

Wait-free Queues with Polylogarithmic Step Complexity

Hossein Naderibeni

supervised by Eric Ruppert

September 16, 2022

Abstract

In this work, we are going to introduce a novel lock-free queue implementation. Linearizability and lock-freedom are standard requirements for designing shared data structures. All existing linearizable, lock-free queues in the literature have a common problem in their worst case called CAS Retry Problem. Our contribution is solving this problem while outperforming the previous algorithms.

Contents

1	Introduction	2
2	Related Work	4
2.1	List-based Queues	4
2.2	Universal Constructions	5
2.3	Attiya Fourier Lower Bound	5
3	Our work	6
3.1	Pseudocode description	16
3.2	Pseudocode	17
3.3	Proof of Correctness	21
3.4	Garbage Collection or Getting rid of the infinite Arrays	36
4	Using Queues to Implement Vectors	37
5	Conclusion	38

1 Introduction

Shared data structures have become an essential field in distributed algorithms research. We are reaching the physical limits of how many transistors we can place on a CPU core. The industry solution to provide more computational power is to increase the number of cores of the CPU. This is why distributed algorithms have become important. It is not hard to see why multiple processes cannot update sequential data structures designed for one process. For example, consider two processes trying to insert some values into a sequential linked list simultaneously. Processes p, q read the same tail node, p changes the next pointer of the tail node to its new node and after that q does the same. In this run, p 's update is overwritten. One solution is to use locks; whenever a process wants to do an update or query on a data structure, the process locks it, and others cannot use it until the lock is released. Using locks has some disadvantages; for example, one process might be slow, and holding a lock for a long time prevents other processes from progressing. Moreover, locks do not allow complete parallelism since only the one process holding the lock can make progress.

The question that may arise is, "What properties matter for a lock-free data structure?", since executions on a shared data structure are different from sequential ones, the correctness conditions also differ. To prove a concurrent object works perfectly, we have to show it satisfies safety and progress conditions. A *safety condition* tells us that the data structure does not return wrong responses, and a *progress property* requires that operations eventually terminate.

The standard safety condition is called *linearizability*, which ensures that for any concurrent execution on a linearizable object, each operation should appear to take effect instantaneously at some moment between its invocation and response. Figure 1 is an example of an execution on a linearizable queue that is initially empty. The arrow shows time, and each rectangle shows the time between the invocation and the termination of an operation. Since `Enqueue(A)` and `Enqueue(B)` are concurrent, `Enqueue(B)` may or may not take effect before `Enqueue(A)`. The execution in Figure 2 is not linearizable since A has been enqueued before B, so it has to be dequeued first.

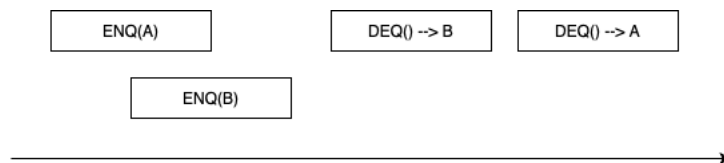


Figure 1: An example of a linearizable execution. Either `Enqueue(A)` or `Enqueue(B)` could take effect first since they are concurrent.

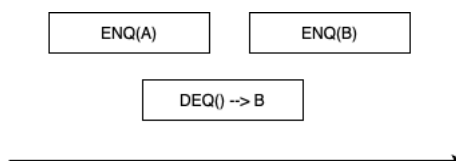


Figure 2: An example of an execution that is not linearizable. Since `Enqueue(A)` has completed before `Enqueue(B)` is invoked the `Dequeue()` should return A or nothing.

There are various progress properties; the strongest is wait-freedom, and the more common is lock-freedom. An algorithm is *wait-free* if each operation terminates after a finite number of its own steps. We call an algorithm *lock-free* if, after a sufficient number of steps, one operation terminates. A wait-free algorithm is also lock-free but not vice versa; in an infinite run of a lock-free algorithm there might be an operation that takes infinitely many steps but never terminates.

In section 2 we talk about previous queues and their common problems. We also talk about polylogarithmic construction of shared objects.

Jayanti [DBLP:conf/podc/Jayanti98a] proved an $\Omega(\log p)$ lower bound on the worst-case shared-access time complexity of p -process universal constructions. He also introduced [DBLP:conf/podc/ChandraJT98] a construction that achieves $O(\log^2 p)$ shared accesses. Here, we first introduce a universal construction using $O(\log p)$ CAS operations [DBLP:conf/fsttcs/JayantiP05]. In section 3 we introduce a polylogarithmic step wait-free universal construction. Our main ideas in of the universal construction also appear in our Queue Algorithm (alg0 [7]). The main short come of our universal construction is using big CAS objects. We

use the universal construction as a stepping stone towards our queue algorithm, so we will not explain it in too much detail.

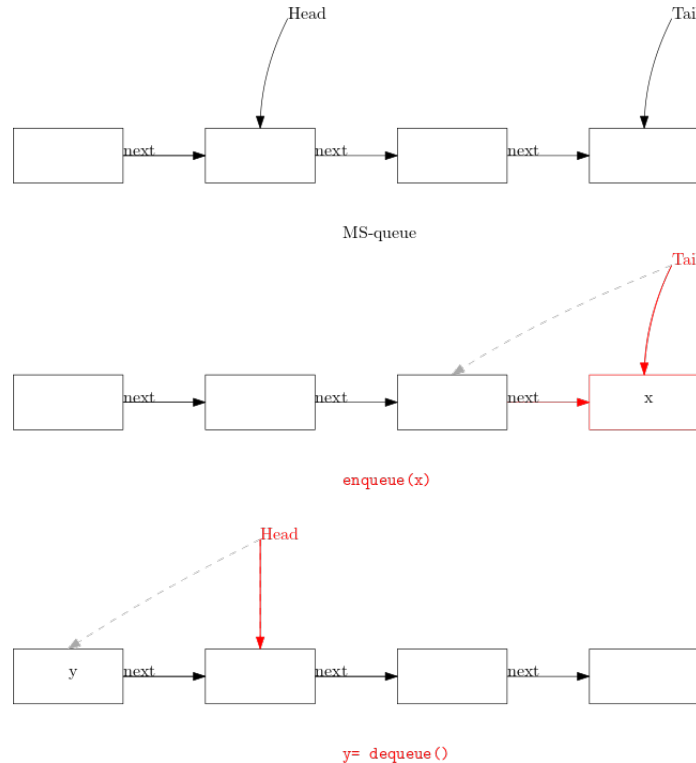
In section 4 we introduce a concurrent wait-free datastructure, to agree on the order of the operations invoked on some processes.

In section 5 we introduce our main work, the queue; prove its linearizability and wait-freeness.

2 Related Work

2.1 List-based Queues

In the following paragraphs, we look at previous lock-free queues. Michael and Scott [\[DBLP:conf/podc/MichaelS96\]](#) introduced a lock-free queue which we refer to as the MS-queue. A version of it is included in the standard Java Concurrency Package. Their idea is to store the queue elements in a singly-linked list (see Figure [3](#)). [\[fig::msq\]](#) Head points to the first node in the linked list that has not been dequeued, and Tail points to the last element in the queue. To insert a node into the linked list, they use atomic primitive operations like LL/SC or CAS. If p processes try to enqueue simultaneously, only one can succeed, and the others have to retry. This makes the amortized number of steps to be $\Omega(p)$ per enqueue. Similarly, dequeue can take $\Omega(p)$ steps.



[\[fig::msq\]](#) Figure 3: MS-queue structure, enqueue and dequeue operations. In the first diagram the first element has been dequeued. Red arrows show new pointers and gray dashed arrows show the old pointers.

Moir, Nussbaum, and Shalev [\[DBLP:conf/spaa/MoirNSS05\]](#) presented a more sophisticated queue by using the elimination technique. The elimination mechanism has the dual purpose of allowing operations to complete in parallel and reducing contention for the queue. An Elimination Queue consists of an MS-queue augmented with an elimination array. Elimination works by allowing opposing pairs of concurrent operations such as an enqueue and a dequeue to exchange values when the queue is empty or when concurrent operations can be linearized to empty the queue. Their algorithm makes it possible for long-running operations to eliminate an opposing operation. The empirical evaluation showed the throughput of their work is better than the MS-queue, but the worst case is still the same; in case there are p concurrent enqueues, their algorithm is not better than MS-queue.

Hoffman, Shalev, and Shavit [\[DBLP:conf/opodis/HoffmanSS07\]](#) tried to make the MS-queue more parallel by introducing the Baskets Queue. Their idea is to allow more parallelism by treating the simultaneous enqueue operations as a basket. Each basket has a time interval in which all its nodes' enqueue operations overlap. Since the operations in a basket are concurrent, we can order them in any way. Enqueues in a basket try to find their order in the basket one by one by using CAS operations. However, like the previous algorithms, if there are still p concurrent enqueue operations in a basket, the amortized step complexity remains $\Omega(p)$ per operation.

Ladan-Mozes and Shavit [\[DBLP:journals/dc/Ladan-MozesS08\]](#) presented an Optimistic Approach to Lock-Free FIFO Queues. They use a doubly-linked list and do fewer CAS operations than MS-queue. But as before, the worst case is when there are p concurrent enqueues which have to be enqueued one by one. The amortized worst-case complexity is still $\Omega(p)$ CASes.

Hendler et al. [\[DBLP:conf/spaa/HendlerIST10\]](#) proposed a new paradigm called flat combining. Their queue is linearizable but not lock-free. Their main idea is

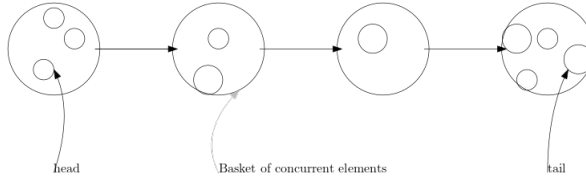


Figure 4: Baskets queue idea. There is a time that all operations in a basket were running concurrently, but only one has succeeded to do CAS. To order the operations in a basket, the mechanism in the algorithm for processes is to CAS again. The successful process will be the next one in the basket and so on.

that with knowledge of all the history of operations, it might be possible to answer queries faster than doing them one by one. In our work we also maintain the whole history. They present experiments that show their algorithm performs well in some situations.

Gidenstam, Sundell, and Tsigas [\[DBLP:conf/opodis/GidenstamST10\]](#) introduced a new algorithm using a linked list of arrays. Global head and tail pointers point to arrays containing the first and last elements in the queue. Global pointers are up to date, but head and tail pointers may be behind in time. An enqueue or a dequeue searches in the head array or tail array to find the first unmarked element or last written element (see Figure [5](#)). Their data structure is lock-free. Still, if the head array is empty and p processes try to enqueue simultaneously, the step complexity remains $\Omega(p)$.

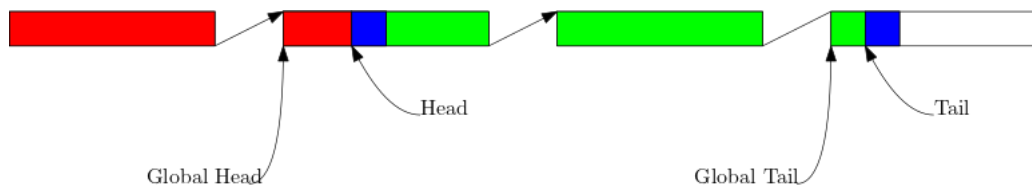


Figure 5: Global pointers point to arrays. Head and Tail elements are blue, dequeued elements are red and current elements of the queue are green.

Kogan and Petrank [\[DBLP:conf/ppopp/KoganP11\]](#) introduced wait-free queues based on the MS-queue and use Herlihy's helping technique to achieve wait-freedom. Their step complexity is $\Omega(p)$ because of the helping mechanism.

In the worst-case step complexity of all the list-based queues discussed above, there is a p term that comes from the case all p processes try to do an enqueue simultaneously. Morrison and Afek call this the *CAS retry problem* [\[DBLP:conf/ppopp/MorrisonA13\]](#). It is not limited to list-based queues and array-based queues share the CAS retry problem as well [\[DBLP:conf/spaa/TsigasZ01, DBLP:conf/icdcp/Shafiei09, DBLP:conf/iceccs/ColvinG05\]](#). We are focusing on seeing if we can implement a queue in sublinear steps in terms of p or not.

2.2 Universal Constructions

Herlihy discussed the possibility of implementing shared objects from other objects [\[10.1145/114005.102808\]](#). A *universal construction* is an algorithm that can implement a shared version of any given sequential object. We can implement a concurrent queue using a universal construction. Jayanti proved an $\Omega(\log p)$ lower bound on the worst-case shared-access time complexity of p -process universal constructions [\[DBLP:conf/podc/Jayanti98a\]](#). He also introduced a construction that achieves $O(\log^2 p)$ shared accesses [\[DBLP:conf/podc/ChandraJT98\]](#). His universal construction can be used to create any data structure, but its implementation is not practical because of using unreasonably large-sized CAS operations.

Ellen and Woelfel introduced an implementation of a Fetch&Inc object with step complexity of $O(\log p)$ using $O(\log n)$ -bit LL/SC objects, where n is the number of operations [\[10.1007/978-3-642-41527-2_20\]](#). Their idea has similarities to Jayanti's construction, and they represent the value of the Fetch&Inc using the history of successful operations.

2.3 Attiya Fourier Lower Bound

3 Our work

Jayanti and Petrovic introduced a wait-free polylogarithmic multi-enqueuer single-dequeueer queue [7]. We are going to design a polylogarithmic multi-enqueuer multi-dequeueer queue using some of their ideas. But we do not use CAS operations with big words and do not put a limit on the number of concurrent operations. We apply two ideas from their work to create a new shared data structure which enables processes to agree on the linearization ordering of the processes. We use the shared tournament tree among p processes (see Figure 6) to agree on one total ordering on the operations invoked by processes. Each process has a leaf which the order of operations invoked by the process is stored in it. When the process wishes to do an operation it appends the operation to its leaf's sequence and after that, the process tries to propagate its new operation up to the tree's root. An ordering of operations propagated up to a node is stored in that node. All processes agree on the sequence stored in the root that is used as the linearization ordering.

