
NOTES ON TU'S INTRODUCTION TO MANIFOLDS

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Euclidean Spaces

1 Smooth Functions on a Euclidean Space

This section is a review of C^∞ functions on \mathbb{R}^n , which will be our primary tool for studying more "exotic" manifolds.

1.1 Definitions

Let us denote the coordinates in \mathbb{R}^n by x^1, x^2, \dots, x^n and let $p = (p^1, p^2, \dots, p^n)$ be a point in an open set U in \mathbb{R}^n . We can now begin with our first definition:

Definition 1.1. Let k be a non-negative integer. A vector-valued function $f : U \rightarrow \mathbb{R}^m$ is C^k at p if all of its components f^1, f^2, \dots, f^m are C^k at p , that is, if all of their partial derivatives of all orders $j \leq k$ exist and are continuous at p . The function is C^∞ at p if it is C^k at p for all $k \geq 0$. We say that the function is C^k on U if it is C^k at every point in U . A function that is C^∞ can be interchangeably called "smooth".

Remark. A C^∞ function need not be real-analytic. A famous example of this is given by the function $f(x)$ on \mathbb{R} defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

By induction, we can show that f is C^∞ on \mathbb{R} with its derivatives $f^{(k)}(0)$ equal to 0 for all $k \geq 0$ (this is left as an exercise to the reader). Thus, the Taylor series of this function at the origin is identically 0, so f is not equal to its Taylor series and is not real-analytic at 0.

1.2 Taylor's Theorem with Remainder

Definition 1.2. An open subset U of \mathbb{R}^n containing p is *star-shaped* with respect to p if for any $x \in U$, the line segment connecting x and p is completely contained inside U , i.e. it does not intersect the boundary of U .

Lemma 1.1. Let f be a smooth function on an open subset U of \mathbb{R}^n star-shaped with respect to $p = (p^1, p^2, \dots, p^n)$ in U . Then, there exists functions $g_1, g_2, \dots, g_n \in C^\infty(U)$ such that

$$f(x) = f(p) + \sum_{i=1}^n (x^i - p^i) g_i(x), \quad g_i(p) = \partial_i f(p).$$

Proof. Since U is star-shaped with respect to p in U , there exists a line segment that lies in U connecting any point x in U to p parametrized by $u(t) = tx + (1-t)p$, $0 \leq t \leq 1$ such that $f(u(t))$ is defined in U . By the chain rule, we have

$$\frac{d}{dt} f(u(t)) = \sum_i \partial_i f(u(t)) u'(t) = \sum_i (x^i - p^i) \partial_i f(u(t))$$

Integrating both sides with respect to t from 0 to 1, we get

$$f(x) - f(p) = \sum_i (x^i - p^i) g_i(x),$$

where

$$g_i(x) = \int_0^1 \partial_i f(u(t)) dt,$$

with

$$g_i(p) = \partial_i f(p)$$

□

Problems

Problem 1.1. Taylor's theorem with remainder to order 2

Prove that if $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is C^∞ , then there exists C^∞ functions g_{11}, g_{12}, g_{22} on \mathbb{R}^2 such that

$$\begin{aligned} f(x, y) = f(0, 0) &+ \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial y}(0, 0)y \\ &+ g_{11}(x, y)x^2 + g_{12}(x, y)xy + g_{22}(x, y)y^2. \end{aligned}$$

Solution. Applying Taylor's theorem with remainder (Lemma 1.1) to $f(x, y)$, there exists functions $g_1(x, y), g_2(x, y)$ with $g_1(0, 0) = \partial_x f(0, 0), g_2(0, 0) = \partial_y f(0, 0)$ so that

$$f(x, y) = f(0, 0) + g_1(x, y)x + g_2(x, y)y.$$

Applying Taylor's theorem again to $g_1(x, y), g_2(x, y)$ yields

$$\begin{aligned} f(x, y) = f(0, 0) &+ \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial y}(0, 0)y \\ &+ g_{11}(x, y)x^2 + g_{12}(x, y)xy + g_{22}(x, y)y^2. \end{aligned}$$

□

2 Tangent Vectors in \mathbb{R}^n as Derivations

2.1 The Directional Derivative

Loosely speaking, the tangent space $T_p(\mathbb{R}^n)$ at p in \mathbb{R}^n can be thought of as the vector space of all arrows emanating from p . Elements of $T_p(\mathbb{R}^n)$ are called *tangent vectors* at p in \mathbb{R}^n . With this, we are ready to define the *directional derivative* in \mathbb{R}^n :

Definition 2.1. Let v be a tangent vector at p in \mathbb{R}^n and let f be smooth in a neighborhood of p . Then, the *directional derivative* of f in the direction v at p , denoted $D_v f$, is defined as

$$D_v f = v^i \partial_i f(p),$$

where Einstein notation is used and a sum over the index i is implied unless specified otherwise.

From the above, we can see that there is a natural association between v and D_v independent of the input function. As such, we may want to characterize the *tangent vectors themselves* as operators on functions, which will be made precise in the next two subsections.

2.2 Germs of Functions

Definition 2.2. Let $p \in \mathbb{R}^n$, U be a neighborhood of p , and $f : U \rightarrow \mathbb{R}$ be smooth, and consider the set of all pairs (f, U) . We can define an equivalent relation $(f, U) \sim (g, V)$ if there exists a neighborhood $W \subset U \cap V$ such that $f = g$ when restricted to W . Then, the equivalence class of (f, U) is called the *germ* of f at p . The set of all germs of C^∞ functions on \mathbb{R}^n at p is then denoted $C_p^\infty(\mathbb{R}^n)$, or simply C_p^∞ if there is no ambiguity.

As an exercise, one may check that C_p^∞ , together with function addition and multiplication, forms an algebra over \mathbb{R} , i.e. C_p^∞ not only is a vector space over \mathbb{R} , but also forms a ring (with function multiplication).

2.3 Derivations at a Point

For any $v \in T_p(\mathbb{R}^n)$, the directional derivative at p gives an \mathbb{R} -linear mapping

$$D_v : C_p^\infty \rightarrow \mathbb{R}$$

that satisfies the Leibniz rule (product rule)

$$D_v(fg) = (D_v f)g(p) + f(p)(D_v g). \quad (2.1)$$

To generalize this notion, we have our next definition:

Definition 2.3. Let $D : C_p^\infty \rightarrow \mathbb{R}$ be a linear map that satisfies the Leibniz rule. Then, D is called a *derivation at p* or a *point derivation* of C_p^∞ . The set of all derivations at p is then denoted by $\mathcal{D}_p(\mathbb{R}^n)$.

One may check that $\mathcal{D}_p(\mathbb{R}^n)$ is a vector space over \mathbb{R} . Since we know that directional derivatives at p are all derivations at p , there exists a (linear) map

$$\begin{aligned} \phi : T_p(\mathbb{R}^n) &\rightarrow \mathcal{D}_p(\mathbb{R}^n) \\ v &\mapsto D_v \end{aligned} \quad (2.2)$$

Lemma 2.1. If D is a derivation at p , then $D(c) = 0$ for any constant function c .

Theorem 2.2. The linear map $\phi : T_p(\mathbb{R}^n) \rightarrow \mathcal{D}_p(\mathbb{R}^n)$ defined in (2.2) is an isomorphism of vector spaces.

Proof. Let us first prove injectivity. Since ϕ is a linear map, suppose $D_v = 0$ for all $v \in T_p(\mathbb{R}^n)$. Then, by definition, we have

$$0 = D_v(x^j) = \sum_i v^i \partial_i x^j = \sum_i v^i \delta_i^j = v^j.$$

Thus, $v = 0$ so ϕ is injective.

To prove surjectivity, let D be a derivation at p and let (f, U) be a germ in C_p^∞ , with U an open ball. By Taylor's theorem with remainder (Lemma 1.1), there exists smooth functions g_i on U such that

$$f(x) = f(p) + \sum_i (x^i - p^i) g_i(x), \quad g(p) = \partial_i f(p).$$

Applying the derivation D on both sides and using Leibniz rule, we obtain

$$Df = \sum_i (Dx^i) g(p) + \sum_i (p^i - p^i) Dg_i(x) = \sum_i (Dx^i) \partial_i f(p).$$

Thus, $D = D_v$ for $v = \langle Dx^i \rangle$. □

This theorem shows that one may identify tangent vectors at p with derivations at p . Moreover, the standard basis $\{e_i\}$ for $T_p(\mathbb{R}^n)$ directly corresponds with the set of partial derivatives $\{\partial_i|_p\}$. Thus, a tangent vector $v = \langle v^i \rangle = \sum v^i e_i$ may be written as

$$v = \sum_i v^i \partial_i|_p. \tag{2.3}$$

2.4 Vector Fields

Definition 2.4. A function X on an open subset U of \mathbb{R}^n is called a *vector field on U* if for each point p in U , X maps p to a tangent vector X_p in $T_p(\mathbb{R}^n)$. Formally, the vector field X is defined by

$$\begin{aligned} X : U &\rightarrow \bigcup_{p \in U} T_p(\mathbb{R}^n) \\ p &\mapsto X_p \in T_p(\mathbb{R}^n) \end{aligned}$$

Since $T_p(\mathbb{R}^n)$ has standard basis $\{\partial_i|_p\}$, the vector X_p may be written as a linear combination

$$X_p = \sum_i a^i(p) \partial_i|_p, \quad p \in U, \quad a^i(p) \in \mathbb{R}. \tag{2.4}$$

Omitting p , we may write

$$X = \sum_i a^i \partial_i. \tag{2.5}$$

We say that a vector field X is C^∞ on U if the coefficient functions a^i are all C^∞ on U .

Remark. Let us denote by $C^\infty(U)$ the ring of smooth functions on U . Then, if $X = \sum a^i \partial_i$ is a smooth vector field and f a smooth function on U , $fX = \sum (fa^i) \partial_i$ is a smooth vector field on U . Thus, the set of all C^∞ vector fields on U , denoted by $\mathfrak{X}(U)$, forms a *module* over $C^\infty(U)$.

2.5 Vectors Fields as Derivations

Let X be a C^∞ vector field on an open subset U of \mathbb{R}^n and f a C^∞ function on U . Then, Xf is defined by

$$Xf = \sum_i a^i \partial_i f.$$

This shows that the vector field X is a linear map

$$\begin{aligned} C^\infty(U) &\rightarrow C^\infty(U) \\ f &\mapsto Xf \end{aligned}$$

Proposition 2.3. $X(fg)$ satisfies the Leibniz rule.

Proof. This is left as an exercise. □

If A is an algebra over a field K , a derivation of A is a K -linear map $D : A \rightarrow A$ that satisfies the Leibniz rule

$$D(ab) = (Da)b + a(Db) \quad \forall a, b \in A.$$

Let us denote by $\text{Der}(A)$ the set of derivations of A . Clearly, $\text{Der}(A)$ is a vector space over K . Thus, one may identify a linear map

$$\begin{aligned} \varphi : \mathfrak{X}(U) &\rightarrow \text{Der}(C^\infty(U)), \\ X &\mapsto (f \mapsto Xf). \end{aligned}$$

In fact, the map above is an isomorphism of vector spaces. Thus, vector fields on an open set U may be identified with derivations of the algebra $C^\infty(U)$.

3 The Exterior Algebra of Multicovectors

Definition 3.1. Let f be a k -linear function and g an l -linear function on a vector space V . Then, their *tensor product* is the $(k+l)$ -linear function $f \otimes g$ defined by

$$(f \otimes g)(v_1, \dots, v_{k+l}) = f(v_1, \dots, v_k) g(v_{k+1}, \dots, v_{k+l})$$

Definition 3.2. The *wedge product* of two alternating multilinear functions $f \in A_k(V)$ and $g \in A_l(V)$ is defined by "alternating" the tensor product:

$$f \wedge g = \frac{1}{k!l!} A(f \otimes g)$$

Explicitly, we have

$$(f \wedge g)(v_1, \dots, v_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) f(v_{\sigma_1}, \dots, v_{\sigma_k}) g(v_{\sigma_{k+1}}, \dots, v_{\sigma_{k+l}}).$$

Problems

Problem 3.1. Tensor products of covectors

Let e_1, \dots, e^n be a basis for a vector space and let $\alpha^1, \dots, \alpha^n$ be its dual basis in V^* . Suppose $[g_{ij}] \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix. Define a bilinear function $f : V \times V \rightarrow \mathbb{R}$ by

$$f(v, w) = \sum_{1 \leq i, j \leq n} g_{ij} v^i w^j$$

for $v = \sum v^i e_i$ and $w = \sum w^j e_j$ in V . Describe f in terms of the tensor products of α^i and α^j , $1 \leq i, j \leq n$.

Solution. Substituting $v^i = \alpha^i v$, $w^j = \alpha^j w$ into the above equation, we obtain

$$f(v, w) = \sum_{i,j} g_{ij} (\alpha^i v) (\alpha^j w) = \sum_{i,j} g_{ij} (\alpha^i \otimes \alpha^j)(v, w).$$

Problem 3.2. Hyperplanes

- Let V be a vector space of dimension n and $f : V \rightarrow \mathbb{R}$ a nonzero linear functional. Show that $\dim \ker f = n - 1$. A linear subspace of V of dimension $n - 1$ is called a *hyperplane* in V .
- Show that a nonzero linear functional on a vector space V is determined up to a multiplicative constant by its kernel, a hyperplane in V . In other words, if f and $g : V \rightarrow \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

Solution.

- By the rank-nullity theorem, we have

$$\dim \ker f = \dim V - \dim \text{im } f = n - 1$$

□

- Choose a basis $\{e_1, \dots, e_{n-1}\}$ for $\ker f = \ker g$. Let $\{e_1, \dots, e_n\}$ be a basis for V and $\{\alpha^1, \dots, \alpha^n\}$

be the dual basis for V^* . Writing $f = \sum f_i \alpha^i, g = \sum g_j \alpha^j$, for any $v = \sum v^i e_i \in \ker V$, we have

$$f(v) = \sum_{i,j}^{n-1} f_i v^j \alpha^i e_j = \sum_{i,j}^{n-1} f_i v^j \delta_j^i = f_i v^i = 0$$

$$g(v) = \cdots = g_i v^i = 0.$$

Thus, $f_i = g_i = 0$ for $1 \leq i \leq n-1$, so $f = f_n \alpha^n$ and $g = g_n \alpha^n$. \square

Problem 3.3. A basis for k -tensors

Let V be a vector space of dimension n with basis e_1, \dots, e_n . Let $\alpha^1, \dots, \alpha^n$ be the dual basis for V^* . Show that a basis for the space $L_k(V)$ of k -linear functions on V is $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ for all multi-indices (i_1, \dots, i_k) . (You may have noticed that this problem is a generalization of Problem 3.1.)

4 Differential Forms on \mathbb{R}^n

4.1 Differential 1-forms and the Differential of a Function

Definition 4.1. The *cotangent space* to \mathbb{R}^n at p , denoted by $T_p^*(\mathbb{R}^n)$, is defined as the dual space of the tangent space $T_p(\mathbb{R}^n)$.

Definition 4.2. A *covector field* or a *differential 1-form* (or *1-form*) on an open subset U of \mathbb{R}^n is a function ω that assigns to each point p in U a *covector* $\omega_p \in T_p^*(\mathbb{R}^n)$,

$$\omega : U \rightarrow \bigcup_{p \in U} T_p^*(\mathbb{R}^n),$$

$$p \mapsto \omega_p \in T_p^*(\mathbb{R}^n).$$

Given any smooth function $f : U \rightarrow \mathbb{R}$, we can construct a 1-form df , called the *differential* of f , as follows

$$(df)_p(X_p) = X_p f$$

Proposition 4.1. The basis $\{(dx^i)_p\}$ of the cotangent space $T_p^*(\mathbb{R}^n)$ is dual to the standard basis $\{\partial_i|_p\}$ of the tangent space $T_p(\mathbb{R}^n)$.

Proof. By definition

$$(dx^i)_p(\partial_j|_p) = \partial_j x^i = \delta_j^i.$$

\square

Proposition 4.2. The differential of a C^∞ function $f : U \rightarrow \mathbb{R}$ on an open set U in \mathbb{R}^n can be written as

$$df = \sum (\partial_i f) dx^i$$

Proof. By definition

$$(df) \partial_i = \partial_i f$$

On the other hand, applying $(df)_p$ to the standard basis $\{\partial_i|_p\}$ of $T_p(\mathbb{R}^n)$ yields

$$(df) \partial_i = \sum_j a_j (dx^j) (\partial_i) = \sum_j a_j \partial_i x^j = \sum_j a_j \delta_i^j = a_i$$

□

4.2 Differential k -forms

Definition 4.3. A *differential form* ω of degree k or a k -*form* on an open subset U of \mathbb{R}^n is a function that assigns each point p in U an alternating k -linear function on the tangent space $T_p(\mathbb{R}^n)$. Formally, we have

$$\begin{aligned} \omega : U &\rightarrow A_k(T_p(\mathbb{R}^n)) \\ p &\mapsto \omega_p \end{aligned}$$

Denote by $\Omega^k(U)$ the vector space of C^∞ k -forms on U . Then, their direct sum, denoted, $\Omega^*(U) = \bigoplus_k \Omega^k(U)$, forms what is called a *graded algebra* over \mathbb{R} .

Definition 4.4. A *graded algebra* is an algebra A that is a direct sum of vector spaces A^k , i.e.

$$A = \bigoplus_k A^k$$

with a multiplication map (\cdot, \cdot) that sends $a_i \in A^i$ and $a_j \in A^j$ to $(a_i, a_j) = a_i a_j \in A^{i+j}$

Example. The ring of polynomials $K[x]$ over a field K is a graded algebra, for we have $K[x] = \bigoplus_k A^k$, where $A^k = \{rx^k | r \in K\}$, and that for any $ax^i \in A^i$, $bx^j \in A^j$, we have $(ax^i)(bx^j) = (ab)x^{i+j} \in A^{i+j}$.

Example. The ring $\Omega^*(U)$ of differential forms on an open subset U of \mathbb{R}^n is a graded algebra over \mathbb{R} with the wedge product as its multiplication map.

4.3 Differential Forms as Multilinear Functions on Vector Fields

If ω is a smooth 1-form and X a smooth vector field on an open set U in \mathbb{R}^n , then the function $\omega(X)$ can be defined point-wise:

$$\omega(X)_p = \omega_p(X_p), \quad p \in U.$$

In coordinate form,

$$\omega = \sum \omega_i dx^i, \quad X = \sum X^j \partial_j,$$

so

$$\omega(X) = \sum \omega_i X^i,$$

which implies that $\omega(X)$ is smooth on U .

Remark. Let $\mathcal{F}(U) = C^\infty(U)$. A 1-form ω on U gives rise to a linear map

$$\begin{aligned} \mathfrak{X}(U) &\rightarrow \mathcal{F}(U), \\ X &\mapsto \omega(X). \end{aligned}$$

Similarly, a k -form ω on U gives rise to a k -linear map

$$\begin{aligned} \underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{k \text{ times}} &\rightarrow \mathcal{F}(U), \\ (X_i, \dots, X_k) &\mapsto \omega(X_i, \dots, X_k). \end{aligned}$$

4.4 The Exterior Derivative

To define the *exterior derivative* on C^∞ k -forms, on an open subset U of \mathbb{R}^n , we first define it on 0-forms: the exterior derivative of a C^∞ function $f \in C^\infty(U)$ is defined to be its differential:

$$df = \sum_i \partial_i f dx^i \in \Omega^1(U)$$

Definition 4.5. For any $k \geq 1$, given a k -form $\omega = \sum_I a_I dx^I \in \Omega^k(U)$, its *exterior derivative* is defined as

$$d\omega = \sum_I da_I \wedge dx^I = \sum_I \left(\sum_j \partial_j a_I dx^j \right) \wedge dx^I \in \Omega^{k+1}(U).$$

Example. Let A be a C^∞ 1-form $A_x dx + A_y dy + A_z dz$ on \mathbb{R}^3 . Then, its exterior derivative is a (C^∞) 2-form given by

$$\begin{aligned} dA &= dA_x \wedge dx + dA_y \wedge dy + dA_z \wedge dz \\ &= (\partial_y A_x dy + \partial_z A_x dz) \wedge dx + (\partial_x A_y dx + \partial_z A_y dz) \wedge dy + (\partial_x A_z dx + \partial_y A_z dy) \wedge dz \\ &= (\partial_x A_y - \partial_y A_x) dx \wedge dy + (\partial_y A_z - \partial_z A_y) dy \wedge dz + (\partial_z A_x - \partial_x A_z) dz \wedge dx \end{aligned}$$

Definition 4.6. Let $A = \bigoplus_{k=0}^\infty A^k$ be a graded algebra over a field K . An *antiderivation* of the graded algebra A is a K -linear map $D : A \rightarrow A$ such that for $a \in A^k$ and $b \in A^l$,

$$D(ab) = (Da)b + (-1)^k a(Db).$$

If there is an integer m such that for any k , the antiderivation D is a map $A^k \rightarrow A^{k+m}$, then we say that it is an antiderivation of *degree* m .

Proposition 4.3.

- (i) The exterior differentiation $d : \Omega^*(U) \rightarrow \Omega^*(U)$ is an antiderivation of degree 1.
- (ii) (*Poincare's lemma*) $d^2 = 0$.
- (iii) For any $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(df)(X) = Xf$

Proof.

