Basic Linear Algebra

1 Matrices

- Rectangular array of data: elements are real numbers.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

2 Basic concepts

- vectors
- norms and distances
- eigenvalues, eigenvectors
- linearly independent vectors, basis
- orthogonal bases
- matrices, orthogonal matrices
- orthogonal matrix decompositions: SVD

3 Review of fundamental concepts

3.1 Matrix-vector multiplication

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{pmatrix} = y$$

- Alternative presentation of matrix-vector multiplication

$$y = Ax = (\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)\, x = \sum_{j=1}^{n} \mathbf{a}_j x_j$$

- The vector y is a linear combination of the columns of A.

3.2 Matrix-matrix multiplication

- Let $A \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{p\times n}$. Then, $C = AB = (c_{ij}) \in \mathbb{R}^{m\times n}$ is defined as follows:

$$c_{ij} = \sum_{k=1}^{s} a_{ik} b_{kj}, \text{ for all } i = 1, \cdots, m, j = 1, \cdots, n.$$

- Each column vector in B is multiplied by A.

3.3 Vector norms

- Measure the "size" of a vector.
  - 1-norm: $||x||_1 = \sum_{i=1}^{n} |x_i|$
  - 2-norm: $||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
  - max-norm: $||x||_\infty = \max_{1\leq i\leq n} |x_i|$
  - All of the above are special cases of the L_p-norm (or p-norm):

$$||x||_p = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}$$

- Two norms $||\cdot||_\alpha$ and $||\cdot||_\beta$ are equivalent if there exist constants C_1 and C_2 such that

$$C_1||x||_\alpha \leq ||x||_\beta \leq C_2||x||_\alpha, \text{ for all } x.$$

  - $||\cdot||_1$ and $||\cdot||_2$ are equivalent with $x \in \mathbb{R}^n$:

$$||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2.$$

- Generally, a vector norm is a mapping $R^n \to R$, with the properties
  - $||x|| \geq 0$, for all x
  - $||x|| = 0$, if and only if $x = 0$
  - $||\alpha x|| = |\alpha|||x||$, $\alpha \in \mathbb{R}$
  - $||x + y|| \leq ||x|| + ||y||$, for all x and y
- How to measure distance between vectors?
  - Obvious answer: the distance between two vectors x and y is $||x - y||$, where $||\cdot||$ is some vector norm.
  - Alternative: use the angle between two vectors x and y to measure the distance between them.
  - How to calculate the angle between two vectors?
- Angle between vectors
  - The inner product between two vectors is defined by $(x, y) = x^T y$.
  - This is associated with the Euclidean norm: $||x||_2 = (x, x)^{1/2}$.

- The angle θ between two vectors x and y is

$$\cos(\theta) = \frac{(x, y)}{||x||_2 ||y||_2}.$$

- The cosine of the angle between two vectors x and y can be used to measure the similarity between the two vectors: if x and y are close, the angle between them is small, and $\cos(\theta) \approx 1$; if x and y are orthogonal, i.e., (x, y) = 0, $\cos(\theta) = 0$ ($\theta = \pi/2$).

- Why not just use the Euclidean distance?
  - Example: term-document matrix
  - Each entry tells how many times a term appears in the document:

	Doc1	Doc2	Doc3
Term1	10	1	0
Term2	10	1	0
Term3	0	0	0

  - Using the Euclidean distance Documents 1 and 2 look dissimilar, and Documents 2 and 3 look similar. This is just due to the length of the documents.
  - Using the cosine of the angle between document vectors Documents 1 and 2 are similar to each other and dissimilar to Document 3.

3.4 Linear independence

- Given a set of vectors $\{v_1, v_2, \cdots, n_n\} \in \mathbb{R}^m$, with $m \geq n$, consider the set of linear combinations $y = \sum_{j=1}^{n} \alpha_j v_j$ for arbitrary coefficients α_j's.
- The vectors $\{v_1, v_2, \cdots, n_n\}$ are linearly independent, if $\sum_{j=1}^{n} \alpha_j v_j = 0$, if and only if $\alpha_j = 0$ for all $j = 1, \cdots, n$.
- A set of m linearly independent vectors of $\mathbb{R}^m$ is called a basis in $\mathbb{R}^m$: any vector in $\mathbb{R}^m$ can be expressed as a linear combination of the basis vectors.

