Assume a 1-dimensional linear regression model $y = w_0 + w_1 x$. The residual sum of squares (RSS) of the training data $\mathcal{D}^{train} = \{(x_1, y_1), \ldots, (x_N, y_N)\}$ can be written as:

$$RSS(w_0, w_1) = \sum_{n=1}^{N} (y_n - w_0 - w_1 x_n)^2$$

We estimate the weights w_0, w_1 by minimizing the above error.

(1) Show that minimizing RSS results in the following closed-form expression:

$$w_1^* = \frac{\sum_{n=1}^{N} x_n y_n - N\left(\frac{1}{N}\sum_{n=1}^{N} x_n\right)\left(\frac{1}{N}\sum_{n=1}^{N} y_n\right)}{\sum_{n=1}^{N} x_n^2 - N\left(\frac{1}{N}\sum_{n=1}^{N} x_n\right)^2}$$

$$w_0^* = \left(\frac{1}{N}\sum_{n=1}^{N} y_n\right) - w_1\left(\frac{1}{N}\sum_{n=1}^{N} x_n\right)$$

Tip: Set the partial derivatives $\frac{\vartheta RSS(w_0,w_1)}{\vartheta w_0}$ and $\frac{\vartheta RSS(w_0,w_1)}{\vartheta w_1}$ equal to 0. Then solve a 2×2 system of linear equations with respect to w_0 and w_1.
We minimize the RSS function with respect to w_0 and w_1

$$\frac{\vartheta RSS(w_0, w_1)}{\vartheta w_0} = 0 \Rightarrow -2\sum_{n=1}^{N}(y_n - w_0 - w_1 x_n) = 0$$

$$\Rightarrow N w_0 + \left(\sum_{n=1}^{N} x_n\right) w_1 = \sum_{n=1}^{N} y_n \quad (1)$$

$$\frac{\vartheta RSS(w_0, w_1)}{\vartheta w_1} = 0 \Rightarrow -2\sum_{n=1}^{N} x_n(y_n - w_0 - w_1 x_n) = 0$$

$$\Rightarrow \left(\sum_{n=1}^{N} x_n\right) w_0 + \left(\sum_{n=1}^{N} x_{n1}^2\right) w_1 = \sum_{n=1}^{N} x_n y_n \quad (2)$$

Combining (1) and (2) we get the 2×2 system of equations

$$\begin{bmatrix} N & \sum_{n=1}^{N} x_n \\ \sum_{n=1}^{N} x_n & \sum_{n=1}^{N} x_n^2 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} N & \sum_{n=1}^{N} y_n \\ \sum_{n=1}^{N} x_n & \sum_{n=1}^{N} x_n y_n \end{bmatrix}$$

We solve the above system using the determinants and we get:

$$w_1^* = \frac{\sum_{n=1}^{N} x_n y_n - N\left(\frac{1}{N}\sum_{n=1}^{N} x_n\right)\left(\frac{1}{N}\sum_{n=1}^{N} y_n\right)}{\sum_{n=1}^{N} x_n^2 - N\left(\frac{1}{N}\sum_{n=1}^{N} x_n\right)^2}$$

$$w_0^* = \left(\frac{1}{N}\sum_{n=1}^{N} y_n\right) - w_1\left(\frac{1}{N}\sum_{n=1}^{N} x_n\right)$$

(2) Show that the above expressions for w_0^* and w_1^* are equivalent to the following:

$$w_1^* = \frac{\sum\limits_{n=1}^{N} (x_n - \bar{x})(y_n - \bar{y})}{\sum\limits_{i=1}^{N} (x_n - \bar{x})^2}$$
$$w_0^* = \bar{y} - w_1 \bar{x}$$

where $\bar{x} = \frac{1}{N}\sum_{n=1}^{N} x_n$ and $\bar{y} = \frac{1}{N}\sum_{n=1}^{N} y_n$ are the sample means of input features and outcome values, respectively.

$$\begin{aligned}
w_1^* &= \frac{\sum\limits_{n=1}^{N} (x_n - \bar{x})(y_n - \bar{y})}{\sum\limits_{n=1}^{N} (x_n - \bar{x})^2} \\
&= \frac{\sum\limits_{n=1}^{N} x_n y_n - \bar{y}\sum\limits_{n=1}^{N} x_n - \bar{x}\sum\limits_{n=1}^{N} y_n + N\bar{x}\bar{y}}{\sum\limits_{n=1}^{N} x_n^2 - 2\bar{x}\sum\limits_{n=1}^{N} x_n + N\bar{x}^2} \\
&= \frac{\sum\limits_{n=1}^{N} x_n y_n - \frac{1}{N}\sum\limits_{n=1}^{N} y_n \sum\limits_{n=1}^{N} x_n - \frac{1}{N}\sum\limits_{n=1}^{N} x_n \sum\limits_{n=1}^{N} y_n + N\frac{1}{N}\sum\limits_{n=1}^{N} x_n \frac{1}{N}\sum\limits_{n=1}^{N} y_n}{\sum\limits_{n=1}^{N} x_n^2 - 2\frac{1}{N}\sum\limits_{n=1}^{N} x_n \sum\limits_{n=1}^{N} x_n + N\left(\frac{1}{N}\sum\limits_{n=1}^{N} x_n\right)^2} \\
&= \frac{\sum\limits_{n=1}^{N} x_n y_n - N\left(\frac{1}{N}\sum\limits_{n=1}^{N} x_n\right)\left(\frac{1}{N}\sum\limits_{n=1}^{N} y_n\right)}{\sum\limits_{n=1}^{N} x_n^2 - N\left(\frac{1}{N}\sum\limits_{n=1}^{N} x_n\right)^2}
\end{aligned}$$

It is straightforward to show for w_1^*.

(3) How would you interpret w_0^*?

The weight w_0^*, which is the bias term of the linear regression line, is equivalent to the mean of the error.