Calculation with matrices 行列で計算

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1 +, -, x and /

1.1 Calculation of real numbers 実数の計算

```
x <- 3
y <- -5
x+y
```

```
## [1] -2
```

```
x-y
```

```
## [1] 8
```

```
x*y
```

```
## [1] -15
```

```
x/y
```

```
## [1] -0.6
```

1.1.1 Important numbers 大事な値

```
zero <- 0
one <- 1
x
```

```
## [1] 3
```

```
x + zero
```

```
## [1] 3
```

```
x * zero
```

```
## [1] 0
```

```
x * one
```

```
## [1] 3
```

```
x.inverse <- 1/x
x * x.inverse
```

```
## [1] 1
```

```
x/y
```

```
## [1] -0.6
```

```
y.inverse <- 1/y
x * y.inverse
```

```
## [1] -0.6
```

1.2 Square matrices 正方行列

1.2.1 2×2 matrices

```
X <- matrix(c(1,2,3,4),2,2)
X
```

```
##      [,1] [,2]
## [1,]    1    3
## [2,]    2    4
```

```
Y <- matrix(c(4,5,6,7),2,2)
Y
```

```
##      [,1] [,2]
## [1,]    4    6
## [2,]    5    7
```

```
X+Y
```

```
##      [,1] [,2]
## [1,]    5    9
## [2,]    7   11
```

```
X-Y
```

```
##      [,1] [,2]
## [1,]   -3   -3
## [2,]   -3   -3
```

```
X %*% Y
```

```
##      [,1] [,2]
## [1,]   19   27
## [2,]   28   40
```

```
Y %*% X
```

```
##      [,1] [,2]
## [1,]   16   36
## [2,]   19   43
```

Calculate yourself without computers. 計算機を使わずに計算すること。

1.2.1.1 Important matrices

```
ONE <- matrix(c(1,0,0,1),2,2)
ONE
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

```
ZERO <- matrix(0,2,2)
ZERO
```

```
##      [,1] [,2]
## [1,]    0    0
## [2,]    0    0
```

```
X.inverse <- solve(X)
X
```

```
##      [,1] [,2]
## [1,]    1    3
## [2,]    2    4
```

```
X.inverse
```

```
##      [,1] [,2]
## [1,]   -2  1.5
## [2,]    1 -0.5
```

```
X %*% X.inverse
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

```
X.inverse %*% X
```

```
##      [,1] [,2]
## [1,]    1    0
## [2,]    0    1
```

```
# x/y
Y.inverse <- solve(Y)
X %*% Y.inverse
```

```
##      [,1] [,2]
## [1,]    4   -3
## [2,]    3   -2
```

```
Y.inverse %*% X
```

```
##      [,1] [,2]
## [1,]  2.5  1.5
## [2,] -1.5 -0.5
```

1.2.2 $n \times n$ matrices $n \times n$ 行列

```
n <- sample(3:5,1)
X <- matrix(sample((-20):20,n^2,replace=TRUE),n,n)
Y <- matrix(sample((-20):20,n^2,replace=TRUE),n,n)
X
```

```
##      [,1] [,2] [,3]
## [1,]    4  -16   12
## [2,]   -6  -10   18
## [3,]    9    0   14
```

```
X + Y
```

```
##      [,1] [,2] [,3]
## [1,]   14  -19   16
## [2,]   -9   10   35
## [3,]    5   12    9
```

```
X - Y
```

```
##      [,1] [,2] [,3]
## [1,]   -6  -13    8
## [2,]   -3  -30    1
## [3,]   13  -12   19
```

```
X %*% Y
```

```
##      [,1] [,2] [,3]
## [1,]   40 -188 -316
## [2,] -102   34 -284
## [3,]   34  141  -34
```

```
X %*% solve(X)
```

```
##               [,1] [,2]          [,3]
## [1,]  1.000000e+00    0 -5.551115e-17
## [2,] -5.551115e-17    1  1.110223e-16
## [3,]  0.000000e+00    0  1.000000e+00
```

```
solve(X) %*% X
```

```
##              [,1]         [,2]         [,3]
## [1,] 1.000000e+00 0.000000e+00 0.000000e+00
## [2,] 5.551115e-17 1.000000e+00 1.110223e-16
## [3,] 0.000000e+00 5.551115e-17 1.000000e+00
```

```
X %*% solve(Y)
```

```
##            [,1]      [,2]       [,3]
## [1,] -0.2447419 0.2370937 -1.7896750
## [2,] -1.3177820 0.6891013 -2.3112811
## [3,]  0.8107075 0.4646272 -0.5717017
```

```
solve(Y) %*% X
```

```
##            [,1]       [,2]       [,3]
## [1,]  0.9915870 -1.7338432  1.8692161
## [2,]  0.6753346 -0.6378585  1.5892925
## [3,] -0.9724665 -0.1437859 -0.4810707
```