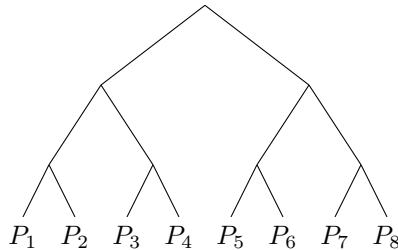


Figure 6: Each process has a leaf and in each node there is an ordering of operations stored. Each node tries to propagate its operations up to the root, which stores the total ordering of all operations.

As we said in each node the sequence of operations is stored. We implement the sequence using an array and appending to the sequence by doing CAS operations on the first null element in the array. In each propagate step, our algorithm uses a subroutine **REFRESH**(n) that aggregates new operations from node n 's both children (that have not already been propagated to n) and tries to append them into n . The general idea is that if we call **REFRESH**(n) twice, the operations in n 's children before the first **REFRESH**(n) are guaranteed to be in n . Because if both of the **Refresh**()es fail there is another instance of **Refresh**() in between which has succeeded to do CAS and has already appended the operations the first **Refresh** was trying to append. This mechanism makes us to do **TryAppends** twice instead of **Appending** to the array. From the next paragraph we explain our ideas which are different from Jayanti.

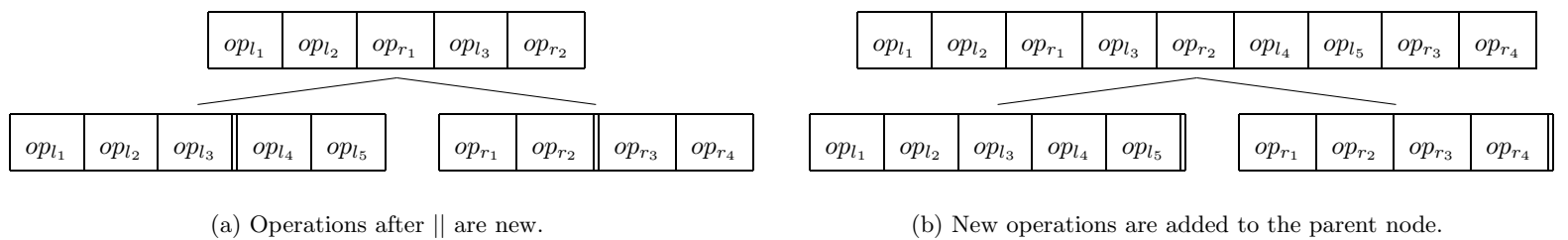


Figure 7: Successful **Refresh**, operations in children after || are new.

Instead of storing operations explicitly in the nodes, we are going to keep track of some statistics of them. This allows us to CAS fixed-size objects in each **REFRESH**(n). To do that, we introduce blocks that only contain the number of operations from the left and the right child in a **Refresh**() procedure and only propagate the statistics block of the new operations. In each **Refresh**() there is at most one operation from each process trying to be propagated. Since one operation cannot invoke two operations concurrently. Also as the operations in a **REFRESH**() step are concurrent we can linearize them among themselves in any order we wish. Note that if two operations are in one **Propagate**() step in a node they are going to be propagated up to the root together. From now on instead of propagating sequence of operations in **Refresh** steps we propagate blocks of operations. The idea is that we can describe a block's contents only using some numbers. Our choice is to put the operations propagated from the left child before the operations propagated from the right child. In this way if we know the number of total operations in a block and the number of operations from the left child we can order

the operations in a unique way which is a complete ordering. We can instead keep track of sets of concurrent operations and create the total ordering of all operations at the root (see Figure ^{fig::set}8). In the next paragraphs we explain the reason behind our choice of ordering.

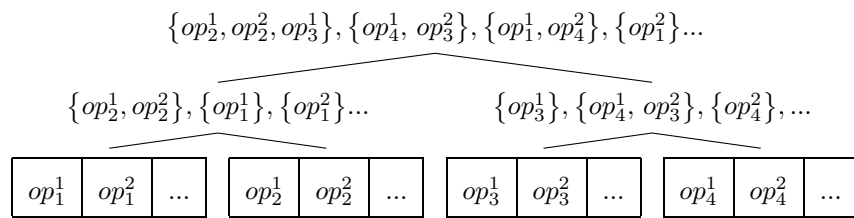


fig::set

Figure 8: In each internal node, we store the set of all the operations propagated together, and one can arbitrarily linearize the sets of concurrent operations among themselves. Since we linearize operations when they are added to the root, ordering the blocks in the root is important.

Previously we talked about storing the sequence of operations in the nodes of the tree. A process may wish to know information about the root ordering. Two functionalities are to get the i th propagated operation and compute the rank of a propagated operation in the linearization. Since our algorithm is aimed for a queue, we make some assumptions here that we only wish to know the order of a dequeue and one only tries to get an enqueue. We will explain in detail in the next but for now let's say **enqueues** and **dequeues** are appended to the tree and when one wants to find the response to a dequeue, it computes the order of the dequeue in linearization and using this information it computes which enqueue is the answer to the dequeue or if the answer is null. If the answer was some enqueue we find the enqueue using **DSearch(i)** and **GetENQ(n,b,i)**. **DSearch(i)** finds the block containing the i th enqueue in the root and **GetENQ(n,b,i)** finds its sub-block recursively to reach a leaf. **Index()** is similar but more complicated, finding super-blocks from a leaf to the root. The main challenge in each level of **Get(i)** and **Index(op)** is that it should take polylogarithmic steps with respect to p . After appending operation **op** to the root, processes can find out information about the linearization ordering using **Get(i)** and **Index(op)**. Each block stores an extra constant amount of information (like prefix sums) to allow binary searches to find the required block in a node quickly.

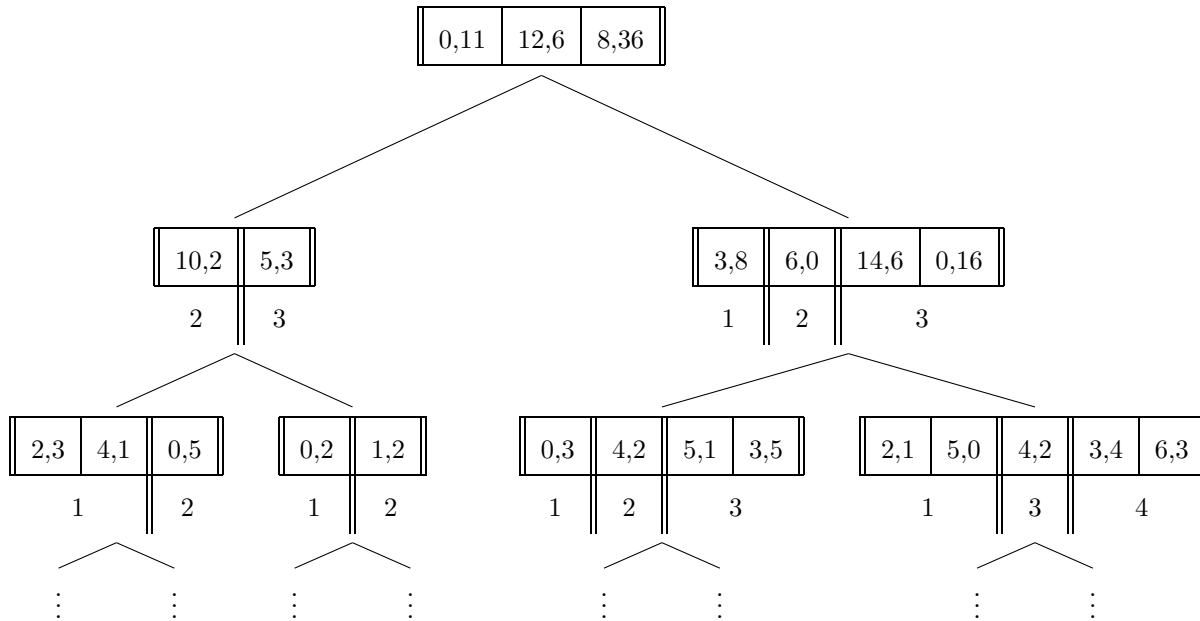


fig:block

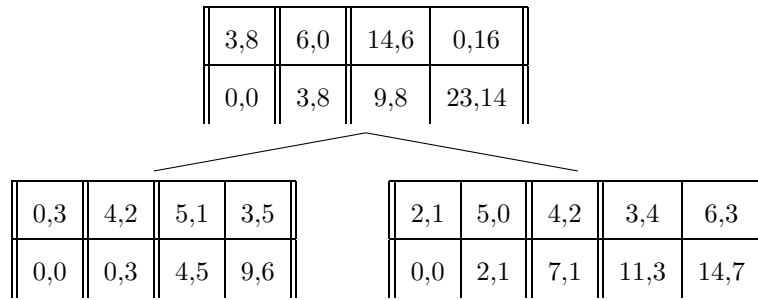
Figure 9: Showing concurrent operation sets with blocks. Each block consists of a pair(left, right) indicating the number of operations from the left and the right child, respectively. Block (12,6) in the root contains blocks (10,2) from the left child and (6,0) from the right child. Blocks between two lines || are propagated together to the parent. For example, Blocks (2,3) and (4,1) from the leftmost leaf and (0,2) from its sibling are propagated together into the block (10,2) in their parent. The number underneath a group of blocks in a node indicates which block in the node's parent those blocks were propagated to.

The definition of linearizability allows concurrent operations to be reordered arbitrarily. Thus, a group of concurrent operations can be appended to our root sequence as one block without specifying the order among the operations.

In the original algorithm we differ between the enqueues and the dequeues in a block. Later we discuss about the reason.

Each block b in node n is the aggregation of blocks in the children of n that are newly read by the `PROPAGATE()` step that created block b . For example, the third block in the root (8,36) is created by merging block (5,3) from the left child and (14,6) and (0,16) from the right child. Block (5,3) also points to elements from blocks (0,5) and (1,2).

We choose to linearize operations in a block from the left child before those from the right child as a convention. Operations within a block of the root can be ordered in any way that is convenient. In effect, this means that if there are concurrent new blocks in a `REFRESH()` step from several processes we linearize them in the order of their process ids. So for example operations aggregated in block (10,2) are in the order (2,3),(4,1),(0,2). All blocks from the left child will come before the right child and the order of blocks of each child is preserved among themselves.



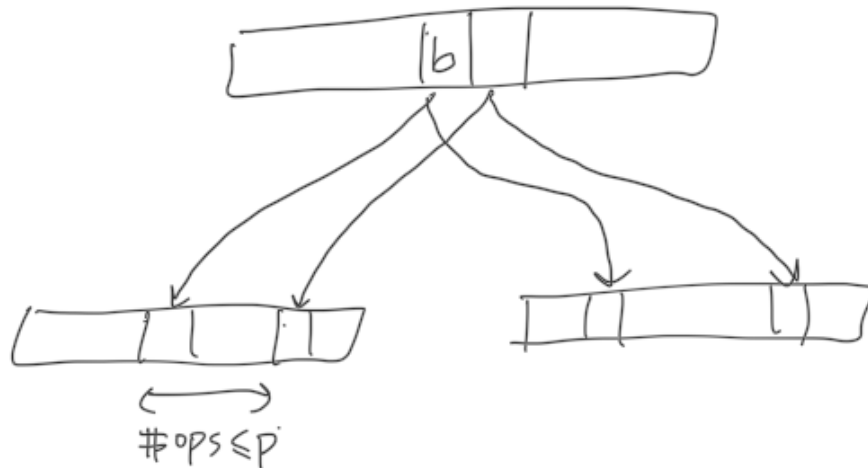
ig:prefix

Figure 10: Using Prefix sums in blocks. When we want to find block b elements in its children, we can use binary search. The number below each block shows the count of elements in the previous blocks.

In a `PROPAGATE()` invocation path from a leaf to root, there will be `REFRESH()` steps with merges from $2, 4, 8, \dots, p$ processes. So in a complete propagation, at most $2p$ blocks are merged into one block. (maybe useful for analysis)

`GETINDEX(i)` returns the i th operation stored in the block tree sequence. We do that by finding the block b_i containing i th element in the root, and then recursively finding the subblock of b_i which contains i th element. To make this recursive search faster, instead of iterating over all elements in sequence of blocks we store prefix sum of number of elements in the blocks sequence and pointers to make `BSearch` faster.

Furthermore, in each block, we store the prefix sum of left and right elements. Moreover, for each block, we store two pointers to the last left and right subblock of it (see fig [fig:point](#) and [fig:prefix](#)).



:::pointer

Figure 11: Block have pointers to the starting block of theirs for each child.

Starting from the root, $\text{GETINDEX}(i)$ BSearches i in the prefix sum array to find block containing i th operation, then continues recursively calling $\text{GETELEMENT}(b, i)$ to find i th element of block b . From lemma [lem:block_size](#) we know a block size is at most p . So BSearch takes at most $O(\log p)$, since with knowing pointers of a block and its previous block we can determine the base (domain ?) to search and its size is $O(p)$.

Design of a Queue Each process is assigned to a leaf in a shared tournament tree. Thus, for example, the leaf node for process p_i contains an array of elements by p_i in the order they were invoked. Each internal node of the tree contains an array of blocks of elements. Block b in node n is created in a $\text{PROPAGATE}()$ step and is merged block of new blocks at the time of $\text{PROPAGATE}()$ reading n 's children blocks. Each block consists of pointers left and right, to the last block merged into itself from left and right child in that order. Moreover, two numbers, left and right, indicate the count of elements in the blocks from the left and right child consecutively. Furthermore, prefix left, and right can be computed from the prefix sum of left and right values. Elements of block b can be determined recursively ($\text{GETELEMENTS}(b)$). The i th element in the sequence can be determined in $O(\log^2 p)$ steps by recursively finding i th element in block b ($\text{GETELEMENT}(i)$). After element e is propagated (appended to a block into the root), its index can be computed with $\text{GETINDEX}(op)$.

