

# 1 Pseudocode

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## Algorithm Tree Fields Description

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### ◇ Shared

- A binary tree of Nodes with one leaf for each process. root is the root node.

### ◇ Local

- *Node* leaf: process's leaf in the tree.

### ◇ Structures

#### ► Node

- *\*Node* left, right, parent : initialized when creating the tree.
- *BlockList*
- *int* head= 1: #blocks in blocks. blocks[0] is a block with all integer fields equal to zero.
- *int* num<sub>propagated</sub>= 0 : # groups of blocks that have been propagated from the node to its parent. Since it is incremented after propagating, it may be behind by 1.

#### ► Block

- *int* group : the value read from num<sub>propagated</sub> when appending this block to the node.

#### ► LeafBlock extends Block

- *Object* element : Each block in a leaf represents a single operation. If the operation is enqueue(x) then element=x, otherwise element=null.
- *int* sum<sub>enq</sub>, sum<sub>deq</sub> : # enqueue, dequeue operations in the prefix for the block

#### ► InternalBlock extends Block

- *int* end<sub>left</sub>, end<sub>right</sub> : indices of the last subblock of the block in the left and right child
- *int* sum<sub>enq-left</sub> : # enqueue operations in the prefix for left.blocks[end<sub>left</sub>]
- *int* sum<sub>deq-left</sub> : # dequeue operations in the prefix for left.blocks[end<sub>left</sub>]
- *int* sum<sub>enq-right</sub> : # enqueue operations in the prefix for right.blocks[end<sub>right</sub>]
- *int* sum<sub>deq-right</sub> : # dequeue operations in the prefix for right.blocks[end<sub>right</sub>]

#### ► RootBlock extends InternalBlock

- *int* size : size of the queue after performing all operations in the prefix for this block
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### Abbreviations:

- blocks[b].sum<sub>x</sub>=blocks[b].sum<sub>x-left</sub>+blocks[b].sum<sub>x-right</sub> (for b≥0 and x ∈ {enq, deq})
- blocks[b].sum=blocks[b].sum<sub>enq</sub>+blocks[b].sum<sub>deq</sub> (for b≥0)
- blocks[b].num<sub>x</sub>=blocks[b].sum<sub>x</sub>-blocks[b-1].sum<sub>x</sub>  
(for b>0 and x ∈ {∅, enq, deq, enq-left, enq-right, deq-left, deq-right})

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## Algorithm Queue

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```
201: void ENQUEUE(Object e) ▷ Creates a block with element e and adds it to the tree.
202:   block newBlock= NEW(LeafBlock)
203:   newBlock.element= e
204:   newBlock.sumenq= leaf.blocks[leaf.head].sumenq+1
205:   newBlock.sumdeq= leaf.blocks[leaf.head].sumdeq
206:   leaf.APPEND(newBlock)
207: end ENQUEUE

208: Object DEQUEUE() ▷ Creates a block with null value element, appends it to the tree, computes its order among operations, and returns its response.
209:   block newBlock= NEW(LeafBlock)
210:   newBlock.element= null
211:   newBlock.sumenq= leaf.blocks[leaf.head].sumenq
212:   newBlock.sumdeq= leaf.blocks[leaf.head].sumdeq+1
213:   leaf.APPEND(newBlock)
214:   <b, i>= INDEXDEQ(leaf.head, 1)
215:   output= FINDRESPONSE(b, i)
216:   return output
217: end DEQUEUE

218: <int, int> FINDRESPONSE(int b, int i)
219:   if root.blocks[b-1].size + root.blocks[b].numenq - i < 0 then
220:     return null
221:   else
222:     e= i - root.blocks[b-1].size + root.blocks[b-1].sumenq
223:     return root.GetENQ(root.DSEARCH(e, b))
224:   end if
225: end FINDRESPONSE
```

deqRest

checkEmpty

computeE

findAnswer

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## Algorithm Node

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301: void PROPAGATE()
firstRefresh 302:   if not REFRESH() then
secondRefresh 303:     REFRESH()
304:   end if
305:   if this is not root then
306:     parent.PROPAGATE()
307:   end if
308: end PROPAGATE

309: boolean REFRESH()
readHead 310:   h= head
makeCreateBlock 311:   <new, npleft, npright>= CREATEBLOCK(h)  ▷ npleft, npright are the
values read from the children's numpropagated field.
addOP 312:   if new.num==0 then return true  ▷ The block contains nothing.
cas 313:   else if blocks.tryAppend(new, h) then
okcas 314:     for each dir in {left, right} do
setSuper 315:       CAS(dir.super[npdir], null, h)  ▷ Write would work too.
incNP 316:       CAS(dir.numpropagated, npdir, npdir+1)
317:     end for
incrementHead1 318:   CAS(head, h, h+1)
319:   return true
320: else
321:   CAS(head, h, h+1)  ▷ Even if another process wins, help
computeLength 322:   to increase the head. The winner might have fallen sleep before increasing
incrementHead2 323:   head.
324:   return false
325: end if
326: end REFRESH

↔ Precondition: blocks[start..end] contains a block with field f ≥ i
325: int BSEARCH(field f, int i, int start, int end)
▷ Does binary search for the value
i of the given prefix sum field. Returns the index of the leftmost block in
blocks[start..end] whose field f is ≥ i.
326: end BSEARCH