1.2.2.1 Important matrices

```
ONE <- diag(rep(1,n))
ONE
```

```
##      [,1] [,2] [,3]
## [1,]    1    0    0
## [2,]    0    1    0
## [3,]    0    0    1
```

```
ZERO <- matrix(0,n,n)
ZERO
```

```
##      [,1] [,2] [,3]
## [1,]    0    0    0
## [2,]    0    0    0
## [3,]    0    0    0
```

```
X.inverse <- solve(X)
X
```

```
##      [,1] [,2] [,3]
## [1,]    4  -16   12
## [2,]   -6  -10   18
## [3,]    9    0   14
```

```
X.inverse
```

```
##             [,1]        [,2]       [,3]
## [1,]  0.04098361 -0.06557377 0.04918033
## [2,] -0.07201405  0.01522248 0.04215457
## [3,] -0.02634660  0.04215457 0.03981265
```

2 Power and exponential

2.1 Power to integer

```
x <- 3
p <- 2
x^p
```

```
## [1] 9
```

```
x * x
```

```
## [1] 9
```

```
p <- 3
x^3
```

```
## [1] 27
```

```
x * x * x
```

```
## [1] 27
```

```
X <- matrix(c(1,2,3,4),2,2)
p <- 2
X %*% X
```

```
##      [,1] [,2]
## [1,]    7   15
## [2,]   10   22
```

```
p <- 3
X %*% X %*% X
```

```
##      [,1] [,2]
## [1,]   37   81
## [2,]   54  118
```

2.2 Power to non-integer

```
x <- 2
p <- 2
x. <- x^p
x.
```

```
## [1] 4
```

```
p.inverse <- 1/p
p.inverse
```

```
## [1] 0.5
```

```
x.^p.inverse
```

```
## [1] 2
```

2.3 Power of diagonal matrix

```
X <- matrix(c(2,0,0,3),2,2) # Diagonal matrix
X
```

```
##      [,1] [,2]
## [1,]    2    0
## [2,]    0    3
```

```
diag(c(2,3))
```

```
##      [,1] [,2]
## [1,]    2    0
## [2,]    0    3
```

```
X %*% X
```

```
##      [,1] [,2]
## [1,]    4    0
## [2,]    0    9
```

```
p <- 2
p.inverse <- 1/p
diag.X <- diag(X)
diag.X
```

```
## [1] 2 3
```

```
diag(diag.X^p)
```

```
##      [,1] [,2]
## [1,]    4    0
## [2,]    0    9
```

```
diag(diag.X^p.inverse)
```

```
##          [,1]     [,2]
## [1,] 1.414214 0.000000
## [2,] 0.000000 1.732051
```

