Displacements and Loads

- Partition the stiffness matrix as indicated by Eq. 14–18. Expansion then leads to

$$\mathbf{Q}_k = \mathbf{K}_{11}\mathbf{D}_u + \mathbf{K}_{12}\mathbf{D}_k$$
$$\mathbf{Q}_u = \mathbf{K}_{21}\mathbf{D}_u + \mathbf{K}_{22}\mathbf{D}_k$$

The unknown displacements $\mathbf{D}_u$ are determined from the first of these equations. Using these values, the support reactions $\mathbf{Q}_u$ are computed from the second equation. Finally, the internal loadings $\mathbf{q}$ at the ends of the members can be computed from Eq. 16–7, namely

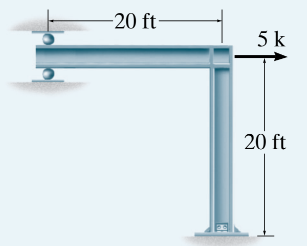
$$\mathbf{q} = \mathbf{k}'\mathbf{T}\mathbf{D}$$

If the results of any of the unknowns are calculated as negative quantities, it indicates they act in the negative coordinate directions.

16

EXAMPLE 16.1

Determine the loadings at the joints of the two-member frame shown in Fig. 16–4*a*. Take $I = 500 \text{ in}^4$, $A = 10 \text{ in}^2$, and $E = 29(10^3)$ ksi for both members.

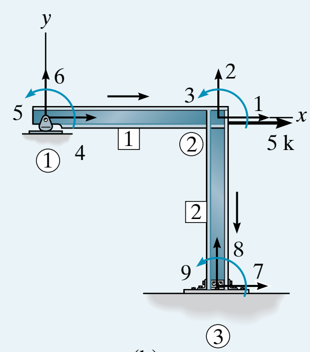


(a)

SOLUTION

Notation. By inspection, the frame has two elements and three nodes, which are identified as shown in Fig. 16–4*b*. The origin of the global coordinate system is located at ①. The code numbers at the nodes are specified with the *unconstrained degrees of freedom numbered first*. From the constraints at ① and ③, and the applied loading, we have

$$\mathbf{D}_k = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}\begin{array}{c} 6 \\ 7 \\ 8 \\ 9 \end{array} \qquad \mathbf{Q}_k = \begin{array}{c} \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$



(b)

Fig. 16–4

Structure Stiffness Matrix. The following terms are common to both element stiffness matrices:

$$\frac{AE}{L} = \frac{10[29(10^3)]}{20(12)} = 1208.3 \text{ k/in.}$$

$$\frac{12EI}{L^3} = \frac{12[29(10^3)(500)]}{[20(12)]^3} = 12.6 \text{ k/in.}$$

$$\frac{6EI}{L^2} = \frac{6\left[29(10^3)(500)\right]}{\left[20(12)\right]^2} = 1510.4 \text{ k}$$

$$\frac{4EI}{L} = \frac{4\left[29(10^3)(500)\right]}{20(12)} = 241.7(10^3) \text{ k} \cdot \text{in.}$$

$$\frac{2EI}{L} = \frac{2\left[29(10^3)(500)\right]}{20(12)} = 120.83(10^3) \text{ k} \cdot \text{in.}$$

Member 1:

$$\lambda_x = \frac{20 - 0}{20} = 1 \qquad \lambda_y = \frac{0 - 0}{20} = 0$$

Substituting the data into Eq. 16–10, we have

$$\mathbf{k}_1 = \begin{array}{c} \begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccccc} 1208.3 & 0 & 0 & -1208.3 & 0 & 0 \\ 0 & 12.6 & 1510.4 & 0 & -12.6 & 1510.4 \\ 0 & 1510.4 & 241.7(10^3) & 0 & -1510.4 & 120.83(10^3) \\ -1208.3 & 0 & 0 & 1208.3 & 0 & 0 \\ 0 & -12.6 & -1510.4 & 0 & 12.6 & -1510.4 \\ 0 & 1510.4 & 120.83(10^3) & 0 & -1510.4 & 241.7(10^3) \end{array}\right] \end{array} \begin{array}{c} \\ 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{array}$$

The rows and columns of this 6×6 matrix are identified by the three x, y, z code numbers, first at the near end and followed by the far end, that is, 4, 6, 5, 1, 2, 3, respectively, Fig. 16–4*b*. This is done for later assembly of the elements.

