
Algorithm Fields description

◇ *Shared*

- *Tree tree* : A binary tree of Nodes. *root* is a pointer to the root node.

◇ *Local*

- **Node leaf* : a pointer to the process's leaf in the tree.

◇ *Structures*► *Node*

- **Node left, right, parent* : initialized when creating the tree.
- *BlockList blocks*
- *int size= 1* : #blocks in blocks.
- *int num_{propagated}= 0* : # groups of blocks that have been propagated from the node to its parent. Since it is incremented after propagating, it may be behind by 1.
- *int[] super*: *super[i]* stores an approximate index of the superblock of the blocks in *blocks* whose *group* field have value *i*.

► *Root extends Node*

- *PBRT blocks*
BlockList is implemented with a persistent red-black tree.

► *NonRootNode extends Node*

- *Block[] blocks*
BlockList is implemented with an array.

► *Leaf extends NonRootNode*

- *int last_{done}*
Stores the index of the block in the root such that the process that owns this leaf has most recently finished the. A block is finished if all of its operations are finished. *enqueue(e)* is finished if *e* is returned by some *dequeue()* and *dequeue()* is finished when it computes its response. *put the definitions before the pseudocode*

► *Block* ▷ For a *block* in a *blocklist* we define *the prefix for the block* to be the blocks in the *BlockList* up to and including the *block*. *put the definitions before the pseudocode*

- *int group* : the value read from *num_{propagated}* when appending this block to the node.

► *LeafBlock extends Block*

- *Object element* : Each block in a leaf represents a single operation. For enqueue operations *element* is the input of the enqueue and for dequeue operations it is null.
- *Object response* : stores the response of the operation in the *LeafBlock*.
- *int sum_{enq}, sum_{deq}* : # enqueue, dequeue operations in the prefix for the block

► *InternalBlock extends Block*

- *int end_{left}, end_{right}* : index of the last subblock of the block in the left and right child
- *int sum_{enq-left}* : # enqueue operations in the prefix for *left.blocks[end_{left}]*
- *int sum_{deq-left}* : # dequeue operations in the prefix for *left.blocks[end_{left}]*
- *int sum_{enq-right}* : # enqueue operations in the prefix for *right.blocks[end_{right}]*
- *int sum_{deq-right}* : # dequeue operations in the prefix for *right.blocks[end_{right}]*

► *RootBlock extends InternalBlock*

- *int length* : length of the queue after performing all operations in the prefix for this block
- *counter num_{finished}* : number of finished operations in the block
- *int order* : the index of the block in the *BlockList* containing the block.

Variable naming:

- *b_{op}*: index of the block containing operation *op*
- *r_{op}*: rank of operation *op* i.e. the ordering among the operations of its type according to linearization ordering

Abbreviations:

- *blocks[b].sum_x=blocks[b].sum_{x-left}+blocks[b].sum_{x-right}* (for $b \geq 0$ and $x \in \{\text{enq}, \text{deq}\}$)
- *blocks[b].sum=blocks[b].sum_{enq}+blocks[b].sum_{deq}* (for $b \geq 0$)
- *blocks[b].num_x=blocks[b].sum_x-blocks[b-1].sum_x*
(for $b > 0$ and $x \in \{\emptyset, \text{enq}, \text{deq}, \text{enq-left}, \text{enq-right}, \text{deq-left}, \text{deq-right}\}$, *blocks[0].num_x=0*)

Algorithm Queue

```
201: void ENQUEUE(Object e) ▷ Creates a block with element e and appends it to the tree.
202:   block newBlock= NEW(LeafBlock)
203:   newBlock.element= e
204:   newBlock.sumenq= leaf.blocks[leaf.size].sumenq+1
205:   newBlock.sumdeq= leaf.blocks[leaf.size].sumdeq
206:   leaf.APPEND(newBlock)
207: end ENQUEUE

208: Object DEQUEUE()
209:   block newBlock= NEW(LeafBlock) ▷ Creates a block
    with null value element, appends it to the tree, computes its order among
    operations, then computes and returns its response.
210:   newBlock.element= null
211:   newBlock.sumenq= leaf.blocks[leaf.size].sumenq
212:   newBlock.sumdeq= leaf.blocks[leaf.size].sumdeq+1
213:   leaf.APPEND(newBlock)
214:   return leaf.HELPDEQUEUE()

215: end DEQUEUE

216: int, int FINDRESPONSE(int i, int b) ▷ Computes the rank and
    index of the block in the root of the enqueue that is the response of the ith
    dequeue in the root's bth block. Returns <-1,--> if the queue is empty.
217:   if root.blocks[b-1].length + root.blocks[b].numenq - i < 0 then
218:     return <-1,-->
219:   else
    ▷ We call the dequeues that
    return a value non-null dequeues. rth non-null dequeue returns the element
    of th rth enqueue. We can compute # non-null dequeues in the prefix for
    a block this way: #non-null dequeues= length - #enqueuees. Note that the
    ith dequeue in the given block is not a non-null dequeue.
220:     renq= root.blocks[b-1].sumenq- root.blocks[b-1].length + i
221:     return <root.blocks.get(enq, renq).order, renq>
222:   end if
223: end FINDRESPONSE
```