3.5 Matrix rank

- The rank of a matrix is the maximum number of linearly independent column vectors.
- A square matrix $A \in \mathbb{R}^{n \times n}$ with rank n is called nonsingular.
- A nonsingular matrix A has an inverse A^{-1} satisfying

$$AA^{-1} = A^{-1}A = I_n.$$

- What is the rank of an out-product matrix $xy^T \in \mathbb{R}^{m \times n}$ with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$?
- Let $A \in \mathbb{R}^{n \times n}$ be nonsingular, and let $B = A + uv^T$ with $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$. Then, $B^{-1} = A^{-1} - \frac{A^{-1}uv^T A^{-1}}{1+v^T A^{-1} u}$.

3.6 Range and Null Space

- V is a subspace of $\mathbb{R}^m$, if and only if $\alpha v_1 + \beta v_2 \in V$, for any $v_1, v_2 \in V$ and any scalars α and β.
  - Let W be the set of all points, (x, y), from $\mathbb{R}^2$ in which $x \geq 0$. Is this a subspace of $\mathbb{R}^2$?
  - Let W be the set of all points from $\mathbb{R}^3$ of the form $(0, x_2, x_3)$. Is this a subspace of $\mathbb{R}^3$?
  - Let W be the set of all points from $\mathbb{R}^3$ of the form $(1, x_2, x_3)$. Is this a subspace of $\mathbb{R}^3$?
- The range of a matrix $A \in \mathbb{R}^{m\times n}$ is defined by

$$\mathrm{ran}(A) = \{y \in \mathbb{R}^m : y = Ax \text{ for some } x \in \mathbb{R}^n\}.$$

- The null space of A is defined by

$$\mathrm{null}(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$

- It follows from the definition that $\mathrm{rank}(A) = \dim(\mathrm{ran}(A))$.
  - The dimension of a space S, denoted as $\dim(S)$ denotes the maximum number of linearly independent vectors in S.
- Show that

$$\dim(\mathrm{null}(A)) + \mathrm{rank}(A) = n.$$

3.7 Eigenvalues and eigenvectors

- Let A be a $n \times n$ matrix. The vector $v \neq 0$ that satisfies

$$Av = \lambda v$$

  for some scalar λ is called the eigenvector of A and λ is the eigenvalue corresponding to the eigenvector v.
- An example: $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$

$$Av = \lambda v \rightarrow (A - \lambda I_n)v = 0 \rightarrow |A - \lambda I_n| = 0 \rightarrow \left| \begin{pmatrix} 2-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} \right| = 0$$

  Two eigenvalues $\lambda_1 = 3.62$ and $\lambda_2 = 1.38$. and two eigenvectors:

$$v_1 = \begin{pmatrix} 0.52 \\ 0.85 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0.85 \\ -0.52 \end{pmatrix}$$

- Suppose that λ is an eigenvalue of the matrix A with corresponding eigenvector x. Then if k is a positive integer λ^k is an eigenvalue of the matrix A^k with corresponding eigenvector x.
- Suppose $A = (a_{ij})$ is an $n \times n$ triangular matrix. Compute its eigenvalues.
- Suppose that A is a square matrix and further suppose that there exists an invertible matrix P (of the same size as A of course) such that $P^{-1}AP$ is a diagonal matrix. In such a case we call A diagonalizable and say that P diagonalizes A.
- Let P be an invertible matrix. Show that A and $P^{-1}AP$ contain the same set of eigenvalues.