```

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## Algorithm Root

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↔ Precondition: root.blocks[end].sumenq ≥ e
801: <int, int> DSEARCH(int e, int end)  ▷ Returns <b,i> if Eroot,e = Eroot,b,i.
802:   start= end-1
803:   while root.blocks[start].sumenq ≥ e do
doubling 804:     start= max(start-(end-start), 0)
805:   end while
806:   b= root.BSearch(sumenq, e, start, end)
hComputei 807:   i= e- root.blocks[b-1].sumenq
808:   return <b,i>
809: end DSEARCH

```

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Algorithm Node	
	<pre> <math>\leadsto</math> Precondition: <code>blocks[b].num<sub>enq</sub> ≥ i ≥ 1</code> </pre>
	<pre> 401: <i>element</i> GETENQ(<i>int</i> b, <i>int</i> i) <span style="float: right;">▷ Returns the element of <math>E_{this,b,i}</math>.</span> 402:   <b>if</b> this is leaf <b>then</b> 403:     <b>return</b> blocks[b].element 404:   <b>else if</b> <math>i \leq \text{blocks}[b].\text{num}_{\text{enq-left}}</math> <b>then</b> <span style="float: right;">▷ <math>E_{this,b,i}</math> is in the left child of this node.</span> 405:     subBlock= left.BSEARCH(sum<sub>enq</sub>, i+blocks[b-1].sum<sub>enq-left</sub>, blocks[b-1].end<sub>left</sub>+1, blocks[b].end<sub>left</sub>) 406:     <b>return</b> left.GETENQ(subBlock, i) 407:   <b>else</b> 408:     i= i-blocks[b].num<sub>enq-left</sub> 409:     subBlock= right.BSEARCH(sum<sub>enq</sub>, i+right.blocks[b-1].sum<sub>enq-right</sub>, blocks[b-1].end<sub>right</sub>+1, blocks[b].end<sub>right</sub>) 410:     <b>return</b> right.GETENQ(subBlock, i) 411:   <b>end if</b> 412: <b>end</b> GETENQ </pre>
tBaseCase	
ftOrRight	
tChildGet	
	<pre> <math>\leadsto</math> Precondition: bth block of the node has propagated up to the root and <code>blocks[b].num<sub>enq</sub> ≥ i</code>. </pre>
	<pre> 413: &lt;<i>int</i>, <i>int</i>&gt; INDEXDEQ(<i>int</i> b, <i>int</i> i) <span style="float: right;">▷ Returns &lt;x, y&gt; if <math>D_{this,b,i} = D_{root,x,y}</math>.</span> 414:   <b>if</b> this is root <b>then</b> 415:     <b>return</b> &lt;b, i&gt; 416:   <b>else</b> 417:     dir= (parent.left==n)? left: right <span style="float: right;">▷ check if this node is a left or a right child</span> 418:     superBlock= parent.BSEARCH(sum<sub>deq-dir</sub>, i+blocks[b-1].sum<sub>deq</sub>, super[blocks[b].group]-p, super[blocks[b].group]+p) <span style="float: right;">▷ superblock's group has at most <math>p</math> difference with the value stored in <code>super[]</code>.</span> 419:     <b>if</b> dir is right <b>then</b> 420:       i+= blocks[superBlock].num<sub>deq-left</sub> <span style="float: right;">▷ consider the dequeues from the right child</span> 421:     <b>end if</b> 422:     <b>return</b> this.parent.INDEXDEQ(superBlock, i) 423:   <b>end if</b> 424: <b>end</b> INDEXDEQ </pre>
xBaseCase	
puteSuper	
iderRight	
Algorithm Leaf	
	<pre> 601: <i>void</i> APPEND(<i>block</i> blk) <span style="float: right;">▷ Append is only called by the owner of the leaf.</span> 602:   head+=1 603:   blk.group= head 604:   blocks[head]= blk 605:   parent.PROPAGATE() 606: <b>end</b> APPEND </pre>
appendEnd	
pendStart	
Algorithm BlockList	
	<p>▷ : Supports two operations <code>blocks.tryAppend(Block b)</code>, <code>blocks[i]</code>. Initially empty, when <code>blocks.tryAppend(b, n)</code> returns true b is appended to <code>blocks[n]</code> and <code>blocks[i]</code> returns <math>i</math>th block in the blocks. If some instance of <code>blocks.tryAppend(b, n)</code> returns false there is a concurrent instance of <code>blocks.tryAppend(b', n)</code> which has returned true.blocks[0] contains an empty block with all fields equal to 0 and <code>end<sub>left</sub></code>, <code>end<sub>right</sub></code> pointers to the first block of the corresponding children.</p> <p><i>block[]</i> blocks: array of blocks</p> <p><i>int[]</i> super: super[i] stores an approximate index of the superblock of the blocks in blocks whose group field have value i.</p> <pre> 701: <i>boolean</i> TRYAPPEND(<i>block</i> blk, <i>int</i> n) 702:   <b>return</b> CAS(blocks[n], null, blk) 703: <b>end</b> TRYAPPEND </pre>

## 2 Proof of Linearizability

**TEST** Fix the logical order of definitions (cyclic references).

**TEST** Is it better to show  $\text{ops}(\text{EST}_n, t)$  with  $\text{EST}_n, t$ ?

**Question** A good notation for *the index of the b*?

**Question** How to remove the notion of time? To say  $\text{pre}(n, i)$  contains  $n.\text{blocks}[0..i]$  instead of  $\text{EST}(n, t)$  which  $\text{head}=i$  at time  $t$ . Is it good? Furthermore, can we remove the notion of established blocks?

**Definition 1** (Block). A block is an object storing some statistics, as described in Algorithm Queue. A block in a node's blocklist implicitly represents a set of operations. If  $n.\text{blocks}[i] = b$  we call  $i$  the *index* of block  $b$ . Block  $b$  is before block  $b'$  in node  $n$  if and only if the index of the  $b$  is smaller than the index of the  $b'$ 's. For a block in a `BlockList` we define *the prefix for the block* to be the blocks in the `BlockList` up to and including the block.

**Invariant 2** (headPosition). If the value of  $n.\text{head}$  is  $h$  then,  $n.\text{blocks}[i] = \text{null}$  for  $i > h$  and  $n.\text{blocks}[i] \neq \text{null}$  for  $i < h$ .

*Proof.* The invariant is true initially since 1 is assigned to  $n.\text{head}$  and  $n.\text{blocks}[x]$  is null for every  $x$ . The truth of the invariant may be affected by writing into  $n.\text{blocks}$  or incrementing  $n.\text{head}$ . We show the invariant still holds after these two changes.

