NOTES ON TU'S INTRODUCTION TO MANIFOLDS

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Contents

Chapter 1

Euclidean Spaces

1 Smooth Functions on a Euclidean Space

This section is a review of C^∞ functions on $\mathbb{R}^n$, which will be our primary tool for studying more "exotic" manifolds.

1.1 Definitions

Let us denote the coordinates in $\mathbb{R}^n$ by $x^1, x^2, \ldots, x^n$ and let $p = \left(p^1, p^2, \ldots, p^n\right)$ be a point in an open set U in $\mathbb{R}^n$. We can now begin with our first definition:

Definition 1.1. Let k be a non-negative integer. A vector-valued function $f : U \to \mathbb{R}^m$ is C^k at p if all of its components $f^1, f^2, \ldots, f^m$ are C^k at p, that is, if all of their partial derivatives of all orders $j \leq k$ exist and are continuous at p. The function is C^∞ at p if it is C^k at p for all $k \geq 0$. We say that the function is C^k on U if it is C^k at every point in U. A function that is C^∞ can be interchangeably called "smooth".

Remark. A C^∞ function need not be real-analytic. A famous example of this is given by the function $f(x)$ on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

By induction, we can show that f is C^∞ on $\mathbb{R}$ with its derivatives $f^{(k)}(0)$ equal to 0 for all $k \geq 0$ (this is left as an exercise to the reader). Thus, the Taylor series of this function at the origin is identically 0, so f is not equal to its Taylor series and is not real-analytic at 0.

1.2 Taylor's Theorem with Remainder

Definition 1.2. An open subset U of $\mathbb{R}^n$ containing p is *star-shaped* with respect to p if for any $x \in U$, the line segment connecting x and p is completely contained inside U, i.e. it does not intersect the boundary of U.

Lemma 1.1. Let f be a smooth function on an open subset U of $\mathbb{R}^n$ star-shaped with respect to $p = \left(p^1, p^2, \ldots, p^n\right)$ in U. Then, there exists functions $g_1, g_2, \ldots, g_n \in C^\infty(U)$ such that

$$f(x) = f(p) + \sum_{i=1}^{n} \left(x^i - p^i\right) g_i(x), \quad g_i(p) = \partial_i f(p).$$

Proof. Since U is star-shaped with respect to p in U, there exists a line segment that lies in U connecting any point x in U to p parametrized by $u(t) = tx + (1-t)p$, $0 \leq t \leq 1$ such that $f(u(t))$ is defined in U. By the chain rule, we have

$$\frac{d}{dt} f(u(t)) = \sum_i \partial_i f(u(t)) u'(t) = \sum_i \left(x^i - p^i\right) \partial_i f\left(u(t)\right)$$

Integrating both sides with respect to t from 0 to 1, we get

$$f(x) - f(p) = \sum_i \left(x^i - p^i\right) g_i(x),$$

where

$$g_i(x) = \int_0^1 \partial_i f(u(t)) dt,$$

with

$$g_i(p) = \partial_i f(p)$$

□

Problems

Problem 1.1. Taylor's theorem with reaminder to order 2
Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is C^∞, then there exists C^∞ functions g_{11}, g_{12}, g_{22} on $\mathbb{R}^2$ such that

$$\begin{aligned} f(x, y) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)\, x + \frac{\partial f}{\partial y}(0, 0)\, y \\ + g_{11}(x, y) x^2 + g_{12}(x, y)\, xy + g_{22}(x, y)\, y^2. \end{aligned}$$

Solution. Applying Taylor's theorem with remainder (Lemma 1.1) to $f(x, y)$, there exists functions $g_1(x, y), g_2(x, y)$ with $g_1(0, 0) = \partial_x f(0, 0), g_2(0, 0) = \partial_y f(0, 0)$ so that

$$f(x, y) = f(0, 0) + g_1(x, y)\, x + g_2(x, y)\, y.$$

Applying Taylor's theorem again to $g_1(x, y), g_2(x, y)$ yields

$$\begin{aligned} f(x, y) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)\, x + \frac{\partial f}{\partial y}(0, 0)\, y \\ + g_{11}(x, y) x^2 + g_{12}(x, y)\, xy + g_{22}(x, y)\, y^2. \end{aligned}$$

□

2 Tangent Vectors in $\mathbb{R}^n$ as Derivations

2.1 The Directional Derivative

Loosely speaking, the tangent space $T_p(\mathbb{R}^n)$ at p in $\mathbb{R}^n$ can be thought of as the vector space of all arrows emanating from p. Elements of $T_p(\mathbb{R}^n)$ are called *tangent vectors* at p in $\mathbb{R}^n$. With this, we are ready to define the *directional derivative* in $\mathbb{R}^n$:

Definition 2.1. Let v be a tangent vector at p in $\mathbb{R}^n$ and let f be smooth in a neighborhood of p. Then, the *directional derivative* of f in the direction v at p, denoted $D_v f$, is defined as

$$D_v f = v^i \partial_i f(p),$$

where Einstein notation is used and a sum over the index i is implied unless specified otherwise.

From the above, we can see that there is a natural association between v and D_v independent of the input function. As such, we may want to characterize the *tangent vectors themselves* as operators on functions, which will be made precise in the next two subsections.

2.2 Germs of Functions

Definition 2.2. Let $p \in \mathbb{R}^n$, U be a neighborhood of p, and $f : U \to \mathbb{R}$ be smooth, and consider the set of all pairs (f, U). We can define a equivalent relation $(f, U) \sim (g, V)$ if there exists a neighborhood $W \subset U \cap V$ such that $f = g$ when restricted to W. Then, the equivalence class of (f, U) is called the *germ* of f at p. The set of all germs of C^∞ functions on $\mathbb{R}^n$ at p is then denoted $C_p^\infty(\mathbb{R}^n)$, or simply C_p^∞ if there is no ambiguity.

As an exercise, one may check that C_p^∞, together with function addition and multiplication, forms an algebra over $\mathbb{R}$, i.e. C_p^∞ not only is a vector space over $\mathbb{R}$, but also forms a ring (with function multiplication).

2.3 Derivations at a Point

For any $v \in T_p(\mathbb{R}^n)$, the directional derivative at p gives an $\mathbb{R}$-linear mapping

$$D_v : C_p^\infty \to \mathbb{R}$$

that satisfies the Leibniz rule (product rule)

$$D_v (fg) = (D_v f)\, g(p) + f(p)\, (D_v g)\,. \tag{2.1}$$

To generalize this notion, we have our next definition:

Definition 2.3. Let $D : C_p^\infty \to \mathbb{R}$ be a linear map that satisfies the Leibniz rule. Then, D is called a *derivation at* p or a *point derivation* of C_p^∞. The set of all derivations at p is then denoted by $\mathcal{D}_p(\mathbb{R}^n)$.

One may check that $\mathcal{D}_p(\mathbb{R}^n)$ is a vector space over $\mathbb{R}$. Since we know that directional derivatives at p are all derivations at p, there exists a (linear) map

$$\begin{aligned} \phi : T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n) \\ v &\mapsto D_v \end{aligned} \tag{2.2}$$

Lemma 2.1. If D is a derivation at p, then $D(c) = 0$ for any constant function c.

Theorem 2.2. The linear map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ defined in (2.2) is an isomorphism of vector spaces.

Proof. Let us first prove injectivity. Since ϕ is a linear map, suppose $D_v = 0$ for all $v \in T_p(\mathbb{R}^n)$. Then, by definition, we have

$$0 = D_v\left(x^j\right) = \sum_i v^i \partial_i x^j = \sum_i v^i \delta_i^j = v^j.$$

Thus, $v = 0$ so ϕ is injective.
To prove surjectivity, let D be a derivation at p and let (f, U) be a germ in C_p^∞, with U an open ball. By Taylor's theorem with remainder (Lemma 1.1), there exists smooth functions g_i on U such that

$$f(x) = f(p) + \sum_i \left(x^i - p^i\right) g_i(x),\ g(p) = \partial_i f(p).$$

Applying the derivation D on both sides and using Leibniz rule, we obtain

$$Df = \sum_i \left(Dx^i\right) g(p) + \sum_i \left(p^i - p^i\right) Dg_i(x) = \sum_i \left(Dx^i\right) \partial_i f(p).$$

Thus, $D = D_v$ for $v = \left\langle Dv^i \right\rangle$. $\square$

This theorem shows that one may identify tangent vectors at p with derivations at p. Moreover, the standard basis $\{e_i\}$ for $T_p(\mathbb{R}^n)$ directly corresponds with the set of partial derivatives $\{\partial_i|_p\}$. Thus, a tangent vector $v = \left\langle v^i \right\rangle = \sum v^i e_i$ may be written as

$$v = \sum_i v^i \partial_i|_p. \tag{2.3}$$

2.4 Vector Fields

Definition 2.4. A function X on an open subset U of $\mathbb{R}^n$ is called a *vector field on U* if for each point p in U, X maps p to a tangent vector X_p in $T_p(\mathbb{R}^n)$. Formally, the vector field X is defined by

$$X : U \to \bigcup_{p \in U} T_p(\mathbb{R}^n)$$
$$p \mapsto X_p \in T_p(\mathbb{R}^n)$$

Since $T_p(\mathbb{R}^n)$ has standard basis $\{\partial_i|_p\}$, the vector X_p may be written as a linear combination

$$X_p = \sum_i a^i(p) \partial_i|_p, \quad p \in U, \quad a^i(p) \in \mathbb{R}. \tag{2.4}$$

Omitting p, we may write

$$X = \sum_i a^i \partial_i. \tag{2.5}$$

We say that a vector field X is C^∞ on U if the coefficient functions a^i are all C^∞ on u.

Remark. Let us denote by $C^\infty(U)$ the ring of smooth functions on U. Then, if $X = \sum a^i \partial_i$ is a smooth vector field and f a smooth function on U, $fX = \sum \left(fa^i\right) \partial_i$ is a smooth vector field on U. Thus, the set of all C^∞ vector fields on U, denoted by $\mathfrak{X}(U)$, forms a *module* over $C^\infty(U)$.

2.5 Vectors Fields as Derivations

Let X be a C^∞ vector field on an open subset U of $\mathbb{R}^n$ and f a C^∞ function on U. Then, Xf is defined by

$$Xf = \sum_i a^i \partial_i f.$$

This shows that the vector field X is a linear map

$$\begin{aligned} C^\infty(U) &\to C^\infty(U) \\ f &\mapsto Xf \end{aligned}$$

Proposition 2.3. $X(fg)$ satisfies the Leibniz rule.

Proof. This is left as an exercise. ☐

If A is an algebra over a field K, a derivation of A is a K-linear map $D : A \to A$ that satisfies the Leibniz rule

$$D(ab) = (Da)\, b + a\, (Db) \ \forall a, b \in A.$$

Let us denote by $\mathrm{Der}\,(A)$ the set of derivations of A. Clearly, $\mathrm{Der}\,(A)$ is a vector space over K. Thus, one may identify a linear map

$$\begin{aligned} \varphi : \mathfrak{X}(U) &\to \mathrm{Der}\left(C^\infty(U)\right), \\ X &\mapsto (f \mapsto Xf)\,. \end{aligned}$$

In fact, the map above is an isomorphism of vector spaces. Thus, vector fields on an open set U may be identified with derivations of the algebra $C^\infty(U)$.

3 The Exterior Algebra of Multicovectors

Definition 3.1. Let f be a k-linear function and g an l-linear function on a vector space V. Then, their *tensor product* is the $(k+l)$-linear function $f \otimes g$ defined by

$$(f \otimes g)\,(v_1, \ldots, v_{k+l}) = f\,(v_1, \ldots, v_k)\, g\,(v_{k+1}, \ldots, v_{k+l})$$

Definition 3.2. The *wedge product* of two alternating multilinear functions $f \in A_k(V)$ and $g \in A_l(V)$ is defined by "alternating" the tensor product:

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g)$$

Explicitly, we have

$$(f \wedge g)(v_1, \dots v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) f(v_{\sigma_1}, \dots v_{\sigma_k}) g\left(v_{\sigma_{k+1}}, \dots v_{\sigma_{k+l}}\right).$$

Problems

Problem 3.1. Tensor products of covectors
Let $e_1, \dots, e^n$ be a basis for a vector space and let $\alpha^1, \dots, \alpha^n$ be its dual basis in V^*. Suppose $[g_{ij}] \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix. Define a bilinear function $f : V \times V \to \mathbb{R}$ by

$$f(v, w) = \sum_{1 \le i,j \le n} g_{ij} v^i w^j$$

for $v = \sum v^i e_i$ and $w = \sum w^j e_j$ in V. Describe f in terms of the tensor products of α^i and α^j, $1 \le i, j \le n$.

Solution. Substituting $v^i = \alpha^i v, w^j = a^j w$ into the above equation, we obtain

$$f(v, w) = \sum_{i,j} g_{ij} \left(\alpha^i v\right)\left(\alpha^j w\right) = \sum_{i,j} g_{ij} \left(\alpha^i \otimes \alpha^j\right)(v, w).$$

Problem 3.2. Hyperplanes

(a) Let V be a vector space of dimension n and $f : V \to \mathbb{R}$ a nonzero linear functional. Show that $\dim \ker f = n - 1$. A linear subspace of V of dimension $n - 1$ is called a *hyperplane* in V.

(b) Show that a nonzero linear functional on a vector space V is determined up to a multiplicative constant by its kernel, a hyperplane in V. In other words, if f and $g : V \to \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

Solution.

(a) By the rank-nullity theorem, we have

$$\dim \ker f = \dim V - \dim \operatorname{im} f = n - 1$$

□

(b) Choose a basis $\{e_1, \dots, e_{n-1}\}$ for $\ker f = \ker g$. Let $\{e_1, \dots e_n\}$ be a basis for V and $\{\alpha^1, \dots, \alpha^n\}$

be the dual basis for V^*. Writing $f = \sum f_i \alpha^i, g = \sum g_j \alpha^j$, for any $v = \sum v^i e_i \in \ker V$, we have

$$f(v) = \sum_{i,j}^{n-1} f_i v^j \alpha^i e_j = \sum_{i,j}^{n-1} f_i v^j \delta_j^i = f_i v^i = 0$$
$$g(v) = \cdots = g_i v^i = 0.$$

Thus, $f_i = g_i = 0$ for $1 \leq i \leq n-1$, so $f = f_n \alpha^n$ and $g = g_n \alpha^n$. □

Problem 3.3. A basis for k-tensors
Let V be a vector space of dimensino n with basis $e_1, \ldots, e_n$. Let $\alpha^1, \ldots, \alpha^n$ be the dual basis for V^*. Show that a basis for the space $L_k(V)$ of k-linear functions on V is $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ for all multi-indices $(i_1, \ldots, i_k)$. (You may have noticed that this problem is a generalization of Problem 3.1.)

4 Differential Forms on $\mathbb{R}^n$

4.1 Differential 1-forms and the Differential of a Function

Definition 4.1. The *cotangent space* to $\mathbb{R}^n$ at p, denoted by $T_p^*(\mathbb{R}^n)$, is defined as the dual space of the tangent space $T_p(\mathbb{R}^n)$.

Definition 4.2. A *covector field* or a *differential* 1*-form* (or 1*-form*) on an open subset U of $\mathbb{R}^n$ is a function ω that assigns to each point p in U a *covector* $\omega_p \in T_p^*(\mathbb{R}^n)$,

$$\omega : U \to \bigcup_{p \in U} T_p^*(\mathbb{R}^n),$$
$$p \mapsto \omega_p \in T_p^*(\mathbb{R}^n).$$

Given any smooth function $f : U \to \mathbb{R}$, we can construct a 1-form df, called the *differential* of f, as follows

$$(df)_p(X_p) = X_p f$$

Proposition 4.1. The basis $\{(dx^i)_p\}$ of the cotangent space $T_p^*(\mathbb{R}^n)$ is dual to the standard basis $\{\partial_i|_p\}$ of the tangent space $T_p(\mathbb{R}^n)$.

Proof. By definition

$$(dx^i)_p(\partial_j|_p) = \partial_j x^i = \delta_j^i.$$

□

Proposition 4.2. The differential of a C^∞ function $f : U \to \mathbb{R}$ on an open set U in $\mathbb{R}$ can be written as

$$df = \sum (\partial_i f)\, dx^i$$

Proof. By definition

$$(df)\, \partial_i = \partial_i f$$

On the other hand, applying $(df)_p$ to the standard basis $\{\partial_i|_p\}$ of $T_p(\mathbb{R}^n)$ yields

$$(df)\, \partial_i = \sum_j a_j \left(dx^j\right)(\partial_i) = \sum_j a_j \partial_i x^j = \sum_j a_j \delta_i^j = a_i$$

□

4.2 Differential k-forms

Definition 4.3. A *differential form ω of degree k* or a *$k - form$* on an open subset U of $\mathbb{R}^n$ is a function that assigns each point p in U an alternating k-linear function on the tangent space $T_p(\mathbb{R}^n)$. Formally, we have

$$\begin{aligned} \omega : U &\to A_k\left(T_p\left(\mathbb{R}^n\right)\right) \\ p &\mapsto \omega_p \end{aligned}$$

Denote by $\Omega^k(U)$ the vector space of C^∞ k-forms on U. Then, their direct sum, denoted, $\Omega^*(U) = \bigoplus_k \Omega^k(U)$, forms what is called a *graded algebra* over $\mathbb{R}$.

Definition 4.4. A *graded algebra* is an algebra A that is a direct sum of vector spaces A^k, i.e.

$$A = \bigoplus_k A^k$$

with a multiplication map $(\cdot, \cdot)$ that sends $a_i \in A^i$ and $a_j \in A^j$ to $(a_i, a_j) = a_i a_j \in A^{i+j}$

Example. The ring of polynomials $K[x]$ over a field K is a graded algebra, for we have $K[x] = \bigoplus_k A^k$, where $A^k = \{rx^k | r \in K\}$, and that for any $ax^i \in A^i, bx^j \in A^j$, we have $\left(ax^i\right)\left(bx^j\right) = (ab)\, x^{i+j} \in A^{i+j}$.

Example. The ring $\Omega^*(U)$ of differential forms on an open subset U of $\mathbb{R}^n$ is a graded algebra over $\mathbb{R}$ with the wedge product as its multiplication map.

4.3 Differential Forms as Multilinear Functions on Vector Fields

If ω is a smooth 1-form and X a smooth vector field on an open set U in $\mathbb{R}^n$, then the function $\omega(X)$ can be defined point-wise:

$$\omega(X)_p = \omega_p(X_p), \quad p \in U.$$

In coordinate form,

$$\omega = \sum \omega_i dx^i, \quad X = \sum X^j \partial_j,$$

so

$$\omega(X) = \sum \omega_i X^i,$$

which implies that $\omega(X)$ is smooth on U.

Remark. Let $\mathcal{F}(U) = C^\infty(U)$. A 1-form ω on U gives rise to a linear map

$$\begin{aligned} \mathfrak{X}(U) &\to \mathcal{F}(U), \\ X &\mapsto \omega(X). \end{aligned}$$

Similarly, a k-form ω on U gives rise to a k-linear map

$$\begin{aligned} \underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{k \text{ times}} &\to \mathcal{F}(U), \\ (X_i, \ldots, X_k) &\mapsto \omega(X_i, \ldots, X_k). \end{aligned}$$

4.4 The Exterior Derivative

To define the *exterior derivative* on C^∞ k-forms, on an open subset U of $\mathbb{R}^n$, we first define it on 0-forms: the exterior derivative of a C^∞ function $f \in C^\infty(U)$ is defined to be its diffential:

$$df = \sum_i \partial_i f dx^i \in \Omega^1(U)$$

Definition 4.5. For any $k \geq 1$, given a k-form $\omega = \sum_I a_I dx^I \in \Omega^k(U)$, its *exterior derivative* is defined as

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_j \partial_j a_I dx^j \right) \wedge dx^I \in \Omega^{k+1}(U).$$

Example. Let A be a C^∞ 1-form $A_x dx + A_y dy + A_z dz$ on $\mathbb{R}^3$. Then, its exterior derivative is a (C^∞) 2-form given by

$$\begin{aligned} dA &= dA_x \wedge dx + dA_y \wedge dy + dA_z \wedge dz \\ &= (\partial_y A_x dy + \partial_z A_x dz) \wedge dx + (\partial_x A_y dx + \partial_z A_y dz) \wedge dy + (\partial_x A_z dx + \partial_y A_z dy) \wedge dz \\ &= (\partial_x A_y - \partial_y A_x) dx \wedge dy + (\partial_y A_z - \partial_z A_y) dy \wedge dz + (\partial_z A_x - \partial_x A_z) dz \wedge dx \end{aligned}$$

Definition 4.6. Let $A = \bigoplus_{k=0}^\infty A^k$ be a graded algebra over a field K. An *antiderivation* of the graded algebra A is a K-linear map $D : A \to A$ such that for $a \in A^k$ and $b \in A^l$,

$$D(ab) = (Da) b + (-1)^k a (Db).$$

If there is an integer m such that for any k, the antiderivation D is a map $A^k \to A^{k+m}$, then we say that it is an antiderivation of *degree* m.

Proposition 4.3.

(i) The exterior differentiation $d : \Omega^* (U) \to \Omega^* (U)$ is an antiderivation of degree 1.

(ii) *(Poincare's lemma)* $d^2 = 0$.

(iii) For any $f \in C^\infty (U)$ and $X \in \mathfrak{X} (U)$, $(df) (X) = Xf$

Proof.