3.8 Matrix norms

- Let $||\cdot||$ be a vector norm and $A \in \mathbb{R}^{m\times n}$. The corresponding matrix norm is

$$||A|| = \sup_{x\in\mathbb{R}^n : x\neq 0} \frac{||Ax||}{||x||} = \sup_{x\in\mathbb{R}^n : ||x||=1} ||Ax||.$$

- Show that

$$\begin{aligned} ||A+B|| &\leq ||A|| + ||B|| \\ ||Ax|| &\leq ||A||\ ||x|| \\ ||AB|| &\leq ||A||\ ||B|| \end{aligned}$$

- $||A||_2 = \left(\max_i \lambda_i(A^T A)\right)^{1/2}$: square root of the largest eigenvalue of $A^T A$.
- $||A||_1 = \max_j \sum_{i=1}^m |a_{ij}|$: maximum over columns.
- $||A||_\infty = \max_i \sum_{j=1}^n |a_{ij}|$: maximum over rows.
- Frobenius norm: does not correspond to any vector norm.

$$||A||_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}$$

  - Define $\text{trace}(B) = \sum_{i=1}^n b_{ii}$ for any matrix $B = (b_{ij}) \in \mathbb{R}^{n\times n}$.
  - Show that $||A||_F^2 = \text{trace}(AA^T)$.

3.9 Condition number

- Define the condition number of a matrix to be

$$\kappa(A) = ||A||\ ||A^{-1}||.$$

- Consider a matrix $A = \begin{pmatrix} a & 1 \\ 1 & 1 \end{pmatrix}$ with $a \neq 1$. A is nonsingular with

$$A^{-1} = \frac{1}{a-1}\begin{pmatrix} 1 & -1 \\ -1 & a \end{pmatrix}$$

- Nonsingularity is not always enough, as the norm of A^{-1} tends to infinity, as $a \to 1$.

3.10 Orthogonality

- Two vectors x and y are orthogonal, if $x^T y = 0$.
- Given a set of orthogonal vectors $\{v_1, v_2, \cdots, n_n\} \in \mathbb{R}^m$, with $m \geq n$, i.e., $v_i^T v_j = 0$, for $i \neq j$, then they are linearly independent. Why?

- Let the set of orthogonal vectors v_j, $j = 1, \cdots, m$ in $\mathbb{R}^m$ be normalized, i.e., $||q|| = 1$. Then they are orthonormal, and constitute an orthonormal basis in $\mathbb{R}^m$.
- A matrix $V = [v_1, v_2, \cdots, v_m]$ is called an orthogonal matrix, if its columns are orthonormal. Prove the following properties of an orthogonal matrix:
  - An orthogonal matrix $Q \in \mathbb{R}^{m\times m}$ has rank m.
  - $Q^{-1} = Q^T$, that is, $Q^TQ = I_m$, and $QQ^T = I_m$.
  - The Euclidean length of a vector $x \in \mathbb{R}^m$ is invariant under an orthogonal transformation Q, that is,
  $$||Qx||_2^2 = ||x||_2^2.$$
  - The product of two orthogonal matrices Q and P is orthogonal.
- In the previous example we determined the eigenvectors of the matrix:
  $$v_1 = \begin{pmatrix} 0.52 \\ 0.85 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0.85 \\ -0.52 \end{pmatrix}$$
  The vectors v_1 and v_2 are orthogonal: $v_1^T v_2 = 0$.

3.11 Eigenvalues and eigenvectors of a symmetric matrix

- The eigenvectors of a symmetric matrix are mutually orthogonal and its eigenvalues are real.
  - $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix, if $A = A^T$.
- A symmetric matrix $A \in \mathbb{R}^{n\times n}$ can be written in the form $A = U\Lambda U^T$, where the columns of U (U is an orthogonal matrix) are the eigenvectors of A and Λ is a diagonal matrix, the diagonal elements of Λ which are the corresponding eigenvalues of A. This is called the eigendecomposition of A.
- A square matrix A is said to be idempotent if $A^2 = A$.
  - The eigenvalues of a symmetric matrix A are all either 0 or 1 if and only if A is idempotent. Why?
  - Assume a symmetric and idempotent matrix A is of rank r. What is the trace of A?
- Let A and B be both symmetric and of the same size. Then $AB = BA$ if and only if there exists an orthogonal transformation P such that $P^TAP = D_1$, $P^TBP = D_2$, where D_1 and D_2 are diagonal matrices.
- Example of symmetric matrices: graphs
  - The adjacency matrix of an undirected graph is a symmetric matrix.
- **Courant-Fischer Min-max Theorem**:
  - If $A \in \mathbb{R}^{n\times n}$ is symmetric, then
  $$\lambda_k(A) = \max_{\dim(S)=k} \min_{0\neq y\in S} \frac{y^TAy}{y^Ty}.$$

