B.\text{super}$  is equal to or one less than the index of the superblock of  $B$ .  $\square$

We prove  $\text{IndexDequeue}$ 's correctness using Corollary 37 on each step of the  $\text{IndexDequeue}$ .

**Lemma 38** ( $\text{IndexDequeue}$  correctness). *If  $1 \leq i \leq n.\text{blocks}[b].\text{num}_{\text{deq}}$  then  $n.\text{IndexDequeue}(b, i)$  returns  $\langle x, y \rangle$  such that  $D_i(n, b) = D_y(\text{root}, x)$ .*

*Proof.* We will prove this by induction on the distance of  $n$  from the  $\text{root}$ . The base case where  $n$  is  $\text{root}$  is trivial (see Line 109). For the non-root nodes  $n.\text{IndexDequeue}(b, i)$  computes  $\text{superblockIndex}$ , the index of the superblock of the  $b$ th block in  $n$ , in Line 112 by Corollary 37. After that, the position of  $D_i(n, b)$  in  $D(n.\text{parent}, \text{superblockIndex})$  is computed in Lines 113–118. By Definition 14,  $\text{Dequeues}$  in a block are ordered based on the order of its subblocks from left to right. If  $D_i(n, b)$  was propagated from the left child, the number of dequeues in the left subblocks of  $n.\text{parent.blocks}[\text{superblockIndex}]$  before  $n.\text{blocks}[b]$  is considered in Line 114 (see Figure 20). Otherwise, if  $D_i(n, b)$  was propagated from the right child, the number of dequeues in the subblocks from the left child is considered to be ahead of the computed index (Line 115) (see

Figure 21). Finally, `IndexDequeue` is called on `n.parent` recursively, and it returns the correct response by the induction hypothesis.  $\square$

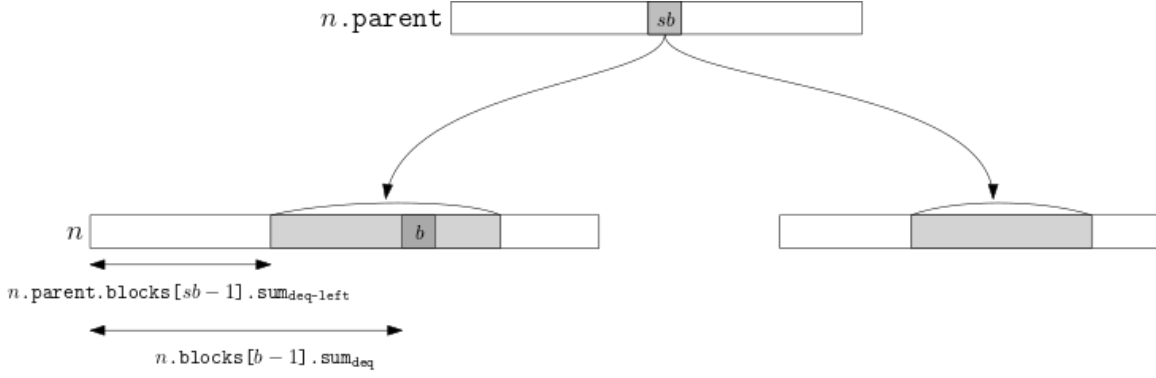


Figure 20: The number of `Dequeue` operations before  $D_i(n, b)$  shown in the case where  $n$  is a left child. The index of the superblock is shown with  $sb$ .