In order to compute elements of a block faster we store prefix-sum blocks (block i has tuple (right-sum=#right ops in previous block, left-sum=#left ops in previous blocks)) [See Figure [fig:prefix](#)]. Here is the algorithm to get elements of a block.

Specification A Queue is a shared data structure that stores a sequence of elements. It has two methods $\text{Enqueue}(e)$ and $\text{Dequeue}()$. $\text{Enqueue}(e)$ adds e to the end of the sequence. $\text{Dequeue}()$ returns the first element stored in the sequence and removes it from the sequence.

CreateBlock() **CreateBlock(n)** returns a block containing new operations of **n**'s children. **b'.end_{left}** stores the index of the rightmost subblock of left child of **b**'s previous block. Other attributes are assigned values followed by definition.

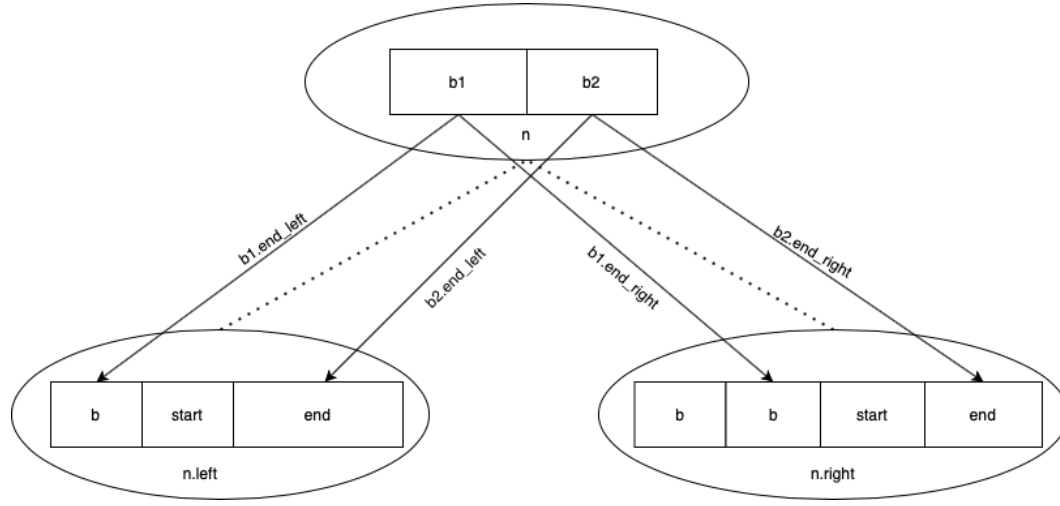


Figure 12: Snapshot of a **CreateBlock()**

CreateBlock()

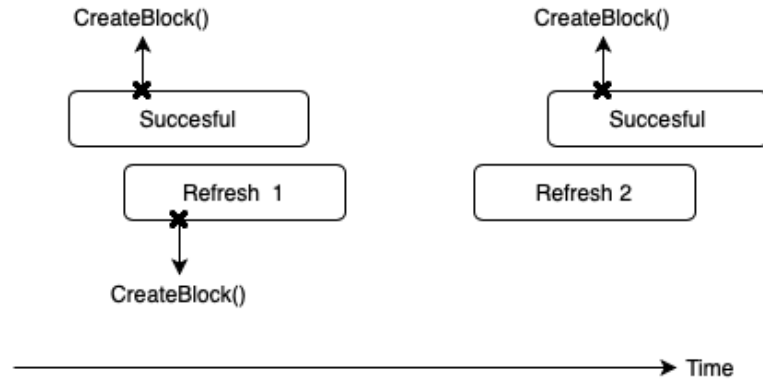


Figure 13: The second failed Refresh is assuredly concurrent to a Successful Refresh() with **CreateBlock** line after first failed Refresh's **CreateBlock()**.

Computing $\text{Get}(n, b, i)$ To find the i th element in block b of node n , we search among subblocks of b that is bounded by p . Subblocks of a block are within the start and end block of the $\text{CreateBlock}()$ procedure of it.

How $\text{Refresh}(n)$ works.

1. Read n 's counter and head
2. Create block b
3. CAS b into n
4. If previous succeed:
 - (a) Update sup of b 's ending subblocks
 - (b) Increment children's counters

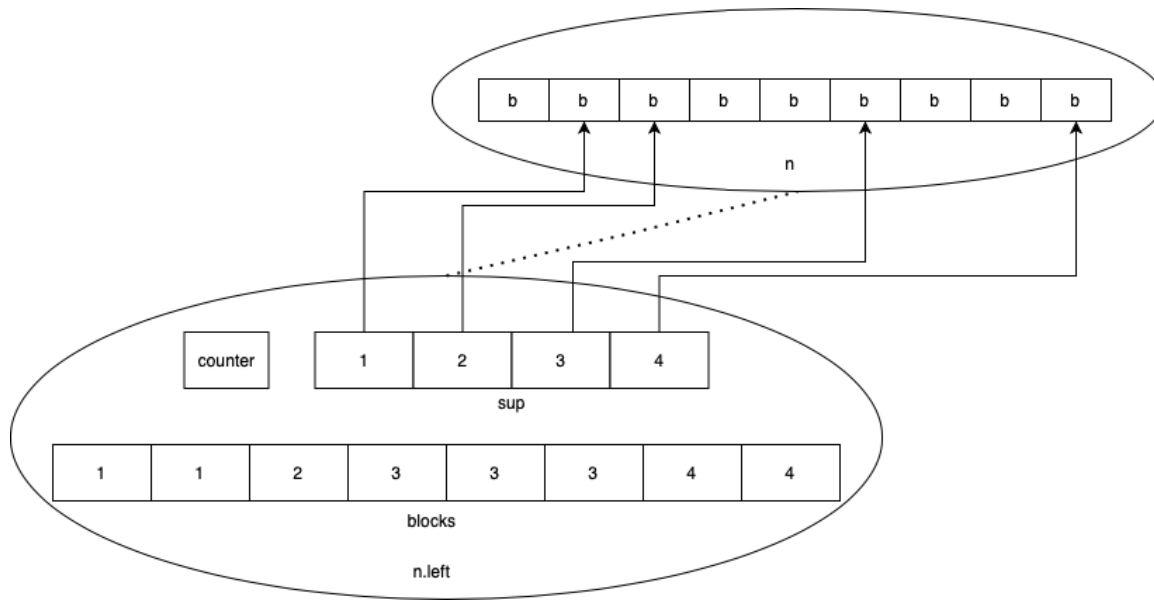


Figure 14: Sup and timer in a node, numbers on blocks are their time values.

Computing superblock

Implementing Queue using Block Tree In this work, we design a queue with $O(\log^2 p + \log n)$ steps per operation, where n is the number of total operations invoked. We avoid the $\Omega(p)$ worst-case step complexity of existing shared queues based on linked lists or arrays (CAS Retry Problem). A queue stores a sequence of elements and supports two operations, enqueue and dequeue. **Enqueue(e)** appends element e to the sequence stored. **Dequeue()** removes and returns the first element among in the sequence. If the queue is empty it returns **null**. Knowing index i is the tail of the queue, we can return the dequeue response using **Get(i)**. So in the rest we modify block tree to compute i for each **Dequeue()** to achieve a FIFO queue.

Next, we describe how to use block tree to implement queues. The block tree, maintains the history of all operations, not only the current state of the queue. Now consider the following history of operations. What should each **Dequeue()** return? We can implement Enqueue and Dequeue using our block tree. An **Enqueue(e)** appends an operation with input argument e in the block tree. To do a **Dequeue()**, process p first appends a **DEQ** operation to the tree. Then p finds the rank of the **DEQ** using **Index()**, the rank of the **DEQ** and the information stored in the root about the queue p computes the rank of the **ENQ** having the answer of the **DEQ**. Finally p returns the argument of that **ENQ** using **Get(i)**.

ENQ(5)	ENQ(2)	DEQ()	ENQ(3)	DEQ()	DEQ()	DEQ()	ENQ(4)	ENQ(6)	DEQ()
--------	--------	-------	--------	-------	-------	-------	--------	--------	-------

Table 1: An example histoy of operations on the queue

A non-null dequeue is one that returns a non-null value. In the example above, `Dequeue()` operations return 5, 2, 3, null, 4 in order. Before `ENQ(4)` the queue gets empty so the last `DEQ()` returns null. If the queue is non-empty and r `Dequeue()` operations have returned a non-null response, then i th `Dequeue()` returns the input of the $r + 1$ th `Enqueue()`. So, in order to answer a Dequeue, it's sufficient to know the size of the queue and the number of previous non-null dequeues.

In the Block Tree, we did not store the sequence of operations explicitly but instead stored blocks of concurrent operations to optimize `Propagate()` steps and increase parallelism. So now the problem is to find the result of each Dequeue. From lemma [lem:block_size](#) we know we can linearize operations in a block in any order; here, we choose to decide to put Enqueue operations in a block before Dequeue operations. In the next example, operations in a cell are concurrent. `DEQ()` operations return null, 5, 2, 1, 3, 4, null respectively. We will next describe how these values can be computed efficiently.

DEQ()	ENQ(5), ENQ(2), ENQ(1), DEQ()	ENQ(3), DEQ()	ENQ(4), DEQ(), DEQ(), DEQ(), DEQ()
-------	-------------------------------	---------------	------------------------------------

Table 2: An example history of operation blocks on the queue

Now, we claimed that by knowing the current size of the queue and the number of non-null dequeue operations before the current dequeue, we could compute the index of the resulting `Enqueue()`. We apply this approach to blocks; if we store the size of the queue after each block of operations happens and the number of non-null dequeues till a block, we can compute each dequeue's index of result in $O(1)$ steps.

	DEQ()	ENQ(5), ENQ(2), ENQ(1), DEQ()	ENQ(3), DEQ()	ENQ(4), DEQ(), DEQ(), DEQ(), DEQ()
#enqueues	0	3	1	1
#dequeues	1	1	1	4
#non-null dequeues	0	1	2	5
size	0	2	2	0

Table 3: Augmented history of operation blocks on the queue

Size and the number of non-null dequeues for b th block could be computed this way:

`size[b] = max(size[b-1] +enqueues[b] -dequeues[b], 0)`

`non-null dequeues[b] = non-null dequeues[b-1] +dequeues[b] -size[b-1] -enqueues[b]`

Given DEQ is in block b , `response(DEQ)` would be:

`(size[b-1] - index of DEQ in the block's dequeues >= 0) ? ENQ[non-null dequeues[b-1] + index of DEQ in the block's dequeues]
: null;`

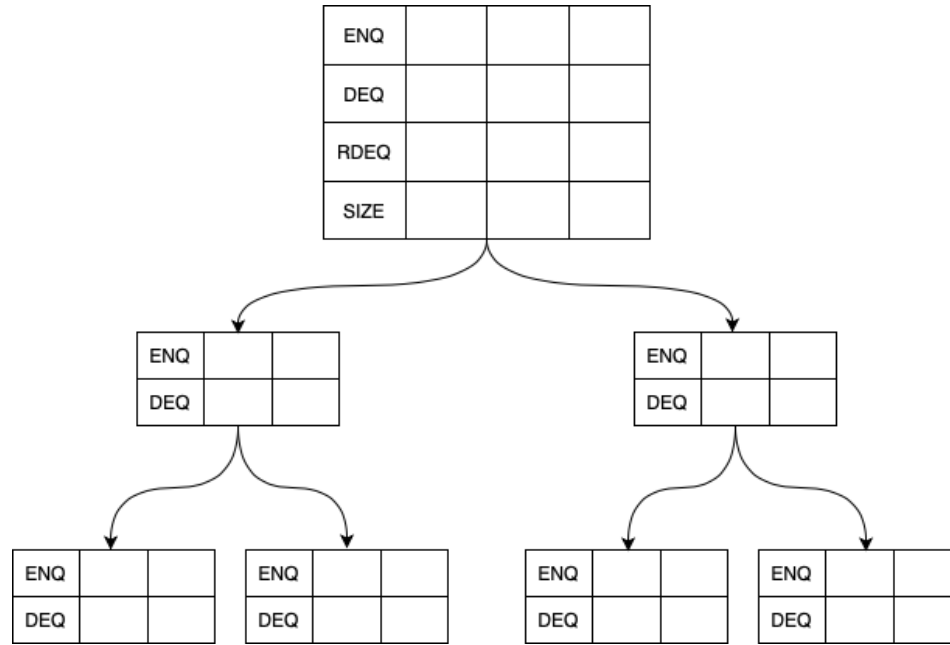


Figure 15: Fields stored in the Queue nodes.

ig::queue

3.1 Pseudocode description

Tree In order to reach an agreement on the order of operations among p processes, we use a Tournament Tree. Leaf l_i is assigned to a process i . Each process adds op to its leaf. In each internal node an ordering of operations in its subtree is stored. All processes agree on the total ordering of all operations stored in the root. This ordering will be the linearization of the operations.

Implicit Storing Blocks For efficiency, instead of storing explicit sequence of operations in nodes of the Tournament Tree, we use Blocks. A Block is a constant size object that implicitly represents a sequence of operations. In each node there is an array of Blocks.

Block b contains subblocks in the left and right children. WLOG left subblocks of b are some consecutive blocks in the left child starting from where previous block of b has ended to the the end of b . See Figure [fig::createBlock](#) 12.

