A Wait-free Queue with Poly-logarithmic Step Complexity

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Abstract

In this work, we introduce a novel linearizable wait-free queue implementation. Linearizability and lock-freedom are standard requirements for designing shared data structures. To the best of our knowledge, all of the existing linearizable lock-free queues in the literature have a common problem in their worst case called the CAS Retry Problem. We show that our algorithm avoids this problem and has worst case running time better than prior lock-free queues. The amortized number of steps for an `Enqueue` or `Dequeue` in our algorithm, is $O(\log^2 p + \log q)$, where p is the number of processes and q is the size of the queue when the operation is linearized.

Contents

1 Introduction

Shared data structures have become an essential field in distributed algorithms research. We are reaching the physical limits of how many transistors we can place on a CPU core. The industry solution to provide more computational power is to increase the number of cores of the CPU. It is not hard to see why multiple processes cannot share a sequential data structure to work with. For example, consider two processes trying to append to a sequential linked list simultaneously. Processes p, q read the same tail node, p changes the next pointer of the tail node to its new node and after that q does the same. In this run, p's update is overwritten. One solution is to use locks; whenever a process wants to do an update or query on a data structure, the process locks the data structure, and others have to wait until the lock is released to read or update the data structure. Using locks has some disadvantages; for example, one process might be slow, and holding a lock for a long time prevents other processes from progressing. Moreover, locks do not allow complete parallelism since only the process holding the lock can make progress at the time the data structure is locked. For these reasons, we are interested in the design of an efficient shared queue without using locks.

A sequential queue stores a sequence of elements and supports two operations, enqueue and dequeue. `Enqueue(e)` appends element `e` to the sequence stored. `Dequeue` removes and returns the first element in the queue. If the queue is empty it returns `null`. In a concurrent version of the queue data structure, operations do not happen one at a time. The question that may arise is "What properties matter for implementation of a shared data structure?", since executions on a shared data structure are different from sequential ones, the correctness conditions also differ. To prove a concurrent object works perfectly, we have to show it satisfies safety and progress conditions. A *safety condition* tells us that the data structure does not return wrong responses, and a *progress property* requires that operations eventually terminate.

A system is called *asynchronous* when processes in the system run at arbitrarily varying speeds, i.e., the scheduling of each process is independent from the scheduling of other processes. Our model is an asynchronous shared-memory distributed system of p processes. Herlihy [?] showed that one cannot implement concurrent versions of all data structures with multi-reader multi-writer registers alone and introduced a hierarchy on how powerful objects are to solve the consensus problem among processes. Objects like LL/SC, CAS can be used to reach consensus among any number of processors. A CAS object provides an atomic Compare & Swap(`new, old`) operation: if the value stored in the object is `old`, it updates the value to `new` and returns `true`, otherwise it returns `false`. In this work we use Compare & Swap (`CAS`) objects to synchronize among processes.

The standard safety condition is called *linearizability* [?], which ensures that for any concurrent execution on a linearizable object, each operation should appear to take effect instantaneously at some moment between its invocation and response. Figure ?? depicts an example of an execution on a linearizable queue that is initially empty. The arrow shows time, and each rectangle shows the time between the invocation and the termination of an operation. Since `Enqueue(A)` and `Enqueue(B)` are concurrent, `Enqueue(B)` may or may not take effect before `Enqueue(A)`. The execution in Figure ?? is not linearizable since `A` has been enqueued before `B`, so it has to be dequeued first.

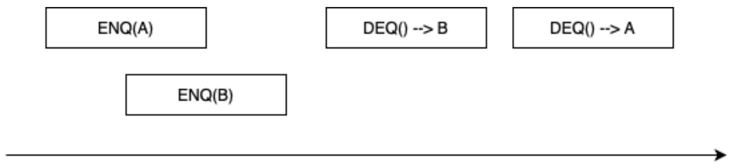


Figure 1: An example of a linearizable execution. Either `Enqueue(A)` or `Enqueue(B)` could take effect first since they are concurrent.

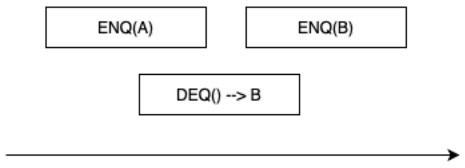


Figure 2: An example of an execution on an empty queue that is not linearizable. `Enqueue(A)` has terminated before `Enqueue(B)` is invoked and the `Dequeue` can occurr before termination of `Enqueue(A)` and return `null` or can occurr after termination of `Enqueue(A)` and return `A`.

There are various progress properties; the strongest is wait-freedom, and the more common is lock-freedom. An algorithm is *wait-free* if each operation terminates after a finite number of its own steps. We call an algorithm *lock-free* if, after a sufficient number of steps, one operation terminates. A wait-free algorithm is also lock-free but not vice versa; in an infinite run of a lock-free algorithm there might be an operation that takes infinitely many steps but never terminates.

We will present a wait-free linearizable queue, which to the best of our knowledge is the first wait-free queue whose operations run in a poly-logarithmic number of steps. We design a tournament tree in which each process tries to propagate its operations along a path from the process's leaf up to the root to be linearized. Processes can do queries to get information about the linearization stored in the root. We create a helping mechanism for when a process is propagating its operation to a node, such that new operations are ensured to be propagated after at most 2 `CAS` invocations. The amortized number of steps for an `Enqueue` or `Dequeue`, is $O(\log^2 p + \log q)$, where q is the size of the queue when the operation is linearized. `CAS` instructions cost more than other instructions for processor. Each operation in our queue does $\Theta(\log p)$ `CAS` operations compared to $\Omega(p)$ `CAS` steps for previous queues in the worst case. We use unbounded memory space in our queue, we present a way to reduce the memory size to the total number of operations. We do not address garbage collection but we describe a way we think it can be handled.

Thesis outline The rest of the thesis is organized as follows. Chapter 2 gives an outline of related work done in the area and the motivation of our implementation. Previous lock-free queues and their common problem is presented in Section 2.1. In Section 2.2 we mention some restricted lock-free queues. Section 2.3 talks about poly-logarithmic constructions of shared objects. Section 2.4 ends with a lower bound on the amortized time complexity of shared queues.

In Chapter 3 we introduce a poly-logarithmic step wait-free queue. In Section 3.1 we give a high-level description of our implementation. We also talk about the motivation and requirements in our design to achieve poly-log time. Section 3.2 contains the algorithm itself.

We prove the correctness of our queue in Chapter 4 by showing it is linearizable.

Chapter 5 is devoted to performance analysis for our implementation. We analyze the number of `CAS` instructions our algorithm invokes and the worst-case and amortized running time of the queue. In the end we prove our queue is wait-free.

Finally, we give some concluding remarks in Chapter 6. It contains how we can improve our algorithm's memory usage and how to use its idea in designing other wait-free data structures.

2 Related Work

In this section, we look at previous lock-free queues.

2.1 List-based Queues

Michael and Scott [?] introduced a lock-free queue which we refer to as the MS-queue. A version of it is included in the standard Java Concurrency Package. Their idea is to store the queue elements in a singly-linked list (see Figure ??). A shared variable Head points to the first node in the linked list that has not been dequeued, and Tail points to the last element in the queue. To insert a node into the linked list, they use atomic primitive operations like `LL/SC` or `CAS`. If p processes try to enqueue simultaneously, only one can succeed, and the others have to retry. This makes the amortized number of steps $\Omega(p)$ per enqueue. Similarly, dequeue can take $\Omega(p)$ steps.

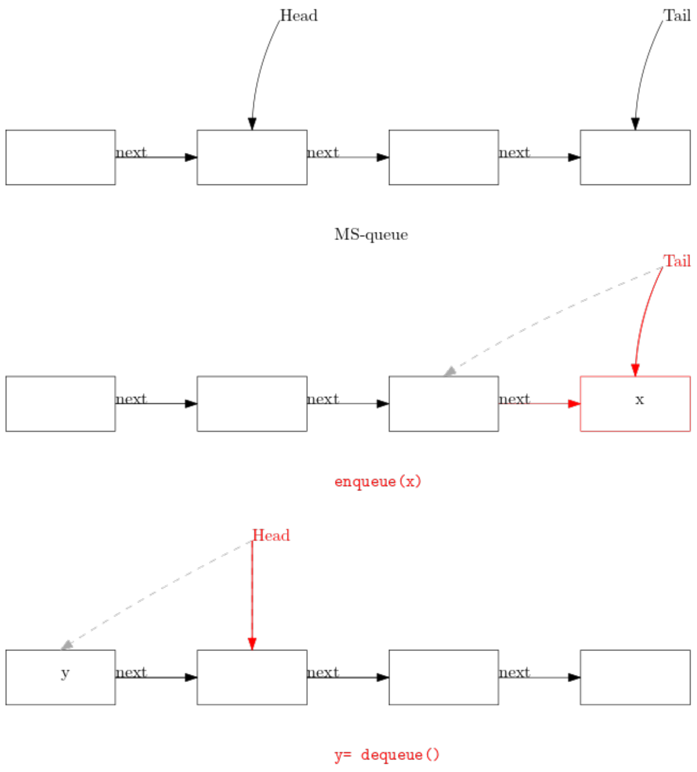


Figure 3: MS-queue structure, enqueue and dequeue operations. In the first diagram the first element has been dequeued. Red arrows show new pointers and gray dashed arrows show the old pointers.

Moir, Nussbaum, and Shalev [?] presented a more sophisticated queue by using the *elimination* technique. The elimination mechanism has the dual purpose of allowing operations to complete in parallel and reducing contention for the queue. An Elimination Queue consists of an MS-queue augmented with an elimination array. Elimination works by allowing opposing pairs of concurrent operations such as an enqueue and a dequeue to exchange values when the queue is empty or when concurrent operations can be linearized to empty the queue. Their algorithm makes it possible for long-running operations to eliminate an opposing operation. The empirical evaluation showed the throughput of their work is better than the MS-queue, but the worst case is still the same; in case there are p concurrent enqueues, their algorithm is not better than MS-queue.

Hoffman, Shalev, and Shavit [?] tried to make the MS-queue more parallel by introducing the Baskets Queue. Their idea is to allow more parallelism by treating the simultaneous enqueue operations as a basket. Each basket has a time interval in which all its nodes' enqueue operations overlap. Since the operations in a basket are concurrent, we can order them in any way. Enqueues in a basket try to find their order in the basket one by one by using `CAS` operations. However, like the previous algorithms, if there are p concurrent enqueue operations in a basket, the amortized step complexity remains $\Omega(p)$ per operation.

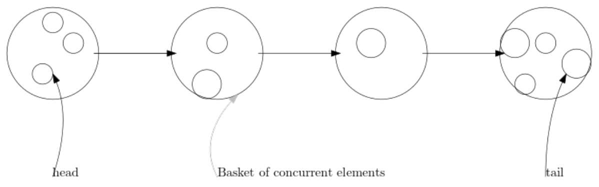


Figure 4: Baskets queue idea. There is a time that all operations in a basket were running concurrently, but only one has succeeded to do `CAS`. To order the operations in a basket, the mechanism in the algorithm for processes is to `CAS` again. The successful process will be the next one in the basket and so on.

Ladan-Mozes and Shavit [?] presented an optimistic approach to implement a queue. MS-queue uses two `CAS`es to do an enqueue: one to change the tail to the new node and another one to change the next pointer of the previous added node to the new node. They use a doubly-linked list to do fewer `CAS` operations in an `Enqueue` than MS-queue. As in previous algorithms, the worst case happens in the case where the contention is high: when p concurrent enqueues happen, their nodes have to be appended to the linked list one by one. The amortized worst-case complexity is still $\Omega(p)$ `CAS`es.

Hendler et al. [?] proposed a new paradigm called flat combining. The key idea behind flat combining is to allow a combiner who has acquired the global lock on the data structure to learn about the requests of threads on the queue, combine them and apply the combined results on the data structure. Their queue is linearizable but not lock-free and they present experiments that show their algorithm performs well in some situations.

Gidenstam, Sundell, and Tsigas [?] introduced a new algorithm using a linked list of arrays. The queue is stored in a shared array where head and tail pointers point to the current elements in the queue. When the array is full, an empty array is linked to the array and tail pointers are updated. A global head points to the array containing the first element in the queue, and each process has a local head index that points to the first element in that array. Global tail and local tail pointers are similar (see Figure **??**). A process updates the position of the pointers after it does an operation. One process might go to sleep before setting the pointers, so the pointers might be behind their real places. They mention how to scan the arrays to update pointers while doing an operation. A process writes an element in the location head by a `CAS` instruction, so if p processes try to enqueue simultaneously, the step (and `CAS`) complexity remains $\Omega(p)$. Their design is lock-free but not wait-free.

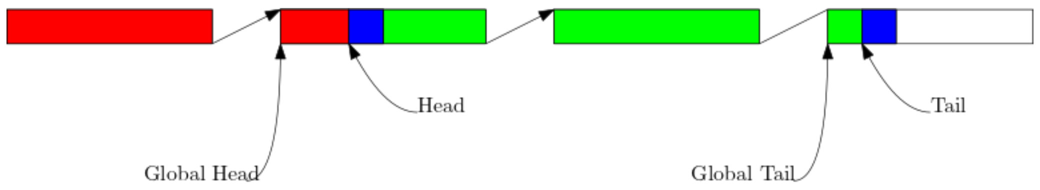


Figure 5: Global pointers point to arrays. Head and Tail elements are blue, dequeued elements are red and current elements of the queue are green.

Kogan and Petrank [?] introduced wait-free queues based on the MS-queue and use Herlihy's helping technique to achieve wait-freedom. Their step complexity is $\Omega(p)$ because of the helping mechanism.

Milman et al. [?] designed a lock-free queue supporting futures. In their queue, operations return future objects instead of responses. Later when the response is needed it can be evaluated from the future object. They also define a weaker linearizabilty condition such that each operation can be linearized between its invocation and when its future is evaluated. Their idea of batching allows a sequence of operations to be submitted as a batch for later execution on the MS-queue. They use some properties of the size of the queue before and after a batch which is similar to a part of our work. Their queue is not wait-free, in fact if the batch sizes are 1 then the queue is like MS-queue.

Nikolaev and Ravindran [?] present a wait-free queue that uses the fast-path slow-path methodology. Their work is based on a circular queue using bounded memory. When a process wishes to do an enqueue or a dequeue, it starts two paths. One is faster but its running time is not bounded and the other one is slower but has a time bound. They show that these two paths do not effect each other and the queue remains consistent. If a process makes no progress other processes help its slow path to finish. The helping phase suffers from CAS Retry Problem, because processes compete in a `CAS` loop to decide which succeeds to help. Because of this, the worst-case complexity cannot be better than $\Omega(p)$.

In the worst-case, step complexity of all the list-based queues discussed above, includes an $\Omega(p)$ term that comes from the case where all p processes try to do an enqueue simultaneously. Morrison and Afek call this the *CAS retry problem* [?]. It is not limited to list-based queues and array-based queues share the CAS retry problem as well [?, ?, ?]. Our motivation is to overcome this problem and present a wait-free sublinear queue.