- (i) For any $\omega = f dx^I, \tau = g dx^J$, we have

$$\begin{aligned}
 d(\omega \wedge \tau) &= d(f g dx^I \wedge dx^J) \\
 &= \sum_i \partial_i (fg) dx^i \wedge dx^I \wedge dx^J \\
 &= \sum_i (\partial_i f) dx^i \wedge dx^I \wedge g dx^J + \sum_j f (\partial_j g) dx^j \wedge dx^I \wedge dx^J \\
 &= \sum_i ((\partial_i f) dx^i \wedge dx^I) \wedge g dx^J + \sum_j f dx^I \wedge ((-1)^{\deg \omega} (\partial_j g) dx^j \wedge dx^J) \\
 &= d\omega \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau.
 \end{aligned}$$

□

- (ii) *Exercise*

- (iii) This is just the definition of the exterior derivative on a smooth function as its differential.

□

Proposition 4.4. Characterization of the exterior derivative. The three properties of Proposition 4.3 uniquely characterize exterior differentiation on an open subset U in \mathbb{R}^n . Explicitly, if $D : \Omega^*(U) \rightarrow \Omega^*(U)$ satisfies the three properties:

- (i) D is of degree 1,
 - (ii) $D^2 = 0$,
 - (iii) For any $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, $(Df)(X) = Xf$,
- then $D = d$.

Proof. By linearity, it suffices to prove $D = d$ on k -forms of type $\omega = f dx^I$. By the definition of exterior differentiation, we have $dx^i = D x^i$. This implies that $D dx^i = D(D x^i) = 0$ (by property (ii)).

Thus, we have $Ddx^I = 0$ for any multi-index I of degree k . Applying D to the k -form ω , we obtain

$$\begin{aligned} D\omega &= D(fdx^I) \\ &= (Df) \wedge dx^I + fD(dx^I) \\ &= df \wedge dx^I \\ &= d(fdx^I) \\ &= d\omega \end{aligned}$$

□

4.5 Closed Forms and Exact Forms

Definition 4.7. A k -form ω on U is *closed* if $d\omega = 0$; it is *exact* if there exists a $k+1$ -form τ such that $d\tau = \omega$.

Remark. Every exact form is closed, for if $\omega = d\tau$ for some τ then $d\omega = d(d\tau) = 0$.

Problems

Problem 4.1. At each point $p \in \mathbb{R}^3$, define a bilinear function ω_p on $T_p(\mathbb{R}^3)$ by

$$\omega_p(\mathbf{a}, \mathbf{b}) = \omega_p \left(\begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix} \right) = p^3 \det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix},$$

for tangent vectors $\mathbf{a}, \mathbf{b} \in T_p(\mathbb{R}^3)$, where p^3 is the third component of $p = (p^1, p^2, p^3)$. Since ω_p is an alternating bilinear function on $T_p(\mathbb{R}^3)$, ω is a 2-form on \mathbb{R}^3 . Write ω in terms of the standard basis $dx^i \wedge dx^j$ at each point.

Solution. For $1 \leq i < j \leq 3$, we have

$$(\omega_p)_{ij} = \omega_p(e^i, e^j) = p^3 \det \begin{bmatrix} \delta_i^1 & \delta_j^1 \\ \delta_i^2 & \delta_j^2 \end{bmatrix} = p^3 (\delta_i^1 \delta_j^2 - \delta_j^1 \delta_i^2) = p^3 \epsilon_{ij},$$

where ϵ_{ij} denotes the Levi-Civita symbol. Thus, ω in terms of the standard basis is given by

$$\omega = p^3 dx^1 \wedge dx^2$$

Problem 4.2. Suppose the standard coordinates on \mathbb{R}^2 are called r and θ (this \mathbb{R}^2 is the (r, θ) -plane, not the (x, y) -plane). If $x = r \cos \theta$ and $y = r \sin \theta$, calculate dx, dy , and $dx \wedge dy$ in terms of dr and $d\theta$.

Solution. Taking the exterior derivative of $x = r \cos \theta$ and $y = r \sin \theta$ yields

$$\begin{aligned} dx &= \cos \theta dr - r \sin \theta d\theta \\ dy &= \sin \theta dr + r \cos \theta d\theta \end{aligned}$$

Taking the wedge product of the above expressions, we obtain

$$dx \wedge dy = r \cos^2 \theta dr \wedge d\theta - (r \sin^2 \theta) d\theta \wedge dr = r dr \wedge d\theta.$$

Problem 4.3. Suppose the standard coordinates on \mathbb{R}^3 are called ρ, ϕ , and θ . If $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$, calculate dx, dy, dz , and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\phi$, and $d\theta$.

Solution. Taking the exterior derivatives of x, y, z yields

$$\begin{aligned} dx &= \sin \phi \cos \theta d\rho + \rho \cos \phi \cos \theta d\phi - \rho \sin \phi \sin \theta d\theta \\ dy &= \sin \phi \sin \theta d\rho + \rho \cos \phi \sin \theta d\phi + \rho \sin \phi \cos \theta d\theta \\ dz &= \cos \phi d\rho - \rho \sin \phi d\phi \end{aligned}$$

Taking the wedge product $dx \wedge dy \wedge dz$, we obtain

$$\begin{aligned} dx \wedge dy \wedge dz &= (\rho^2 \sin^3 \phi \cos^2 \theta + \rho^2 \sin \phi^3 \sin^2 \theta + \rho^2 \sin \phi \cos^2 \phi \cos^2 \theta \\ &\quad + \rho^2 \sin \phi \cos^2 \phi \sin^2 \theta) d\rho \wedge d\phi \wedge d\theta = \rho^2 \sin \phi d\rho \wedge d\phi \wedge d\theta. \end{aligned}$$

Manifolds

In this chapter, we will go over what it means for a topological space to be a topological *manifold*. To put it succinctly, a *topological manifold* is a topological space that is Hausdorff, second countable, and locally Euclidean. To understand what any of those terms mean, let us revisit a few definitions from point set topology.

5 Review of Point Set Topology

5.1 Topological Spaces

Definition 5.1. A *topology* on a set S is a collection \mathcal{T} of subsets containing both the empty set \emptyset and the set S such that \mathcal{T} is closed under arbitrary unions and finite intersections. Formally, \mathcal{T} obeys the following axioms

- (i) $\emptyset \in \mathcal{T}$ and $S \in \mathcal{T}$.
- (ii) For any number of indexed subsets $A_i \in \mathcal{T}$, their union $\bigcup A_i \in \mathcal{T}$.
- (iii) If $A_1, \dots, A_n \in \mathcal{T}$ then $\bigcap A_i \in \mathcal{T}$.

The elements of \mathcal{T} are called *open sets* and the pair (S, \mathcal{T}) is then called a *topological space*. Sometimes, we may refer to the pair (S, \mathcal{T}) as just the topological space S when the choice of topology is implied. If \mathcal{T}_1 and \mathcal{T}_2 are two topologies on a set S and $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say that \mathcal{T}_1 is *coarser* than \mathcal{T}_2 , or that \mathcal{T}_2 is *finer* than \mathcal{T}_1 .

Just as we can define a basis on a vector space, a topology \mathcal{T} on a set S can also be defined.

5.2 Bases for a Topological Space

Definition 5.2. A subcollection \mathcal{B} of a topology \mathcal{T} on a topological space S is a *basis for the topology* \mathcal{T} if given any open set U and a point $p \in U$, there is an open set $B \in \mathcal{B}$ such that $p \in B \subset U$. We also say that \mathcal{B} *generates* the topology \mathcal{T} or that \mathcal{B} is a *basis for the topological space* S .

Proposition 5.1. A collection \mathcal{B} of open sets of S is a basis if and only if every open set in S is a union of sets in \mathcal{B} .

Proof. Let \mathcal{B} be a basis for the topology \mathcal{T} of S . By definition, for any point $p \in U \subset S$, there exists $B \in \mathcal{B}$ such that $p \in B$. Thus, the open set U in S is a union of sets in \mathcal{B} .

Conversely, suppose every open set in S is a union of sets in \mathcal{B} and let U be such an open set. Then, $U = \bigcup_{i \in I} B_i$ for a collection of subsets $B_i \in \mathcal{B}$ indexed by $i \in I$. For any point $p \in U$, our previous statement implies that $p \in B_i \subset U$ for some $i \in I$. Hence, \mathcal{B} is a basis for S . \square

Proposition 5.2. A collection \mathcal{B} of subsets of a set S is a basis for some topology \mathcal{T} on S if and only if

- (i) S is the union of all the sets in \mathcal{B} , and
- (ii) given any two sets B_1 and B_2 in \mathcal{B} and a point $p \in B_1 \cap B_2$, there is a set $B \in \mathcal{B}$ such that $p \in B \subset B_1 \cap B_2$.

Proof. Let \mathcal{B} be a basis for some topology \mathcal{T} on S . By Proposition 5.1, S is a union of sets in \mathcal{B} . (ii) is also trivial from the definition of a basis, for $B_1 \cap B_2$ is an open set in S , and the definition of a basis \mathcal{B} requires that there exists a set $B \in \mathcal{B}$ that satisfies the above property.

Conversely, let us assume points (i) and (ii) and let \mathcal{T} be a collection of all sets that are unions of sets in \mathcal{B} . Clearly, \emptyset and S are both in \mathcal{T} . By construction, \mathcal{T} is also closed under arbitrary union. To show that \mathcal{B} is closed under finite intersection, Let $U = \bigcup_i B_i$ and $V = \bigcup_j B_j$ for some collection of sets $B_i, B_j \in \mathcal{B}$. Then,

$$U \cap V = \bigcup_{i,j} (B_i \cap B_j).$$

Thus, any point $p \in U \cap V$ is in $B_i \cap B_j$ for some i, j . (ii) then implies that there is a set $B_p \in \mathcal{B}$ such that $p \in B_p \subset B_i \cap B_j$. Therefore,

$$U \cap V = \bigcup_{p \in U \cap V} B_p \in \mathcal{T}.$$

□

5.3 Subspace Topology

Definition 5.3. (Subspace topology) Let (S, \mathcal{T}) be a topological space and A a subset of S . Define \mathcal{T}_A to be the collection of subsets

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}$$

It is easy to see that \mathcal{T}_A defines a topology on A . \mathcal{T}_A is then called the *subspace topology* or the *relative topology* of A in S , and elements of \mathcal{T}_A are said to be *open in A* .

Proposition 5.3. Let $\mathcal{B} = \{B_i\}$ be a basis for a topological space S , and A be a subspace of S . Then $\{B_i \cap A\}$ is a basis for A .

Proof. Let U_A be an open set in A and p be any point in U_A . By definition of the subspace topology, $U_A = U \cap A$ for some open set U in S . Thus, $p \in U$, which implies that there exists $B \in \mathcal{B}$ such that $p \in B_p \subset U$. p is also in A , so we have

$$p \in B_p \cap A \subset U \cap A = U_A.$$

Thus, $\{B_i \cap A\}$ is a basis for A . □

5.4 Second-Countability

Definition 5.4. A topological space is said to be *second countable* if it has a countable basis.

Proposition 5.4. A subspace A of a second-countable space S is second countable.

Proof. By Proposition 5.3, given a countable basis $\mathcal{B} = \{B_i\}$ for a space S , $\{B_i \cap A\}$ forms a countable basis for the subspace A . \square

5.5 The Hausdorff Condition

Definition 5.5. A topological space S is *Hausdorff* if given any two distinct points x, y in S , there exist disjoint open sets U, V in S such that $x \in U$ and $y \in V$. A Hausdorff space is *normal* if given any two disjoint closed sets F, G in S , there exists open sets U, V such that $F \subset U$ and $G \subset V$.

Proposition 5.5. Any subspace A of a Hausdorff space S is Hausdorff.

Proof. Let x and y be distinct points in A . Since S is Hausdorff, there exist disjoint open sets U and V in S that contain x and y respectively. Then, $U \cap A$ and $V \cap A$ are disjoint open sets in A that contain x and y respectively. \square

5.6 Product Topology

Definition 5.6. (Product topology) Given two topological spaces X and Y , let us consider the collection \mathcal{B} of subsets of $X \times Y$, the cartesian product of X and Y , of the form $U \times V$ (called basic open sets), where U and V are open sets in X and Y respectively. It is easy to see that \mathcal{B} forms a topology for $X \times Y$. In fact, from Proposition 5.2, \mathcal{B} satisfies the conditions for a basis and generates a topology on $X \times Y$. This topology is called the *product topology*.

Proposition 5.6. Let $\{A_i\}$ and $\{B_j\}$ be bases for topological spaces X and Y . Then, $\{A_i \times B_j\}$ is a basis for $X \times Y$.

Proof. Let W be an open set in $X \times Y$ and a point $p = (x, y) \in W$. Then, there exist open sets $U \in X$ and $V \in Y$ such that $p \in U \times V$, with $U \times V \subset W$ a basic open set. Thus, $x \in A_i \subset U$ and $y \in B_j \subset V$ for some i, j and $p = (x, y) \in A_i \times B_j \subset U \times V \subset W$. \square

Corollary 5.7. The product of two second-countable spaces is second countable.

Proposition 5.8. The product of two Hausdorff spaces X and Y is Hausdorff.

Proof. Given any two distinct points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ in $X \times Y$. WLOG, assume that $x_1 \neq x_2$. Since X is Hausdorff, there exist disjoint open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$. Then, $U_1 \times Y$ and $U_2 \times Y$ are two disjoint neighborhoods of p_1 and p_2 , so $X \times Y$ is Hausdorff. \square

5.7 Continuity

Definition 5.7. Let $f : X \rightarrow Y$ be a function of topological spaces. We say that f is *continuous* at a point p in X if for every neighborhood V of $f(p)$ in Y , there is a neighborhood U of p such that $f(U) \subset V$. We say that f is *continuous on X* if it is continuous at every point in X .

Proposition 5.9. A function $f : X \rightarrow Y$ is continuous if and only if the inverse image (preimage) of any open set is open.

Proof. Suppose f is continuous. Let V be an open set in Y and let $p \in f^{-1}(V)$. Then, $f(p) \in V$, and by definition of continuity of f , there is a neighborhood U of p such that $f(U) \subset V$. Thus, $p \in U \subset f^{-1}(V)$, so $f^{-1}(V)$ is open.

Conversely, let $p \in X$ and V be a neighborhood of $f(p)$. Then, $f^{-1}(V)$ is open in X and $f^{-1}(V)$ is a neighborhood of p . Thus, $U = f^{-1}(V)$ is a neighborhood of p such that $f(U) = f(f^{-1}(V)) \subset V$, so f is continuous at p . \square

Proposition 5.10. The composition of continuous maps is continuous.

Proof. Exercise. \square

5.8 Compactness

Definition 5.8. Let S be a topological space. A collection $\{U_\alpha\}$ of open sets in S is said to *cover S* or to be an *open cover* of S if $S \subset \bigcup_\alpha U_\alpha$. A *subcover* of an open cover is a subcollection whose union still contains S . The topological space S is then said to be *compact* if every open cover of S has a finite subcover.

Proposition 5.11. A subspace A of a topological space S is compact if and only if every open cover of A has a finite subcover.

Proof. Suppose A is compact and let a collection of open sets in S , $\{U_\alpha\}$, be an open cover of A . Clearly, $\{U_\alpha \cap A\}$ is also an open cover of A , and since A is compact, it has a finite subcover given by $\{U_i \cap A\}_{i=1}^n$. Thus, $\{U_i\}_{i=1}^n$ is a finite subcover of A . The converse is easily seen from the definition of compactness. \square

Proposition 5.12. A closed subset F of a compact topological space S is compact.

Proof. Let $\{U_\alpha\}$ be an open cover of F in S . Clearly, $S \subset (\bigcup_\alpha U_\alpha) \cup \bar{f} = \bigcup_\alpha (U_\alpha \cup \bar{f})$. On the other hand, F is closed, so \bar{f} is open, and thus $\{U_\alpha \cup \bar{f}\}$ is an open cover for S . By compactness, there exists a finite subcover $\{U_i \cup \bar{f}\}_{i=1}^n$ for S . Hence, $\{U_i\}_{i=1}^n$ is a finite subcover for F and F is compact. \square

Proposition 5.13. The image of a compact set under a continuous map is compact.

Proof. Let $f : X \rightarrow Y$ be a continuous map and K a compact subset of X . Suppose $\{V_\alpha\}$ is an open cover of $f(K)$, with V_α open sets in Y . By continuity of f , the preimages $f^{-1}(V_\alpha)$ are all open. Additionally,

$$K \subset f^{-1}(f(K)) \subset f^{-1}\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha f^{-1}(V_\alpha).$$

Thus, $\{f^{-1}(V_\alpha)\}$ is an open cover of K . By the compactness of K , there is a finite subcover $\{f^{-1}(V_i)\}_{i=1}^n$ of K , so

$$K \subset \bigcup_i f^{-1}(V_i) = f^{-1}\left(\bigcup_i V_i\right).$$

Therefore, $f(K) \subset \bigcup_i V_i$, so $f(K)$ has a finite subcover, and thus is compact. \square

Theorem 5.14. (The Tychonoff Theorem). The product of any collection of compact spaces is compact in the product topology.

5.9 Boundedness in \mathbb{R}^n

Definition 5.9. A subset A of \mathbb{R}^n is said to be *bounded* if it is contained in some open ball $B(p, r)$; it is *unbounded* otherwise.

Lemma 5.15. A subset A of \mathbb{R}^n is bounded if and only if for any $p \in \mathbb{R}^n$, there exists $r \in \mathbb{R}^+$ such that $A \subset B(p, r)$

Proof. Suppose A is bounded. By definition, there exists $p_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{R}^+$ such that $A \subset B(p_0, r_0)$. Now, let p be any point in \mathbb{R}^n and $r \in \mathbb{R}^+$ be given by $r = \|p - p_0\| + r_0$. We claim that $B(p_0, r_0) \subset B(p, r)$. Indeed, for any point $x \in B(p_0, r_0)$, we have

$$d(x, p) = \|x - p\| = \|x - p_0 + p_0 - p\| \leq \|x - p_0\| + \|p_0 - p\| < r_0 + \|p - p_0\| = r.$$

Thus, $A \subset B(p, r)$. The converse is clearly true, so we are done. \square

Corollary 5.16. A subset A of \mathbb{R}^n is bounded if and only if for any $p \in \mathbb{R}^n$, there exists a hypercube centered at p that contains A .

Proposition 5.17. The closed interval $[a, b] \subset \mathbb{R}$ is compact.

Proof. Let $\mathcal{A} = \{U_\alpha\}$ be an open cover for $[a, b]$ and let S be the set of points $y \in (a, b]$ such that the interval $[a, y]$ can be covered by a finite number of elements of \mathcal{A} .

First, we claim that S is non-empty; that is, there exists at least one element $y \in (a, b]$ such that $[a, y]$ has a finite cover. Choose an element U_{α_0} that contains a . Since U_{α_0} is open, $U_{\alpha_0} \supset [a, y]$ for some $y \in (a, b]$. Again, choose an element U_{α_1} that contains y . Then, $[a, y] = [a, y) \cup \{y\} \subset U_{\alpha_0} \cup U_{\alpha_1}$, so $y \in S$.

Now, we claim that $c = \sup S \in S$. Again, choose an element $U_c \in \mathcal{A}$ that contains c . Since U_c is open, $U_c \supset (d, c]$ for some $d \in (a, c)$. If $c \notin S$, then there exists a point $p \in S$ that is in (d, c) , for otherwise, $d < c = \sup S$ would be an upper bound on S . Since $p \in S$, $[a, p]$ can be covered by at most n elements of \mathcal{A} . This subcollection, in addition to U_c , would then admit a finite cover of $[a, c]$, contradicting the assumption that $c \notin S$.

Finally, we will show that $c = b$, thus proving the proposition. Suppose $c < b$ and let S' be the set of points $y' \in (c, b]$ such that $[c, y']$ can be covered by a finite number of elements of \mathcal{A} . Similar to S , S' is non-empty, so there is an element $y' \in (c, b]$ such that $[c, y']$ can be covered by a finite number of elements of \mathcal{A} . This implies that

$$[a, y'] = [a, c] \cup [c, y']$$

can also be finitely covered, contradicting the fact that c is an upper bound on S . \square

Corollary 5.18. The n -hypercube given by $A = [a_1, a_1 + d] \times \cdots \times [a_n, a_n + d]$ is compact.

Theorem 5.19. (The Heine-Borel Theorem). A subset of \mathbb{R}^n is compact if and only if it is closed and bounded.

Proof. Let K be a compact subset of \mathbb{R}^n . We first prove that K is closed by showing its complement, \overline{K} , is open.

Let $x \in \overline{K}$. For every point $y_\alpha \in K$, construct an open ball $B(x, d_\alpha)$, where $d_\alpha = \|y_\alpha - x\|$. By construction, $y_\alpha \notin B(x, d_\alpha)$ so $y_\alpha \in \overline{B(x, d_\alpha)}$ for every $y_\alpha \in K$. Thus, $K \subset \bigcup_\alpha \overline{B(x, d_\alpha)}$, so $\{\overline{B(x, d_\alpha)}\}$ is an open cover for K . By the compactness of K , there exists a finite subcover $\{\overline{B(x, d_i)}\}_{i=1}^n$ for K , with $d_i \leq d_j$ whenever $i < j$. Notice that

$$\overline{B(x, d_1)} \supset \cdots \supset \overline{B(x, d_n)},$$

So $K \subset \overline{B(x, d_1)}$, and thus $x \in B(x, d_1) \subset \overline{K}$. This implies that \overline{K} is open, so K is closed. Now, let us prove K is bounded. Let $r_0 \in K$ and for every $m \in \mathbb{Z}^+$, construct an open ball $B(r_0, m)$. Notice that $\{B(r_0, m)\}_{m=1}^\infty$ is an open cover for \mathbb{R} , so it is an open cover for K . By the compactness of K , there exists a finite subcover $\{B(r_0, m)\}_{m \in A}$, where A is some subset of \mathbb{Z}^+ . By the well-ordering principle, there is a largest element m_0 in A , i.e. $B(r_0, m) \subset B(r_0, m_0)$ for every $m \in A - \{m_0\}$. Thus, $K \subset B(r_0, m_0)$, so K is bounded.