We store ordering among operations in the tournament tree constructed by nodes. In each node we store pointers to its relatives, an array of blocks and an index to the first empty block. Furthermore in leaf nodes there is an array of operations where each operation is stored in one cell with the same index in blocks. There is a counter in each node incrementing after a successful Refresh() step. It means after that some bunch of blocks in a node have propagated into the parent then the counter increases. Each new block added to a node sets its time regarding counter. This helps us to know which blocks have aggregated together to a block, not precisely though. We also store the index of the aggregated block of a block with time i in `super[i]`.

In each block we store 4 essential stats that implicitly summarize which operations are in the block `numenq-left`, `numdeq-left`, `numenq-right`, `numdeq-right`. In order to make BSearch()es faster we store prefix sums as well and there are some more general stats that help to make pseudocode more readable but not necessary.

To compute the head of the queue before a dequeue two more fields are stored in the root `size` and `sumnon-null deq`. `size` in a block shows the number of elements after the block has finished and `sumnon-null deq` is the total number of non-null dequeues till the block.

Enqueue(e) just appends an operation with element e to the root. Dequeue() appends an operation to the root and computes its ordering and the enqueue operation containing the head before it calling ComputeHead() and then gets and returns the operation's element.

Append(op) adds op to the invoking process's leaf's ops and blocks, propagates it up to the root and if the op is a dequeue returns its order in residing block in the root and the block's index. As we said later Propagate() assuredly aggregates new blocks to a block in the parent by calling Refresh() two times. Refresh(n) creates a block, tries to CAS it into the pn 's blocks and if it was successful updates `super` and `counter` in both of n 's children.

We only want to know the element of enqueue operations and compute ordering for dequeue operations. That's the reason here Get() searches between enqueues only and Index() returns ordering of a dequeue among dequeues. Get(n, b, i) decides the requested element is in which child of n and continues to search recursively. index(n, i, b) calculates the ordering of the given operation in n 's parent each step and finally returns the result among total ordering.

3.2 Pseudocode

Algorithm Tree Fields Description

◇ Shared

- A binary tree of Nodes with one leaf for each process. root is the root node.

◇ Local

- *Node* leaf: process's leaf in the tree.

► Node

- **Node* left, right, parent : initialized when creating the tree.
- *BlockList*
- *int* head= 1: #blocks in blocks. blocks[0] is a block with all integer fields equal to zero.

► Block

- *int* super : approximate index of the superblock, read from parent.head when appending the block to the node

► LeafBlock extends Block

- *Object* element : Each block in a leaf represents a single operation. If the operation is enqueue(x) then element=x, otherwise element=null.
- *int* sum_{enq}, sum_{deq} : # enqueue, dequeue operations in the prefix for the block

► InternalBlock extends Block

- *int* end_{left}, end_{right} : indices of the last subblock of the block in the left and right child
- *int* sum_{enq-left} : # enqueue operations in the prefix for left.blocks[end_{left}]
- *int* sum_{deq-left} : # dequeue operations in the prefix for left.blocks[end_{left}]
- *int* sum_{enq-right} : # enqueue operations in the prefix for right.blocks[end_{right}]
- *int* sum_{deq-right} : # dequeue operations in the prefix for right.blocks[end_{right}]

► RootBlock extends InternalBlock

- *int* size : size of the queue after performing all operations in the prefix for this block
-

Abbreviations:

- blocks[b].sum_x=blocks[b].sum_{x-left}+blocks[b].sum_{x-right} (for b≥0 and x ∈ {enq, deq})
- blocks[b].sum=blocks[b].sum_{enq}+blocks[b].sum_{deq} (for b≥0)
- blocks[b].num_x=blocks[b].sum_x-blocks[b-1].sum_x
(for b>0 and x ∈ {∅, enq, deq, enq-left, enq-right, deq-left, deq-right})

Algorithm Queue

```
201: void ENQUEUE(Object e) ▷ Creates a block with element e and adds it to the tree.
202:   block newBlock= NEW(LeafBlock)
203:   newBlock.element= e
204:   newBlock.sumenq= leaf.blocks[leaf.head].sumenq+1
205:   newBlock.sumdeq= leaf.blocks[leaf.head].sumdeq
206:   leaf.APPEND(newBlock)
207: end ENQUEUE

208: Object DEQUEUE() ▷ Creates a block with null value element, appends it to the tree, computes its order among operations, and returns its response.
209:   block newBlock= NEW(LeafBlock)
210:   newBlock.element= null
211:   newBlock.sumenq= leaf.blocks[leaf.head].sumenq
212:   newBlock.sumdeq= leaf.blocks[leaf.head].sumdeq+1
213:   leaf.APPEND(newBlock)
214:   <b, i>= INDEXDEQ(leaf.head, 1)
215:   output= FINDRESPONSE(b, i)
216:   return output
217: end DEQUEUE

218: <int, int> FINDRESPONSE(int b, int i)
219:   if root.blocks[b-1].size + root.blocks[b].numenq - i < 0 then
220:     return null
221:   else
222:     e= i - root.blocks[b-1].size + root.blocks[b-1].sumenq
223:     return root.GetENQ(root.DSEARCH(e, b))
224:   end if
225: end FINDRESPONSE
```

deqRest

Algorithm BlockList

▷ : Supports two operations `blocks.tryAppend(Block b)`, `blocks[i]`. Initially empty, when `blocks.tryAppend(b, n)` returns true `b` is appended to `blocks[n]` and `blocks[i]` returns *i*th block in the blocks. If some instance of `blocks.tryAppend(b, n)` returns false there is a concurrent instance of `blocks.tryAppend(b', n)` which has returned true. `blocks[0]` contains an empty block with all fields equal to 0 and `endleft`, `endright` pointers to the first block of the corresponding children.

`block[] blocks`: array of blocks

```
701: boolean TRYAPPEND(block blk, int n)
702:   return CAS(blocks[n], null, blk)
703: end TRYAPPEND
```

Algorithm Node

```

301: void PROPAGATE()
302:   if not REFRESH() then
303:     REFRESH()
304:   end if
305:   if this is not root then
306:     parent.PROPAGATE()
307:   end if
308: end PROPAGATE

309: boolean REFRESH()
310:   h= head
311:   for each dir in {left, right} do
312:     hdir= dir.head
313:     if dir.blocks[hdir] != null then
314:       dir.ADVANCE(hdir, h)
315:     end if
316:   end for
317:   new= CREATEBLOCK(h)
318:   if new.num==0 then return true
319:   end if
320:   result= blocks[h].CAS(null, new)
321:   hp= parent.head
322:   this.ADVANCE(h, hp)
323:   return result
324: end REFRESH

325: void ADVANCE(int h, int hp)
326:   blocks[h].super.CAS(null, hp)
327:   head.CAS(h, h+1)
328: end ADVANCE

329: int BSEARCH(field f, int i, int start, int end)
330: end BSEARCH

331: <Block, int, int> CREATEBLOCK(int i) ▷ Creates and returns the block
    to be inserted as ith block in blocks.

332:   block newBlock= NEW(block)
333:   for each dir in {left, right} do
334:     indexlast= dir.head-1
335:     indexprev= blocks[i-1].enddir
336:     newBlock.enddir= indexlast
337:     blocklast= dir.blocks[indexlast]
338:     blockprev= dir.blocks[indexprev]
339:     ▷ newBlock includes dir.blocks[indexprev+1..indexlast].
340:     newBlock.sumenq-dir= blocks[i-1].sumenq-dir + blocklast.sumenq
    - blockprev.sumenq
341:     newBlock.sumdeq-dir= blocks[i-1].sumdeq-dir + blocklast.sumdeq
    - blockprev.sumdeq
342:   end for
343:   if this is root then
344:     newBlock.size = max(root.blocks[i-1].size + newBlock.numenq
    - newBlock.numdeq, 0)
345:   end if
346:   return <b, nleft, nright>
347: end CREATEBLOCK

```

Algorithm Root

```

301: <int, int> DSEARCH(int e, int end)
302:   start= end-1
303:   while root.blocks[start].sumenq ≥ e do
304:     start= max(start-(end-start), 0)
305:   end while
306:   b= root.BSearch(sumenq, e, start, end)
307:   i= e- root.blocks[b-1].sumenq
308:   return <b,i>
309: end DSEARCH

310: <int, int> DSEARCH(int e, int end)
311:   start= end-1
312:   while root.blocks[start].sumenq ≥ e do
313:     start= max(start-(end-start), 0)
314:   end while
315:   b= root.BSearch(sumenq, e, start, end)
316:   i= e- root.blocks[b-1].sumenq
317:   return <b,i>
318: end DSEARCH

319: <int, int> DSEARCH(int e, int end)
320:   start= end-1
321:   while root.blocks[start].sumenq ≥ e do
322:     start= max(start-(end-start), 0)
323:   end while
324:   b= root.BSearch(sumenq, e, start, end)
325:   i= e- root.blocks[b-1].sumenq
326:   return <b,i>
327: end DSEARCH

```

Algorithm Node	
	\rightsquigarrow Precondition: $\text{blocks}[b].\text{num}_{\text{enq}} \geq i \geq 1$
	401: <i>element</i> GETENQ(<i>int</i> b, <i>int</i> i) \triangleright Returns the element of $E_i(\text{this}, b)$.
	402: if this is leaf then
tBaseCase	403: return blocks[b].element
ftOrRight	404: else if $i \leq \text{blocks}[b].\text{num}_{\text{enq-left}}$ then $\triangleright E_i(\text{this}, b)$ is in the left child of this node.
tChildGet	405: subBlock= left.BSEARCH(sum_{enq} , $i + \text{blocks}[b-1].\text{sum}_{\text{enq-left}}$, $\text{blocks}[b-1].\text{end}_{\text{left}} + 1$, $\text{blocks}[b].\text{end}_{\text{left}}$)
	406: return left.GETENQ(subBlock, i)
	407: else
	408: $i = i - \text{blocks}[b].\text{num}_{\text{enq-left}}$
tChildGet	409: subBlock= right.BSEARCH(sum_{enq} , $i + \text{right.blocks}[b-1].\text{sum}_{\text{enq-right}}$, $\text{blocks}[b-1].\text{end}_{\text{right}} + 1$, $\text{blocks}[b].\text{end}_{\text{right}}$)
	410: return right.GETENQ(subBlock, i)
	411: end if
	412: end GETENQ
	\rightsquigarrow Precondition: bth block of the node has propagated up to the root and $\text{blocks}[b].\text{num}_{\text{enq}} \geq i$.
	413: <i><int, int></i> INDEXDEQ(<i>int</i> b, <i>int</i> i) \triangleright Returns <i><x, y></i> if $D_{\text{this}, b, i} = D_{\text{root}, x, y}$.
	414: if this is root then
xBaseCase	415: return <i><b, i></i>
	416: else
	417: $\text{dir} = (\text{parent.left} == \text{n}) ? \text{left} : \text{right}$ \triangleright check if this node is a left or a right child
puteSuper	418: superBlock= parent.BSEARCH($\text{sum}_{\text{deq-dir}}$, $i + \text{blocks}[b-1].\text{sum}_{\text{deq}}$, $\text{blocks}[b].\text{super} - 2$, $\text{blocks}[b].\text{super} + 2$)
	\triangleright superblock's group has at most p difference with the value stored in super[].
	419: if dir is left then
viousLeft	420: $i += \text{blocks}[b-1].\text{sum}_{\text{enq}} - \text{blocks}[\text{superBlock} - 1].\text{sum}_{\text{enq-left}}$ \triangleright consider the enqueues in the previous blocks from the left child
	421: end if
	422: if dir is right then
iousRight	423: $i += \text{blocks}[b-1].\text{sum}_{\text{enq}} - \text{blocks}[\text{superBlock} - 1].\text{sum}_{\text{enq-right}}$ \triangleright consider the enqueues in the previous blocks from the right child
foreRight	424: $i += \text{blocks}[\text{superBlock}].\text{num}_{\text{deq-left}}$ \triangleright consider the dequeues from the right child
	425: end if
	426: return this.parent.INDEXDEQ(superBlock, i)
	427: end if
	428: end INDEXDEQ

Algorithm Leaf	
	601: <i>void</i> APPEND(<i>block</i> blk) \triangleright Append is only called by the owner of the leaf.
pendStart	602: blk.group= head
	603: blocks[head]= blk
appendEnd	604: head+=1
	605: parent.PROPAGATE()
	606: end APPEND

3.3 Proof of Correctness

TEST Fix the logical order of definitions (cyclic references).

TEST Is it better to show $\text{ops}(\text{EST}_n, t)$ with EST_n, t ?

Question A good notation for *the index of the b*?

Question How to remove the notion of time? To say $\text{pre}(n, i)$ contains $n.\text{blocks}[0..i]$ instead of $\text{EST}(n, t)$ which $\text{head}=i$ at time t . Is it good? Furthermore, can we remove the notion of established blocks?

Definition 1 (Block). A block is an object storing some statistics, as described in Algorithm Queue. A block in a node's blocklist implicitly represents a set of operations. If $n.\text{blocks}[i] = b$ we call i the *index* of block b . Block b is before block b' in node n if and only if the index of the b is smaller than the index of the b' 's. For a block in a `BlockList` we define *the prefix for the block* to be the blocks in the `BlockList` up to and including the block.

Lemma 2 (head Increment). *Let R be an instance of Refresh on node n that reaches Line 320. After R terminates $n.\text{head}$ is greater than h , the value read in line 310 of R .*

Proof. If Line 310 or 311 are successful then the claim holds, otherwise another process has incremented the head from h to $h+1$. \square

Invariant 3 (headPosition). If the value of $n.\text{head}$ is h then, $n.\text{blocks}[i] = \text{null}$ for $i > h$ and $n.\text{blocks}[i] \neq \text{null}$ for $i < h$.

Proof. The invariant is true initially since 1 is assigned to $n.\text{head}$ and $n.\text{blocks}[x]$ is null for every x . The truth of the invariant may be affected by writing into $n.\text{blocks}$ or incrementing $n.\text{head}$. We show the invariant still holds after these two changes.