(i) For any $\omega = f dx^I, \tau = g dx^J$, we have

$$\begin{aligned} d (\omega \wedge \tau) &= d \left(f g dx^I \wedge dx^J\right) \\ &= \sum_i \partial_i (fg) \, dx^i \wedge dx^I \wedge dx^J \\ &= \sum_i (\partial_i f) \, dx^i \wedge dx^I \wedge g dx^J + \sum_j f (\partial_j g) \, dx^j \wedge dx^I \wedge dx^J \\ &= \sum_i \left((\partial_i f) \, dx^i \wedge dx^I\right) \wedge g dx^J + \sum_j f dx^I \wedge \left((-1)^{\deg \omega} (\partial_i g) \, dx^j \wedge dx^J\right) \\ &= d\omega \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau. \end{aligned}$$

□

(ii) *Exercise*

(iii) This is just the definition of the exterior derivative on a smooth function as its differential.

□

Proposition 4.4. Characterization of the exterior derivative. The three properties of Proposition 4.3 uniquely characterize exterior differentiation on an open subset U in $\mathbb{R}$. Explicitly, if $D : \Omega^* (U) \to \Omega^* (U)$ satisfies the three properties:

(i) D is of degree 1,

(ii) $D^2 = 0$,

(iii) For any $f \in C^\infty (U)$ and $X \in \mathfrak{X} (U), (Df) (X) = Xf$,

then $D = d$.

Proof. By linearity, it suffices to prove $D = d$ on k-forms of type $\omega = f dx^I$. By the definition of exterior differentiation, we have $dx^i = Dx^i$. This implies that $Ddx^i = D \left(Dx^i\right) = 0$ (by property (ii)).

Thus, we have $Ddx^I = 0$ for any multi-index I of degree k. Applying D to the k-form ω, we obtain

$$\begin{aligned} D\omega &= D\left(f dx^I\right) \\ &= (Df) \wedge dx^I + f D\left(dx^I\right) \\ &= df \wedge dx^I \\ &= d\left(f dx^I\right) \\ &= d\omega \end{aligned}$$

□

4.5 Closed Forms and Exact Forms

Definition 4.7. A k-form ω on U is *closed* if $d\omega = 0$; it is *exact* if there exists a $k+1$-form τ such that $d\tau = \omega$.

Remark. Every exact form is closed, for if $\omega = d\tau$ for some τ then $d\omega = d\left(d\tau\right) = 0$.

Problems

Problem 4.1. At each point $p \in \mathbb{R}^3$, define a bilinear function ω_p on $T_p\left(\mathbb{R}^3\right)$ by

$$\omega_p\left(\mathbf{a}, \mathbf{b}\right) = \omega_p\left(\begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix}\right) = p^3 \det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix},$$

for tangent vectors $\mathbf{a}, \mathbf{b} \in T_p\left(\mathbb{R}^3\right)$, where p^3 is the third component of $p = \left(p^1, p^2, p^3\right)$. Since ω_p is an alterning bilinear function on $T_p\left(\mathbb{R}^3\right)$, ω is a 2-form on $\mathbb{R}^3$. Write ω in terms of the standard basis $dx^i \wedge dx^j$ at each point.

Solution. For $1 \leq i < j \leq 3$, we have

$$\left(\omega_p\right)_{ij} = \omega_p\left(e^i, e^j\right) = p^3 \det \begin{bmatrix} \delta_i^1 & \delta_j^1 \\ \delta_i^2 & \delta_j^2 \end{bmatrix} = p^3\left(\delta_i^1 \delta_j^2 - \delta_j^1 \delta_i^2\right) = p^3 \epsilon_{ij},$$

where ϵ_{ij} denotes the Levi-Civita symbol. Thus, ω in terms of the standard basis is given by

$$\omega = p^3 dx^1 \wedge dx^2$$

Problem 4.2. Suppose the standard coordinates on $\mathbb{R}^2$ are called r and θ (this $\mathbb{R}^2$ is the (r, θ)-plane, not the (x, y)-plane). If $x = r\cos\theta$ and $y = r\sin\theta$, calculate dx, dy, and $dx \wedge dy$ in terms of dr and $d\theta$.

Solution. Taking the exterior derivative of $x = r\cos\theta$ and $y = r\sin\theta$ yields

$$dx = \cos\theta dr - r\sin\theta d\theta$$
$$dy = \sin\theta dr + r\cos\theta d\theta$$

Taking the wedge product of the above expressions, we obtain

$$dx \wedge dy = r\cos^2\theta dr \wedge d\theta - \left(r\sin^2\theta\right) d\theta \wedge dr = r dr \wedge d\theta.$$

Problem 4.3. Suppose the standard coordinates on $\mathbb{R}^3$ are called ρ, ϕ, and θ. If $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, calculate dx, dy, dz, and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\phi$, and $d\theta$.

Solution. Taking the exterior derivatives of x, y, z yields

$$dx = \sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta$$
$$dy = \sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta$$
$$dz = \cos\phi d\rho - \rho\sin\phi d\phi$$

Taking the wedge product $dx \wedge dy \wedge dz$, we obtain

$$\begin{aligned} dx \wedge dy \wedge dz = (&\rho^2\sin^3\phi\cos^2\theta + \rho^2\sin\phi^3\sin^2\theta + \rho^2\sin\phi\cos^2\phi\cos^2\theta \\ &+ \rho^2\sin\phi\cos^2\phi\sin^2\theta) d\rho \wedge d\phi \wedge d\theta = \rho^2\sin\phi d\rho \wedge d\phi \wedge d\theta. \end{aligned}$$

Chapter 2

Manifolds

In this chapter, we will go over what it means for a topological space to be a topological *manifold*. To put it succinctly, a *topological manifold* is a topological space that is Hausdorff, second countable, and locally Euclidean. To understand what any of those terms mean, let us revisit a few definitions from point set topology.

5 Review of Point Set Topology

5.1 Topological Spaces

Definition 5.1. A *topology* on a set S is a collection $\mathcal{T}$ of subsets containing both the empty set $\emptyset$ and the set S such that $\mathcal{T}$ is closed under arbitrary unions and finite intersections. Formally, $\mathcal{T}$ obeys the following axioms

(i) $\emptyset \in \mathcal{T}$ and $S \in \mathcal{T}$.

(ii) For any number of indexed subsets $A_i \in \mathcal{T}$, their union $\bigcup A_i \in \mathcal{T}$.

(iii) If $A_1, \ldots, A_n \in \mathcal{T}$ then $\bigcap A_i \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called *open sets* and the pair $(S, \mathcal{T})$ is then called a *topological space*. Sometimes, we may refer to the pair $(S, \mathcal{T})$ as just the topological space S when the choice of topology is implied. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on a set S and $\mathcal{T}_1 \subset T_2$, then we say that $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$, or that $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$.

Just as we can define a basis on a vector space, a topology $\mathcal{T}$ on a set S can also be defined.

5.2 Bases for a Topological Space

Definition 5.2. A subcollection $\mathcal{B}$ of a topology $\mathcal{T}$ on a topological space S is a *basis for the topology* $\mathcal{T}$ if given any open set U and a point $p \in U$, there is an open set $B \in \mathcal{B}$ such that $p \in B \subset U$. We also say that $\mathcal{B}$ *generates* the topology $\mathcal{T}$ or that $\mathcal{B}$ is a *basis for the topological space* S.

Proposition 5.1. A collection $\mathcal{B}$ of open sets of S is a basis if and only if every open set in S is a union of sets in $\mathcal{B}$.

Proof. Let $\mathcal{B}$ be a basis for the topology $\mathcal{T}$ of S. By definition, for any point $p \in U \subset S$, there exists $B \in \mathcal{B}$ such that $p \in B$. Thus, the open set U in S is a union of sets in $\mathcal{B}$.

Conversely, suppose every open set in S is a union of sets in $\mathcal{B}$ and let U be such an open set. Then, $U = \bigcup_{i \in I} B_i$ for a collection of subsets $B_i \in \mathcal{B}$ indexed by $i \in I$. For any point $p \in U$, our previous statement implies that $p \in B_i \subset U$ for some $i \in I$. Hence, $\mathcal{B}$ is a basis for S. $\square$

Proposition 5.2. A collection $\mathcal{B}$ of subsets of a set S is a basis for some topology $\mathcal{T}$ on S if and only if

(i) S is the union of all the sets in $\mathcal{B}$, and

(ii) given any two sets B_1 and B_2 in $\mathcal{B}$ and a point $p \in B_1 \cap B_2$, there is a set $B \in \mathcal{B}$ such that $p \in B \subset B_1 \cap B_2$.

Proof. Let $\mathcal{B}$ be a basis for some topology $\mathcal{T}$ on S. By Proposition 5.1, S is a union of sets in $\mathcal{B}$. (ii) is also trivial from the definition of a basis, for $B_1 \cap B_2$ is an open set in S, and the definition of a basis $\mathcal{B}$ requires that there exists a set $B \in \mathcal{B}$ that satisfies the above property.

Conversely, let us assume points (i) and (ii) and let $\mathcal{T}$ be a collection of all sets that are unions of sets in $\mathcal{B}$. Clearly, $\emptyset$ and S are both in $\mathcal{T}$. By construction, $\mathcal{T}$ is also closed under arbitrary union. To show that $\mathcal{B}$ is closed under finite intersection, Let $U = \bigcup_i B_i$ and $V = \bigcup_j B_j$ for some collection of sets $B_i, B_j \in \mathcal{B}$. Then,

$$U \cap V = \bigcup_{i,j} (B_i \cap B_j) \,.$$

Thus, any point $p \in U \cap V$ is in $B_i \cap B_j$ for some i, j. (ii) then implies that there is a set $B_p \in \mathcal{B}$ such that $p \in B_p \subset B_i \cap B_j$. Therefore,

$$U \cap V = \bigcup_{p \in U \cap V} B_p \in \mathcal{T}.$$

□

5.3 Subspace Topology

Definition 5.3. (Subspace topology) Let $(S, \mathcal{T})$ be a topological space and A a subset of S. Define $\mathcal{T}_A$ to be the collection of subsets

$$\mathcal{T}_A = \{U \cap A | U \in \mathcal{T}\}$$

It is easy to see that $\mathcal{T}_A$ defines a topology on A. $\mathcal{T}_A$ is then called the *subspace topology* or the *relative topology* of A in S, and elements of $\mathcal{T}_A$ are said to be *open in* A.

Proposition 5.3. Let $\mathcal{B} = \{B_i\}$ be a basis for a topological space S, and A be a subspace of S. Then $\{B_i \cap A\}$ is a basis for A.

Proof. Let U_A be an open set in A and p be any point in U_A. By definition of the subspace topology, $U_A = U \cap A$ for some open set U in S. Thus, $p \in U$, which implies that there exists $B \in \mathcal{B}$ such that $p \in B_p \subset U$. p is also in A, so we have

$$p \in B_p \cap A \subset U \cap A = U_A.$$

Thus, $\{B_i \cap A\}$ is a basis for A. □

5.4 Second-Countability

Definition 5.4. A topological space is said to be *second countable* if it has a countable basis.

Proposition 5.4. A subspace A of a second-countable space S is second countable.

Proof. By Proposition 5.3, given a countable basis $\mathcal{B} = \{B_i\}$ for a space S, $\{B_i \cap A\}$ forms a countable basis for the subspace A. $\square$

5.5 The Hausdorff Condition

Definition 5.5. A topological space S is *Hausdorff* if given any two distinct points x, y in S, there exist disjoint open sets U, V in S such that $x \in U$ and $y \in V$. A Hausdorff space is *normal* if given any two disjoint closed sets F, G in S, there exists open sets U, V such that $F \subset U$ and $G \subset V$.

Proposition 5.5. Any subspace A of a Hausdorff space S is Hausdorff.

Proof. Let x and y be distinct points in A. Since S is Hausdorff, there exist disjoint open sets U and V in S that contain x and y respectively. Then, $U \cap A$ and $V \cap A$ are disjoint open sets in A that contain x and y respectively. $\square$

5.6 Product Topology

Definition 5.6. (Product topology) Given two topological spaces X and Y, let us consider the collection $\mathcal{B}$ of subsets of $X \times Y$, the cartesian product of X and Y, of the form $U \times V$ (called basic open sets), where U and V are open sets in X and Y respectively. It is easy to see that $\mathcal{B}$ forms a topology for $X \times Y$. In fact, from Proposition 5.2, $\mathcal{B}$ satisfies the conditions for a basis and generates a topology on $X \times Y$. This topology is called the *product topology*.

Proposition 5.6. Let $\{A_i\}$ and $\{B_j\}$ be bases for topological spaces X and Y. Then, $\{A_i \times B_j\}$ is a basis for $X \times Y$.

Proof. Let W be an open set in $X \times Y$ and a point $p = (x, y) \in W$. Then, there exist open sets $U \in X$ and $V \in Y$ such that $p \in U \times V$, with $U \times V \subset W$ a basic open set. Thus, $x \in A_i \subset U$ and $y \in B_j \subset V$ for some i, j and $p = (x, y) \in A_i \times B_j \subset U \times V \subset W$. $\square$

Corollary 5.7. The product of two second-countable spaces is second countable.

Proposition 5.8. The product of two Hausdorff spaces X and Y is Hausdorff.

Proof. Given any two distinct points $p_1 = (x_1, y_1)$ and $p_2(x_2, y_2)$ in $X \times Y$. WLOG, assume that $x_1 \neq x_2$. Since X is Hausdorff, there exist disjoint open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$. Then, $U_1 \times Y$ and $U_2 \times Y$ are two disjoint neighborhoods of p_1 and p_2, so $X \times Y$ is Hausdorff. □

5.7 Continuity

Definition 5.7. Let $f : X \to Y$ be a function of topological spaces. We say that f is *continuous at a point* p in X if for every neighborhood V of $f(p)$ in Y, there is a neighborhood U of p such that $f(U) \subset V$. We say that f is *continuous on* X if it is continuous at every point in X.

Proposition 5.9. A function $f : X \to Y$ is continuous if and only if the inverse image (preimage) of any open set is open.

Proof. Suppose f is continuous. Let V be an open set in Y and let $p \in f^{-1}(V)$. Then, $f(p) \in V$, and by definition of continuity of f, there is a neighborhood U of p such that $f(U) \subset V$. Thus, $p \in U \subset f^{-1}(V)$, so $f^{-1}(V)$ is open.

Conversely, let $p \in X$ and V be a neighborhood of $f(p)$. Then, $f^{-1}(V)$ is open in X and $f^{-1}(V)$ is a neighborhood of p. Thus, $U = f^{-1}(V)$ is a neighborhood of p such that $f(U) = f\left(f^{-1}(V)\right) \subset V$, so f is continuous at p. □

Proposition 5.10. The composition of continuous maps is continuous.

Proof. Exercise. □

5.8 Compactness

Definition 5.8. Let S be a topological space. A collection $\{U_\alpha\}$ of open sets in S is said to *cover* S or to be an *open cover* of S if $S \subset \bigcup_\alpha U_\alpha$. A *subcover* of an open cover is a subcollection whose union still contains S. The topological space S is then said to be *compact* if every open cover of S has a finite subcover.

Proposition 5.11. A subspace A of a topological space S is compact if and only if every open cover of A has a finite subcover.

Proof. Suppose A is compact and let a collection of open sets in S, $\{U_\alpha\}$, be an open cover of A. Clearly, $\{U_\alpha \cap A\}$ is also an open cover of A, and since A is compact, it has a finite subcover given by $\{U_i \cap A\}|_{i=1}^n$. Thus, $\{U_i\}|_{i=1}^n$ is a finite subcover of A. The converse is easily seen from the definition of compactness. □

Proposition 5.12. A closed subset F of a compact topological space S is compact.

Proof. Let $\{U_\alpha\}$ be an open cover of F in S. Clearly, $S \subset (\bigcup_\alpha U_\alpha) \cup \bar{f} = \bigcup_\alpha \left(U_\alpha \cup \bar{f}\right)$. On the other hand, F is closed, so $\bar{f}$ is open, and thus $\{U_\alpha \cup \bar{f}\}$ is an open cover for S. By compactness, there exists a finite subcover $\{U_i \cup \bar{f}\}_{i=1}^n$ for S. Hence, $\{U_i\}_{i=1}^n$ is a finite subcover for F and F is compact. $\square$

Proposition 5.13. The image of a compact set under a continuous map is compact.

Proof. Let $f : X \to Y$ be a continuous map and K a compact subset of X. Suppose $\{V_\alpha\}$ is an open cover of $f(K)$, with V_α open sets in Y. By continuity of f, the preimages $f^{-1}(V_\alpha)$ are all open. Additionally,

$$K \subset f^{-1}(f(K)) \subset f^{-1}\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha f^{-1}(V_\alpha).$$

Thus, $\{f^{-1}(V_\alpha)\}$ is an open cover of K. By the compactness of K, there is a finite subcover $\{f^{-1}(V_i)\}|_{i=1}^n$ of K, so

$$K \subset \bigcup_i f^{-1}(V_i) = f^{-1}\left(\bigcup_i V_i\right).$$

Therefore, $f(K) \subset \bigcup_i V_i$, so $f(K)$ has a finite subcover, and thus is compact. $\square$

Theorem 5.14. (The Tychonoff Theorem). The product of any collection of compact spaces is compact in the product topology.

5.9 Boundedness in $\mathbb{R}^n$

Definition 5.9. A subset A of $\mathbb{R}^n$ is said to be *bounded* if it is contained in some open ball $B(p,r)$; it is *unbounded* otherwise.

Lemma 5.15. A subset A of $\mathbb{R}^n$ is bounded if and only if for any $p \in \mathbb{R}^n$, there exists $r \in \mathbb{R}^+$ such that $A \subset B(p,r)$

Proof. Suppose A is bounded. By definition, there exists $p_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{R}^+$ such that $A \subset B(p_0, r_0)$. Now, let p be any point in $\mathbb{R}^n$ and $r \in \mathbb{R}^+$ be given by $r = ||p - p_0|| + r_0$. We claim that $B(p_0, r_0) \subset B(p,r)$. Indeed, for any point $x \in B(p_0, r_0)$, we have

$$d(x,p) = ||x - p|| = ||x - p_0 + p_0 - p|| \leq ||x - p_0|| + ||p_0 - p|| < r_0 + ||p - p_0|| = r.$$

Thus, $A \subset B(p,r)$. The converse is clearly true, so we are done. $\square$

Corollary 5.16. A subset A of $\mathbb{R}^n$ is bounded if and only if for any $p \in \mathbb{R}^n$, there exists a hypercube centered at p that contains A.

Proposition 5.17. The closed interval $[a,b] \subset \mathbb{R}$ is compact.

Proof. Let $\mathcal{A} = \{U_\alpha\}$ be an open cover for $[a, b]$ and let S be the set of points $y \in (a, b]$ such that the interval $[a, y]$ can be covered by a finite number of elements of $\mathcal{A}$.

First, we claim that S is non-empty; that is, there exists at least one element $y \in (a, b]$ such that $[a, y]$ has a finite cover. Choose an element U_{α_0} that contains a. Since U_{α_0} is open, $U_{\alpha_0} \supset [a, y)$ for some $y \in (a, b]$. Again, choose an element U_{α_1} that contains y. Then, $[a, y] = [a, y) \cup \{y\} \subset U_{\alpha_0} \cup U_{\alpha_1}$, so $y \in S$.

Now, we claim that $c = \sup S \in S$. Again, choose an element $U_c \in A$ that contains c. Since U_c is open, $U_c \supset (d, c]$ for some $d \in (a, c)$. If $c \notin S$, then there exists a point $p \in S$ that is in (d, c), for otherwise, $d < c = \sup S$ would be a upper bound on S. Since $p \in S$, $[a, p]$ can be covered by at most n elements of $\mathcal{A}$. This subcollection, in addition to U_c, would then admit a finite cover of $[a, c]$, contradicting the assumption that $c \notin S$.

Finally, we will show that $c = b$, thus proving the proposition. Suppose $c < b$ and let S' be the set of points $y' \in (c, b]$ such that $[c, b]$ can be covered by a finite number of elements of $\mathcal{A}$. Similar to S, S' is non-empty, so there is an element $y' \in (c, b]$ such that $[c, y']$ can be covered by a finite number of elements of $\mathcal{A}$. This implies that

$$[a, y'] = [a, c] \cup [c, y']$$

can also be finitely covered, contradicting the fact that c is an upper bound on S. □

Corollary 5.18. The n-hypercube given by $A = [a_1, a_1 + d] \times \cdots \times [a_n, a_n + d]$ is compact.

Theorem 5.19. (The Heine-Borel Theorem). A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

Proof. Let K be a compact subset of $\mathbb{R}^n$. We first prove that K is closed by showing its complement, $\overline{K}$, is open.
Let $x \in \overline{K}$. For every point $y_\alpha \in K$, construct an open ball $B(x, d_\alpha)$, where $d_\alpha = ||y_\alpha - x||$. By construction, $y_\alpha \notin B(x, d_\alpha)$ so $y_\alpha \in \overline{B(x, d_\alpha)}$ for every $y_\alpha \in K$. Thus, $K \in \bigcup_\alpha \overline{B(x, d_\alpha)}$, so $\{\overline{B(x, d_\alpha)}\}$ is an open cover for K. By the compactness of K, there exists a finite subcover $\{\overline{B(x, d_i)_{i=1}^{n}}\}$ for K, with $d_i \leq d_j$ whenever $i < j$. Notice that

$$\overline{B(x, d_1)} \supset \cdots \supset \overline{B(x, d_n)},$$

So $K \subset \overline{B(x, d_1)}$, and thus $x \in B(x, d_1) \subset \overline{K}$. This implies that $\overline{K}$ is open, so K is closed. Now, let us prove K is bounded. Let $r_0 \in K$ and for every $m \in \mathbb{Z}^+$, construct an open ball $B(r_0, m)$. Notice that $\{B(r_0, m)\}_{m=1}^{\infty}$ is an open cover for $\mathbb{R}$, so it is an open cover for K. By the compactness of K, there exists a finite subcover $\{B(r_0, m)\}_{m \in A}$, where A is some subset of $\mathbb{Z}^+$. By the well-ordering principle, there is a largest element m_0 in A, i.e. $B(r_0, m) \subset B(r_0, m_0)$ for every $m \in A - \{m_0\}$. Thus, $K \in B(r_0, m_0)$, so K is bounded.