The converse is a direct consequence of Corollary 5.16, Corollary 5.18, and Proposition 5.12. \square

5.10 Connectedness

Definition 5.10. A topological space S is *disconnected* if it is a union of two disjoint non-empty open sets. It is *connected* if it is not disconnected. A subset A of S is *disconnected* if it is disconnected in the subspace topology.

Proposition 5.20. A subset A of a topological space S is disconnected if and only if there are open sets U and V in S such that

- (i) $U \cap A \neq \emptyset, V \cap A \neq \emptyset$,
- (ii) $U \cap V \cap A = \emptyset$,
- (iii) $A \subset U \cup V$.

A pair of open sets in S with these properties is called a *separation* of A .

Proof. Suppose A is disconnected and $A = U \cup V$ with $U \cap V = \emptyset$. By definition, $U = U' \cap A$ and $V = V' \cap A$ for some open sets U', V' in S . It is easy to see that U' and V' satisfy the above conditions.

Now, suppose there are open sets U and V in S that satisfy the above conditions. Condition (i) tells us A is nonempty. Then, from condition (iii), $A = (U \cap A) \cup (V \cap A)$. From condition (ii), we also have $(U \cap A) \cap (V \cap A) = \emptyset$, so we are done. \square

Proposition 5.21. The image of a connected space X under a continuous map $f : X \rightarrow Y$ is connected.

Proof. Suppose $f(X)$ is disconnected. Then, there are open sets U, V in Y that separate $f(X)$. By Proposition 5.9, $f^{-1}(U)$ and $f^{-1}(V)$ are open in X . Clearly, $f^{-1}(U) \cup f^{-1}(V) = X$. We also have $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U) \cap f^{-1}(V) \cap X = f^{-1}(U) \cap f^{-1}(V) \cap f^{-1}(A) = f^{-1}(U \cap V \cap A) = f^{-1}(\emptyset) = \emptyset$, so X is disconnected, which is a contradiction. \square

Proposition 5.22. The union of a collection of connected subsets A_α of a topological space S having a common point p is connected.

Proof. Suppose $\bigcup_\alpha A_\alpha = U \cup V$, where U and V are two disjoint non-empty open subsets of $\bigcup_\alpha A_\alpha$, and suppose without loss of generality that $p \in U$. Now, let $p' \in V$. Then, $p' \in A_{\alpha_0}$ for some α_0 . Clearly, $\{U, V\}$ separates A_{α_0} , which contradicts the connectedness of A_{α_0} . Thus, $\bigcup_\alpha A_\alpha$ is connected. \square

5.11 Connected Components

Definition 5.11. Let x be a point in a topological space S . Then, the union of all connected subsets of S containing x , denoted C_x , is called the *connected component* of S containing x .

Remark. By Proposition 5.22, C_x is connected.

Proposition 5.23. Let C_x be a connected component of x of a topological space S . Then, a connected subset A of S is either disjoint from C_x or is contained entirely in C_x .

Proof. Suppose A and C_x have a point in common. Then, by Proposition 5.22, $A \cup C_x$ is connected, and thus is a connected set containing x . Hence, $A \cup C_x \subset C_x$, so $A \subset C_x$. \square

Corollary 5.24. For any two points x, y in a topological space S , the connected components C_x and C_y are either disjoint or coincide.

Proof. Obvious. \square

5.12 Closure

Definition 5.12. Let S be a topological space and A a subset of S . The *closure* of A in S , denoted by $\text{cl}(A)$ or $\text{cl}_S(A)$, is defined to be the intersection of all the closed sets containing A .

Remark. As an intersection of closed sets, $\text{cl}(A)$ is a closed set. It is the smallest closed set containing A in the sense that any closed set containing A must contain $\text{cl}(A)$.

Proposition 5.25. (Local characterization of closure). Let A be a subset of a topological space S . A point $p \in S$ is in the closure $\text{cl}(A)$ if and only if every neighborhood of p contains a point of A .

Proof. We will proceed by contraposition. Let $p \in S$ and suppose $p \notin \text{cl}(A)$. We want to show that there is a neighborhood of p that is disjoint from A . Indeed, since $p \notin \text{cl}(A)$, there is a closed set C containing A that does not contain p . Thus, $p \in \overline{C^c}$, an open set disjoint from A .

Now, suppose there is a neighborhood of p that is disjoint from A and let U be such a neighborhood. Then, \overline{U} is a closed set in S that contains A but does not contain p . Thus, $p \notin \text{cl}(A)$. \square

Definition 5.13. A point p in a topological space S is an *accumulation point* or a *limit point* of a subset A of S if every neighborhood of p in S contains a point of A other than p . The set of all accumulation points of A is denoted by $\text{ac}(A)$.

If U is a neighborhood of p in S , the set $U - \{p\}$ is called a *deleted neighborhood* of p . Notice that the above definition is equivalent to saying that every deleted neighborhood of p contains a point of A .

Example. If $A = [0, 1) \cup \{2\}$ in \mathbb{R} , then the closure of A is $[0, 1] \cup \{2\}$, but the set of accumulation points of A is only the closed interval $[0, 1]$.

Proposition 5.26. Let A be a subset of a topological space S . Then,

$$\text{cl}(A) = A \cup \text{ac}(A)$$

Proof. By definition, $A \subset \text{cl}(A)$. On the other hand, by Proposition 5.25, $\text{ac}(A) \subset \text{cl}(A)$, so $A \cup \text{ac}(A) \subset \text{cl}(A)$.

Now, suppose $p \in \text{cl}(A)$ and $p \notin A$. Again, by Proposition 5.25, a deleted neighborhood $U - \{p\}$ of p must contain a point of A (since $p \notin A$). By definition, p is an accumulation point of A and thus $p \in \text{ac}(A)$. \square

Proposition 5.27. A set A is closed if and only if $A = \text{cl}(A)$.

Proof. If $A = \text{cl}(A)$ then since $\text{cl}(A)$ is closed, A is closed. Conversely, if A is closed, then A is a closed set containing A , so $\text{cl}(A) \subset A \subset \text{cl}(A)$. Thus, $A = \text{cl}(A)$. \square

Proposition 5.28. If $A \subset B$ in a topological space S , then $\text{cl}(A) \subset \text{cl}(B)$.

Proof. By definition, $\text{cl}(B)$ contains B , and thus contains A . Since $\text{cl}(B)$ is a closed set containing A , it must contain $\text{cl}(A)$. \square

5.13 Convergence

Let us recall the definition of a *sequence*:

Definition 5.14. Let S be a topological space. A *sequence* in S is a map from the set \mathbb{Z}^+ of positive integers to S . We write a sequence as $\langle x_i \rangle$ or $x_1, x_2, x_3 \dots$.

Then, convergence of a sequence in a topological space can be defined as follows:

Definition 5.15. Let S be a topological space and $\langle x_i \rangle$ be a sequence in S . The sequence $\langle x_i \rangle$ *converges* to p if for every neighborhood U of p , there is a positive integer N such that for all $i \geq N$, $x_i \in U$. We say that p is a *limit* of the sequence $\langle x_i \rangle$ and write $x_i \rightarrow p$ or $\lim_{i \rightarrow \infty} x_i = p$.

Proposition 5.29. (Uniqueness of the limit). In a Hausdorff space S , if a sequence $\langle x_i \rangle$ converges to p and to q , then $p = q$.

Proof. Let p and q be limits of a sequence $\langle x_i \rangle$ in S and suppose $p \neq q$. Since S is Hausdorff, there exists disjoint open sets U and V in S that contains p and q respectively. By the definition of a limit, there are positive integers M and N such that for $i \geq M$ and $j \geq N$, $x_i \in U$ and $x_j \in V$. This implies that for all $k \geq \max(M, N)$, $x_k \in U \cap V = \emptyset$. By contradiction, $p = q$. \square

Proposition 5.30. (The sequence lemma). Let S be a topological space and A a subset of S . If there is a sequence $\langle a_i \rangle$ in A that converges to p , then $p \in \text{cl}(A)$. The converse is also true if S is first-countable.

Proof. Suppose $a_i \rightarrow p$ with $a_i \in A$ for all i . By the definition of convergence, every neighborhood U of p must contain all but finitely many points of $\langle a_i \rangle$. In particular, U must contain a point in A , so $p \in \text{cl}(A)$ by Proposition 5.25.

Conversely, suppose $p \in \text{cl}(A)$. Since S is first-countable, there is a countable basis of neighborhoods $\{U_i\}$ around p such that

$$U_1 \supset U_2 \supset \cdots.$$

By Proposition 5.25, there is a point $a_i \in A$ in each neighborhood U_i . We claim that the sequence $\langle a_i \rangle$ is a sequence that converges to p , since given any neighborhood U of p , there exists a positive integer N such that $p \in U_N \subset U$. Then, for all $i \geq N$, we have

$$a_i \in U_i \subset U_N \subset U,$$

so p is a limit of $\langle a_i \rangle$. □

Problems

Problem 5.1. Closed sets

Let S be a topological space. Prove the following two statements.

- (a) If $\{F_i\}_{i=1}^n$ is a finite collection of closed sets in S , then $\bigcup_{i=1}^n F_i$ is closed.
- (b) If $\{F_\alpha\}$ is arbitrary collection of closed sets in S , then $\bigcap_\alpha F_\alpha$ is closed.

Solution. A closed set F_i in S is the complement of some open set U_i in S . Thus, given a finite collection of closed sets, $\{F_i\}_{i=1}^n$, we have

$$\bigcup_{i=1}^n F_i = \bigcup_{i=1}^n \overline{U_i} = \overline{\bigcap_{i=1}^n U_i},$$

so $\{F_i\}_{i=1}^n$ is closed.

Similarly, given an arbitrary collection of closed sets $\{F_\alpha\}$, we have

$$\bigcap_\alpha F_i = \bigcap_\alpha \overline{U_i} = \overline{\bigcup_\alpha U_i},$$

so $\{F_i\}_{i=1}^n$ is closed.

Problem 5.2. The $\varepsilon - \delta$ criterion for continuity

Prove that a function $f : A \rightarrow \mathbb{R}^n$ is continuous at $p \in A$ if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in A$ satisfying $d(x, p) < \delta$, one has $d(f(x), f(p)) < \varepsilon$.

Proof. Suppose f is continuous at $p \in A$. Let $\varepsilon > 0$ and $B(f(p), \varepsilon)$ be an open ball centered at $f(p)$. By the continuity of f at p , there exists a neighborhood U of p such that $f(U) \subset B(f(p), \varepsilon)$. Since open balls in \mathbb{R}^n are a basis for \mathbb{R}^n , there exists an open ball $B(p, \delta) \in \mathbb{R}^n$ such that $B(p, \delta) \cap A \subset U$ via the subspace topology. Thus, $f(B(p, \delta) \cap A) \subset f(U) \subset B(f(p), \varepsilon)$, which is what we need.

Now, let us prove the converse. The statement can be equivalently stated in "topological terms" as follows: for any $\varepsilon > 0$, one can construct an open ball $B(p, \delta)$ such that $f(B(p, \delta)) \subset B(f(p), \varepsilon)$. This immediately implies the continuity of f since for any neighborhood V , we can choose ε such that $B(f(p), \varepsilon) \subset V$, after which the above argument follows. □

Problem 5.3. Let A and B be subsets of a topological space S . Prove the following:

- (a) $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$ (the closure of the union is the union of the closure), and
- (b) $\text{cl}(A \cap B) \subset \text{cl}(A) \cap \text{cl}(B)$ (the closure of the intersection is contained in the intersection of the closure).

Proof.

- (a) Since $\text{cl}(A \cup B)$ is a closed set containing $A \cup B$, and thus containing A and B , it contains $\text{cl}(A)$ and $\text{cl}(B)$. Conversely, let $p \in \text{cl}(A \cup B)$. If $p \in A \cup B$ then p is in either in A or B , and thus in $\text{cl}(A) \cup \text{cl}(B)$. If $p \notin A \cup B$ then $p \in \text{ac}(A \cup B)$, and thus every deleted neighborhood of p contains a point p' of $A \cup B$. Thus, p' is either in A or B , so $p \in \text{cl}(A) \cup \text{cl}(B)$.
- (b) Observe that $\text{cl}(A)$ and $\text{cl}(B)$ are both closed sets containing $A \cap B$, so their intersection contains $\text{cl}(A \cap B)$

□

6 Manifolds

6.1 Topological Manifolds

Definition 6.1. A topological space M is *locally Euclidean of dimension n* if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of \mathbb{R}^n . We call the pair $(U, \phi : U \rightarrow \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood* or a *coordinate open set*, and ϕ a *coordinate map* or a *coordinate system* on U . We say that a chart (U, ϕ) is *centered* at $p \in U$ if $\phi(p) = 0$.

Definition 6.2. A *topological manifold* is a Hausdorff, second countable, locally Euclidean space. It is said to be of *dimension n* if it is locally Euclidean of dimension n .

Remark. For any $n \neq m$, an open subset of \mathbb{R}^n is not homeomorphic to an open subset of \mathbb{R}^m . This result is called *invariance of dimension*, and it will be proven later on for *smooth* manifolds. The general result is not covered in this book.

Example. The Euclidean space \mathbb{R}^n is covered by a single chart $(\mathbb{R}^n, \mathbf{1})$, where $\mathbf{1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the identity map. Every open subset U of \mathbb{R}^n is also a topological manifold, with chart $(U, \mathbf{1}_U)$.

Remark. Recall that a subspace of a Hausdorff space is Hausdorff and a subspace of a second countable space is also second countable. Thus, any subspace of \mathbb{R}^n is automatically Hausdorff and second countable.

Example. The graph of $y = x^{2/3}$ in \mathbb{R}^2 is a topological manifold. Since it is a subspace of \mathbb{R}^2 , it is Hausdorff and second countable. It is also locally Euclidean, for it is homeomorphic to \mathbb{R} via the coordinate map $(x, x^{2/3}) \mapsto x$.

6.2 Compatible Charts

Definition 6.3. Two charts $(U, \phi : U \rightarrow \mathbb{R}^n)$ and $(V, \Psi : V \rightarrow \mathbb{R}^n)$ of a topological manifold are C^∞ -compatible (or *compatible* for short) if the two maps

$$\phi \circ \Psi^{-1} : \Psi(U \cap V) \rightarrow \phi(U \cap V), \quad \Psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \Psi(U \cap V)$$

are C^∞ . These two maps are called the *transition functions* between the charts. If $U \cap V$ is empty, then the two charts are automatically C^∞ -compatible. To simplify notations, we can write $U_{\alpha\beta}$ for $U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma}$ for $U_\alpha \cap U_\beta \cap U_\gamma$.

Definition 6.4. A C^∞ atlas or simply an *atlas* on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞ -compatible charts that cover M , i.e. such that $\bigcup_\alpha U_\alpha = M$. We say that a chart (V, Ψ) is *compatible with an atlas* $\{(U_\alpha, \phi_\alpha)\}$ if it is compatible with all the charts (U_α, ϕ_α) in the atlas.

Lemma 6.1. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space. If two charts (V, Ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.

Proof. For any $p \in V \cap W$, $p \in U_\alpha$ for some α . By construction, the maps $\Psi \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \sigma^{-1}$ are C^∞ on $U_\alpha \cap V$ and $U_\alpha \cap W$ respectively. Thus, the map $\Psi \circ \sigma^{-1} = (\Psi \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \sigma^{-1})$ is C^∞ on $U_\alpha \cap V \cap W$. Since this is true for any $p \in V \cap W$, $\Psi \circ \sigma^{-1}$ is C^∞ on $V \cap W$, and the same is true for $\sigma \circ \Psi^{-1}$. \square

6.3 Smooth Manifolds

Definition 6.5. A *smooth* or C^∞ manifold is a topological manifold M together with a maximal atlas. The maximal atlas is also called a *differentiable structure* on M . A manifold is said to have dimension n if all of its connected components have dimension n . A 1-dimensional manifold is also called a *curve*, a 2-dimensional manifold a *surface*, and an n -dimensional manifold an *n -manifold*.

To check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of any atlas on M will do, for we have the following proposition.

Proposition 6.2. Any atlas $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ on a locally Euclidean space is contained in a unique maximal atlas.

Proof. For any atlas \mathfrak{U} , adjoin to \mathfrak{U} all charts that are compatible with \mathfrak{U} . By Lemma 6.1, these charts are all compatible with one another, so the new collection of charts \mathfrak{M} is an atlas. It is also maximal, for any chart compatible with it must also be compatible with \mathfrak{U} , and thus lies in \mathfrak{M} . \mathfrak{M} is also unique, for suppose there is another maximal atlas \mathfrak{M}' that contains \mathfrak{U} then any chart in \mathfrak{M}' must be compatible with \mathfrak{U} . By construction, this implies $\mathfrak{M}' \subset \mathfrak{M}$, and thus $\mathfrak{M}' = \mathfrak{M}$. \square

In summary, to show a topological space M is a smooth manifold, it suffices to check that

- (i) M is Hausdorff and second countable, and
- (ii) M has a (not necessarily maximal) C^∞ atlas.

6.4 Examples of Smooth Manifolds

Example. (Euclidean space). The Euclidean space \mathbb{R}^n is a smooth manifold with a single chart $\{(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})\}$.

Example. Any open subset U of a manifold M is also a manifold, for if $\{U_\alpha, \phi_\alpha\}$ is an atlas for M then $\{U_\alpha \cap U, \phi_\alpha|_{U_\alpha \cap U}\}$ is an atlas for U .

Example. (Graph of a smooth function). For a subset $A \subset \mathbb{R}^n$ and a function $f : A \rightarrow \mathbb{R}^m$, the graph of f is defined to be the subset

$$\Gamma(f) = \{(x, f(x)) \in A \times \mathbb{R}^m\}.$$

If U is an open subset of \mathbb{R}^n and $f : U \rightarrow \mathbb{R}^m$ is C^∞ , then the two maps

$$\begin{aligned} \phi : \Gamma(f) &\rightarrow U, & (1, f) : U &\rightarrow \Gamma(f), \\ (x, f(x)) &\mapsto x, & x &\mapsto (x, f(x)), \end{aligned}$$

are continuous and inverse to each other, and thus are homeomorphisms. The graph $\Gamma(f)$ of a C^∞ function $f : U \rightarrow \mathbb{R}^m$ has a single chart $\{(\Gamma(f), \phi)\}$, so it is a C^∞ manifold.

Example. (General linear groups). The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ is a manifold, for it is the preimage of the determinant function $\det : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, which is continuous, and thus an open subset of $\mathbb{R}^{n \times n} \simeq \mathbb{R}^{n^2}$. Similarly, the *complex general linear group* $\mathrm{GL}(n, \mathbb{C})$ is an open subset of $\mathbb{C}^{n \times n} \simeq \mathbb{R}^{2n^2}$, so it is a manifold of dimension $2n^2$.

Proposition 6.3. If $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_i, \Psi_i)\}$ are C^∞ atlases for the manifolds M and N of dimension m and n respectively, then the collection

$$\{(U_\alpha \times V_i, \phi_\alpha \times \Psi_i : U_\alpha \times V_i \rightarrow \mathbb{R}^m \times \mathbb{R}^n)\}$$

is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a smooth manifold of dimension $m + n$.

Proof. It suffices to show that any two charts $\{(U_\alpha \times V_i, \phi_\alpha \times \Psi_i)\}$ and $\{(U_\beta \times V_j, \phi_\beta \times \Psi_j)\}$ are compatible. Since ϕ_α and Ψ_i are homeomorphisms, the product map $\phi_\alpha \times \Psi_i : (x, y) \mapsto (f(x), f(y)) \in \mathbb{R}^m \times \mathbb{R}^n$ is also a homeomorphism, and thus so is its inverse. Thus, $(\phi_\alpha \times \Psi_i)(\phi_\beta \times \Psi_j)^{-1}$ and $(\phi_\beta \times \Psi_j)(\phi_\alpha \times \Psi_i)^{-1}$ are homeomorphisms, so the product manifold is a smooth manifold of dimension $m + n$. \square

Problems

Problem 6.1. A sphere with hair. Show that a sphere with hair (**Figure 6.1**) is *not* a topological manifold by showing it is not locally Euclidean at q .

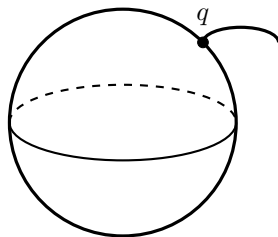


Figure. 6.1: A sphere with hair.

Solution. Suppose the sphere with hair is locally Euclidean of dimension n at q . Then, q has a neighborhood U homeomorphic to an open ball $B := B(0, r) \subset \mathbb{R}^n$. Then, the homeomorphism $\phi : U \rightarrow B$ restricts to the homeomorphism $\phi : U - \{q\} \rightarrow B - \{0\}$. Since $U - \{q\}$ has two connected components B must be an open ball of dimension $n = 1$. This implies $U - \{q\}$ also consists of two connected components of dimension 1, contradicting the invariance of dimension since the sphere with the point q removed still has dimension 3.