In the algorithm, some value is appended to $n.\text{blocks}[]$ by writing into $n.\text{blocks}[\text{head}]$ only in Line 313. Writing into $n.\text{blocks}[\text{head}]$ preserves the invariant, since the claim does not talk about $n.\text{blocks}[\text{head}]$. The value of $n.\text{head}$ is modified only in lines 311 and 312. Depending on whether the `TryAppend()` in Line 320 succeeded or not, we show that the claim holds after the increment of $n.\text{head}$ in either case. If $n.\text{head}$ is incremented to h it is sufficient to show $n.\text{blocks}[h] \neq \text{null}$ to prove the invariant still holds. In the first case the process applied a successful `TryAppend(new, h)` in line 313, which means $n.\text{blocks}[h]$ is not null anymore. Note that whether 311 or 312 return true or false, after they finish we know that $n.\text{head}$ has been incremented from the value read in Line 310 (Lemma 2). The failure case is also the same since it means some non-null value has been written into $n.\text{blocks}[\text{head}]$ by some process. \square

Observation 4. $n.\text{head}$ is non-decreasing over time.

Proof. The claim follows trivially from the code since $n.\text{head}$ is only incremented in Line 311 of `Refresh()`. \square

Lemma 5 (headProgress). *If $n.\text{blocks}[i] \neq \text{null}$ and $i > 0$ then $n.\text{blocks}[i].\text{end}_{\text{left}} \geq n.\text{blocks}[i-1].\text{end}_{\text{left}}$ and $n.\text{blocks}[i].\text{end}_{\text{right}} \geq n.\text{blocks}[i-1].\text{end}_{\text{right}}$.*

Proof. Consider the block b written into $n.\text{blocks}[i]$ by `TryAppend()` at Line 320. It is created by the `CreateBlock(i)` called at Line 317. Prior to this call to `CreateBlock(i)`, $n.\text{head}=i$ at Line 310, so $n.\text{blocks}[i-1]$ is already a non-null value b' by Invariant 3. Thus the `CreateBlock(i-1)` that creates b' terminates before `CreateBlock(i)` that creates b is invoked. The value written into $b.\text{end}_{\text{left}}$ at Line 336 of `CreateBlock(i)` was read from $n.\text{left.head}-1$ at Line 334 of `CreateBlock(i)`. Similarly, the value in $n.\text{blocks}[i-1].\text{end}_{\text{left}}$ was read from $n.\text{left.head}-1$ during the call to `CreateBlock(i-1)`. Since $n.\text{left.head}$ is non-decreasing $b'.\text{end}_{\text{left}} \leq b.\text{end}_{\text{left}}$. The proof for $\text{end}_{\text{right}}$ is similar. \square

Definition 6 (Subblock). Block b is a *direct subblock* of $n.\text{blocks}[i]$ if it is in $n.\text{left.blocks}[n.\text{blocks}[i-1].\text{end}_{\text{left}}+1..n.\text{blocks}[i].\text{end}_{\text{left}}] \cup n.\text{right.blocks}[n.\text{blocks}[i-1].\text{end}_{\text{right}}+1..n.\text{blocks}[i].\text{end}_{\text{right}}]$. Block b is a subblock of $n.\text{blocks}[i]$ if b is a direct subblock of $n.\text{blocks}[i]$ or a subblock of a direct subblock of $n.\text{blocks}[i]$.

Corollary 7 (No Duplicates). *If op is in $n.\text{blocks}[i]$ then there is no $j \neq i$ such that $op \in \text{ops}(n.\text{blocks}[j])$.*

Proof. Operation `op` is invoked only one time in an execution because every operations invoked is distinct. Since there is node `n` which `op` is in two different blocks of `n`, there is node `n'` that is the lowest height node in the tree that contains `op` in two of its blocks `b1, b2`. By Definition ^{def::subblock}~~6~~, ~~b1~~ and ~~b2~~ have distinct subblocks(not only direct subblocks) and since `op` is in only one leaf block, then it cannot be in both `b1` and `b2`. \square

Definition 8 (Superblock). Block `b` is *direct superblock* of block `c` if `c` is a direct subblock of `b`. Block `b` is *superblock* of block `c` if `c` is a subblock of `b`.

def::ops

Definition 9 (Operations of a block). A leaf block `b` in a leaf represents `enqueue(x)` if `b.element=x≠null`. Else if `b.element=null` `b` represents a `dequeue()`. The set of operations of block `b` are the operations in the subblocks of `b`. We denote the set of operations of block `b` by `ops(b)`.

We say block `b` is *propagated to node n* if `b` is in `n.blocks` or is a subblock of a block in `n.blocks`. We also say `b` contains `op` if `op∈ops(b)`.

Definition 10. A block `b` in `n.blocks` is *established* at time `t` if `n.head>` index of `b` at time `t`. $EST_{n, t}$ is the set of established blocks of node `n` at time `t`.

head

Observation 11. Once a block `b` is written in `n.blocks[i]` then `n.blocks[i]` never changes.

Lemma 12. Every block has at most one direct superblock.

Proof. To show this we are going to refer to the way `n.blocks[]` is partitioned while propagating blocks up to `n.parent`. `n.CreateBlock(i)` merges the blocks in `n.left.blocks[n.blocks[i-1].endleft..n.blocks[i].endleft]` and `n.right.blocks[n.blocks[i-1].endright..n.blocks[i].endright]` (Lines ^{lastLine}~~334, 335~~). Since `endleft, endright` are non-decreasing (`n.blocks[i].endleft|right>n.blocks[i-1].endleft|right`), so the range of the subblocks of `n.blocks[i]` which is `(n.blocks[i-1].enddir+1..n.blocks[i].enddir)` does not overlap with the range of the subblocks of `n.blocks[i-1]`. \square

shedOrder

Lemma 13 (establishedOrder). If time `t < time t'`, then $ops(EST_{n, t}) \subseteq ops(EST_{n, t'})$.

Proof. Blocks are only appended (not modified) with CAS to `n.blocks[n.head]` and `n.head` is non-decreasing, so the set of operations in established blocks of a node can only grow. \square

useless?

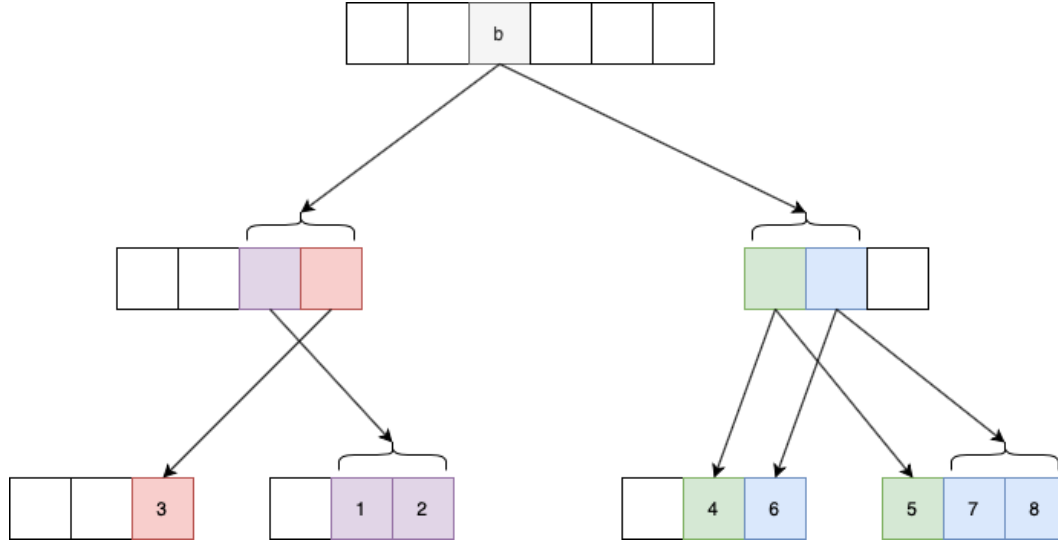


Figure 16: Order of elements in `b`: elements in leaves are ordered with numerical order in the drawing.

► Processes are numbered from 1 to p and leaves of the tree are assigned from left to right. We will show in Lemma [24](#) that there is at most one operation from each process in a given block.

Definition 14 (Ordering of operations inside the nodes). • The prefix of an operation op in the sequence of operations S is the sequence of operations strictly before op .

- $E(n, b)$ is the sequence of enqueue operations in $\text{ops}(n.\text{blocks}[b])$ defined recursively as follows. $E(\text{leaf}, b)$ is the single enqueue operation in $\text{ops}(\text{leaf}.\text{blocks}[b])$ or an empty sequence if $\text{leaf}.\text{blocks}[b].\text{num}_{\text{enq}}=0$. If n is an internal node, then

$$E(n, b) = E(n.\text{left}, n.\text{blocks}[b-1].\text{end}_{\text{left}} + 1) \cdot E(n.\text{left}, n.\text{blocks}[b-1].\text{end}_{\text{left}} + 2) \cdots E(n.\text{left}, n.\text{blocks}[b].\text{end}_{\text{left}}) \cdot \\ E(n.\text{right}, n.\text{blocks}[b-1].\text{end}_{\text{right}} + 1) \cdot E(n.\text{right}, n.\text{blocks}[b-1].\text{end}_{\text{right}} + 2) \cdots E(n.\text{right}, n.\text{blocks}[b].\text{end}_{\text{right}})$$

- $E_i(n, b)$ is the i th enqueue in $E(n, b)$.
- The order of the enqueue operations in the node n is $E(n) = E(n, 1) \cdot E(n, 2) \cdot E(n, 3) \cdots$
- $E_i(n)$ is the i th enqueue in $E(n)$.
- $D(n, b)$ is the sequence of dequeue operations in $\text{ops}(n.\text{blocks}[b])$ defined recursively as follows. $D(\text{leaf}, b)$ is the single dequeue operation in $\text{ops}(\text{leaf}.\text{blocks}[b])$ or an empty sequence if $\text{leaf}.\text{blocks}[b].\text{num}_{\text{deq}}=0$. If n is an internal node, then

$$D(n, b) = D(n.\text{left}, n.\text{blocks}[b-1].\text{end}_{\text{left}} + 1) \cdot D(n.\text{left}, n.\text{blocks}[b-1].\text{end}_{\text{left}} + 2) \cdots D(n.\text{left}, n.\text{blocks}[b].\text{end}_{\text{left}}) \cdot \\ D(n.\text{right}, n.\text{blocks}[b-1].\text{end}_{\text{right}} + 1) \cdot D(n.\text{right}, n.\text{blocks}[b-1].\text{end}_{\text{right}} + 2) \cdots D(n.\text{right}, n.\text{blocks}[b].\text{end}_{\text{right}})$$

- $D_i(n, b)$ is the i th dequeue in $D(n, b)$.
- The order of the dequeue operations in the node n : $D(n) = D(n, 1) \cdot D(n, 2) \cdot D(n, 3) \cdots$
- $D_i(n)$ is the i th dequeue in $D(n)$.

Definition 15 (Linearization). $L = E(\text{root}, 1).D(\text{root}, 1).E(\text{root}, 2).D(\text{root}, 2).E(\text{root}, 3).D(\text{root}, 3) \cdots$

► In the non-root nodes, we only need ordering of enqueues and dequeues among the operations of their own type. Since `GetENQ()` only searches among enqueues and `IndexDEQ()` works with dequeues.

ueRefresh

Lemma 16 (trueRefresh). *Let t_i be the time an instance R of $n.Refresh()$ is invoked and t_t be the time it terminates. If the $TryAppend(new, s)$ of R returns **true**, then $ops(EST_{n.left, t_i}) \cup ops(EST_{n.right, t_i}) \subseteq ops(EST_{n, t_t})$.*

Proof. Since $TryAppend$ returns true a block **new** is written into $n.blocks[h]$ in Line ^{cas}320.

We show $ops(EST_{n.left, t_i}) \subseteq ops(EST_{n, t_t})$. Let h be the value $n.Refresh()$ reads from $n.head$ at line ^{readHead}310, $h_{left,i}$ be the value of $n.left.head$ at t_i and $h_{left,read}$ be the value read from $n.left.head-1$ at line ^{lastLine}334. end_{left} field of the block returned by $CreateBlock(i)$ is $h_{left,read}$. By lines ^{prevLine}335 and ^{lastLine}334 the new block in $n.blocks[h]$ contains $n.left.blocks[n.blocks[h-1].end_{left}+1..h_{left,read}]$. Since $left.head$ is read after t_i then $h_{left,read} > h_{left,i}$ which means $ops(EST_{n.left, t_i}) \subseteq ops(n.left.blocks[0..h_{left,read}])$. After the successful $TryAppend$ in line ^{cas}320 we know all blocks in $n.left.blocks[0..h_{left,read}-1]$ are subblocks of $n.blocks[0..h]$ by the definition of subblock. At t_t we have $n.head > h$ by Lemma ^{lem::headProgress}5. So $n.blocks[1..h]$ are in EST_{n,t_t} by definition of EST . Note that after line ^{incrementHead2}?? we are sure that the head is incremented by Lemma ^{lem::headInc}2) which means $n.head = h+1$ at t_t so the new block is established at t_t and the new block contains the new operations which is what we wanted to show. The proof for $ops(EST_{n.right, t_i}) \subseteq ops(EST_{n, t_t})$ is the same. □

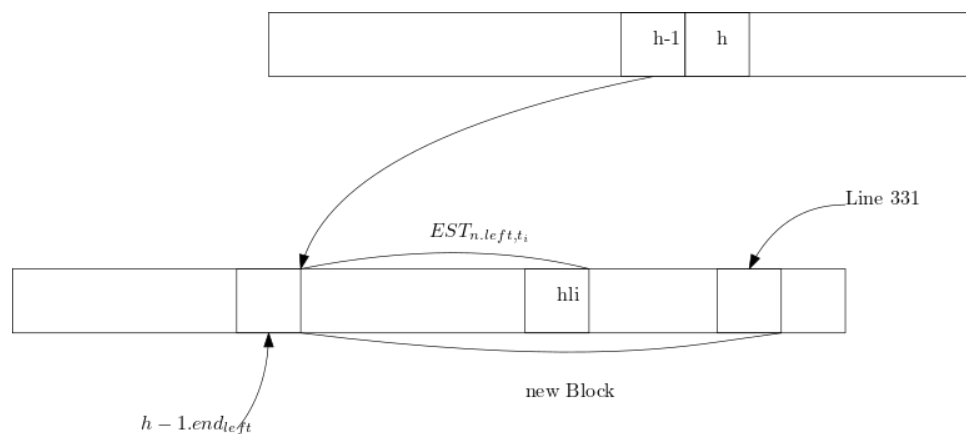


Figure 17: New established operations of the left child are in the new block.

ueRefresh

Lemma 17 (Stronger True Refresh). *Let t_i be the time an instance of $n.Refresh()$ read the head (Line ^{readHead}310) and t_t be the time its $TryAppend(new, s)$ terminates with and returns **true** (Line ^{cas}320). We have $ops(EST_{n.left, t_i}) \cup ops(EST_{n.right, t_i}) \subseteq ops(n.blocks)$.*

Definition 18. An instance of $Refresh()$ is successful iff its $TryAppend(new, s)$ terminates with and returns **true**.