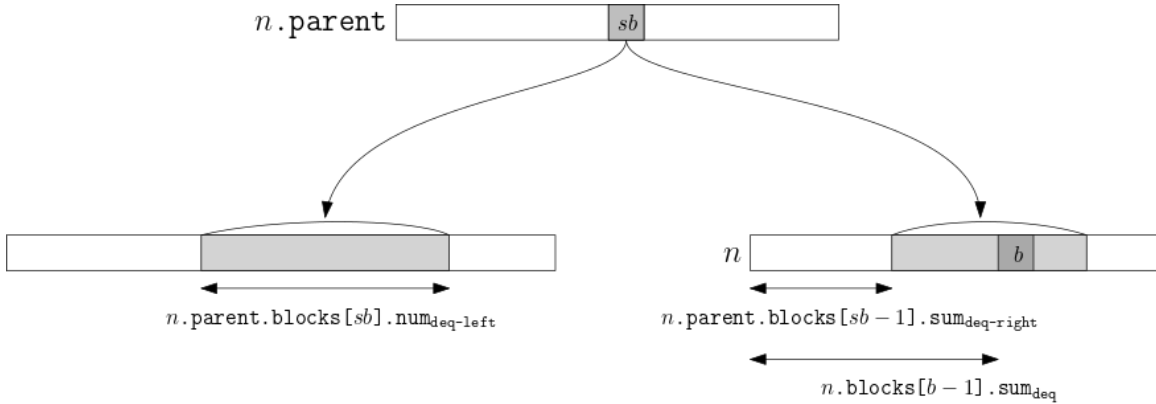


Figure 21: The number of `Dequeue` operations before  $D_i(n, b)$  shown in the case where  $n$  is a right child. The index of the superblock is shown with  $sb$ .

## 4.6 Linearizability

We now prove the two properties needed for linearizability.

**Lemma 39.**  *$L$  is a legal linearization ordering.*

*Proof.* We must show for any execution that every operation that terminates is in  $L$  exactly once.

Also, if  $op_1$  terminates before  $op_2$  in starts in the execution, then  $op_1$  is before  $op_2$  in the lineariza-

tion. The first claim is directly reasoned from Corollary 26. For the latter, if  $op_1$  terminates before  $op_2$  starts,  $op_1.\text{Append}$  has terminated before  $op_2.\text{Append}$  started. From Corollary 25,  $op_1$  is in `root.blocks` before  $op_2$  starts to propagate. By definition of  $L$ ,  $op_1$  is linearized before  $op_2$ .  $\square$

Once some operations are aggregated in one block, they will get propagated up to the root together, and they can be linearized in any order among themselves. We have chosen to put `Enqueues` in a block before `Dequeues` (see Definition 14).

**Definition 40.** If a `Dequeue` operation returns `null` it is called a *null Dequeue*, otherwise it is called *non-null Dequeue*.

Next, we define the responses that `Dequeues` should return, according to the linearization.

**Definition 41.** Assume the operations in `root.blocks` are applied sequentially on an empty queue in the order of  $L$ .  $\text{Resp}(d) = e.\text{element}$  if the element of `Enqueue`  $e$  is the response to `Dequeue`  $d$ . Otherwise if  $d$  is a *null Dequeue* then  $\text{Resp}(d) = \text{null}$ .

In the next lemma, we show that the `size` field in each `root block` is computed correctly.

**Lemma 42.** `root.blocks[b].size` is the size of the queue after the operations in `root.blocks[0...b]` are applied in the order of  $L$ .

*Proof.* We prove the claim by induction on  $b$ . The base case when  $b = 0$  is trivial since the queue is initially empty and `root.blocks[0]` contains an empty block with `size` field equal to 0. We are going to show the correctness when  $b = i$  assuming correctness when  $b = i - 1$ . By Definition 14 `Enqueue` operations come before `Dequeue` operations in a block in  $L$ . By Lemma 17 `numenq` and `numdeq` fields in a block show the number of `Enqueue` and `Dequeue` operations in it. If there are more than `root.blocks[i - 1].size + root.blocks[i].numenq` dequeue operations in `root.blocks[i]` then the queue would become empty after `root.blocks[i]`. Otherwise, the size of the queue after the  $b$ th block in the root is `root.blocks[b - 1].size + root.blocks[b].numenq - root.blocks[b].numdeq`. In both cases, this is the same as the assignment on Line 91.  $\square$

The next lemma is useful to compute the number of non-null dequeues.