In the algorithm, some value is appended to  $n.\text{blocks}[]$  by writing into  $n.\text{blocks}[\text{head}]$  only in Line 313. It is obvious that writing into  $n.\text{blocks}[\text{head}]$  preserves the invariant, since the claim does not talk about  $n.\text{blocks}[\text{head}]$ . The value of  $n.\text{head}$  is modified only in lines <sup>incrementHead1</sup>318, <sup>incrementHead2</sup>321. Depending on whether the `TryAppend()` in Line <sup>cas</sup>313 succeeded or not we show that the claim holds after the increment lines of  $n.\text{head}$  in either case. If  $n.\text{head}$  is incremented to  $h$  it is sufficient to show  $n.\text{blocks}[h] \neq \text{null}$  to prove the invariant still holds. In the first case the process applied a successful `TryAppend(new, h)` in line <sup>okcas</sup>314, which means  $n.\text{blocks}[h]$  is not null anymore. Note that whether <sup>incrementHead1</sup>318 returns true or false after Line  $n.\text{head}$  we know has been incremented from Line <sup>readHead</sup>310. The failure case is also the same since it means some value is written into  $n.\text{blocks}[\text{head}]$  by some process.  $\square$

*Explain More*

**Lemma 3** (headProgress).  $n.\text{head}$  is non-decreasing over time and  $n.\text{blocks}[i].\text{end}_{\text{left}} \geq n.\text{blocks}[i-1].\text{end}_{\text{left}}$  and  $n.\text{blocks}[i].\text{end}_{\text{right}} \geq n.\text{blocks}[i-1].\text{end}_{\text{right}}$ .

*Proof.* The first claim follows trivially from the pseudocode since  $n.\text{head}$  is only incremented in the pseudocode in the lines <sup>incrementHead1</sup>318 and <sup>incrementHead2</sup>321 of the `Refresh()`.

Consider the block  $b$  written into  $n.\text{blocks}[i]$  by `TryAppend()` at Line <sup>cas</sup>313. It is created by the `CreateBlock()` called at Line <sup>invokeCreateBlock</sup>311. Prior to this call to `CreateBlock()`,  $n.\text{head}=i$  at Line <sup>readHead</sup>310, so  $n.\text{blocks}[i-1]$  is already a non-null value  $b'$  by invariant <sup>lem::headPosition</sup>2. Thus the `CreateBlock(i-1)` that creates  $b'$  terminates before `CreateBlock(i)` that creates  $b$  is invoked. The value written into  $b.\text{end}_{\text{left}}$  at Line <sup>endDefLine</sup>333 of `CreateBlock()` was read from  $n.\text{left}.head$  at Line <sup>lastLine</sup>331 of `CreateBlock`. Similarly, the value in  $n.\text{blocks}[i-1].\text{end}_{\text{left}}$  was read from  $n.\text{left}.head$ . Since  $n.\text{left}.head$  is non-decreasing  $b'.\text{end}_{\text{left}} \leq b.\text{end}_{\text{left}}$ . The proof for  $\text{end}_{\text{right}}$  is similar.  $\square$

**Definition 4** (Subblock). Block  $b$  is a *direct subblock* of  $n.\text{blocks}[i]$  if it is in  $n.\text{left}.blocks[n.\text{blocks}[i-1].\text{end}_{\text{left}}+1..n.\text{blocks}[i].\text{end}_{\text{left}}] \cup n.\text{right}.blocks[n.\text{blocks}[i-1].\text{end}_{\text{right}}+1..n.\text{blocks}[i].\text{end}_{\text{right}}]$  (The  $\text{end}_{\text{left}}$ ,  $\text{end}_{\text{right}}$  fields are written in <sup>endDefLine</sup>333 of `CreateBlock()`). Block  $b$  is a subblock of  $n.\text{blocks}[i]$  if  $b$  is a direct subblock of  $n.\text{blocks}[i]$  or a subblock of a direct subblock of  $n.\text{blocks}[i]$ .

**Definition 5** (Superblock). Block  $b$  is *direct superblock* of block  $c$  if  $c$  is a direct subblock of  $b$ . Block  $b$  is *superblock* of block  $c$  if  $c$  is a subblock of  $b$ .

**Definition 6** (Operations of a block). A leaf block  $b$  in a leaf represents operation  $x$  if  $b.\text{element}=x \neq \text{null}$ . The set of operations of block  $b$  are the operations in the subblocks of  $b$ . We denote the set of operations of block  $b$  by  $\text{ops}(b)$ .

We say block  $b$  is *propagated to node n* if  $b$  is in  $n.\text{blocks}$  or is a subblock of a block in  $n.\text{blocks}$ . We also say  $b$  contains  $op$  if  $op \in \text{ops}(b)$ .

**Definition 7.** A block  $b$  in  $n.blocks$  is *established* at time  $t$  if the last value written into  $n.head$  before  $t$  is greater than the index of  $b$  in  $n.blocks$  at time  $t$ .  $EST_{n, t}$  is the set of established blocks of node  $n$  at time  $t$ .

**head** **Observation 8.** Once a block  $b$  is written in  $n.blocks[i]$  then  $n.blocks[i]$  never changes.

**Lemma 9.** Every block has most one direct superbblock.

*Proof.* To show this we are going to refer to the way  $n.blocks[]$  is partitioned while propagating blocks up to  $n.parent$ .  $n.CreateBlock(i)$  merges the blocks in  $n.left.blocks[n.blocks[i-1].end_{left}..n.blocks[i].end_{left}]$  and  $n.right.blocks[n.blocks[i-1].end_{right}..n.blocks[i].end_{right}]$  (Lines lastLine, pr). Since  $end_{left}, end_{right}$  are non-decreasing, so the range of the subblocks of  $n.blocks[i]$  which is  $(n.blocks[i-1].end_{dir}+1..n.blocks[i].end_{dir})$  does not overlap with the range of the subblocks of  $n.blocks[i-1]$ .  $\square$

**append** **Corollary 10** (No Duplicates). If  $op$  is in  $n.blocks[i]$  then there is no  $j \neq i$  such that  $op \in ops(n.blocks[j])$ .

**shedOrder** **Lemma 11** (establishedOrder). If time  $t < time t'$ , then  $ops(EST_{n, t}) \subseteq ops(EST_{n, t'})$ .

*Proof.* Blocks are only appended (not modified) with CAS to  $n.blocks[n.head]$  and  $n.head$  is non-decreasing, so the set of operations in established blocks of a node can only grows.  $\square$