2.2 Restricted Queues

David introduced the first sublinear concurrent queue [?]. Even though his algorithm does $O(1)$ steps for each operation, it is a single-enqueuer single-dequeuer queue and uses infinite memory. The author states that to reduce memory usage to be bounded the time per operation increases to linear.

Jayanti and Petrovic introduced a wait-free poly-logarithmic multi-enqueuer single-dequeuer queue [?]. We use their idea of having a tournament tree among processes to agree on the linearization of operations to design a poly-logarithmic multi-enqueuer multi-dequeuer queue. Unlike their work, our algorithm does not put a limit on the number of concurrent dequeuers.

2.3 Universal Constructions and other Poly-log Time Data Structures

A *universal construction* is an algorithm that can implement a shared version of any given sequential object, introduced by Herlihy [?]. We can implement a concurrent queue using a universal construction. Jayanti proved an $\Omega(\log p)$ lower bound on the worst-case shared-access time complexity of p-process universal constructions [?]. He also mentions that the universal construction by Afek, Dauber, and Touitou [?] can be modified to $O(\log p)$ worst case step complexity, using atomic access to $\Omega(p \log p)$ bit words. Chandra, Jayanti and Tan introduced a semi-universal construction that achieves $\mathrm{O}(\log^2 p)$ shared accesses using reasonably sized words [?]. Their universal construction cannot be used to create any data structure, but a set of objects called closed objects (a queue is not a closed object). We mention a non-practical universal construction with a poly-log number of `CAS` instructions on page 13.

Ellen and Woelfel introduced an implementation of a Fetch&Inc object with step complexity of $O(\log p)$ using $O(\log n)$-bit `LL/SC` objects, where n is the number of operations [?]. Their idea to achieve logarithmic complexity is to use a tree storing the Fetch&Inc operations invoked by processes. When a process wants to do Fetch&Inc it adds a Fetch&Inc to the tree and returns the number of elements in the tree. There are some similarities between designing a queue and a Fetch&Inc object. A Fetch&Inc object can be constructed from a queue. The algorithm by Ellen and Woelfel is interesting because it is the first wait-free data structure achieving poly-log implementation after [**?**].

2.4 Attiya–Fouren Lower Bound

Because of the `CAS` retry problem in previous list-based queues one might guess the $\Omega(p)$ term is inherent in the time complexity of concurrent queues. Attiya and Fouren gave a lower bound on amortized complexity of lock-free queues with regard to c, the number of concurrent processes. Their result says if c is $O(\log\log p)$, any implementation of queues using reads, writes and conditional operations like `CAS` has $O(\sqrt{\log\log p})$ amortized complexity [**?**]. The surprising point is that their result does not contradict ours and we manage to reach poly-log time complexity.

3 Queue Implementation

In our model there are p processes doing `Enqueue` and `Dequeue` operations on a queue concurrently. We design a linearizable wait-free queue with $O(\log^2 p + \log q)$ steps per operation, where q is the number of elements in the queue at the time of linearization. We avoid the $\Omega(p)$ worst-case step complexity of existing shared queues based on linked lists or arrays, which suffer from the CAS Retry Problem.

There is a shared binary tree among the processes (see Figure **??**) to agree on one total ordering of the operations invoked by processes. Each process has a leaf in which the operations invoked by the process are stored in order. When a process wishes to do an operation it appends the operation to its leaf and tries to propagate its new operation up to the tree's root. Each node of the tree keeps an ordering of operations propagated up to it. All processes agree on the sequence of operations in the root and this ordering is used as the linearization ordering.

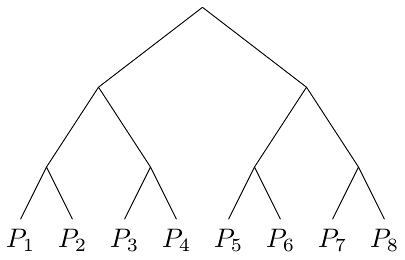
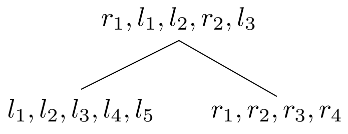


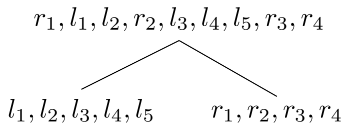
Figure 6: Each of the processes $P_1, P_2, ..., P_p$ has a leaf and in each node there is an ordering of operations stored. Each process tries to propagate its operations up to the root, which stores a total ordering of all operations.

To propagate operations to a node n in the tree, a process observes the operations in both of n's children that are not already in n, merges them to create an ordering and then tries to append the ordering to the sequence stored in n. We call this procedure n`.Refresh()` (see Figure **??**). A `Refresh` on n with a successful append helps to propagate their operations up to the n. We shall prove that if a process invokes `Refresh` on the node n two times and fails to append the new operations to n both times, the operations that were in n's children before the first `Refresh` are guaranteed to be in n after the second failed `Refresh`. We sketch the argument here.

We use `CAS` (Compare & Swap) instructions to implement the `Refresh`'s attempt to append described in the previous paragraph. The second failed `Refresh` of P is assuredly concurrent with a successful `Refresh`



(a) Before the `Refresh`.


$r_1, l_1, l_2, r_2, l_3, l_4, l_5, r_3, r_4$
l_1, l_2, l_3, l_4, l_5
r_1, r_2, r_3, r_4

(b) New operations are appended.

Figure 7: Before and after a n.`Refresh` with a successful append. Operations propagating from the left child are labelled with l and from the right child with r.

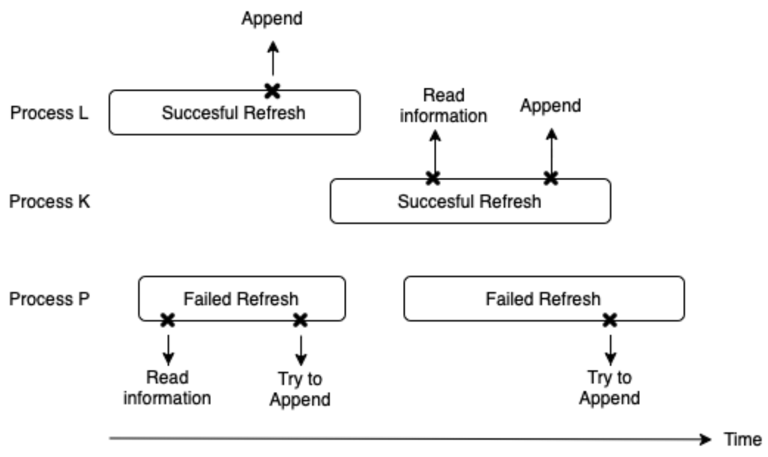


Figure 8: Time relations between the concurrent successful `Refresh`es and the two consecutive `Refresh`es.

that has read its information after the invocation of the first failed `Refresh` (see Figure **??**). This is because some process L does a successful append during P's first failed attempt, and some process K performs a `Refresh` that reads its information after L's append and then performs a successful append during P's second failed `Refresh`. Process K's `Refresh` helps to append the new operations in n's children before P's first failed `Refresh`, in case they were not already appended. After a process appends its operation into its leaf it can call `Refresh` on the path up to the root at most two times on each node. So, with $O(\log p)$ `CAS`es an operation can ensure it appears in the linearization. This cooperative solution allows us to overcome the CAS Retry Problem.

It is not efficient to store the sequence of operations in each node explicitly because each operation would have to be copied all the way up to the root; doing this would not be possible in poly-logarithmic time. Instead we use an implicit representation of the operations propagated together. Furthermore, we do not need to maintain an ordering on operations propagated together in a node until they have reached the root. It is sufficient to only keep track of sets of operations propagated together in each `Refresh` and then define the linearization ordering only in the root (see Figure **??**). Achieving a constant-sized implicit representation of operations in a `Refresh` allows us to do `CAS` instruction on fixed-size objects in each `Refresh`. To do that, we introduce *block*s. A block stores information about the operations propagated by a `Refresh`. It contains the number of operations from the left and the right child propagated to the node by the `Refresh` procedure. See Figure **??** for an example. A node stores an array of blocks of operations propagated up to it. A propagate step aggregates the new blocks in children into a new block and puts it in the parent's blocks. We call the aggregated blocks *subblocks* of the new block and the new block the *superblock* of them. In each `Refresh` there is at most one operation from each process trying to be propagated, because one process cannot invoke two operations concurrently. Thus, there are at most p operations in a block. Furthermore, since the operations in a `Refresh` step are concurrent we can linearize them among themselves in any order we wish, because if two operations are read in one successful `Refresh` step in a node they are going to be propagated up to the root together. Our choice is to put the operations propagated from the left child before the operations propagated from the right child. In this way if we know the number of operations from the left child and the number of operations from the right child in a block we have a complete ordering on the operations.

So far, we have a shared tree that processes use to agree on the implicit ordering stored in its root. With this agreement on linearization ordering we can design a universal construction; for a given object we can

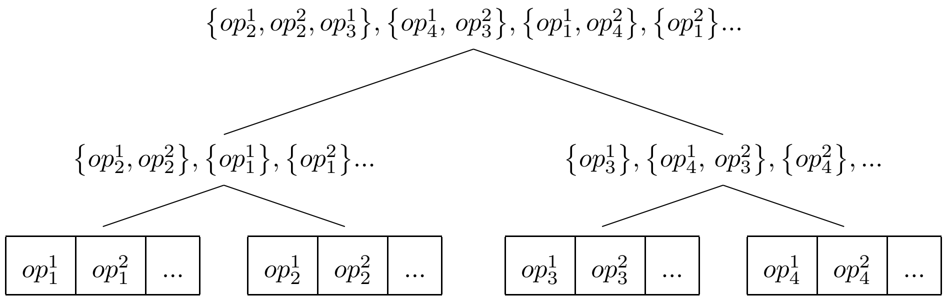

$\{op_2^1, op_2^2, op_3^1\}, \{op_4^1, op_3^2\}, \{op_1^1, op_4^2\}, \{op_1^2\}$...
$\{op_2^1, op_2^2\}, \{op_1^1\}, \{op_1^2\}$...
$\{op_3^1\}, \{op_4^1, op_3^2\}, \{op_4^2\}$, ...
op_1^1 op_1^2 ...
op_2^1 op_2^2 ...
op_3^1 op_3^2 ...
op_4^1 op_4^2 ...

Figure 9: Leaves are for processes P_1 to P_4 from left to right. In each internal node one can arbitrarily linearize the sets of concurrent operations propagated together in a `Refresh`. For example op_4^1 and op_3^2 have propagated together in one `Propagate` step and they will be propagated up to the root together. Since their execution time intervals overlap, they can be linearized in any order.

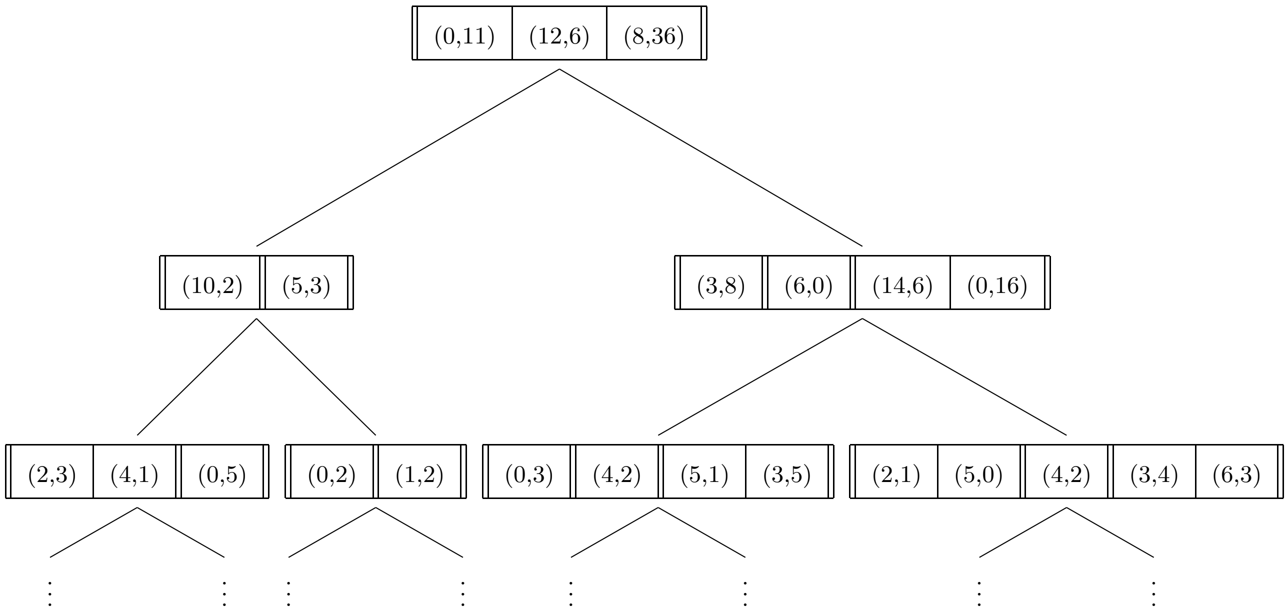


Figure 10: Using blocks to represent operations. Blocks between two lines || are propagated together to the parent. Each block consists of a pair (left, right) indicating the number of operations from the left and the right child, respectively. For example, (12,6) in the root contains (10,2) from the left child and (6,0) from the right child. The third block in the root (8,36) is created by merging (5,3) from the left child and (14,6) and (0,16) from the right child. (5,3) is superblock of (0,5) and (1,2) and (5,1),(3,5) and (4,2) are subblocks of (14,6).

perform an operation *op* by applying all the operations up until *op* in the root on a local copy of the object and then returning the response for *op*. However, this approach is not enough for an efficient queue. We show that we can build an efficient queue if we can compute two things about the ordering in the root: (1) the ith propagated operation and (2) the rank of a propagated operation in the linearization. We explain how to implement (1) and (2) in poly-logarithmic steps.

After propagating an operation *op* to the root, processes can find out information about the linearization ordering using (1) and (2). To get the ith operation in the root, we find the block B containing the ith operation in the root, and then recursively find the subblocks of B in the descendent of the root that contain that ith operation. When we reach a block in a leaf, the operation is explicitly stored there. To make this search faster, instead of iterating over all blocks in the node, we store the prefix sum of the number of elements in the blocks sequence to permit a binary search for the required block. We also store pointers to determine the range of subblocks of a block to make the binary search faster. In each block, we store the prefix sum of operations from the left child and from the right child. Moreover, for each block, we store two pointers to the last left and right subblock of it (see Figure **??**). We know a block size is at most p, so binary search takes at most $O(\log p)$ time, since the pointers of a block and its previous block reduce the search range size to $O(p)$.

To compute the rank in the root of an operation in the leaf, we need to find the superblock of the block that operation is in. After a block is installed in a node we store the approximate index of its superblock in it to make this faster.