**Lemma 43.** *If operations in the root are applied in the order of  $L$ , the number of non-null Dequeues in  $\text{root.blocks}[0 \cdots b]$  is  $\text{root.blocks}[b].\text{sum}_{\text{enq}} - \text{root.blocks}[b].\text{size}$ .*

*Proof.* There are  $\text{root.blocks}[b].\text{sum}_{\text{enq}}$  Enqueue operations in  $\text{root.blocks}[0 \cdots b]$  by Corollary 18. The size of the queue after doing  $\text{root.blocks}[0 \cdots b]$  in the order of  $L$  is the number of enqueues in  $\text{root.blocks}[0 \cdots b]$  minus the number of non-null Dequeues in  $\text{root.blocks}[0 \cdots b]$ . By the correctness of the `size` field from Lemma 42 and `sumenq` field from Lemma 17, the number of non-null Dequeues is  $\text{root.blocks}[b].\text{sum}_{\text{enq}} - \text{root.blocks}[b].\text{size}$ .  $\square$

**Corollary 44.** *If operations in the root are applied in the order of  $L$ , the number of non-null dequeues in  $\text{root.blocks}[b]$  is  $\text{root.blocks}[b].\text{num}_{\text{enq}} - \text{root.blocks}[b].\text{size} + \text{root.blocks}[b-1].\text{size}$ .*

**Lemma 45.**  *$\text{Resp}(D_i(\text{root}, b))$  is null iff  $\text{root.blocks}[b-1].\text{size} + \text{root.blocks}[b].\text{num}_{\text{enq}} - i < 0$ .*

*Proof.* Immediately from Corollary 44 and Lemma 17.  $\square$

**Lemma 46.**  *$\text{FindResponse}(b, i)$  returns  $\text{Resp}(D_i(\text{root}, b))$ .*

*Proof.*  $D_i(\text{root}, b)$  is  $D_{\text{root.blocks}[b-1].\text{sum}_{\text{deq}}+i}(\text{root})$  by Definition 14 and Lemma 18.  $D_i(\text{root}, b)$  returns null at Line 20 if  $\text{root.blocks}[b-1].\text{size} + \text{root.blocks}[b].\text{num}_{\text{enq}} - i < 0$  and  $\text{Resp}(D_i(\text{root}, b)) = \text{null}$  in this case by Lemma 45. Otherwise, if  $D_i(\text{root}, b)$  is the  $e$ th non-null Dequeue in  $L$  it should return the  $e$ th enqueued value. By Lemma 43 there are  $\text{root.blocks}[b-1].\text{sum}_{\text{enq}} - \text{root.blocks}[b-1].\text{size}$  non-null Dequeue operations in  $\text{root.blocks}[0 \cdots b-1]$ . The Dequeues in  $\text{root.blocks}[b]$  before  $D_i(\text{root}, b)$  are non-null Dequeues. So  $D_i(\text{root}, b)$  is the  $e$ th non-null Dequeue where  $e = i + \text{root.blocks}[b-1].\text{sum}_{\text{deq}} - \text{root.blocks}[b-1].\text{size}$  (Line 22). See Figure 22.

After computing  $e$  at Line 22, the code finds  $b, i$  such that  $E_i(\text{root}, b) = E_e(\text{root})$  using `DoublingSearch` and then finds its `element` using `GetEnqueue` (Line 23). Correctness of `DoublingSearch` and `GetEnqueue` routines are shown in Lemmas 29 and 30.  $\square$