The converse is a direct consequence of Corollary 5.16, Corollary 5.18, and Proposition 5.12. □

5.10 Connectedness

Definition 5.10. A topological space S is *disconnected* if it is a union of two disjoint non-empty open sets. It is *connected* if it is not disconnected. A subset A of S is *disconnected* if it is disconnected in the subspace topology.

Proposition 5.20. A subset A of a topological space S is disconnected if and only if there are open sets U and V in S such that

(i) $U \cap A \neq \emptyset, V \cap A \neq \emptyset$,

(ii) $U \cap V \cap A = \emptyset$,

(iii) $A \subset U \cup V$.

A pair of open sets in S with these properties is called a *separation* of A.

Proof. Suppose A is disconnected and $A = U \cup V$ with $U \cap V = \emptyset$. By definition, $U = U' \cap A$ and $V = V' \cap A$ for some open sets U', V' in S. It is easy to see that U' and V' satisfy the above conditions.

Now, suppose there are open sets U and V in S that satisfy the above conditions. Condition (i) tells us A is nonempty. Then, from condition (iii), $A = (U \cap A) \cup (V \cap A)$. From condition (ii), we also have $(U \cap A) \cap (V \cap A) = \emptyset$, so we are done. ☐

Proposition 5.21. The image of a connected space X under a continuous map $f : X \to Y$ is connected.

Proof. Suppose $f(X)$ is disconnected. Then, there are open sets U, V in Y that separate $f(X)$. By Proposition 5.9, $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Clearly, $f^{-1}(U) \cup f^{-1}(V) = X$. We also have $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U) \cap f^{-1}(V) \cap X = f^{-1}(U) \cap f^{-1}(V) \cap f^{-1}(A) = f^{-1}(U \cap V \cap A) = f^{-1}(\emptyset) = \emptyset$, so X is disconnected, which is a contradiction. ☐

Proposition 5.22. The union of a collection of connected subsets A_α of a topological space S having a common point p is connected.

Proof. Suppose $\bigcup_\alpha A_\alpha = U \cup V$, where U and V are two disjoint non-empty open subsets of $\bigcup_\alpha A_\alpha$, and suppose without loss of generality that $p \in U$. Now, let $p' \in V$. Then, $p' \in A_{\alpha_0}$ for some α_0. Clearly, $\{U, V\}$ separates A_{α_0}, which contradicts the connectedness of A_α. Thus, $\bigcup_\alpha A_\alpha$ is connected. ☐

5.11 Connected Components

Definition 5.11. Let x be a point in a topological space S. Then, the union of all connected subsets of S containing x, denoted C_x, is called the *connected component* of S containing x.

Remark. By Proposition 5.22, C_x is connected.

Proposition 5.23. Let C_x be a connected component of x of a topological space S. Then, a connected subset A of S is either disjoint from C_x or is contained entirely in C_x.

Proof. Suppose A and C_x have a point in common. Then, by Proposition 5.22, $A \cup C_x$ is connected, and thus is a connected set containing x. Hence, $A \cup C_x \subset C_x$, so $A \subset C_x$. $\square$

Corollary 5.24. For any two points x, y in a topological space S, the connected components C_x and C_y are either disjoint or coincide.

Proof. Obvious. $\square$

5.12 Closure

Definition 5.12. Let S be a topological space and A a subset of S. The *closure* of A in S, denoted by $\mathrm{cl}\,(A)$ or $\mathrm{cl}_S\,(A)$, is defined to be the intersection of all the closed sets containing A.

Remark. As an intersection of closed sets, $\mathrm{cl}\,(A)$ is a closed set. It is the smallest closed set containing A in the sense that any closed set containing A must contain $\mathrm{cl}\,(A)$.

Proposition 5.25. (Local characterization of closure). Let A be a subset of a topological space S. A point $p \in S$ is in the closure $\mathrm{cl}\,(A)$ if and only if every neighborhood of p contains a point of A.

Proof. We will proceed by contraposition. Let $p \in S$ and suppose $p \notin \mathrm{cl}\,(A)$. We want to show that there is a neighborhood of p that is disjoint from A. Indeed, since $p \notin \mathrm{cl}\,(A)$, there is a closed set C containing A that does not contain p. Thus, $p \in \overline{C}$, an open set disjoint from A.

Now, suppose there is a neighborhood of p that is disjoint from A and let U be such a neighborhood. Then, $\overline{U}$ is a closed set in S that contains A but does not contain p. Thus, $p \in \mathrm{cl}\,(A)$. $\square$

Definition 5.13. A point p in a topological space S is an *accumulation point* or a *limit point* of a subset A of S if every neighborhood of p in S contains a point of A other than p. The set of all accumulation points of A is denoted by $\mathrm{ac}\,(A)$.

If U is a neighborhood of p in S, the set $U - \{p\}$ is called a *deleted neighborhood* of p. Notice that the above definition is equivalent to saying that every deleted neighborhood of p contains a point of A.

Example. If $A = [0, 1) \cup \{2\}$ in $\mathbb{R}$, then the closure of A is $[0, 1] \cup \{2\}$, but the set of accumulation points of A is only the closed interval $[0, 1]$.

Proposition 5.26. Let A be a subset of a topological space S. Then,

$$\mathrm{cl}\,(A) = A \cup \mathrm{ac}\,(A)$$

Proof. By definition, $A \subset \mathrm{cl}(A)$. On the other hand, by Proposition 5.25, $\mathrm{ac}(A) \subset \mathrm{cl}(A)$, so $A \cup \mathrm{ac}(A) \subset \mathrm{cl}(A)$.

Now, suppose $p \in \mathrm{cl}(A)$ and $p \notin A$. Again, by Proposition 5.25, a deleted neighborhood $U - \{p\}$ of p must contain a point of A (since $p \notin A$). By definition, p is an accumulation point of A and thus $p \in \mathrm{ac}(A)$. □

Proposition 5.27. A set A is closed if and only if $A = \mathrm{cl}(A)$.

Proof. If $A = \mathrm{cl}(A)$ then since $\mathrm{cl}(A)$ is closed,A is closed. Conversely, if A is closed, then A is a closed set containing A, so $\mathrm{cl}(A) \subset A \subset \mathrm{cl}(A)$. Thus, $A = \mathrm{cl}(A)$. □

Proposition 5.28. If $A \subset B$ in a topological space S, then $\mathrm{cl}(A) \subset \mathrm{cl}(B)$.

Proof. By definition, $\mathrm{cl}(B)$ contains B, and thus contains A. Since $\mathrm{cl}(B)$ is a closed set containing A, it must contain $\mathrm{cl}(A)$. □

5.13 Convergence

Let us recall the definition of a *sequence*:

Definition 5.14. Let S be a topological space. A *sequence* in S is a map from the set $\mathbb{Z}^+$ of positive integers to S. We write a sequence as $\langle x_i \rangle$ or $x_1, x_2, x_3 \ldots$

Then, convergence of a sequence in a topological space can be defined as follows:

Definition 5.15. Let S be a topological space and $\langle x_i \rangle$ be a sequence in S. The sequence $\langle x_i \rangle$ *converges* to p if for every neighborhood U of p, there is a positive integer N such that for all $i \geq N$, $x_i \in U$. We say that p is a *limit* of the sequence $\langle x_i \rangle$ and write $x_i \to p$ or $\lim_{i\to\infty} x_i = p$.

Proposition 5.29. (Uniqueness of the limit). In a Hausdorff space S, if a sequence $\langle x_i \rangle$ converges to p and to q, then $p = q$.

Proof. Let p and q be limits of a sequence $\langle x_i \rangle$ in S and suppose $p \neq q$. Since S is Hausdorff, there exists disjoint open sets U and V in S that contains p and q respectively. By the definition of a limit, there are positive integers M and N such that for $i \geq M$ and $j \geq N$, $x_i \in U$ and $x_j \in V$. This implies that for all $k \geq \max(M, N)$, $x_k \in U \cap V = \emptyset$. By contradiction, $p = q$. □

Proposition 5.30. (The sequence lemma). Let S be a topological space and A a subset of S. If there is a sequence $\langle a_i \rangle$ in A that converges to p, then $p \in \mathrm{cl}(A)$. The converse is also true if S is first-countable.

Proof. Suppose $a_i \to p$ with $a_i \in A$ for all i. By the definition of convergence, every neighborhood U of p must contain all but finitely many points of $\langle a_i \rangle$. In particular, U must contain a point in A, so $p \in \mathrm{cl}(A)$ by Proposition 5.25.

Conversely, suppose $p \in \mathrm{cl}\,(A)$. Since S is first-countable, there is a countable basis of neighborhoods $\{U_i\}$ around p such that
$$U_1 \supset U_2 \supset \cdots .$$
By Proposition 5.25, there is a point $a_i \in A$ in each neighborhood U_i. We claim that the sequence $\langle a_i \rangle$ is a sequence that converges to p, since given any neighborhood U of p, there exists a positive integer N such that $p \in U_N \subset U$. Then, for all $i \geq N$, we have

$$a_i \in U_i \subset U_N \subset U,$$

so p is a limit of $\langle a_i \rangle$. □

Problems

Problem 5.1. Closed sets
Let S be a topological space. Prove the following two statements.

(a) If $\{F_i\}_{i=1}^n$ is a finite collection of closed sets in S, then $\bigcup_{i=1}^n F_i$ is closed.

(b) If $\{F_\alpha\}$ is arbitrary collection of closed sets in S, then $\bigcap_\alpha F_\alpha$ is closed.

Solution. A closed set F_i in S is the complement of some open set U_i in S. Thus, given a finite collection of closed sets, $\{F_i\}_{i=1}^n$, we have

$$\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} \overline{U_i} = \overline{\bigcap_{i=1}^{n} U_i},$$

so $\{F_i\}_{i=1}^n$ is closed.

Similarly, given an arbitrary collection of closed sets $\{F_\alpha\}$, we have

$$\bigcap_{\alpha} F_i = \bigcap_{\alpha} \overline{U_i} = \overline{\bigcup_{\alpha} U_i},$$

so $\{F_i\}_{i=1}^n$ is closed.

Problem 5.2. The $\varepsilon - \delta$ criterion for continuity
Prove that a function $f : A \to R^n$ is continuous at $p \in A$ if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in A$ satisfying $d(x, p) < \delta$, one has $d(f(x), f(p)) < \varepsilon$.

Proof. Suppose f is continuous at $p \in A$. Let $\varepsilon > 0$ and $B\,(f(p), \varepsilon)$ be an open ball centered at $f(p)$. By the continuity of f at p, there exists a neighborhood U of p such that $f(U) \subset B\,(f(p), \varepsilon)$. Since open balls in $\mathbb{R}^n$ are a basis for $\mathbb{R}^n$, there exists an open ball $B(p, \delta) \in \mathbb{R}^n$ such that $B\,(p, \delta) \cap A \subset U$ via the subspace topology. Thus, $f\,(B\,(p, \delta) \cap A) \subset f(U) \subset B\,(f(p), \varepsilon)$, which is what we need.

Now, let us prove the converse. The statement can be equivalently stated in "topological terms" as follows: for any $\varepsilon > 0$, one can construct an open ball $B\,(p, \delta)$ such that $f(B\,(p, \delta)) \subset B\,(f(p), \varepsilon)$. This immediately implies the continuity of f since for any neighborhood V, we can choose ε such that $B\,(f(p), \varepsilon) \subset V$, after which the above argument follows. □

Problem 5.3. Let A and B be subsets of a topological space S. Prove the following:

(a) $\mathrm{cl}\,(A \cup B) = \mathrm{cl}\,(A) \cup \mathrm{cl}\,(B)$ (the closure of the union is the union of the closure), and

(b) $\mathrm{cl}\,(A \cap B) \subset \mathrm{cl}\,(A) \cap \mathrm{cl}\,(B)$ (the closure of the intersection is contained in the intersection of the closure).

Proof.

(a) Since $\mathrm{cl}\,(A \cup B)$ is a closed set containing $A \cup B$, and thus containing A and B, it contains $\mathrm{cl}\,(A)$ and $\mathrm{cl}\,(B)$. Conversely, let $p \in \mathrm{cl}\,(A \cup B)$. If $p \in A \cup B$ then p is in either in A or B, and thus in $\mathrm{cl}\,(A) \cup \mathrm{cl}\,(B)$. If $p \notin A \cup B$ then $p \in \mathrm{ac}\,(A \cup B)$, and thus every deleted neighborhood of p contains a point p' of $A \cup B$. Thus, p' is either in A or B, so $p \in \mathrm{cl}\,(A) \cup \mathrm{cl}\,(B)$.

(b) Observe that $\mathrm{cl}\,(A)$ and $\mathrm{cl}\,(B)$ are both closed sets containing $A \cap B$, so their intersection contains $\mathrm{cl}\,(A \cap B)$

$\square$

6 Manifolds

6.1 Topological Manifolds

Definition 6.1. A topological space M is *locally Euclidean of dimension n* if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood* or a *coordinate open set*, and ϕ a *coordinate map* or a *coordinate system* on U. We say that a chart (U, ϕ) is *centered* at $p \in U$ if $\phi(p) = 0$.

Definition 6.2. A *topological manifold* is a Hausdorff, second countable, locally Euclidean space. It is said to be of *dimension n* if it is locally Euclidean of dimension n.

Remark. For any $n \neq m$, an open subset of $\mathbb{R}^n$ is not homeomorphic to an open subset of $\mathbb{R}^m$. This result is called *invariance of dimension*, and it will be proven later on for *smooth* manifolds. The general result is not covered in this book.

Example. The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathbf{1})$, where $\nvDash : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. Every open subset U of $\mathbb{R}^n$ is also a topological manifold, with chart $(U, \mathbf{1}_U)$.

Remark. Recall that a subspace of a Hausdorff space is Hausdorff and a subspace of a second countable space is also second countable. Thus, any subspace of $\mathbb{R}^n$ is automatically Hausdorff and second countable.

Example. The graph of $y = x^{2/3}$ in $\mathbb{R}^2$ is a topological manifold. Since it is a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is also locally Euclidean, for it is homeomorphic to $\mathbb{R}$ via the coordinate map $\left(x, x^{2/3}\right) \mapsto x$.

6.2 Compatible Charts

Definition 6.3. Two charts $(U, \phi : U \to \mathbb{R}^n)$ and $(V, \Psi : V \to \mathbb{R}^n)$ of a topological manifold are C^∞-*compatible* (or *compatible* for short) if the two maps

$$\phi \circ \Psi^{-1} : \Psi(U \cap V) \to \phi(U \cap V), \quad \Psi \circ \phi^{-1} : \phi(U \cap V) \to \Psi(U \cap V)$$

are C^∞. These two maps are called the *transition functions* between the charts. If $U \cap V$ is empty, then the two charts are automatically C^∞-compatible. To simplify notations, we can write $U_{\alpha\beta}$ for $U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma}$ for $U_\alpha \cap U_\beta \cap U_\gamma$.

Definition 6.4. A C^∞ *atlas* or simply an *atlas* on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M, i.e. such that $\bigcup_\alpha U_\alpha = M$. We say that a chart (V, Ψ) is *compatible with an atlas* $\{(U_\alpha, \phi_\alpha)\}$ if it is compatible with all the charts (U_α, ϕ_α) in the atlas.

Lemma 6.1. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space. If two charts (V, Ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.

Proof. For any $p \in V \cap W$, $p \in U_\alpha$ for some α. By construction, the maps $\Psi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \sigma^{-1}$ are C^∞ on $U_\alpha \cap V$ and $U_\alpha \cap W$ respectively. Thus, the map $\Psi \circ \sigma^{-1} = \left(\Psi \circ \phi_\alpha^{-1}\right) \circ \left(\phi_\alpha \circ \sigma^{-1}\right)$ is C^∞ on $U_\alpha \cap V \cap W$. Since this is true for any $p \in V \cap W$, $\Psi \circ \sigma^{-1}$ is C^∞ on $V \cap W$, and the same is true for $\sigma \circ \Psi^{-1}$. □

6.3 Smooth Manifolds

Definition 6.5. A *smooth* or C^∞ *manifold* is a topological manifold M together with a maximal atlas. The maximal atlas is also called a *differentiable structure* on M. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is also called a *curve*, a 2-dimensional manifold a *surface*, and an n-dimensional manifold an $n-$*manifold*.

To check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of any atlas on M will do, for we have the following proposition.

Proposition 6.2. Any atlas $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ on a locally Euclidean space is contained in a unique maximal atlas.

Proof. For any atlas $\mathfrak{U}$, adjoin to $\mathfrak{U}$ all charts that are compatible with $\mathfrak{U}$. By Lemma 6.1, these charts are all compatible with one another, so the new collection of charts $\mathfrak{M}$ is an atlas. It is also maximal, for any chart compatible with it must also be compatible with $\mathfrak{U}$, and thus lies in $\mathfrak{M}$. $\mathfrak{M}$ is also unique, for suppose there is another maximal atlas $\mathfrak{M}'$ that contains $\mathfrak{U}$ then any chart in $\mathfrak{M}'$ must be compatible with $\mathfrak{U}$. By construction, this implies $\mathfrak{M}' \subset \mathfrak{M}$, and thus $\mathfrak{M}' = \mathfrak{M}$. □

In summary, to show a topological space M is a smooth manifold, it suffices to check that

(i) M is Hausdorff and second countable, and

(ii) M has a (not necessarily maximal) C^∞ atlas.

6.4 Examples of Smooth Manifolds

Example. (*Euclidean space*). The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $\{(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})\}$.

Example. Any open subset U of a manifold M is also a manifold, for if $\{U_\alpha, \phi_\alpha\}$ is an atlas for M then $\{U_\alpha \cap U, \phi_\alpha|_{U_\alpha \cap U}\}$ is an atlas for U.

Example. (*Graph of a smooth function*). For a subset $A \subset \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^m$, the graph of f is defined to be the subset

$$\Gamma(f) = \{(x, f(x)) \in A \times \mathbb{R}^n\}.$$

If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^m$ is C^∞, then the two maps

$$\begin{aligned} \phi : \Gamma(f) &\to U, & (1, f) : U &\to \Gamma(f), \\ (x, f(x)) &\mapsto x, & x &\mapsto (x, f(x)), \end{aligned}$$

are continuous and inverse to each other, and thus are homeomorphisms. The graph $\Gamma(f)$ of a C^∞ function $f : U \to \mathbb{R}^m$ has a single chart $\{(\Gamma(f), \phi)\}$, so it is a C^∞ manifold.

Example. (*General linear groups*). The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ is a manifold, for it is the preimage of the determinant function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$, which is continuous, and thus an open subset of $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$. Similarly, the *complex general linear group* $\mathrm{GL}(n, \mathbb{C})$ is an open subset of $\mathbb{C}^{n\times n} \simeq \mathbb{R}^{2n^2}$, so it is a manifold of dimension $2n^2$.

Proposition 6.3. If $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_i, \Psi_i)\}$ are C^∞ atlases for the manifolds M and N of dimension m and n respectively, then the collection

$$\{(U_\alpha \times V_i, \phi_\alpha \times \Psi_i : U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$$

is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a smooth manifold of dimension $m + n$.

Proof. It suffirces to show that any two charts $\{(U_\alpha \times V_i, \phi_\alpha \times \Psi_i)\}$ and $\{(U_\beta \times V_j, \phi_\alpha \times \Psi_j)\}$ are compatible. Since ϕ_α and Ψ_i are homeomorphisms, the product map $\phi_\alpha \times \Psi_i : (x, y) \mapsto (f(x), f(y)) \in \mathbb{R}^m \times \mathbb{R}^n$ is also a homeomorphism, and thus so is its inverse. Thus, $(\phi_\alpha \times \Psi_i)(\phi_\beta \times \Psi_j)^{-1}$ and $(\phi_\beta \times \Psi_j)(\phi_\alpha \times \Psi_i)^{-1}$ are homeomorphisms, so the product manifold is a smooth manifold of dimension $m + n$. ☐

Problems

Problem 6.1. A sphere with hair. Show that a sphere with hair (**Figure 6.1**) is *not* a topological manifold by showing it is not locally Euclidean at q.

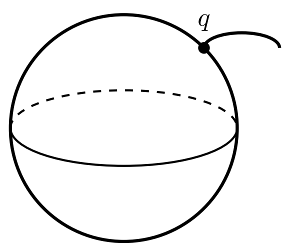

Figure. 6.1: A sphere with hair.

Solution. Suppose the sphere with hair is locally Euclidean of dimension n at q. Then, q has a neighborhood U homeomorphic to an open ball $B := B(0, r) \subset \mathbb{R}^n$. Then, the homeomorphism $\phi : U \to B$ restricts to the homeomorphism $\phi : U - \{q\} \to B - \{0\}$. Since $U - \{q\}$ has two connected components B must be an open ball of dimension $n = 1$. This implies $U - \{q\}$ also consists of two connected components of dimension 1, contradicting the invariance of dimension since the sphere with the point q removed still has dimension 3.

Problem 6.2. Existence of a coordinate neighborhood. Let $\mathfrak{M} = \{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas on a manifold M. For any open set U in M and a point $p \in U$, prove the existence of a coordinate open set U_α such that $p \in U_\alpha \subset U$.