- If $A \in \mathbb{R}^{n \times n}$ is symmetric, then

$$\lambda_1(A) = \max_{0 \neq y \in \mathbb{R}^n} \frac{y^T A y}{y^T y}, \quad \lambda_n(A) = \min_{0 \neq y \in \mathbb{R}^n} \frac{y^T A y}{y^T y}.$$

- Show that $||A||_2 = \left(\max_i \lambda_i(A^T A)\right)^{1/2}$: square root of the largest eigenvalue of $A^T A$.

- If $A \in \mathbb{R}^{n \times n}$ is symmetric, then

$$||A||_2 = \max(|\lambda_1(A)|, |\lambda_n(A)|).$$

- Assume both A and E are n-by-n symmetric matrix. Show that

$$\lambda_k(A) + \lambda_n(E) \leq \lambda_k(A + E) \leq \lambda_k(A) + \lambda_1(E).$$

- Assume both A and E are n-by-n symmetric matrix. Show that

$$|\lambda_k(A + E) - \lambda_k(A)| \leq ||E||_2.$$

3.12 Positive Semi-definite Matrix and Positive Definite Matrix

- A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite, if and only if $x^T A x \geq 0$, for any $x \in \mathbb{R}^n$.
  - All eigenvalues of A are non-negative.
  - $X^T A X$ for any $X \in \mathbb{R}^{n \times m}$ is positive semi-definite.
- A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive definite, if and only if $x^T A x > 0$, for any $0 \neq x \in \mathbb{R}^n$.
  - All eigenvalues of A are positive.
  - All principal submatrices of A are positive definite.
  - All diagonal entries of A are positive.
- Show that a symmetric and idempotent matrix A is positive semi-definite.

3.13 Matrix Differentiation

Let $f = \Psi(x)$ where $f \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$. The symbol

$$\frac{\partial f}{\partial x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

denotes the first-order partial derivative of the function. Such a matrix is also called the Jacobian matrix of the function Ψ.

- Let $f = w^T x$ where $w \in \mathbb{R}^n$, then
$$\frac{\partial f}{\partial x} = w.$$

- Let $f = Ax$ where $A \in \mathbb{R}^{m \times n}$, and A does not depend on x, then
$$\frac{\partial f}{\partial x} = A.$$
Note: $f_i = \sum_{k=1}^{n} a_{ik} x_k$.

- Let the scalar α be defined as $\alpha = y^T A x$, where $y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, then
$$\frac{\partial \alpha}{\partial x} = A^T y$$
and
$$\frac{\partial \alpha}{\partial y} = Ax.$$

- Let the scalar α be defined as $\alpha = x^T A x$, where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, then
$$\frac{\partial \alpha}{\partial x} = Ax + A^T x = (A + A^T)x.$$

4 Taylor Expansion

- Single variable Taylor expansion: Let $f(x)$ be an infinitely differentiable function, then
$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots && (1) \\ &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(k+1)}(\xi)}{(k+1)!}(x-a)^{k+1} && (2) \end{aligned}$$
where ξ is a number between a and x.

- Multi-variable Taylor expansion: Let $f(x)$ be a multivariate function where x is a vector, then
$$f(x) = f(a) + (x-a)^T f(a)' + (x-a)^T f''(a)(x-a) + \cdots$$

Acknowledgements

This document is prepared based on materials provided by Dr. Jieping Ye (University of Michigan).