2.4 Decomposition of matrix 行列の分解

Eigen value decomposition　固有値分解

$$X = VSV^{-1}$$

$$X^2 = (VSV^{-1})(VSV^{-1}) = (VS)(V^{-1}V)(SV^{-1}) = V(S^2)V^{-1}$$

$$X^p = V(S^p)V^{-1}$$

```
X <- matrix(c(2,3,4,5),2,2)
eigen.out <- eigen(X)
eigen.out
```

```
## eigen() decomposition
## $values
## [1]  7.2749172 -0.2749172
##
## $vectors
##            [,1]       [,2]
## [1,] -0.6042272 -0.8692521
## [2,] -0.7968121  0.4943691
```

```
V <- eigen.out[[2]]
S <- diag(eigen.out[[1]])
V %*% S %*% solve(V)
```

```
##      [,1] [,2]
## [1,]    2    4
## [2,]    3    5
```

```
X %*% X
```

```
##      [,1] [,2]
## [1,]   16   28
## [2,]   21   37
```

```
V %*% diag(eigen.out[[1]]^2) %*% solve(V)
```

```
##      [,1] [,2]
## [1,]   16   28
## [2,]   21   37
```

```
X. <- V %*% diag(eigen.out[[1]]^(1/2)) %*% solve(V)
X.
```

```
##      [,1] [,2]
## [1,]  NaN  NaN
## [2,]  NaN  NaN
```

```
s <- complex(real=eigen.out[[1]],imaginary=0)
s^(1/2)
```

```
## [1] 2.697205+0.0000000i 0.000000+0.5243255i
```

```
X.. <- V %*% diag(s^(1/2)) %*% solve(V)
X..
```

```
##                        [,1]                 [,2]
## [1,] 0.8127223+0.3663357i 1.429014-0.277794i
## [2,] 1.0717608-0.2083458i 1.884483+0.157990i
```

```
X.. %*% X..
```

```
##      [,1] [,2]
## [1,] 2+0i 4+0i
## [2,] 3+0i 5+0i
```

2.5 Matrix exponential

指数関数の性質

$$\frac{d}{dx}e^x = e^x$$

$$e^0 = 1$$

指数関数の定義式

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

$$e^{VSV^{-1}} = \sum_{k=0}^{\infty} \frac{1}{k!} VS^kV^{-1} = V(\sum_{k=0}^{\infty} \frac{1}{k!} S^k)V^{-1}$$

対角成分ごとにe^xを計算できる

```
X <- matrix(c(2,3,4,5),2,2)
eigen.out <- eigen(X)
s <- eigen.out[[1]]
V <- eigen.out[[2]]
s
```

```
## [1]  7.2749172 -0.2749172
```

```
V
```

```
##            [,1]       [,2]
## [1,] -0.6042272 -0.8692521
## [2,] -0.7968121  0.4943691
```

```
V %*% diag(exp(s))%*% solve(V)
```

```
##          [,1]      [,2]
## [1,] 435.5261  764.4523
## [2,] 573.3392 1008.8653
```

3 非正方行列の積 Multiplication of non-square matrices

$M_{n,m}$, $n \times m$ matrix can be multiplied by $M_{k,n}$ from its left and by $M_{m,k}$ from its right. The dimension of the products are $k \times n$ and $m \times k$, respectively.

$n \times m$ 行列 $M_{n,m}$ は、左から$M_{l,n}$行列を掛けることができ、右から$M_{m,k}$行列を掛けることができる。生じる行列はそれぞれ、$k \times n$、$m \times k$ である。

```
n <- 4
m <- 3
k <- 2

M1 <- matrix(1:(n*m),n,m)
M1
```

```
##      [,1] [,2] [,3]
## [1,]    1    5    9
## [2,]    2    6   10
## [3,]    3    7   11
## [4,]    4    8   12
```

```
dim(M1)
```

```
## [1] 4 3
```

```
M2 <- matrix(1:(k*n),k,n)
M2
```

```
##      [,1] [,2] [,3] [,4]
## [1,]    1    3    5    7
## [2,]    2    4    6    8
```

```
dim(M2)
```

```
## [1] 2 4
```

```
M21 <- M2 %*% M1
M21
```

```
##      [,1] [,2] [,3]
## [1,]   50  114  178
## [2,]   60  140  220
```

```
dim(M21)
```

```
## [1] 2 3
```

```
M3 <- matrix(1:(m*k),m,k)
M13 <- M1 %*% M3
M13
```

```
##      [,1] [,2]
## [1,]   38   83
## [2,]   44   98
## [3,]   50  113
## [4,]   56  128
```

```
dim(M13)
```

```
## [1] 4 2
```