Problem 6.2. Existence of a coordinate neighborhood. Let $\mathfrak{M} = \{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas on a manifold M . For any open set U in M and a point $p \in U$, prove the existence of a coordinate open set U_α such that $p \in U_\alpha \subset U$.

Solution. Since $\{U_\alpha\}$ covers \mathfrak{M} , choose a chart (U_p, ϕ_p) such that $p \in U_p$. Then, since \mathfrak{M} is maximal, the chart $(U_p \cap U, \phi_p|_{U_p \cap U})$ is contained in \mathfrak{M} and that $p \in U_p \cap U \subset U$.

7 Smooth Maps on a Manifold

7.1 Smooth Functions on a Manifold

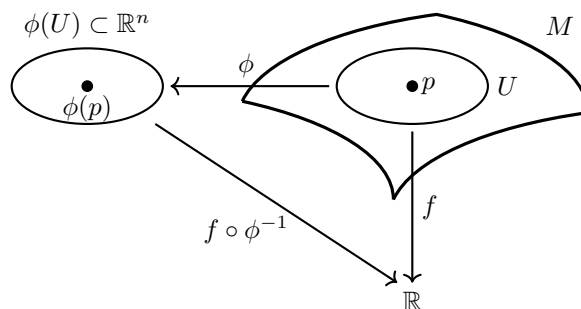


Figure. 7.1: Checking if a function f is C^∞ at p by pulling back to \mathbb{R}^n .

Definition 7.1. Let M be a manifold of dimension n . A function $f : M \rightarrow \mathbb{R}$ is said to be C^∞ or *smooth* at a point p in M if there is a chart (U, ϕ) about p in M such that the function $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is C^∞ at $\phi(p)$ (see **Figure 7.1**). The function f is then said to be C^∞ on M if it is C^∞ at every point in M .

Remark. The definition of the smoothness of a function f at a point is independent of the chart (U, ϕ) , for if $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$ and (V, Ψ) is any other chart about p in M then on $\Psi(U \cap V)$,

$$f \circ \Psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \Psi^{-1}),$$

which is C^∞ at $\Psi(p)$ through the transition function $\phi \circ \Psi^{-1}$.

Notice that in the above definition, we did not assume the continuity of f . However, if f is C^∞ at p in M , then $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$, being a C^∞ function at $\phi(p)$ in an open subset U of \mathbb{R}^n , is continuous at $\phi(p)$. Thus, $f = (f \circ \phi^{-1}) \circ \phi$ is continuous at p . Since we are only interested in smooth functions on open sets, there is no loss of generality when we assume that f is continuous.

Proposition 7.1. (Smoothness of a real-valued function). Let M be a manifold of dimension n , and $f : M \rightarrow \mathbb{R}$ a real-valued function on M . Then, the following are equivalent:

- (i) The function $f : M \rightarrow \mathbb{R}$ is C^∞ .
- (ii) The manifold M has an atlas such that for every chart (U, ϕ) in the atlas, $f \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ is C^∞ .
- (iii) For every chart (V, ϕ) on M , the function $f \circ \Psi^{-1} : \Psi(V) \rightarrow \mathbb{R}$ is C^∞ .

Proof. We will prove the proposition through the following implications:

(ii) \Rightarrow (i): This follows directly from the definition above.

(i) \Rightarrow (iii): From the above remark, for any point $p \in V$, $f \circ \Psi^{-1}$ is C^∞ at $\Psi(p)$, so it is smooth on $\Psi(V)$.

(iii) \Rightarrow (ii): Obvious. □

Definition 7.2. Let $F : N \rightarrow M$ be a map and h a function on M . The *pullback* of h by F , denoted F^*h , is the composition $h \circ F$.

In this terminology, a function f on M is C^∞ on a chart (U, ϕ) if and only if its pullback $(\phi^{-1})^* f$ by ϕ^{-1} is C^∞ on the subset $\phi(U)$ of \mathbb{R}^n .

7.2 Smooth Maps Between Manifolds

Unless otherwise specified, a manifold will always mean a C^∞ manifold. A manifold M will always have dimension n , unless we are talking about two different manifolds N and M simultaneously, in which case N will have dimension n and M dimension m .

Definition 7.3. Let N and M be manifolds of dimension n and m , respectively. A continuous map $F : N \rightarrow M$ is C^∞ at a point p in N if there are charts (V, Ψ) about $F(p)$ in M and (U, ϕ) about p in N such that the composition $\Psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \mathbb{R}^m$, is C^∞ at $\phi(p)$ (see **Figure 7.2**). The continuous map $F : N \rightarrow M$ is then said to be C^∞ if it is C^∞ at every point of N .

Notice that in the above definition, the map $F : N \rightarrow M$ is assumed to be continuous. This is to ensure that $F^{-1}(V)$ is an open set in N . Thus, smooth maps between manifolds are by definition

$$\begin{array}{ccc}
 N \supset U & \xrightarrow{F} & V \subset M \\
 \downarrow \phi & & \downarrow \Psi \\
 \mathbb{R}^n \supset \phi(U) & \xrightarrow{\Psi \circ F \circ \phi^{-1}} & \Psi(V) \subset \mathbb{R}^m
 \end{array}$$

Figure. 7.2: Checking the map $F : N \rightarrow M$ is C^∞ at p .

continuous.

Remark. (Smooth maps into \mathbb{R}^m). In the case $M = \mathbb{R}^m$, we can take $(\mathbb{R}^m, \mathbf{1})$ to be the chart about p in \mathbb{R}^m . According to the above definition, $F : N \rightarrow \mathbb{R}^m$ is C^∞ at $p \in N$ if and only if there is a chart (U, ϕ) about p such that $F \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^m$ is C^∞ at $\phi(p)$. Letting $m = 1$, we recover Definition 7.1 of a function being C^∞ at a point.

Similar to how the smoothness of a function $N \rightarrow \mathbb{R}$ at a point p in N is independent of the choice of a chart on N about p , the definition of the smoothness of a map $F : N \rightarrow M$ at a point is also independent of the choice of charts.

Proposition 7.2. Suppose $F : N \rightarrow M$ is C^∞ at $p \in N$. If (U, ϕ) is any chart about p in N and (V, Ψ) any chart about $F(p)$ in M , then $\Psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

Proof. Since F is C^∞ at $p \in N$, there are charts (U_α, ϕ_α) about p in N and (V_β, Ψ_β) about $F(p)$ in M such that $\Psi_\beta \circ F \circ \phi_\alpha^{-1}$ is C^∞ at $\phi_\alpha(p)$. By the C^∞ compatibility of charts, both $\phi_\alpha \circ \phi^{-1}$ and $\Psi \circ \Psi_\beta^{-1}$ are C^∞ on open subsets of Euclidean spaces. Thus, the composition

$$\Psi \circ F \circ \phi^{-1} = (\Psi \circ \Psi_\beta^{-1}) \circ (\Psi_\beta \circ F \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi^{-1})$$

is C^∞ at $\phi(p)$. □

Proposition 7.3. (Smoothness of a map in terms of charts). Let N and M be smooth manifolds, and $F : N \rightarrow M$ a continuous map. The following are equivalent:

- (i) The map $F : N \rightarrow M$ is C^∞ .
- (ii) There are atlases \mathfrak{U} for N and \mathfrak{V} for M such that for every chart (U, ϕ) in \mathfrak{U} and (V, Ψ) in \mathfrak{V} , the map

$$\Psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \mathbb{R}^m$$

is C^∞ .

- (iii) For every chart (U, ϕ) on N and (V, Ψ) on M the map

$$\Psi \circ F \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \mathbb{R}^m$$

is C^∞ .

Proof. Similar to Proposition 7.1, we will prove the following implications:

- (ii) \Rightarrow (i): This follows immediately from the definition of smooth maps between manifolds.
- (i) \Rightarrow (iii): From Proposition 7.2, for any charts (U, ϕ) on N and (V, Ψ) on M with $p \in U \cap F^{-1}(V) \neq \emptyset$, the composition $\Psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$, and thus is C^∞ on $\phi(U \cap F^{-1}(V))$.
- (iii) \Rightarrow (ii): Obvious. □

Proposition 7.4. (Composition of C^∞ maps). If $F : N \rightarrow M$ and $G : M \rightarrow P$ are C^∞ maps of manifolds, then $G \circ F : N \rightarrow P$ is also C^∞ .

Proof. Let (U, ϕ) , (V, Ψ) , and (W, σ) be charts on N , M , and P , respectively. Then, the composition

$$\sigma \circ (G \circ F) \circ \phi^{-1} = (\sigma \circ G \circ \Psi^{-1}) \circ (\Psi \circ F \circ \phi^{-1})$$

is a composition of C^∞ maps on open subsets of Euclidean spaces (by Proposition 7.3), and thus is also C^∞ . Thus, the composite map $G \circ F$ is C^∞ (also by Proposition 7.3). □

7.3 Diffeomorphisms

Definition 7.4. A *diffeomorphism* of manifolds is a bijective C^∞ map $F : N \rightarrow M$ whose inverse F^{-1} is also C^∞ .

Remark. According to the next two propositions, coordinate maps are diffeomorphisms. Conversely, every diffeomorphism of an open subset of a manifold with an open subset of an Euclidean space can serve as a coordinate map.

Proposition 7.5. If (U, ϕ) is a chart on a manifold M of dimension n , then the coordinate map $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.

Proof. By definition, ϕ is a homeomorphism, so it is bijective. Thus, it suffices to show that ϕ and ϕ^{-1} are C^∞ . Choose the atlases $\{(U, \phi)\}$ for U and $\{(\phi(U)), \mathbf{1}\}$ for $\phi(U)$. Since $\mathbf{1} \circ \phi \circ \phi^{-1} = \mathbf{1}$ is the

identity map, it is C^∞ . By Proposition 7.3, ϕ is C^∞ . Similarly, using the same atlases as above, we see that $\phi \circ \phi^{-1} \circ 1$ is also C^∞ . Again, by Proposition 7.3, ϕ^{-1} is C^∞ . \square

Proposition 7.6. Let U be an open subset of a manifold M of dimension n . If $F : U \rightarrow F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of \mathbb{R}^n , then (U, F) is a chart in the differentiable structure of M .

Proof. For any chart (U_α, ϕ_α) in the maximal atlas of M , both ϕ_α and ϕ_α^{-1} are C^∞ by Proposition 7.5. Thus, as compositions of C^∞ maps, both $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are C^∞ . Hence, (U, F) is compatible with the maximal atlas, and by the maximality of the atlas, (U, F) is also in the atlas. \square

7.4 Smoothness in Terms of Components

Proposition 7.7. (Smoothness of a vector-valued function). Let N be a manifold and $F : N \rightarrow \mathbb{R}^m$ a continuous map. Then, the following are equivalent:

- (i) The map $F : N \rightarrow \mathbb{R}^m$ is C^∞ .
- (ii) The manifold N has an atlas such that for every chart (U, ϕ) in the atlas, the map $F \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^m$ is C^∞ .
- (iii) For every chart (U, ϕ) on N , the map $F \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^m$ is C^∞ .

Proof. (ii) \Rightarrow (i): Take the atlas $\{(\mathbb{R}^m, 1)\}$ consisting of a single chart on \mathbb{R}^m . Then, by Proposition 7.3(ii), F is C^∞ .

(i) \Rightarrow (iii): Again, in Proposition 7.3(iii), let (V, Ψ) be the chart $(\mathbb{R}^m, 1)$ on \mathbb{R}^m .

(iii) \rightarrow (ii): Obvious. \square

Proposition 7.8. (Smoothness in terms of components). Let N be a manifold. A vector-valued function $F : N \rightarrow \mathbb{R}^m$ is C^∞ if and only if its component functions $F^1, \dots, F^m : N \rightarrow \mathbb{R}$ are all C^∞ .

Proof. The map $F : N \rightarrow \mathbb{R}^m$ is C^∞ .

\Leftrightarrow For every chart (U, ϕ) on N , the function $F \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^m$ is C^∞ (Proposition 7.7).

\Leftrightarrow For every chart (U, ϕ) on N , the functions $F^i \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}$ are all C^∞ (by the definition of smooth maps between Euclidean spaces).

\Leftrightarrow The component functions $F^i : N \rightarrow \mathbb{R}$ are all C^∞ (Proposition 7.1). \square

Example. (Smoothness of a map to a circle). The map $F : \mathbb{R} \rightarrow S^1, F(t) = (\cos t, \sin t)$ is C^∞ since the component functions $\cos t, \sin t$ are all C^∞ .

Proposition 7.9. (Smoothness of a map in terms of vector-valued functions). Let N and M be manifolds of dimension n and m , respectively, and $F : N \rightarrow M$ be a continuous map. Then, the following are equivalent:

- (i) The map $F : N \rightarrow M$ is C^∞ .
- (ii) The manifold M has an atlas such that for every chart $(V, \Psi) = (V, y^1, \dots, y^m)$ in the atlas, the vector-valued function $\Psi \circ F : F^{-1}(V) \rightarrow \mathbb{R}^m$ is C^∞ .
- (iii) For every chart $(V, \Psi) = (V, y^1, \dots, y^m)$ on M , the vector-valued function $\Psi \circ F : F^{-1}(V) \rightarrow \mathbb{R}^m$ is C^∞ .

Proof. (ii) \rightarrow (i): By Proposition 7.7(i) \Rightarrow (iii), for every chart (U, ϕ) on N , the map

$$(\Psi \circ F) \circ \phi^{-1} : \phi(U \cap F^{-1}(V)) \rightarrow \mathbb{R}^m$$

is C^∞ . Thus, by Proposition 7.3, $F : N \rightarrow M$ is C^∞ .

(i) \Rightarrow (iii): As a coordinate map, Ψ is C^∞ (Proposition 7.5). Thus, as a composition of C^∞ maps, $\Psi \circ F$ is C^∞ .

(iii) \Rightarrow (ii): Obvious. □

By Proposition 7.8, the smoothness criterion of a map in terms of vector-valued functions can be translated into the smoothness criterion in terms of components

Proposition 7.10. (Smoothness of a map in terms of components). Let $F : N \rightarrow M$ be a continuous map between two manifolds of dimension n and m , respectively. Then, the following are equivalent:

- (i) The map $F : N \rightarrow M$ is C^∞ .
- (ii) The manifold M has an atlas such that for every chart $(V, \Psi) = (V, y^1, \dots, y^m)$ in the atlas, the component coordinates $y^i \circ F : F^{-1}(V) \rightarrow \mathbb{R}$ of F relative to the chart are all C^∞ .
- (iii) For every chart $(V, \Psi) = (V, y^1, \dots, y^m)$ on M , the component coordinates $y^i \circ F : F^{-1}(V) \rightarrow \mathbb{R}$ of F relative to the chart are all C^∞ .

7.5 Examples of Smooth Maps

Example. (Smoothness of a projection map). Let M and N be manifolds and $\pi : M \times N \rightarrow M$, $\pi(p, q) = p$ be the projection to the first factor. Then, π is a C^∞ map.

Proof. Let (p, q) be an arbitrary point in $M \times N$ and suppose (U, ϕ) and (V, Ψ) are coordinate neighborhoods of p and q in M and N , respectively. By Proposition 6.3, $(U \times V, \phi \times \Psi)$ is a coordinate neighborhood of (p, q) . Observe that

$$(\phi \circ \pi \circ (\phi \times \Psi)^{-1})(\phi(p), \Psi(q)) = \phi(p)$$

is a C^∞ map of subsets of Euclidean spaces. Thus, π is C^∞ at (p, q) , and therefore is C^∞ on $M \times N$. □

Example. Let M_1 , M_2 , and N be manifolds of dimension m_1 , m_2 , and n respectively. Then, the map $(f_1, f_2) : N \rightarrow M_1 \times M_2$ is C^∞ if and only if the components $f_i : N \rightarrow M_i$ are both C^∞ .

Proof. $(f_1, f_2) : N \rightarrow M_1 \times M_2$ is C^∞

\Leftrightarrow For every chart $(V = V_1 \times V_2, \Psi = \Psi_1 \times \Psi_2) = (V, y^1, \dots, y^{m_1}, y^{m_1+1}, \dots, y^{m_1+m_2})$ on $M_1 \times M_2$, the component coordinates $y^i \circ (f_1, f_2) : (f_1, f_2)^{-1}(V) \rightarrow \mathbb{R}$ of (f_1, f_2) relative to the chart are all C^∞ (Proposition 7.10).

\Leftrightarrow The vector-valued functions $\Psi_1 \circ f_1 : f_1^{-1}(V_1) \rightarrow \mathbb{R}^{m_1}$ and $\Psi_2 \circ f_2 : f_2^{-1}(V_2) \rightarrow \mathbb{R}^{m_2}$ are both C^∞ (Definition of smooth maps between subsets of Euclidean spaces).

\Leftrightarrow The components $f_i : N \rightarrow M_i$ are both C^∞ (Proposition 7.9). \square

Now that we have the definition of a smooth map between manifolds, we can define a *Lie group*.

Definition 7.5. A *Lie group* is a C^∞ manifold G having a group structure such that the multiplication map

$$\mu : G \times G \rightarrow G$$

and the inverse map

$$\iota : G \rightarrow G, \iota(x) = x^{-1},$$

are both C^∞ .

Similarly, a *topological group* is a topological space having a group structure such that the multiplication map and the inverse map are both continuous. Note that a topological group need not be a topological manifold, only a topological space.

Example.

- (i) The Euclidean space \mathbb{R}^n is a Lie group under addition.
- (ii) The set \mathbb{C}^\times of nonzero complex numbers is a Lie group under multiplication.
- (iii) The unit circle S^1 in \mathbb{C}^\times is a Lie group under multiplication.
- (iv) The Cartesian product $G_1 \times G_2$ of two Lie groups (G_1, μ_1) and (G_2, μ_2) is a Lie group under componentwise multiplication $\mu_1 \times \mu_2$.

Example. (General linear group). As an open subset of $\mathbb{R}^{n \times n}$, the general linear group $\text{GL}(n, \mathbb{R})$ is a manifold. It is also a Lie group, since the multiplication map and the inverse map are both C^∞ .

7.6 Partial Derivatives

Let M be a manifold of dimension n , $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart on M and f be a C^∞ function. Denote by r^1, \dots, r^n the standard coordinates on \mathbb{R}^n . Then, we have $x^i = r^i \circ \phi$.

Definition 7.6. For any $p \in U$, the *partial derivative* $\partial f / \partial x^i$ of f with respect to x^i at p is defined by

$$\left. \frac{\partial}{\partial x^i} \right|_p f := \left. \frac{\partial}{\partial r^i} \right|_{\phi(p)} (f \circ \phi^{-1})$$

In 'simple' terms, the partial derivative of f with respect to x^i at p is given by the partial derivative of the *pullback* $(\phi^{-1})^* f = f \circ \phi^{-1}$ with respect to r^i at $\phi(p)$ (**Figure 7.1**). Since $p = \phi^{-1}(\phi(p))$, we can rewrite the above equation into

$$\left(\frac{\partial f}{\partial x^i} \circ \phi^{-1} \right) (\phi(p)) = \frac{\partial (f \circ \phi^{-1})}{\partial r^i} (\phi(p))$$

Thus, as a function on $\phi(U)$, the pullback $\phi^* (\partial f / \partial x^i)$,

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial (f \circ \phi^{-1})}{\partial r^i},$$

is C^∞ on $\phi(U)$. Hence, the partial derivative $\partial f / \partial x^i$ is C^∞ on U .

Proposition 7.11. Suppose (U, x^1, \dots, x^n) is a chart on a manifold. Then,

$$\frac{\partial x^i}{\partial x^j} = \delta_j^i.$$

Proof. By definition, at any point $p \in U$, we have

$$\frac{\partial x^i}{\partial x^j}(p) = \frac{\partial (x^i \circ \phi^{-1})}{\partial r^j} (\phi(p)) = \frac{\partial r^i}{\partial r^j} (\phi(p)) = \delta_j^i.$$

□

Definition 7.7. Let $F : N \rightarrow M$ be a smooth map, and let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \Psi) = (V, y^1, \dots, y^m)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by

$$F^i := y^i \circ F = r^i \circ \Psi \circ F : U \rightarrow \mathbb{R}$$

the i th component of F in the chart (V, Ψ) . Then, the matrix $[\partial F^i / \partial x^j]$ is called the *Jacobian matrix* of F relative to the charts (U, ϕ) and (V, Ψ) . In case N and M have the same dimension, the determinant $\det [\partial F^i / \partial x^j]$ is called the *Jacobian determinant* of F relative to the two charts. The Jacobian determinant is also written as $\partial (F^1, \dots, F^n) / \partial (x^1, \dots, x^n)$.

When N and M are open subsets of Euclidean spaces and the charts are $(U, \langle r^j \rangle_{j=1}^n)$ and $(V, \langle r^i \rangle_{i=1}^m)$, the Jacobian matrix $[\partial F^i / \partial r^j]$, where $F^i = r^i \circ F$, is the usual Jacobian matrix from calculus.