$CreateBlock()$  aggregates the blocks in the children that are not already established in the parent into one block. If a  $Refresh()$  procedure returns true it means it has appended the block created by  $CreateBlock()$  into the parent node's sequence. So suppose two  $Refreshes$  fail. Since the first  $Refresh()$  was not successful, it means another CAS operation by a  $Refresh$ , concurrent to the first  $Refresh()$ , was successful before the second  $Refresh()$ . So it means the second failed  $Refresh$  is concurrent with another successful  $Refresh()$  that assuredly has read block before the mentioned line 35. After all it means if any of the  $Refresh()$  attempts were successful the claim is true, and also if both fail the mentioned claim still holds.

**Lemma 12** (head Increment).

Let  $R = n.\text{Refresh}$  be an instance of **Refresh** that reaches Line <sup>cas</sup>313. After  $R$  terminates  $n.\text{head}$  is greater than  $h$ , the value read in line <sup>readHead</sup>310 of  $R$ .

*Proof.* If Line <sup>incrementHead1</sup>318 or <sup>incrementHead2</sup>321 are successful then the claim holds, otherwise another process has incremented the head from  $h$  to  $h+1$ .  $\square$

**Lemma 13** (trueRefresh).

Let  $t_i$  be the time an instance  $R$  of  $n.\text{Refresh}()$  is invoked and  $t_t$  be the time it terminates. Suppose the  $\text{TryAppend}(\text{new}, s)$  of  $R$  returns **true**, then  $\text{ops}(\text{EST}_{n.\text{left}}, t_i) \cup \text{ops}(\text{EST}_{n.\text{right}}, t_i) \subseteq \text{ops}(\text{EST}_n, t_t)$ .

*Proof.* From Lemma <sup>lem::establishedOrder</sup>II we know that  $\text{ops}(\text{EST}_n, t_i) \subseteq \text{ops}(\text{EST}_n, t_t)$ . So it remains to show the operations of  $\text{ops}(\text{EST}_{n.\text{left}}, t_i) \cup \text{ops}(\text{EST}_{n.\text{right}}, t_i) - \text{ops}(\text{EST}_n, t_i)$ , which we call *new operations*, are all in  $\text{ops}(\text{EST}_n, t_t)$ . If  $\text{TryAppend}$  returns **true** a block **new** is written into  $n.\text{blocks}[h]$  (Line <sup>cas</sup>313). We show  $\text{ops}(\text{EST}_{n.\text{left}}, t_i) \subseteq \text{ops}(\text{EST}_n, t_t)$ . The proof for the right child's claim is the same. Let  $n.\text{left}.\text{head}$  at  $t_i$  be  $hli$ . Let  $n.\text{Refresh}()$  read head equal to  $h$  (Line <sup>readHead</sup>310). By the lines <sup>prevLine</sup>332, <sup>useLine</sup>331 the new block in  $n.\text{blocks}[h]$  contains  $n.\text{left}.\text{blocks}[n.\text{blocks}[h-1].\text{end}_{\text{left}}+1..n.\text{left}.\text{head}]$ . Since  $\text{left}.\text{head}$  is read after  $t_i$  then  $\text{ops}(\text{EST}_{n.\text{left}}, t_i) \subseteq \text{ops}(n.\text{left}.\text{blocks}[0..n.\text{left}.\text{head}])$ . By Lemma <sup>lem::establishedOrder</sup>II  $\text{ops}(n.\text{left}.\text{blocks}[0..n.\text{left}.\text{head}]) \subseteq \text{ops}(\text{EST}_n, t_i) \subseteq \text{ops}(\text{EST}_n, t_t)$ . Since after line <sup>incrementHead2</sup>321 we are sure that the head is incremented (Lemma <sup>lem::headInc</sup>12) and  $n.\text{head} = h+1$  at  $t_t$  so the new block is established at  $t_t$  and the new block contains the new operations which is what we wanted to show.  $\square$

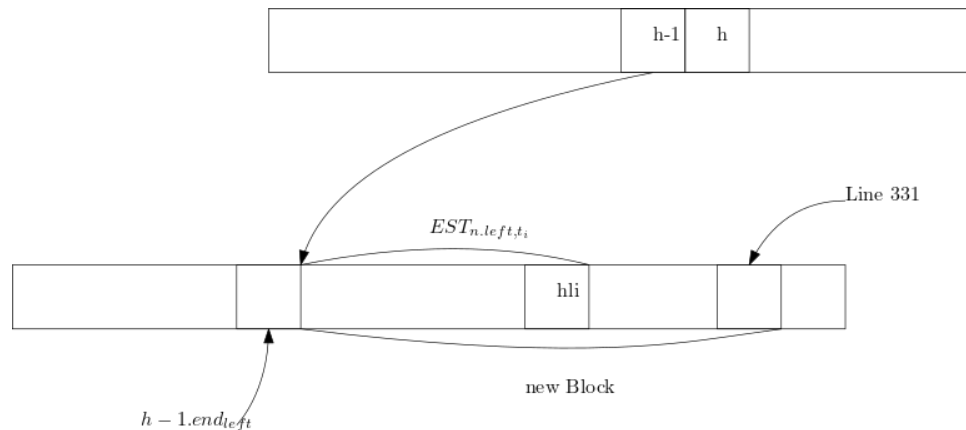


Figure 1: New established operations of the left child are in the new block.

**Lemma 14** (Precise True Refresh).

Let  $t_i$  be the time an instance of  $n.\text{Refresh}()$  read the head (Line <sup>readHead</sup>310) and  $t_t$  be the time its  $\text{TryAppend}(\text{new}, s)$  terminates with and returns **true** (Line <sup>cas</sup>313). We have  $\text{ops}(\text{EST}_{n.\text{left}}, t_i) \cup \text{ops}(\text{EST}_{n.\text{right}}, t_i) \subseteq \text{ops}(n.\text{blocks})$ .

**Lemma 15** (Double Refresh). *Consider two consecutive failed instances  $R_1, R_2$  of  $\text{n.Refresh}()$  by some process. Let  $t_1$  be the time  $R_1$  is invoked and  $t_2$  be the time  $R_2$  terminated. We have  $\text{ops}(\text{EST}_{\text{n.left}}, t_1) \cup \text{ops}(\text{EST}_{\text{n.right}}, t_1) \subseteq \text{ops}(\text{EST}_{\text{n}}, t_2)$ .*

*Proof.*

If Line 313 of  $R_1$  or  $R_2$  returns **true**, then the claim is held by Lemma 13. Let  $R_1$  read  $i$  and  $R_2$  read  $i+1$  from Line 310. If  $R_2$  reads some value greater than  $i+1$  in Line 310 it means a successful instance of  $\text{Refresh}()$  started after Line 310 of  $R_1$  and finished its Line 318 or 321 before 310 of  $R_2$ , from Lemma 13 by the end of this instance  $\text{ops}(\text{EST}_{\text{n.left}}, t_1) \cup \text{ops}(\text{EST}_{\text{n.right}}, t_1)$  has been propagated.

Since  $R_2$ 's  $\text{TryAppend}()$  returns **false** then there is another successful instance  $R'_2$  of  $\text{n.Refresh}()$  that has done  $\text{TryAppend}()$  successfully into  $\text{n.blocks}[i+1]$  before  $R_2$  tries to append. In Figure 1 we see why the block  $R'_2$  is appending contains established block in the  $\text{n}$ 's children at  $t_1$ , since it create a block reading the head after  $t_1$ . By Lemma 14 after  $R'_2$ 's CAS we have  $\text{ops}(\text{EST}_{\text{n.left}}, t_1) \cup \text{ops}(\text{EST}_{\text{n.right}}, t_1) \subseteq \text{ops}(\text{n.blocks})$ . Also by Lemma 12 of  $R'_2$  head is more than  $i+1$  after  $R'_2$ 's 321 line, so the block appended by  $R'_2$  to  $\text{n}$  is established by then. To summarized  $t_1$  is before  $R'_2$ 's read head and  $R'_2$ 's CAS is before  $R_2$ 's termination. So  $\text{ops}(\text{EST}_{\text{n.left}}, t_1) \cup \text{ops}(\text{EST}_{\text{n.right}}, t_1) \subseteq \text{ops}(\text{EST}_{\text{n}}, t_2)$ .  $\square$

*last sentence need more detail and should be earlier. define  $i$  and tell why  $R_2$ prime exists*

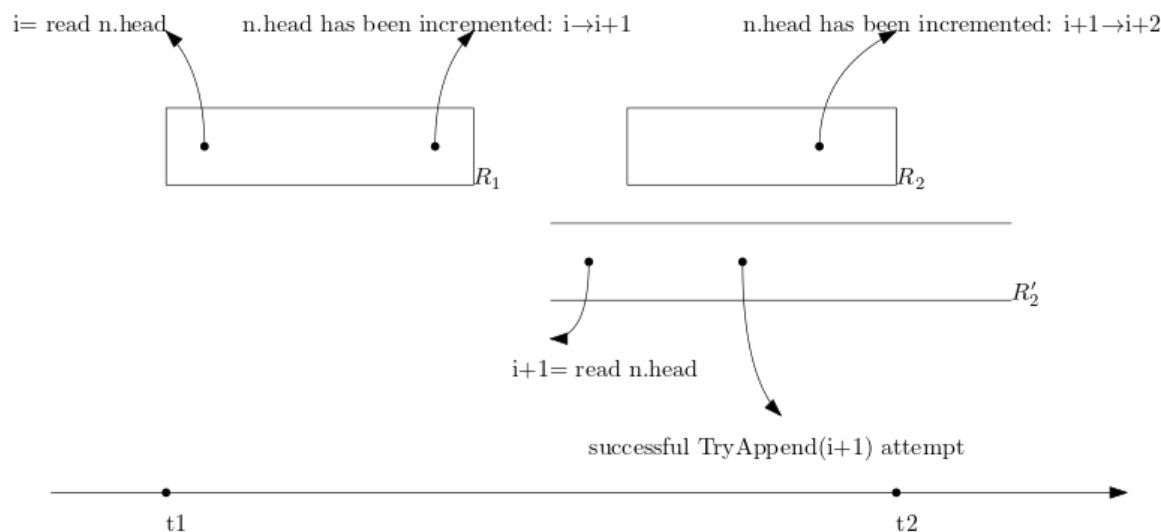