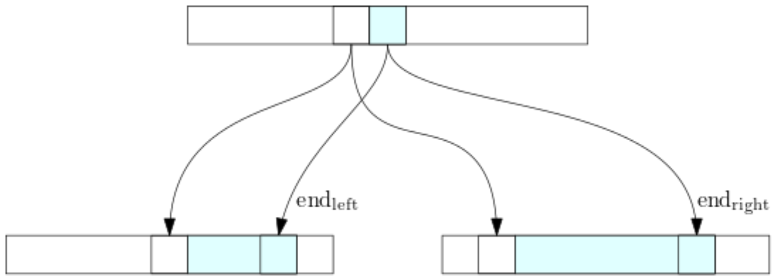


Figure 11: Each block stores the index of its last subblock in each child.

In an execution on a queue where no dequeue operation returns `null`, the kth dequeue returns the argument of the kth enqueue. In the general case a dequeue returns `null` if and only if the size of the queue after the previous operation is 0. We refer to such a dequeue as a *null dequeue*. If the dequeue is the kth non-null dequeue, it returns the argument of the kth enqueue. Having the size of the queue after an operation we can compute the number of non-null dequeues from the number of enqueues before the operation. So, if we store the size of the queue after each block of operations in the root, we can compute the index of the enqueue whose argument is the response to a given dequeue in constant time.

In our case of implementing a queue, a process only needs to compute the rank of a `Dequeue` and get an `Enqueue` with a specific rank. We know we can linearize operations in a block in any order; here, we choose to put `Enqueue` operations in a block before `Dequeue` operations. Consider the following operations, where operations in a cell are concurrent.

`Deq`	`Enq(5)`, `Enq(2)`, `Enq(1)`, `Deq`	`Enq(3)`, `Deq`	`Enq(4)`, `Deq`, `Deq`, `Deq`, `Deq`

The `Dequeue` operations return `null, 5, 2, 1, 3, 4, null`, respectively. Now, we claimed that by knowing the size of the queue, we can compute the rank of the required `Enqueue` for any non-null `Dequeue`. We apply this approach to blocks; if we store the size of the queue after each block of operations, we can compute the index of each `Dequeue`'s result in O(1) steps.

	`Deq`	`Enq(5)`, `Enq(2)`, `Enq(1)`, `Deq`	`Enq(3)`, `Deq`	`Enq(4)`, `Deq`, `Deq`, `Deq`, `Deq`
#`Enqs`	0	3	1	1
#`Deqs`	1	1	1	4
Size at end	0	2	2	0

Table 1: Augmented history of operation blocks on the queue.

The size of the queue after the bth block in the root could be computed as

$$\max\Big(\text{size after } b-1\text{th block} + \#\texttt{Enqueues} \text{ in } b\text{th block} - \#\texttt{Dequeues} \text{ in } b\text{th block}, 0\Big).$$

Moreover, the total number of non-null dequeues in blocks $1, 2, ..., b$ in the root is

$$\sum_{i=1}^{b} \#\texttt{Enqueues} \text{ in } i\text{th block} - \text{size after } b\text{th block}.$$

Given a `Dequeue` is in block B, its response is the argument of the `Enqueue` whose rank is the number of non-null `Dequeue`s in blocks $1, 2, ..., b-1+$ index of the `Dequeue` in B's `Dequeue`s, if $\big($size of the queue after $b-1$th block $+ \#$`Enqueues` in bth block $- \#$index of `Dequeue` in B's `Dequeues`$\big) \geq 0$. Otherwise the response would be `null`.

3.1 Details of the Implementation

Pseudocode for the queue implementation is given in Section ??. It uses the following two types of objects.

Node In each Node we store pointers to its parent and its left and right child, an array of Blocks called blocks and the index head of the first empty entry in blocks.

Block The information stored in a Block depends on whether the Block is in an internal node or a leaf. If it is in a leaf, we use a LeafBlock which simply stores one operation. If a block B is in an internal node n, then it contains subblocks in the left and right children of n. The left subblocks of B are some consecutive blocks in the left child of n starting from where the left subblocks of the block prior to B ended. The right subblocks of B are also defined similar. In each block we store four essential fields that implicitly summarize which operations are in the block $\texttt{sum}_{\texttt{enq-left}}$, $\texttt{sum}_{\texttt{deq-left}}$, $\texttt{sum}_{\texttt{enq-right}}$, $\texttt{sum}_{\texttt{deq-right}}$. The $\texttt{sum}_{\texttt{enq-left}}$ field is the total number of Enqueue operations in the blocks before the last subblock of B in the left child. The other fields' semantics are similar. The $\texttt{end}_{\texttt{left}}$ and $\texttt{end}_{\texttt{right}}$ field store the index of the last subblock of a block in the left and the right child, respectively. The approximate index of the superblock of non-root blocks is stored in their super field. The size field in a block in the root node stores the size of the queue after the operations in the block have been performed.

EnQueue(e) An Enqueue operation does not return a response, so it is sufficient to just propagate the Enqueue operation to the root and then use its position in the linearization for future Dequeue operations. Enqueue(e) creates a LeafBlock with element $= e$, sets its $\texttt{sum}_{\texttt{enq}}$ and $\texttt{sum}_{\texttt{deq}}$ fields and then appends it to the tree.

DeQueue() Dequeue creates a LeafBlock, sets its $\texttt{sum}_{\texttt{enq}}$ and $\texttt{sum}_{\texttt{deq}}$ fields and appends it to the tree. Then, it computes the position of the appended Dequeue operation in the root using IndexDequeue and after that finds the response of the Dequeue by calling FindResponse.

FindResPonse(b, i) To compute the response of the ith Dequeue in the bth block of the root Line ?? computes whether the queue is empty or not. If there are more Dequeues than Enqueues the queue would become empty before the requested Dequeue. If the queue is not empty, Line ?? computes the rank e of the Enqueue whose argument is the response to the Dequeue. Knowing the response is the eth Enqueue in the root (which is before the bth block) we find the block and position containing the `Enqueue` operation using `DSearch` and after that `GetEnqueue` finds its `element`.

`APPend(`B`)` The `head` field is the index of the first empty slot in `blocks` in a `LeafBlock`. There are no multiple write accesses on `head` and `blocks` in a leaf because only the process that the leaf belongs to appends to it. `Append(`B`)` adds B to the end of the `blocks` field in the leaf, increments `head` and then calls `Propagate` on the leaf's `parent`. When `Propagate` terminates it is guaranteed that the appended block is a subblock of a block in the `root`.

`ProPagate()` `Propagate` on node n uses the double refresh idea described earlier and invokes two `Refresh`es on n in Lines **??** and **??**. Then, it invokes `Propagate` on n`.parent` recursively until it reaches the root.

`Refresh()` and `Advance()` The goal of a `Refresh` on node n is to create a block of n's children's new blocks and append it to n`.blocks`. The variable `h` is read from n`.head` at Line **??**. The new block created by `Refresh` will be inserted into n`.blocks[h]`. Lines **??**–**??** of n`.Refresh` help to `Advance` n's children. `Advance` increments the children's `head` if necessary and sets the `super` field of their most recent appended blocks. The reason behind this helping is explained later when we discuss `IndexDequeue`. After helping to `Advance` the children, a new block called `new` is created in Line **??**. Then, if `new` is empty, `Refresh` returns `true` because there are no new operations to propagate and it is unnecessary to add an empty block to the tree. Later we will use the fact that all blocks contain at least one operation. Line **??** tries to install `new`. If it was successful all is good. If not, it means someone else has already put a block in n`.blocks[h]`. In this case, `Refresh` helps advance n`.head` to `h+1` and update the `super` field of n`.blocks[h]` at Line **??**.

`CreateBlock()` n`.CreateBlock(`h`)` is used by `Refresh` to construct a block containing new operations of n's children. The block `new` is created in Line **??** and its fields are filled similarly for both left and right directions. The variable $\texttt{index}_{\texttt{prev}}$ is the index of the block preceding the first subblock in the child in direction `dir` that is aggregated into `new`. Field $\texttt{new.end}_{\texttt{dir}}$ stores the index of the rightmost subblock of `new` in the child. Then $\texttt{sum}_{\texttt{enq-dir}}$ is computed from the sum of the number of `Enqueue` operations in the `new` block from direction `dir` and the value stored in $n$$\texttt{.blocks[h-1].sum}_{\texttt{enq-dir}}$. The field $\texttt{sum}_{\texttt{deq-dir}}$ is computed similarly. Then, if `new` is going to be installed in the `root`, the `size` field is also computed.

`GetEnqueue(`b, i`)` and `DSearch(`e, end`)` We can describe an operation in a node in two ways: the rank of the operation among all the operations in the node or the index of the block containing the operation in the node and the rank of the operation within that block. If we know the block and rank within the block of an operation we can find the subblock containing the operation and the operation's rank within that subblock in poly-log time. To find the response of a `Dequeue`, we know about the rank of the response `Enqueue` in the root (`e` in Line **??**). We also know the `e`th `Enqueue` is in `root.blocks[1..end]`. `DSearch` uses doubling to find the range that contains the answer block (Lines **??**–**??**) and then tries to find the required indices with a binary search (Line **??**). A call to n`.GetEnqueue(`b, i`)` returns the `element` of the ith enqueue in bth the block of n. The range of subblocks of a block is determined using the $\texttt{end}_{\texttt{left}}$ and $\texttt{end}_{\texttt{right}}$ fields of the block and its previous block. Then, the subblock is found using binary search on the $\texttt{sum}_{\texttt{enq}}$ field (Lines **??** and **??**).

`IndexDequeue(`b, i`)` A call to n`.IndexDequeue(`b, i`)` computes the block and the rank within the block in the root of the ith `Dequeue` of the bth block of n. Let R_n be the successful `Refresh` on node n that did a successful `CAS(null,` B`)` into `n.blocks[`b`]`. Let par be n`.parent`. Without loss of generality assume for the rest of this section that n is the left child of par. Let R_{par} be the first successful par`.Refresh` that reads some value greater than b for `left.head` and therefore contains B in its created block in Line **??**. Let j be the index of the block that R_{par} puts in par`.blocks`.

Since the index of the superblock of B is not known until B is propagated, R_n cannot set the `super` field of B while creating it. One approach for R_{par} is to set the `super` field of B after propagating B to par. This solution would not be efficient because there might be up to p subblocks that R_{par} propagated, which need to update their `super` field. However, intuitively, once B is installed, its superblock is going to be close to n`.parent.head` at the time of installation. If we know the approximate position of the superblock of B then we can search for the real superblock when it is needed. Thus, B`.super` does not have to be the exact location of the superblock of B, but we want it to be close to j. We can set B`.super` to par`.head` while creating B, but the problem is that there might be many `Refresh`es on par that could happen after R_n reads par`.head` and before propagating B to par. If R_n sets B`.super` to par`.head` after appending B to n`.blocks` (Line **??**), R_n might go to sleep at some time after installing B and before setting B`.super`. In this case, the next `Refresh`es on n and par help fill in the value of B`.super`.

Block B is appended to n`.blocks[`b`]` on Line **??**. After appending B, B`.super` is set on Line **??** of a call to `Advance` from n`.Refresh` by the same process or another process or by Line **??** of a n`.parent.Refresh`.

We shall show that this is sufficient to ensure that B`.super` differs from the index of B's superblock by at most 1.

3.2 Pseudocode

Algorithm Tree Fields Description

◇ *Shared*

- A binary tree of `Nodes` with one `leaf` for each process. `root` is the root `node`.

◇ *Local*

- *Node* `leaf`: process's leaf in the tree.

▶ *Node*

- **Node* `left, right, parent` : Initialized when creating the tree.
- *Block[]* `blocks` : Initially `blocks[0]` contains an empty block with all fields equal to 0.
- *int* `head= 1`: #blocks in `blocks`. `blocks[0]` is a block with all integer fields equal to zero.

▶ *Block*

- *int* `super` : approximate index of the superblock, read from `parent.head` when appending the block to the node

▶ *InternalBlock* extends *Block*

- *int* $\texttt{end}_{\texttt{left}}$, $\texttt{end}_{\texttt{right}}$: indices of the last subblock of the block in the left and right child
- *int* $\texttt{sum}_{\texttt{enq-left}}$: # enqueues in $\texttt{left.blocks[1..end}_{\texttt{left}}\texttt{]}$
- *int* $\texttt{sum}_{\texttt{deq-left}}$: # dequeues in $\texttt{left.blocks[1..end}_{\texttt{left}}\texttt{]}$
- *int* $\texttt{sum}_{\texttt{enq-right}}$: # enqueues in $\texttt{right.blocks[1..end}_{\texttt{right}}\texttt{]}$
- *int* $\texttt{sum}_{\texttt{deq-right}}$: # dequeues in $\texttt{right.blocks[1..end}_{\texttt{right}}\texttt{]}$

▶ *LeafBlock* extends *Block*

- *Object* `element` : Each block in a leaf represents a single operation. If the operation is `enqueue(x)` then `element=x`, otherwise `element=null`.
- *int* $\texttt{sum}_{\texttt{enq}}$, $\texttt{sum}_{\texttt{deq}}$: # enqueue, dequeue operations in the prefix for the block

▶ *RootBlock* extends *InternalBlock*

- *int* `size` : size of the queue after performing all operations in the prefix for this block

Abbreviations used in the code and the proof of correctness.