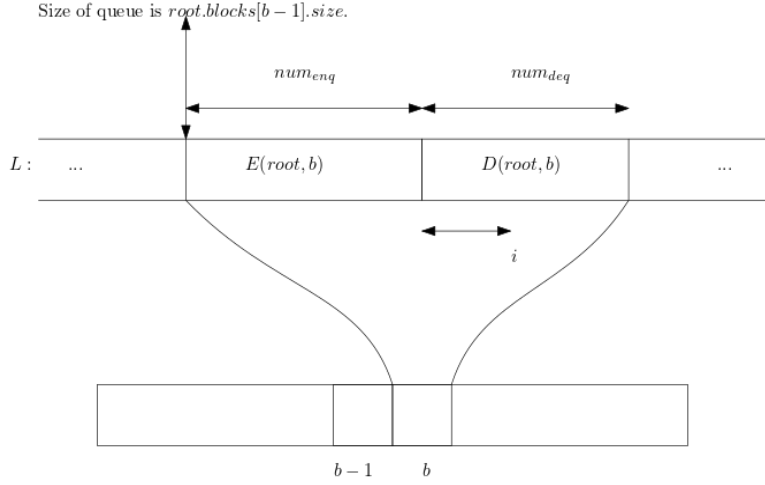


Figure 22: The position of  $D_i(\text{root}, b)$ .

**Lemma 47.** *The responses to operations in our algorithm are the same as in the sequential execution in the order given by  $L$ .*

*Proof.* `Enqueue` operations do not return any value. By Lemma 46, the response of a `Dequeue` in our algorithm is same as its response in the sequential execution of  $L$ .  $\square$

**Theorem 48 (Main).** *The queue implementation is linearizable.*

*Proof.* The theorem follows from Lemmas 39 and 47.  $\square$

**Remark** In fact our algorithm is strongly linearizable as defined in [10]. By Definition 14 the linearization ordering of operations will not change as blocks containing new operations are appended to the root.

## 5 Analysis

In this section, we analyze the number of CAS invocations and the time complexity of our algorithm.

**Proposition 49.** An Enqueue or Dequeue operation does at most  $14 \log p$  CAS operations.

*Proof.* In each level of the tree **Refresh** is invoked at most two times, and every **Refresh** invokes at most seven CASes, one in Line 70 and two from each **Advance** in Line 64 or 71.  $\square$

**Lemma 50** (DoublingSearch Analysis). *If the element enqueued by  $E_i(\text{root}, b) = E_e(\text{root})$  is the response to some Dequeue operation in  $\text{root.blocks}[end]$ , then  $\text{DoublingSearch}(e, end)$  takes  $O(\log(\text{root.blocks}[b].\text{size} + \text{root.blocks}[end].\text{size}))$  steps.*

*Proof.* First we show  $end - b - 1 \leq 2 \times \text{root.blocks}[b - 1].\text{size} + \text{root.blocks}[end].\text{size}$ . There can be at most  $\text{root.blocks}[b].\text{size}$  Dequeues in  $\text{root.blocks}[b + 1 \dots end - 1]$ ; otherwise all elements enqueued by  $\text{root.blocks}[b]$  would be dequeued before  $\text{root.blocks}[end]$ . Furthermore, in the execution of queue operations in the linearization ordering, the size of the queue becomes  $\text{root.blocks}[end].\text{size}$  after the operations of  $\text{root.blocks}[end]$ . The final size of the queue after  $\text{root.blocks}[1 \dots end]$  is  $\text{root.blocks}[end].\text{size}$ . After an execution on a queue, the size of the queue is greater than or equal to  $\#enqueues - \#dequeues$  in the execution. We know the number of dequeues in  $\text{root.blocks}[b + 1 \dots end - 1]$  is less than  $\text{root.blocks}[b].\text{size}$ , therefore in  $\text{root.blocks}[b + 1 \dots end - 1]$  there cannot be more than  $\text{root.blocks}[b].\text{size} + \text{root.blocks}[end].\text{size}$  Enqueues. Overall there can be at most  $2 \times \text{root.blocks}[b].\text{size} + \text{root.blocks}[end].\text{size}$  operations in  $\text{root.blocks}[b + 1 \dots end - 1]$  and since from Line 68 we know that the **num** field of every block in the tree is greater than 0, each block has at least one operation, so there are at most  $2 \times \text{root.blocks}[b].\text{size} + \text{root.blocks}[end].\text{size}$  blocks in between  $\text{root.blocks}[b]$  and  $\text{root.blocks}[end]$ . So,  $end - b - 1 \leq 2 \times \text{root.blocks}[b].\text{size} + \text{root.blocks}[end].\text{size}$ .