Definition 19. Let $R_1 t$ be the time R_1 is invoked and t_{R_2} be the time R_2 terminates. $line t$ is the immediate time before running Line $line$. t_{line} is the immediate time after running Line $line$. $line t^{op}$ is the immediate time before running Line $line$ of operation op . t_{line}^{op} is the immediate time after running Line $line$ of operation op .

leRefresh

Lemma 20 (Double Refresh). *Consider two consecutive instances R_1, R_2 of **Refresh()** on internal node n by a process p . If R_1 and R_2 both fail and return false, then we have $ops(EST_{n.left, R_1 t}) \cup ops(EST_{n.right, R_1 t}) \subseteq ops(EST_n, t_{R_2})$.*

Proof.

If R_2 reads some value greater than $i + 1$ in Line readHead 310 it means a successful instance of **Refresh()** performed its Line readHead 310 after t_{B10} and finished its Line incrementHead 312 before t_{B10} , from Lemma lem::pretrueRefresh 17 by the end of this instance $ops(EST_{n.left, t_1}) \cup ops(EST_{n.right, t_1})$ has been propagated.

Let R_1 read i and R_2 read $i + 1$ from Line readHead 310. As R_2 's **TryAppend()** returns false, there is another successful instance R'_2 of **n.Refresh()** that has done **TryAppend()** successfully into $n.blocks[i+1]$ before R_2 tries to append. Since R'_2 creates the block after reading the value $i + 1$ from $n.head$ (Line readHead 310) and R_1 reads the value i from $n.head$ and the head's value is increasing by Lemma lem::headProgress 5 then $t_{R_2'}^{readHead} > t_{R_1}^{readHead} >_{R_1} t$ (see Figure 18). By Lemma lem::pretrueRefresh 17 after R'_2 's CAS (cas 320) we have $ops(EST_{n.left, t_1}) \cup ops(EST_{n.right, t_1}) \subseteq ops(n.blocks)$. Also by Lemma lem::headInc 2 on R_2 the value of $n.head$ head is more than $i + 1$ after R'_2 terminates, so the block appended by R'_2 to n is established by then ($n.head \geq i + 2 > i + 1$). To summarize, $R_1 t$ is before R'_2 's read of $n.head$ (readHead 310) and R_2 's successful CAS is before R_2 's termination. So, by Lemma lem::pretrueRefresh 17, $ops(EST_{n.left, t_1}) \cup ops(EST_{n.right, t_1}) \subseteq ops(EST_n, t_2)$. \square

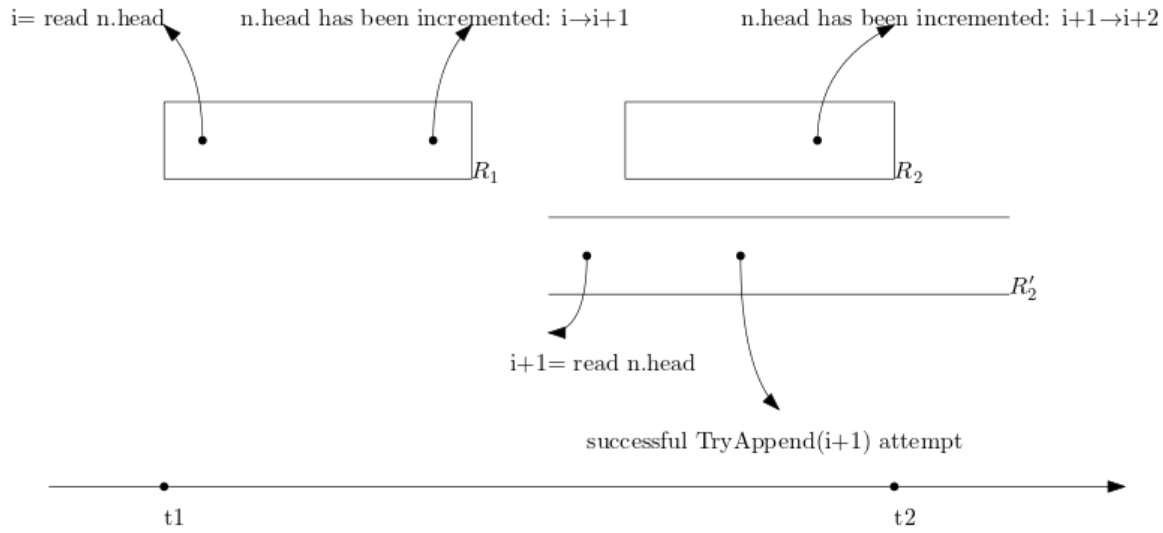


Figure 18: $R_1 t < t_{B10}^{readHead} < \text{incrementing } n.head \text{ from } i \text{ to } i+1 < t_{B10}^{readHead} < t_{B20}^{readHead} < \text{incrementing } n.head \text{ from } i+1 \text{ to } i+2 < t_{R_2}$

leRefresh

Corollary 21. $ops(EST_{n.left, firstRefresh} 302 t) \cup ops(EST_{n.right, firstRefresh} 302 t) \subseteq ops(EST_n, secondRefresh} 303 t)$

Proof. If the first **Refresh()** in line 302 returns true then by Lemma lem::trueRefresh 16 the claim holds. Also if first **Refresh()** failed and the second **Refresh()** succeeded the claim still holds by Lemma lem::trueRefresh 16. Finally, if both failed the claim is satisfied by Lemma doubleRefresh 20. \square

lyRefresh **Corollary 22** (Propagate Step). *All operations in n 's children's established blocks before running line firstRefresh 302 of a `Propagate` routine are guaranteed to be in n 's established blocks after line secondRefresh 303.*

Proof. If firstRefresh 302 or secondRefresh 303 succeed, the claim is true by Lemma Lem::trueRefresh 16. Otherwise Lines firstRefresh 302 and secondRefresh 303 satisfy the preconditions of Lemma doubleRefresh 20. \square

actlyOnce **Corollary 23.** *After `Append(blk)` finishes $\text{ops}(\text{blk}) \subseteq \text{ops}(\text{root.blocks}[x])$ for exactly one x .*

Proof. After `Append(blk)`'s termination, `blk` is in `root.blocks` since `blk` is established in the leaf it has been added to. By applying Lemma doublyRefresh 22 inductively it is propagated up to the root. Finally Lemma append 17 shows only one block in the root contains `blk`. \square

blockSize **Lemma 24** (Block Size Upper Bound). *Each block contains at most one operation of each process.*

Proof. To derive a contradiction, assume there are two operations op_1 and op_2 of process p in block b in node n . Without loss of generality op_1 is invoked earlier than op_2 . A process cannot invoke more than one operations concurrently, so op_1 has to be finished before op_2 . By Corollary Lem::appendExactlyOnce 23, before appending op_2 to the tree op_1 exists in every node on the path from p 's leaf to the root, because op_1 's `Append` is finished before op_2 's `Append` starts. So, there is some block b' before b in n containing op_1 . Existence of op_1 in b and b' contradicts Lemma append 17. \square

locksBound **Lemma 25** (Subblocks Upperbound). *Each block has at most p direct subblocks.*

Proof. The claim follows directly from Lemma blockSize 24 and the observation that each block appended to the tree contains at least one operation, due to the test on Line addOP 318. We can also see the blocks in the leaves have exactly one operation in the `Enqueue()` and `Dequeue()` routines. \square

get

Lemma 26 (Get correctness). *If $n.blocks[b].num_{enq} \geq i$ then $n.GetENQ(b, i)$ returns the element enqueued by $E_i(n, b)$.*

Proof. We are going to prove this lemma by induction on the height of node n . For the base case, n is a leaf. Leaf blocks each contain exactly one operation, so by the hypothesis, only $n.GetENQ(b, 1)$ can be called and only when $n.blocks[b]$ contains an enqueue. At Line 403, $n.GetENQ(b, 1)$ returns the element of the enqueue operation stored in the b th block of leaf n .

For the induction step we prove $n.GetENQ(b, i)$ returns $E_i(n, b)$, assuming $n.child.GetENQ(subblock, i)$ returns $E_i(n.child, b)$. We argue that Line 404 correctly decides whether the i th enqueue in b th block of internal node n is in the left child or right child subblocks of $n.blocks[b]$. From Definition 14 of $E(n, b)$ we know enqueue operations in a block are ordered from left to right and since the leaves of the tree are ordered by process id from left to right, thus operations from the left subblocks come before operations from the right subblocks in a block (See Figure 19). Furthermore the $num_{enq-left}$ field in $n.blocks[b]$ stores the number of enqueue() operations from the blocks's subblocks in the left child of n . So the i th enqueue operation is propagated from the right child if i is greater than $b.num_{enq-left}$. Otherwise we should search for the i th enqueue in the left child. By definition 6 and 9 we need to search in subblocks of $n.blocks[b]$ from the range $n.left.blocks[n.blocks[i-1].end_{left}+1..n.blocks[i].end_{left}] \cup n.right.blocks[n.blocks[i-1].end_{right}+1..n.blocks[i].end_{right}]$.

If the i th enqueue of $n.blocks[b]$ is in the left child it would be i th enqueue in $n.left.blocks[n.blocks[i-1].end_{left}+1..n.blocks[i].end_{left}]$ by Definition 6. Also, we know there are $eb = n.blocks[b-1].sum_{enq-left}$ enqueues in the blocks before this range, so $E_i(n, b)$ is $E_{i+eb}(n.left)$ which is $E_{i'}(n.left, b')$ for some b' and i' . We can compute b' and then search for $i + eb$ th enqueue in $n.left$, where i' is $i+eb-n.left.blocks[b'-1].sum_{enq}$. The parameters in Line 405 are for searching $E_{i+eb}(n.left)$ in $n.left.block$ in the expected range of blocks, so this BSearch returns the index of the subblock containing $E_i(n, b)$.

Otherwise the enqueue we are looking for is in the right child. Then, there are $n.blocks[b].num_{enq-left}$ enqueues ahead of it in $n.blocks[b]$ but not in $n.right.blocks[n.blocks[i-1].end_{right}+1..n.blocks[i].end_{right}]$. So we need to search for $i-n.blocks[b].num_{enq-left}+n.blocks[b-1].sum_{enq-right}$ (Line 409). Other parameters for the left child are chosen similarly to the way they were chosen for the right child.

So, in both cases the direct subblock containing $E_i(n, b)$ is computed in Lines 405 and 409. Finally, $n.child.GetENQ(subblock, i)$ is invoked on the subblock containing $E_i(n, b)$ and it returns $E_i(n, b)$ by the hypothesis of the induction. \square

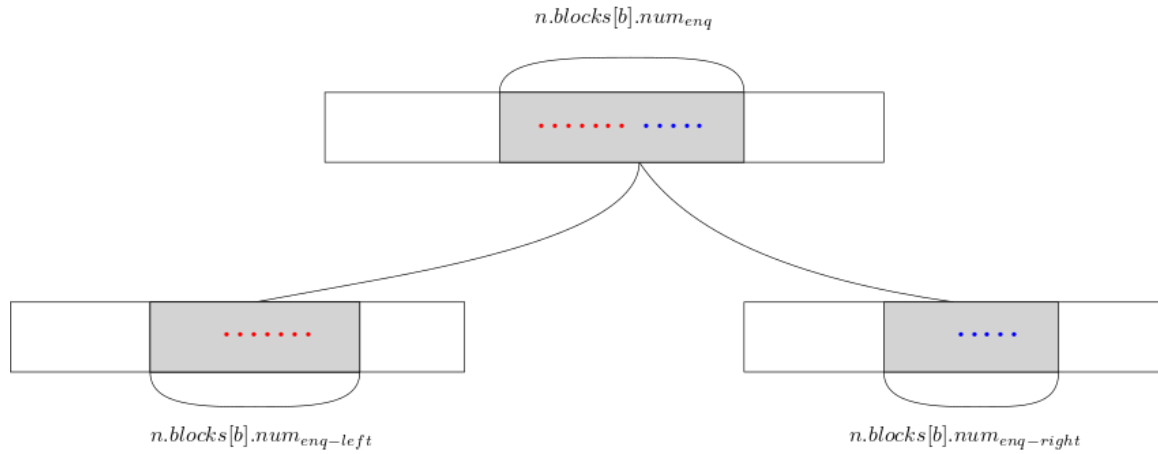


Figure 19: The number and ordering of the enqueue operations propagated from the left and the right child to $n.blocks[b]$. Enqueue operations from the left subblocks (colored red), are ordered before the enqueue operations from the right child (colored blue).

figGet

dsearch

Lemma 27 (DSearch correctness). *If $\text{root.blocks}[\text{end}].\text{sum}_{\text{enq}} \geq e$, $\text{DSearch}(e, \text{end})$ returns $\langle b, i \rangle$ such that $E_i(\text{root}, b) = E_e(\text{root})$.*

Proof. DSearch performs a doubling search from $\text{root.blocks}[\text{end}]$ to $\text{root.blocks}[0]$ to find $E_e(\text{root})$. From Lemma we know sum_{enq} fields of $\text{root.blocks}[]$ are sorted in a non-decreasing order. Since $\text{root.blocks}[0].\text{sum}_{\text{enq}} = 0$ and there is a block in the root with sum_{enq} value greater than e , so there is a b that $\text{root.blocks}[b].\text{sum}_{\text{enq}} \geq e$ but $\text{root.blocks}[b-1].\text{sum}_{\text{enq}} < e$. This block contains $E_i(\text{root}, b)$ and the search on Line ~~802-806~~ ^{dsearchStartEnd} will eventually reach the b .