- $\texttt{blocks[b].sum}_{\texttt{x}}\texttt{=blocks[b].sum}_{\texttt{x-left}}\texttt{+blocks[b].sum}_{\texttt{x-right}}$ (for internal blocks where $\texttt{b}\geq 0$ and $\texttt{x} \in \{\texttt{enq, deq}\}$)
- $\texttt{blocks[b].num}_{\texttt{x}}\texttt{=blocks[b].sum}_{\texttt{x}}\texttt{-blocks[b-1].sum}_{\texttt{x}}$
  (for all blocks where b>0 and $\texttt{x} \in \{$ enq, deq, enq-left, enq-right, deq-left, deq-right$\}$)

Algorithm *Queue*

```
void Enqueue(Object e)                                   ▷ Creates a block with element e and adds it to the tree.
    block newBlock= new(LeafBlock)
    newBlock.element= e
    newBlock.$sum_{enq}$= leaf.blocks[leaf.head].$sum_{enq}$+1
    newBlock.$sum_{deq}$= leaf.blocks[leaf.head].$sum_{deq}$
    leaf.Append(newBlock)
end Enqueue

Object Dequeue()            ▷ Creates a block with null value element, appends it to the tree and returns its response.
    block newBlock= new(LeafBlock)
    newBlock.element= null
    newBlock.$sum_{enq}$= leaf.blocks[leaf.head].$sum_{enq}$
    newBlock.$sum_{deq}$= leaf.blocks[leaf.head].$sum_{deq}$+1
    leaf.Append(newBlock)
    <b, i>= IndexDequeue(leaf.head, 1)
    output= FindResponse(b, i)
    return output
end Dequeue

element FindResponse(int b, int i)          ▷ Returns the response to $D_i(root, b)$, the ith Dequeue in root.blocks[b].
    if root.blocks[b-1].size + root.blocks[b].$num_{enq}$ - i $<$ 0 then                 ▷ Check if the queue is empty.
        return null
    else
        e= i + (root.blocks[b-1].$sum_{enq}$-root.blocks[b-1].size)   ▷ The response is $E_e(root)$, the eth Enqueue in the root.
        return root.GetEnqueue(root.DSearch(e, b))
    end if
end FindResponse
```

Algorithm Root

```
    ⇝ Precondition: root.blocks[end].sum$_{enq}$ $\geq$ e
    ▷ Returns <b,i> such that $E_e(root) = E_i(root, b)$, i.e., the eth Enqueue in the root is the ith Enqueue within block b of the root.
26: <int, int> DSearch(int e, int end)
27:     start= end-1
28:     while root.blocks[start].sum$_{enq}$$\geq$e do
29:         start= max(start-(end-start), 0)
30:     end while
31:     b= root.BinarySearch( e, start, end)
32:     i= e- root.blocks[b-1].sum$_{enq}$
33:     return <b,i>
34: end DSearch
```

Algorithm Leaf

```
35: void Append(block B)                                    ▷ Only called by the owner of the leaf.
36:     blocks[head]= B
37:     head+=1
38:     parent.Propagate()
39: end Append
```

Algorithm *Node*

```
    ▷ n.Propagate propagates operations in this.children up to this when it terminates.
40: void Propagate()
41:     if not Refresh() then
42:         Refresh()
43:     end if
44:     if this is not root then
45:         parent.Propagate()
46:     end if
47: end Propagate

    ▷ Creates a block containing new operations of this.children, and then tries to append it to this.
48: boolean Refresh()
49:     h= head
50:     for each dir in {left, right} do
51:         $h_{dir}$= dir.head
52:         if dir.blocks[$h_{dir}$]!=null then
53:             dir.Advance($h_{dir}$)
54:         end if
55:     end for
56:     new= CreateBlock(h)
57:     if new.num==0 then return true
58:     end if
59:     result= blocks[h].CAS(null, new)
60:     this.Advance(h)
61:     return result
62: end Refresh

63: void Advance(int h)
64:     $h_p$= parent.head
65:     blocks[h].super.CAS(null, $h_p$)
66:     head.CAS(h, h+1)
67: end Advance

    ⇝ Precondition: blocks[start..end] contains a block with $sum_{enq}$ greater than or equal to i
    ▷ Does a binary search for the value i of $sum_{enq}$ field. Returns the index of the leftmost block in blocks[start..end] whose $sum_{enq}$ is $\geq$ i.
68: int BinarySearch( int i, int start, int end)
69:     return min{j: blocks[j].$sum_{enq} \geq$ i}
70: end BinarySearch

    ▷ Creates and returns the block to be installed in blocks[i]. Created block includes left.blocks[$index_{prev}$+1..$index_{last}$] and right.blocks[$index_{prev}$+1..$index_{last}$].
71: Block CreateBlock(int i)
72:     block new= new(InternalBlock)
73:     for each dir in {left, right} do
74:         $index_{prev}$= blocks[i-1].$end_{dir}$
75:         new.$end_{dir}$= dir.head-1
76:         $block_{prev}$= dir.blocks[$index_{prev}$]
77:         $block_{last}$= dir.blocks[new.$end_{dir}$]
78:         new.$sum_{enq-dir}$= blocks[i-1].$sum_{enq-dir}$ +
                        $block_{last}$.$sum_{enq}$ - $block_{prev}$.$sum_{enq}$
79:         new.$sum_{deq-dir}$= blocks[i-1].$sum_{deq-dir}$ +
                        $block_{last}$.$sum_{deq}$ - $block_{prev}$.$sum_{deq}$
80:     end for
81:     if this is root then
82:         new.size = max(root.blocks[i-1].size + new.$num_{enq}$
                    - new.$num_{deq}$, 0)
83:     end if
84:     return new
85: end CreateBlock
```

Algorithm Node

```
    ⇝ Precondition: blocks[b].num_enq ≥ i ≥ 1
86: element GetEnqueue(int b, int i)                      ▷ Returns the element of $E_i(this, b)$.
87:     if this is leaf then
88:         return blocks[b].element
89:     else if i ≤ blocks[b].num_enq-left then            ▷ $E_i(this, b)$ is in the left child of this node.
90:         subBlock= left.BinarySearch( i+blocks[b-1].sum_enq-left, blocks[b-1].end_left+1, blocks[b].end_left)
91:         return left.GetEnqueue(subBlock, i)
92:     else
93:         i= i-blocks[b].num_enq-left
94:         subBlock= right.BinarySearch( i+blocks[b-1].sum_enq-right, blocks[b-1].end_right+1, blocks[b].end_right)
95:         return right.GetEnqueue(subBlock, i)
96:     end if
97: end GetEnqueue

    ⇝ Precondition: bth block of the node has propagated up to the root and blocks[b].num_deq ≥ i.
98: <int, int> IndexDequeue(int b, int i)                  ▷ Returns <x, y> if $D_i(this, b) = D_y(root, x)$.
99:     if this is root then
100:        return <b, i>
101:    else
102:        dir= (parent.left==n ? left: right)
103:        sb= (parent.blocks[blocks[b].super].sum_deq-dir > blocks[b].sum_deq ? blocks[b].super: blocks[b].super+1)
104:        if dir is left then
105:            i+= blocks[b-1].sum_deq-parent.blocks[sb-1].sum_deq-left
106:        else
107:            i+= blocks[b-1].sum_deq-parent.blocks[sb-1].sum_deq-right
108:            i+= parent.blocks[sb].num_deq-left
109:        end if
110:        return this.parent.IndexDequeue(sb, i)
111:    end if
112: end IndexDequeue
```

4 Proof of Correctness

We adopt linearizability as our definition of correctness. In our case, where we create the linearization ordering in the root, we need to prove (1) the ordering is legal, i.e, for every execution on our queue if operation op_1 terminates before operation op_2 then op_1 is linearized before operation op_2 and (2) if we do operations sequentially in their the linearization order, operations get the same results as in our queue. The proof is structured like this. First, we define and prove some facts about blocks and the node's `head` field. Then, we introduce the linearization ordering formally. Next, we prove double `Refresh` on a node is enough to propagate its children's new operations up to the node, which is used to prove (1). After this, we prove some claims about the size and operations of each block, which we use to prove the correctness of `DSearch()`, `GetEnqueue()` and `IndexDequeue()`. Finally, we prove the correctness of the way we compute the response of a dequeue, which establishes (2).

4.1 Basic Properties

In this subsection we talk about some properties of blocks and fields of the tree nodes.

A block is an object storing some statistics, as described in Algorithm Queue. A block in a node implicitly represents a set of operations.

Definition 1 (Ordering of a block in a node)**.** Let b be n`.blocks[`i`]` and b' be n`.blocks[`j`]`. We call i the *index* of block b. Block b is *before* block b' in node n if and only if $i < j$. We define *the prefix for* block b in node n to be the blocks in n`.blocks[0..`i`]`.

Next, we show that the value of `head` in a node can only be increased. By the termination of a `Refresh`, `head` has been incremented by the process doing the `Refresh` or by another process.

Observation 2. *For each node n, n*`.head` *is non-decreasing over time.*

Proof. The claim follows trivially from the code since `head` is only changed by incrementing in Line **??** of `Advance`. ☐

Lemma 3. *Let R be an instance of* `Refresh` *on a node n that its created block* `new` *is not empty (*`new.sum` $\neq$ 0*). After R terminates, n*`.head` *is greater than the value read in line* **??** *of R.*

Proof. If the `CAS` in Line **??** is successful then the claim holds. Otherwise n`.head` has changed from the value that was read in Line **??**. By Observation **??** this means another process has incremented n`.head`. ☐

Now we show n.`blocks[`n.`head]` is either the last block written into node n or the first empty block in n.

Invariant 4 (headPosition)**.** If the value of n.`head` is h then n.`blocks[`i`]` $=$ `null` for $i > h$ and n.`blocks[`i`]` $\neq$ `null` for $0 \leq i < h$.

Proof. Initially the invariant is true since n.`head` $= 1$, n.`blocks[0]` $\neq$ `null` and n.`blocks[`x`]` $=$ `null` for every $x > 0$. The truth of the invariant may be affected by writing into `n.blocks` or incrementing `n.head`. We show that if the invariant holds before such a change then it still holds after the change.

In the algorithm, n.`blocks` is modified only on Line **??**, which updates n.`blocks[`h`]` where h is the value read from n.`head` in Line **??**. Since the `CAS` in Line **??** is successful it means n.`head` has not changed from h before doing the `CAS`: if n.`head` had changed before the `CAS` then it would be greater than h by Observation **??** and hence n.`blocks[`h`]`$\neq$`null` and by the induction hypothesis, so the `CAS` would fail. Writing into n.`blocks[`h`]` when $h = n$.`head` preserves the invariant, since the claim does not talk about the content of n.`blocks[`n.`head]`.

The value of `n.head` is modified only in Line **??** of `Advance`. If n.`head` is incremented to h+1 it is sufficient to show `n.blocks[`h`]`$\neq$`null`. `Advance` is called in Lines **??** and **??**. For Line **??**, n.`blocks[h]` $\neq$ `null` because of the `if` condition in Line **??**. For Line **??**, Line **??** was finished before doing **??**. Whether Line **??** is successful or not, n.`blocks[h]` $\neq$ `null` after the n.`blocks[`h`].CAS`. □

We define the subblocks of a block recursively.

Definition 5 (Subblock)**.** A block is a *direct subblock* of the ith block in node n if it is in

$$n\texttt{.left.blocks[}n\texttt{.blocks[}i-1\texttt{].end}_{\texttt{left}}\texttt{+1}\cdots n\texttt{.blocks[}i\texttt{].end}_{\texttt{left}}\texttt{]}$$

or in

$$n\texttt{.right.blocks[}n\texttt{.blocks[}i-1\texttt{].end}_{\texttt{right}}\texttt{+1}\cdots n\texttt{.blocks[}i\texttt{].end}_{\texttt{right}}\texttt{]}.$$

Block b is a *subblock* of block c if b is a direct subblock of c or a subblock of a direct subblock of c. We say block b is *propagated* to node n if b is in *n.blocks* or is a subblock of a block in n.`blocks`.

The next lemma is used to prove that subblocks of two blocks in a node are disjoint.

Lemma 6. *If* n.`blocks[`i`]` $\neq$ `null` *and* $i > 0$ *then* n.`blocks[`i`].end`$_{\texttt{left}}$ $\geq$ n.`blocks[`$i-1$`].end`$_{\texttt{left}}$ *and* n.`blocks[`i`].end`$_{\texttt{right}}$ $\geq$ n.`blocks[`$i-1$`].end`$_{\texttt{right}}$.

Proof. Consider the block b written into `n.blocks[`i`]` by `CAS` at Line **??**. Block b is created by the `CreateBlock(`i`)` called at Line **??**. Prior to this call to `CreateBlock(`i`)`, `n.head` $= i$ at Line **??**, so `n.blocks[`$i-1$`]` is already a non-null value b' by Invariant **??**. Thus, the `CreateBlock(`$i-1$`)` that created b' terminated before the `CreateBlock(`i`)` that creates b is invoked. The value written into b`.end`$_{\texttt{left}}$ at Line **??** of `CreateBlock(`i`)` was one less than the value of n`.left.head` read at Line **??** of `CreateBlock(`i`)`. Similarly, the value in `n.blocks[`$i-1$`].end`$_{\texttt{left}}$ was one less than the value read from `n.left.head` during the call to `CreateBlock(`$i-1$`)`. By Observation **??**, n`.left.head` is non-decreasing, so b'`.end`$_{\texttt{left}} \leq b$`.end`$_{\texttt{left}}$. The proof for `end`$_{\texttt{right}}$ is similar. ☐

Lemma 7. *Subblocks of any two blocks in node n do not overlap.*

Proof. We are going to prove the lemma by contradiction. Consider the lowest `node` n in the tree that violates the claim. Then subblocks of n`.blocks[`i`]` and n`.blocks[`j`]` overlap for some $i < j$. Since n is the lowest node in the tree violating the claim, direct subblocks of blocks of n`.blocks[`i`]` and n`.blocks[`j`]` have to overlap. Without loss of generality assume left child subblocks of n`.blocks[`i`]` overlap with the left child subblocks of n`.blocks[`j`]`. By Lemma **??** we have n`.blocks[`i`].end`$_{\texttt{left}} \leq n$`.blocks[`$j-1$`].end`$_{\texttt{left}}$, so the ranges $[n$`.blocks[`$i-1$`].end`$_{\texttt{left}}+1 \cdots n$`.blocks[`$i$`].end`$_{\texttt{left}}]$ and $[n$`.blocks[`$j-1$`].end`$_{\texttt{left}}+1 \cdots n$`.blocks[`$j$`].end`$_{\texttt{left}}]$ cannot overlap. Therefore, direct subblocks of n`.blocks[`i`]` and n`.blocks[`j`]` cannot overlap. ☐

Definition 8 (Superblock)**.** Block b is *superblock* of block c if c is a direct subblock of b.

Corollary 9. *Every block has at most one superblock.*

Proof. A block having more than one superblock contradicts Lemma **??**. ☐

Now we can define the operations of a block using the definition of subblocks.

Definition 10 (Operations of a block)**.** A block b in a leaf represents an `Enqueue` if b`.element` $\neq$ `null`. Otherwise, if b`.element` $=$ `null`, b represents a `Dequeue`. The set of operations of block b is the union of the operations in leaf subblocks of b. We denote the set of operations of block b by *ops*(b) and the union of operations of a set of blocks B by *ops*(B). We also say b contains *op* if *op* $\in$ *ops*(b).

The next lemma proves that each operation appears at most once in the blocks of a node.

Lemma 11. *If op is in n.blocks$[i]$ then there is no $j \neq i$ such that op is in n.blocks$[j]$.*

Proof. We prove this claim by contradiction using Lemma **??**. Assume *op* is in the subblocks of both $n.blocks[i]$ and $n.blocks[j]$. From Lemma **??** we know that the subblocks of these blocks are different, so there are two leaf blocks containing *op*. Since each process puts each operation in only one block of its leaf then *op* cannot be in two leaf blocks. This is a contradiction. □

Definition 12. n`.blocks[`i`]` is *established* if n`.head` $> i$. An operation is *established* in node n if it is in an established block of n. EST_t^n is the set of established operations in node n at time t.

Now we want to say that blocks of a node grow over time.