Thus, the doubling search reaches **start** such that the  $\text{root.blocks}[\text{start}].\text{sum}_{\text{enq}}$  is less than  $e$  in  $O(\log(\text{root.blocks}[b].\text{size} + \text{root.blocks}[end].\text{size}))$  steps. See Figure 23. After Line 41,

the binary search that finds  $b$  also takes  $O(\log(\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}))$ . Next,  $i$  is computed via the definition of  $\text{sum}_{\text{enq}}$  in constant time (Line 43).  $\square$

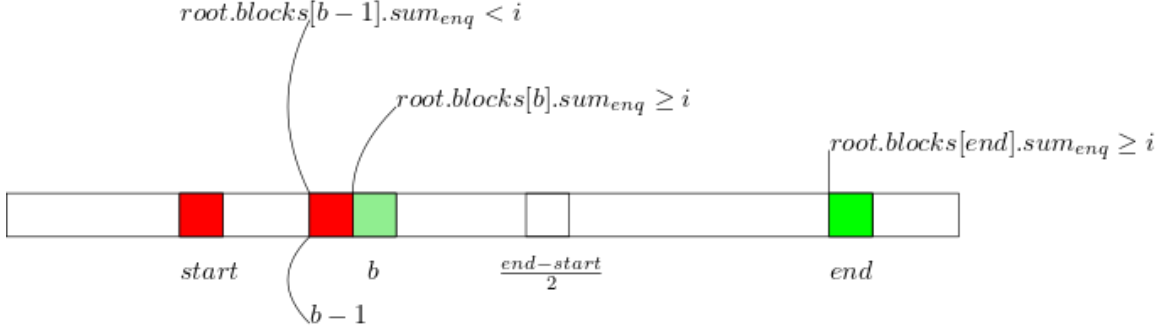


Figure 23: Distance relations between  $\text{start}$ ,  $b$ ,  $\text{end}$ .

**Lemma 51** (Worst Case Time Analysis). *The worst case number of steps for an **Enqueue** is  $O(\log^2 p)$  and for a **Dequeue**, is  $O(\log^2 p + \log q_e + \log q_d)$ , where  $q_d$  is the size of the queue when the **Dequeue** is linearized and  $q_e$  is the size of the queue at the time the response of the **Dequeue** is linearized.*

*Proof.* **Enqueue** consists of creating a block and appending it to the tree. The first part takes constant time. To propagate the operation to the root the algorithm tries at most two **Refreshes** in each node of the path from the leaf to the root (Lines 52, 53). We can see from the code that each **Refresh** takes a constant number of steps and does  $O(1)$  CASes. Since the height of the tree is  $\Theta(\log p)$ , **Enqueue** takes  $O(\log p)$  steps.

A **Dequeue** creates a block whose `element` is `null`, appends it to the tree, computes its rank among non-null dequeues, finds the corresponding enqueue and returns the response. The first two parts are similar to an **Enqueue** operation and take  $O(\log p)$  steps. To compute the rank of a **Dequeue** in  $D(n)$ , the **Dequeue** calls `IndexDequeue()`. `IndexDequeue` does  $O(1)$  steps in each level which takes  $O(\log p)$  steps. If the response to the **Dequeue** is `null`, `FindResponse` returns `null` in  $O(1)$  steps. Otherwise, if the response to a dequeue in `root.blocks[end]` is in `root.blocks[b]` the `DoublingSearch` takes  $\Theta(\log(\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}))$  by Lemma