□

searchTime

Lemma 28 (DSearch Analysis). *Assume $\text{root.blocks}[\text{end}].\text{sum}_{\text{enq}} \geq e$ and $E_e(\text{root})$'s element is the response to some $\text{Dequeue}()$ operation in $\text{root.blocks}[\text{end}]$, then $\text{DSearch}(e, \text{end})$ takes $\Theta(\log \text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size})$ steps.*

Proof. First we show $\text{end} - b \leq 2 \times (\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size} + 1)$. From line ~~318~~ ^{addOP}, we know that num field of the every block in the tree is greater than 0. So, each block in $\text{root.blocks}[b..\text{end}]$ contains at least one Enqueue or at least one Dequeue . Suppose there were more than $\text{root.blocks}[b].\text{size}$ Dequeues in $\text{root.blocks}[b+1..\text{end}-1]$. Then the element in the queue which is the response to the $\text{Dequeue}()$ would become dequeued at some point after $\text{blocks}[b]$'s last operations and before $\text{root.blocks}[\text{end}]$'s first operation. Which means the response to a Dequeue in $\text{root.blocks}[\text{end}]$ could not be in $E(n, b)$. Furthermore since the size of the queue would become $\text{root.blocks}[\text{end}].\text{size}$ after the operations of $\text{root.blocks}[\text{end}]$, there cannot be more than $\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}-1].\text{size}$ Enqueues in $\text{root.blocks}[b+1..\text{end}-1]$, because there can be at most $\text{root.blocks}[b].\text{size}$ Dequeues and the final size of the queue is $\text{root.blocks}[\text{end}-1].\text{size}$. Overall there can be at most $2 \times \text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}$ operations in $\text{root.blocks}[b+1..\text{end}-1]$ and since each block size is ≥ 1 thus there are at most $2 \times \text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}$ blocks in between $\text{root.blocks}[b]$ and $\text{root.blocks}[\text{end}]$. So $\text{end} - b \leq 2 \times \text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size} + 1$. See Figure ~~20~~ ^{fig::doubling}.

Now that we know there are at most $\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}$ blocks in between $\text{root.blocks}[b]$ and $\text{root.blocks}[\text{end}]$ then with doubling search in $\Theta(\log(\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}))$ steps we reach $\text{start} = c$ that the $\text{root.blocks}[c].\text{sum}_{\text{enq}}$ is less than e and $\text{end} - c$ is not more than $2 \times 2 \times (\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size})$. Beause otherwise, then $(\text{end} - c)/2$ satisfied the $\text{root.blocks}[(\text{end} - c)/2].\text{sum}_{\text{enq}} < e$. In line ~~804~~ ^{Doubling} the difference between end and start is doubled. See Figure ~~20~~ ^{fig::doubling}.

After computing b , the value i is computed via the definition of sum_{enq} in constant time (Line ~~807~~ ^{DSearchComputei}). So the whole DSearch routine takes $\Theta(\log(\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}))$ steps.

□

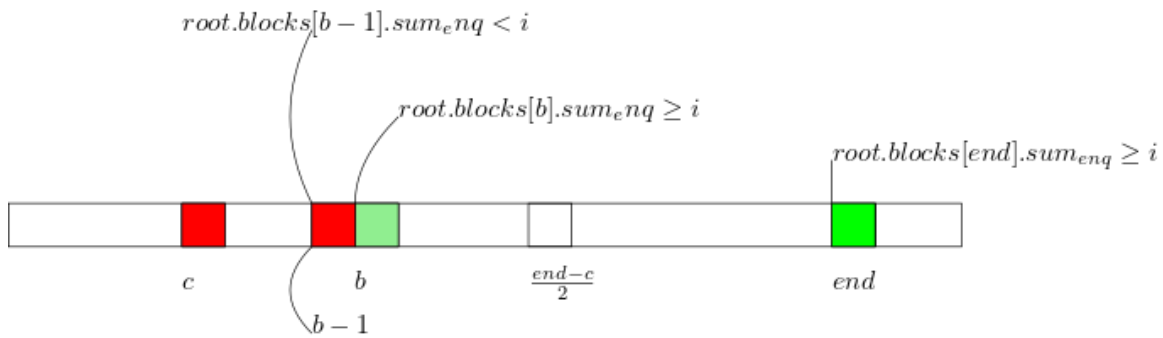


Figure 20: Distance relations between b, c, end

:doubling

erRefresh

Definition 29. A Refresh is *successful* if it performs a successful CAS on Line ^{cas}320. If Refresh instance R_1 does its CAS at Line ^{cas}320 sooner than Refresh instance R_2 we say R_1 *happened* before R_2 .

Let i be the value R_n , a successful instance of Refresh() on the node n , reads from $n.head$. R_n does a successful CAS(null, new) into $n.blocks[h]$, where b is the new. Without loss of generality for the rest of this section assume n is the left child of $n.parent$. From now on we say p as an abbreviation for $n.parent$. Let R_p be the first successful $p.Refresh()$ that reads some value greater than i for $left.head$ and is contains b in its created block s in Line ^{invokeCreateBlock}317. From Lemma ^{lem::appendExactlyOnce}23 we know there could be only one $p.Refresh()$ propagating b . R_p does a successful CAS(null, new) into $p.blocks[j]$, where s is the new.

Although other fields of b are set while creating it, because the index of the superblock of b is not known until it is propagated, R_n cannot set the `super` field of a b while creating it. One approach is to set the `super` field of b after it is propagated by R_b but this would not be efficient because there might be p subblocks in s . However, once b is installed, its superblock is going to be close to $n.parent.head$ at the time of installation. One idea is that if we know the approximate position of the superblock of b then we can search for the real superblock when we wished to know the superblock of b i.e. $b.super$ does not have to be the exact location of the superblock of b , but we want it to be close to j . We can set $b.super$ to $n.parent.head$ while creating b , but the problem is that there might be many $p.Refreshes$ could happen that contain blocks from the right child of p and j could be arbitrarily (right word?) greater than $b.super$. We set $b.super$ to $p.head$ after appending b to $n.blocks$ (Line ^{setSuper1}326). Maybe R_n goes to sleep at some time after installing b and before setting $b.super$. In this case the next Refreshes on n and $n.parent$ help fill in the value of $b.super$.

Block b is appended to $n.blocks[h]$ on Line ^{cas}320. After appending b , $b.super$ is set on Line ^{setSuper1}326 of a call to Advance by the same process or another process's $n.Refresh()$ or maybe an $n.parent.Refresh()$. We want to bound how far $b.super$ is from the index of b 's superblock, which is created by a successful $n.parent.Refresh()$ that propagates b .

reIncHead

Observation 30. After $n.blocks[i].CAS(null, b)$ succeeds, $n.head$ cannot increase from i to $i+1$ unless $b.super$ is set.

Proof. From the Observation ^{nonDecreasingHead}4 we know the only change to $n.head$ is on Line ^{incrementHead}327 which is incrementing. Before an instance of Advance() increments $n.head$ on Line ^{incrementHead}327, Line ^{setSuper1}326 ensures that $n.blocks[head].super$ was set at Line ^{setSuper1}326. \square

Corollary 31. If $n.blocks[i].super$ is null, then $n.head < i$ and $n.blocks[i+1]$ is null.

Proof. If $b.super$ is null then $n.head$ cannot advance so the next $n.Refresh()$ will fail. By the previous corollary, $b.super$ has to be set before the next successful Refresh() on n after R_n . \square

Now let us talk about how the $p.Refreshes$ that took place after the putting b into n , will help to set $b.super$ and propagate b .

Lemma 32. If $b \in n.parent.blocks[i]$ then $b.super \leq i$.

Proof. For R_p to contain block b , it has to read $n.head$ greater than h (see Line ^{lastLine}334). For $n.head$ to be greater than h it means $n.head$ is incremented in Line ^{incrementHead}327 which means $b.super$ was already set in Line ^{setSuper1}326 (see Observation ^{setSupBeforeIncHead}30). So if R_p propagates b it means $b.super$ was already set. Let j be the value written in $b.super$. j has been read in Line ^{readHead}310 or Line ^{readParentHead}321 which both are before calling Advance that sets $b.super$. From Observation ^{nonDecreasingHead}4 we know $p.head$ is non-decreasing so $j \leq i$. The reader may wonder when the case $j = i$ happens, it happens when $p.blocks[j]=null$ while j is read and R_p puts its created block into $n.blocks[j]$. \square

Lemma 33. If $R_n.Refresh()$ puts b in $n.blocks[h]$ at Line ^{cas}320, then the block created by one of the next two successful $p.Refreshes$ according to the Definition ^{orderRefresh}29 contains b and $b.super$ is set before Line ^{invokeCreateBlock}317 of the the second successful $p.Refresh()$.

Proof. It is sufficient to prove one of the two successful $p.Refresh()$ es propagate b . If the first successful $p.Refresh()$ propagated b then the claim is true, so in the remaining part we assume the first $p.Refresh()$ did not propagate b and prove the second $p.Refresh()$ propagates b .

$b.super$ is set by some instance of `Refresh()` on n or p showed by R' and $n.head$ is incremented by some `Refresh()` called R'' . We want to know how great $j - b.super$ can be. $p.head$ is hp when R' reads it. From Lemma lem:headProgress $p.head$ could only increase from hp to $hp+1$ if $p[hp] \neq \text{null}$. In other words there should be a successful $p.Refresh()$ for $p.head$ to increase. We claim there cannot be another successful $p.Refresh()$ after R' reads $p.head$ and before R_p performs Line LastLine 334.

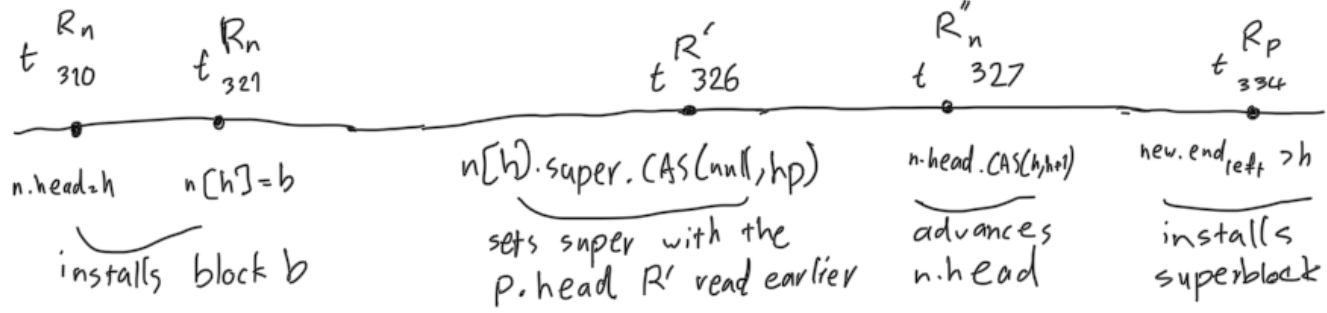


Figure 21: Time relations between R_n, R_p, R', R''

Assume the first successful $p.Refresh()$ after cas 320 did not set $b.super$. It might happen maybe because the value read for h_{left} in Line readChildHead 312 is less than i or maybe $i = h_{left}$ and $left.blocks[h_{left}] = \text{null}$, which means $n.head$ is advanced but b is still not installed in $n.blocks[i]$ which means R_n has not reached to the Line cas 320.

Let the first successful $p.Refresh()$ be R_{p1} and the second next successful $p.Refresh()$ be R_{p2} . If R_{p1} reads x in Line readHead 310, then R_{p2} has to read $x+1$ in Line readHead 310 (induced from lem:headProgress 310, 4). See the timeline in Figure 22 for two consecutive successful `Refresh()` instances R_{p1}, R_{p2} on p .

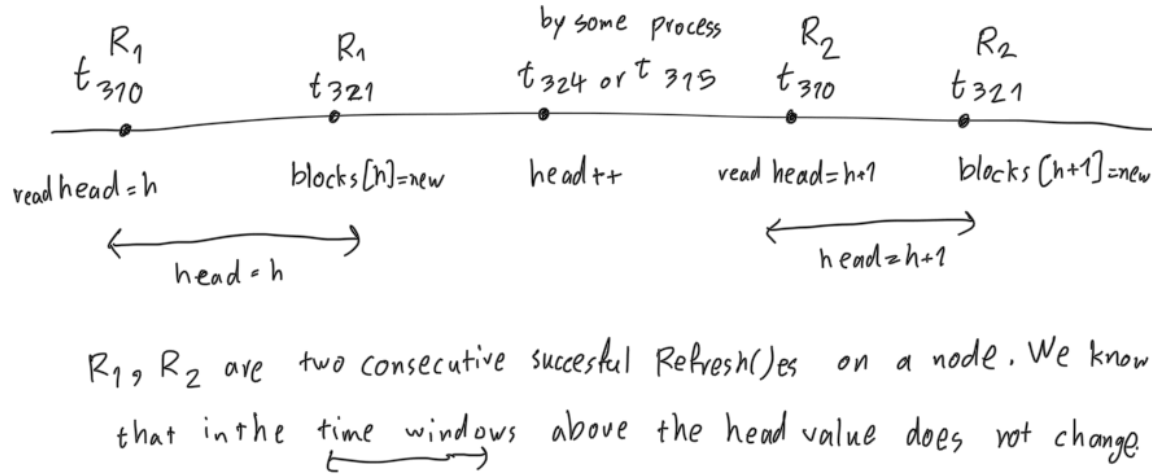


Figure 22: R_{p2} reads $p.head$ after $t_{321}^{R_{p1}}$, which is after $t_{321}^{R_n}$. R_{p2} has to help increment $n.head$ and set $b.super$.