Observation 13. *If time $t <$ time t' (t is before t'), then $ops(n.blocks)$ at time t is a subset of $ops(n.blocks)$ at time t'.*

Proof. Blocks are only appended (not modified) with `CAS` to n`.blocks[`n`.head]`, so the set of blocks of a node after the `CAS` contains the the set of blocks before the `CAS`. □

Corollary 14. *If time $t <$ time t', then $EST_n^t \subseteq EST_n^{t'}$.*

Proof. From Observations **??**, **??**. □

4.2 Ordering Operations

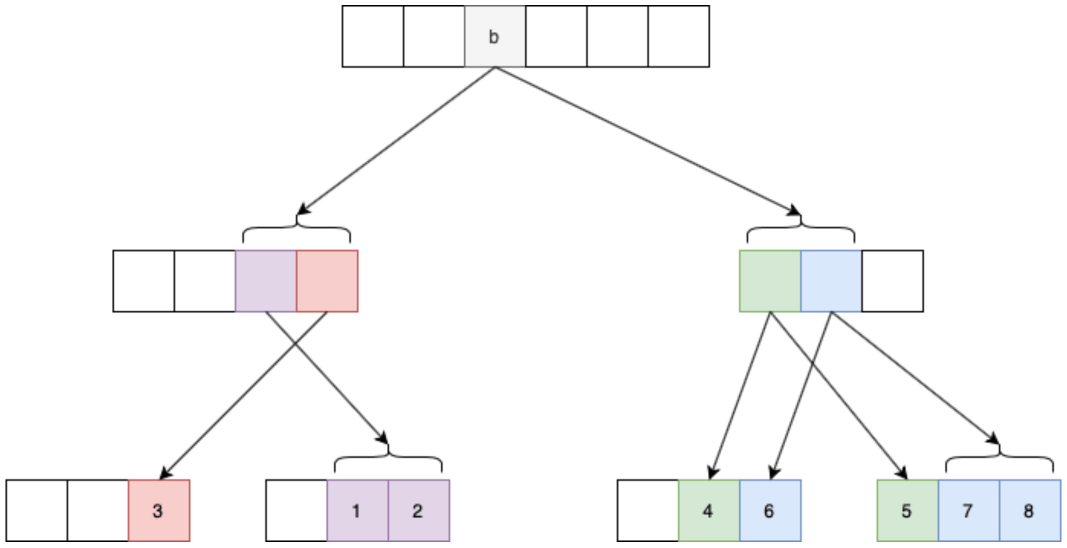


Figure 12: Order of operations in b. Operations in the leaves are ordered with numerical order shown in the drawing.

Now we define the ordering of operations stored in each node. In the non-root nodes we only need to order operations of a type among themselves. Processes are numbered from 1 to p and leaves of the tree are assigned from left to right. We will show in Lemma **??** that there is at most one operation from each process in a given block.

Definition 15 (Ordering of operations inside the nodes)**.**

- $E(n,b)$ is the sequence of enqueue operations in $ops(n\texttt{.blocks[}b\texttt{]})$ defined recursively as follows. $E(leaf,b)$ is the single enqueue operation in $ops(leaf\texttt{.blocks[}b\texttt{]})$ or an empty sequence if $leaf\texttt{.blocks[}b\texttt{]}$ represents a dequeue operation. If n is an internal node, then

$$\begin{aligned}E(n,b) =& E(n\texttt{.left}, n\texttt{.blocks[}b-1\texttt{].end}_{\texttt{left}}+1)\cdots E(n\texttt{.left}, n\texttt{.blocks[}b\texttt{].end}_{\texttt{left}})\cdot\\ & E(n\texttt{.right}, n\texttt{.blocks[}b-1\texttt{].end}_{\texttt{right}}+1)\cdots E(n\texttt{.right}, n\texttt{.blocks[}b\texttt{].end}_{\texttt{right}}).\end{aligned}$$

- $E_i(n,b)$ is the ith enqueue in $E(n,b)$.
- The order of the enqueue operations in the node n is $E(n)=E(n,1)\cdot E(n,2)\cdot E(n,3)\cdots$
- $E_i(n)$ is the ith enqueue in $E(n)$.

- $D(n, b)$ is the sequence of dequeue operations in $\mathit{ops}(n\texttt{.blocks[}b\texttt{]})$ defined recursively as follows. $D(\mathit{leaf}, b)$ is the single dequeue operation in $\mathit{ops}(\mathit{leaf}\texttt{.blocks[}b\texttt{]})$ or an empty sequence if $\mathit{leaf}\texttt{.blocks[}b\texttt{]}$ represents an enqueue operation. If n is an internal node, then

$$\begin{aligned} D(n,b) = & D(n\texttt{.left}, n\texttt{.blocks[}b-1\texttt{].end}_{\texttt{left}}+1)\cdots D(n\texttt{.left}, n\texttt{.blocks[}b\texttt{].end}_{\texttt{left}})\cdot \\ & D(n\texttt{.right}, n\texttt{.blocks[}b-1\texttt{].end}_{\texttt{right}}+1)\cdots D(n\texttt{.right}, n\texttt{.blocks[}b\texttt{].end}_{\texttt{right}}). \end{aligned}$$

- $D_i(n, b)$ is the ith enqueue in $D(n, b)$.
- The order of the dequeue operations in the node n is $D(n) = D(n, 1) \cdot D(n, 2) \cdot D(n, 3)$...
- $D_i(n)$ is the ith dequeue in $D(n)$.

The linearization ordering is given by the order in which operations appear in the blocks in the root.

Definition 16 (Linearization)**.**

$$L = E(\mathit{root}, 1) \cdot D(\mathit{root}, 1) \cdot E(\mathit{root}, 2) \cdot D(\mathit{root}, 2) \cdot E(\mathit{root}, 3) \cdot D(\mathit{root}, 3) \cdots$$

Observation 17. *For any node n and indices $i < j$ of* `blocks` *in in, we have*

$$n\texttt{.blocks[}j\texttt{].sum}_{\texttt{x}} - n\texttt{.blocks[}i\texttt{].sum}_{\texttt{x}} = \sum_{k=i+1}^{j} n\texttt{.blocks[}k\texttt{].num}_{\texttt{x}}$$

where x *in* $\{\texttt{enq}, \texttt{deq}, \texttt{enq-left}, \texttt{enq-right}, \texttt{deq-left}, \texttt{deq-right}\}$.

The next claim is also true if we replace `enq` with `deq` and E with D.

Lemma 18. *Let B, B' be $n\texttt{.blocks[}b\texttt{]}$, $n\texttt{.blocks[}b-1\texttt{]}$ respectively.*

(1) If n is an internal node then $B\texttt{.num}_{\texttt{enq-left}} = \Big|E(n\texttt{.left}, B'\texttt{.end}_{\texttt{left}}+1)\cdots E(n\texttt{.left}, B\texttt{.end}_{\texttt{left}})\Big|$.

(2) If n is an internal node then $B\texttt{.num}_{\texttt{enq-right}} = \Big|E(n\texttt{.right}, B'\texttt{.end}_{\texttt{right}}+1)\cdots E(n\texttt{.right}, B\texttt{.end}_{\texttt{right}})\Big|$.

(3) $B\texttt{.num}_{\texttt{enq}} = \Big|E(n, b)\Big|$.

Proof. We prove the claim by induction on the height of node n. Base case (3) for leaves is trivial and (1) and (2) are vacuously true for leaves. Supposing the claim is true for n's children, we prove the correctness of the claim for n.

$$\begin{aligned}
B\texttt{.num}_{\texttt{enq-left}} =& B.\texttt{sum}_{\texttt{enq-left}} - B'.\texttt{sum}_{\texttt{enq-left}} && \text{Definition of } \texttt{num}_{\texttt{enq}} \\
=& B'.\texttt{sum}_{\texttt{enq-left}} + n\texttt{.left.blocks[}B\texttt{.end}_{\texttt{left}}\texttt{].sum}_{\texttt{enq}} \\
& - n\texttt{.left.blocks[}B'\texttt{.end}_{\texttt{left}}\texttt{].sum}_{\texttt{enq}} - B'.\texttt{sum}_{\texttt{enq-left}} && \texttt{CreateBlock} \\
=& n.\texttt{left.blocks[}B\texttt{.end}_{\texttt{left}}\texttt{].sum}_{\texttt{enq}} - n.\texttt{left.blocks[}B'\texttt{.end}_{\texttt{left}}\texttt{].sum}_{\texttt{enq}} \\
=& \sum_{i=B'\texttt{.end}_{\texttt{left}}+1}^{B\texttt{.end}_{\texttt{left}}} n\texttt{.left.blocks[}i\texttt{].num}_{\texttt{enq}} && \text{Observation ??} \\
=& \Big| E(n.\texttt{left}, B'.\texttt{end}_{\texttt{left}}+1) \cdots E(n.\texttt{left}, B.\texttt{end}_{\texttt{left}}) \Big| && \text{Induction hypothesis (3)}
\end{aligned}$$

The last line holds because of the induction hypothesis (3). (2) is similar to (1). Now we prove (3) starting from the Definition of $E(n, b)$.

$$\begin{aligned}
E(n, b) =& E(n\texttt{.left}, n\texttt{.blocks[}b-1\texttt{].end}_{\texttt{left}}+1) \cdots E(n\texttt{.left}, n\texttt{.blocks[}b\texttt{].end}_{\texttt{left}}) \cdot \\
& E(n\texttt{.right}, n\texttt{.blocks[}b-1\texttt{].end}_{\texttt{right}}+1) \cdots E(n\texttt{.right}, n\texttt{.blocks[}b\texttt{].end}_{\texttt{right}}).
\end{aligned}$$

By (1) and (2) we have $\Big|E(n, b)\Big| = B\texttt{.num}_{\texttt{enq-left}} + B\texttt{.num}_{\texttt{enq-right}} = B\texttt{.num}_{\texttt{enq}}$. □

The next claim is also true if we replace `enq` with `deq` and E with D.

Corollary 19. *Let B be n*`.blocks[`b`]`*.*

(1) If n is an internal node then $B\texttt{.sum}_{\texttt{enq-left}} = \Big|E(n.\texttt{left}, 1) \cdots E(n.\texttt{left}, B.\texttt{end}_{\texttt{left}})\Big|$

(2) If n is an internal node then $B\texttt{.sum}_{\texttt{enq-right}} = \Big|E(n.\texttt{right}, 1) \cdots E(n.\texttt{right}, B.\texttt{end}_{\texttt{right}})\Big|$

(3) $B\texttt{.sum}_{\texttt{enq}} = \Big|E(n, 1) \cdot E(n, 2) \cdots E(n, b)\Big|$

4.3 Propagating Operations to the Root

In this section we explain why two `Refresh`es are enough to propagate a node's operations to its parent.

Definition 20. Let t^{op} be the time *op* is invoked, ${}^{op}t$ be the time *op* terminates, t^{op}_{l} be the time immediately before running Line l of operation *op* and ${}^{op}_{l}t$ be the time immediately after running Line l of operation *op*. We sometimes suppress *op* and write t_l or ${}_lt$ if *op* is clear from the context. In the text v_l is the value of variable `v` immediately after line l for the process we are talking about and v_t is the value of variable `v` at time t.

Definition 21 (Successful Refresh)**.** An instance of `Refresh` is *successful* if its `CAS` in Line **??** returns `true`. If a successful instance of `Refresh` terminates, we say it is *complete.*

In the next two results we show for every successful `Refresh`, all the operations established in the children before the `Refresh` are in the parent after the `Refresh`'s successful `CAS` at Line **??**.

Lemma 22. *If R is a successful instance of n*`.Refresh`*, then we have $EST^{t^R}_{n.\texttt{left}} \cup EST^{t^R}_{n.\texttt{right}} \subseteq ops(n.\texttt{blocks}_{??})$.*

Proof. We show

$$EST^{t^R}_{n.\texttt{left}} = ops(n.\texttt{left.blocks[0..n.left.head}_{t^R} - 1\texttt{]})$$
$$\subseteq ops(n.\texttt{blocks}_{??}) = ops(n.\texttt{blocks[0..}n.\texttt{head}_{??}\texttt{]}).$$

In every node, `blocks[0]` is an empty block without any operations. Line **??** stores a block `new` in n that has $\texttt{end}_{\texttt{left}} = n.\texttt{left.head}_{??} - 1$. Therefore, by Definition **??**, after the successful `CAS` in Line **??** we know all blocks in $n.\texttt{left.blocks[}1\cdots n.\texttt{left.head}_{??} - 1\texttt{]}$ are subblocks of $n.\texttt{blocks[}1\cdots n.\texttt{head}_{??}\texttt{]}$. Because of Observation **??** we have $n.\texttt{left.head}_{t^R} - 1 \leq n.\texttt{left.head}_{??} - 1$ and $n.\texttt{head}_{??} \leq n.\texttt{head}_{??}$. From Observation **??** the claim follows. The proof for the right child is the same. □

Corollary 23. *If R is a complete instance of n*`.Refresh`*, then we have $EST^{t^R}_{n.\texttt{left}} \cup EST^{t^R}_{n.\texttt{right}} \subseteq EST^{{}^Rt}_{n}$.*

Proof. The left hand side is the same as Lemma **??**, so it is sufficient to show when R terminates the established blocks in n are a superset of $n.\texttt{blocks}_{??}$. Line **??** writes the block `new` in $n.\texttt{blocks[}h\texttt{]}$ where h is value of $n.$`head` read at Line **??**. Because of Lemma **??** we are sure that $n.\texttt{head} > h$ when R terminates. So the block `new` appended to n at Line **??** is established at ${}^Rt$. □

In the next lemma we show that if two consecutive instances of `Refresh` by the same process on node n fail, then the blocks established in the children of n before the first `Refresh` are guaranteed to be in n after the second `Refresh`.

Lemma 24. *Consider two consecutive terminating instances R_1, R_2 of* `Refresh` *on internal node n by process p. If neither R_1 nor R_2 is a successful* `Refresh`*, then we have $EST^{t^{R_1}}_{n.\texttt{left}} \cup EST^{t^{R_1}}_{n.\texttt{right}} \subseteq EST^{R_2 t}_{n}$.*

Proof. Let R_1 read i from $n.\texttt{head}$ at Line **??**. By Lemma **??**, R_1 and R_2 both cannot read the same value i. By Observation **??**, R_2 reads a larger value of $n.\texttt{head}$ than R_1.

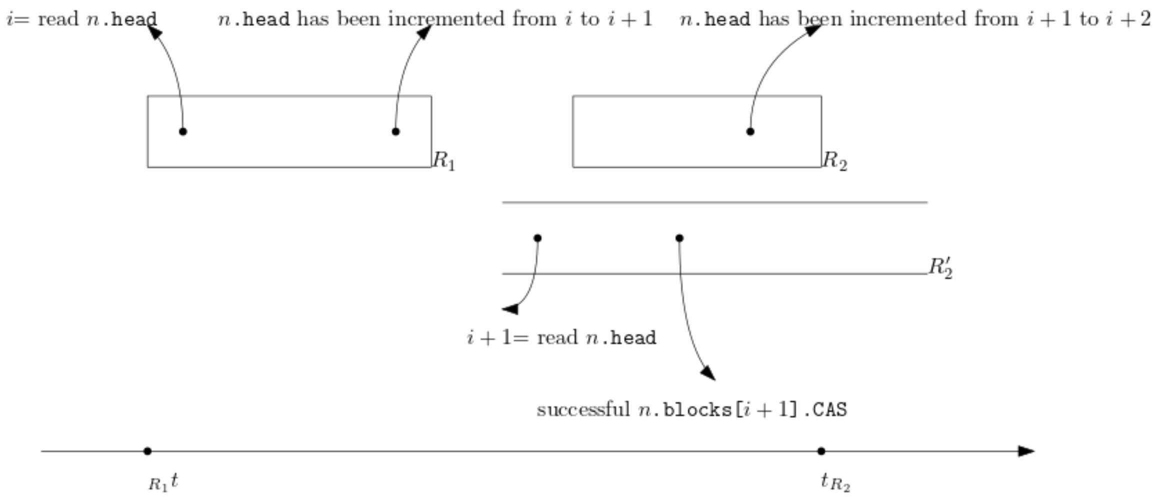
Consider the case where R_1 reads i and R_2 reads $i+1$ from Line **??**. As R_2's `CAS` in Line **??** returns `false`, there is another successful instance R'_2 of $n.\texttt{Refresh}$ that has done a `CAS` successfully into $n.\texttt{blocks}[i+1]$ before R_2 tries to `CAS`. R'_2 creates its block `new` after reading the value $i+1$ from $n.\texttt{head}$ (Line **??**) and R_1 reads the value i from $n.\texttt{head}$. By Observation **??** we have ${}^{R_1}t < t^{R_1}_{??} < t^{R2'}_{??}$ (see Figure **??**). By Lemma **??** we have $EST^{n.\texttt{left}}_{{}^{R'_2}_{??}t} \cup EST^{n.\texttt{right}}_{{}^{R'_2}_{??}t} \subseteq ops(n.\texttt{blocks}_{t^{R'_2}_{??}})$. Also by Lemma **??** on R_2, the value of $n.\texttt{head}$ is more than $i+1$ after R_2 terminates, so the block appended by R'_2 into $n.\texttt{blocks}[i]$ is established by the time R_2 terminates. To summarize, ${}^{R_1}t$ is before R'_2's read of $n.\texttt{head}$ ($t^{R'_2}_{??}$) and R'_2's successful `CAS` ($t^{R'_2}_{??}$) is before R_2's termination (t^{R_2}), so by Observation **??** and Lemma **??** we have **??** $EST^{t^{R_1}}_{n.\texttt{left}} \cup EST^{t^{R_1}}_{n.\texttt{right}} \subseteq ops(n.\texttt{blocks}_{t^{R'_2}_{??}}) \subseteq EST^{R_2 t}_{n}$.

