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Bayesian quantile forecasting via the realized hysteretic GARCH model

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Abstract

This research introduces a new model, a realized hysteretic GARCH, that is similar to a three-regime nonlinear framework combined with daily returns and realized volatility. The setup allows the mean and volatility switching in a regime to be delayed when the hysteresis variable lies in a hysteresis zone. This nonlinear model presents explosive persistence and high volatility in Regime 1 in order to capture extreme cases. We employ the Bayesian Markov chain Monte Carlo (MCMC) procedure to estimate model parameters and to forecast volatility, value at risk (VaR), and expected shortfall (ES). A simulation study highlights the properties of the proposed MCMC methods, as well as their accuracy and satisfactory performance as quantile forecasting tools. We also consider two competing models, the realized GARCH and the realized threshold GARCH, for comparison and carry out Bayesian risk forecasting via predictive distributions on four stock markets. The out-of-sample period covers the recent 4 years by a rolling window approach and includes the COVID-19 pandemic period. Among the realized models, the realized hysteretic GARCH model outperforms at the 1% level in terms of violation rates and backtests.

KEYWORDS

expected shortfall, hysteretic GARCH model, Markov chain Monte Carlo method, realized volatility, threshold GARCH model, value at risk

1 | INTRODUCTION

The GARCH family proposed by Bollerslev (1986) and Engle (1982) utilizes daily returns on an asset to model its volatility and then uses the collected information to predict next period volatility. However, in any market, there are many unexpected changes over a short time period. Thus, standard GARCH models capture that the speed of conditional variance is slow. Realized variance, being the summation of squared intraday returns, has speedily gained recognition as a measure of daily volatility. Realized volatility itself is a highly efficient estimator of return volatility but is limited to microstructure noise and nontrading hours. Several papers provide an expression for the bias and bias correction of realized variance, such as Oomen (2005), Zhang et al. (2005), Hansen and Lunde (2006), Bandi and Russell (2008), and Barndorff-Nielsen et al. (2008) among others. The realized GARCH (R-GARCH) model proposed by Hansen et al. (2012) employs high-frequency financial information in realized measurement, such as the realized kernel. There are some extensions, such as Hansen and Huang (2016) who propose the realized EGARCH model for return series, and Chen and Watanabe (2019) who investigate a realized threshold GARCH (R-TGARCH) that discusses the asymmetry of realized volatility for quantile and volatility forecasts. This present research proposes a realized hysteretic GARCH model, which is a three-regime nonlinear model combined with daily returns and realized volatility.

Hysteresis concepts have been applied in the areas of economics, engineering, epidemiology, and even in criminology; see Truong et al. (2017), Chen et al. (2021), and so on. In economics, it refers to an event that persists into the future, even after the factors that led to that event have been eliminated. To avoid abrupt changes in regime switching, Li et al. (2015) take the joint hysteretic and regime-switching structure and extend it to the hysteretic autoregressive (HAR) model instead of a traditional two-regime threshold autoregressive model. Chen and Truong (2016) integrate a hysteresis zone in the HAR model with GARCH specification. Our research enhances the idea to propose a realized hysteretic GARCH (R-HGARCH) model to forecast tail risks, setting up a nonlinear heteroskedastic model combined with daily returns and realized volatility. The hysteresis model with two thresholds is similar to a three-regime R-GARCH model. When the hysteresis variable lies in a hysteresis zone, this model allows mean and volatility switching in a regime to be delayed. To the best of our knowledge, realized hysteretic models have not yet been considered in the literature.

There are indeed more hysteresis or buffered time series models in the literature, for example, Truong et al. (2017) and Zhu et al. (2017). In addition, there are some extensions in multivariate HAR models, such as Chen et al. (2019), Chen et al. (2019), and Chen et al. (2021). The novelty of this present study compared with the other heteroskedastic models is that we incorporate realized volatility into a nonlinear model and present explosive persistence and dynamic conditional volatility in Regime 1 in order to capture extreme cases. In other words, we allow an explosive regime in the lower regime in the R-HGARCH model so as to better describe the tail risks: value at risk (VaR) and expected shortfall (ES). VaR and ES are widely used to evaluate whether financial institutions have sufficient capital to cover maximum losses, especially in many commercial institutions. Morgan (1996) presents VaR as a measure in risk management, defining the probability of the worst potential loss over a given time period for a given probability. The Basel Committee on Banking Supervision (BCBS) has considered another choice to measure risk, ES, in bank capital regulation. ES is an alternative for measuring risk that denotes the expected value of a return less than VaR under a probability level. It is also called conditional value at risk (CVaR); see Acerbi and Tasche (2002). ES is considered a more helpful risk measure than VaR, because it is coherent; see Föllmer and Schied (2002). It is also a criterion of financial portfolio risk.