Example. (Jacobian matrix of a transition map). Given two overlapping charts $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \Psi) = (V, y^1, \dots, y^n)$ on a manifold M , the transition map $\Psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \Psi(U \cap V)$ is a diffeomorphism of open subsets of \mathbb{R}^n . Hence, its Jacobian matrix $J(\Psi \circ \phi^{-1})$ at $\phi(p)$ is given by

$$\left[\frac{\partial (\Psi \circ \phi^{-1})^i}{\partial r^j} (\phi(p)) \right] = \left[\frac{\partial (r^i \circ \Psi \circ \phi^{-1})}{\partial r^j} (\phi(p)) \right] = \left[\frac{\partial (y^i \circ \phi^{-1})}{\partial r^j} (\phi(p)) \right] = \left[\frac{\partial y^i}{\partial x^j} (p) \right].$$

7.7 The Inverse Function Theorem

By Proposition 7.6, any diffeomorphism $F: U \rightarrow F(U) \subset \mathbb{R}^n$ of an open subset U of a manifold may be thought of as a coordinate system on U . We say that a C^∞ map $F: N \rightarrow M$ is *locally invertible* or a *local diffeomorphism* at $p \in N$ if p has a neighborhood U on which $F|_U: U \rightarrow F(U)$ is a diffeomorphism.

Given n smooth functions F^1, \dots, F^n in a neighborhood of a point p in a manifold N of dimension n , one would like to know whether they form a coordinate system, possibly on a smaller neighborhood of p . This is equivalent to whether $F = (F^1, \dots, F^n): N \rightarrow \mathbb{R}^n$ is a local diffeomorphism at p . The inverse function theorem provides the answer to this question.

Theorem 7.12. (Inverse function theorem for \mathbb{R}^n). Let $F: W \rightarrow \mathbb{R}^n$ be a C^∞ map defined on an open subset W of \mathbb{R}^n . For any point p in W , the map F is locally invertible at p if and only if the Jacobian determinant $\det [\partial F^i / \partial r^j(p)]$ is nonzero.

Since the inverse function theorem for \mathbb{R}^n is a local result, it is easily translatable to manifolds.

Theorem 7.13. (Inverse function theorem for manifolds). Let $F: N \rightarrow M$ be a C^∞ map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U, \phi) = (U, x^1, \dots, x^n)$ about p in N and $(V, \Psi) = (V, y^1, \dots, y^k)$ about $F(p)$ in M , $F(U) \subset V$. Set $F^i = y^i \circ F$. Then, F is locally invertible at p if and only if its Jacobian determinant $\det [\partial F^i / \partial x^j(p)]$ is nonzero.

Proof. The Jacobian matrix of F relative to the charts (U, ϕ) and (V, Ψ) is given by

$$\left[\frac{\partial F^i}{\partial x^j}(p) \right] = \left[\frac{\partial (r^i \circ \Psi \circ F)}{\partial x^j}(p) \right] = \left[\frac{\partial (r^i \circ (\Psi \circ F \circ \phi^{-1}))}{\partial r^j}(\phi(p)) \right] = J(\Psi \circ F \circ \phi^{-1})$$

As $\Psi \circ F \circ \phi^{-1}: \phi(U) \rightarrow \Psi(V)$ is a map on an open subset of \mathbb{R}^n , by the inverse function theorem for \mathbb{R}^n , the Jacobian determinant

$$\det \left[\frac{\partial F^i}{\partial x^j}(p) \right] = \det [J(\Psi \circ F \circ \phi^{-1})]$$

is nonzero if and only if $\Psi \circ F \circ \phi^{-1}$ is locally invertible at $\phi(p)$. This is equivalent to the local invertibility of F at p (see **Figure 7.2**). \square

Corollary 7.14. Let N be a manifold of dimension n . A set of n smooth functions F^1, \dots, F^n defined on a coordinate neighborhood U, x^1, \dots, x^n of a point $p \in N$ forms a coordinate system about p if and only if the Jacobian determinant $\det [\partial F^i / \partial x^j(p)]$ is nonzero.

Proof. Let $F = (F^1, \dots, F^n): U \rightarrow \mathbb{R}^n$. Then

$$\det [\partial F^i / \partial x^j(p)] \neq 0$$

$\Leftrightarrow F: U \rightarrow \mathbb{R}^n$ is locally invertible at p (inverse function theorem).

\Leftrightarrow There is a neighborhood W of p in N such that $F: W \rightarrow F(W)$ is a diffeomorphism (definition of local invertibility).

$\Leftrightarrow (W, F^1, \dots, F^n)$ is a coordinate chart about p in the differentiable structure of N (Proposition 7.6). \square

Example. (Polar coordinates). The set of polar coordinates (r, θ) in \mathbb{R}^2 can be written in terms of the standard Cartesian coordinates via the map $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$F(x, y) = \left(\sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right) \right).$$

By the inverse function theorem, the map F can serve as a coordinate map in a neighborhood of p if and only if the Jacobian determinant of F

$$\frac{\partial(F^1, F^2)}{\partial(x, y)} = \det \begin{bmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ -\frac{y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{bmatrix} = \frac{1}{\sqrt{x^2 + y^2}}$$

is nonzero. Thus, F is a local diffeomorphism at p if and only if $p \neq (0, 0)$, which checks out, as we can recall that at $(0, 0)$, the angular coordinate θ is not well-defined.

Problems

Problem 7.1. Smoothness of an inclusion map. Let M and N be manifolds and let q_0 be a point in N . Show that the inclusion map $i_{q_0}: M \rightarrow M \times N, i_{q_0}(p) = (p, q_0)$, is C^∞ .

Solution. Let $p \in M$, $(U, \phi) = (U, x^1, \dots, x^m)$ be a chart about p in M , and $(V, \Psi) = (V, y^1, \dots, y^n)$ be a chart about q_0 in N . By Proposition 6.3, $(U \times V, \phi \times \Psi)$ is a chart about (p, q_0) . Given a point $r \in \phi(U \cap F^{-1}(V))$,

$$((\phi \times \Psi) \circ i_{q_0} \circ \phi^{-1})(r) = (r, q_0).$$

Thus, $(\phi \times \Psi) \circ i_{q_0} \circ \phi^{-1}$ is smooth on $\phi(U \cap F^{-1}(V))$. By Proposition 7.3, i_{q_0} is smooth on M .

8 Quotients

8.1 The Quotient Topology

Let S be a set. Denote by S/\sim the set of equivalence classes and call the set the *quotient* of S by the equivalence relation \sim . Then, there is a natural *projection map* $\pi: S \rightarrow S/\sim$ that sends $x \in S$ to its equivalence class $[x]$.

Now, assume S is a topological space. We define a topology on S/\sim by declaring a set U in S/\sim to be *open* if and only if $\pi^{-1}(U)$ is open in S . Clearly, both the empty set \emptyset and the entire quotient S/\sim are open. Furthermore, since the collection of open sets in S/\sim are closed under arbitrary unions and finite intersections, they form a topology. This is called the *quotient topology* on S/\sim . With this topology, S/\sim is called the *quotient space* of S by the equivalence relation \sim . With the quotient topology on S/\sim , the projection map $\pi: S \rightarrow S/\sim$ is automatically continuous, since the preimage of any open set in S/\sim is by definition open in S .

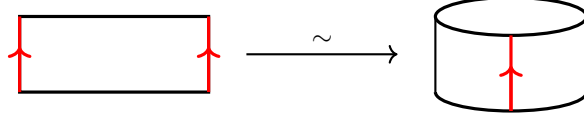


Figure. 8.1: Glueing the ends of the strip $[0, 3] \times [0, 1]$ into a cylinder via the equivalence relation $(0, y) \sim (3, y)$.

8.2 Continuity of a Map on a Quotient

Let \sim be an equivalence relation on a topological space S and give S/\sim the quotient topology. Suppose a function $f: S \rightarrow Y$ from S to another topological space Y is constant on each equivalence class. Then, it induces a map $\bar{f}: S/\sim \rightarrow Y$ by

$$\bar{f}([p]) = f(p)$$

In other words, there is a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{f} & Y \\ \pi \downarrow & \nearrow \bar{f} & \\ S/\sim & & \end{array}$$

Proposition 8.1. The induced map $\bar{f}: S/\sim \rightarrow Y$ is continuous if and only if the map $f: S \rightarrow Y$ is continuous.

Proof. If \bar{f} is continuous, then $f = \bar{f} \circ \pi$ is a composite of continuous maps, and thus it is continuous. Conversely, if f is continuous, then for any $V \in Y$, $f^{-1}(V) = \pi^{-1}(\bar{f}^{-1}(V))$ is open in S . By the definition of the quotient topology, $\bar{f}^{-1}(V)$ is open in S/\sim . Hence, \bar{f} is continuous. \square

Remark. This proposition gives us a useful way to check whether a function \bar{f} on a quotient space S/\sim is continuous by simply checking the continuity of the lifted map $f := \bar{f} \circ \pi$ on S .

8.3 Identification of a Subset to a Point

If A is a subspace of a topological space S , we can define an equivalence relation \sim on S by declaring $x \sim x$ for all $x \in S$ and $x \sim y$ for all $x, y \in A$. We say that the quotient space S/\sim is obtained from S by *identifying A to a point*.

Example. Let I be the unit interval $[0, 1]$ and I/\sim the quotient space obtained from I by identifying the two points $\{0, 1\}$ to a single point. Denote by S^1 the unit circle in the complex plane. The function $f: I \rightarrow S^1$, $f(x) = \exp(2\pi i x)$, assumes the same values at 0 and 1, and thus induces a function $\bar{f}: I/\sim \rightarrow S^1$. It can be shown that \bar{f} is a homeomorphism.

8.4 A Necessary Condition for a Hausdorff Quotient

The quotient construction does not, in general, preserve the Hausdorff property or second countability. Indeed, there is a necessary condition, which is shown by the following proposition

Proposition 8.2. If the quotient space S/\sim is Hausdorff, then the equivalence class $[p]$ of any point p in S is closed in S .

Proof. Suppose the quotient space S/\sim is Hausdorff. Since any singleton set in a Hausdorff space is closed, the image $\{\pi(p)\}$ of any point $p \in S$ is closed in S/\sim . By the continuity of π , the preimage $\pi^{-1}(\pi(p)) = [p]$ must be closed in S . \square

8.5 Open Equivalence Relations

In this section, we will derive the conditions under which a quotient space is Hausdorff or second countable. Recall that a map $f: X \rightarrow Y$ of topological spaces is *open* if for any open set U in X , the image $f(U)$ is open in Y .

Definition 8.1. An equivalence relation \sim on a topological space S is said to be *open* if the projection map $\pi: S \rightarrow S/\sim$ is open.

In other words, the equivalence relation \sim on S is open if and only if for every open set U in S , the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

of all points equivalent to some point in U is open.

Example. In general, the projection map to a quotient space is not open. For example, let \sim be the equivalence relation on \mathbb{R} that identifies the two points 1 and -1 . Then, the projection map $\pi: \mathbb{R} \rightarrow \mathbb{R}/\sim$ is not open since while the interval $V = (0, 2)$ is open in \mathbb{R} ,

$$\pi^{-1}(\pi(V)) = (0, 2) \cup \{1\}$$

is not open in \mathbb{R} .

Definition 8.2. Let \sim be an equivalence relation on S and let R be the subset of $S \times S$ that defines the relation

$$R = \{(x, y) \in S \times S \mid x \sim y\}.$$

Then, R is called the *graph* of the equivalence relation \sim .

Theorem 8.3. Suppose \sim is an open equivalence relation on a topological space S . Then, the quotient space S/\sim is Hausdorff if and only if the graph R of \sim is closed in $S \times S$.

Proof. R is closed in $S \times S$.

$\Leftrightarrow \overline{R}$ is open in $S \times S$.

\Leftrightarrow For any point $(x, y) \in \overline{R}$, there is a basic open set $U \times V$ containing (x, y) such that $U \times V \cap R = \emptyset$.

\Leftrightarrow For every pair $x \not\sim y$ in S , there exists neighborhoods U of x and V of y such that no element of U is equivalent to an element of V .

\Leftrightarrow For any two points $[x] \neq [y]$ in S/\sim , there exists neighborhoods U of x and V of y in S such that $\pi(U) \cap \pi(V) = \emptyset$. (*)

We will now show that $(*)$ is equivalent to S/\sim being Hausdorff. First, assume $(*)$. Since π is open, $\pi(U)$ and $\pi(V)$ are disjoint open sets in S/\sim that separates $[x]$ and $[y]$. Thus, S/\sim is Hausdorff.

Now, suppose S/\sim is Hausdorff. Then, for any $[x] \neq [y]$ in S/\sim , there exists disjoint open sets A and B in S/\sim such that $[x] \in A$ and $[y] \in B$. Let $U = \pi^{-1}(A)$ and $V = \pi^{-1}(B)$. Then, $x \in U$ and $y \in V$. Thus, U and V are neighborhoods of x and y in S respectively such that $\pi(U) \cap \pi(V) = \pi(\pi^{-1}(A)) \cap \pi(\pi^{-1}(B)) \subset A \cap B = \emptyset$, so $\pi(U) \cap \pi(V) = \emptyset$. \square

Corollary 8.4. A topological space S is Hausdorff if and only if the diagonal Δ in $S \times S$ is closed.

Proof. By simply letting the equivalence relation \sim be equality, the quotient space S/\sim is then S itself, and thus the graph R of S is simply the diagonal

$$\Delta = \{(x, x) \in S \times S \mid x \in S\}$$

of $S \times S$. \square

Theorem 8.5. Let \sim be an open equivalence relation on a topological space S with projection $\pi: S \rightarrow S/\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for S , then its image $\{\pi(B_\alpha)\}$ is a basis for S/\sim .

Proof. Since π is an open map, $\{\pi(B_\alpha)\}$ is a collection of open sets in S/\sim . Now, let U be an open set in S/\sim and let $[x] \in U$. Then, $x \in \pi^{-1}(U)$, and since \mathcal{B} is a basis for S , there exists an open set $B_\alpha \in \mathcal{B}$ such that

$$x \in B_\alpha \subset \pi^{-1}(U).$$

Hence,

$$[x] = \pi(x) \in \pi(B_\alpha) \subset \pi(\pi^{-1}(U)) \subset U,$$

and thus $\{\pi(B_\alpha)\}$ is a basis for S/\sim . \square

Corollary 8.6. If \sim is an open equivalence relation on a second-countable space S , then the quotient space S/\sim is second countable.

Problems

Problem 8.1. Orbit space of a continuous group action. Suppose the right action of a topological group G on a topological space S is continuous, i.e., the map $S \times G \rightarrow S$ describing the action is continuous. Define two points x, y to be equivalent if they are in the same orbit, i.e., there is an element $g \in G$ such that $y = xg$. Let S/G be the quotient space; it is called the *orbit space* of the action. Prove that the projection map $\pi: S \rightarrow S/G$ is an open map.

Proof. Let U be an open set in S . For any $g \in G$, since right multiplication by g is bijective (since any $g \in G$ has an inverse) and continuous, it is a homeomorphism. Hence, the set Ug is open in S . Thus,

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} Ug$$

is a union of open sets in S , and therefore it is open in S . By definition of the quotient topology, π is an open map. \square

The Tangent Space

9 The Tangent Space

9.1 The Tangent Space at a point

Just as for \mathbb{R}^n , we define a *germ* of a C^∞ function at p in M to be an equivalence class of C^∞ functions defined in a neighborhood of p in M , two such functions being equivalent if they agree on some, possibly smaller, neighborhood of p . The set of germs of C^∞ real-valued functions at p in M is then denoted by $C_p^\infty(M)$. The addition and multiplication of functions make $C_p^\infty(M)$ a ring; with scalar multiplication by real numbers, $C_p^\infty(M)$ becomes an algebra over \mathbb{R} .

Generalizing a derivation at a point in \mathbb{R}^n , we define a *derivation at a point* in a manifold M , or a *point-derivation* of $C_p^\infty(M)$, to be a linear map $D: C_p^\infty(M) \rightarrow \mathbb{R}$ that satisfies the Leibniz rule:

$$D(fg) = (Df)g(p) + f(p)(Dg).$$

Definition 9.1. A *tangent vector* at a point p in a manifold M is a derivation at p .

Just as for \mathbb{R}^n , the tangent vectors at p form a vector space $T_p(M)$, called the *tangent space* of M at p . We also write $T_p M$ instead of $T_p(M)$.

Remark. (Tangent space to an open subset). If U is an open set containing p in M , then the algebra $C_p^\infty(U)$ of germs of C^∞ functions in U at p is the same as $C_p^\infty(M)$. Hence, $T_p U = T_p M$.

Given a coordinate neighborhood $(U, \phi) = (U, x^1, \dots, x^n)$ about a point p in a manifold M , we recall the definition of the partial derivative $\partial/\partial x^i$. Let r^1, \dots, r^n be the standard coordinates on \mathbb{R}^n . Then

$$x^i = r^i \circ \phi: U \rightarrow \mathbb{R}.$$

If f is a C^∞ function in a neighborhood of p , we set

$$\left. \frac{\partial}{\partial x^i} \right|_p f = \left. \frac{\partial}{\partial r^i} \right|_{\phi(p)} (f \circ \phi^{-1}).$$

Clearly, $\left. \frac{\partial}{\partial x^i} \right|_p: C_p^\infty(M) \rightarrow \mathbb{R}$ is linear and satisfies the Leibniz rule; thus, it is a derivation and so is a tangent vector at p .

9.2 The Differential of a Map

Let $F: N \rightarrow M$ be a C^∞ map between two manifolds. At each point $p \in N$, the map induces a linear map of vector spaces, called its *differential* at p ,

$$F_*: T_p N \rightarrow T_{F(p)} M$$

as follows (**Figure 9.1**). If $X \in T_p N$, then $F_*(X_p)$ is the tangent vector in $T_{F(p)} M$ defined by

$$(F_*(X_p)) f = X_p (f \circ F) \in \mathbb{R} \text{ for } f \in C_p^\infty(M).$$

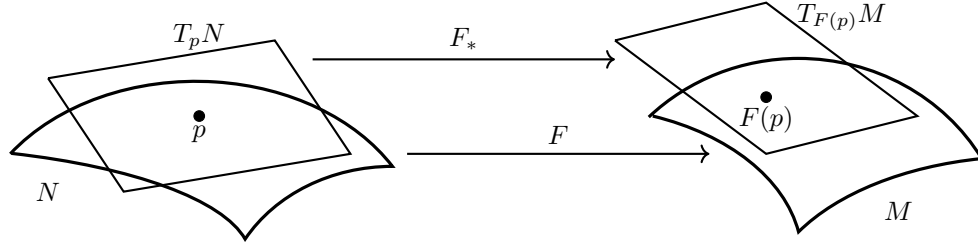


Figure 9.1: $F: N \rightarrow M$ induces a map $F_*: T_p N \rightarrow T_{F(p)} M$.

Example. (Differential of a map between Euclidean spaces). Suppose $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is C^∞ and p is a point in \mathbb{R}^n . Let x^1, \dots, x^n be the coordinates on \mathbb{R}^n and y^1, \dots, y^m the coordinates on \mathbb{R}^m . Then, the tangent vectors $\{\partial/\partial x^j|_p\}$ and $\{\partial/\partial y^i|_{F(p)}\}$ form the bases for the tangent spaces $T_p(\mathbb{R}^n)$ and $T_{F(p)}(\mathbb{R}^m)$ respectively. The linear map $F_*: T_p(\mathbb{R}^n) \rightarrow T_{F(p)}(\mathbb{R}^m)$ is described by a matrix $[a_j^i]$ relative to these two bases:

$$F_* \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \sum_k a_j^k \frac{\partial}{\partial y^k} \Big|_{F(p)}.$$

Let $F^i = y^i \circ F$ be the i th component of F . Then,

$$a_j^i = F_* \left(\frac{\partial}{\partial x^j} \Big|_p \right) y^i = \frac{\partial}{\partial x^j} \Big|_p (y^i \circ F) = \frac{\partial F^i}{\partial x^j}(p).$$

Thus, the matrix describing F_* relative to the bases $\{\partial/\partial x^j|_p\}$ and $\{\partial/\partial y^i|_{F(p)}\}$ is $[\partial F^i/\partial x^j|_p]$, which is precisely the Jacobian matrix of the derivative of F at p . Hence, the differential of a map between manifolds generalizes the derivative of a map between Euclidean spaces.

9.3 The Chain Rule

Let $F: N \rightarrow M$ and $G: M \rightarrow P$ be smooth maps of manifolds, and $p \in N$. The differentials of F at p and G at $F(p)$ are linear maps

$$T_p N \xrightarrow{F_{*,p}} T_{F(p)} M \xrightarrow{G_{*,F(p)}} T_{G(F(p))} P.$$

Theorem 9.1. (The chain rule). If $F: N \rightarrow M$ and $G: M \rightarrow P$ are smooth maps of manifolds and $p \in N$, then

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}.$$

Proof. Let $X_p \in T_p N$ and f be a smooth function at $(G \circ F)(p)$ in P . Then,

$$((G \circ F)_{*,p} X_p) f = X_p (f \circ (G \circ F)).$$

We also have

$$((G_{*,F(p)} \circ F_{*,p}) X_p) f = (G_{*,F(p)}(F_{*,p}(X_p))) f = F_{*,p}(X_p) (f \circ G) = X_p (f \circ G \circ F).$$

□

Remark. The differential of the identity map $\mathbb{1}_M: M \rightarrow M$ is the identity map

$$\mathbb{1}_{T_p M}: T_p M \rightarrow T_p M$$

since for any $X_p \in T_p M$ and $f \in C_p^\infty(M)$,

$$((\mathbb{1}_M)_*(X_p))(f) = X_p(f \circ \mathbb{1}_M) = X_p(f).$$

Corollary 9.2. If $F: N \rightarrow M$ is a diffeomorphism of manifolds and $p \in N$, then $F_*: T_p N \rightarrow T_{F(p)} M$ is an isomorphism of vector spaces.