Figure 2:  $t_1 < r_1$  reading head  $<$  incrementing  $\text{n.head}$  from  $i$  to  $i+1 < R'_2$  reading head  $<$   $\text{TryAppend}(i+1) <$  incrementing  $\text{n.head}$  from  $i+1$  to  $i+2 < t_2$

*this chain with more depth should be in the proof*



lyRefresh

**Corollary 16** (Propagate Step). *All operations in  $n$ 's children's established blocks before running line 302 of a **Propagate** routine are guaranteed to be in  $n$ 's established blocks after line 303.*

*Proof.* If 302 or 303 succeed, the claim is true by 13. Otherwise Lines 302 and 303 satisfy the preconditions of Lemma 15.  $\square$

**Corollary 17.** *After **Append**(blk) finishes  $\text{ops}(\text{blk}) \subseteq \text{ops}(\text{root.blocks}[x])$  for some  $x$  and only one  $x$ .*

*Proof.* After **Append**(blk)'s termination, blk is in **root.blocks** since blk is established in the leaf it has been added to. By induction on 16 it is propagated up to the root. Finally 10 shows only one block in the root contains blk.  $\square$

**blockSize** **Lemma 18** (Block Size Upper Bound). *Each block contains at most one operation from each process.*

*Proof.* By proof of contradiction, assume there are more than one operation from process  $p$  in block  $b$  in node  $n$ . A process cannot invoke more than one operations concurrently, so one operation has to be finished before the other starts. From  $p$ 's operations in  $b$ , let  $op_1$  be the operation invoked first, before  $op_2$ . Note that  $op_1$  has terminated before  $op_2$  started. So by Corollary 17 before appending  $op_2$  to the tree  $op_1$  exists in every node from the path of  $p$ 's leaf to the root. Because  $op_1$ 's **Append** is finished before  $op_2$ 's **Append** starts. So there is some block  $b'$  before  $b$  in  $n$  containing  $op_1$ .  $op_1$  existing in  $b$  and  $b'$  contradicts with append 10.  $\square$

**blocksBound** **Lemma 19** (Subblocks Upperbound). *Each block has at most  $p$  direct subblocks.*

*Proof.* It follows directly from Lemma blockSize 18 and the observation that each block appended to the tree contains at least one operation, induced from Line addOP 312.  $\square$

**Definition 20** (Ordering of operations inside the nodes). ► Note that processes are numbered from 1 to  $p$ , left to right in the leaves of the tree and from Lemma 118 we know there is at most one operation from each process in a given block.

- We call operations strictly before  $op$  in the sequence of operations  $S$ , prefix of the  $op$ .
- $E(n, b)$  is the sequence of enqueue operations  $\in \text{ops}(\mathbf{n.blocks}[b])$  ordered by their process id.
- $E_{n,b,i}$  is the  $i$ th enqueue in  $E(n, b)$ .
- $D(n, b)$  is the sequence of dequeue operations  $\in \text{ops}(\mathbf{n.blocks}[b])$  ordered by their process id.
- $D_{n,b,i}$  is the  $i$ th dequeue in  $D(n, b)$ .
- Order of the enqueue operations in  $n$ :  $E(n) = E(n, 1).E(n, 2).E(n, 3)\dots$
- $E_{n,i}$  is the  $i$ th enqueue in  $E(n)$ .
- Order of the dequeue operations in  $n$ :  $D(n) = D(n, 1).D(n, 2).D(n, 3)\dots$
- $D_{n,i}$  is the  $i$ th dequeue in  $D(n)$ .
- Linearization:  $L = E(\text{root}, 1).D(\text{root}, 1).E(\text{root}, 2).D(\text{root}, 2).E(\text{root}, 3).D(\text{root}, 3)\dots$

*Note that in the non-root nodes we only order enqueues and dequeues among the operations of their own type. Since `GetENQ()` only searches among enqueues and `IndexDEQ()` works on dequeues.*

**Lemma 21** (Get correctness). *If  $n.blocks[b].num_{enq} \geq i$  then  $n.GetENQ(b, i)$  returns  $E_{n,b,i}$ .*

*Proof.* We are going to prove this lemma by induction on the height of node  $n$ . The base case, where  $n$  is a leaf, is straightforward (Line getBaseCase 403). Leaf blocks each contain exactly one operation, so only  $n.GetENQ(b, 1)$  can be called where  $n$  is a leaf.  $n.GetENQ(b, 1)$  returns the operation stored in the  $b$ th block of leaf  $n$ .

For the induction step we prove  $n.GetENQ(b, i)$  returns  $E_{n,b,i}$ , if  $n.child.GetENQ(b, i)$  returns  $E_{n.child,b,i}$ . For non-leaf nodes it is decided that the  $i$ th enqueue in block  $b$  of internal node  $n$  is in the  $n.blocks[b]$ 's subblocks in the left child of  $n$  or in the  $n.blocks[b]$ 's subblocks in the right child of  $n$  (line leftOrRight 404). From Definition ordering 20 we know enqueue operations in a block are ordered by their process id and since in leaves of the tree are ordered by process id from left to right, thus operations from the left subblocks come before operations from the right subblocks in a block (See Figure figGet 3). Furthermore  $b.num_{enq-left}$  stores the number of `enqueue()` operations from the  $n.blocks[b]$ 's subblocks in the left child of  $n$ . So if  $i$  is greater than  $b.num_{enq-left}$  it means  $i$ th operation is propagated from the right child, otherwise we should search for the  $i$ th enqueue in the left child. By definition def::opdef::subblock 6 and 4 we need to search in subblocks of  $n.blocks[b]$  from the range  $n.left.blocks[n.blocks[i-1].end_{left}+1..n.blocks[i].end_{left}] \cup n.right.blocks[n.blocks[i-1].end_{right}+1..n.blocks[i].end_{right}]$ .

If the  $i$ th enqueue of  $n.blocks[b]$  is in the left child it would be  $i$ th enqueue in  $n.left.blocks[n.blocks[i-1].end_{left}+1..n.blocks[i].end_{left}]$  by Definition def::subblock 4. Also we know there are  $eb = n.blocks[b-1].sum_{enq-left}$  enqueues in the blocks before this range, so  $E_{n,b,i}$  is  $E_{n.left,i+eb}$ , which is  $E_{n.left,b',i'}$  for some  $b'$  and  $i'$ . We can compute  $b'$  search for  $i + eb$ th enqueue in  $n.left$  and  $i'$  is  $i+eb-n.left.blocks[b'-1].sum_{enq}$ . The parameters in leftChildGet 405 are for searching  $E_{n.left,i+eb}$  in  $n.left.block$  in the expected range of blocks, so this BSearch returns the index of the subblock containing  $E_{n,b,i}$ .