Algorithm Node

```

301: void PROPAGATE()
302:   if not REFRESH() then
303:     REFRESH()
304:   end if
305:   if this is not root then
306:     parent.PROPAGATE()
307:   end if
308: end PROPAGATE

309: boolean REFRESH()
310:   s= size
311:   <new, npleft, npright>= CREATEBLOCK(h)
312:   if new.num==0 then return true
313:   else if blocks.tryAppend(new, s) then
314:     for each dir in {left, right} do
315:       CAS(dir.super[npdir], null, h+1)
316:       CAS(dir.numpropagated, npdir, npdir+1)
317:     end for
318:     CAS(size, h, h+1)
319:     return true
320:   else
321:     CAS(size, h, h+1)
322:     return false
323:   end if
324: end REFRESH

325: int BSEARCH(field f, int i, int start, int end)
326: end BSEARCH

327: <Block, int, int> CREATEBLOCK(int i)
328:   block newBlock= NEW(block)
329:   newBlock.group= numpropagated
330:   newBlock.order= i
331:   for each dir in {left, right} do
332:     indexlast= dir.size
333:     indexprev= blocks[i-1].enddir
334:     newBlock.enddir= indexlast
335:     blocklast= dir.blocks[indexlast]
336:     blockprev= dir.blocks[indexprev]
337:     thisdir= dir.numpropagated
338:     newBlock.sumenq-dir= blocks[i-1].sumenq-dir + blocklast.sumenq
339:     - blockprev.sumenq
340:     newBlock.sumdeq-dir= blocks[i-1].sumdeq-dir + blocklast.sumdeq
341:     - blockprev.sumdeq
342:   end for
343:   newBlock.length= max(root.blocks[i-1].length + b.numenq -
344:   b.numdeq, 0)
345:   return <b, npleft, npright>
346: end CREATEBLOCK

```

↪ Precondition: blocks[start..end] contains a block with field $f \geq i$

325: int BSEARCH(field f, int i, int start, int end)

↪ Does binary search for the value i of the given prefix sum field. Returns the index of the leftmost block in blocks[start..end] whose field f is $\geq i$.

326: end BSEARCH

Algorithm Node

↪ Precondition: $\text{blocks}[b].\text{num}_{\text{enq}} \geq i$

```
401: element GETENQ(int b, int i)
402:   if this is leaf then
403:     return blocks[b].element
404:   else if  $i \leq \text{blocks}[b].\text{num}_{\text{enq-left}}$  then ▷ i exists in the left child of this node
405:     subBlock= left.BSEARCH(sumenq, i, blocks[b-1].endleft+1, blocks[b].endleft) ▷ Search range of left child's subblocks of blocks[b].
406:     return left.GET(i-left.blocks[subBlock-1].sumenq, subBlock)
407:   else
408:     i= i-blocks[b].numenq-left
409:     subBlock= right.BSEARCH(sumenq, i, blocks[b-1].endright+1, blocks[b].endright) ▷ Search range of right child's subblocks of blocks[b].
410:     return right.GET(i-right.blocks[subBlock-1].sumenq, subBlock)
411:   end if
412: end GETENQ
```

↪ Precondition: b th block of the node has propagated up to the root and $\text{blocks}[b].\text{num}_{\text{enq}} \geq i$.

```
413: <int, int> INDEXDEQ(int b, int i) ▷ Returns the rank of i-th dequeue in the bth block of the node, among the dequeues in the root.
414:   if this is root then
415:     return <b, i>
416:   else
417:     dir= (parent.left==n)? left: right ▷ check if a left or a right child
418:     superBlock= parent.BSEARCH(sumdeq-dir, i, super[blocks[b].group]-p, super[blocks[b].group]+p)
▷ superblock's group has at most p difference with the value stored in super[].
419:     if dir is right then
420:       i+= blocks[superBlock].sumdeq-left ▷ consider the dequeues from the right child
421:     end if
422:     return this.parent.INDEXDEQ(superBlock, i)
423:   end if
424: end INDEX
```

Algorithm Leaf

501: void APPEND(block blk) ▷ Append is only called by the owner of the leaf.