Proof. By definition, F has a C^∞ inverse $G: M \rightarrow N$ such that $G \circ F = \mathbb{1}_N$ and $F \circ G = \mathbb{1}_M$. By the chain rule,

$$\begin{aligned} G_* \circ F_* &= (G \circ F)_* = (\mathbb{1}_N)_* = \mathbb{1}_{T_p N} \\ F_* \circ G_* &= (F \circ G)_* = (\mathbb{1}_M)_* = \mathbb{1}_{T_{F(p)} M}. \end{aligned}$$

□

Corollary 9.3. (Invariance of dimension). If an open set $U \subset \mathbb{R}^n$ is diffeomorphic to an open set $V \subset \mathbb{R}^m$, then $n = m$.

Proof. Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a diffeomorphism. Since U and V are diffeomorphic, for any point $p \in U$, $T_p U$ and $T_{F(p)} V$ are isomorphic vector spaces. Thus, $\mathbb{R}^n \simeq T_p(U) \simeq T_{F(p)}(V) \simeq \mathbb{R}^m$, and therefore, $n = m$. □

9.4 Bases for the Tangent Space at a Point

Denote by r^1, \dots, r^n the standard coordinates on \mathbb{R}^n and let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart about a point p in a manifold M of dimension n . Since $\phi: U \rightarrow \phi(U) \subset \mathbb{R}^n$ is diffeomorphism (Proposition 7.5), by Corollary 9.2, the differential

$$\phi_*: T_p M \rightarrow T_{\phi(p)} \mathbb{R}^n$$

is an isomorphism of vector spaces. Thus, the tangent space $T_p M$ has the same dimension n as the manifold M .

Proposition 9.4. Let $(U, \phi) = (U, x^1, \dots, x^n)$ be a chart about a point p in a manifold M . Then,

$$\phi_* \left(\left. \frac{\partial}{\partial x^i} \right|_p \right) = \left. \frac{\partial}{\partial r^i} \right|_{\phi(p)}.$$

Proof. For any $f \in C_p^\infty(M)$,

$$\begin{aligned} \phi_* \left(\frac{\partial}{\partial x^i} \Big|_p \right) (f) &= \left(\frac{\partial}{\partial x^i} \Big|_p \right) (f \circ \phi) \\ &= \left(\frac{\partial}{\partial r^i} \Big|_{\phi(p)} \right) ((f \circ \phi) \circ \phi^{-1}) \\ &= \left(\frac{\partial}{\partial r^i} \Big|_{\phi(p)} \right) (f). \end{aligned}$$

□

Proposition 9.5. If $(U, \phi) = (U, x^1, \dots, x^n)$ is a chart containing p , then the tangent space $T_p M$ has a basis

$$\frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p$$

Proof. Since an isomorphism of vector spaces sends a basis to a basis, by Proposition 9.4, $\{\partial/\partial x^i|_p\}$ is a basis for $T_p M$. □

Proposition 9.6. (Transition matrix for coordinate vectors). Suppose (U, x^1, \dots, x^n) and (V, y^1, \dots, y^n) are two coordinate charts in a manifold M . Then,

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$$

on $U \cap V$.

Proof. Obvious. □

9.5 A Local Expression for the Differential

Proposition 9.7. Given a smooth map $F: N \rightarrow M$ of manifolds and a point $p \in N$, let (U, x^1, \dots, x^n) and (V, y^1, \dots, y^n) be coordinate charts about p in N and $F(p)$ in M respectively. Relative to the bases $\{\partial/\partial x^j|_p\}$ for $T_p N$ and $\{\partial/\partial y^i|_p\}$ for $T_{F(p)} M$, the differential $F_{*,p}: T_p N \rightarrow T_{F(p)} M$ is represented by the matrix $[\partial F^i / \partial x^j(p)]$, where $F^i = y^i \circ F$.

Proof. The differential $F_{*,p}$ is completely determined by the matrix $[a_j^i]$, where

$$F_{*,p} \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \sum_k a_j^k \frac{\partial}{\partial y^k} \Big|_{F(p)}.$$

Applying both sides to y^i yields

$$a_j^i = F_{*,p} \left(\frac{\partial}{\partial x^j} \Big|_p \right) (y^i) = \left(\frac{\partial}{\partial x^j} \Big|_p \right) (y^i \circ F) = \frac{\partial F^i}{\partial x^j}.$$

□

Remark. (Inverse function theorem). In terms of the differential, the inverse function theorem for manifolds (Theorem 7.13) has a coordinate-free description: a C^∞ map $F: N \rightarrow M$ between two manifolds of the same dimension is locally invertible at a point $p \in N$ if and only if its differential $F_{*,p}: T_p N \rightarrow T_{F(p)} M$ at p is an isomorphism.

9.6 Curves in a Manifold

A *smooth curve* in a manifold M is by definition a smooth map $c: (a, b) \rightarrow M$ from some open interval (a, b) into M . Usually, we assume $0 \in (a, b)$ and say that c is a *curve starting at p* if $c(0) = p$. The *velocity vector* $c'(t_0)$ of the curve c at time $t_0 \in (a, b)$ is defined to be

$$c'(t_0) := c_* \left(\frac{d}{dt} \Big|_{t_0} \right) \in T_{c(t_0)} M.$$

We also say that $c'(t_0)$ is the velocity of c at the point $c(t_0)$. Alternatively, we may also write

$$\frac{dc}{dt}(t_0) \quad \text{and} \quad \frac{d}{dt} \Big|_{t_0} c.$$

Proposition 9.8. (Velocity of a curve in local coordinates). Let $c: (a, b) \rightarrow M$ be a smooth curve, and let (U, x^1, \dots, x^n) be a chart about $c(t)$. Write $c^i = x^i \circ c$ for the i th component of c in the chart. Then, $c'(t)$ is given by

$$c'(t) = \sum_{i=1}^n \frac{dc^i}{dt} \frac{\partial}{\partial x^i} \Big|_{c(t)}.$$

Proof. Applying $c'(t)$ to the component x^i of the coordinate map yields

$$(c'(t))^i = c_* \left(\frac{d}{dt} \Big|_t \right) (x^i) = \left(\frac{d}{dt} \Big|_t \right) (x^i \circ c) = \frac{dc^i}{dt}.$$

□

Proposition 9.9. (Existence of a curve with a given initial vector). For any point p in a manifold M and any tangent vector $X_p \in T_p M$, there are $\varepsilon > 0$ and a smooth curve $c: (-\varepsilon, \varepsilon) \rightarrow M$ such that $c(0) = p$ and $c'(0) = X_p$.

Proof. Let (U, ϕ) be a chart centered at p , i.e., $\phi(p) = 0 \in \mathbb{R}^n$. Suppose $X_p = \sum a^i \partial / \partial x^i|_p$ and let α be a curve in \mathbb{R}^n such that $\alpha(0) = 0$ and $\alpha'(0) = \sum a^i \partial / \partial x^i|_0$. By Proposition 9.8, α can be written

as

$$\alpha(t) = (a^1 t, \dots, a^n t), \quad t \in (-\varepsilon, \varepsilon),$$

where ε is chosen so that $\alpha(t)$ lies completely in $\phi(U)$. Define $c = \phi^{-1} \circ \alpha: (-\varepsilon, \varepsilon) \rightarrow M$. Then,

$$c(0) = \phi^{-1}(\alpha(0)) = \phi(0)^{-1} = p,$$

and by the chain rule,

$$c'(0) = \phi_*^{-1} \alpha_* \left(\frac{d}{dt} \Big|_{t_0} \right) = \phi_*^{-1} \left(\sum_i a^i \frac{\partial}{\partial r^i} \Big|_0 \right) = \sum_i a^i \frac{\partial}{\partial x^i} \Big|_{\phi^{-1}(0)=p} = X_p.$$

□

Proposition 9.10. Suppose X_p is a tangent vector at a point p of a manifold M and $f \in C_p^\infty(M)$. If $c: (-\varepsilon, \varepsilon) \rightarrow M$ is a smooth curve starting at p with $c'(0) = X_p$, then

$$X_p f = \frac{d}{dt} \Big|_0 (f \circ c).$$

Proof. By definition,

$$X_p f = c'(0)(f) = c_* \left(\frac{d}{dt} \Big|_0 \right) (f) = \frac{d}{dt} \Big|_0 (f \circ c).$$

□

9.7 Computing the Differential Using Curves

Proposition 9.11. Let $F: N \rightarrow M$ be a smooth map of manifolds, $p \in N$, and $X_p \in T_p N$. If c is a smooth curve starting at p in N with velocity X_p at p , then

$$F_{*,p}(X_p) = \frac{d}{dt} \Big|_0 (F \circ c)(t).$$

In other words, $F_{*,p}(X_p)$ is the velocity of the image curve $F \circ c$ starting at $F(p)$.

Proof. By hypothesis, $c(0) = p$ and $c'(0) = X_p$. Then

$$\begin{aligned} F_{*,p}(X_p) &= F_{*,p} \left(c_* \left(\frac{d}{dt} \Big|_0 \right) \right) \\ &= (F_*, p \circ c_*) \left(\frac{d}{dt} \Big|_0 \right) \\ &= (F \circ c)_* \left(\frac{d}{dt} \Big|_0 \right) \\ &= \frac{d}{dt} \Big|_0 (F \circ c)(t). \end{aligned}$$

□

9.8 Immersions and Submersions

Just as the derivative of a map between Euclidean spaces is a linear map that best approximates the given map at a point, the differential at a point also serves the same purpose for a C^∞ map between manifolds. Two cases are especially important. A C^∞ map $F: N \rightarrow M$ is said to be an *immersion at* $p \in N$ if its differential $F_{*,p}: T_p N \rightarrow T_{F(p)} M$ is injective, and a *submersion at* p if $F_{*,p}$ is surjective. We call F an *immersion* if it is an immersion at every $p \in N$ and a *submersion* if it is a submersion at every $p \in N$.

Remark. Suppose N and M are manifolds of dimension n and m respectively. Then, $\dim T_p N = n$ and $\dim T_{F(p)} M = m$. The injectivity of the differential $F_{*,p}: T_p N \rightarrow T_{F(p)} M$ immediately implies that $n \leq m$. Similarly, the surjectivity of $F_{*,p}$ immediately implies that $n \geq m$. Thus, if $F: N \rightarrow M$ is an immersion at a point of N , then $n \leq m$; if F is a submersion at a point of N , then $n \geq m$.

Example. The prototype of an immersion is the inclusion of \mathbb{R}^n in a higher-dimensional \mathbb{R}^m :

$$i(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0).$$

Similarly, the prototype of a submersion is the projection of \mathbb{R}^n onto a lower-dimensional \mathbb{R}^m :

$$\pi(x^1, \dots, x^m, x^{m+1}, \dots, x^n) = (x^1, \dots, x^m)$$

Example. If U is an open subset of a manifold M , then the inclusion $i: U \rightarrow M$ is both an immersion and a submersion. Particularly, this shows that a submersion need not be onto.

Later on, we will show that every immersion is locally an inclusion and every submersion is locally a projection.

9.9 Rank, and Critical and Regular Points

Consider a smooth map $F: N \rightarrow M$ of manifolds. Its *rank* at a point p in N , denoted $\text{rk } F(p)$, is defined as the rank of the differential $F_{*,p}: T_p N \rightarrow T_{F(p)} M$. Relative to the coordinate neighborhoods (U, x^1, \dots, x^n) at p and (V, y^1, \dots, y^m) at $F(p)$, the differential is represented by the Jacobian matrix $[\partial F^i / \partial x^j(p)]$, so

$$\text{rk } F(p) = \text{rk} \left[\frac{\partial F^i}{\partial x^j}(p) \right].$$

Since the differential of a map is independent of coordinate charts, so is the rank of the Jacobian matrix.

Definition 9.2. A point p in N is a *critical point* of F if the differential

$$F_{*,p}: T_p N \rightarrow T_{F(p)} M$$

fails to be surjective. It is a *regular point* of F if the differential $F_{*,p}$ is surjective. In other words, p is a regular point of the map F if and only if F is a submersion at p . A point in M is a *critical value* if it is the image of a critical point; otherwise it is a *regular value*.

Remark.

- (i) We do *not* define a regular value to be the image of a regular point. In fact, a regular value need not be in the image of F at all. Any point of M not in the image of F is automatically a regular value because it is not the image of a critical point.
- (ii) A point c in M is a critical value if and only if *some* point in the preimage $F^{-1}(\{c\})$ is a critical point. A point c in the image of F is a regular value if and only if *every* point in the preimage $F^{-1}(\{c\})$ is a regular point.

Proposition 9.12. For a real-valued function $f: M \rightarrow \mathbb{R}$, a point p in M is a critical point if and only if relative to some chart (U, x^1, \dots, x^n) containing p , all the partial derivatives satisfy

$$\frac{\partial f}{\partial x^j}(p) = 0, \quad j = 1, \dots, n$$

Proof. By Proposition 9.7, the differential $f_{*,p}: T_p M \rightarrow T_{f(p)} \mathbb{R} \simeq \mathbb{R}$ is represented by the matrix

$$\left[\frac{\partial f}{\partial x^1}(p) \cdots \frac{\partial f}{\partial x^n}(p) \right].$$

Since $\text{im } f_{*,p}$ is a linear subspace of \mathbb{R} , it is either zero-dimensional or one-dimensional. In other words, $f_{*,p}$ is either the zero map or a surjective map. Hence, $f_{*,p}$ fails to be surjective if and only if all the partial derivatives $\partial f / \partial x^j(p)$ are zero. \square

Problems

Problem 9.1. Let $F: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the map

$$(u, v, w) = F(x, y) = (x, y, xy).$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_*(\partial/\partial x|_p)$ as a linear combination of $\partial/\partial u$, $\partial/\partial v$, and $\partial/\partial w$ at $F(p)$.

Solution.

$$F_* \left(\frac{\partial}{\partial x} \Big|_p \right) = \frac{\partial F_u}{\partial x} \frac{\partial}{\partial u} + \frac{\partial F_v}{\partial x} \frac{\partial}{\partial v} + \frac{\partial F_w}{\partial x} \frac{\partial}{\partial w} = \frac{\partial}{\partial u} + y \frac{\partial}{\partial w}$$

Problem 9.2. Fix a real number α and define: $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = (u, v) = F(x, y) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let $X = -y\partial/\partial x + x\partial/\partial y$ be a vector field on \mathbb{R}^2 . If $p = (x, y) \in \mathbb{R}^2$ and $F_*(X_p) = (a\partial/\partial u + b\partial/\partial v)|_{F(p)}$, find a and b in terms of x, y , and α .

Solution. The differential F_* is given by the Jacobian matrix

$$\left[\frac{\partial(F^1, F^2)}{\partial(x, y)} \right] = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

Thus,

$$\begin{aligned} F_*(X) &= -y \left(\frac{\partial F^u}{\partial x} \frac{\partial}{\partial u} + \frac{\partial F^v}{\partial x} \frac{\partial}{\partial v} \right) + x \left(\frac{\partial F^u}{\partial y} \frac{\partial}{\partial u} + \frac{\partial F^v}{\partial y} \frac{\partial}{\partial v} \right) \\ &= \left(-y \frac{\partial F^u}{\partial x} + x \frac{\partial F^u}{\partial y} \right) \frac{\partial}{\partial u} + \left(-y \frac{\partial F^v}{\partial x} + x \frac{\partial F^v}{\partial y} \right) \frac{\partial}{\partial v} \\ &= -(x \sin \alpha + y \cos \alpha) \frac{\partial}{\partial u} + (x \cos \alpha - y \sin \alpha) \frac{\partial}{\partial v} \end{aligned}$$

Problem 9.3. Differential of a linear map

Let $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$, there is a canonical identification $T_p(\mathbb{R}^n) \simeq \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i} \Big|_p \mapsto \mathbf{a} = \langle a^1, \dots, a^n \rangle.$$

Show that the differential $L_{*,p}: T_p(\mathbb{R}^n) \rightarrow T_{L(p)}(\mathbb{R}^m)$ is the map $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ itself, with the identification of the tangent spaces as above.

Solution. Let (x^1, \dots, x^n) and (y^1, \dots, y^m) be the standard coordinates on \mathbb{R}^n and \mathbb{R}^m respectively and suppose $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is represented in those basis by the matrix $[a_j^i]$, where

$$F^i = \sum_j a_j^i x^j.$$

Then, F_* is represented by the Jacobian matrix $[\partial F^i / \partial x^j]_p$, where

$$\frac{\partial F^i}{\partial x^j} = \frac{\partial (y^i \circ F)}{\partial x^j} = \frac{\partial}{\partial x^j} \sum_k a_k^i x^k = a_k^i \delta_j^k = a_j^i.$$

Problem 9.4. Differential of multiplication and inverse

Let G be a Lie group with multiplication map $\mu: G \times G \rightarrow G$, inverse map $\iota: G \rightarrow G$, and identity element e .

(a) Show that the differential at the identity of the multiplication map μ is addition:

$$\begin{aligned} \mu_{*,(e,e)}: T_e G \times T_e G &\rightarrow T_e G, \\ (X_e, Y_e) &\mapsto X_e + Y_e. \end{aligned}$$

(b) Show that the differential at the identity of the inverse map ι is the negation:

$$\begin{aligned} \iota_{*,e}: T_e G &\rightarrow T_e G, \\ X_e &\mapsto -X_e. \end{aligned}$$

Proof.

- (a) Let $c(t)$ be a curve starting at e in G such that $c'(0) = X_e$. Then, $\alpha(t) = (c(t), e)$ is a curve starting at (e, e) in $G \times G$ with $\alpha'(0) = (X_e, 0)$. By Proposition 9.11,

$$\begin{aligned}\mu_{*,(e,e)}(X_e, 0) &= \mu_{*,(e,e)}(\alpha'(0)) \\ &= \left. \frac{d}{dt} \right|_0 (\mu \circ \alpha)(t) \\ &= \left. \frac{d}{dt} \right|_0 \mu(c(t), e) \\ &= \left. \frac{d}{dt} \right|_0 (c(t)) \\ &= c'(0) \\ &= X_e\end{aligned}$$

Similarly,

$$\mu_{*,(e,e)}(0, Y_e) = Y_e,$$

so

$$\mu_{*,(e,e)}(X_e, Y_e) = X_e + Y_e.$$

- (b) Let $c(t)$ be a curve starting at e in G such that $c'(0) = X_e$. Then,

$$(\iota \circ c)'(0) = (\iota \circ c)_* \left(\left. \frac{d}{dt} \right|_0 \right) = \iota_{*,e}(c'(0)) = \iota_{*,e}(X_e).$$

Thus, $\alpha(t) = (c(t), (\iota \circ c)(t))$ is a curve starting at (e, e) in $G \times G$ with $\alpha'(0) = (X_e, \iota_{*,e}(X_e))$. Observe that

$$\begin{aligned}X_e + \iota_{*,e}(X_e) &= \mu_{*,(e,e)}(X_e, \iota_{*,e}(X_e)) \\ &= \mu_{*,(e,e)}(\alpha'(0)) \\ &= \left. \frac{d}{dt} \right|_0 (\mu \circ \alpha)(t) \\ &= \left. \frac{d}{dt} \right|_0 \mu(c(t), (\iota \circ c)(t)) \\ &= \left. \frac{d}{dt} \right|_0 e \\ &= 0\end{aligned}$$

Hence,

$$\iota_{*,e}(X_e) = -X_e$$

□

Problem 9.5. Local maxima

A real-valued function $f: M \rightarrow \mathbb{R}$ on a manifold is said to have a *local maximum* at $p \in M$ if there is a neighborhood U of p such that $f(p) \geq f(q)$ for all $q \in U$.

- (a) Prove that if a differentiable function $f: I \rightarrow \mathbb{R}$ defined on an open interval I has a local maximum at $p \in I$, then $f'(p) = 0$.
- (b) Prove that a local maximum of a C^∞ function $f: M \rightarrow \mathbb{R}$ is a critical point of f .

Proof.

- (a) By definition, there is a neighborhood U of p in I such that $f(p) \geq f(x)$ for all $x \in U$. Then,

$$\lim_{x \rightarrow p^-} \frac{f(x) - f(p)}{x - p} \geq 0$$

and

$$\lim_{x \rightarrow p^+} \frac{f(x) - f(p)}{x - p} \leq 0.$$

Since f is differentiable, its derivative at p exists, where

$$f'(p) = \lim_{x \rightarrow p^-} \frac{f(x) - f(p)}{x - p} = \lim_{x \rightarrow p^+} \frac{f(x) - f(p)}{x - p} = 0$$

- (b) Suppose $f: M \rightarrow \mathbb{R}$ has a local maximum at p . Let X_p be a tangent vector in $T_p M$ and let $c(t): (-\varepsilon, \varepsilon) \rightarrow M$ be a curve starting at p with initial vector X_p . Then, $f \circ c: \mathbb{R} \rightarrow \mathbb{R}$ is a real-valued function with a local maximum at $0 \in (-\varepsilon, \varepsilon)$, so by part (a), $(f \circ c)'(0) = 0$. Thus, $f_{*,p}(X_p) = f_{*,p}(c'(0)) = 0$ so p is a critical point of f .

□

10 Submanifolds

We now have two ways of determining if a topological space is a manifold:

- (a) by directly checking that it is a Hausdorff, second countable, and has a C^∞ atlas;
- (b) by showing it is an appropriate quotient space.

In this section, we introduce the concept of a *regular submanifold* of a manifold. Using the inverse function theorem, we derive a criterion, called the *regular level set theorem*, that can be used to show that a level set of a C^∞ map of manifolds is a regular submanifold, and therefore a manifold.