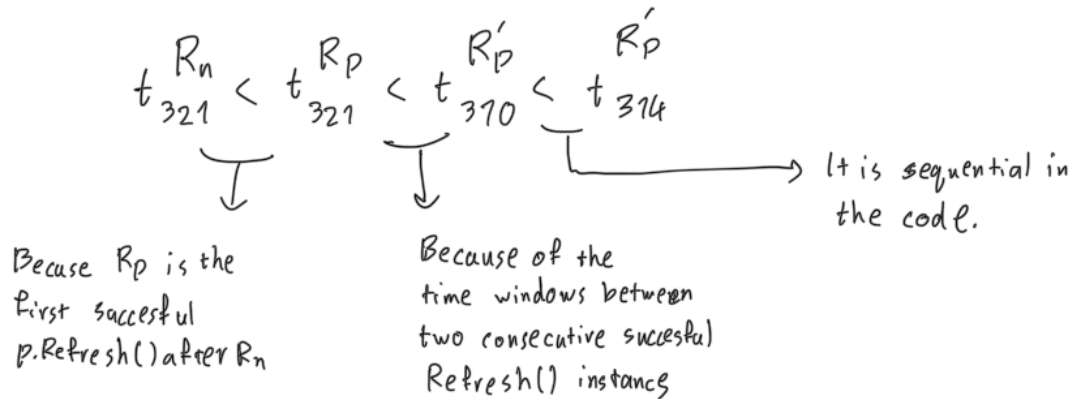


Figure 23: The second `Refresh` on p contains b and reads $n.head > i$.

So $b.super$ has set by some process before the second next successful $p.Refresh()$ on Line ~~326~~^{setSuper1}. Since i is read in the Line ~~310~~^{readHead} then the $CreateBlock()$ in Line ~~317~~^{invokeCreateBlock} is going to read some value for $left.head$ greater than h and propagates b to p . So if b was not propagated already we are sure the second next successful $p.Refresh()$ propagates b . \square

Corollary 34. *If b has propagated to f , then $b.super$ has at most 1 difference with the index of the superblock of b in p .*

superBlock

Lemma 35 (Computing SuperBlock). *For the `superblock` value computed in line 418 of `n.IndexDEQ(b,i)` we have `n.parent.blocks[superblock]` contains $D_{n,b,i}$.*

Proof. First we show the value read for `super[b.group]` in line 418 is not null. Values `npdir` read in lines 417, `super` are set before incrementing in lines 419, 420. So before incrementing `numpropagated`, `super[numpropagated]` is set so it cannot be null while reading. Then by Lemma 34 if we search in the range p , we can find the superblock. □

Lemma 36 (Index correctness). *If `n.blocks[b].numdeq` $\geq i$ then `n.IndexDEQ(b,i)` returns the rank in $D(\text{root})$ of $D_{n,b,i}$.*

Proof. We will prove this by induction on the distance of `n` from the `root`. We can see the base case where `n` is root is trivial (Line 415). In the non-root nodes `n.IndexDEQ(b,i)` computes the superblock of the i th Dequeue in the b th block of `n` in `n.parent` by Lemma 35 (Line 418). After that the order in $D(n.parent, \text{superblock})$ is computed. Note that by Lemma 24 in each block there is at most one operation from each process and operations of one type are ordered based on the order in the subblocks (See Figure 24). Finally `index()` is called on `n.parent` recursively and it returns the correct response from induction hypothesis. If the operation was propagated from the right child the number of dequeues from the left child are added to it (Line 421), because the left child operations come before the right child operations (Definition 14). □

Make sure to show preconditions of all invocation of `BSearch` are satisfied.

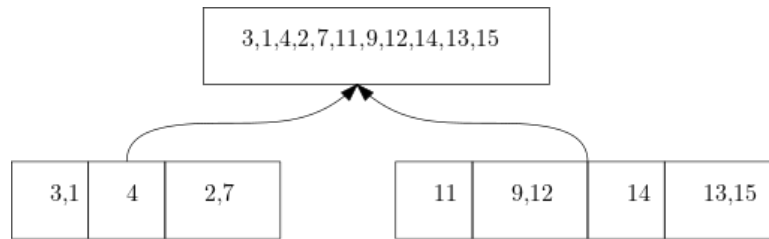


Figure 24: Relation of ordering of operations of a block from its subblocks

Subblocks

Definition 37. Assume the operations in L are applied on an empty queue. If element of `enqueue e` is the response to `dequeue d` then we say $R(d)=e$. If d 's response is `null` (queue is empty) then $R(d)=\text{null}$.

Definition 38. In an execution on a queue, the dequeue operations that return some value are called *non-null dequeues*.

Observation 39. In a sequential execution on a queue, k th non-null dequeue returns the `element` of k th enqueue.

Lemma 40. `root.blocks[b].size` is the size of the queue if the operations in the prefix for the b th block in the root are applied with the order of L .

Proof. need to say? :: If the size of a queue is greater than 0 then a `Dequeue()` would decrease the size of the queue, otherwise the size of the queue remains 0. By definition [ordering](#) [ll4](#) enqueue operations come before dequeue operations in a block in L .

We prove the claim by induction on b . Base case $b=0$ is trivial since the queue is initially empty and `root.blocks[0].size=0`. For $b=i$ we are going to use the hypothesis for $b=i-1$. If there are more than `root.blocks[i-1].size+ root.blocks[i].sum_enq` dequeue operations in `root.blocks[i]` then the queue would become empty after `root.blocks[i]`. Otherwise we can compute the size of the queue after b th block using with this equality `root.blocks[b].size= root.blocks[b-1].size+ root.blocks[b].sum_enq- root.blocks[b].sum_deq` (Line [computeLength](#) [qhhistory](#) [344](#)). See Table [4](#) for an example of running some blocks of operations on an empty queue. \square

Lemma 41 (Duality of #non-null dequeues and `block.size`). If the operations are applied with the order of L , the number of non-null dequeues in the prefix for a block b is `b.sum_enq-b.size`

Proof. There are `b.sum_enq` enqueue operations in the prefix for b , then the size of the queue after the prefix for b is `#enqs - #non-null dequeues` in the prefix for b , by Observation 35. So `#non-null dequeues` is `b.sum_enq-b.size`. The correctness of the `block.size` field is shown in Lemma [sizeCorrectness](#) [40](#). \square

Lemma 42. $R(D_{\text{root},b,i})$ is null iff `root.blocks[b-1].size + root.blocks[b].num_enq- i < 0`.

Lemma 43 (Computing Response). `FindResponse(b,i)` returns $R(D_{\text{root},b,i}).\text{element}$.

Proof. First note that by Definition [ordering](#) [ll4](#) the linearization ordering of operations will not change as new operations come so instead of talking about the linearization of operations before the $E_i(\text{root},b)$ we talk about what if the whole operation in the linearization are applied on a queue.

$D_{\text{root},b,i}$ is $D_{\text{root},\text{root.blocks}[b-1].\text{sum_deq}+i}$ from the definition [ordering](#) [ll4](#) and `sum_enq`. $D_{\text{root},b,i}$ returns null if `root.blocks[b-1].size + root.blocks[b].num_enq- i < 0` by Lemma [nullReturnCheckEmpty](#) [42](#) (Line [220](#)). Otherwise if it is d' th non-null dequeue in L it returns d' th enqueue by Observation [responseToADeq](#) [39](#). By Lemma [numberOfNND](#) [41](#) there are `root.blocks[b-1].sum_enq - root.blocks[b-1].size` non-null dequeue operations before prefix for `root.blocks[b-1]`. Note that the dequeues in `root.blocks[b]` before the i th dequeue are non-null dequeues. So the response is $E_{i-\text{root.blocks}[b-1].\text{size}+\text{root.blocks}[b-1].\text{sum_deq}}(\text{root})$ (Line [computeE](#) [222](#)). See figure [computeResponseDetail](#) [25](#).

After computing e we can find b,i such that $E_i(\text{root},b) = E_e(\text{root})$ using `DSearch` and then find its `element` using `GetEnq` (Line [findAnswer](#) [223](#)). \square

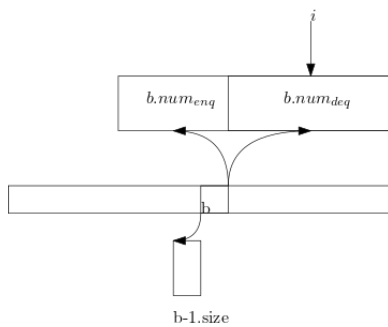


Figure 25: The position of $E_i(\text{root},b)$.

	DEQ()	ENQ(5), ENQ(2), ENQ(1), DEQ()	ENQ(3), DEQ()	ENQ(4), DEQ(), DEQ(), DEQ(), DEQ()
#enqueues	0	3	1	1
#dequeues	1	1	1	4
#non-null dequeues	0	1	2	5
size	0	2	2	0

Table 4: An example of root blocks fields. Blocks are from left to right and operations in the blocks are also from the left to right.

qhistory

Theorem 44 (Main). *The queue implementation is linearizable.*

Proof. We choose L in Definition [14](#) to be linearization ordering of operations and prove if we linearize operations as L the queue works consistently. □

Lemma 45 (satisfiability). *L can be a linearization ordering.*

Proof. To show this we need to say if in an execution, op_1 terminates before op_2 starts then op_1 is linearized before op_2 . If op_1 terminates before op_2 starts it means $op_1.\text{Append}()$ is terminated before $op_2.\text{Append}()$ starts. From Lemma [7](#) op_1 is in `root.blocks` before op_2 propagates so op_1 is linearized before op_2 by Definition [14](#). □

Once some operations are aggregated in one block they will be propagated together up to the root and we can linearize them in any order among themselves. Furthermore in L we arbitrary choose the order to be by process id, since it makes computations in the blocks faster. □

Lemma 46 (correctness). *If operations are applied as L on a sequential queue, the sequence of the responses would be the same as our algorithm.*

Proof. *Old parts to review* We show that the ordering L stored in the root, satisfies the properties of a linearizable ordering.

1. If op_1 ends before op_2 begins in E , then op_1 comes before op_2 in T .
 - This is followed by Lemma [7](#). The time op_1 ends it is in root, before op_2 , by Definition [14](#) op_1 is before op_2 .
2. Responses to operations in E are same as they would be if done sequentially in order of L .
 - Enqueue operations do not have any response so it does no matter how they are ordered. It remains to prove Dequeue d returns the correct response according to the linearization order. By Lemma [13](#) it is deduced that the head of the queue at time of the linearization of d is computed properly. If the Queue is not empty by Lemma [26](#) we know that the returning response is the computed index element.

□

Lemma 47 (Amortized time analysis). **Enqueue()** and **Dequeue()**, each take $O(\log^2 p + \log q)$ steps in amortized analysis. Where p is the number of processes and q is the size of the queue at the time of invocation of operation.

Proof. **Enqueue(x)** consists of creating a **block(x)** and appending it to the tree. The first part takes constant time. To propagate x to the root the algorithm tries two **Refreshes** in each node of the path from the leaf to the root (Lines ~~502, 503~~^{firstRefresh, secondRefresh}). We can see from the code that each **Refresh** takes constant number of steps since creating a block is done in constant time and does $O(1)$ CASes. Since the height of the tree is $\Theta(\log p)$, **Enqueue(x)** takes $O(\log p)$ steps.

A **Dequeue()** creates a block with null value element, appends it to the tree, computes its order among enqueue operations, and returns the response. The first two part is similar to an **Enqueue** operation. To compute the order of a **dqueue** in $D(n)$ there are some constant steps and **IndexDeq()** is called. **IndexDeq** does a search with range p in each level (Lemma ~~17~~^{superRange}) which takes $O(\log^2 p)$ in the tree. In the **FindResponse()** routine **DSearch()** in the root takes $\Theta(\log(\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}))$ by Lemma ~~27~~^{dsearch}, which is $O(\log \text{size of the queue when enqueue is invoked}) + \log \text{size of the queue when dequeue is invoked})$. Each search in **GetEnq()** takes $O(\log p)$ since there are $\leq p$ subblocks in a block (Lemma ~~25~~^{subBlocksBound}), so **GetEnq()** takes $O(\log^2 p)$ steps.

If we split **DSearch** time cost between the corresponding **Enqueue**, **Dequeue**, in amortized we have **Enqueue** takes $O(\log p + q)$ and **Dequeue** takes $O(\log^2 p + q)$ steps. □

Lemma 48 (CASes invoked). An **Enqueue()** or **Dequeue()** operation, does at most $4 \log p$ CAS operations.

Proof. In each height of the tree at most 2 times **Refresh()** is invoked and every **Refresh()** has 2 CASes, one in Line ~~520~~^{cas} and one in Lines ~~521, 522~~^{incrementHeight, decrementHeight}. □

3.4 Garbage Collection or Getting rid of the infinite Arrays

4 Using Queues to Implement Vectors

Supporting Append, Read, Write in PolyLog time by modifying Get(Enq) Method. Create a Universal Construction Using our vector

5 Conclusion

possible directions for work

Maybe Stacks

Characterize what datastructure can be used for this approach, we already know: queue, fetch & Inc, Vectors