Member 2:

$$\lambda_x = \frac{20 - 20}{20} = 0 \qquad \lambda_y = \frac{-20 - 0}{20} = -1$$

Substituting the data into Eq. 16–10 yields

$$\mathbf{k}_2 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{cccccc} 12.6 & 0 & 1510.4 & -12.6 & 0 & 1510.4 \\ 0 & 1208.3 & 0 & 0 & -1208.3 & 0 \\ 1510.4 & 0 & 241.7(10^3) & -1510.4 & 0 & 120.83(10^3) \\ -12.6 & 0 & -1510.4 & 12.6 & 0 & -1510.4 \\ 0 & -1208.3 & 0 & 0 & 1208.3 & 0 \\ 1510.4 & 0 & 120.83(10^3) & -1510.4 & 0 & 241.7(10^3) \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array}$$

As usual, column and row identification is referenced by the three code numbers in x, y, z sequence for the near and far ends, respectively, that is, 1, 2, 3, then 7, 8, 9, Fig. 16–4*b*.

The structure stiffness matrix is determined by assembling $\mathbf{k}_1$ and $\mathbf{k}_2$. The result, shown partitioned, as $\mathbf{Q} = \mathbf{KD}$, is

$$\left[\begin{array}{c} 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{array}\right] = \begin{array}{c} \begin{array}{ccccc|cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccccc|cccc} 1220.9 & 0 & 1510.4 & -1208.3 & 0 & 0 & -12.6 & 0 & 1510.4 \\ 0 & 1220.9 & -1510.4 & 0 & -1510.4 & -12.6 & 0 & -1208.3 & 0 \\ 1510.4 & -1510.4 & 483.3(10^3) & 0 & 120.83(10^3) & 1510.4 & -1510.4 & 0 & 120.83(10^3) \\ -1208.3 & 0 & 0 & 1208.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1510.4 & 120.83(10^3) & 0 & 241.7(10^3) & 1510.4 & 0 & 0 & 0 \\ \hline 0 & -12.6 & 1510.4 & 0 & 1510.4 & 12.6 & 0 & 0 & 0 \\ -12.6 & 0 & -1510.4 & 0 & 0 & 0 & 12.6 & 0 & -1510.4 \\ 0 & -1208.3 & 0 & 0 & 0 & 0 & 0 & 1208.3 & 0 \\ 1510.4 & 0 & 120.83(10^3) & 0 & 0 & 0 & -1510.4 & 0 & 241.7(10^3) \end{array}\right] \end{array} \left[\begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ \hline 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \qquad (1)$$

Displacements and Loads. Expanding to determine the displacements yields

$$\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1220.9 & 0 & 1510.4 & -1208.3 & 0 \\ 0 & 1220.9 & -1510.4 & 0 & -1510.4 \\ 1510.4 & -1510.4 & 483.3(10^3) & 0 & 120.83(10^3) \\ -1208.3 & 0 & 0 & 1208.3 & 0 \\ 0 & -1510.4 & 120.83(10^3) & 0 & 241.7(10^3) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving, we obtain

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{bmatrix} = \begin{bmatrix} 0.696 \text{ in.} \\ -1.55(10^{-3}) \text{ in.} \\ -2.488(10^{-3}) \text{ rad} \\ 0.696 \text{ in.} \\ 1.234(10^{-3}) \text{ rad} \end{bmatrix}$$

Using these results, the support reactions are determined from Eq. (1) as follows:

$$\begin{bmatrix} Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{bmatrix} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & -12.6 & 1510.4 & 0 & 1510.4 \\ -12.6 & 0 & -1510.4 & 0 & 0 \\ 0 & -1208.3 & 0 & 0 & 0 \\ 1510.4 & 0 & 120.83(10^3) & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} 0.696 \\ -1.55(10^{-3}) \\ -2.488(10^{-3}) \\ 0.696 \\ 1.234(10^{-3}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.87 \text{ k} \\ -5.00 \text{ k} \\ 1.87 \text{ k} \\ 750 \text{ k} \cdot \text{in.} \end{bmatrix} \qquad \textit{Ans.}$$

The internal loadings at node ② can be determined by applying Eq. 16–7 to member 1. Here $\mathbf{k}_1'$ is defined by Eq. 16–1 and $\mathbf{T}_1$ by Eq. 16–3. Thus,

$$\mathbf{q}_1 = \mathbf{k}_1\mathbf{T}_1\mathbf{D} = \begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \\ \end{array}$$

$$\left[\begin{array}{cccccc} 1208.3 & 0 & 0 & -1208.3 & 0 & 0 \\ 0 & 12.6 & 1510.4 & 0 & -12.6 & 1510.4 \\ 0 & 1510.4 & 241.7(10^3) & 0 & -1510.4 & 120.83(10^3) \\ -1208.3 & 0 & 0 & 1208.3 & 0 & 0 \\ 0 & -12.6 & -1510.4 & 0 & 12.6 & -1510.4 \\ 0 & 1510.4 & 120.83(10^3) & 0 & -1510.4 & 241.7(10^3) \end{array}\right] \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \left[\begin{array}{c} 0.696 \\ 0 \\ 1.234(10^{-3}) \\ 0.696 \\ -1.55(10^{-3}) \\ -2.488(10^{-3}) \end{array}\right] \end{array} \begin{array}{c} 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{array}$$

Note the appropriate arrangement of the elements in the matrices as indicated by the code numbers alongside the columns and rows. Solving yields

$$\begin{bmatrix} q_4 \\ q_6 \\ q_5 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.87\text{ k} \\ 0 \\ 0 \\ 1.87\text{ k} \\ -450\text{ k}\cdot\text{in.} \end{bmatrix} \qquad \textit{Ans.}$$

The above results are shown in Fig. 16–4*c*. The directions of these vectors are in accordance with the positive directions defined in Fig. 16–1. Furthermore, the origin of the local x', y', z' axes is at the near end of the member. In a similar manner, the free-body diagram of member 2 is shown in Fig. 16–4*d*.