10.1 Submanifolds

Definition 10.1. A subset S of a manifold N of dimension n is a *regular submanifold* of dimension k if for every $p \in S$ there is a coordinate neighborhood $(U, \phi) = (U, x^1, \dots, x^n)$ of p in the maximal atlas of N such that $U \cap S$ is defined by the vanishing of $n - k$ of the coordinate functions. By renumbering the coordinates, we may assume these $n - k$ coordinate functions are x^{k+1}, \dots, x^n .

We call such a chart (U, ϕ) in N an *adapted chart* relative to S . On $U \cap S$, $\phi = (x^1, \dots, x^k, 0, \dots, 0)$. Let

$$\phi_S: U \cap S \rightarrow \mathbb{R}^k$$

be the restriction of the first k components of ϕ to $U \cap S$, that is, $\phi_S = (x^1, \dots, x^k)$. Note that $(U \cap S, \phi_S)$ is a chart for S in the subspace topology.

Definition 10.2. If S is a regular submanifold of dimension k in a manifold N of dimension n , then $n - k$ is said to be the *codimension* of S in N .

Remark. As a topological space, a regular submanifold of N is required to have the subspace topology.

Example. The dimension k of a regular submanifold may be equal to n . In that case, $U \cap S$ is defined by the vanishing of none of the coordinate functions and so $U \cap S = U$. Therefore, an open subset of a manifold is a regular submanifold of the same dimension.

Remark. There are other types of submanifolds, but unless otherwise specified, a “submanifold” will always mean a “regular submanifold”.

Proposition 10.1. Let S be a regular submanifold of N and $\mathfrak{U} = \{(U, \phi)\}$ a collection of compatible adapted charts of N that covers S . Then $\mathfrak{U}_S = \{(U \cap S, \phi_S)\}$ is an atlas for S . Therefore, a regular submanifold is itself a manifold. If N has dimension n and S is locally defined by the vanishing of $n - k$ coordinates, then $\dim S = k$.

Proof. Let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \Psi) = (V, y^1, \dots, y^n)$ be two adapted charts in \mathfrak{U} . Renumber the coordinates so that the last $n - k$ coordinates vanish on S ; then, assuming U and V intersect, for $p \in U \cap V \cap S$,

$$\phi_S(p) = (x^1, \dots, x^k) \quad \text{and} \quad \Psi_S(p) = (y^1, \dots, y^k)$$

so

$$\Psi_S \circ \phi_S^{-1}(x^1, \dots, x^k) = (y^1, \dots, y^k).$$

Thus, the transition function $\Psi_S \circ \phi_S^{-1}$ is C^∞ . Similarly, $\phi_S \circ \Psi_S^{-1}$ is also C^∞ . Hence, any two charts in $\mathfrak{U}_S = \{(U \cap S, \phi_S)\}$ are C^∞ compatible. Since $\{U \cap S\}_{U \in \mathfrak{U}}$ covers S , the collection \mathfrak{U}_S is a C^∞ atlas for S . This also shows that $\dim S = k$. \square

10.2 Level Sets of a Function

Definition 10.3. A *level set* of a map $F: N \rightarrow M$ is a subset

$$F^{-1}(\{c\}) = \{p \in N \mid F(p) = c\}$$

for some $c \in M$. We also write $F^{-1}(c)$ instead of $F^{-1}(\{c\})$. The value $c \in M$ is called the *level* of the level set $F^{-1}(c)$. If $F: N \rightarrow \mathbb{R}^m$, then $Z(F) := F^{-1}(\mathbf{0})$ is the *zero set* of F .

Remark. Recall that c is a regular value of F if and only if either c is not in the image of F or at every point $p \in F^{-1}(c)$, the differential $F_{*,p}: T_p N \rightarrow T_{F(p)} M$ is surjective. The inverse image $F^{-1}(c)$ of a

regular value c is called a *regular level set*. If the zero set $F^{-1}(\mathbf{0})$ is a regular level set of $F: N \rightarrow \mathbb{R}^m$, it is called a *regular zero set*.

Remark. If a regular level set $F^{-1}(c)$ is nonempty, say $p \in F^{-1}(c)$, then the map $F: N \rightarrow M$ is a submersion at p . Hence, $\dim N \geq \dim M$.

Lemma 10.2. Let $g: N \rightarrow \mathbb{R}$ be a C^∞ function. A regular level set $g^{-1}(c)$ of level c of the function g is the regular zero set of the function $f = g - c$.

Proof. For any $p \in N$,

$$g(p) = c \iff f(p) = g(p) - c = 0.$$

Hence, $g^{-1}(c) = f^{-1}(0)$. Let $S = f^{-1}(0)$. Since the differential $f_{*,p} = g_{*,p}$ at every point $p \in N$, the functions f and g have exactly the same critical points. Since g has no critical points in N , neither does f . \square

Theorem 10.3. Let $g: N \rightarrow \mathbb{R}$ be a C^∞ function on the manifold N . Then, a nonempty regular level set $S = g^{-1}(c)$ is a regular submanifold of N of codimension 1.

Proof. Let $f = g - c$. By the preceding lemma, $S = f^{-1}(0)$ and is a regular level set of f . Let $p \in S$. Since p is a regular point of f , relative to any chart (U, x^1, \dots, x^n) about p , $(\partial f / \partial x^i)(p) \neq 0$ for some i . WLOG, we may assume $(\partial f / \partial x^1)(p) \neq 0$. Then, the Jacobian determinant $\partial(f, x^2, \dots, x^n) / \partial(x^1, \dots, x^n)$ at p is $\partial f / \partial x^1(p) \neq 0$, so by the inverse function theorem, there is a coordinate neighborhood U_p of p on which f, x^2, \dots, x^n forms a coordinate system. Relative to the chart $(U_p, f, x^2, \dots, x^n)$, the level set $U_p \cap S$ is defined by setting the coordinate f equal to 0, so $(U_p, f, x^2, \dots, x^n)$ is an adapted chart relative to S . Hence, S is a regular submanifold of dimension $n - 1$ in N . \square

10.3 The Regular Level Set Theorem

Theorem 10.4. (Regular level set theorem). Let $F: N \rightarrow M$ be a C^∞ map of manifolds, with $\dim N = n$ and $\dim M = m$. Then, a nonempty regular level set $F^{-1}(c)$, $c \in M$, is a regular submanifold of N of dimension $n - m$.

Proof. Let $(V, \Psi) = (V, y^1, \dots, y^m)$ be a chart of M centered at c , i.e., $\Psi(c) = \mathbf{0} \in \mathbb{R}^m$. Then, $F^{-1}(V)$ is an open set in N that contains $F^{-1}(c)$. Moreover, in $F^{-1}(V)$, $F(c)^{-1} = (\Psi \circ F)^{-1}(\mathbf{0})$, so the level set $F^{-1}(c)$ is the zero set of $\Psi \circ F$. Let $F^i = y^i \circ F = r^i \circ (\Psi \circ F)$. Then, $F^{-1}(c)$ is also the zero set of the functions F^1, \dots, F^m on $F^{-1}(V)$.

By hypothesis, $F^{-1}(c)$ is nonempty, so $n \geq m$. Choose a point $p \in F^{-1}(c)$ and let $(U, \phi) = (U, x^1, \dots, x^n)$ be a coordinate neighborhood of p in N . Since $F^{-1}(c)$ is a regular level set, $p \in F^{-1}(c)$ is a regular point of F . Thus, the $m \times n$ Jacobian matrix $[\partial F^i / \partial x^j(p)]$ has rank m . WLOG, we may assume the first $m \times m$ block $[\partial F^i / \partial x^j]_{1 \leq i, j \leq m}$ is nonsingular.

Now, replace the first m coordinates x^1, \dots, x^m of the chart (U, ϕ) by F^1, \dots, F^m . We claim that there is a neighborhood U_p of p such that $(U_p, F^1, \dots, F^m, x^{m+1}, \dots, x^n)$ is a chart in the atlas of N .

Indeed, the Jacobian determinant

$$\frac{\partial (F^1, \dots, F^m, x^{m+1}, \dots, x^n)}{\partial (x^1, \dots, x^n)} = \det \left[\frac{\partial F^i}{\partial x^j}(p) \right]_{1 \leq i, j \leq m} \neq 0,$$

so by the inverse function theorem, $F^1, \dots, F^m, x^{m+1}, \dots, x^n$ forms a coordinate neighborhood of p .

In the chart $(U_p, F^1, \dots, F^m, x^{m+1}, \dots, x^n)$, the set $S := F^{-1}(c)$ is obtained by setting the first m coordinate functions F^1, \dots, F^m equal to 0. Thus, $(U_p, F^1, \dots, F^m, x^{m+1}, \dots, x^n)$ is an adapted chart for N relative to S . Hence S is a regular submanifold of dimension $n - m$. \square

The proof for the regular level set theorem gives the following useful lemma.

Lemma 10.5. Let $F: N \rightarrow \mathbb{R}^m$ be a C^∞ map on a manifold N of dimension n and let S be the level set $F^{-1}(\mathbf{0})$. If relative to some coordinate chart (U, x^1, \dots, x^n) about $p \in S$, the Jacobian determinant $\partial (F^1, \dots, F^m) / \partial (x^{j_1}, \dots, x^{j_m})(p) \neq 0$, then in some neighborhood of p one may replace x^{j_1}, \dots, x^{j_m} by F^1, \dots, F^m to obtain an adapted chart for N relative to S .

Remark. The regular level set theorem gives a sufficient but not necessary condition for a level set to be a regular submanifold. For example, if $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ is the map $f(x, y) = y^2$ then the zero set $Z(f) = Z(y^2)$ is the x -axis, a regular submanifold of \mathbb{R}^2 . However, since $\partial f / \partial x = 0$ and $\partial f / \partial y = 2y = 0$ on the x -axis, every point in $Z(f)$ is a critical point of f . Thus, although $Z(f)$ is a regular submanifold of \mathbb{R}^2 , it is not a regular level set of f .

10.4 Examples of Regular Submanifolds

Example. (Hypersurface). The solution set S of $x^n + y^n + z^n = 1$ ($n \in \mathbb{Z}$) in \mathbb{R}^3 is a manifold of dimension 2, for suppose $f(x, y, z) = x^n + y^n + z^n$, then $S = f^{-1}(1)$. Since $\partial f / \partial x = nx^{n-1}$, $\partial f / \partial y = ny^{n-1}$, and $\partial f / \partial z = nz^{n-1}$, f has no critical points in S . Thus, 1 is a regular value of $f: \mathbb{R}^3 \rightarrow \mathbb{R}$. By the regular level set theorem, S is a regular submanifold of \mathbb{R}^3 of dimension 2.

Problems

Problem 10.1. Define $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = x^3 - 6xy + y^2.$$

Find all values $c \in \mathbb{R}$ for which the level set $f^{-1}(c)$ is a regular submanifold of \mathbb{R}^2 .

Solution. Since $\partial f / \partial x = 3x^2 - 6y$ and $\partial f / \partial y = -6x + 2y$, the only critical points of f are $(0, 0)$ and $(6, 18)$. We want the critical points to lie outside of $f^{-1}(c)$, so $c \neq f(0, 0) = 0$ and $c \neq f(6, 18) = -108$.

Problem 10.2. Regular submanifolds

Suppose that a subset S of \mathbb{R}^2 has the property that locally on S one of the coordinates is a C^∞ function of the other coordinate. Show that S is a regular submanifold of \mathbb{R}^2 .

Solution. Let $p = (x, y)$ be a point on S . By hypothesis, there is a neighborhood U of p in \mathbb{R}^2 such that, wlog, $y = f(x)$ on $U \cap S$ for some C^∞ function $f: \mathbb{R} \supset A \rightarrow B \subset \mathbb{R}$ and $V := A \times B \subset U$. Let

$F: V \rightarrow \mathbb{R}^2$ be given by $F(x, y) = (x, y - f(x))$. Clearly, F is C^∞ , so it is a diffeomorphism onto its image. Thus, F can be used as a coordinate map. Moreover, $(V, F) = (V, x, y - f(x))$ is an adapted chart relative to S for \mathbb{R}^2 , since in the chart (V, F) , $V \cap S \subset U \cap S$, is defined by the vanishing of the coordinate function $y - f(x)$. Hence, S is a regular submanifold of \mathbb{R}^2 .

Problem 10.3. Graph of a smooth function

Show that the graph $\Gamma(f)$ of a smooth function $f: \mathbb{R}^n \rightarrow \mathbb{R}$,

$$\Gamma(f) = \{(x^1, \dots, x^n, f(x^1, \dots, x^n)) \in \mathbb{R}^{n+1}\},$$

is a regular submanifold of \mathbb{R}^{n+1} .

Solution. Clearly, $(\mathbb{R}^{n+1}, x^1, \dots, x^n, x^{n+1} - f(x^1, \dots, x^n))$ is an adapted chart for \mathbb{R}^{n+1} relative to $\Gamma(f)$, so $\Gamma(f)$ is a regular submanifold of \mathbb{R}^{n+1} .

11 Categories and Functors

11.1 Categories

Definition 11.1. A *category* consists of a collection of elements, called *objects*, and for any two objects A and B , a set $\text{Mor}(A, B)$ of elements, called *morphisms* from A to B , such that given any morphism $f \in \text{Mor}(A, B)$ and any morphism $g \in \text{Mor}(A, B)$, the *composite* $g \circ f \in \text{Mor}(A, B)$ is defined. Furthermore, the composition of morphisms is required to satisfy two properties:

- (i) The identity axiom: for each object A , there is an identity morphism $\mathbb{1}_A \in \text{Mor}(A, A)$ such that for any $f \in \text{Mor}(A, B)$ and $g \in \text{Mor}(B, A)$,

$$f \circ \mathbb{1}_A = f \quad \text{and} \quad \mathbb{1}_A \circ g = g;$$

- (ii) The associative axiom: for $f \in \text{Mor}(A, B)$, $g \in \text{Mor}(B, C)$, and $h \in \text{Mor}(C, D)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

If $f \in \text{Mor}(A, B)$, we often write $f: A \rightarrow B$.

Example. The collection of groups and group homomorphisms forms a category in which the objects are groups and for any two groups A and B , $\text{Mor}(A, B)$ is the set of group homomorphisms from A to B .

Example. The collection of all vector spaces over \mathbb{R} and \mathbb{R} -linear maps forms a category in which the objects are real vector spaces and for any two real vector spaces V and W , $\text{Mor}(V, W)$ is the set $\text{Hom}(V, W)$ of linear maps from V to W .

Example. The collection of all topological spaces together with continuous maps between them is called the *continuous category*.

Example. The collection of smooth manifolds together with smooth maps between them is called the *smooth category*.

Example. We call a pair (M, q) , where M is a manifold and q a point in M , a *pointed manifold*. Given any two such pairs (N, p) and (M, q) , let $\text{Mor}((N, p), (M, q))$ be the set of all smooth maps $F: N \rightarrow M$ such that $F(p) = q$. This gives rise to the *category of pointed manifolds*.

Definition 11.2. Two objects A and B in a category are said to be *isomorphic* if there are morphisms $f: A \rightarrow B$ and $g: B \rightarrow A$ such that

$$g \circ f = \mathbb{1}_A \quad \text{and} \quad f \circ g = \mathbb{1}_B.$$

In this case both f and g are called *isomorphisms*.

11.2 Functors

Definition 11.3. A (*covariant*) *functor* \mathcal{F} from one category \mathcal{C} to another category \mathcal{D} is a map that associates to each object A in \mathcal{C} an object $\mathcal{F}(A)$ in \mathcal{D} and to each morphism $f: A \rightarrow B$ a morphism $\mathcal{F}(f): \mathcal{F}(A) \rightarrow \mathcal{F}(B)$ such that

- (i) $\mathcal{F}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{F}(A)}$,
- (ii) $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

Example. The tangent space construction is a functor from the category of pointed manifolds to the category of vector spaces. To each pointed manifold (N, p) we associate the tangent space $T_p N$ and to each smooth map $f: (N, p) \rightarrow (M, f(p))$ we associate the differential $f_{*,p}: T_p N \rightarrow T_{f(p)} M$. The functorial property (i) holds because if $1: N \rightarrow N$ is the identity map, then its differential $\mathbb{1}_{*,p}: T_p N \rightarrow T_p N$ is also the identity map. The functorial property (ii) holds because in this context it is the chain rule

$$(g \circ f)_{*,p} = g_{*,f(p)} \circ f_{*,p}.$$

Proposition 11.1. Let $\mathcal{F}: \mathcal{C} \rightarrow \mathcal{D}$ be a functor from a category \mathcal{C} to a category \mathcal{D} . If $f: A \rightarrow B$ is an isomorphism in \mathcal{C} , then $\mathcal{F}(f): \mathcal{F}(A) \rightarrow \mathcal{F}(B)$ is an isomorphism in \mathcal{D} .

Proof. Since f is an isomorphism in \mathcal{C} , there is a morphism $g: B \rightarrow A$ in \mathcal{C} such that

$$g \circ f = \mathbb{1}_A \quad \text{and} \quad f \circ g = \mathbb{1}_B.$$

Applying the functor \mathcal{F} to both equations, noting the functorial properties, yields

$$\mathcal{F}(g) \circ \mathcal{F}(f) = \mathbb{1}_{\mathcal{F}(A)} \quad \text{and} \quad \mathcal{F}(f) \circ \mathcal{F}(g) = \mathbb{1}_{\mathcal{F}(B)}.$$

Hence, $\mathcal{F}(f)$ and $\mathcal{F}(g)$ are isomorphisms in \mathcal{D} . □

Remark. We can recast Corollary 9.2 in a more functorial form: Suppose $f: N \rightarrow M$ is a diffeomorphism. Then (N, p) and $(M, f(p))$ are isomorphic objects in the category of pointed manifolds. By

Proposition 11.1, the tangent spaces $T_p N$ and $T_{f(p)} M$ are isomorphic as vector spaces and therefore have the same dimension. It follows that the dimension of a manifold is invariant under a diffeomorphism.

If in the definition of a covariant functor we reverse the direction of the arrow for the morphism $\mathcal{F}(f)$, then we obtain a *contravariant functor*. More precisely, the definition is as follows.

Definition 11.4. A *contravariant functor* \mathcal{F} from one category \mathcal{C} to another category \mathcal{D} is a map that associates to each object A in \mathcal{C} an object $\mathcal{F}(A)$ in \mathcal{D} and to each morphism $f: A \rightarrow B$ a morphism $\mathcal{F}(f): \mathcal{F}(B) \rightarrow \mathcal{F}(A)$ such that

- (i) $\mathcal{F}(1_A) = 1_{\mathcal{F}(A)}$;
- (ii) $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$ (Note the reversal of order).

Example. Smooth functions on a manifold give rise to a contravariant functor that associates to each manifold M the algebra $\mathcal{F}(M) = C^\infty(M)$ of C^∞ functions on M and to each smooth map $F: N \rightarrow M$ of manifolds the pullback map $\mathcal{F}(F) = F^*: C^\infty(M) \rightarrow C^\infty(N)$, $F^*h = h \circ F$ for $h \in C^\infty(M)$. It is easy to verify that the pullback satisfies the two (contravariant) functorial properties:

- (i) $(1_M)^* = 1_{C^\infty(M)}$;
- (ii) if $F: N \rightarrow M$ and $G: M \rightarrow P$ are C^∞ maps, then $(G \circ F)^* = F^* \circ G^*: C^\infty(P) \rightarrow C^\infty(N)$.

Another example of a contravariant functor is the dual of a vector space, which we will review in the next subsection.

11.3 The Dual Functor and the Multivector Functor

Let V be a real vector space. Recall that its dual space V^* is the vector space of all *linear functionals* on V , i.e., linear functions $\alpha: V \rightarrow \mathbb{R}$. We also write

$$V^* = \text{Hom}(V, \mathbb{R}).$$

If V is a finite-dimensional vector space with basis $\{e_1, \dots, e_n\}$, then its dual space V^* has as a basis a collection of linear functionals $\{\alpha^1, \dots, \alpha^n\}$ defined by

$$\alpha^i(e_j) = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

Since a linear function on V is determined by what it does on a basis of V , this set of equations defines α^i uniquely.

A linear map $L: V \rightarrow W$ of vector spaces induces a linear map L^* , called the *dual* of L , as follows. To every linear functional $\alpha: W \rightarrow \mathbb{R}$, the dual L^* associates the linear functional

$$V \xrightarrow{L} W \xrightarrow{\alpha} \mathbb{R}.$$

Thus, the dual map $L^*: W^* \rightarrow V^*$ is given by

$$L^*(\alpha) = \alpha \circ L \text{ for } \alpha \in W^*.$$

Note that the dual of L reverse the direction of the arrow.

Proposition 11.2. (Functorial properties of the dual). Suppose V , W , and S are real vector spaces.

- (i) If $\mathbb{1}_V: V \rightarrow V$ is the identity map on V , then $\mathbb{1}_V^*: V^* \rightarrow V^*$ is the identity map on V^* .
- (ii) If $f: V \rightarrow W$ and $g: W \rightarrow S$ are linear maps, then $(g \circ f)^* = f^* \circ g^*$.

Proof. Given a linear functional α ,

- (i) $\mathbb{1}_V^*(\alpha) = \alpha \circ \mathbb{1}_V = \alpha$, and
- (ii) $(f^* \circ g^*)(\alpha) = f^*(\alpha \circ g) = \alpha \circ (g \circ f) = (g \circ f)^*(\alpha)$.