If R_2 reads some value greater than $i+1$ in Line **??** it means $n.\texttt{head}$ has been incremented at least two times since ${}^{R_1}_{??}t$. By Invariant **??**, when $n.\texttt{head}$ is incremented from $i+1$ to $i+2$, $n.\texttt{blocks}[i+1]$ is non-null. Let R_3 be the `Refresh` on n that has put the block in $n.\texttt{blocks}[i+1]$. R_3 read $n.\texttt{head} = i+1$ at Line **??** and has put its block in $n.\texttt{blocks}[i+1]$ before R_2's read of $n.\texttt{head}$ at Line **??**. So we have $t^{R_1} <^{R_3}_{??} t <^{R_3}_{??} t < t^{R_2}_{??} <^{R_2} t$. From Observation **??** on the operations before and after R_3's `CAS` and Lemmas **??**, **??** on R_3 the claim holds. ☐

Corollary 25. $EST^{??t}_{n.\texttt{left}} \cup EST^{??t}_{n.\texttt{right}} \subseteq EST^{t??}_{n}$

Proof. If the first `Refresh` in line **??** returns `true` then by Corollary **??** the claim holds. If the first `Refresh` failed and the second `Refresh` succeeded the claim still holds by Corollary **??**. Otherwise both failed and the claim is satisfied by Lemma **??**. ☐

Now we show that after `Append(`b`)` on a leaf finishes, the operation contained in b will be established in `root`.



i= read n.head
n.head has been incremented from i to $i+1$
n.head has been incremented from $i+1$ to $i+2$
R_1
R_2
R_2'
$i+1$= read n.head
successful n.blocks[$i+1$].CAS
${}_{R_1}t$
t_{R_2}

Figure 13: ${}_{R_1}t < t^{R_1}_{??} <$ incrementing n.head from i to $i+1 < t^{R_2'}_{??} < t^{R_2'}_{??} <$ incrementing n.head from $i+1$ to $i+2 < t_{R_2}$

Corollary 26. *For $A = l$.*`Append(`b`)` *we have $ops(b) \subseteq EST_n^{t^A}$ for each node n in the path from l to* `root`*.*

Proof. A adds b to the assigned leaf of the process, establishes it at Line **??** and then calls `Propagate` on the parent of the leaf where it appended b. For every node n, n.`Propagate` appends b to n, establishes it in n by Corollary **??** and then calls n.`parent.Propagate` until n is `root`. □

Corollary 27. *After l.*`Append(`b`)` *finishes, b is subblock of exactly one block in each node along the path from l to the* `root`*.*

Proof. By the previous corollary and Lemma **??** there is exactly one block in each node containing b. □

4.4 Correctness of GetEnqueue

First we prove some claims about the size and operations of a block. These lemmas will be used later for the correctness and analysis of `GetEnqueue()`.

Lemma 28. *Each block contains at most one operation of each process, and therefore at most p operations in total.*

Proof. To derive a contradiction, assume there are two operations op_1 and op_2 of process p in block b in node n. Without loss of generality op_1 is invoked earlier than op_2. Process p cannot invoke more than one operation concurrently, so op_1 has to be finished before op_2 begins. By Corollary **??**, before op_2 calls `Append`, op_1 exists in every node of the tree on the path from p's leaf to the root. Since b contains op_2, it must be created after op_2 is invoked. This means there is some block b' before b in n containing op_1. The existence of op_1 in b and b' contradicts Lemma **??**. ☐

Lemma 29. *Each block has at most p direct subblocks.*

Proof. The claim follows directly from Lemma **??** and the observation that each block appended to an internal node contains at least one operation, due to the test on Line **??**. We can also see the blocks in the leaves have exactly one operation in the `Enqueue` and `Dequeue` routines. ☐

`DSearch(`e`, `end`)` returns a pair `<`b`, `i`>` such that the ith `Enqueue` in the bth block of the root is the eth `Enqueue` in the sequence stored in the root.

Lemma 30 (`DSearch` Correctness)**.** *If* `root.blocks[`end`]` $\neq$ `null` *and* $1 \leq e \leq$ `root.blocks[`end`].`$\texttt{sum}_{\texttt{enq}}$, `DSearch(`$e$`, `$end$`)` *returns* `<`$b$`, `$i$`>` *such that* $E_i(root, b) = E_e(root)$.

Proof. From Lines **??** and **??** we know the $\texttt{sum}_{\texttt{enq-left}}$ and $\texttt{sum}_{\texttt{enq-right}}$ fields of `blocks` in each node are sorted in non-decreasing order. Since $\texttt{sum}_{\texttt{enq}} = \texttt{sum}_{\texttt{enq-left}} + \texttt{sum}_{\texttt{enq-right}}$, the $\texttt{sum}_{\texttt{enq}}$ values of `root.blocks[`$0 \cdot\cdot end$`]` are also non-decreasing. Furthermore, since `root.blocks[0].`$\texttt{sum}_{\texttt{enq}} = 0$ and `root.blocks[`end`].`$\texttt{sum}_{\texttt{enq}} \geq e$, there is a b such that `root.blocks[`b`].`$\texttt{sum}_{\texttt{enq}} \geq e$ and `root.blocks[`$b-1$`].`$\texttt{sum}_{\texttt{enq}} < e$ by Corollary **??**. Block `root.blocks[`b`]` contains $E_i(root, b)$. Lines **??**–**??** doubles the search range in Line **??** and will eventually reach `start` such that `root.blocks[start].`$\texttt{sum}_{\texttt{enq}} \leq e \leq$ `root.blocks[end].`$\texttt{sum}_{\texttt{enq}}$. Then, in Line **??**, the binary search finds the b such that `root.blocks[`$b-1$`].`$\texttt{sum}_{\texttt{enq}} < e \leq$`root.blocks[`$b$`].`$\texttt{sum}_{\texttt{enq}}$. By Corollary **??**, `root.blocks[`b`]` is the block that contains $E_e(root)$. Finally i is computed using the definition of $\texttt{sum}_{\texttt{enq}}$ and Corollary **??**. ☐

Lemma 31 (GetEnqueue correctness). *If* $1 \leq i \leq n$.blocks[b].num$_{\texttt{enq}}$ *then* n.GetEnqueue(b, i) *returns* $E_i(n,b)$.element.

Proof. We are going to prove this lemma by induction on the height of node n. For the base case, suppose n is a leaf. Leaf blocks each contain exactly one operation, n.blocks[b].sum$_{\texttt{enq}} \leq 1$, which means only n.GetEnqueue(b,1) can be called when n is a leaf and contains and Enqueue operation. Line **??** of n.GetEnqueue(b, 1) returns the element of the Enqueue operation stored in the bth block of leaf n, as required.

For the induction step we prove if n.dir.GetEnqueue(b', i) returns $E_i(n.\texttt{dir}, b')$ then n.GetEnqueue(b, i) returns $E_i(n,b)$. From Definition **??** of $E(n,b)$ we know that operations from the left subblocks come before the operations from the right subblocks in a block (see Figure **??**). By Lemma **??**, the num$_{\texttt{enq-left}}$ field in n.blocks[b] is the number of Enqueue operations from the blocks's subblocks in the left child of n. So the ith Enqueue operation in n.blocks[b] is propagated from the right child if and only if i is greater than n.blocks[b].num$_{\texttt{enq-left}}$. Line **??** decides whether the ith enqueue in the bth block of internal node n is in the left child or right child subblocks of n.blocks[b]. By Definitions **??** and **??** to find an operation in the subblocks of n.blocks[i] we need to search in the range

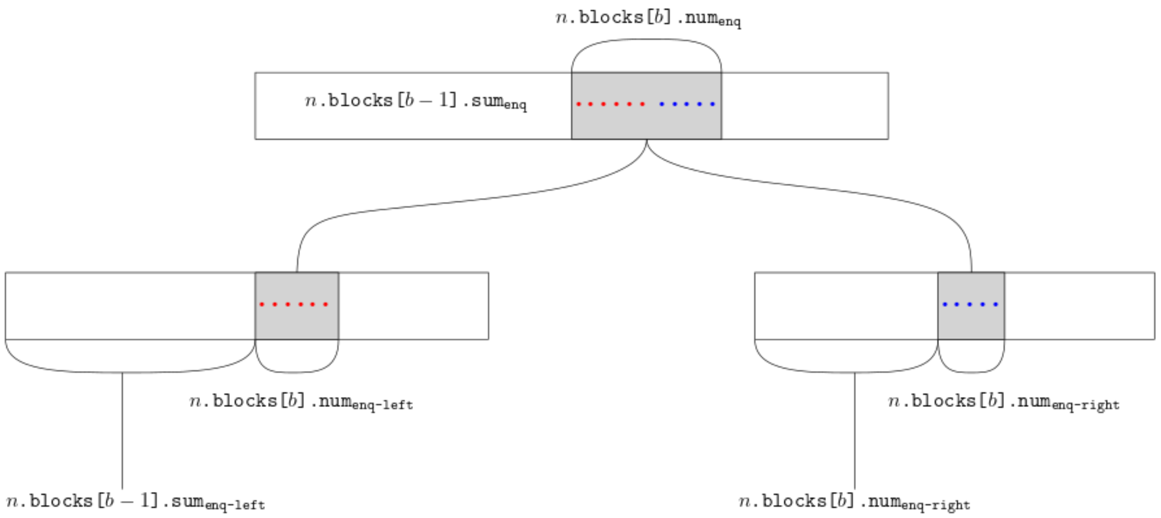
n.left.blocks[n.blocks[i-1].end$_{\texttt{left}}$+1..n.blocks[i].end$_{\texttt{left}}$] or

n.right.blocks[n.blocks[i-1].end$_{\texttt{right}}$+1..n.blocks[i].end$_{\texttt{right}}$].

First we consider the case where the Enqueue we are looking for is in the left child. There are eb = n.blocks[$b-1$].sum$_{\texttt{enq-left}}$ Enqueues in the blocks of n.left before the left subblocks of n.blocks[b], so $E_i(n,b)$ is $E_{i+eb}(n.\texttt{left})$ which is $E_{i'}(n.\texttt{left}, b')$ for some b' and i'. We can compute b' and then search for the i'th Enqueue in n.left.blocks[b'], where i' is $i+eb-n$.left.blocks[$b'-1$].sum$_{\texttt{enq}}$. The parameters in Line **??** are for searching $E_{i+eb}(n.\texttt{left})$ in n.left.blocks in the range of left subblocks of n.blocks[b], so this BinarySearch returns the index of the subblock containing $E_i(n,b)$.

Otherwise, the enqueue we are looking for is in the right child. Because Enqueues from the left subblocks are ordered before the Enqueues from the right subblocks, there are n.blocks[b].num$_{\texttt{enq-left}}$ enqueues ahead of $E_i(n,b)$ from the left child. So we need to search for $i - n$.blocks[b].num$_{\texttt{enq-left}}$ + n.blocks[$b-$1].sum$_{\texttt{enq-right}}$ in the right child (Line **??**). Other parameters for the right child are chosen similarly to the left child.

So, in both cases the direct subblock containing $E_i(n,b)$ is computed in Line **??** or **??**. Let *subblock* be the block in n.`dir` containing $E_i(n,b)$. Finally, n`.child.GetEnqueue(`*subblock*`,` i`)` is invoked and it returns $E_i(n,b)$`.element` by the hypothesis of the induction. $\square$



n.blocks[b].num$_{\text{enq}}$
n.blocks[$b-1$].sum$_{\text{enq}}$
n.blocks[b].num$_{\text{enq-left}}$
n.blocks[b].num$_{\text{enq-right}}$
n.blocks[$b-1$].sum$_{\text{enq-left}}$
n.blocks[b].num$_{\text{enq-right}}$

Figure 14: The number and ordering of the enqueue operations propagated from the left and the right child to n`.blocks[`b`]`. Both n`.blocks[`b`]` and its subblocks are shown in grey. `Enqueue` operations from the left subblocks (colored red), are ordered before the `Enqueue` operations from the right child (colored blue).

4.5 Correctness of IndexDequeue

The next few results show that the `super` field of a block is accurate within one of the actual index of the block's superblock in the parent node. Then we explain how it is used to compute the rank of a given `Dequeue` in the root.

Definition 32. If a `Refresh` instance R_1 does its `CAS` at Line **??** earlier than `Refresh` instance R_2 we say R_1 has *happened before* R_2.

Observation 33. *After* n`.blocks[`i`].CAS(null,` B`)` *succeeds,* n`.head` *cannot increase from* i *to* $i+1$ *until* B`.super` *is set.*

Proof. From Observation **??** we know the n`.head` changes only by the increment on Line **??**. Before an instance of `Advance` increments `n.head` on Line **??**, Line **??** ensures that `n.blocks[head].super` was set at Line **??**. □

Corollary 34. *If* n`.blocks[`i`].super` *is* `null`*, then* n`.head` $\leq i$ *and* n`.blocks[`$i+1$`]` *is* `null`.

Proof. By Invariant **??** and Observation **??**. □

Now let us consider how the `Refresh`es that took place on the parent of node n after block B was stored in n will help to set B`.super` and propagate B to the parent.

Observation 35. *If the block created by an instance* R_p *of* n`.parent.Refresh` *contains block* $B =$ n`.blocks[`b`]` *then* R_p *reads a value greater than* b *from* n`.head` *in Line* **??**.

Lemma 36. *If* $B =$ n`.blocks[`b`]` *is a direct subblock of* n`.parent.blocks[`sb`]` *then* B`.super` $\leq sb$.