Else if the enqueue we are looking for is in the right child then there are  $n.blocks[b].num_{enq-left}$  enqueues ahead of it in  $n.blocks[b]$  but not in  $n.right.blocks[n.blocks[i-1].end_{right}+1..n.blocks[i].end_{right}]$ . So we need to search for  $i - n.blocks[b].num_{enq-left} + n.blocks[b-1].sum_{enq-right}$  (Line rightChildGet 409). Other parameters are assigned similar for the left child. So in both cases the direct subblock containing  $E_{n,b,i}$  is computed in Lines leftChildGet 405 and rightChildGet 409.

Finally,  $n.child.GetENQ()$  is invoked on the subblock containing  $E_{n,b,i}$  which returns  $E_{n,b,i}$  by the hypothesis of the induction.  $\square$

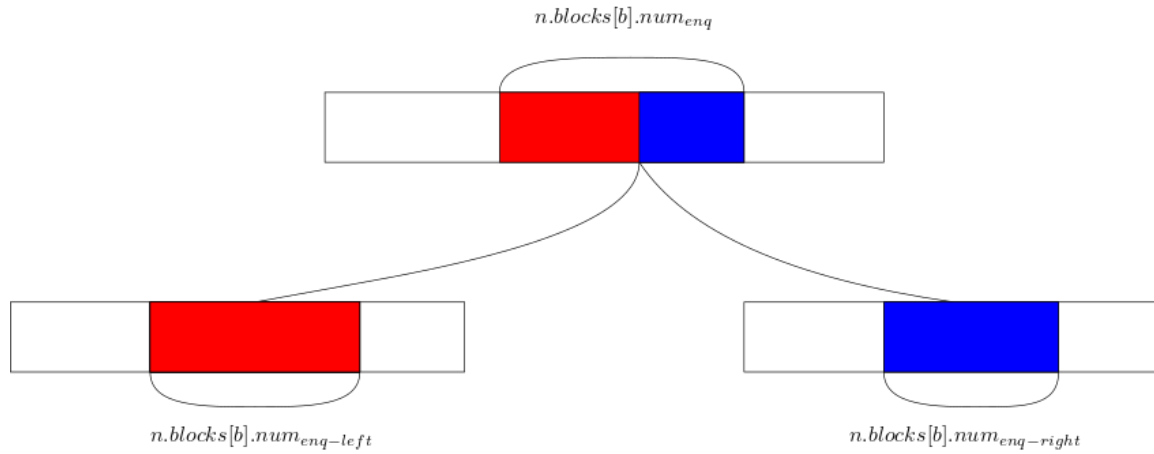


Figure 3: The number and ordering of the enqueues propagated from the left and the right child to  $n.blocks[b]$ . Enqueue operations from the left subblocks, colored red, are ordered before the enqueue operations from the right child, colored blue.

dsearch

**Lemma 22** (DSearch correctness). Assume  $\text{root.blocks}[\text{end}].\text{sum}_{\text{enq}} \geq e$  and  $E_{\text{root},e}$ 's element is the response to some `Dequeue()` operation in  $\text{root.blocks}[\text{end}]$ .  $\text{DSearch}(e, \text{end})$  returns  $\langle b, i \rangle$  such that  $E_{\text{root},b,i} = E_{\text{root},e}$ .

*Proof.* It is trivial to see that the doubling search from  $\text{root.blocks}[\text{end}]$  to  $\text{root.blocks}[0]$  will find  $E_{\text{root},e}$  eventually. Because  $\text{root.blocks}[].\text{sum}_{\text{enq}}$  is an increasing value from 0 to some value greater than  $e$ . So there is a  $b$  that  $\text{root.blocks}[b].\text{sum}_{\text{enq}} > e$  but  $\text{root.blocks}[b-1].\text{sum}_{\text{enq}} < e$ .

First we show  $\text{end}-b \leq 2 \times \text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size} + 1$ . From line <sup>addOP</sup> 312, we know that size of the every block in the tree is greater than 0. So each block in  $\text{root.blocks}[b..\text{end}]$  contains at least one `Enqueue` or at least one `Dequeue`. Suppose there were more than  $\text{root.blocks}[b].\text{size}$  `Dequeues` in  $\text{root.blocks}[b+1..\text{end}-1]$ . Then the queue would become empty at some point after  $\text{blocks}[b]$ 's last operations and before  $\text{root.blocks}[\text{end}]$ 's first operation. Which means the response to a `Dequeue` in  $\text{root.blocks}[\text{end}]$  could not be in  $E(n,b)$ . Furthermore since the size of the queue would become  $\text{root.blocks}[\text{end}].\text{size}$  after the  $\text{root.blocks}[\text{end}]$ , there cannot be more than  $\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}$  `Enqueues`. Because there can be at most  $\text{root.blocks}[b].\text{size}$  `Dequeues` and the final size is  $\text{root.blocks}[\text{end}].\text{size}$ . Overall there can be at most  $2 \times \text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}$  operations in  $\text{root.blocks}[b+1..\text{end}-1]$  and since each block size is  $\geq 1$  thus there are at most  $2 \times \text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}$  blocks in between  $\text{root.blocks}[b]$  and  $\text{root.blocks}[\text{end}]$ . So  $\text{end}-b \leq 2 \times \text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size} + 1$ . See Figure <sup>end-b</sup> 5.

Now that we know there are at most  $\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}$  blocks in between  $\text{root.blocks}[b]$  and  $\text{root.blocks}[\text{end}]$  then with doubling search in  $\Theta(\log(\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}))$  steps we reach  $\text{start}=c$  that the  $\text{root.blocks}[c].\text{sum}_{\text{enq}}$  is less than  $e$  and  $\text{end}-c$  is not more than  $2 \times \text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}$ . Beause otherwise, then  $(\text{end}-c)/2$  satisfied the  $\text{root.blocks}[(\text{end}-c)/2].\text{sum}_{\text{enq}} < e$ . In line <sup>doubling</sup> 804 the differenece between  $\text{end}$  and  $\text{start}$  is doubled. See Figure <sup>fig::doubling</sup> 4.

After computing  $b$ , the value  $i$  is computed via the definition of  $\text{sum}_{\text{enq}}$  in constant time (Line <sup>DSearchCompute i</sup> 807). So the routine non constant part is the binary search which takes  $\Theta(\log \text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size})$  steps from the first paragraph.

□

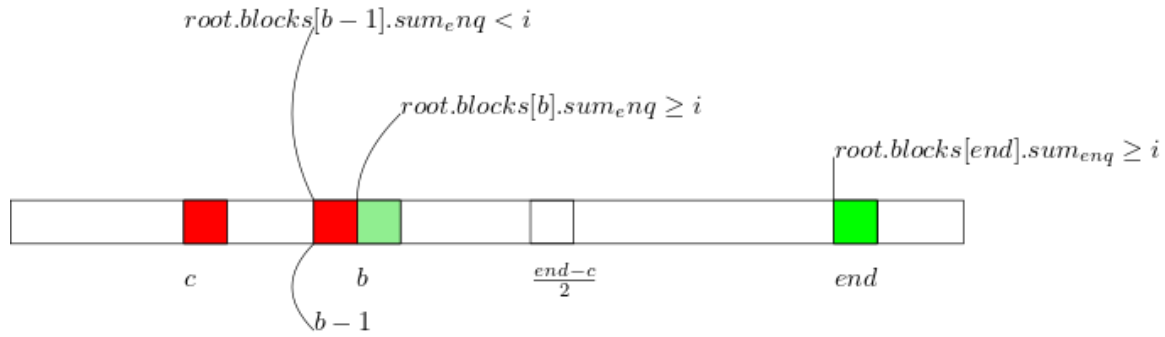


Figure 4: Distance relations between  $b, c, \text{end}$

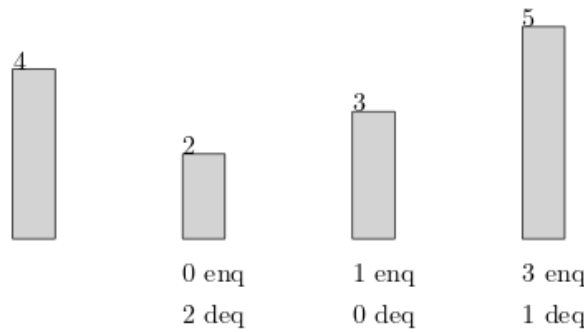


Figure 5: The number written on top of the bars is the queue size. the first block is  $b$  and the last block is  $\text{end}$ .

end-b

**Lemma 23** (Index correctness).  $n.\text{IndexDEQ}(b, i)$  returns the rank in  $D(\text{root})$  of  $D_{n,b,i}$ .

*Proof.* We will prove this by induction on the distance of  $n$  from the root. We can see the base case  $\text{root}.\text{IndexDEQ}(b, i)$  is trivial (Line indexBaseCase 415). In the non-root nodes  $n.\text{IndexDEQ}(b, i)$  computes the superblock of the  $i$ th Dequeue in the  $b$ th block of  $n$  in  $n.\text{parent}$  by Lemma superBlockComputeSuper 24 (Line 418). After that the order in  $D(n.\text{parent}, \text{superblock})$  is computed and  $\text{index}()$  is called on  $n.\text{parent}$  recursively. Then if the operation was propagated from the right child the number of dequeues from the left child are added to it (Line considerRight 420), because the left child operations come before the right child operations (Definition ordering 20).  $\square$