The integral of ES is a complex issue when the error term follows a non-Gaussian distribution. To solve this challenge, we employ the Bayesian Markov chain Monte Carlo (MCMC) procedure to estimate all model parameters efficiently and forecast volatility, VaR, and ES. Many papers use Bayesian quantile tools to provide VaR forecasts; see Chen et al. (2012), Chen et al. (2017), and Chen and Gerlach (2016). The advantages of Bayesian estimation are as follows. (i) Through prior probability distribution, we can constrain the parameter space. (ii) This approach solves high-dimension problems and has effective estimate parameters in the mathematical operation. (iii) It allows one to make inferences on all unknown parameters simultaneously.

We carry out a simulation study and empirical study using Bayesian MCMC methods. Our simulation study highlights the properties of the proposed MCMC methods and their accuracy and satisfactory performance as quantile forecasting tools. As our empirical investigation, Bayesian risk forecasting via predictive distributions for four stock markets covers Nikkei 225, KOSPI Composite Index (KOSPI), FTSE 100 Index (FTSE 100), and Dow Jones Industrial Average (DJIA). We also consider two competing models, R-GARCH, and R-TGARCH, for comparing their quantile forecasts. The out-of-sample period covers the recent 4 years and includes the COVID-19 pandemic period.

Our research evaluates the volatility forecasts and quantile forecasts and utilizes the violation rate (VRate) to analyze VaR and ES. VRate is a common way to know the performance of VaR. We employ backtesting methods for the accuracy of VaR estimation. There are three hypothesis-testing methods: The first is the unconditional coverage (UC) test of Kupiec (1995); the second is the conditional coverage (CC) test of Christoffersen (1998); the third is the dynamic conditional quantile (DQ) test of Engle and Manganelli (2004). The UC test evaluates whether the true loss exceeds VaR. The CC test evaluates whether VaR estimates have accurate coverage at each time, by combining the likelihood ratio test and the UC test. It is notable that the DQ test is more influential than the CC test. Chen et al. (2012) consider the ES rate to be similar to VRate. We follow Embrechts et al. (2005) who use two measures to evaluate ES. According to Patton (2011), we employ mean squared error (MSE) and quasi-likelihood (QLIKE) to evaluate volatility forecasts to measure the performance of the target models. Empirical data in Patton and Sheppard (2009) and Hansen (2005) prove that MSE and QLIKE have a better ability to find good volatility prediction models.

The rest of this research runs as follows. Section 2 presents the R-GARCH, R-TGARCH, and the hysteretic GARCH models. Section 3 illustrates the Bayesian inference through the MCMC methods by focusing on the hysteretic GARCH models and provides Bayesian forecasting of VaR and ES. Section 4 demonstrates the performance of Bayesian estimation and forecasting via a simulation study. Section 5 validates the empirical examples. Section 6 provides conclusions and future directions.

2 | REALIZED GARCH-TYPE MODELS

Andersen et al. (2001) observe that realized volatility has the capability to estimate volatility on returns and offers more information than typically squared returns. Realized volatility applies high-frequency data to estimate daily volatility. Suppose that we have m intraday returns during day t, $\{r_{t,i},\ i=1,\ldots,m\}$, a model-free estimator of the true volatility σ_t^2 is as follows:

$$RV_t=\sum_{i=1}^{m} r_{t,i}^2.$$

There are some drawbacks to using high-frequency data for calculating the realized volatility estimator, which include market microstructure noise and nontrading hours. Barndorff-Nielsen et al. (2008) propose a realized kernel estimator for lessening the effects of market microstructure noise on the realized volatility estimators. We discuss three GARCH-type models incorporating realized kernel (x_t) along these lines.

