(1) Let variable $\mathcal{X}$ follow a Bernoulli distribution with parameter p, i.e., the probability of x is $f(x) = p^x(1-p)^{1-x}$. If the set of independent and identically distributed data (i.i.d.) samples $\mathcal{X} = \{x_1, \ldots, x_N\}$ follows the Bernoulli distribution, derive the maximum likelihood estimate (MLE) for p^{MLE}.

$$\ell(p) = \prod_{n=1}^{N} f(x_n)$$

$$\mathcal{L}(p) = \log \ell(p) = \log \prod_{n=1}^{N} f(x_n) = \sum_{n=1}^{N} \log f(x_n) =$$

$$= \sum_{n=1}^{N} \log\left[p^{x_n}(1-p)^{1-x_n}\right] = \sum_{n=1}^{N}\left[\log p^{x_n} + \log(1-p)^{1-x_n}\right]$$

$$= \sum_{n=1}^{N}\left[x_n \log p + (1-x_n)\log(1-p)\right]$$

$$= (\log p)\left(\sum_{n=1}^{N} x_n\right) + \left[\log(1-p)\right]\sum_{n=1}^{N}(1-x_n)$$

$$\frac{\partial \mathcal{L}(p)}{\partial p} = \frac{1}{p}\left(\sum_{n=1}^{N} x_n\right) + \left(-\frac{1}{1-p}\right)\sum_{n=1}^{N}(1-x_n) = \frac{(1-p)\sum_{n=1}^{N} x_n - p\sum_{n=1}^{N}(1-x_n)}{p(1-p)} =$$

$$= \frac{\sum_{n=1}^{N} x_n - p\sum_{n=1}^{N} x_n - pN + p\sum x_n}{p(1-p)} = \frac{\sum_{n=1}^{N} x_n - pN}{p(1-p)}$$

$$\frac{\partial \mathcal{L}(p)}{\partial p} = 0 \Rightarrow \sum_{n=1}^{N} x_n - pN = 0 \Rightarrow \boxed{\hat{p} = \frac{1}{N}\sum_{n=1}^{N} x_n}$$

(2) Let variable $\mathcal{X}$ follow an exponential distribution with parameter λ, i.e., the probability of x is $f(x) = \lambda e^{-\lambda x}$. If the set of independent and identically distributed data (i.i.d.) samples $\mathcal{X} = \{x_1, \ldots, x_N\}$ follows the exponential distribution, derive the maximum likelihood estimate (MLE) for λ^{MLE}.

$$\ell(\lambda) = \prod_{n=1}^{N} f(x_n)$$

$$\mathcal{L}(\lambda) = \log \prod_{n=1}^{N} f(x_n) = \sum_{n=1}^{N} \log f(x_n) = \sum_{n=1}^{N} \log\left(\lambda e^{-\lambda x_n}\right)$$

$$= \sum_{n=1}^{N}\left(\log\lambda + \log e^{-\lambda x_n}\right) = \sum_{n=1}^{N}\left(\log\lambda - \lambda x_n\right)$$

$$= N \cdot \log\lambda - \lambda \sum_{n=1}^{N} x_n$$

$$\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda} = \frac{N}{\lambda} - \sum_{n=1}^{N} x_n = 0 \Rightarrow \boxed{\hat{\lambda} = \frac{1}{N}\sum_{n=1}^{N} x_n}$$