502: size+=1

503: blk.group= size

504: blocks[size]= blk

505: parent.PROPAGATE()

506: end APPEND

507: Object HELPDEQUEUE()

508: <b_{deq}, r_{deq}>= INDEXDEQ(leaf.size, 1) ▷ r is the rank among the dequeues of the dequeue of the b_{deq}th block in the root containing.

509: <r_{enq}, r_{enq}>= FINDRESPONSE(r_{deq}, b_{deq}) ▷ r_{enq} is the rank of the enqueue whose element is the response to the dequeue in the block containing it and

 b_{deq} is the index of that block of it in the blocklist. If the response is null then r_{deq} is -1.

510: if r_{enq}== -1 then

511: output= null

512: root.blocks[b_{deq}].numfinished.inc() ▷ shared counter

513: if root.blocks[b_{deq}].numfinished==root.blocks[b_{deq}].num then

514: lastdone= b_{deq}

515: end if

516: else

517: output= GETENQ(b_{enq}, r_{enq}) ▷ getting the reponse's element.

518: root.blocks[b_{enq}].numfinished.inc()

519: root.blocks[b_{enq}].numfinished.inc()

520: if root.blocks[b_{deq}].numfinished==root.blocks[b_{deq}].num then

521: lastdone= b_{deq}

522: else if root.blocks[b_{enq}].numfinished==root.blocks[b_{enq}].num then

523: lastdone= b_{enq}

524: end if

525: end if

526: return output

527: end DEQUEUE

Algorithm BlockList

▷ : Supports two operations `blocks.tryAppend(Block b)`, `blocks[i]`. Initially empty, when `blocks.tryAppend(b,`

`n)` returns true `b` is appended to `blocks[n]` and `blocks[i]` returns i th block in the blocks. If some instance of `blocks.tryAppend(b, n)` returns false there is a concurrent instance of `blocks.tryAppend(b', n)` which has returned true. `blocks[0]` contains an empty block with all fields equal to 0 and `endleft`, `endright` pointers to the first block of the corresponding children.

◇ *PBRT implementation*

A persistent red-black tree supporting `append(b, key)`, `get(key=i)`, `split(j)`. `append(b, key)` returns true in case successful. Since order, `sumenq` are both strictly increasing we can use one of them for another.

`root`: pointer to the root of the PBRT

601: `boolean TRYAPPEND(block blk, int n)`

▷ adds block `b` to the `root.blocks[n]`

602: **if** `root.size % p2 == 0` **then**

▷ Help every often p^2 operations appended to the root.

603: `Help()`

604: `CollectGarbage()`

605: **end if**

606: `blk.numfinished = 0`

607: `*oldRoot = &root.blocks.root`

608: `*newRoot = root.blocks.Append(blk).root`

609: **return** `CAS(root, oldRoot, newRoot)`

610: **end TRYAPPEND**

◇ *Array implementation*

`blocks[]`: array of blocks

611: `boolean TRYAPPEND(block blk, int n)`

612: **return** `CAS(blocks[n], null, blk)`

613: **end TRYAPPEND**

801: `void HELP`

▷ Helps pending operations

802: **for** leaf `l` **in** `leaves` **do**

814: `Block FINDMOSTRECENTDONE(b)`

803: `last = l.size - 1` ▷ `l.blocks[last]` can not be null because size increases after appending, see lines 603-602.

815: **for** leaf `l` **in** `leaves` **do**

816: `max = Max(l.maxOld, max)`

804: **if** `l.blocks[last].element == null` **then** ▷ operation is dequeue

817: **end for**

805: `l.blocks[last].response = l.HELPDEQUEUE()`

818: **return** `max`

▷ This snapshot suffices.

806: **end if**

819: **end FINDMOSTRECENTDONE**

807: **end for**

808: **end HELP**

820: `response FALLBACK(op i)` ▷ how to use as exception handling? by adding try catch in all the methods reading the root?

809: `void COLLECTGARBAGE` ▷ Collects the root blocks that are done.

821: **if** `root.blocks.get(numenq), i` is null **then** ▷ this enqueue was already finished

810: `s = FindMostRecentDone(Root.Blocks.root)` ▷ Lemma: If block `b` is done after helping then all blocks before `b` are done as well.

822: **return** `this.leaf.response(block.order)`

811: `t1, t2 = RBT.split(order, s)`

823: **end if**

812: `RBTRoot.CAS(t2.root)`

824: **end FALLBACK**

813: **end COLLECTGARBAGE**