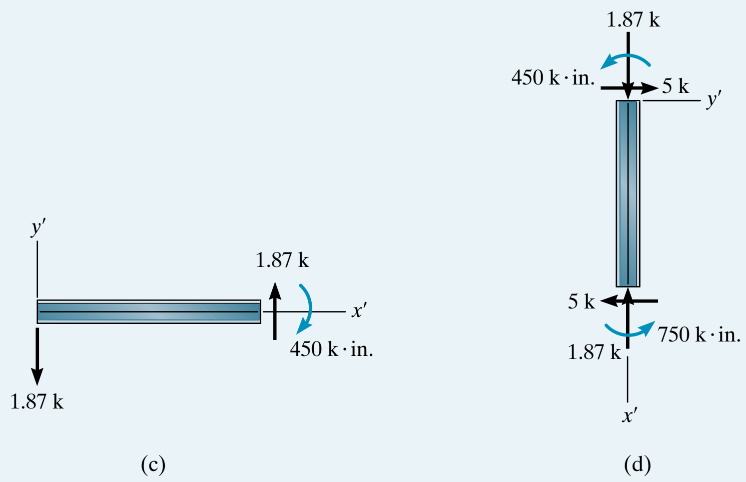
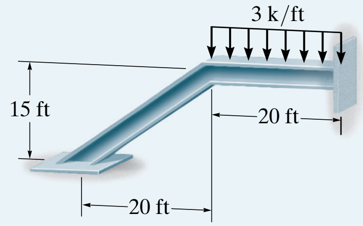


Fig. 16–4

EXAMPLE 16.2

Determine the loadings at the ends of each member of the frame shown in Fig. 16–5*a*. Take $I = 600 \text{ in}^4$, $A = 12 \text{ in}^2$, and $E = 29(10^3)$ ksi for each member.



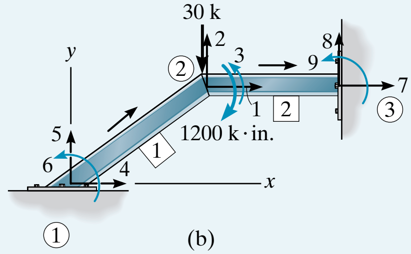
(a)

Fig. 16–5

SOLUTION

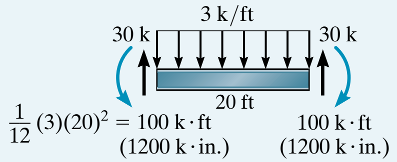
Notation. To perform a matrix analysis, the distributed loading acting on the horizontal member will be replaced by equivalent end moments and shears computed from statics and the table listed on the inside back cover. (Note that no external force of 30 k or moment of 1200 k · in. is placed at ③ since the reactions at code numbers 8 and 9 *are to be unknowns* in the load matrix.) Then using superposition, the results obtained for the frame in Fig. 16–5*b* will be modified for this member by the loads shown in Fig. 16–5*c*.

As shown in Fig. 16–5*b*, the nodes and members are numbered and the origin of the global coordinate system is placed at node ①. As usual, the code numbers are specified with numbers assigned first to the unconstrained degrees of freedom. Thus,



(b)

+



(c)

$$
\mathbf{D}_k = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \qquad \mathbf{Q}_k = \begin{bmatrix} 0 \\ -30 \\ -1200 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}
$$

Structure Stiffness Matrix

Member 1:

$$\frac{AE}{L} = \frac{12[29(10^3)]}{25(12)} = 1160 \text{ k/in.}$$

$$\frac{12EI}{L^3} = \frac{12[29(10^3)]600}{[25(12)]^3} = 7.73 \text{ k/in.}$$

$$\frac{6EI}{L^2} = \frac{6[29(10^3)]600}{[25(12)]^2} = 1160 \text{ k}$$

$$\frac{4EI}{L} = \frac{4[29(10^3)]600}{25(12)} = 232(10^3) \text{ k} \cdot \text{in.}$$

$$\frac{2EI}{L} = \frac{2[29(10^3)]600}{25(12)} = 116(10^3) \text{ k} \cdot \text{in.}$$

$$\lambda_x = \frac{20 - 0}{25} = 0.8 \qquad \lambda_y = \frac{15 - 0}{25} = 0.6$$