□

Fix a positive integer k . For any linear map $L: V \rightarrow W$ of vector spaces, define the *pullback* map $L^*: A_k(W) \rightarrow A_k(V)$, where $A_k(V)$ denotes the set of alternating k -linear functions on V , to be

$$(L^*f)(v_1, \dots, v_k) = f(L(v_1), \dots, L(v_k))$$

for $f \in A_k(W)$ and $v_1, \dots, v_k \in V$. It is easy to see that L^* is a linear map: $L^*(af + bg) = aL^*f + bL^*g$ for $a, b \in \mathbb{R}$ and $f, g \in A_k(W)$.

Proposition 11.3. The pullback of covectors by a linear maps satisfies the two functorial properties:

- (i) If $\mathbb{1}_V: V \rightarrow V$ is the identity map on V , then $\mathbb{1}_V^* = \mathbb{1}_{A_k(V)}$ is the identity map on $A_k(V)$.
- (ii) If $K: U \rightarrow V$ and $L: V \rightarrow W$ are linear maps of vector spaces, then

$$(L \circ K)^* = K^* \circ L^*: A_k(W) \rightarrow A_k(U).$$

Proof. Obvious. □

To each vector space V , we associate the vector space $A_k(V)$ of all k -covectors on V , and to each linear map $L: V \rightarrow W$ of vector spaces, we associate the pullback $A_k(L) = L^*: A_k(W) \rightarrow A_k(V)$. Then, $A_k()$ is a contravariant functor from the category of vector spaces and linear maps to itself. When $k = 1$, it is easy to see that the functor simplifies to the dual map. Thus, the multicovector functor $A_k()$ generalizes the dual functor $()^*$.

Problems

Problem 11.1. If $F: N \rightarrow M$ is a diffeomorphism of manifolds and $p \in N$, prove that $F_{*,F(p)}^{-1} = (F_{*,p})^{-1}$.

Proof. By the chain rule, $F_{*,F(p)}^{-1} \circ F_{*,p} = (F^{-1} \circ F)_{*,p} = \mathbb{1}_{*,p}$. Hence, $F_{*,F(p)}^{-1} = (F_{*,p})^{-1}$. □

Problem 11.2. Show that if $L: V \rightarrow V$ is a linear operator on a vector space V of dimension n , then the pullback $L^*: A_n(V) \rightarrow A_n(V)$ is multiplication by the determinant of L .

Proof. Let $\{e_1, \dots, e_n\}$ be a basis for V . Since any $f \in A_n(V)$ is n -linear, it suffices to compute $(L^*f)(e_1, \dots, e_n)$. Observe that $(L^*f)(e_1, \dots, e_n) = f(L(e_1), \dots, L(e_n)) = \det(L)$ is precisely the determinant of L defined as the unique alternating n -linear function on the columns of (the matrix representation) of L . \square

12 The rank of a Smooth Map

12.1 Constant Rank Theorem

Suppose $f: N \rightarrow M$ is a C^∞ map of manifolds and we want to show that the level set $f(c)^{-1}$ is a manifold for some c in M . In order to apply the level set theorem, we need the differential f_* to have maximal rank at every point of $f(c)^{-1}$. Sometimes, this may not be true; even if it is true, it may be difficult to show. In such cases, the constant-rank level set theorem can be helpful.

Before we prove the constant-rank level set theorem, we must first revisit the constant rank theorem for Euclidean spaces.

Theorem 12.1. (Constant rank theorem for Euclidean spaces). If $f: \mathbb{R}^n \supset U \rightarrow \mathbb{R}^m$ has constant rank k in a neighborhood of a point $p \in U$, then after a suitable change of coordinates near $p \in U$ and $f(p)$ in \mathbb{R}^m , the map f assumes the form

$$(x^1, \dots, x^n) \mapsto (x^1, \dots, x^k, 0, \dots, 0).$$

More precisely, there are diffeomorphisms G of a neighborhood of p in U sending p to the origin in \mathbb{R}^n and a diffeomorphism F of a neighborhood of $f(p)$ in \mathbb{R}^m sending $f(p)$ to the origin in \mathbb{R}^m such that

$$(F \circ f \circ G^{-1})(x^1, \dots, x^n) = (x^1, \dots, x^k, 0, \dots, 0).$$

Proof. Let U be a neighborhood of p in \mathbb{R}^n and suppose $f = (f^1, \dots, f^m): U \rightarrow \mathbb{R}^m$ has constant rank k in U . Without loss of generality, we can reorder the component functions f^1, \dots, f^m such that

$$\det \left[\frac{\partial f^i}{\partial x^j}(p) \right]_{1 \leq i, j \leq k} \neq 0.$$

Now, define $G: U \rightarrow \mathbb{R}^n$ by the replacement of the first k coordinates x^1, \dots, x^k by f^1, \dots, f^k , i.e.,

$$G(x^1, \dots, x^n) = (f^1, \dots, f^k, x^{k+1}, \dots, x^n).$$

Then, the Jacobian determinant of G

$$\frac{\partial (f^1, \dots, f^k, x^{k+1}, \dots, x^n)}{\partial (x^1, \dots, x^n)} = \det \left[\frac{\partial f^i}{\partial x^j}(p) \right]_{1 \leq i, j \leq k} \neq 0,$$

so by the inverse function theorem, there are neighborhoods $U_1 \subset U$ of p in \mathbb{R}^n and V_1 of $G(p)$ in \mathbb{R}^n such that $G: U_1 \rightarrow V_1$ is a diffeomorphism.

On V_1 , we have

$$\begin{aligned} (x^1, \dots, x^n) &= (G \circ G^{-1})(x^1, \dots, x^n) \\ &= ((f^1, \dots, f^k) \circ G^{-1}, (x^{k+1}, \dots, x^n) \circ G^{-1})(x^1, \dots, x^n). \end{aligned}$$

Hence, $x^i = f^i \circ G^{-1}$ for $1 \leq i \leq k$, and so

$$\begin{aligned} (f \circ G^{-1})(x^1, \dots, x^n) &= ((f^1, \dots, f^k) \circ G^{-1}, (f^{k+1}, \dots, f^m) \circ G^{-1})(x^1, \dots, x^n) \\ &= (x^1, \dots, x^k, ((f^{k+1}, \dots, f^m) \circ G^{-1})(x^1, \dots, x^n)) \\ &= (x^1, \dots, x^k, g^{k+1}, \dots, g^m), \end{aligned}$$

where $g^j = f^j \circ G^{-1}$, $k+1 \leq j \leq m$.

Since $G: U_1 \rightarrow V_1$ is a diffeomorphism and f has constant rank k on U , $f \circ G^{-1}$ has constant rank k on V_1 . Its Jacobian matrix is

$$J(f \circ G^{-1}) = \left[\begin{array}{c|c} \frac{\partial x^i}{\partial x^j} & \frac{\partial x^i}{\partial x^\beta} \\ \hline \frac{\partial g^\alpha}{\partial x^j} & \frac{\partial g^\alpha}{\partial x^\beta} \end{array} \right] = \left[\begin{array}{c|c} I_{k \times k} & \mathbf{0}_{k \times (n-k)} \\ \hline \frac{\partial g^\alpha}{\partial x^j} & \frac{\partial g^\alpha}{\partial x^\beta} \end{array} \right],$$

where $1 \leq i, j \leq k$, $k+1 \leq \alpha \leq m$ and $k+1 \leq \beta \leq n$. For this matrix to have rank k , all $(k+1) \times (k+1)$ minors must have determinant 0. Particularly,

$$\det \begin{bmatrix} I_{k \times k} & \mathbf{0}_{k \times 1} \\ \frac{\partial g^\alpha}{\partial x^j} & \frac{\partial g^\alpha}{\partial x^\beta} \end{bmatrix} = \frac{\partial g^\alpha}{\partial x^\beta} = 0$$

for all $k+1 \leq \alpha \leq m$ and $k+1 \leq \beta \leq n$. Thus, the functions g^α are functions of x^1, \dots, x^k only.

Finally, let $F: \mathbb{R}^m \rightarrow \mathbb{R}^m$ be given by

$$F(x^1, \dots, x^m) = (x^1, \dots, x^k, x^{k+1} - g^{k+1}, \dots, x^m - g^m).$$

Then,

$$(F \circ f \circ G^{-1})(x^1, \dots, x^n) = F(x^1, \dots, x^k, g^{k+1}, \dots, g^m) = (x^1, \dots, x^k, 0, \dots, 0).$$

□

Theorem 12.2. (Constant rank theorem). Let N and M be manifolds of dimensions n and m respectively. Suppose $f: N \rightarrow M$ has constant rank k in a neighborhood of a point p in N . Then, there are charts (U, ϕ) centered at p in N and (V, Ψ) centered at $f(p)$ in M such that for (r^1, \dots, r^n) in $\phi(U)$,

$$(\Psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Proof. Let (U_0, ϕ_0) be a chart centered at p in N and (V_0, Ψ_0) a chart centered at $f(p)$ in M . Then,

the map $\Psi \circ f \circ \phi^{-1}$ is a smooth map between Euclidean spaces with constant rank k in $\phi(U_0)$. By the constant rank theorem for Euclidean spaces (Theorem 12.1), there are diffeomorphisms $F: V_0 \supset V \rightarrow \mathbb{R}^m$ and $G: U_0 \supset U \rightarrow \mathbb{R}^n$ into their images such that

$$(F \circ (\Psi_0 \circ f \circ \phi_0^{-1}) \circ G^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Let $\phi = G \circ \phi_0$ and $\Psi = F \circ \Psi_0$, and we are done. \square

Theorem 12.3. (Constant-rank level set theorem). Let $f: N \rightarrow M$ be a C^∞ map of manifolds and $c \in M$. If f has constant rank k in a neighborhood of the level set $f^{-1}(c)$ in N , then $f^{-1}(c)$ is a regular submanifold of N of codimension k .

Proof. Let p be a point in $f^{-1}(c)$. By the constant rank theorem (Theorem 12.2), there are charts $(U, \phi) = (U, x^1, \dots, x^n)$ centered at p in N and (V, Ψ) centered at $f(p) = c$ in M such that

$$(\Psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^k, 0, \dots, 0).$$

Thus, the level set $(\Psi \circ f \circ \phi^{-1})^{-1}(0)$ is obtained by the vanishing of the coordinates (r^1, \dots, r^k) .

Now, observe that

$$\phi(f^{-1}(c)) = \phi(f^{-1}(\Psi^{-1}(0))) = (\Psi \circ f \circ \phi^{-1})^{-1}(0).$$

Thus, the level set $f^{-1}(c)$ is defined by the vanishing of the coordinate functions (x^1, \dots, x^k) , where $x^i = r^i \circ \phi$. Hence, $f^{-1}(c)$ is a regular submanifold of N of codimension k . \square

12.2 The Immersion and Submersion Theorems

Proposition 12.4. Let N and M be manifolds of dimensions n and m respectively. If a C^∞ map $f: N \rightarrow M$ is an immersion at a point $p \in N$, then it has constant rank n in a neighborhood of p . If a C^∞ map $f: N \rightarrow M$ is a submersion at a point $p \in N$, then it has constant rank m in a neighborhood of p .

Proof. Let (U, ϕ) be a chart about p in N and (V, Ψ) a chart about $f(p)$ in M . Then, relative to the charts (U, ϕ) and (V, Ψ) , the linear map $f_{*,p}$ is represented by the matrix $[\partial f^i / \partial x^j(p)]$. Clearly,

$$\begin{aligned} f_{*,p} \text{ is injective} &\iff n \leq m, \text{rk} [\partial f^i / \partial x^j(p)] = n, \\ f_{*,p} \text{ is surjective} &\iff n \geq m, \text{rk} [\partial f^i / \partial x^j(p)] = m. \end{aligned}$$

Hence, an immersion or submersion at p is equivalent to the maximality of $\text{rk} [\partial f^i / \partial x^j(p)]$. Moreover, if f has maximal rank at p , then it has maximal rank at all points in some neighborhood of p . This proves the proposition. \square

By Proposition 12.4, the following theorems follow directly from the constant rank theorem.

Proposition 12.5. Let $f: N \rightarrow M$ be manifolds of dimensions n and m respectively.

- (i) **(Immersion theorem)** Suppose $f: N \rightarrow M$ is an immersion at $p \in N$. Then, there are charts (U, ϕ) centered at p in N and (V, Ψ) centered at $f(p)$ in M such that in a neighborhood of $\phi(p)$,

$$(\Psi \circ f \circ \phi^{-1})(r^1, \dots, r^n) = (r^1, \dots, r^n, 0, \dots, 0).$$

- (ii) **(Submersion theorem)** Suppose $f: N \rightarrow M$ is submersion at $p \in N$. Then, there are charts (U, ϕ) centered at p in N and (V, Ψ) centered at $f(p)$ in M such that in a neighborhood of $\phi(p)$,

$$(\Psi \circ f \circ \phi^{-1})(r^1, \dots, r^m, r^{m+1}, \dots, r^n) = (r^1, \dots, r^m).$$

Corollary 12.6. A submersion $f: N \rightarrow M$ of manifolds is an open map.

Proof. Let W be an open subset of N . Choose a point $f(p)$ in $f(W)$, with $p \in W$. By the submersion theorem, f is locally a projection. Since a projection is an open map, there is an open neighborhood U of p in W such that $f(U)$ is open in M . Moreover,

$$f(p) \in f(U) \subset f(W),$$

so $f(W)$ is open in M . □

12.3 Images of Smooth Maps

Definition 12.1. A C^∞ map $f: N \rightarrow M$ is called an *embedding* if

- (i) it is a one-to-one immersion and
- (ii) the image $f(N)$ with the subspace topology is homeomorphic to N under f .

Remark. The nomenclature of submanifolds may vary across different literatures. Many authors give the image $f(N)$ of a one-to-one immersion $f: N \rightarrow M$ not the subspace topology, but the topology inherited from f ; i.e., a subset $f(U)$ of $f(N)$ is said to be open if and only if U is open in N . With this topology, $f(N)$ is by definition homeomorphic to N . These authors then define a submanifold to be the image of any one-to-one immersion with the topology and differentiable structure inherited from f . Such a set is sometimes called an *immersed submanifold* of M . If the underlying set of an immersed submanifold is given the subspace topology instead, then the resulting space may not even be a manifold at all.

Theorem 12.7. If $f: N \rightarrow M$ is an embedding, then its image $f(N)$ is a regular submanifold of M .

Proof. Let $p \in N$. By the immersion theorem, there are charts (U, x^1, \dots, x^n) about p in N and

(V, y^1, \dots, y^m) about $f(p)$ in M such that $f: U \rightarrow V$ has the form

$$(x^1, \dots, x^n) \mapsto (x^1, \dots, x^n, 0, \dots, 0).$$

Thus, $f(U)$ is defined by the vanishing of the coordinates y^{n+1}, \dots, y^m .

By the definition of an embedding, with the subspace topology, $f(N)$ is homeomorphic to N . Thus, the image $f(U)$ is open in $f(N)$. By definition of the subspace topology, there exists an open set V' in M such that $(V' \cap f(N)) = f(U)$. In $V \cap V'$,

$$(V \cap V') \cap f(N) = V \cap f(U) = f(U),$$

and since $f(U)$ is defined by the vanishing of the coordinates y^{n+1}, \dots, y^m , $(V \cap V', y^1, \dots, y^m)$ is an adapted chart for $f(N)$ containing $f(p)$. Since $f(p)$ is an arbitrary point in $f(N)$, this shows that $f(N)$ is a regular submanifold of M . \square

Theorem 12.8. If N is a regular submanifold of M , then the inclusion $i: N \rightarrow M$, $i(p) = p$, is an embedding.

Proof. Clearly, $i: N \rightarrow i(N)$ is a homeomorphism. Thus, it remains to show that $i: N \rightarrow M$ is an immersion.

Let $p \in N$. Since N is a regular submanifold of M , there exists an adapted chart (V, y^1, \dots, y^m) for M about p such that $V \cap N$ is the zero set of y^{n+1}, \dots, y^m . Then, relative to the charts $(V \cap N, y^1, \dots, y^n)$ for N and (V, y^1, \dots, y^m) , the inclusion i is given by,

$$(y^1, \dots, y^n) \mapsto (y^1, \dots, y^n, 0, \dots, 0),$$

which shows that i is an inclusion. \square

Remark. In the literature, the image of an embedding is often called an *embedded submanifold*. Theorems 12.7 and 12.8 show that an embedded submanifold and a regular submanifold are one and the same.

12.4 Smooth Maps into a Submanifold

Suppose $f: N \rightarrow M$ is a C^∞ map whose image $f(N)$ lies in a subset $S \subset M$. Even if S is manifold, the induced map $\tilde{f}: N \rightarrow S$ may or may not be C^∞ depending on whether S is a regular submanifold or an immersed submanifold of M .

Theorem 12.9. Suppose $f: N \rightarrow M$ is C^∞ and the image of f lies in a subset S of M . If S is a regular submanifold of M , then the induced map $\tilde{f}: N \rightarrow S$ is C^∞ .

Proof. Let $p \in N$ and denote by n , m , and s the dimensions of N , M , and S respectively. Since S is a regular submanifold of M of dimension s , there exists an adapted chart (V, y^1, \dots, y^m) for M about p such that $S \cap V$ is defined by the vanishing of the coordinates y^{s+1}, \dots, y^m with coordinate map $\Psi_S = (y^1, \dots, y^s)$. By the continuity of f , there exists a neighborhood U of p such that $f(U) \subset V$ is

open in M . Then, $f(U) \subset V \cap S$, so that for $q \in U$,

$$(\Psi \circ f)(q) = (y^1(f(q)), \dots, y^s(f(q)), 0, \dots, 0).$$

It follows that on U ,

$$\Psi_S \circ \tilde{f} = (y^1 \circ f, \dots, y^s \circ f),$$

and since $y^1 \circ f, \dots, y^s \circ f$ are C^∞ on U , \tilde{f} is C^∞ on U and hence at p . As p was an arbitrary point in N , it follows that the map $\tilde{f}: N \rightarrow S$ is C^∞ . \square

12.5 The Tangent Plane to a Surface in \mathbb{R}^3

Suppose $f(x^1, x^2, x^3)$ is a real-valued function on \mathbb{R}^3 with no critical points on its zero set $N = f^{-1}(0)$. By the regular level set theorem, N is a regular submanifold of \mathbb{R}^3 . By Theorem 12.8, the inclusion $i: N \rightarrow \mathbb{R}^3$ is an embedding, so at any point p in N , $i_{*,p}: T_p N \rightarrow T_p \mathbb{R}^3$ is injective. We may therefore think of the tangent plane $T_p N$ as a plane in $T_p \mathbb{R}^3 \simeq \mathbb{R}^3$ and we would like to find the equation of this plane.

Suppose $v = \sum v^i \partial/\partial x^i|_p$ is a vector in $T_p N$. Under the linear isomorphism $T_p \mathbb{R}^3 \simeq \mathbb{R}^3$, we identify v with the vector $\langle v^1, v^2, v^3 \rangle$ in \mathbb{R}^3 . Now, let $c(t)$ be a curve lying in N starting at p with initial vector v , i.e., $c(0) = p$ and $c'(0) = v = \langle v^1, v^2, v^3 \rangle$. Since $c(t)$ lies in N , $f(c(t)) = 0$ for all t . By the chain rule,

$$0 = \frac{d}{dt} f(c(t)) = \sum_{i=1}^3 \frac{\partial f}{\partial x^i}(c(t)) (c^i)'(t).$$

At $t = 0$,

$$0 = \sum_{i=1}^3 \frac{\partial f}{\partial x^i}(c(0)) (c^i)'(0) = \sum_{i=1}^3 \frac{\partial f}{\partial x^i}(p) v^i.$$

Since the vector $v = \langle v^1, v^2, v^3 \rangle$ represents an arrow from $p = (p^1, p^2, p^3)$ to $x = (x^1, x^2, x^3)$ in the tangent plane, one usually makes the substitution $v^i = x^i - p^i$. Thus, the tangent plane to N at p is defined by the equation

$$\sum_{i=1}^3 \frac{\partial f}{\partial x^i}(p) (x^i - p^i) = 0.$$

An interpretation of this equation is that the gradient vector of f at p is normal to any vector in the tangent plane.

Problems

Problem 12.1. The unit sphere S^n in \mathbb{R}^{n+1} is defined by the equation $\sum_{i=1}^{n+1} (x^i)^2 = 1$. For $p = (p^1, \dots, p^{n+1}) \in S^n$, show that a necessary and sufficient condition for

$$X_p = \sum a^i \frac{\partial}{\partial x^i} \Big|_p \in T_p(\mathbb{R}^{n+1})$$

to be tangent to S^n at p is $\sum a^i p^i = 0$.

Solution. Clearly, S^n is the zero set of the function $f: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ given by

$$f(x^1, \dots, x^{n+1}) = \sum_{i=1}^{n+1} (x^i)^2 - 1.$$

Now, identify $T_p(\mathbb{R}^{n+1}) \simeq \mathbb{R}^{n+1}$. Then, we can identify $X_p \mapsto v = \langle a^1, \dots, a^{n+1} \rangle$, and thus the tangent plane to S^n at p is defined by the equation

$$\sum_{i=1}^{n+1} \frac{\partial f}{\partial x^i}(p) a^i = 0,$$

or equivalently,

$$\sum_{i=1}^{n+1} p^i a^i = 0.$$

Problem 12.2. Critical points of a smooth map on a compact manifold

Show that a smooth map f from a compact manifold N to \mathbb{R}^m has a critical point.

Solution.

13 The Tangent Bundle

References

- [1] L. W. Tu, *Introduction to Manifolds*, 2nd ed. Springer, 2010.