Solution. Since $\{U_\alpha\}$ covers $\mathfrak{M}$, choose a chart (U_p, ϕ_p) such that $p \in U_p$. Then, since $\mathfrak{M}$ is maximal, the chart $\left(U_p \cap U, \phi_p|_{U_p \cap U}\right)$ is contained in $\mathfrak{M}$ and that $p \in U_p \cap U \subset U$.

7 Smooth Maps on a Manifold

7.1 Smooth Functions on a Manifold

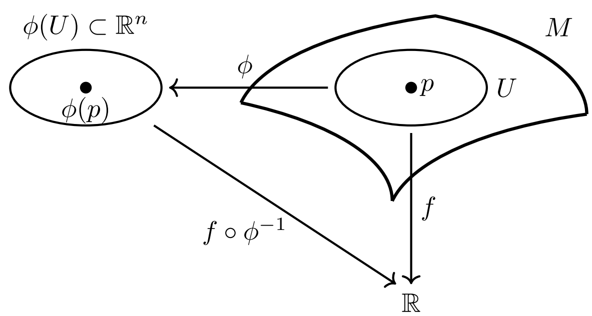

Figure. 7.1: Checking if a function f is C^∞ at p by pulling back to $\mathbb{R}^n$.

Definition 7.1. Let M be a manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be C^∞ or *smooth at a point* p in M if there is a chart (U, ϕ) about p in M such that the function $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is C^∞ at $\phi(p)$ (see **Figure 7.1**). The function f is then said to be C^∞ on M if it is C^∞ at every point in M.

Remark. The definition of the smoothness of a function f at a point is independent of the chart (U, ϕ), for if $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$ and (V, Ψ) is any other chart about p in M then on $\Psi(U \cap V)$,

$$f \circ \Psi^{-1} = \left(f \circ \phi^{-1}\right) \circ \left(\phi \circ \Psi^{-1}\right),$$

which is C^∞ at $\Psi(p)$ through the transition function $\phi \circ \Psi^{-1}$.

Notice that in the above definition, we did not assume the continuity of f. However, if f is C^∞ at p in M, then $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$, being a C^∞ function at $\phi(p)$ in an open subset U of $\mathbb{R}^n$, is continuous at $\phi(p)$. Thus, $f = \left(f \circ \phi^{-1}\right) \circ \phi$ is continuous at p. Since we are only interested in smooth functions on open sets, there is no loss of generality when we assume that f is continuous.

Proposition 7.1. (Smoothness of a real-valued function). Let M be a manifold of dimension n, and $f : M \to \mathbb{R}$ a real-valued function on M. Then, the following are equivalent:

(i) The function $f : M \to \mathbb{R}$ is C^∞.

(ii) The manifold M has an atlas such that for every chart (U, ϕ) in the atlas, $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is C^∞.

(iii) For every chart (V, ϕ) on M, the function $f \circ \Psi^{-1} : \Psi(V) \to \mathbb{R}$ is C^∞.

Proof. We will prove the proposition through the following implications:
(ii) $\Rightarrow$ (i): This follows directly from the definition above.
(i) $\Rightarrow$ (iii): From the above remark, for any point $p \in V$, $f \circ \Psi^{-1}$ is C^∞ at $\Psi(p)$, so it is smooth on $\Psi(V)$.
(iii) $\Rightarrow$ (ii): Obvious. $\square$

Definition 7.2. Let $F : N \to M$ be a map and h a function on M. The *pullback* of h by F, denoted F^*h, is the composition $h \circ F$.

In this terminology, a function f on M is C^∞ on a chart (U, ϕ) if and only if its pullback $\left(\phi^{-1}\right)^* f$ by ϕ^{-1} is C^∞ on the subset $\phi(U)$ of $\mathbb{R}^n$.

7.2 Smooth Maps Between Manifolds

Unless otherwise specified, a manifold will always mean a C^∞ manifold. A manifold M will always have dimension n, unless we are talking about two different manifolds N and M simultaneously, in which case N will have dimension n and M dimension M.

Definition 7.3. Let N and M be manifolds of dimension n and m, respectively. A continuous map $F : N \to M$ is C^∞ *at a point* p in N if there are charts (V, Ψ) about $F(p)$ in M and (U, ϕ) about p in N such that the composition $\Psi \circ F \circ \phi^{-1} : \phi\left(F^{-1}(V) \cap U\right) \to \mathbb{R}^m$, is C^∞ at $\phi(p)$ (see **Figure 7.2**). The continuous map $F : N \to M$ is then said to be C^∞ if it is C^∞ at every point of N.

Notice that in the above definition, the map $F : N \to M$ is assumed to be continuous. This is to ensure that $F^{-1}(V)$ is an open set in F. Thus, smooth maps between manifolds are by definition

$$
\begin{tikzcd}
N \supset U \arrow[r, "F"] \arrow[d, "\phi"'] & V \subset M \arrow[d, "\Psi"] \\
\mathbb{R}^n \supset \phi(U) \arrow[r, "\Psi \circ F \circ \phi^{-1}"'] & \Psi(V) \subset \mathbb{R}^m
\end{tikzcd}
$$

Figure. 7.2: Checking the map $F : N \to M$ is C^∞ at p.

continuous.

Remark. (*Smooth maps into* $\mathbb{R}^m$). In the case $M = \mathbb{R}^m$, we can take $(\mathbb{R}^m, \mathbb{1})$ to be the chart about p in $\mathbb{R}^m$. According to the above definition, $F : N \to \mathbb{R}^m$ is C^∞ at $p \in N$ if only if there is a chart (U, ϕ) about p such that $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ at $\phi(p)$. Letting $m = 1$, we recover Definition 7.1 of a function being C^∞ at a point.

Similar to how the smoothness of a function $N \to \mathbb{R}$ at a point p in N is independent of the choice of a chart on N about p, the definition of the smoothness of a map $F : N \to M$ at a point is also independent of the choice of charts.

Proposition 7.2. Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, ϕ) is any chart about p in N and (V, Ψ) any chart about $F(p)$ in M, then $\Psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

Proof. Since F is C^∞ at $p \in N$, there are charts (U_α, ϕ_α) about p in N and (V_β, Ψ_β) about $F(p)$ in M such that $\Psi_\beta \circ F \circ \phi_\alpha^{-1}$ is C^∞ at $\phi_\alpha(p)$. By the C^∞ compatibility of charts, both $\phi_\alpha \circ \phi^{-1}$ and $\Psi \circ \Psi_\beta^{-1}$ are C^∞ on open subsets of Euclidean spaces. Thus, the composition

$$\Psi \circ F \circ \phi^{-1} = (\Psi \circ \Psi_\beta^{-1}) \circ \left(\Psi_\beta \circ F \circ \phi_\alpha^{-1}\right) \circ (\phi_\alpha \circ \phi^{-1})$$

is C^∞ at $\phi(p)$. $\square$

Proposition 7.3. (Smoothness of a map in terms of charts). Let N and M be smooth manifolds, and $F : N \to M$ a continuous map. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) There are atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M such that for every chart (U, ϕ) in $\mathfrak{U}$ and (V, Ψ) in $\mathfrak{V}$, the map
$$\Psi \circ F \circ \phi^{-1} : \phi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$
is C^∞.

(iii) For every chart (U, ϕ) on N and (V, Ψ) on M the map
$$\Psi \circ F \circ \phi^{-1} : \phi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$
is C^∞.

Proof. Similar to Proposition 7.1, we will prove the following implications:
(ii) $\Rightarrow$ (i): This follows immediately from the definition of smooth maps between manifolds.
(i) $\Rightarrow$ (iii): From Proposition 7.2, for any charts (U, ϕ) on N and (V, Ψ) on M with $p \in U \cap F^{-1}(V) \neq \emptyset$, the composition $\Psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$, and thus is C^∞ on $\phi\left(U \cap F^{-1}(V)\right)$.
(iii) $\Rightarrow$ (ii): Obvious. $\square$

Proposition 7.4. (Composition of C^∞ maps). If $F : N \to M$ and $G : M \to P$ are C^∞ maps of manifolds, then $G \circ F : N \to P$ is also C^∞.

Proof. Let (U, ϕ), (V, Ψ), and (W, σ) be charts on N, M, and P, respectively. Then, the composition
$$\sigma \circ (G \circ F) \circ \phi^{-1} = \left(\sigma \circ G \circ \Psi^{-1}\right) \circ \left(\Psi \circ F \circ \phi^{-1}\right)$$
is a composition of C^∞ maps on open subsets of Euclidean spaces (by Proposition 7.3), and thus is also C^∞. Thus, the composite map $G \circ F$ is C^∞ (also by Proposition 7.3). $\square$

7.3 Diffeomorphisms

Definition 7.4. A *diffeomorphism* of manifolds is a bijective C^∞ map $F : N \to M$ whose inverse F^{-1} is also C^∞.

Remark. According to the next two propositions, coordinate maps are diffeomorphisms. Conversely, every diffeomorphism of an open subset of a manifold with an open subset of an Euclidean space can serve as a coordinate map.

Proposition 7.5. If (U, ϕ) is a chart on a manifold M of dimension n, then the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.

Proof. By definition, ϕ is a homeomorphism, so it is bijective. Thus, it suffices to show that ϕ and ϕ^{-1} are C^∞. Choose the atlases $\{(U, \phi)\}$ for U and $\{(\phi(U)), \mathbb{1}\}$ for $\phi(U)$. Since $\mathbb{1} \circ \phi \circ \phi^{-1} = \mathbb{1}$ is the identity map, it is C^∞. By Proposition 7.3, ϕ is C^∞. Similarly, using the same atlases as above, we see that $\phi \circ \phi^{-1} \circ \mathbb{1}$ is also C^∞. Again, by Proposition 7.3, ϕ^{-1} is C^∞. □

Proposition 7.6. Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a chart in the differentiable structure of M.

Proof. For any chart (U_α, ϕ_α) in the maximal atlas of M, both ϕ_α and ϕ_α^{-1} are C^∞ by Proposition 7.5. Thus, as compositions of C^∞ maps, both $F \circ \phi^{-1}$ and $\phi \circ F^{-1}$ are C^∞. Hence, (U, F) is compatible with the maximal atlas, and by the maximality of the atlas, (U, F) is also in the atlas. □

7.4 Smoothness in Terms of Components

Proposition 7.7. (Smoothness of a vector-valued function). Let N be a manifold and $F : N \to \mathbb{R}^m$ a continuous map. Then, the following are equivalent:

(i) The map $F : N \to \mathbb{R}^m$ is C^∞.

(ii) The manifold N has an atlas such that for every chart (U, ϕ) in the atlas, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.

(iii) For every chart (U, ϕ) on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.

Proof. (ii) $\Rightarrow$ (i): Take the atlas $\{(\mathbb{R}^m, \mathbb{1})\}$ consisting of a single chart on $\mathbb{R}^m$. Then, by Proposition 7.3(ii), F is C^∞.
(i) $\Rightarrow$ (iii): Again, in Proposition 7.3(iii), let (V, Ψ) be the chart $(\mathbb{R}^m, \mathbb{1})$ on $\mathbb{R}^m$.
(iii) $\to$ (ii): Obvious. □

Proposition 7.8. (Smoothness in terms of components). Let N be a manifold. A vector-valued function $F : N \to \mathbb{R}^m$ is C^∞ if and only if its component functions $F^1, \ldots, F^m : N \to \mathbb{R}$ are all C^∞.

Proof. The map $F : N \to \mathbb{R}^m$ is C^∞.
$\Leftrightarrow$ For every chart (U, ϕ) on N, the function $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ (Proposition 7.7.
$\Leftrightarrow$ For every chart (U, ϕ) on N, the functions $F^i \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ are all C^∞ (by the definition of smooth maps between Euclidean spaces).
$\Leftrightarrow$ The component functions $F^i : N \to \mathbb{R}$ are all C^∞ (Proposition 7.1). □

Example. (*Smoothness of a map to a circle*). The map $F : \mathbb{R} \to S^1, F(t) = (\cos t, \sin t)$ is C^∞ since the component functions $\cos t, \sin t$ are all C^∞.

Proposition 7.9. (Smoothness of a map in terms of vector-valued functions). Let N and M be manifolds of dimension n and m, respectively, and $F : N \to M$ be a continuous map. Then, the following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) The manifold M has an atlas such that for every chart $(V, \Psi) = \left(V, y^1, \ldots, y^m\right)$ in the atlas, the vector-valued function $\Psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞.

(iii) For every chart $(V, \Psi) = \left(V, y^1, \ldots, y^m\right)$ on M, the vector-valued function $\Psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞.

Proof. (ii) $\to$ (i): By Proposition 7.7(i) $\Rightarrow$ (iii), for every chart (U, ϕ) on N, the map

$$(\Psi \circ F) \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$

is C^∞. Thus, by Proposition 7.3, $F : N \to M$ is C^∞.
(i) $\Rightarrow$ (iii): As a coordinate map, Ψ is C^∞ (Proposition 7.5). Thus, as a composition of C^∞ maps, $\Psi \circ F$ is C^∞.
(iii) $\Rightarrow$ (ii): Obvious. $\square$

By Proposition 7.8, the smoothness criterion of a map in terms of vector-valued functions can be translated into the smoothness criterion in terms of components

Proposition 7.10. (Smoothness of a map in terms of components). Let $F : N \to M$ be a continuous map between two manifolds of dimension n and m, respectively. Then, the following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) The manifold M has an atlas such that for every chart $(V, \Psi) = \left(V, y^1, \ldots, y^m\right)$ in the atlas, the component coordinates $y^i \circ F : F^{-1}(V) \to \mathbb{R}$ of F relative to the chart are all C^∞.

(iii) For every chart $(V, \Psi) = \left(V, y^1, \ldots, y^m\right)$ on M, the component coordinates $y^i \circ F : F^{-1}(V) \to \mathbb{R}$ of F relative to the chart are all C^∞.

7.5 Examples of Smooth Maps

Example. (*Smoothness of a projection map*). Let M and N be manifolds and $\pi : M \times N \to M$, $\pi(p, q) = p$ be the projection to the first factor. Then, π is a C^∞ map.

Proof. Let (p, q) be an arbitrary point in $M \times N$ and suppose (U, ϕ) and (V, Ψ) are coordinate neighborhoods of p and q in M and N, respectively. By Proposition 6.3, $(U \times V, \phi \times \Psi)$ is a coordinate neighborhood of (p, q). Observe that

$$\left(\phi \circ \pi \circ (\phi \times \Psi)^{-1}\right)(\phi(p), \Psi(q)) = \phi(p)$$

is a C^∞ map of subsets of Euclidean spaces. Thus, π is C^∞ at (p, q), and therefore is C^∞ on $M \times N$. $\square$

Example. Let M_1, M_2, and N be manifolds of dimension m_1, m_2, and n respectively. Then, the map $(f_1, f_2) : N \to M_1 \times M_2$ is C^∞ if and only if the components $f_i : N \to M_i$ are both C^∞.

Proof. $(f_1, f_2) : N \to M_1 \times M_2$ is C^∞
$\Leftrightarrow$ For every chart $(V = V_1 \times V_2, \Psi = \Psi_1 \times \Psi_2) = (V, y^1, \ldots, y^{m_1}, y^{m_1+1}, \ldots, y^{m_1+m_2})$ on $M_1 \times M_2$, the component coordinates $y^i \circ (f_1, f_2) : (f_1, f_2)^{-1}(V) \to \mathbb{R}$ of (f_1, f_2) relative to the chart are all C^∞ (Proposition 7.10).
$\Leftrightarrow$ The vector-valued functions $\Psi_1 \circ f_1 : f_1^{-1}(V_1) \to \mathbb{R}^{m_1}$ and $\Psi_2 \circ f_2 : f_2^{-1}(V_2) \to \mathbb{R}^{m_2}$ are both C^∞ (Definition of smooth maps between subsets of Euclidean spaces).
$\Leftrightarrow$ The components $f_i : N \to M_i$ are both C^∞ (Proposition 7.9). $\square$

Now that we have the definition of a smooth map between manifolds, we can define a *Lie group*.

Definition 7.5. A *Lie group* is a C^∞ manifold G having a group structure such that the multiplication map

$$\mu : G \times G \to G$$

and the inverse map

$$\iota : G \to G,\ \iota(x) = x^{-1},$$

are both C^∞.

Similarly, a *topological group* is a topological space having a group structure such that the multiplication map and the inverse map are both continuous. Note that a topological group need not be a topological manifold, only a topological space.

Example.

(i) The Euclidean space $\mathbb{R}^n$ is a Lie group under addition.

(ii) The set $\mathbb{C}^\times$ of nonzero complex numbers is a Lie group under multiplication.

(iii) The unit circle S^1 in $\mathbb{C}^\times$ is a Lie group under multiplication.

(iv) The Cartesian product $G_1 \times G_2$ of two Lie groups (G_1, μ_1) and (G_2, μ_2) is a Lie group under componentwise multiplication $\mu_1 \times \mu_2$.

Example. (*General linear group*). As an open subset of $\mathbb{R}^{n\times n}$, the general linear group $\mathrm{GL}(n, \mathbb{R})$ is a manifold. It is also a Lie group, since the multiplication map and the inverse map are both C^∞.

7.6 Partial Derivatives

Let M be a manifold of dimension n, $(U, \phi) = \left(U, x^1, \ldots, x^n\right)$ be a chart on M and f be a C^∞ function. Denote by $r^1, \ldots, r^n$ the standard coordinates on $\mathbb{R}^n$. Then, we have $x^i = r^i \circ \phi$.

Definition 7.6. For any $p \in U$, the *partial derivative $\partial f/\partial x^i$ of f with respect to x^i at p* is defined by

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} \left(f \circ \phi^{-1}\right)$$

In 'simple' terms, the partial derivative of f with respect to x^i at p is given by the partial derivative of the *pullback* $\left(\phi^{-1}\right)^* f = f \circ \phi^{-1}$ with respect to r^i at $\phi(p)$ (**Figure 7.1**).
Since $p = \phi^{-1}(\phi(p))$, we can rewrite the above equation into

$$\left(\frac{\partial f}{\partial x^i} \circ \phi^{-1}\right)(\phi(p)) = \frac{\partial\left(f \circ \phi^{-1}\right)}{\partial r^i}(\phi(p))$$

Thus, as a function on $\phi(U)$, the pullback $\phi^*\left(\partial f/\partial x^i\right)$,

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial\left(f \circ \phi^{-1}\right)}{\partial r^i},$$

is C^∞ on $\phi(U)$. Hence, the partial derivative $\partial f/\partial x^i$ is C^∞ on U.

Proposition 7.11. Suppose $\left(U, x^1, \ldots, x^n\right)$ is a chart on a manifold. Then,

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j.$$

Proof. By definition, at any point $p \in U$, we have

$$\frac{\partial x^i}{\partial x^j}(p) = \frac{\partial\left(x^i \circ \phi^{-1}\right)}{\partial r^j}(\phi(p)) = \frac{\partial r^i}{\partial r^j}(\phi(p)) = \delta^i_j.$$

□

Definition 7.7. Let $F : N \to M$ be a smooth map, and let $(U, \phi) = \left(U, x^1, \ldots, x^n\right)$ and $(V, \Psi) = \left(V, y^1, \ldots, y^m\right)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by

$$F^i :\ = y^i \circ F = r^i \circ \Psi \circ F : U \to \mathbb{R}$$

the ith component of F in the chart (V, Ψ). Then, the matrix $\left[\partial F^i/\partial x^j\right]$ is called the *Jacobian matrix* of F relative to the charts (U, ϕ) and (V, Ψ). In case N and M have the same dimension, the determinant $\det\left[\partial F^i/\partial x^j\right]$ is called the *Jacobian determinant* of F relative to the two charts. The Jacobian determinant is also written as $\partial\left(F^1, \ldots, F^n\right)/\partial\left(x^1, \ldots, x^n\right)$.

When N and M are open subsets of Euclidean spaces and the charts are $\left(U, \left\langle r^j\right\rangle_{j=1}^n\right)$ and $\left(V, \left\langle r^i\right\rangle_{i=1}^m\right)$, the Jacobian matrix $\left[\partial F^i/\partial r^j\right]$, where $F^i = r^i \circ F$, is the usual Jacobian matrix from calculus.

Example. (*Jacobian matrix of a transition map*). Given two overlapping charts $(U, \phi) = \left(U, x^1, \ldots, x^n\right)$ and $(V, \Psi) = \left(V, y^1, \ldots, y^n\right)$ on a manifold M, the transition map $\Psi \circ \phi^{-1} : \phi(U \cap V) \to \Psi(U \cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$. Hence, its Jacobian matrix $J\left(\Psi \circ \phi^{-1}\right)$ at $\phi(p)$ is given by

$$\left[\frac{\partial\left(\Psi \circ \phi^{-1}\right)^i}{\partial r^j}(\phi(p))\right] = \left[\frac{\partial\left(r^i \circ \Psi \circ \phi^{-1}\right)}{\partial r^j}(\phi(p))\right] = \left[\frac{\partial\left(y^i \circ \phi^{-1}\right)}{\partial r^j}(\phi(p))\right] = \left[\frac{\partial y^i}{\partial x^i}(p)\right].$$

7.7 The Inverse Function Theorem

By Proposition 7.6, any diffeomorphism $F\colon U \to F(U) \subset \mathbb{R}^n$ of an open subset U of a manifold may be thought of as a coordinate system on U. We say that a C^∞ map $F\colon N \to M$ is *locally invertible* or a *local diffeomorphism* at $p \in N$ if p has a neighborhood U on which $F|_U\colon U \to F(U)$ is a diffeomorphism.