Proof. Let R_p be the instance of n`.parent.Refresh` that does a successful `CAS` on n`.parent.blocks[`sb`]`. By Observation **??** if R_p propagates B it has to read a greater value than b from n`.head`, which means n`.head` was incremented from b to $b+1$ in Line **??**. By Observation **??** B`.super` was already set in Line **??**. The value written in B`.super`, was read in Line **??** before the `CAS` that sets B`.super` in Line **??**. From Observation **??** we know n`.parent.head` is non-decreasing so B`.super` $\leq sb$, since n`.parent.head` is still equal to sb when R_p executes its `CAS` at Line **??** by Lemma **??**. The reader may wonder when the case b`.super` $= sb$ happens. This can happen when n`.parent.blocks[`B`.super]` $=$ `null` when B`.super` is written and R_p puts its created block into n`.parent.blocks[`B`.super]` afterwards. □

Lemma 37. *Let R_n be a* `Refresh` *that puts B in n*`.blocks[`b`]` *at Line* **??**. *Then, the block created by one of the next two successful n*`.parent.Refresh`*es according to Definition* **??** *contains B and B*`.super` *is set when the second successful n*`.parent.Refresh` *reaches Line* **??**.

Proof. Let R_{p1} be the first successful n`.parent.Refresh` after R_n and R_{p2} be the second next successful n`.parent.Refresh`. To derive a contradiction assume B was neither propagated to n`.parent` by R_{p1} nor by R_{p2}.

Since R_{p2}'s created block does not contain B, by Observation **??** the value R_{p2} reads from n`.head` in Line **??** is at most b. From Observation **??** the value R_{p2} reads in Line **??** is also at most b.

R_n puts B into n`.blocks[`b`]` so R_n reads the value b from n`.head`. Since R_{p2}'s `CAS` into n`.parent.blocks` is successful there should be a `Refresh` instance R'_p on n`.parent` that increments n`.parent.head` (Line **??**) after R_{p1}'s Line **??** and before R_{p2}'s Line **??**. We assumed $t^{R_n}_{??} < t^{R_{p1}}_{??} < t^{R_{p2}}_{??}$ by Definition **??**. Finally, Line **??** is after Line **??** and R_{p2}'s Line **??** is after R'_p's Line **??**, which is after R_n's n`.blocks.CAS`.

$$\left.\begin{array}{c} {}^{R_n}_{??}t <^{R_{p1}}_{??} t \\ {}^{R_{p1}}_{??}t <^{R_{p'}}_{??} t <^{R_{p2}}_{??} t \\ {}^{R_{p2}}_{??}t <^{R_{p2}}_{??} t \end{array}\right\} \Longrightarrow^{R_n}_{??} t <^{R_{p2}}_{??} t$$

So R_{p2} reads a value greater than or equal to b for n`.head` by Observation **??**.

Therefore R_{p2} reads n`.head` $= b$. R_{p2} calls n`.Advance` at Line Line **??**, which ensures n`.head` is incremented from b. So the value R_{p2} reads in Line **??** of `CreateBlock` is greater than b and R_{p2}'s created block contains B. This is in contradiction with our hypothesis.

Furthermore, if B`.super` was not set earlier it is set by R_{p2}'s call to n`.Advance` invoked from Line **??**. ☐

Corollary 38. *If $B = n$*`.blocks[`b`]` *is propagated to n*`.parent`, *then B*`.super` *is equal to or one less than the index of the superblock of B.*

Proof. Let R_n be the n`.Refresh` that put B in n`.blocks` and let R_{p1} be the first successful n`.parent.Refresh` after R_n and R_{p2} be the second next successful n`.parent.Refresh`. Before B can be propagated to n's `parent`, n`.head` must be greater than b, so by Observation **??** B`.super` is set. From the previous Lemma we know that B is propagated by the second next successful `Refresh`'s `CAS` on n`.parent.blocks`. To summarize we have $n.\texttt{parent.head}_{{}^{R_{p2}}_{??}t} = n.\texttt{parent.head}_{{}^{R_{p1}}_{??}t} + 1$ and by Definition **??** and Observation **??** $n.\texttt{parent.head}_{{}^{R_{p1}}_{??}t} \leq n.\texttt{parent.head}_{{}^{R_n}_{??}t}$. The value that is set in B`.super` is read from n`.parent.head` after ${}^{R_n}_{??}t$. So B`.super` is equal to or one less than the index of the superblock of B. ☐

Now using Corollary ?? on each step of the `IndexDequeue` we prove its correctness.

Lemma 39 (`IndexDequeue` correctness)**.** *If* $1 \le i \le n$`.blocks[`b`].num`$_{\texttt{deq}}$ *then* n`.IndexDequeue(`b`,`i`)` *returns* $< x, y >$ *such that* $D_i(n, b) = D_y(\texttt{root}, x)$.

Proof. We will prove this by induction on the distance of n from the `root`. The base case where n is `root` is trivial (see Line ??). For the non-root nodes n`.IndexDequeue(`b`, `i`)` computes sb, the index of the superblock of the bth block in n, in Line ?? by Corollary ??. After that, the position of $D_i(n, b)$ in $D(n.\texttt{parent}, sb)$ is computed in Lines ??–??. By Definition ??, `Dequeue`s in a block are ordered based on the order of its subblocks from left to right. If $D_i(n, b)$ was propagated from the left child, the number of dequeus in the left subblocks of n`.parent.blocks[`sb`]` before n`.blocks[`b`]` is considered in Line ?? (see Figure ??). Otherwise, if $D_i(n, b)$ was propagated from the right child, the number of dequeues in the subblocks from the left child is considered to be ahead of the computed index (Line ??) (see Figure ??). Finally `IndexDequeue` is called on n`.parent` recursively and it returns the correct response by the induction hypothesis. □

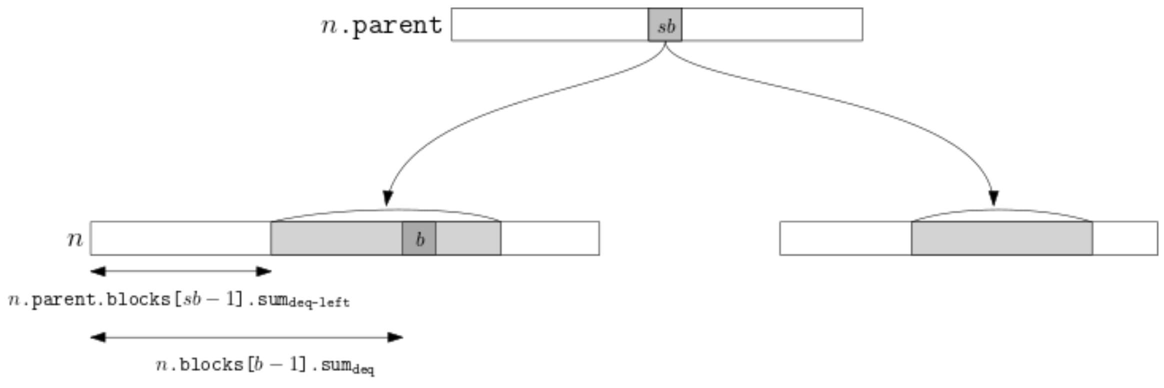
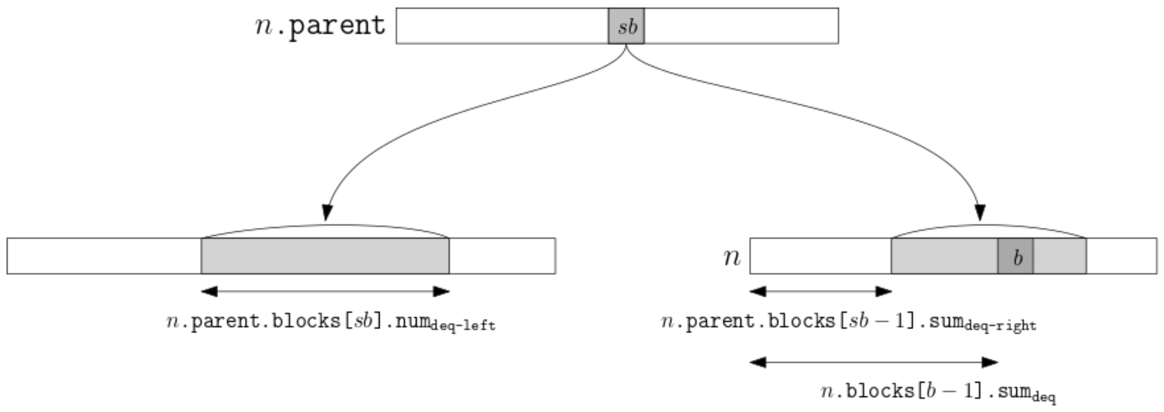

n.parent
sb
n
b
n.parent.blocks[$sb-1$].sum$_{\texttt{deq-left}}$
n.blocks[$b-1$].sum$_{\texttt{deq}}$

Figure 15: The number of `Dequeue` operations before $E_i(n, b)$ shown in the case where n is a left child.



n.parent
sb
n
b
n.parent.blocks[sb].num$_{\text{deq-left}}$
n.parent.blocks[$sb-1$].sum$_{\text{deq-right}}$
n.blocks[$b-1$].sum$_{\text{deq}}$

Figure 16: The number of `Dequeue` operations before $E_i(n, b)$ shown in the case where n is a right child.

4.6 Linearizability

We now prove the two properties needed for linearizability.

Lemma 40. *L is a legal linearization ordering.*

Proof. We must show that, every operation that terminates is in L exactly once and if op_1 terminates before op_2 starts in an execution then op_1 is before op_2 in the linearization. The first claim is directly reasoned from Corollary **??**. For the latter, if op_1 terminates before op_2 starts, op_1`.Append` has terminated before op_2`.Append` started. From Corollary **??**, op_1 is in `root.blocks` before op_2 starts to propagate. By definition of L, op_1 is linearized before op_2. □

Once some operations are aggregated in one block, they will get propagated up to the root together and they can be linearized in any order among themselves. We have chosen to put `Enqueue`s in a block before `Dequeue`s (see Definition **??**).

Definition 41. If a `Dequeue` operation returns `null` it is called a *null* `Dequeue`, otherwise it is called *non-null* `Dequeue`.

Next we define the responses that `Dequeue`s should return, according to the linearization.

Definition 42. Assume the operations in `root.blocks` are applied sequentially on an empty queue in the order of L. $Resp(d) = e$`.element` if the element of `Enqueue` e is the response to `Dequeue` d. Otherwise if d is a `null` dequeue then $Resp(d) =$ `null`.

In the next lemma we show that the `size` field in each `root block` is computed correctly.

Lemma 43. `root.blocks[`b`].size` *is the size of the queue if the operations in* `root.blocks[`$0\cdots b$`]` *are applied in the order of* L.

Proof. We prove the claim by induction on b. The base case when $b = 0$ is trivial since the queue is initially empty and `root.blocks[0]` contains an emprty block with `size` field equal to 0. We are going to show the correctness when $b = i$ assuming correctness when $b = i - 1$. By Definition **??** `Enqueue` operations come before `Dequeue` operations in a block. By Lemma **??** $\texttt{num}_{\texttt{enq}}$ and $\texttt{num}_{\texttt{deq}}$ fields in a block show ther number of `Enqueue` and `Dequeue` operations in it. If there are more than `root.blocks[`$i-1$`].size` $+$ `root.blocks[`i`].`$\texttt{num}_{\texttt{enq}}$ dequeue operations in `root.blocks[`i`]` then the queue would become empty after `root.blocks[`i`]`. Otherwise the size of the queue after the bth block in the root is `root.blocks[`$b-1$`].size`$+$ `root.blocks[`b`].`$\texttt{num}_{\texttt{enq}}$ $-$ `root.blocks[`b`].`$\texttt{num}_{\texttt{deq}}$. In both cases, this is same as the assignment on Line **??**. ☐

The next lemma is useful to compute the number of non-null dequeues.

Lemma 44. *If operations in the root are applied in the order of L, the number of non-null* `Dequeue`*s in* `root.blocks[`$0\cdots b$`]` *is* `root.blocks[`b`].`$\texttt{sum}_{\texttt{enq}}$ $-$ `root.blocks[`b`].size`.

Proof. There are `root.blocks[`b`].`$\texttt{sum}_{\texttt{enq}}$ `Enqueue` operations in `root.blocks[`$0\cdots b$`]` by Corollary **??**. The size of the queue after doing `root.blocks[`$0\cdots b$`]` in the order of L is the number of *enqueues* in `root.blocks[`$0\cdots b$`]` minus the number of *non-null* `Dequeue`*s* in `root.blocks[`$0\cdots b$`]`. By the correctness of the `size` field from Lemma **??** and $\texttt{sum}_{\texttt{enq}}$ field from Lemma **??**, the number of *non-null* `Dequeue`*s* is `root.blocks[`b`].`$\texttt{sum}_{\texttt{enq}}$ $-$ `root.blocks[`b`].size`. ☐

Corollary 45. *If operations in the root are applied in the order of L, the number of non-null dequeues in* `root.blocks[`b`]` *is* `root.blocks[`b`].`$\texttt{num}_{\texttt{enq}}$ $-$ `root.blocks[`b`].size` $+$ `root.blocks[`$b-1$`].size`.

Lemma 46. *$Resp(D_i($*`root`*$, b))$ is* `null` *iff* `root.blocks[`$b-1$`].size` $+$ `root.blocks[`b`].`$\texttt{num}_{\texttt{enq}}$$-i$ < 0.

Proof. From Corollary **??** and Lemma **??**. ☐

Lemma 47. `FindResponse(`b`, `i`)` *returns $Resp(D_i(root, b))$.*

Proof. $D_i(root, b)$ is $D_{\texttt{root.blocks[}b-1\texttt{].sum}_{\texttt{deq}}+i}(root)$ by Definition **??** and Lemma **??**. $D_i(root, b)$ returns `null` at Line **??** if `root.blocks[`$b-1$`].size` $+$ `root.blocks[`b`].`$\texttt{num}_{\texttt{enq}}$ $- i < 0$ and $Resp(D_i(root, b)) =$ `null` in this case by Lemma **??**. Otherwise, if $D_i(root, b)$ is the eth non-null dequeue in L it should return the eth enqueued value. By Lemma ?? there are `root.blocks[`$b-1$`].sum`$_{\texttt{enq}}$ $-$ `root.blocks[`$b-1$`].size` non-null `Dequeue` operations in `root.blocks[`$0\cdots b-1$`]`. The `Dequeue`s in `root.blocks[`b`]` before $D_i(root, b)$ are non-null dequeues. So $D_i(root, b)$ is the eth non-null `Dequeue` where $e = i +$ `root.blocks[`$b-1$`].sum`$_{\texttt{deq}}$ $-$ `root.blocks[`$b-1$`].size` (Line ??). See Figure ??.