3.1 ベクトルの内積 Inner product of vectors

A vector with n elements can be considered $n \times 1$ matrix or $1 \times n$ matrix.

要素数 n のベクトルは$n \times 1$行列、$1 \times n$行列とみなせる。

```
n <- 3
v1 <- c(3,5,6)
v2 <- c(1,2,4)
V1 <- matrix(v1,nrow=n)
V1
```

```
##      [,1]
## [1,]    3
## [2,]    5
## [3,]    6
```

```
V2 <- matrix(v2,nrow=1)
V2
```

```
##      [,1] [,2] [,3]
## [1,]    1    2    4
```

$n \times 1$ matrix can be multiplied by $1 \times n$ matrix from its left. The product is 1×1 matrix, or a scaler value.

$n \times 1$ 行列には、その左側から$1 \times n$行列を掛けることができる。その積は$1 1 $ 行列、もしくは、スカラー値である。

```
V2 %*% V1
```

```
##      [,1]
## [1,]   37
```

The following returns the same value. 次のような計算と同じである。

```
sum(v1*v2)
```

```
## [1] 37
```

4 Differential equation 微分方程式

4.1 Single differential equation 1つの微分方程式

$$\frac{dx(t)}{dt} = ax$$

解

$$x(t) = b \times e^a t$$

4.2 System of differential equation 連立微分方程式

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The solution of this system is known as, この連立微分方程式の解は以下であると知られている。

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{t \times A} \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$
$$A = VSV^{-1}$$
$$S = diag(e^{\lambda_A})$$
$$\Sigma(t) = diag(e^{t \times \lambda_A})$$
$$e^{t \times A} = V\Sigma(t)V^{-1}$$

```
A <- matrix(c(0.5,1,-1,-0.3),2,2)
A
```

```
##      [,1] [,2]
## [1,]  0.5 -1.0
## [2,]  1.0 -0.3
```