Given n smooth functions $F^1, \ldots, F^n$ in a neighborhood of a point p in a manifold N of dimension n, one would like to know whether they form a coordinate system, possibly on a smaller neighborhood of p. This is equivalent to whether $F = \left(F^1, \ldots, F^n\right) : N \to \mathbb{R}^n$ is a local diffeomorphism at p. The inverse function theorem provides the answer to this question.

Theorem 7.12. (Inverse function theorem for $\mathbb{R}^n$). Let $F\colon W \to \mathbb{R}^n$ be a C^∞ map defined on an open subset W of $\mathbb{R}^n$. For any point p in W, the map F is locally invertible at p if and only if the Jacobian determinant $\det\left[\partial F^i/\partial r^j(p)\right]$ is nonzero.

Since the inverse function theorem for $\mathbb{R}^n$ is a local result, it is easily translatable to manifolds.

Theorem 7.13. (Inverse function theorem for manifolds). Let $F\colon N \to M$ be a C^∞ map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U, \phi) = \left(U, x^1, \ldots, x^n\right)$ about p in N and $(V, \Psi) = \left(\Psi, y^1, \ldots, y^k\right)$ about $F(p)$ in M, $F(U) \in V$. Set $F^i = y^i \circ F$. Then, F is locally invertible at p if and only if its Jacobian determinant $\det\left[\partial F^i/\partial x^j(p)\right]$ is nonzero.

Proof. The Jacobian matrix of F relative to the charts (U, ϕ) and (V, Ψ) is given by

$$\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \left[\frac{\partial\left(r^i \circ \Psi \circ F\right)}{\partial x^j}(p)\right] = \left[\frac{\partial\left(r^i \circ \left(\Psi \circ F \circ \phi^{-1}\right)\right)}{\partial r^j}(\phi(p))\right] = J\left(\Psi \circ F \circ \phi^{-1}\right)$$

As $\Psi \circ F \circ \phi^{-1}\colon \phi(U) \to \Psi(V)$ is a map on an open subset of $\mathbb{R}^n$, by the inverse function theorem for $\mathbb{R}^n$, the Jacobian determinant

$$\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \det\left[J\left(\Psi \circ F \circ \phi^{-1}\right)\right]$$

is nonzero if and only if $\Psi \circ F \circ \phi^{-1}$ is locally invertible at $\phi(p)$. This is equivalent to the local invertibility of F at p (see **Figure 7.2**). $\square$

Corollary 7.14. Let N be a manifold of dimension n. A set of n smooth functions $F^1, \ldots, F^n$ defined on a coordinate neighborhood $U, x^1, \ldots, x^n$ of a point $p \in N$ forms a coordinate system about p if and only if the Jacobian determinant $\det\left[\partial F^i/\partial x^j(p)\right]$ is nonzero.

Proof. Let $F = \left(F^1, \ldots, F^n\right) : U \to \mathbb{R}^n$. Then
$\det\left[\partial F^i/\partial x^j(p)\right] \neq 0$
$\Leftrightarrow$ $F\colon U \to \mathbb{R}^n$ is locally invertible at p (inverse function theorem).
$\Leftrightarrow$ There is a neighborhood W of p in N such that $F\colon W \to F(W)$ is a diffeomorphism (definition of local invertibility).

$\Leftrightarrow \left(W, F^1, \ldots, F^n\right)$ is a coordinate chart about p in the differentiable structure of N (Proposition 7.6). $\square$

Example. (*Polar coordinates*). The set of polar coordinates (r, θ) in $\mathbb{R}^2$ can be written in terms of the standard Cartesian coordinates via the map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F\left(x, y\right) = \left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)\right).$$

By the inverse function theorem, the map F can serve as a coordinate map in a neighborhood of p if and only if the Jacobian determinant of F

$$\frac{\partial\left(F^1, F^2\right)}{\partial\left(x, y\right)} = \det\begin{bmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{bmatrix} = \frac{1}{\sqrt{x^2+y^2}}$$

is nonzero. Thus, F is a local diffeomorphism at p if and only if $p \neq (0,0)$, which checks out, as we can recall that at $(0,0)$, the angular coordinate θ is not well-defined.

Problems

Problem 7.1. Smoothness of an inclusion map. Let M and N be manifolds and let q_0 be a point in N. Show that the inclusion map $i_{q_0}\colon M \to M \times N, i_{q_0}(p) = (p, q_0)$, is C^∞.

Solution. Let $p \in M$, $(U, \phi) = \left(U, x^1, \ldots, x^m\right)$ be a chart about p in M, and $(V, \Psi) = \left(V, y^1, \ldots, y^n\right)$ be a chart about q_0 in N. By Proposition 6.3, $(U \times V, \phi \times \Psi)$ is a chart about (p, q_0). Given a point $r \in \phi(U \cap F^{-1}(V))$,

$$\left((\phi \times \Psi) \circ i_{q_0} \circ \phi^{-1}\right)(r) = (r, q_0).$$

Thus, $(\phi \times \Psi) \circ i_{q_0} \circ \phi^{-1}$ is smooth on $\phi(U \cap F^{-1}(V))$. By Proposition 7.3, i_{q_0} is smooth on M.

8 Quotients

8.1 The Quotient Topology

Let S be a set. Denote by $S/\!\sim$ the set of equivalence classes and call the set the *quotient* of S by the equivalence relation $\sim$. Then, there is a natural *projection map* $\pi\colon S \to S/\!\sim$ that sends $x \in S$ to its equivalence class $[x]$.

Now, assume S is a topological space. We define a topology on $S/\!\sim$ by declaring a set U in $S/\!\sim$ to be *open* if and only if $\pi^{-1}(U)$ is open in S. Clearly, both the empty set $\emptyset$ and the entire quotient $S/\!\sim$ are open. Furthermore, since the collection of open sets in $S/\!\sim$ are closed under arbitrary unions and finite intersections, they form a topology. This is called the *quotient topology* on $S/\sim$. With this topology, $S/\!\sim$ is called the *quotient space* of S by the equivalence relation $\sim$. With the quotient topology on $S/\!\sim$, the projection map $\pi\colon S \to S/\!\sim$ is automatically continuous, since the preimage of any open set in $S/\!\sim$ is by definition open in S.

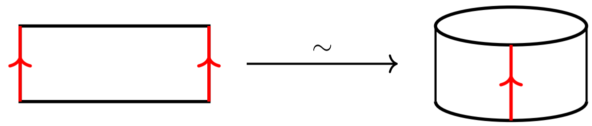

Figure. 8.1: Glueing the ends of the strip $[0,3] \times [0,1]$ into a cylinder via the equivalence relation $(0,y) \sim (3,y)$.

8.2 Continuity of a Map on a Quotient

Let $\sim$ be an equivalence relation on a topological space S and give $S/\sim$ the quotient topology. Suppose a function $f\colon S \to Y$ from S to another topological space Y is constant on each equivalence class. Then, it induces a map $\bar{f}\colon S/\sim \to Y$ by

$$\bar{f}([p]) = f(p)$$

In other words, there is a commutative diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ S/\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Proposition 8.1. The induced map $\bar{f}\colon S/\sim \to Y$ is continuous if and only if the map $f\colon S \to Y$ is continuous.

Proof. If $\bar{f}$ is continuous, then $f = \bar{f} \circ \pi$ is a composite of continuous maps, and thus it is continuous. Conversely, if f is continuous, then for any $V \in Y$, $f^{-1}(V) = \pi^{-1}(\bar{f}^{-1}(V))$ is open in S. By the definition of the quotient topology, $\bar{f}^{-1}(V)$ is open in $S/\sim$. Hence, $\bar{f}$ is coninuous. $\square$

Remark. This proposition gives us a useful way to check whether a function $\bar{f}$ on a quotient space $S/\sim$ is continuous by simply checking the continuity of the lifted map $f := \bar{f} \circ \pi$ on S.

8.3 Identification of a Subset to a Point

If A is a subspace of a topological space S, we can define an equivalence relation $\sim$ on S by declaring $x \sim x$ for all $x \in S$ and $x \sim y$ for all $x, y \in A$. We say that the quotient space $S/\sim$ is obtained from S by *identifying A to a point.*

Example. Let I be the unit interval $[0,1]$ and $I/\sim$ the quotient space obtained from I by identifying the two points $\{0,1\}$ to a single point. Denote by S^1 the unit circle in the complex plane. The function $f\colon I \to S^1$, $f(x) = \exp(2\pi i x)$, assumes the same values at 0 and 1, and thus induces a function $\bar{f}\colon I/\sim \to S^1$. It can be shown that $\bar{f}$ is a homeomorphism.

8.4 A Necessary Condition for a Hausdorff Quotient

The quotient construction does not, in general, preserve the Hausdorff property or second countability. Indeed, there is a necessary condition, which is shown by the following proposition

Proposition 8.2. If the quotient space $S/\sim$ is Hausdorff, then the equivalence class $[p]$ of any point p in S is closed in S.

Proof. Suppose the quotient space $S/\sim$ is Hausdorff. Since any singleton set in a Hausdorff space is closed, the image $\{\pi(p)\}$ of any point $p \in S$ is closed in $S/\sim$. By the continuity of π, the preimage $\pi^{-1}(\pi(p)) = [p]$ must be closed in S. $\square$

8.5 Open Equivalence Relations

In this section, we will derive the conditions under which a quotient space is Hausdorff or second countable. Recall that a map $f\colon X \to Y$ of topological spaces is *open* if for any open set U in X, the image $f(U)$ is open in Y.

Definition 8.1. An equivalence relation $\sim$ on a topological space S is said to be *open* if the projection map $\pi\colon S \to S/\sim$ is open.

In other words, the equivalence relation $\sim$ on S is open if and only if for every open set U in S, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

of all points equivalent to some point in U is open.

Example. In general, the projection map to a quotient space is not open. For example, let $\sim$ be the equivalence relation on $\mathbb{R}$ that identifies the two points 1 and -1. Then, the projection map $\pi\colon \mathbb{R} \to \mathbb{R}/\sim$ is not open since while the interval $V = (0, 2)$ is open in $\mathbb{R}$,

$$\pi^{-1}(\pi(V)) = (0, 2) \cup \{1\}$$

is not open in $\mathbb{R}$.

Definition 8.2. Let $\sim$ be an equivalence relation on S and let R be the subset of $S \times S$ that defines the relation

$$R = \{(x, y) \in S \times S \mid x \sim y\}.$$

Then, R is called the *graph* of the equivalence relation $\sim$.

Theorem 8.3. Suppose $\sim$ is an open equivalence relation on a topological space S. Then, the quotient space $S/\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $S \times S$.

Proof. R is closed in $S \times S$.
$\Leftrightarrow$ $\overline{R}$ is open in $S \times S$.
$\Leftrightarrow$ For any point $(x, y) \in \overline{R}$, there is a basic open set $U \times V$ containing (x, y) such that $U \times V \cap R = \emptyset$.
$\Leftrightarrow$ For every pair $x \not\sim y$ in S, there exists neighborhoods U of x and V of y such that no element of U is equivalent to an element of V.
$\Leftrightarrow$ For any two points $[x] \neq [y]$ in $S/\sim$, there exists neighborhoods U of x and V of y in S such that $\pi(U) \cap \pi(V) = \emptyset$. $(*)$

We will now show that $(*)$ is equivalent to $S/\sim$ being Hausdorff. First, assume $(*)$. Since π is open, $\pi(U)$ and $\pi(V)$ are disjoint open sets in $S/\sim$ that separates $[x]$ and $[y]$. Thus, $S/\sim$ is Hausdorff.

Now, suppose $S/\sim$ is Hausdorff. Then, for any $[x] \neq [y]$ in $S/\sim$, there exists disjoint open sets A and B in $S/\sim$ such that $[x] \in A$ and $[y] \in B$. Let $U = \pi^{-1}(A)$ and $V = \pi^{-1}(B)$. Then, $x \in U$ and $y \in V$. Thus, U and V are neighborhoods of x and y in S respectively such that $\pi(U) \cap \pi(V) = \pi(\pi^{-1}(A)) \cap \pi(\pi^{-1}(B)) \subset A \cap B = \emptyset$, so $\pi(U) \cap \pi(V) = \emptyset$. □

Corollary 8.4. A topological space S is Hausdorff if and only if the diagonal Δ in $S \times S$ is closed.

Proof. By simply letting the equivalence relation $\sim$ be equality, the quotient space $S/\sim$ is then S itself, and thus the graph R of S is simply the diagonal

$$\Delta = \{(x, x) \in S \times S \mid x \in S\}$$

of $S \times S$. □

Theorem 8.5. Let $\sim$ be an open equivalence relation on a topological space S with projection $\pi\colon S \to S/\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for S, then its image $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$.

Proof. Since π is an open map, $\{\pi(B_\alpha)\}$ is a collection of open sets in $S/\sim$. Now, let U be an open set in $S/\sim$ and let $[x] \in U$. Then, $x \in \pi^{-1}(U)$, and since $\mathcal{B}$ is a basis for S, there exists an open set $B_\alpha \in \mathcal{B}$ such that

$$x \in B_\alpha \subset \pi^{-1}(U).$$

Hence,

$$[x] = \pi(x) \in \pi(B_\alpha) \subset \pi(\pi^{-1}(U)) \subset U,$$

and thus $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$. □

Corollary 8.6. If $\sim$ is an open equivalence relation on a second-countable space S, then the quotient space $S/\sim$ is second countable.

Problems

Problem 8.1. Orbit space of a continuous group action. Suppose the right action of a topological group G on a topological space S is continuous, i.e., the map $S \times G \to S$ describing the action is continuous. Define two points x, y to be equivalent if they are in the same orbit, i.e., there is an element $g \in G$ such that $y = xg$. Let S/G be the quotient space; it is called the *orbit space* of the action. Prove that the projection map $\pi\colon S \to S/G$ is an open map.

Proof. Let U be an open set in S. For any $g \in G$, since right multiplication by g is bijective (since any $g \in G$ has an inverse) and continuous, it is a homeomorphism. Hence, the set Ug is open in S. Thus,

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} Ug$$

is a union of open sets in S, and therefore it is open in S. By definition of the quotient topology, π is an open map. $\square$

Chapter 3

The Tangent Space

9 The Tangent Space

9.1 The Tangent Space at a point

Just as for $\mathbb{R}^n$, we define a *germ* of a C^∞ function at p in M to be an equivalence class of C^∞ functions defined in a neighborhood of p in M, two such functions being equivalent if they agree on some, possibly smaller, neighborhood of p. The set of germs of C^∞ real-valued functions at p in M is then denoted by $C_p^\infty(M)$. The addition and multiplication of functions make $C_p^\infty(M)$ a ring; with scalar multiplication by real numbers, $C_p^\infty(M)$ becomes an algebra over $\mathbb{R}$.

Generalizing a derivation at a point in $\mathbb{R}^n$, we define a *derivation at a point* in a manifold M, or a *point-derivation* of $C_p^\infty(M)$, to be a linear map $D\colon C_p^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule:

$$D(fg) = (Df)g(p) + f(p)(Dg).$$

Definition 9.1. A *tangent vector* at a point p in a manifold M is a derivation at p.

Just as for $\mathbb{R}^n$, the tangent vectors at p form a vector space $T_p(M)$, called the *tangent space of M at p*. We also write T_pM instead of $T_p(M)$.

Remark. (*Tangent space to an open subset*). If U is an open set containing p in M, then the algebra $C_p^\infty(U)$ of germs of C^∞ functions in U at p is the same as $C_p^\infty(M)$. Hence, $T_pU = T_pM$.

Given a coordinate neighborhood $(U, \phi) = \left(U, x^1, \ldots, x^n\right)$ about a point p in a manifold M, we recall the definition of the partial derivative $\partial/\partial x^i$. Let $r^1, \ldots, r^n$ be the standard coordinates on $\mathbb{R}^n$. Then

$$x^i = r^i \circ \phi\colon U \to \mathbb{R}.$$

If f is a C^∞ function in a neighborhood of p, we set

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} \left(f \circ \phi^{-1}\right).$$

Clearly, $\left.\frac{\partial}{\partial x^i}\right|_p \colon C_p^\infty(M) \to \mathbb{R}$ is linear and satisfies the Leibniz rule; thus, it is a derivation and so is a tangent vector at p.

9.2 The Differential of a Map

Let $F\colon N \to M$ be a C^∞ map between two manifolds. At each point $p \in N$, the map induces a linear map of vector spaces, called its *differential at p*,

$$F_*\colon T_pN \to T_{F(p)}M$$

as follows (**Figure 9.1**). If $X \in T_pN$, then $F_*(X_p)$ is the tangent vector in $T_{F(p)}M$ defined by

$$\left(F_*(X_p)\right) f = X_p\left(f \circ F\right) \in \mathbb{R} \text{ for } f \in C_p^\infty(M).$$

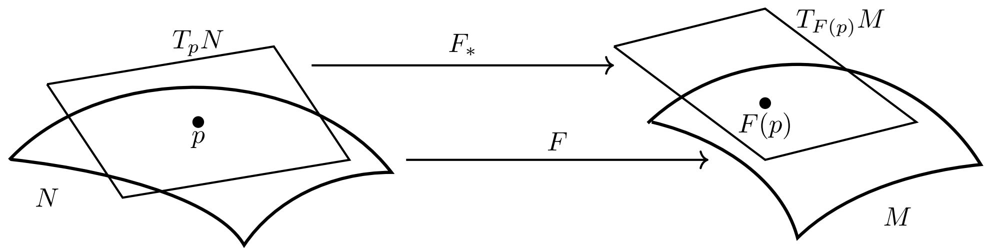

Figure. 9.1: $F\colon N \to M$ induces a map $F_*\colon T_pN \to T_{F(p)}M$.

Example. (*Differential of a map between Euclidean spaces*). Suppose $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is C^∞ and p is a point in $\mathbb{R}^n$. Let $x^1, \dots, x^n$ be the coordinates on $\mathbb{R}^n$ and $y^1, \dots, y^n$ the coordinates on $\mathbb{R}^m$. Then, the tangent vectors $\{\partial/\partial x^j|_p\}$ and $\{\partial/\partial y^i|_{F(p)}\}$ form the bases for the tangent spaces $T_p(\mathbb{R}^n)$ and $T_{F(p)}(\mathbb{R}^m)$ respectively. The linear map $F_*\colon T_p(\mathbb{R}^n) \to T_{F(p)}(\mathbb{R}^m)$ is described by a matrix $[a^i_j]$ relative to these two bases:

$$F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_k a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)}.$$

Let $F^i = y^i \circ F$ be the ith component of F. Then,

$$a^i_j = F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) y^i = \left.\frac{\partial}{\partial x^j}\right|_p \left(y^i \circ F\right) = \frac{\partial F^i}{\partial x^j}(p).$$

Thus, the matrix describing F_* relative to the bases $\{\partial/\partial x^j|_p\}$ and $\{\partial/\partial y^i|_{F(p)}\}$ is $\left[\partial F^i/\partial x^j|_p\right]$, which is precisely the Jacobian matrix of the derivative of F at p. Hence, the differential of a map between manifolds generalizes the derivative of a map between Euclidean spaces.

9.3 The Chain Rule

Let $F\colon N \to M$ and $G\colon M \to P$ be smooth maps of manifolds, and $p \in N$. The differentials of F at p and G at $F(p)$ are linear maps

$$T_pN \xrightarrow{F_{*,p}} T_{F(p)}M \xrightarrow{G_{*,F(p)}} T_{G(F(p))}P.$$

Theorem 9.1. (The chain rule). If $F\colon N \to M$ and $G\colon M \to P$ are smooth maps of manifolds and $p \in N$, then

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}.$$

Proof. Let $X_p \in T_pN$ and f be a smooth function at $(G \circ F)(p)$ in P. Then,

$$((G \circ F)_{*,p} X_p)f = X_p\left(f \circ (G \circ F)\right).$$

We also have

$$(\left(G_{*,F(p)} \circ F_{*,p}\right) X_p)f = \left(G_{*,F(p)}(F_{*,p}(X_P))\right) f = F_{*,p}(X_P)\left(f \circ G\right) = X_p\left(f \circ G \circ F\right).$$

□

Remark. The differential of the identity map $\mathbb{1}_M\colon M \to M$ is the identity map

$$\mathbb{1}_{T_pM}\colon T_pM \to T_pM$$

since for any $X_p \in T_pM$ and $f \in C_p^\infty(M)$,

$$\left(\left(\mathbb{1}_M\right)_* (X_p)\right)(f) = X_p\left(f \circ \mathbb{1}_M\right) = X_p(f).$$

Corollary 9.2. If $F\colon N \to M$ is a diffeomorphism of manifolds and $p \in N$, then $F_*\colon T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

Proof. By definition, F has a C^∞ inverse $G\colon M \to N$ such that $G \circ F = \mathbb{1}_N$ and $F \circ G = \mathbb{1}_M$. By the chain rule,

$$\begin{aligned} G_* \circ F_* &= \left(G \circ F\right)_* = (\mathbb{1}_N)_* = \mathbb{1}_{T_pN} \\ F_* \circ G_* &= \left(F \circ G\right)_* = (\mathbb{1}_M)_* = \mathbb{1}_{T_{F(p)}M}. \end{aligned}$$

□

Corollary 9.3. (Invariance of dimension). If an open set $U \subset \mathbb{R}^n$ is diffeomorphic to an open set $V \subset \mathbb{R}^m$, then $n = m$.