*Do I need to talk about the computation of the order in the parent which is based on the definition of ordering of dequeues in a block?*

*Make sure to show preconditions of all invocation of BSearch are satisfied.*

**Lemma 24** (Computing SuperBlock). After computing line computeSuper 418 of  $n.\text{IndexDEQ}(b, i)$ ,  $n.\text{parent}.\text{blocks}[\text{superblock}]$  contains  $D(n, b, i)$ .

*Proof.* Lemmas 28,29,30,31.  $\square$

**Lemma 25.** Value read for  $\text{super}[b.\text{group}]$  in line 418 is not null.

*Proof.* Values  $\text{np}_{\text{dir}}$  read in lines setNP 337,  $\text{super}$  are set before incrementing in lines setSuperNP 315,316. So before incrementing  $\text{num}_{\text{propagated}}$ ,  $\text{super}[\text{num}_{\text{propagated}}]$  is set so it cannot be null while reading.  $\square$

**Lemma 26.**  $\text{super}[]$  preserves order from child to parent; i.e. if in node  $n$  block  $b$  is before  $c$  then  $b.\text{group} \leq c.\text{group}$

*Proof.* Line setGroup 329. Since  $\text{num}_{\text{propagated}}$  is increasing.  $\square$

**Lemma 27.** Let  $b, c$  be in node  $n$ , if  $b.\text{group} \leq c.\text{group}$  then  $\text{super}[b.\text{group}] \leq \text{super}[c.\text{group}]$

*Proof.* Line setSuper 315.  $\square$

**Lemma 28.** The number of the blocks with  $\text{group}=i$  in a node is  $\leq p$ .

*Proof.* For the sake of simplicity we assumed all the blocks are propagated from the left child.  $\square$

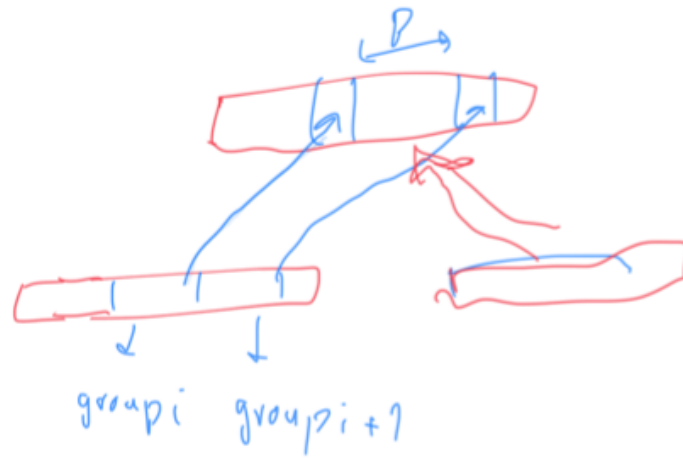


**Lemma 29.**  $\text{super}[i+1] - \text{super}[i] \leq p$

*Proof.* In a Refresh with successful CAS in line 46,  $\text{super}$  and  $\text{counter}$  are set for each child in lines 48,49. Assume the current value of the counter in node  $n$  is  $i+1$  and still  $\text{super}[i+1]$  is not set. If an instance of successful  $\text{Refresh}(n)$  finishes  $\text{super}[i+1]$  is set a new value and a block is added after  $n.\text{parent}[\text{super}[i]]$ . There could be at most  $p$  successful unfinished concurrent instances of  $\text{Refresh}()$  that have not reached line 49. So the distance between  $\text{super}[i+1]$  and  $\text{super}[i]$  is less than  $p$ .  $\square$

**Lemma 30** (super property). If  $\text{super}[i] \neq \text{null}$  in node  $n$ , then  $\text{super}[i]$  is the index of the superblock of a block with  $\text{time}=i$  in  $n.\text{parent}.\text{blocks}$ .

**Lemma 31.** Superblock of  $b$  is within range  $\pm 2p$  of the  $\text{super}[b.\text{time}]$ .



*Proof.*  $\text{super}[i]$  is the index of the superblock of a block containing block  $b$ , followed by Lemma  <sup>$\text{superCounter}$</sup> 30.  $\text{super}(b)$  is the real superblock of  $b$ .  $\text{super}(t)$  is the index of the superblock of the last block with time  $t$ . If  $b.\text{time}$  is  $t$  we have:

$$\text{super}[t] - p \leq \text{super}[t - 1] \leq \text{super}(t - 1) \leq \text{super}(b) \leq \text{super}(t + 1) \leq \text{super}(t + 1) \leq \text{super}[t] + p$$

□

**Lemma 32.** Search in each level of `IndexDeq()` takes  $O(\log p)$  steps.

*Proof.* Show preconditions are satisfied and the range is  $p$ .

□

**Definition 33.** In an execution on a queue, the dequeue operations that return some value are called *non-null dequeues*.

**Observation 34.**  $k$ th non-null dequeue in an execution returns the element of  $k$ th enqueue.

**Lemma 35.**  $\text{root.blocks}[b].\text{size}$  is the size of the queue if the operations in the prefix for the  $b$ th block in the root are applied with the order of  $L$ .

*Proof.* If the size of a queue is greater than 0 then a  $\text{Dequeue}()$  would decrease the size of the queue, otherwise the size of the queue remains 0. By definition of  $L$  in definition <sup>ordering</sup> 20 enqueue operations come before dequeue operations in a block. If the queue won't become empty after  $b$ th block we can compute the size of the queue after  $b$ th block using with this equality  $\text{root.blocks}[b].\text{size} = \text{root.blocks}[b-1].\text{size} + \text{root.blocks}[b].\text{sum}_{\text{enq}} - \text{root.blocks}[b].\text{sum}_{\text{deq}}$  (Line <sup>computeLength</sup> 342). Otherwise  $\text{root.blocks}[b-1].\text{size} + \text{root.blocks}[b].\text{sum}_{\text{enq}} - \text{root.blocks}[b].\text{sum}_{\text{deq}}$  would become negative. Since there are more dequeue operations than the size of the queue. See table <sup>history</sup> II.  $\square$

**Lemma 36.** If the operations are applied with the order of  $L$ , the number of non-null dequeues in the prefix for a block  $b$  is  $\text{b.sum}_{\text{enq}} - \text{b.size}$

*Proof.* If there are  $\text{enqs}$  enqueue operations in the prefix for  $b$ , then the size of the queue after prefix for  $b$  is  $\text{enqs} - \text{non-null deqs}$  by Observation 35. The correctness of the feild size is showed in Lemma <sup>sizeCorrectness</sup> 35.  $\square$

**Lemma 37.**  $D_{\text{root},b,i}$  returns null iff  $\text{root.blocks}[b-1].\text{size} + \text{root.blocks}[b].\text{num}_{\text{enq}} - i < 0$ .

**Lemma 38** (Computing Response). Let  $S$  be the state of an empty queue if the operations in  $L$  are applied on it.  $\text{FindResponse}(b,i)$  returns the element that is the the response to  $D_{\text{root},b,i}$  in  $S$ . If the queue is empty in  $S$  then it returns null.

*Proof.* First note that by Definition <sup>ordering</sup> 20 the linearization ordering of operations will not change as new operations come so instead of talking about the linearization of operations before the  $E_{\text{root},b,i}$  we talk about what if the whole operation in the linearization are applied on a queue.