After computing e at Line ??, the code finds `b,i` such that $E_i(root, b) = E_e(root)$ using `DSearch` and then finds its `element` using `GetEnqueue` (Line ??). Correctness of `DSearch` and `GetEnqueue` routines are shown in Lemmas ?? and ??. □

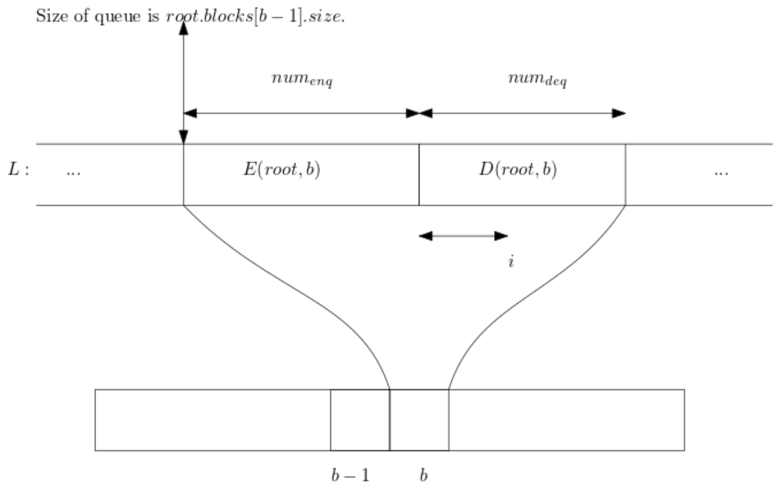


Figure 17: The position of $D_i(root, b)$.

Lemma 48. *The responses to operations in our algorithm are the same as in the sequential execution in the order given by L.*

Proof. `Enqueue` operations do not return any value. By Lemma ??, the response of a `Dequeue` in our algorithm is same as the response from the sequential execution of L. □

Theorem 49 (Main)**.** *The queue implementation is linearizable.*

Proof. The theorem follows from Lemmas ?? and ??. □

Remark In fact our algorithm is strongly linearizable as defined in [?]. By Definition ?? the linearization ordering of operations will not change as blocks containing new operations are appended to the root.

5 Analysis

In this section we analyze the number of `CAS` invocations and the time complexity of our algorithm.

Proposition 50. An `Enqueue` or `Dequeue` operation does at most $14 \log p$ `CAS` operations.

Proof. In each level of the tree `Refresh` is invoked at most two times and every `Refresh` invokes at most 7 `CAS`es, one in Line **??** and two from each `Advance` in Line **??** or **??**. $\square$

Lemma 51 (`DSearch` Analysis)**.** *If the* `element` *enqueued by* $E_i(root, b) = E_e(root)$ *is the response to some* `Dequeue` *operation in* `root.blocks[`end`]`*, then* `DSearch(`e`, `end`)` *takes* $O\big(\log($`root.blocks[`b`].size` $+$ `root.blocks[`end`].size`$)\big)$ *steps.*

Proof. First we show $end - b - 1 \leq 2 \times$ `root.blocks[`$b-1$`].size` $+$ `root.blocks[`end`].size`. There can be at most `root.blocks[`b`].size` `Dequeue`s in `root.blocks[`$b+1 \cdots end-1$`]`; otherwise all elements enqueued by `root.blocks[`b`]` would be dequeued before `root.blocks[`end`]`. Furthermore in the execution of queue operations in the linearization ordering, the size of the queue becomes `root.blocks[`end`].size` after the operations of `root.blocks[`end`]`. The final size of the queue after `root.blocks[`$1 \cdots end$`]` is `root.blocks[`end`].size`. After an execution on a queue the *size* of the queue is greater than or equal to $\#enqueues - \#dequeues$ in the execution. We know the number of dequeues in `root.blocks[`$b + 1 \cdots end - 1$`]` is less than `root.blocks[`b`].size`, therefore in `root.blocks[`$b+1 \cdots end-1$`]` cannot be more than `root.blocks[`b`].size` $+$ `root.blocks[`end`].size` `Enqueue`s. Overall there can be at most $2 \times$ `root.blocks[`b`].size` $+$ `root.blocks[`end`].size` operations in `root.blocks[`$b+1 \cdots end-1$`]` and since from Line **??** we know that the `num` field of the every block in the tree is greater than 0, each block has at least one operation, there are at most $2 \times$ `root.blocks[`b`].size` $+$ `root.blocks[`end`].size` blocks in between `root.blocks[`b`]` and `root.blocks[`end`]`. So $end - b - 1 \leq 2 \times$ `root.blocks[`b`].size` $+$ `root.blocks[`end`].size`.

So, the doubling search reaches `start` such that the `root.blocks[start].sum`$_{\texttt{enq}}$ is less than e in $O\big(\log($`root.blocks[`b`].size` $+$ `root.blocks[`end`].size`$)\big)$ steps. See Figure **??**. After Line **??**, the binary search that finds b also takes $O\big(\log($`root.blocks[`b`].size` $+$ `root.blocks[`end`].size`$)\big)$. Next, `i` is computed via the definition of `sum`$_{\texttt{enq}}$ in constant time (Line **??**). $\square$

Lemma 52 (Worst Case Time Analysis)**.** *The worst case number of steps for an* `Enqueue` *is* $O(\log^2 p)$ *and for a* `Dequeue`*, is* $O(\log^2 p + \log q_e + \log q_d)$*, where* q_d *is the size of the queue when the* `Dequeue` *is linearized*

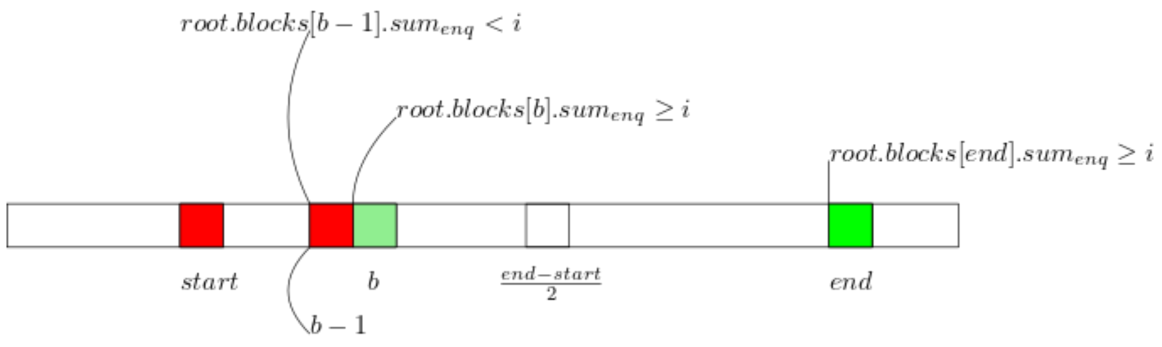

$root.blocks[b-1].sum_{enq} < i$
$root.blocks[b].sum_{enq} \geq i$
$root.blocks[end].sum_{enq} \geq i$
$start$
b
$\frac{end-start}{2}$
end
$b-1$

Figure 18: Distance relations between `start`, b, end.

and q_e is the size of the queue at time the response of the `Dequeue` *is linearized.*

Proof. `Enqueue` consists of creating a block and appending it to the tree. The first part takes constant time. To propagate the operation to the root the algorithm tries at most two `Refresh`es in each node of the path from the leaf to the root (Lines **??**, **??**). We can see from the code that each `Refresh` takes a constant number of steps and does $O(1)$ `CAS`es. Since the height of the tree is $\Theta(\log p)$, `Enqueue` takes $O(\log p)$ steps.

A `Dequeue` creates a block whose `element` is `null`, appends it to the tree, computes its rank among non-null dequeues, finds the corresponding enqueue and returns the response. The first two parts are similar to an `Enqueue` operation and take $O(\log p)$ steps. To compute the rank of a `Dequeue` in $D(n)$ the dequeue calls `IndexDequeue()`. `IndexDequeue` does $O(1)$ steps in each level which takes $O(\log p)$ steps. If the response to the dequeue is `null`, `FindResponse` returns `null` in $O(1)$ steps. Otherwise, if the response to a dequeue in `root.blocks[end]` is in `root.blocks[b]` the `DSearch` takes $\Theta(\log($`root.blocks[b].size+root.blocks[end].size`) by Lemma **??**, which is O(log size of the queue when `Enqueue` is linearized) + log size of the queue when `Dequeue` is linearized). Each search in `GetEnqueue()` takes $O(\log p)$ since there are $\leq p$ subblocks in a block (Lemma **??**), so `GetEnqueue()` takes $O(\log^2 p)$ steps. □

Lemma 53 (Amortized Time Analysis)**.** *The amortized number of steps for an* `Enqueue` *or* `Dequeue`*, is $O(\log^2 p + \log q)$, where q is the size of the queue when the operation is linearized.*

Proof. If we split the `DSearch` time cost between the corresponding `Enqueue` and `Dequeue`, each operation takes $O(\log^2 p + q)$ steps. □

Theorem 54. *The queue implementation is wait-free.*

Proof. To prove the claim, it is sufficient to show that every `Enqueue` and `Dequeue` operation terminates after a finite number of its own steps. This is directly concluded from Lemma **??**. □

6 Future Directions

We designed a tree to achieve agreement on a linearization of operations invoked by p processes in an asynchronous model, which we will call a *block tree*. We implemented two queries to know information about the ordering agreed in the block tree. Then we used the tree to implement a queue where the number of steps per operation is poly-logarithmic with respect to the size of the queue and the number of processes. Block trees can be used as a mechanism to achieve agreement among processes to construct more poly-logarithmic wait-free linearizable objects. In the next paragraphs we talk about possible improvements on block trees and the data structures that we can implement with block trees.

Reducing Space Usage The `blocks` arrays defined in our algorithm are unbounded. To use $O(n)$ space in each node where n is the total number of operations, instead of unbounded arrays we could use the memory model of the wait-free vector introduced by Feldman, Valera-Leon and Damian [?]. We can create an array called `arr` of pointers to array segments (see Figure **??**). When a process wishes to write into location `head` it checks whether `arr[`$\lfloor\log$ `head`$\rfloor$`]` points to an array or not. If not, it creates a shared array of size $2^{\lfloor \log \texttt{head} \rfloor}$ and tries to `CAS` a pointer to the created array into `arr[`$\lfloor\log$ `head`$\rfloor$`]`. Whether the `CAS` is successful or not, `arr[`$\lfloor\log$ `head`$\rfloor$`]` points to an array. When a process wishes to access the ith element it looks up `arr[`$\lfloor\log i\rfloor$`][`$i - 2^{\lfloor\log i\rfloor}$`]`, which takes $O(1)$ steps. The CAS Retry Problem does not happen here because if n elements are appended to the array then only $O(p \times \log n)$ `CAS` steps have happened on the array `arr`. Furthermore, at most p arrays with size $2^{\lfloor\log i\rfloor}$ are allocated by processes while processes try to to the `CAS` on `arr[`i`]`. Jayanti and Shun [?] present a way to initialize wait-free arrays in constant steps. The time taken to allocate arrays in an execution containing n operations is $O(\frac{p\log n}{n})$ per operation, which is negligible if $n >> p$. The vector implementation also has a mechanism for doubling `arr` when necessary, but this happens very rarely, since increasing `arr` from s to $2s$ increases the capacity of the vector from 2^s to 2^{2s}.

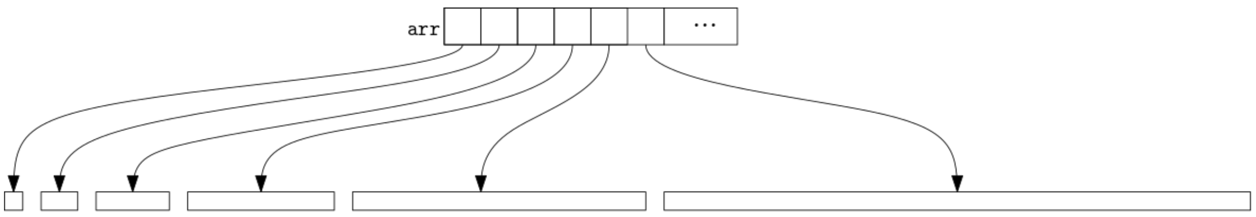


Figure 19: Array Segments

Garbage Collection We did not handle garbage collection: `Enqueue` operations remain in the nodes even after their elements have been dequeued. We can keep track of the `blocks` in the `root` whose operations are all terminated, i.e, all of its enqueues have been dequeued and the responses of all of its dequeues have been computed. We call these blocks *finished blocks*. If we help the operations of all processes to compute their responses, then we can say if block B is finished then all blocks before B are also finished. Knowing the most recent finished block in a node, we can reclaim the memory taken by finished blocks. To throw the garbage in the blocks away we cannot use arrays (or vectors). We need a data structure that supports `tryAppend()`, `read(i)`, `write(i)` and `split(i)` operations in $O(\log n)$, where `split(i)` removes all the indices less than `i`. If each process tries to do the garbage collection every p^2 operation on the queue then the amortized complexity remains the same. We can use a concurrent implementation of persistent red black trees for this [?]. Bashari and Woelfel [?] used persistent red black trees in a similar way.

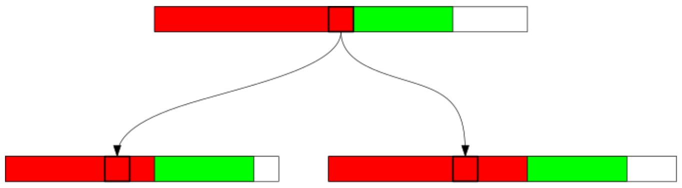


Figure 20: Finished blocks are shown with red color and unfinished blocks are shown with green color. All the subblocks of a finished block are also finished.

Poly-logarithmic Wait-free Sequences Consider a data structure storing a sequence that supports three operations `append(e)`, `get(i)` and `index(e)`. An `append(e)` adds `e` to the end of the sequence, a `get(i)` gets the `i`th element in the sequence and an `index(e)` computes the position of element `e` in the sequence. We can modify our queue to design such data structure. An `append(e)` is implemented like `Enqueue(e)`, `get(i)` is done by calling `DSearch` but with a `BinarySearch` on the entire `root.blocks` array and `index(e)` is done similarly to `IndexDequeue` (except operating on enqueues instead of dequeues). We achieve this with poly-logarithmic steps for each operation with respect to the number of `append`s done.

Other Poly-log Wait-free Data structures There are two reasons the block tree worked well to implement a queue. Firstly, to respond to a `Dequeue` we do not need to look at the entire history of operations: if a `Dequeue` does not return `null`, we can compute the index of the `Enqueue` that is its response in $O(\log n)$ time if we keep the number of enqueues and the size. Secondly, the operations we need to search to respond to the Dequeue is not very far from it in the sequence of operations: the distance is at most linear in the size of the queue. It may be possible to create wait-free poly-logarithmic implementaion of other objects whose operations satisfy these two conditions.

6.1 Attiya–Fouren Lower Bound

As we talked earlier in Section 2.4 the Attiya–Fouren lower bound says that a concurrent implementation of queues using reads, writes and conditional operations like CAS has $O(\sqrt{\log\log p})$ amortized complexity [**?**] when the number of concurrent processes is $O(\log\log p)$. Our amortized complexity is $O(log^2p + logq)$. It is an open problem to the reduce the gap between our algorithm and the Attiya–Fouren lower bound.