Proof. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a diffeomorphism. Since U and V are diffeomorphic, for any point $p \in U$, T_pU and $T_{F(p)}V$ are isomorphic vector spaces. Thus, $\mathbb{R}^n \simeq T_p(U) \simeq T_{F(p)}(V) \simeq \mathbb{R}^m$, and therefore, $n = m$. □

9.4 Bases for the Tangent Space at a Point

Denote by $r^1, \dots, r^n$ the standard coordinates on $\mathbb{R}^n$ and let $(U, \phi) = \left(U, x^1, \dots, x^n\right)$ be a chart about a point p in a manifold M of dimension n. Since $\phi\colon U \to \phi(U) \subset \mathbb{R}^n$ is diffeomorphism (Proposition 7.5), by Corollary 9.2, the differential

$$\phi_*\colon T_pM \to T_{\phi(p)}\mathbb{R}^n$$

is an isomorphism of vector spaces. Thus, the tangent space T_pM has the same dimension n as the manifold M.

Proposition 9.4. Let $(U, \phi) = \left(U, x^1, \dots, x^n\right)$ be a chart about a point p in a manifold M. Then,

$$\phi_*\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)}.$$

Proof. For any $f \in C_p^\infty(M)$,

$$\begin{aligned}\phi_*\left(\left.\frac{\partial}{\partial x^i}\right|_p\right)(f) &= \left(\left.\frac{\partial}{\partial x^i}\right|_p\right)(f\circ\phi) \\ &= \left(\left.\frac{\partial}{\partial r^i}\right|_{\phi(p)}\right)\left((f\circ\phi)\circ\phi^{-1}\right) \\ &= \left(\left.\frac{\partial}{\partial r^i}\right|_{\phi(p)}\right)(f).\end{aligned}$$

□

Proposition 9.5. If $(U,\phi) = \left(U, x^1, \ldots, x^n\right)$ is a chart containing p, then the tangent space T_pM has a basis

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$$

Proof. Since an isomorphism of vector spaces sends a basis to a basis, by Proposition 9.4, $\{\partial/\partial x^i|_p\}$ is a basis for T_pM. □

Proposition 9.6. (Transition matrix for coordinate vectors). Suppose $\left(U, x^1, \ldots, x^n\right)$ and $\left(V, y^1, \ldots, y^n\right)$ are two coordinate charts in a manifold M. Then,

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i}$$

on $U \cap V$.

Proof. Obvious. □

9.5 A Local Expression for the Differential

Proposition 9.7. Given a smooth map $F\colon N \to M$ of manifolds and a point $p \in N$, let $\left(U, x^1 \ldots, x^n\right)$ and $\left(V, y^1, \ldots, y^n\right)$ be coordinate charts about p in N and $F(p)$ in M respectively. Relative to the bases $\{\partial/\partial x^j|_p\}$ for T_pN and $\{\partial/\partial y^i|_p\}$ for $T_{F(p)}M$, the differential $F_{*,p}\colon T_pN \to T_{F(p)}M$ is represented by the matrix $\left[\partial F^i/\partial x^j(p)\right]$, where $F^i = y^i \circ F$.

Proof. The differential $F_{*,p}$ is completely determined by the matrix $[a^i_j]$, where

$$F_{*,p}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_k a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)}.$$

Applying both sides to y^i yields

$$a^i_j = F_{*,p}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right)(y^i) = \left(\left.\frac{\partial}{\partial x^j}\right|_p\right)\left(y^i \circ F\right) = \frac{\partial F^i}{\partial x^j}.$$

□

Remark. (*Inverse function theorem*). In terms of the differential, the inverse function theorem for manifolds (Theorem 7.13) has a coordinate-free description: a C^∞ map $F\colon N \to M$ between two manifolds of the same dimension is locally invertible at a point $p \in N$ if and only if its differential $F_{*,p}\colon T_pN \to T_{F(p)}M$ at p is an isomorphism.

9.6 Curves in a Manifold

A *smooth curve* in a manifold M is by definition a smooth map $c\colon (a,b) \to M$ from some open interval (a,b) into M. Usually, we assume $0 \in (a,b)$ and say that c is a *curve starting at* p if $c(0) = p$. The *velocity vector* $c'(t_0)$ of the curve c at time $t_0 \in (a,b)$ is defined to be

$$c'(t_0)\colon = c_*\left(\left.\frac{d}{dt}\right|_{t_0}\right) \in T_{c(t_0)}M.$$

We also say that $c'(t_0)$ is the velocity of c at the point $c(t_0)$. Alternatively, we may also write

$$\frac{dc}{dt}(t_0) \quad \text{and} \quad \left.\frac{d}{dt}\right|_{t_0} c.$$

Proposition 9.8. (Velocity of a curve in local coordinates). Let $c\colon (a,b) \to M$ be a smooth curve, and let $\left(U, x^1, \ldots, x^n\right)$ be a chart about $c(t)$. Write $c^i = x^i \circ c$ for the ith component of c in the chart. Then, $c'(t)$ is given by

$$c'(t) = \sum_{i=1}^n \frac{dc^i}{dt}\left.\frac{\partial}{\partial x^i}\right|_{c(t)}.$$

Proof. Applying $c'(t)$ to the component x^i of the coordinate map yields

$$(c'(t))^i = c_*\left(\left.\frac{d}{dt}\right|_t\right)(x^i) = \left(\left.\frac{d}{dt}\right|_t\right)\left(x^i \circ c\right) = \frac{dc^i}{dt}.$$

□

Proposition 9.9. (Existence of a curve with a given initial vector). For any point p in a manifold M and any tangent vector $X_p \in T_pM$, there are $\varepsilon > 0$ and a smooth curve $c\colon (-\varepsilon, \varepsilon) \to M$ such that $c(0) = p$ and $c'(0) = X_p$.

Proof. Let (U, ϕ) be a chart centered at p, i.e., $\phi(p) = 0 \in \mathbb{R}^n$. Suppose $X_p = \sum a^i \partial/\partial x^i|_p$ and let α be a curve in $\mathbb{R}^n$ such that $\alpha(0) = 0$ and $\alpha'(0) = \sum a^i \partial/\partial r^i|_0$. By Proposition 9.8, α can be written as

$$\alpha(t) = \left(a^1 t, \ldots, a^n t\right), \quad t \in (-\varepsilon, \varepsilon),$$

where ε is chosen so that $\alpha(t)$ lies completely in $\phi(U)$. Define $c = \phi^{-1} \circ \alpha \colon (-\varepsilon, \varepsilon) \to M$. Then,

$$c(0) = \phi^{-1}(\alpha(0)) = \phi(0)^{-1} = p,$$

and by the chain rule,

$$c'(0) = \phi_*^{-1} \alpha_* \left(\left. \frac{d}{dt} \right|_{t_0} \right) = \phi_*^{-1} \left(\sum_i a^i \left. \frac{\partial}{\partial r^i} \right|_0 \right) = \sum_i a^i \left. \frac{\partial}{\partial x^i} \right|_{\phi^{-1}(0)=p} = X_p.$$

□

Proposition 9.10. Suppose X_p is a tangent vector at a point p of a manifold M and $f \in C_p^\infty(M)$. If $c \colon (-\varepsilon, \varepsilon) \to M$ is a smooth curve starting at p with $c'(0) = X_p$, then

$$X_p f = \left. \frac{d}{dt} \right|_0 (f \circ c).$$

Proof. By definition,

$$X_p f = c'(0)(f) = c_* \left(\left. \frac{d}{dt} \right|_0 \right)(f) = \left. \frac{d}{dt} \right|_0 (f \circ c).$$

□

9.7 Computing the Differential Using Curves

Proposition 9.11. Let $F \colon N \to M$ be a smooth map of manifolds, $p \in N$, and $X_p \in T_p N$. If c is a smooth curve starting at p in N with velocity X_p at p, then

$$F_{*,p}(X_p) = \left. \frac{d}{dt} \right|_0 (F \circ c)(t).$$

In other words, $F_{*,p}(X_p)$ is the velocity of the image curve $F \circ c$ starting at $F(p)$.

Proof. By hypothesis, $c(0) = p$ and $c'(0) = X_p$. Then

$$\begin{aligned} F_{*,p}(X_p) &= F_{*,p} \left(c_* \left(\left. \frac{d}{dt} \right|_0 \right) \right) \\ &= (F_*, p \circ c_*) \left(\left. \frac{d}{dt} \right|_0 \right) \\ &= (F \circ c)_* \left(\left. \frac{d}{dt} \right|_0 \right) \\ &= \left. \frac{d}{dt} \right|_0 (F \circ c)(t). \end{aligned}$$

□

9.8 Immersions and Submersions

Just as the derivative of a map between Euclidean spaces is a linear map that best approximates the given map at a point, the differential at a point also serves the same purpose for a C^∞ map between manifolds. Two cases are especially important. A C^∞ map $F\colon N \to M$ is said to be an *immersion at* $p \in N$ if its differential $F_{*,p}\colon T_pN \to T_{F(p)}M$ is injective, and a *submersion at* p if $F_{*,p}$ is surjective. We call F an *immersion* if it is an immersion at every $p \in N$ and a *submersion* if it is a submersion at every $p \in N$.

Remark. Suppose N and M are manifolds of dimension n and m respectively. Then, $\dim T_pN = n$ and $\dim T_{F(p)}M = m$. The injectivity of the differential $F_{*,p}\colon T_pN \to T_{F(p)}M$ immediately implies that $n \le m$. Similarly, the surjectivity of $F_{*,p}$ immediately implies that $n \ge m$. Thus, if $F\colon N \to M$ is an immersion at a point of N, then $n \le m$; if F is a submersion at a point of N, then $n \ge m$.

Example. The prototype of an immersion is the inclusion of $\mathbb{R}^n$ in a higher-dimensional $\mathbb{R}^m$:

$$i\left(x^1, \dots, x^n\right) = \left(x^1, \dots, x^n, 0, \dots, 0\right).$$

Similarly, the prototype of a submersion is the projection of $\mathbb{R}^n$ onto a lower-dimensional $\mathbb{R}^m$:

$$\pi\left(x^1, \dots, x^m, x^{m+1}, \dots, x^n\right) = \left(x^1, \dots, x^m\right)$$

Example. If U is an open subset of a manifold M, then the inclusion $i\colon U \to M$ is both an immersion and a submersion. Particularly, this shows that a submersion need not be onto.

Later on, we will show that every immersion is locally an inclusion and every submersion is locally a projection.

9.9 Rank, and Critical and Regular Points

Consider a smooth map $F\colon N \to M$ of manifolds. Its *rank* at a point p in N, denoted $\operatorname{rk} F(p)$, is defined as the rank of the differential $F_{*,p}\colon T_pN \to T_{F(p)}M$. Relative to the coordinate neighborhoods $\left(U, x^1, \dots, x^n\right)$ at p and $\left(V, y^1, \dots, y^m\right)$ at $F(p)$, the differential is represented by the Jacobian matrix $\left[\partial F^i/\partial x^j(p)\right]$, so

$$\operatorname{rk} F(p) = \operatorname{rk}\left[\frac{\partial F^i}{\partial x^j}(p)\right].$$

Since the differential of a map is independent of coordinate charts, so is the rank of the Jacobian matrix.

Definition 9.2. A point p in N is a *critical point* of F if the differential

$$F_{*,p}\colon T_pN \to T_{F(p)}M$$

fails to be surjective. It is a *regular point* of F if the differential $F_{*,p}$ is surjective. In other words, p is a regular point of the map F if and only if F is a submersion at p. A point in M is a *critical value* if it is the image of a critical point; otherwise it is a *regular value*.

Remark.

(i) We do *not* define a regular value to be the image of a regular point. In fact, a regular value need not be in the image of F at all. Any point of M not in the image of F is automatically a regular value because it is not the image of a critical point.

(ii) A point c in M is a critical value if and only if *some* point in the preimage $F^{-1}(\{c\})$ is a critical point. A point c in the image of F is a regular value if and only if *every* point in the preimage $F^{-1}(\{c\})$ is a regular point.

Proposition 9.12. For a real-valued function $f\colon M \to \mathbb{R}$, a point p in M is a critical point if and only if relative to some chart $\left(U, x^1, \ldots, x^n\right)$ containing p, all the partial derivatives satisfy

$$\frac{\partial f}{\partial x^j}(p) = 0, \quad j = 1, \ldots, n$$

Proof. By Proposition 9.7, the differential $f_{*,p}\colon T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$ is represented by the matrix

$$\left[\frac{\partial f}{\partial x^1}(p) \cdots \frac{\partial f}{\partial x^n}(p)\right].$$

Since $\operatorname{im} F_{*,p}$ is a linear subspace of $\mathbb{R}$, it is either zero-dimensional or one-dimensional. In other words, $f_{*,p}$ is either the zero map or a surjective map. Hence, $f_{*,p}$ fails to be surjective if and only if all the partial derivatives $\partial f/\partial x^j(p)$ are zero. $\square$

Problems

Problem 9.1. Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$(u, v, w) = F(x, y) = (x, y, xy).$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_*\left(\partial/\partial x|_p\right)$ as a linear combination of $\partial/\partial_u, \partial/\partial v$, and $\partial/\partial w$ at $F(p)$.

Solution.

$$F_*\left(\left.\frac{\partial}{\partial x}\right|_p\right) = \frac{\partial F_u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial F_v}{\partial x}\frac{\partial}{\partial v} + \frac{\partial F_w}{\partial x}\frac{\partial}{\partial w} = \frac{\partial}{\partial u} + y\frac{\partial}{\partial w}$$

Problem 9.2. Fix a real number α and define: $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = (u, v) = F(x, y) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let $X = -y\partial/\partial x + x\partial/\partial y$ be a vector field on $\mathbb{R}^2$. If $p = (x, y) \in \mathbb{R}^2$ and $F_*(X_p) = (a\partial/\partial u + b\partial/\partial v)\,|_{F(p)}$, find a and b in terms of x, y, and α.

Solution. The differential F_* is given by the Jacobian matrix

$$\left[\frac{\partial(F^1, F^2)}{\partial(x, y)}\right] = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Thus,

$$\begin{aligned} F_*(X) &= -y\left(\frac{\partial F^u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial F^v}{\partial x}\frac{\partial}{\partial v}\right) + x\left(\frac{\partial F^u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial F^v}{\partial y}\frac{\partial}{\partial v}\right) \\ &= \left(-y\frac{\partial F^u}{\partial x} + x\frac{\partial F^u}{\partial y}\right)\frac{\partial}{\partial u} + \left(-y\frac{\partial F^v}{\partial x} + x\frac{\partial F^v}{\partial y}\right)\frac{\partial}{\partial v} \\ &= -\left(x\sin\alpha + y\cos\alpha\right)\frac{\partial}{\partial u} + \left(x\cos\alpha - y\sin\alpha\right)\frac{\partial}{\partial v} \end{aligned}$$

Problem 9.3. Differential of a linear map

Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$, there is a canonical identification $T_p(\mathbb{R}^n) \simeq \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto \mathbf{a} = \left\langle a^1, \ldots, a^n \right\rangle.$$

Show that the differential $L_{*,p}\colon T_p\left(\mathbb{R}^n\right) \to T_{L(p)}\left(\mathbb{R}^m\right)$ is the map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ itself, with the identification of the tangent spaces as above.

Solution. Let $\left(x^1, \ldots, x^n\right)$ and $\left(y^1, \ldots, y^m\right)$ be the standard coordinates on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and suppose $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is represented in those basis by the matrix $\left[a^i_j\right]$, where

$$F^i = \sum_j a^i_j x^j.$$

Then, F_* is represented by the Jacobian matrix $\left[\partial F^i/\partial x^j|_p\right]$, where

$$\frac{\partial F^i}{\partial x^j} = \frac{\partial\left(y^i \circ F\right)}{\partial x^j} = \frac{\partial}{\partial x^j}\sum_k a^i_k x^k = a^i_k \delta^k_j = a^i_j.$$

Problem 9.4. Differential of multiplication and inverse

Let G be a Lie group with multiplication map $\mu\colon G \times G \to G$, inverse map $\iota\colon G \to G$, and identity element e.

(a) Show that the differential at the identity of the multiplication map μ is addition:

$$\begin{aligned} \mu_{*,(e,e)}\colon T_eG \times T_eG &\to T_eG, \\ (X_e, Y_e) &\mapsto X_e + Y_e. \end{aligned}$$

(b) Show that the differential at the identity of the inverse map ι is the negation:

$$\begin{aligned} \iota_{*,e}\colon T_eG &\to T_eG, \\ X_e &\mapsto -X_e. \end{aligned}$$

Proof.

(a) Let $c(t)$ be a curve starting at e in G such that $c'(0) = X_e$. Then, $\alpha(t) = (c(t), e)$ is a curve starting at (e, e) in $G \times G$ with $\alpha'(0) = (X_e, 0)$. By Proposition 9.11,

$$\begin{aligned} \mu_{*,(e,e)}\left(X_e, 0\right) &= \mu_{*,(e,e)}\left(\alpha'(0)\right) \\ &= \left.\frac{d}{dt}\right|_0 \left(\mu \circ \alpha\right)(t) \\ &= \left.\frac{d}{dt}\right|_0 \mu(c(t), e) \\ &= \left.\frac{d}{dt}\right|_0 (c(t)) \\ &= c'(0) \\ &= X_p \end{aligned}$$

Similarly,

$$\mu_{*,(e,e)}\left(0, Y_e\right) = Y_e,$$

so

$$\mu_{*,(e,e)}(X_e, Y_e) = X_e + Y_e.$$

(b) Let $c(t)$ be a curve starting at e in G such that $c'(0) = X_e$. Then,

$$\left(\iota \circ c\right)'(0) = \left(\iota \circ c\right)_* \left(\left.\frac{d}{dt}\right|_0\right) = \iota_{*,e}(c'(0) = \iota_{*,e}(X_p).$$

Thus, $\alpha(t) = (c(t), (\iota \circ c)(t))$ is a curve starting at (e, e) in $G \times G$ with $\alpha'(0) = (X_e, \iota_*(X_p))$. Observe that

$$\begin{aligned} X_e + \iota_{*,e}(X_e) &= \mu_{*,(e,e)}\left(X_e, \iota_*(X_p)\right) \\ &= \mu_{*,(e,e)}\left(\alpha'(0)\right) \\ &= \left.\frac{d}{dt}\right|_0 \left(\mu \circ \alpha\right)(t) \\ &= \left.\frac{d}{dt}\right|_0 \mu\left(c(t), (\iota \circ c)(t)\right) \\ &= \left.\frac{d}{dt}\right|_0 e \\ &= 0 \end{aligned}$$

Hence,

$$\iota_{*,e}(X_e) = -X_e$$

$\square$

Problem 9.5. Local maxima
A real-valued function $f\colon M \to \mathbb{R}$ on a manifold is said to have a *local maximum* at $p \in M$ if there is a neighborhood U of p such that $f(p) \geq f(q)$ for all $q \in U$.

(a) Prove that if a differentiable function $f : I \to \mathbb{R}$ defined on an open interval I has a local maximum at $p \in I$, then $f'(p) = 0$.

(b) Prove that a local maximum of a C^∞ function $f\colon M \to \mathbb{R}$ is a critical point of f.

Proof.

(a) By definition, there is a neighborhood U of p in I such that $f(p) \geq f(x)$ for all $x \in U$. Then,

$$\lim_{x \to p^-} \frac{f(x) - f(p)}{x - p} \geq 0$$

and

$$\lim_{x \to p^+} \frac{f(x) - f(p)}{x - p} \leq 0.$$

Since f is differentiable, its derivative at p exists, where

$$f'(p) = \lim_{x \to p^-} \frac{f(x) - f(p)}{x - p} = \lim_{x \to p^+} \frac{f(x) - f(p)}{x - p} = 0$$

(b) Suppose $f\colon M \to \mathbb{R}$ has a local maximum at p. Let X_p be a tangent vector in T_pM and let $c(t)\colon (-\varepsilon, \varepsilon) \to M$ be a curve starting at p with initial vector X_p. Then, $f \circ c\colon \mathbb{R} \to \mathbb{R}$ is a real-valued function with a local maximum at $0 \in (-\varepsilon, \varepsilon)$, so by part (a), $(f \circ c)'(0) = 0$. Thus, $f_{*,p}(X_p) = f_{*,p}(c'(0)) = 0$ so p is a critical point of f.

□

10 Submanifolds

We now have two ways of determining if a topological space is a manifold:

(a) by directly checking that it is a Hausdorff, second countable, and has a C^∞ atlas;

(b) by showing it is an appropriate quotient space.

In this section, we introduce the concept of a *regular submanifold* of a manifold. Using the inverse function theorem, we derive a criterion, called the *regular level set theorem*, that can be used to show that a level set of a C^∞ map of manifolds is a regular submanifold, and therefore a manifold.

10.1 Submanifolds

Definition 10.1. A subset S of a manifold N of dimension n is a *regular submanifold* of dimension k if for every $p \in S$ there is a coordinate neighborhood $(U, \phi) = \left(U, x^1, \ldots, x^n\right)$ of p in the maximal atlas of N such that $U \cap S$ is defined by the vanishing of $n - k$ of the coordinate functions. By renumbering the coordinates, we may assume these $n - k$ coordinate functions are $x^{k+1}, \ldots, x^n$.

We call such a chart (U, ϕ) in N an *adapted chart* relative to S. On $U \cap S$, $\phi = \left(x^1, \ldots, x^k, 0, \ldots, 0\right)$. Let

$$\phi_S \colon U \cap S \to \mathbb{R}^k$$

be the restriction of the first k components of ϕ to $U \cap S$, that is, $\phi_S = \left(x^1, \ldots, x^k\right)$. Note that $(U \cap S, \phi_S)$ is a chart for S in the subspace topology.

Definition 10.2. If S is a regular submanifold of dimension k in a manifold N of dimension n, then $n-k$ is said to be the *codimension* of S in N.

Remark. As a topological space, a regular submanifold of N is required to have the subspace topology.