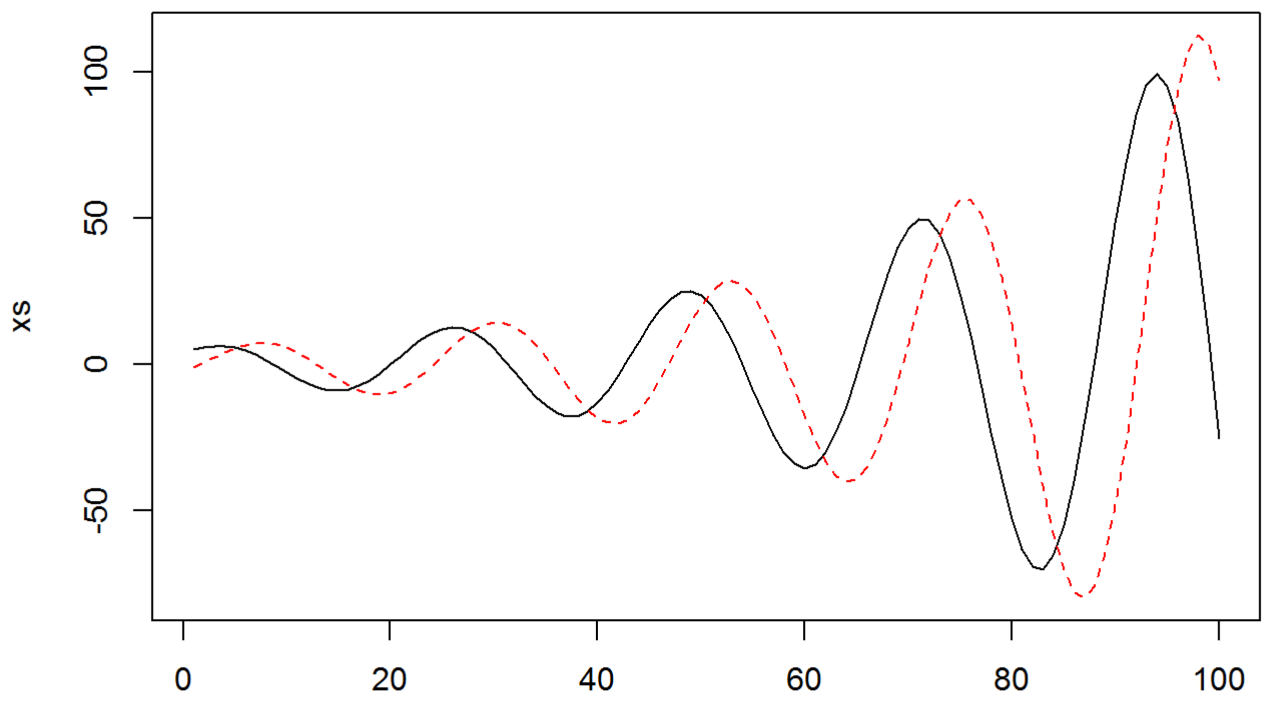
```
eigen.out <- eigen(A)
S <- diag(eigen.out[[1]])
V <- eigen.out[[2]]

t <- seq(from=0,to=30,length=100)
x0 <- c(5,-1)
xs <- matrix(0,length(t),2)
for(i in 1:length(t)){
  etA <- V %*% diag(exp(t[i]*diag(S))) %*% solve(V)
  xs[i,] <- etA %*% x0
}
matplot(xs,type="l")
```