50, which is  $O(\log q_e + \log q_d)$ . Each search in `GetEnqueue()` takes  $O(\log p)$  steps since there are at most  $p$  subblocks in a block (Lemma 28), so `GetEnqueue()` takes  $O(\log^2 p)$  steps.  $\square$

**Lemma 52** (Amortized Worst-case Analysis). *The amortized number of steps for an `Enqueue` or `Dequeue` in the worst-case, is  $O(\log^2 p + \log q)$ , where  $q$  is the size of the queue when the operation is linearized.*

*Proof.* If we split the `DoublingSearch` time cost between the corresponding `Enqueue` and `Dequeue`, each operation takes  $O(\log^2 p + q)$  steps.  $\square$

**Corollary 53.** *If the number of concurrent processes in an execution is less than  $c$ , then each `GetEnqueue` takes  $O(\log c)$  steps because each block has at most  $c$  operations. So the total amortized worst-case step complexity would be  $O(\log p \log c + \log q)$ . Furthermore, in a sequential execution where  $c = 1$ , the step complexity of our algorithm would be  $\Theta(\log p + \log q)$ .*

**Theorem 54.** *The queue implementation is wait-free.*

*Proof.* To prove the claim, it is sufficient to show that every `Enqueue` and `Dequeue` operation terminates after a finite number of its own steps. This is directly concluded from Lemma 51.  $\square$

## 6 Future Directions

We designed a tree to achieve agreement on a linearization of operations invoked by  $p$  processes in an asynchronous model, which we will call a *block tree*. We implemented two queries to compute information about the ordering agreed in the block tree. Then we used the tree to implement a queue where the number of steps per operation is poly-logarithmic with respect to the size of the queue and the number of processes. Block trees can be used as a mechanism to achieve agreement among processes to construct more poly-logarithmic wait-free linearizable objects. In the next paragraphs, we talk about possible improvements on block trees and the data structures that we can implement with block trees.

**Reducing Space Usage** The `blocks` arrays defined in our algorithm are unbounded. To use  $O(n)$  space in each node where  $n$  is the total number of operations, instead of unbounded arrays, we could use the memory model of the wait-free vector introduced by Feldman, Valera-Leon and Damian [8]. We can create an array called `arr` of pointers to array segments (see Figure 24). When a process wishes to write into location `head` it checks whether `arr[⌊log head⌋]` points to an array or not. If not, it creates a shared array of size  $2^{\lceil \log \text{head} \rceil}$  and tries to CAS a pointer to the created array into `arr[⌊log head⌋]`. Whether the CAS is successful or not, `arr[⌊log head⌋]` points to an array. When a process wishes to access the  $i$ th element it looks up `arr[⌊log i⌋][i - 2⌊log i⌋]`, which takes  $O(1)$  steps. The CAS Retry Problem does not happen here because if  $n$  elements are appended to the array, then only  $O(p \times \log n)$  CAS steps have happened on the array `arr`. Furthermore, at most  $p$  arrays with size  $2^{\lceil \log i \rceil}$  are allocated by processes while processes try to do the CAS on `arr[i]`. Jayanti and Shun [17] present a way to initialize wait-free arrays in constant steps. The time taken to allocate arrays in an execution containing  $n$  operations is  $O(\frac{p \log n}{n})$  per operation, which is negligible if  $n \gg p$ . The vector implementation also has a mechanism for doubling `arr` when necessary, but this happens very rarely since increasing `arr` from  $s$  to  $2s$  increases the capacity of the vector from  $2^s$  to  $2^{2s}$ .