Example. The dimension k of a regular submanifold may be equal to n. In that case, $U \cap S$ is defined by the vanishing of none of the coordinate functions and so $U \cap S = U$. Therefore, an open subset of a manifold is a regular submanifold of the same dimension.

Remark. There are other types of submanifolds, but unless otherwise specified, a "submanifold" will always mean a "regular submanifold".

Proposition 10.1. Let S be a regular submanifold of N and $\mathfrak{U} = \{(U, \phi)\}$ a collection of compatible adapted charts of N that covers S. Then $\mathfrak{U} = \{(U \cap S, \phi_S)\}$ is an atlas for S. Therefore, a regular submanifold is itself a manifold. If N has dimension n and S is locally defined by the vanishing of $n-k$ coordinates, then $\dim S = k$.

Proof. Let $(U, \phi) = \left(U, x^1, \ldots, x^n\right)$ and $(V, \Psi) = \left(V, y^1, \ldots, y^n\right)$ be two adapted charts in $\mathfrak{U}$. Renumber the coordinates so that the last $n-k$ coordinates vanish on S; then, assuming U and V intersect, for $p \in U \cap V \cap S$,

$$\phi_S(p) = \left(x^1, \ldots, x^k\right) \quad \text{and} \quad \Psi_S(p) = \left(y^1, \ldots, y^n\right)$$

so

$$\Psi_S \circ \phi_S^{-1}(x_1, \ldots, x^n) = (y^1, \ldots, y^n).$$

Thus, the transition function $\Psi_S \circ \phi_S^{-1}$ is C^∞. Similarly, $\phi_S \circ \Psi_S^{-1}$ is also C^∞. Hence, any two charts in $\mathfrak{U}_S = \{(U \cap S, \phi_S)\}$ are C^∞ compatible. Since $\{U \cap S\}_{U \in \mathfrak{U}}$ covers S, the collection $\mathfrak{U}_S$ is a C^∞ atlas for S. This also shows that $\dim S = k$. $\square$

10.2 Level Sets of a Function

Definition 10.3. A *level set* of a map $F \colon N \to M$ is a subset

$$F^{-1}\left(\{c\}\right) = \{p \in N \mid F(p) = c\}$$

for some $c \in M$. We also write $F^{-1}(c)$ instead of $F^{-1}\left(\{c\}\right)$. The value $c \in M$ is called the *level* of the level set $F^{-1}(c)$. If $F \colon N \to \mathbb{R}^m$, then $Z(F) \colon = F^{-1}(\mathbf{0})$ is the *zero set* of F.

Remark. Recall that c is a regular value of F if and only if either c is not in the image of F or at every point $p \in F^{-1}(c)$, the differential $F_{*,p} \colon T_pN \to T_{F(p)}M$ is surjective. The inverse image $F^{-1}(c)$ of a regular value c is called a *regular level set*. If the zero set $F^{-1}(\mathbf{0})$ is a regular level set of $F\colon N \to \mathbb{R}^m$, it is called a *regular zero set*.

Remark. If a regular level set $F^{-1}(c)$ is nonempty, say $p \in F^{-1}(c)$, then the map $F\colon N \to M$ is a submersion at p. Hence, $\dim N \geq \dim M$.

Lemma 10.2. Let $g\colon N \to \mathbb{R}$ be a C^∞ function. A regular level set $g^{-1}(c)$ of level c of the function g is the regular zero set of the function $f = g - c$.

Proof. For any $p \in N$,

$$g(p) = c \quad \Leftrightarrow f(p) = g(p) - c = 0.$$

Hence, $g^{-1}(c) = f^{-1}(0)$. Let $S = f^{-1}(0)$. Since the differential $f_{*,p} = g_{*,p}$ at every point $p \in N$, the functions f and g have exactly the same critical points. Since g has no critical points in N, neither does f. ☐

Theorem 10.3. Let $g\colon N \to \mathbb{R}$ be a C^∞ function on the manifold N. Then, a nonempty regular level set $S = g^{-1}(c)$ is a regular submanifold of N of codimension 1.

Proof. Let $f = g - c$. By the preceding lemma, $S = f^{-1}(0)$ and is a regular level set of f. Let $p \in S$. Since p is a regular point of f, relative to any chart $\left(U, x^1, \ldots, x^n\right)$ about p, $\left(\partial f/\partial x^i\right)(p) \neq 0$ for some i. WLOG, we may assume $(\partial f/\partial x^1)(p) \neq 0$. Then, the Jacobian determinant $\partial\left(f, x^2, \ldots, x^n\right)/\partial\left(x^1, \ldots, x^n\right)$ at p is $\partial f/\partial x^1(p) \neq 0$, so by the inverse function theorem, there is a coordinate neighborhood U_p of p on which $f, x^2, \ldots, x^n$ forms a coordinate system. Relative to the chart $\left(U_p, f, x^2, \ldots, x^n\right)$, the level set $U_p \cap S$ is defined by setting the coordinate f equal to 0, so $\left(U_p, f, x^2, \ldots, x^n\right)$ is an adapted chart relative to S. Hence, S is a regular submanifold of dimension $n-1$ in N. ☐

10.3 The Regular Level Set Theorem

Theorem 10.4. (Regular level set theorem). Let $F\colon N \to M$ be a C^∞ map of manifolds, with $\dim N = n$ and $\dim M = m$. Then, a nonempty regular level set $F^{-1}(c)$, $c \in M$, is a regular submanifold of N of dimension $n - m$.

Proof. Let $(V, \Psi) = \left(V, y^1, \ldots, y^m\right)$ be a chart of M centered at c, i.e., $\Psi(c) = \mathbf{0} \in \mathbb{R}^m$. Then, $F^{-1}(V)$ is an open set in N that contains $F^{-1}(c)$. Moreover, in $F^{-1}(V)$, $F(c)^{-1} = (\Psi \circ F)^{-1}(\mathbf{0})$, so the level set $F^{-1}(c)$ is the zero set of $\Psi \circ F$. Let $F^i = y^i \circ F = r^i \circ (\Psi \circ F)$. Then, $F^{-1}(c)$ is also the zero set of the functions $F^1, \ldots, F^m$ on $F^{-1}(V)$.

By hypothesis, $F^{-1}(c)$ is nonempty, so $n \geq m$. Choose a point $p \in F^{-1}(c)$ and let $(U, \phi) = \left(U, x^1, \ldots, x^n\right)$ be a coordinate neighborhood of p in N. Since $F^{-1}(c)$ is a regular level set, $p \in F^{-1}(c)$ is a regular point of F. Thus, the $m \times n$ Jacobian matrix $\left[\partial F^i/\partial x^j(p)\right]$ has rank m. WLOG, we may assume the first $m \times m$ block $\left[\partial F^i/\partial x^j\right]_{1 \leq i,j \leq m}$ is nonsingular.

Now, replace the first m coordinates $x^1, \ldots, x^m$ of the chart (U, ϕ) by $F^1, \ldots, F^m$. We claim that there is a neighborhood U_p of p such that $\left(U_p, F^1, \ldots, F^m, x^{m+1}, \ldots, x^n\right)$ is a chart in the atlas of N.

Indeed, the Jacobian determinant

$$\frac{\partial\left(F^1,\ldots,F^m,x^{m+1},\ldots,x^n\right)}{\partial\left(x^1,\ldots,x^n\right)}=\det\left[\frac{\partial F^i}{\partial x^j}(p)\right]_{1\le i,j\le m}\neq 0,$$

so by the inverse function theorem, $F^1,\ldots,F^m,x^{m+1},\ldots,x^n$ forms a coordinate neighborhood of p.

In the chart $\left(U_p,F^1,\ldots,F^m,x^{m+1},\ldots,x^n\right)$, the set $S\colon=F^{-1}(c)$ is obtained by setting the first m coordinate functions $F^1,\ldots,F^m$ equal to 0. Thus, $\left(U_p,F^1,\ldots,F^m,x^{m+1},\ldots,x^n\right)$ is an adapted chart for N relative to S. Hence S is a regular submanifold of dimension $n-m$. $\square$

The proof for the regular level set theorem gives the following useful lemma.

Lemma 10.5. Let $F\colon N\to\mathbb{R}^m$ be a C^∞ map on a manifold N of dimension n and let S be the level set $F^{-1}(\mathbf{0})$. If relative to some coordinate chart $\left(U,x^1,\ldots,x^n\right)$ about $p\in S$, the Jacobian determinant $\partial\left(F^1,\ldots,F^m\right)/\partial\left(x^{j_1},\ldots,x^{j_m}\right)(p)\neq 0$, then in some neighborhood of p one may replace $x^{j_1},\ldots,x^{j_m}$ by $F^1,\ldots,F^m$ to obtain an adapted chart for N relative to S.

Remark. The regular level set theorem gives a sufficient but not necessary condition for a level set to be a regular submanifold. For example, if $f\colon\mathbb{R}^2\to\mathbb{R}$ is the map $f(x,y)=y^2$ then the zero set $Z(f)=Z(y^2)$ is the x-axis, a regular submanifold of $\mathbb{R}^2$. However, since $\partial f/\partial x=0$ and $\partial f/\partial y=2y=0$ on the x-axis, every point in $Z(f)$ is a critical point of f. Thus, although $Z(f)$ is a regular submanifold of $\mathbb{R}^2$, it is not a regular level set of f.

10.4 Examples of Regular Submanifolds

Example. (*Hypersurface*). The solution set S of $x^n+y^n+z^n=1$ ($n\in\mathbb{Z}$) in $\mathbb{R}^3$ is a manifold of dimension 2, for suppose $f(x,y,z)=x^n+y^n+z^n$, then $S=f^{-1}(1)$. Since $\partial f/\partial x=nx^{n-1}$, $\partial f/\partial y=ny^{n-1}$, and $\partial f/\partial z=nz^{n-1}$, f has no critical points in S. Thus, 1 is a regular value of $f\colon\mathbb{R}^3\to\mathbb{R}$. By the regular level set theorem, S is a regular submanifold of $\mathbb{R}^3$ of dimension 2.

Problems

Problem 10.1. Define $f\colon\mathbb{R}^2\to\mathbb{R}$ by

$$f(x,y)=x^3-6xy+y^2.$$

Find all values $c\in\mathbb{R}$ for which the level set $f^{-1}(c)$ is a regular submanifold of $\mathbb{R}^2$.

Solution. Since $\partial f/\partial x=3x^2-6y$ and $\partial f/\partial y=-6x+2y$, the only critical points of f are $(0,0)$ and $(6,18)$. We want the critical points to lie outside of $f^{-1}(c)$, so $c\neq f(0,0)=0$ and $c\neq f(6,18)=-108$.

Problem 10.2. Regular submanifolds
Suppose that a subset S of $\mathbb{R}^2$ has the property that locally on S one of the coordinates is a C^∞ function of the other coordinate. Show that S is a regular submanifold of $\mathbb{R}^2$.

Solution. Let $p=(x,y)$ be a point on S. By hypothesis, there is a neighborhood U of p in $\mathbb{R}^2$ such that, wlog, $y=f(x)$ on $U\cap S$ for some C^∞ function $f\colon\mathbb{R}\supset A\to B\subset\mathbb{R}$ and $V\colon=A\times B\subset U$. Let

$F\colon V \to \mathbb{R}^2$ be given by $F(x,y) = (x, y - f(x))$. Clearly, F is C^∞, so it is a diffeomorphism onto its image. Thus, F can be used as a coordinate map. Moreover, $(V, F) = (V, x, y - f(x))$ is an adapted chart relative to S for $\mathbb{R}^2$, since in the chart (V, F), $V \cap S \subset U \cap S$, is defined by the vanishing of the coordinate function $y - f(x)$. Hence, S is a regular submanfold of $\mathbb{R}^2$.

Problem 10.3. Graph of a smooth function
Show that the graph $\Gamma(f)$ of a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$,

$$\Gamma(f) = \{(x^1, \dots, x^n, f(x^1, \dots, x^n)) \in \mathbb{R}^{n+1}\},$$

is a regular submanifold of $\mathbb{R}^{n+1}$.

Solution. Clearly, $\big(\mathbb{R}^{n+1}, x^1, \dots, x^n, x^{n+1} - f(x^1, \dots, x^n)\big)$ is an adapted chart for $\mathbb{R}^{n+1}$ relative to $\Gamma(f)$, so $\Gamma(f)$ is a regular submanifold of $\mathbb{R}^{n+1}$.

11 Categories and Functors

11.1 Categories

Definition 11.1. A *category* consists of a collection of elements, called *objects*, and for any two objects A and B, a set $\mathrm{Mor}(A,B)$ of elements, called *morphisms* from A to B, such that given any morphism $f \in \mathrm{Mor}(A,B)$ and any morphism $g \in \mathrm{Mor}(A,B)$, the *composite* $g \circ f \in \mathrm{Mor}(A,B)$ is defined. Furthermore, the composition of morphisms is required to satisfy two properties:

(i) The identity axiom: for each object A, there is an identity morphism $\mathbb{1}_A \in \mathrm{Mor}(A,A)$ such that for any $f \in \mathrm{Mor}(A,B)$ and $g \in \mathrm{Mor}(B,A)$,

$$f \circ \mathbb{1}_A = f \quad \text{and} \quad \mathbb{1}_A \circ g = g;$$

(ii) The associative axiom: for $f \in \mathrm{Mor}\,(A,B)$, $g \in \mathrm{Mor}\,(B,C)$, and $h \in \mathrm{Mor}\,(C,D)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

If $f \in \mathrm{Mor}\,(A,B)$, we often write $f\colon A \to B$.

Example. The collection of groups and group homomorphisms forms a category in which the objects are groups and for any two groups A and B, $\mathrm{Mor}\,(A,B)$ is the set of group homomorphisms from A to B.

Example. The collection of all vector spaces over $\mathbb{R}$ and $\mathbb{R}$-linear maps forms a category in which the objects are real vector spaces and for any two real vector spaces V and W, $\mathrm{Mor}\,(V,W)$ is the set $\mathrm{Hom}(V,W)$ of linear maps from V to W.

Example. The collection of all topological spaces together with continuous maps between them is called the *continuous category*.

Example. The collection of smooth manifolds together with smooth maps between them is called the *smooth category.*

Example. We call a pair (M, q), where M is a manifold and q a point in M, a *pointed manifold.* Given any two such pairs (N, p) and (M, q), let $\operatorname{Mor}((N, p), (M, q))$ be the set of all smooth maps $F\colon N \to M$ such that $F(p) = q$. This gives rise to the *category of pointed manifolds.*

Definition 11.2. Two objects A and B in a category are said to be *isomorphic* if there are morphisms $f\colon A \to B$ and $g\colon B \to A$ such that

$$g \circ f = \mathbb{1}_A \quad \text{and} \quad f \circ g = \mathbb{1}_B.$$

In this case both f and g are called *isomorphisms.*

11.2 Functors

Definition 11.3. A *(covariant) functor* $\mathcal{F}$ from one category $\mathcal{C}$ to another category $\mathcal{D}$ is a map that associates to each object A in $\mathcal{C}$ an object $\mathcal{F}(A)$ in $\mathcal{D}$ and to each morphism $f\colon A \to B$ a morphism $\mathcal{F}(f)\colon \mathcal{F}(A) \to \mathcal{F}(B)$ such that

(i) $\mathcal{F}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{F}(A)}$,

(ii) $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

Example. The tangent space construction is a functor from the category of pointed manifolds to the category of vector spaces. To each pointed manifold (N, p) we associate the tangent space T_pN and to each smooth map $f\colon (N, p) \to (M, f(p))$ we associate the differential $f_{*,p}\colon T_pN \to T_{f(p)}M$. The functorial property (i) holds because if $1\colon N \to N$ is the identity map, then its differential $\mathbb{1}_{*,p}\colon T_pN \to T_pN$ is also the identity map. The functorial property (ii) holds because in this context it is the chain rule

$$(g \circ f)_{*,p} = g_{*,f(p)} \circ f_{*,p}.$$

Proposition 11.1. Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$. If $f\colon A \to B$ is an isomorphism in $\mathcal{C}$, then $\mathcal{F}(f)\colon \mathcal{F}(A) \to \mathcal{F}(B)$ is an isomorphism in $\mathcal{D}$.

Proof. Since f is an isomorphism in $\mathcal{C}$, there is a morphism $g\colon B \to A$ in $\mathcal{C}$ such that

$$g \circ f = \mathbb{1}_A \quad \text{and} \quad f \circ g = \mathbb{1}_B.$$

Applying the functor $\mathcal{F}$ to both equations, noting the functorial properties, yields

$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathbb{1}_{\mathcal{F}(A)} \quad \text{and} \quad \mathcal{F}(f) \circ \mathcal{F}(g) = \mathbb{1}_{\mathcal{F}(B)}.$$

Hence, $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are isomorphisms in $\mathcal{D}$. □

Remark. We can recast Corollary 9.2 in a more functorial form: Suppose $f\colon N \to M$ is a diffeomorphism. Then (N, p) and $(M, f(p))$ are isomorphic objects in the category of pointed manifolds. By

Proposition 11.1, the tangent spaces T_pN and $T_{f(p)}M$ are isomorphic as vector spaces and therefore have the same dimension. It follows that the dimension of a manifold is invariant under a diffeomorphism.

If in the definition of a covariant functor we reverse the direction of the arrow for the morphism $\mathcal{F}(f)$, then we obtain a *contravariant functor*. More precisely, the definition is as follows.

Definition 11.4. A *contravariant functor* $\mathcal{F}$ from one category $\mathcal{C}$ to another category $\mathcal{D}$ is a map that associates to each object A in $\mathcal{C}$ an object $\mathcal{F}(A)$ in $\mathcal{D}$ and to each morphism $f\colon A \to B$ a morphism $\mathcal{F}(f)\colon \mathcal{F}(B) \to \mathcal{F}(A)$ such that

(i) $\mathcal{F}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{F}(A)}$;

(ii) $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$ (Note the reversal of order).

Example. Smooth functions on a manifold give rise to a contravariant functor that associates to each manifold M the algebra $\mathcal{F}(M) = C^\infty(M)$ of C^∞ functions on M and to each smooth map $F\colon N \to M$ of manifolds the pullback map $\mathcal{F}(F) = F^*\colon C^\infty(M) \to C^\infty(N)$, $F^*h = h \circ F$ for $h \in C^\infty(M)$. It is easy to verify that the pullback satisfies the two (contravariant) functorial properties:

(i) $(\mathbb{1}_M)^* = \mathbb{1}_{C^\infty(M)}$;

(ii) if $F\colon N \to M$ and $G\colon M \to P$ are C^∞ maps, then $(G \circ F)^* = F^* \circ G^*\colon C^\infty(P) \to C^\infty(N)$.

Another example of a contravariant functor is the dual of a vector space, which we will review in the next subsection.

11.3 The Dual Functor and the Multicovector Functor

Let V be a real vector space. Recall that its dual space V^* is the vector space of all *linear functionals* on V, i.e., linear functions $\alpha\colon V \to \mathbb{R}$. We also write

$$V^* = \operatorname{Hom}(V, \mathbb{R}).$$

If V is a finite-dimensional vector space with basis $\{e_1, \dots, e_n\}$, then its dual space V^* has as a basis a collection of linear functionals $\{\alpha^1, \dots, \alpha^n\}$ defined by

$$\alpha^i(e_j) = \delta^i_j,\ 1 \le i, j \le n.$$

Since a linear function on V is determined by what it does on a basis of V, this set of equations defines α^i uniquely.

A linear map $L\colon V \to W$ of vector spaces induces a linear map L^*, called the *dual* of L, as follows. To every linear functional $\alpha\colon W \to \mathbb{R}$, the dual L^* associates the linear functional

$$V \xrightarrow{L} W \xrightarrow{\alpha} \mathbb{R}.$$

Thus, the dual map $L^*\colon W^* \to V^*$ is given by

$$L^*(\alpha) = \alpha \circ L \text{ for } \alpha \in W^*.$$

Note that the dual of L reverse the direction of the arrow.

Proposition 11.2. (Functorial properties of the dual). Suppose V, W, and S are real vector spaces.

(i) If $\mathbb{1}_V : V \to V$ is the identity map on V, then $\mathbb{1}_V^* : V^* \to V^*$ is the identity map on V^*.

(ii) If $f : V \to W$ and $g : W \to S$ are linear maps, then $(g \circ f)^* = f^* \circ g^*$.

Proof. Given a linear functional α,

(i) $\mathbb{1}_V^*(\alpha) = \alpha \circ \mathbb{1}_V = \alpha$, and

(ii) $(f^* \circ g^*)(\alpha) = f^*(\alpha \circ g) = \alpha \circ (g \circ f) = (g \circ f)^*(\alpha)$.

□

Fix a positive integer k. For any linear map $L : V \to W$ of vector spaces, define the *pullback* map $L^* : A_k(W) \to A_k(V)$, where $A_k(V)$ denotes the set of alternating k-linear functions on V, to be

$$(L^* f)(v_1, \ldots, v_k) = f(L(v_1), \ldots, L(v_k))$$

for $f \in A_k(W)$ and $v_1, \ldots, v_k \in V$. It is easy to see that L^* is a linear map: $L^*(af + bg) = aL^*f + bL^*g$ for $a, b \in \mathbb{R}$ and $f, g \in A_k(W)$.

Proposition 11.3. The pullback of covectors by a linear maps satisfies the two functorial properties:

(i) If $\mathbb{1}_V : V \to V$ is the identity map on V, then $\mathbb{1}_V^* = \mathbb{1}_{A_k(V)}$ is the identity map on $A_k(V)$.