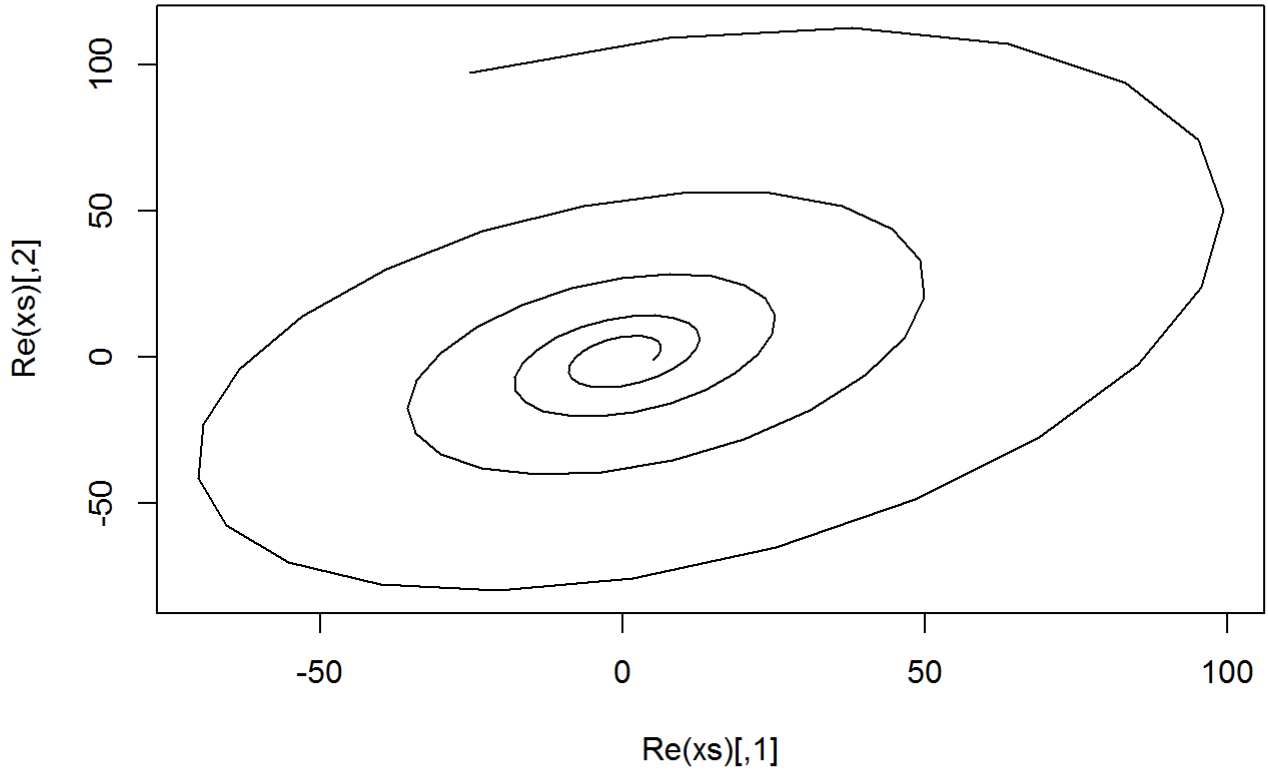
```
## Warning in xy.coords(x, y, xlabel, ylabel, log = log): 複素数の虚部は、コネ
## クションで捨てられました
```

```
## Warning in xy.coords(x, y, xlabel, ylabel, log): 複素数の虚部は、コネクショ
## ンで捨てられました
```

```
## Warning in xy.coords(x, y): 複素数の虚部は、コネクションで捨てられました
```




```
plot(Re(xs),type="l") # state space plot
```



5 Exercises

5.1 Exercise 1-1

正方行列の和を計算する関数は次のように作れる。 それにならって、行列の積(非正方行列を含む)を計算する関数を作成、それが正しいことを確かめよ。

The following function calculate sum of matrices. Make functions of production of (non-square) matrices in the similar way.

```
my.matrix.sum <- function(x,y){
  dm <- dim(x)
  ret <- matrix(0,dm[1],dm[2])
  for(i in 1:dm[1]){
    for(j in 1:dm[2]){
      ret[i,j] <- x[i,j]+y[i,j]
    }
  }
  return(ret)
}

x <- matrix(c(2,3,4,5),2,2)
y <- matrix(c(4,5,6,7),2,2)
x + y
```

```
##      [,1] [,2]
## [1,]    6   10
## [2,]    8   12
```

```
my.matrix.sum(x,y)
```

```
##      [,1] [,2]
## [1,]    6   10
## [2,]    8   12
```

5.2 Exercise 1-2

行列の冪x^pを計算する関数を作成せよ

Make a function of power of matrix.

5.3 Exercise 1-3

行列の指数関数$e^{t \times X}$を計算する関数を作成せよ

Make a function of exponential of matix.

5.4 Exercise 1-4

色々な2×2 行列を作り、それに基づく２変量連立微分方程式に関する解をプロットせよ。matplotと状態空間co-plotをせよ。カーブが多様であるように行列を選べ。

Make various $2 2 $ matrices and plot their answers of the system of differential equations of 2 variables. Plot them in two ways (matplot-way and co-plot-way in state space). Select matrices so that the curves of the variables are heterogeneous.

5.5 Exercise 1-5

3×3 行列による3変量の場合を同様に行え。matplot-wayと3次元状態空間プロットを行え。 さまざまなカーブを描くように、行列を選べ。

Make various 3×3 matrices for three variables’ system of differential equations.

plot them in two ways (matplot-way and 3D-plot-way in state space).

Select matrices so that the curves or the variables are heterogeneous.