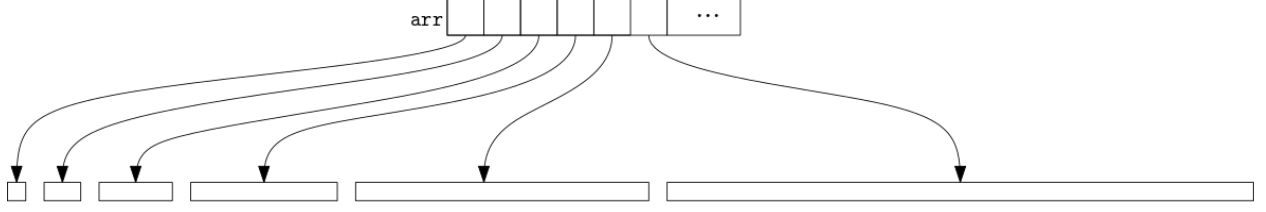


Figure 24: Array Segments

**Garbage Collection** We did not handle garbage collection: **Enqueue** operations remain in the nodes even after their elements have been dequeued. We can keep track of the **blocks** in the **root** whose operations are all terminated, i.e., all enqueues have been dequeued, and the responses of all dequeues have been computed. We call these blocks *finished blocks*. If we help the operations of all processes to compute their responses, then we can say if block  $B$  is finished, then all blocks before  $B$  are also finished. Knowing the most recent finished block in a node, we can reclaim the memory taken by finished blocks. We cannot use arrays (or vectors) to throw the garbage blocks away. We need a data structure that supports **tryAppend()**, **read(i)**, **write(i)** and **split(i)** operations in  $O(\log n)$  time, where **split(i)** removes all the indices less than  $i$ . If each process tries to do the garbage collection once every  $p^2$  operations on the queue, then the amortized complexity remains the same. We can use a concurrent implementation of a persistent red-black trees for this [26]. Bashari and Woelfel [3] used persistent red-black trees in a similar way.

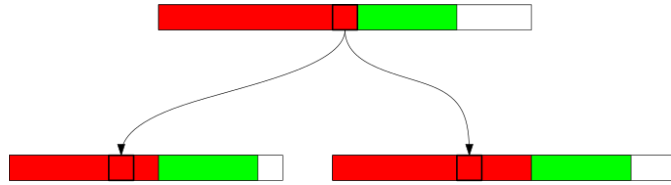


Figure 25: Finished blocks are shown with red color and unfinished blocks are shown with green color. All the subblocks of a finished block are also finished.

**Poly-logarithmic Wait-free Sequences** Consider a data structure storing a sequence that supports three operations **append(e)**, **get(i)** and **index(e)**. An **append(e)** adds  $e$  to the end of the sequence, a **get(i)** gets the  $i$ th element in the sequence and an **index(e)** computes the

position of element `e` in the sequence. We can modify our queue to design such a data structure. An `append(e)` is implemented like `Enqueue(e)`, `get(i)` is done by calling `DoublingSearch` but with a `BinarySearch` on the entire `root.blocks` array and `index(e)` is done similarly to `IndexDequeue` (except operating on enqueues instead of dequeues). We achieve this with poly-logarithmic steps for each operation with respect to the number of `appends` done.

**Other Poly-log Wait-free Data structures** There are two reasons the block tree worked well to implement a queue. Firstly, to respond to a `Dequeue` we do not need to look at the entire history of operations: if a `Dequeue` does not return `null`, we can compute the index of the `Enqueue` that is its response in  $O(\log n)$  time if we keep the number of enqueues and the size. Secondly, the operations we need to search to respond to the `Dequeue` are not very far from it in the sequence of operations: the distance is at most linear in the size of the queue. Creating a wait-free poly-logarithmic implementation of other objects whose operations satisfy these two conditions may be possible.

**Attiya–Fouren Lower Bound** As discussed in Section 2.4 the Attiya–Fouren lower bound says that a concurrent implementation of queues using reads, writes and conditional operations like `CAS` has  $\Omega(c)$  amortized complexity [2] when  $c$ , the number of concurrent processes, is  $O(\log \log p)$ . Our amortized worst-case step complexity is  $\Theta(\log^2 p + \log q)$ . It is an open problem to reduce the gap between our algorithm and the Attiya–Fouren lower bound.

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