(ii) If $K : U \to V$ and $L : V \to W$ are linear maps of vector spaces, then

$$(L \circ K)^* = K^* \circ L^* : A_k(W) \to A_k(U).$$

Proof. Obvious. □

To each vector space V, we associate the vector space $A_k(V)$ of all k-covectors on V, and to each linear map $L : V \to W$ of vector spaces, we associate the pullback $A_k(L) = L^* : A_k(W) \to A_k(V)$. Then, $A_k(\,)$ is a contravariant functor from the category of vector spaces and linear maps to itself. When $k = 1$, it is easy to see that the functor simplifies to the dual map. Thus, the multicovector functor $A_k(\,)$ generalizes the dual functor $(\,)^*$.

Problems

Problem 11.1. If $F : N \to M$ is a diffeomorphism of manifolds and $p \in N$, prove that $F^{-1}_{*,F(p)} = (F_{*,p})^{-1}$.

Proof. By the chain rule, $F^{-1}_{*,F(p)} \circ F_{*,p} = \left(F^{-1} \circ F\right)_{*,p} = \mathbb{1}_{*,p}$. Hence, $F^{-1}_{*,F(p)} = (F_{*,p})^{-1}$. □

Problem 11.2. Show that if $L\colon V \to V$ is a linear operator on a vector space V of dimension n, then the pullback $L^*\colon A_n(V) \to A_n(V)$ is multiplication by the determinant of L.

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis for V. Since any $f \in A_n(V)$ is n-linear, it suffices to compute $(L^*f)(e_1, \ldots, e_n)$. Observe that $(L^*f)(e_1, \ldots, e_n) = f(L(e_1), \ldots, L(e_n)) = \det(L)$ is precisely the determinant of L defined as the unique alternating n-linear function on the columns of (the matrix representation) of L. $\square$

12 The rank of a Smooth Map

12.1 Constant Rank Theorem

Suppose $f\colon N \to M$ is a C^∞ map of manifolds and we want to show that the level set $f(c)^{-1}$ is a manifold for some c in M. In order to apply the level set theorem, we need the differential f_* to have maximal rank at every point of $f(c)^{-1}$. Sometimes, this may not be true; even if it is true, it may be difficult to show. In such cases, the constant-rank level set theorem can be helpful.

Before we prove the constant-rank level set theorem, we must first revisit the constant rank theorem for Euclidean spaces.

Theorem 12.1. (Constant rank theorem for Euclidean spaces). If $f\colon \mathbb{R}^n \supset U \to \mathbb{R}^m$ has constant rank k in a neighborhood of a point $p \in U$, then after a suitable change of coordinates near $p \in U$ and $f(p)$ in $\mathbb{R}^m$, the map f assumes the form

$$\left(x^1, \ldots, x^n\right) \mapsto \left(x^1, \ldots, x^k, 0, \ldots, 0\right).$$

More precisely, there are diffeomorphisms G of a neighborhood of p in U sending p to the origin in $\mathbb{R}^n$ and a diffeomorphism F of a neighborhood of $f(p)$ in $\mathbb{R}^m$ sending $f(p)$ to the origin in $\mathbb{R}^m$ such that

$$\left(F \circ f \circ G^{-1}\right)\left(x^1, \ldots, x^n\right) = \left(x^1, \ldots, x^k, 0, \ldots, 0\right).$$

Proof. Let U be a neighborhood of p in $\mathbb{R}^n$ and suppose $f = \left(f^1, \ldots, f^m\right) : U \to \mathbb{R}^m$ has constant rank k in U. Without loss of generality, we can reorder the component functions $f^1, \ldots, f^m$ such that

$$\det\left[\frac{\partial f^i}{\partial x^j}(p)\right]_{1 \le i,j \le k} \neq 0.$$

Now, define $G\colon U \to \mathbb{R}^n$ by the replacement of the first k coordinates $x^1, \ldots, x^k$ by $f^1, \ldots, f^k$, i.e.,

$$G\left(x^1, \ldots, x^n\right) = \left(f^1, \ldots, f^k, x^{k+1}, \ldots, x^n\right).$$

Then, the Jacobian determinant of G

$$\frac{\partial\left(f^1, \ldots, f^k, x^{k+1}, \ldots, x^n\right)}{\partial\left(x^1, \ldots, x^n\right)} = \det\left[\frac{\partial f^i}{\partial x^j}(p)\right]_{1 \le i,j \le k} \neq 0,$$

so by the inverse function theorem, there are neighborhoods $U_1 \subset U$ of p in $\mathbb{R}^n$ and V_1 of $G(p)$ in $\mathbb{R}^n$ such that $G\colon U_1 \to V_1$ is a diffeomorphism.

On V_1, we have

$$\begin{aligned}\left(x^1, \ldots, x^n\right) &= \left(G \circ G^{-1}\right)\left(x^1, \ldots, x^n\right) \\ &= \left(\left(f^1, \ldots, f^k\right) \circ G^{-1}, \left(x^{k+1}, \ldots, x^n\right) \circ G^{-1}\right)\left(x^1, \ldots, x^n\right).\end{aligned}$$

Hence, $x^i = f^i \circ G^{-1}$ for $1 \le i \le k$, and so

$$\begin{aligned}\left(f \circ G^{-1}\right)\left(x^1, \ldots, x^n\right) &= \left(\left(f^1, \ldots, f^k\right) \circ G^{-1}, \left(f^{k+1}, \ldots, f^m\right) \circ G^{-1}\right)\left(x^1, \ldots, x^n\right) \\ &= \left(x^1, \ldots, x^k, \left(\left(f^{k+1}, \ldots, f^m\right) \circ G^{-1}\right)\left(x^1, \ldots, x^n\right)\right) \\ &= \left(x^1, \ldots, x^k, g^{k+1}, \ldots, g^m\right),\end{aligned}$$

where $g^j = f^j \circ G^{-1}$, $k+1 \le j \le m$.

Since $G\colon U_1 \to V_1$ is a diffeomorphism and f has constant rank k on U, $f \circ G^{-1}$ has constant rank k on V_1. Its Jacobian matrix is

$$J\left(f \circ G^{-1}\right) = \left[\begin{array}{c|c} \dfrac{\partial x^i}{\partial x^j} & \dfrac{\partial x^i}{\partial x^\beta} \\ \hline \dfrac{\partial g^\alpha}{\partial x^j} & \dfrac{\partial g^\alpha}{\partial x^\beta} \end{array}\right] = \left[\begin{array}{c|c} I_{k\times k} & \mathbf{0}_{k\times(n-k)} \\ \hline \partial g^\alpha/\partial x^j & \partial g^\alpha/\partial x^\beta \end{array}\right],$$

where $1 \le i, j \le k$, $k+1 \le \alpha \le m$ and $k+1 \le \beta \le n$. For this matrix to have rank k, all $(k+1)\times(k+1)$ minors must have determinant 0. Particularly,

$$\det\begin{bmatrix} I_{k\times k} & \mathbf{0}_{k\times 1} \\ \partial g^\alpha/\partial x^j & \partial g^\alpha/\partial x^\beta \end{bmatrix} = \frac{\partial g^\alpha}{\partial x^\beta} = 0$$

for all $k+1 \le \alpha \le m$ and $k+1 \le \beta \le n$. Thus, the functions g^α are functions of $x^1, \ldots, x^k$ only.

Finally, let $F\colon \mathbb{R}^m \to \mathbb{R}^m$ be given by

$$F\left(x^1, \ldots, x^m\right) = \left(x^1, \ldots, x^k, x^{k+1} - g^{k+1}, \ldots, x^m - g^m\right).$$

Then,

$$\left(F \circ f \circ G^{-1}\right)\left(x^1, \ldots, x^n\right) = F\left(x^1, \ldots, x^k, g^{k+1}, \ldots, g^m\right) = \left(x^1, \ldots, x^k, 0, \ldots, 0\right).$$

□

Theorem 12.2. (Constant rank theorem). Let N and M be manifolds of dimensions n and m respectively. Suppose $f\colon N \to M$ has constant rank k in a neighborhood of a point p in N. Then, there are charts (U, ϕ) centered at p in N and (V, Ψ) centered at $f(p)$ in M such that for $\left(r^1, \ldots, r^n\right)$ in $\phi(U)$,

$$\left(\Psi \circ f \circ \phi^{-1}\right)\left(r^1, \ldots, r^n\right) = \left(r^1, \ldots, r^k, 0, \ldots, 0\right).$$

Proof. Let (U_0, ϕ_0) be a chart centered at p in N and (V_0, Ψ_0) a chart centered at $f(p)$ in M. Then, the map $\Psi \circ f \circ \phi^{-1}$ is a smooth map between Euclidean spaces with constant rank k in $\phi(U_0)$. By the constant rank theorem for Euclidean spaces (Theorem 12.1), there are diffeomorphisms $F\colon V_0 \supset V \to \mathbb{R}^m$ and $G\colon U_0 \supset U \to \mathbb{R}^n$ into their images such that

$$\left(F \circ \left(\Psi_0 \circ f \circ \phi_0^{-1}\right) \circ G^{-1}\right)\left(r^1, \ldots, r^n\right) = \left(r^1, \ldots, r^k, 0, \ldots, 0\right).$$

Let $\phi = G \circ \phi_0$ and $\Psi = F \circ \Psi_0$, and we are done. □

Theorem 12.3. (Constant-rank level set theorem). Let $f\colon N \to M$ be a C^∞ map of manifolds and $c \in M$. If f has constant rank k in a neighborhood of the level set $f^{-1}(c)$ in N, then $f^{-1}(c)$ is a regular submanifold of N of codimension k.

Proof. Let p be a point in $f^{-1}(c)$. By the constant rank theorem (Theorem 12.2), there are charts $(U, \phi) = \left(U, x^1, \ldots, x^n\right)$ centered at p in N and (V, Ψ) centered at $f(p) = c$ in M such that

$$\left(\Psi \circ f \circ \phi^{-1}\right)\left(r^1, \ldots, r^n\right) = \left(r^1, \ldots, r^k, 0, \ldots, 0\right).$$

Thus, the level set $\left(\Psi \circ f \circ \phi^{-1}\right)^{-1}(0)$ is obtained by the vanishing of the coordinates $\left(r^1, \ldots, r^k\right)$.

Now, observe that

$$\phi(f^{-1}(c)) = \phi(f^{-1}(\Psi^{-1}(0))) = \left(\Psi \circ f \circ \phi^{-1}\right)^{-1}(0).$$

Thus, the level set $f^{-1}(c)$ is defined by the vanishing of the coordinate functions $\left(x^1, \ldots, x^k\right)$, where $x^i = r^i \circ \phi$. Hence, $f^{-1}(c)$ is a regular submanifold of N of codimension k. □

12.2 The Immersion and Submersion Theorems

Proposition 12.4. Let N and M be manifolds of dimensions n and m respectively. If a C^∞ map $f\colon N \to M$ is an immersion at a point $p \in N$, then it has constant rank n in a neighborhood of p. If a C^∞ map $f\colon N \to M$ is a submersion at a point $p \in N$, then it has constant rank m in a neighborhood of p.

Proof. Let (U, ϕ) be a chart about p in N and (V, Ψ) a chart about $f(p)$ in M. Then, relative to the charts (U, ϕ) and (V, Ψ), the linear map $f_{*,p}$ is represented by the matrix $\left[\partial f^i / \partial x^j(p)\right]$. Clearly,

$$\begin{aligned} f_{*,p} \text{ is injective} \quad &\iff \quad n \leq m, \operatorname{rk}\left[\partial f^i / \partial x^j(p)\right] = n, \\ f_{*,p} \text{ is surjective} \quad &\iff \quad n \geq m, \operatorname{rk}\left[\partial f^i / \partial x^j(p)\right] = m. \end{aligned}$$

Hence, an immersion or submersion at p is equivalent to the maximality of rk $\left[\partial f^i / \partial x^j(p)\right]$. Moreover, if f has maximal rank at p, then it has maximal rank at all points in some neighborhood of p. This proves the proposition. □

By Proposition 12.4, the following theorems folow directly from the constant rank theorem.

Proposition 12.5. Let $f\colon N \to M$ be manifolds of dimensions n and m respectively.

(i) **(Immersion theorem)** Suppose $f\colon N \to M$ is an immersion at $p \in N$. Then, there are charts (U, ϕ) centered at p in N and (V, Ψ) centered at $f(p)$ in M such that in a neighborhood of $\phi(p)$,

$$\left(\Psi \circ f \circ \phi^{-1}\right)\left(r^1, \ldots, r^n\right) = \left(r^1, \ldots, r^n, 0, \ldots, 0\right).$$

(ii) **(Submersion theorem)** Suppose $f\colon N \to M$ is submersion at $p \in N$. Then, there are charts (U, ϕ) centered at p in N and (V, Ψ) centered at $f(p)$ in M such that in a neighborhood of $\phi(p)$,

$$\left(\Psi \circ f \circ \phi^{-1}\right)\left(r^1, \ldots, r^m, r^{m+1}, \ldots, r^n\right) = \left(r^1, \ldots, r^m\right).$$

Corollary 12.6. A submersion $f\colon N \to M$ of manifolds is an open map.

Proof. Let W be an open subset of N. Choose a point $f(p)$ in $f(W)$, with $p \in W$. By the submersion theorem, f is locally a projection. Since a projection is an open map, there is an open neighborhood U of p in W such that $f(U)$ is open in M. Moreover,

$$f(p) \in f(U) \subset f(W),$$

so $f(W)$ is open in M. □

12.3 Images of Smooth Maps

Definition 12.1. A C^∞ map $f\colon N \to M$ is called an *embedding* if

(i) it is a one-to-one immersion and

(ii) the image $f(N)$ with the subspace topology is homeomorphic to N under f.

Remark. The nomenclature of submanifolds may vary across different literatures. Many authors give the image $f(N)$ of a one-to-one immersion $f\colon N \to M$ not the subspace topology, but the topology inherited from f; i.e., a subset $f(U)$ of $f(N)$ is said to be open if and only if U is open in N. With this topology, $f(N)$ is by definition homeomorphic to N. These authors then define a submanifold to the the image of any one-to-one immersion with the topology and differentiable structure inherited from f. Such a set is sometimes called an *immersed submanifold* of M. If the underlying set of an immersed submanifold is given the subspace topology instead, then the resulting space may not even be a manifold at all.

Theorem 12.7. If $f\colon N \to M$ is an embedding, then its image $f(N)$ is a regular submanifold of M.

Proof. Let $p \in N$. By the immersion theorem, there are charts $\left(U, x^1, \ldots, x^n\right)$ about p in N and

$\left(V, y^1, \ldots, y^m\right)$ about $f(p)$ in M such that $f\colon U \to V$ has the form

$$\left(x^1, \ldots, x^n\right) \mapsto \left(x^1, \ldots, x^n, 0, \ldots, 0\right).$$

Thus, $f(U)$ is defined by the vanishing of the coordinates $y^{n+1}, \ldots, y^m$.

By the definition of an embedding, with the subspace topology, $f(N)$ is homeomorphic to N. Thus, the image $f(U)$ is open in $f(N)$. By definition of the subspace topology, there exists an open set V' in M such that $(V' \cap f(N)) = f(U)$. In $V \cap V'$,

$$(V \cap V') \cap f(N) = V \cap f(U) = f(U),$$

and since $f(U)$ is defined by the vanishing of the coordinates $y^{n+1,\ldots,y^m}$, $\left(V \cap V', y^1, \ldots, y^m\right)$ is an adapted chart for $f(N)$ containing $f(p)$. Since $f(p)$ is an arbitrary point in $f(N)$, this shows that $f(N)$ is a regular submanifold of M. □

Theorem 12.8. If N is a regular submanifold of M, then the inclusion $i\colon N \to M$, $i(p) = p$, is an embedding.

Proof. Clearly, $i\colon N \to i(N)$ is a homeomorphism. Thus, it remains to show that $i\colon N \to M$ is an immersion.

Let $p \in N$. Since N is a regular submanifold of M, there exists an adapted chart $\left(V, y^1, \ldots, y^m\right)$ for M about p such that $V \cap N$ is the zero set of $y^{n+1}, \ldots, y^m$. Then, relative to the charts $\left(V \cap N, y^1, \ldots, y^n\right)$ for N and $\left(V, y^1, \ldots, y^m\right)$, the inclusion i is given by,

$$\left(y^1, \ldots, y^n\right) \mapsto \left(y^1, \ldots, y^n, 0, \ldots, 0\right),$$

which shows that i is an inclusion. □

Remark. In the literature, the image of an embedding is often called an *embedded submanifold*. Theorems 12.7 and 12.8 show that an embedded submanifold and a regular submanifold are one and the same.

12.4 Smooth Maps into a Submanifold

Suppose $f\colon N \to M$ is a C^∞ map whose image $f(N)$ lies in a subset $S \subset M$. Even if S is manifold, the induced map $\tilde{f}\colon N \to S$ may or may not be C^∞ depending on whether S is a regular submanifold or an immersed submanifold of M.

Theorem 12.9. Suppose $f\colon N \to M$ is C^∞ and the image of f lies in a subset S of M. If S is a regular submanifold of M, then the induced map $\tilde{f}\colon N \to S$ is C^∞.

Proof. Let $p \in N$ and denoted by n, m, and s the dimensions of N, M, and S respectively. Since S is a regular submanifold of M of dimension s, there exists an adapted chart $\left(V, y^1, \ldots, y^m\right)$ for M about p such that $S \cap V$ is defined by the vanishing of the coordinates $y^{s+1}, \ldots, y^m$ with coordinate map $\Psi_S = \left(y^1, \ldots, y^s\right)$. By the continuity of f, there exists a neighborhood U of p such that $f(U) \subset V$ is open in M. Then, $f(U) \subset V \cap S$, so that for $q \in U$,

$$(\Psi \circ f)(q) = \left(y^1(f(q)), \ldots, y^s(f(q)), 0, \ldots, 0\right).$$

It follow that on U,

$$\Psi_S \circ \tilde{f} = \left(y^1 \circ f, \ldots, y^s \circ f\right),$$

and since $y^1 \circ f, \ldots, y^s \circ f$ are C^∞ on U, $\tilde{f}$ is C^∞ on U and hence at p. As p was an arbitrary point in N, it follows that the map $\tilde{f} \colon N \to S$ is C^∞. $\square$

12.5 The Tangent Plane to a Surface in $\mathbb{R}^3$

Suppose $f\left(x^1, x^2, x^3\right)$ is a real-valued function on $\mathbb{R}^3$ with no critical points on its zero set $N = f^{-1}(0)$. By the regular level set theorem, N is a regular submanifold of $\mathbb{R}^3$. By Theorem 12.8, the inclusion $i \colon N \to \mathbb{R}^3$ is an embedding, so at any point p in N, $i_{*,p} \colon T_pN \to T_p\mathbb{R}^3$ is injective. We may therefore think of the tangent plane T_pN as a plane in $T_p\mathbb{R}^3 \simeq \mathbb{R}^3$ and we would like to find the equation of this plane.

Suppose $v = \sum v^i \partial/\partial x^i|_p$ is a vector in T_pN. Under the linear isomorphism $T_p\mathbb{R}^3 \simeq \mathbb{R}^3$, we identify v with the vector $\left\langle v^1, v^2, v^3\right\rangle$ in $\mathbb{R}^3$. Now, let $c(t)$ be a curve lying in N starting at p with initial vector v, i.e., $c(0) = p$ and $c'(0) = v = \left\langle v^1, v^2, v^3\right\rangle$. Since $c(t)$ lies in N, $f(c(t)) = 0$ for all t. By the chain rule,

$$0 = \frac{d}{dt} f(c(t)) = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(c(t)) \left(c^i\right)'(t).$$

At $t = 0$,

$$0 = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(c(0)) \left(c^i\right)'(0) = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(p) v^i.$$

Since the vector $v = \left\langle v^1, v^2, v^3\right\rangle$ represents an arrow from $p = \left(p^1, p^2, p^3\right)$ to $x = \left(x^1, x^2, x^3\right)$ in the tangent plane, one usually makes the substitution $v^i = x^i - p^i$. Thus, the tangent plane to N at p is defined by the equation

$$\sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(p)\left(x^i - p^i\right) = 0.$$

An intepretation of this equation is that the gradient vector of f at p is normal to any vector in the tangent plane.

Problems

Problem 12.1. The unit sphere S^n in $\mathbb{R}^{n+1}$ is defined by the equation $\sum_{i=1}^{n+1}(x^i)^2 = 1$. For $p = \left(p^1, \ldots, p^{n+1}\right) \in S^n$, show that a necessary and sufficient conditaion for

$$X_p = \sum a^i \left.\frac{\partial}{\partial x^i}\right|_p \in T_p\left(\mathbb{R}^{n+1}\right)$$

to be tangent to S^n at p is $\sum a^i p^i = 0$.

Solution. Clearly, S^n is the zero set of the function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ given by

$$f\left(x^1, \ldots, x^{n+1}\right) = \sum_{i=1}^{n+1} (x^i)^2 - 1.$$

Now, identity $T_p\left(\mathbb{R}^{n+1}\right) \simeq \mathbb{R}^{n+1}$. Then, we can identify $X_p \mapsto v = \left\langle a^1, \ldots, a^{n+1}\right\rangle$, and thus the tangent plane to S^n at p is defined by the equation

$$\sum_{i=1}^{n+1} \frac{\partial f}{\partial x^i}(p) a^i = 0,$$

or equivalently,

$$\sum_{i=1}^{n+1} p^i a^i = 0.$$

Problem 12.2. Critical points of a smooth map on a compact manifold
Show that a smooth map f from a compact manifold N to $\mathbb{R}^m$ has a critical point.

Solution.

13 The Tangent Bundle

13.1 The Topology of the Tangent Bundle

References

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