$D_{\text{root},b,i}$  is  $D_{\text{root},\text{root.blocks}[b-1].\text{sum}_{\text{deq}}+i}$  from the definition <sup>ordering</sup> 20 and  $\text{sum}_{\text{enq}}$ .  $D_{\text{root},b,i}$  returns null if  $\text{root.blocks}[b-1].\text{size} + \text{root.blocks}[b].\text{num}_{\text{enq}} - i < 0$  by Lemma <sup>nullReturnCheckEmpty</sup> 37 (Line 220). Otherwise if it is  $d'$ th non-null dequeue in  $L$  it returns  $d'$ th enqueue by Observation <sup>responseToADeq</sup> 34. By Lemma <sup>numberOfNND</sup> 36 there are  $\text{root.blocks}[b-1].\text{sum}_{\text{enq}} - \text{root.blocks}[b-1].\text{size}$  non-null dequeue operations before prefix for  $\text{root.blocks}[b-1]$ . Note that the dequeues in  $\text{root.blocks}[b]$  before the  $i$ th dequeue are non-null dequeues. So the response is  $E_{\text{root},i-\text{root.blocks}[b-1].\text{size}+\text{root.blocks}[b-1].\text{sum}_{\text{deq}}}$  (Line <sup>computeE</sup> 222). See figure <sup>computeResponseDetail</sup> 6.

After computing  $e$  we can find  $b,i$  such that  $E_{\text{root},b,i} = E_{\text{root},e}$  using  $\text{DSearch}$  and then find its element using  $\text{GetEnq}$  (Line <sup>findAnswer</sup> 223).  $\square$

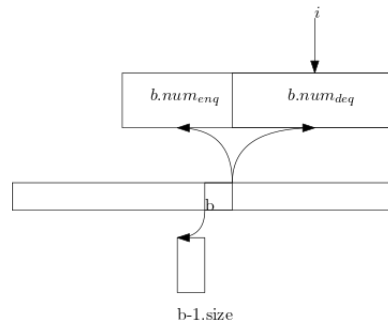


Figure 6: The position of  $E_{\text{root},b,i}$ .



	DEQ()	ENQ(5), ENQ(2), ENQ(1), DEQ()	ENQ(3), DEQ()	ENQ(4), DEQ(), DEQ(), DEQ(), DEQ()
#enqueues	0	3	1	1
#dequeues	1	1	1	4
#non-null dequeues	0	1	2	5
size	0	2	2	0

Table 1: An example of root blocks fields. Blocks are from left to right and operations in the blocks are also from the left to right.

qhistory

**Theorem 39** (Main). *The queue implementation is linearizable.*

*Proof.* We choose  $L$  in Definition [20](#) to be linearization ordering of operations and prove if we linearize operations as  $L$  the queue works consistently.  $\square$

**Lemma 40** (satisfiability).  *$L$  can be a linearization ordering.*

*Proof.* To show this we need to say if in an execution,  $op_1$  terminates before  $op_2$  starts then  $op_1$  is linearized before  $op_2$ . If  $op_1$  terminates before  $op_2$  starts it means  $op_1.\text{Append}()$  is terminated before  $op_2.\text{Append}()$  starts. From Lemma [10](#)  $op_1$  is in `root.blocks` before  $op_2$  propagates so  $op_1$  is linearized before  $op_2$  by Definition [20](#).  $\square$

Once some operations are aggregated in one block they will be propagated together up to the root and we can linearize them in any order among themselves. Furthermore in  $L$  we arbitrary choose the order to be by process id, since it makes computations in the blocks faster.  $\square$

**Lemma 41** (correctness). *If operations are applied as  $L$  on a sequential queue, the sequence of the responses would be the same as our algorithm.*

*Proof. Old parts to review* We show that the ordering  $L$  stored in the root, satisfies the properties of a linearizable ordering.

1. If  $op_1$  ends before  $op_2$  begins in  $E$ , then  $op_1$  comes before  $op_2$  in  $T$ .
  - This is followed by Lemma [10](#). The time  $op_1$  ends it is in root, before  $op_2$ , by Definition [20](#)  $op_1$  is before  $op_2$ .
2. Responses to operations in  $E$  are same as they would be if done sequentially in order of  $L$ .
  - Enqueue operations do not have any response so it does no matter how they are ordered. It remains to prove Dequeue  $d$  returns the correct response according to the linearization order. By Lemma [38](#) it is deduced that the head of the queue at time of the linearization of  $d$  is computed properly. If the Queue is not empty by Lemma [21](#) we know that the returning response is the computed index element.

$\square$

**Lemma 42** (Amortized time analysis). **Enqueue()** and **Dequeue** take  $O(\log^2 p + q)$  steps (amortized analysis), which  $p$  is the number of processes and  $q$  is the size of the queue at the time of invocation.

*Proof.* **Enqueue(x)** consists of creating a **block(x)** and appending it to the tree. The first part takes constant time. To propagate **x** to the root the algorithm tries two **Refreshes** in each node of the path from the leaf to the root (Lines ~~302, 303~~ <sup>firstRefresh, Refresh</sup>). Each **Refresh** takes  $O(1)$  steps since creating a block is done in constant time and does  $O(1)$  CASes. Since the height of the tree is  $O(\log p)$ , **Enqueue(x)** takes  $O(\log p)$  steps.

A **Dequeue()** creates a block with null value element, appends it to the tree, computes its order among operations, and returns the response. The first two part is similar to an **Enqueue** operation. To compute the order there are some constant steps and **IndexDeq** is called. **IndexDeq** does a search with range  $p$  in each level (Lemma ~~31~~ <sup>superRange</sup>) which takes  $O(\log^2 p)$  in the tree. In the **FindResponse()** routine **DSearch()** in the root takes  $\Theta(\log(\text{root.blocks}[b].\text{size} + \text{root.blocks}[\text{end}].\text{size}))$  by Lemma ~~22~~ <sup>dsearch</sup>, which is  $O(\log \text{size of the queue when enqueue is invoked} + \log \text{size of the queue when dequeue is invoked})$ . Each search in **GetEnq()** takes  $O(\log p)$  since there are  $\leq p$  subblocks in a block (Lemma ~~19~~ <sup>subBlocksBound</sup>), so **GetEnq()** takes  $O(\log^2 p)$  steps.

If we split **DSearch** time cost between the corresponding **Enqueue**, **Dequeue**, in amortized we have **Enqueue** takes  $O(\log p + q)$  and **Dequeue** takes  $O(\log^2 p + q)$  steps. □