The R-GARCH model by Hansen et al. (2012) is

$$\begin{aligned} r_t &= \mu_t+\sigma_t\zeta_t,\ \ \zeta_t \overset{i.i.d.}{\sim} D(0,1),\\ \log x_t &= \xi+\psi\log\sigma_t^2+\tau(\zeta_t)+u_t,\ \ u_t \overset{i.i.d.}{\sim} N(0,\sigma_u^2),\\ \log\sigma_t^2 &= \alpha_0+\alpha_1\log x_{t-1}+\beta_1\log\sigma_{t-1}^2, \end{aligned} \tag{1}$$

where r_t and x_t are daily return and realized kernel of the reference asset, respectively; $D(0,1)$ stands for a standardized distribution; and $\{\zeta_t\}$ and $\{u_t\}$ are independent. Here, $\sigma_t^2=\mathrm{var}(r_t|F_{t-1})$ with $F_t=\sigma(r_t,x_t,r_{t-1},x_{t-1},\ldots)$, and μ_t can be a constant term or the lagged AR(1) effect. These equations are the return equation, the GARCH (volatility) equation, and the measurement equation, respectively. The measurement equation makes the realized measurement have latent volatility in a function. Moreover, $\tau(\zeta_t)=\tau_1\zeta_t+\tau_2(\zeta_t^2-1)$ is the leverage function that captures the asymmetric response, reacting to a return shock and volatility at the same time. The measurement equation reflects true volatility. Therefore, we expect ξ to be near zero and ψ to be near one.

Chen and Watanabe (2019) investigate the realized two-regime threshold GARCH model. We call it the R-TGARCH model for short, which is

$$\begin{aligned} r_t &= \mu_t+\sigma_t\zeta_t,\ \zeta_t \overset{i.i.d.}{\sim} D(0,1),\\ \mu_t &= \begin{cases} \phi_0^{(1)}+\phi_1^{(1)}r_{t-1}, & \text{if } z_{t-1}\le\gamma,\\ \phi_0^{(2)}+\phi_1^{(2)}r_{t-1}, & \text{if } z_{t-1}>\gamma,\end{cases}\\ \log x_t &= \xi+\psi\log\sigma_t^2+\tau_1\zeta_t+\tau_2(\zeta_t^2-1)+u_t,\\ & \quad u_t \overset{i.i.d.}{\sim} N(0,\sigma_u^2),\\ \log\sigma_t^2 &= \begin{cases} \alpha_0^{(1)}+\alpha_1^{(1)}\log x_{t-1}+\beta_1^{(1)}\log\sigma_{t-1}^2, & \text{if } z_{t-1}\le\gamma,\\ \alpha_0^{(2)}+\alpha_1^{(2)}\log x_{t-1}+\beta_1^{(2)}\log\sigma_{t-1}^2, & \text{if } z_{t-1}>\gamma,\end{cases} \end{aligned} \tag{2}$$

where z_t is the threshold variable, and the delay lag equals 1, which impacts greatly on the dynamic structure of r_t. We choose z_t as the observation of r_t, which can be other exogenous variable. The lagged AR(1) effect mainly helps judge the mean reversion and market efficiency in a time series.

Inspired by the R-TGARCH model of Chen and Watanabe (2019), we propose the R-HGARCH model. The R-HGARCH model via a switching mechanism is more flexible under different conditions. Let S_t be a regime indicator. We specify the R-HGARCH model as follows:

$$r_t=\mu_t+\sigma_t\zeta_t,\ \ \zeta_t \overset{i.i.d.}{\sim} D(0,1), \tag{3}$$

$$\begin{aligned} \mu_t &= \begin{cases} \phi_0^{(1)}+\phi_1^{(1)}r_{t-1}, & \text{if } S_t=1,\\ \phi_0^{(2)}+\phi_1^{(2)}r_{t-1}, & \text{if } S_t=2,\end{cases}\\ \log x_t &= \xi+\psi\log\sigma_t^2+\tau_1\zeta_t+\tau_2(\zeta_t^2-1)+u_t,\\ & \quad u_t \overset{i.i.d.}{\sim} N(0,\sigma_u^2), \end{aligned} \tag{4}$$

$$\begin{aligned} \log\sigma_t^2 &= \begin{cases} \alpha_0^{(1)}+\alpha_1^{(1)}\log x_{t-1}+\beta_1^{(1)}\log\sigma_{t-1}^2, & \text{if } S_t=1,\\ \alpha_0^{(2)}+\alpha_1^{(2)}\log x_{t-1}+\beta_1^{(2)}\log\sigma_{t-1}^2, & \text{if } S_t=2,\end{cases}\\ \text{where } S_t &= \begin{cases} 1 & \text{if } r_{t-1}\le c_L,\\ 2 & \text{if } r_{t-1}>c_U,\\ S_{t-1} & \text{if } c_L<r_{t-1}\le c_U,\end{cases} \end{aligned} \tag{5}$$

and where S_t is the hysteresis variable. Here, S_t adjusts to different hysteresis regimes, while c_L and c_U are threshold parameters bounded on a hysteresis regime.

We organize $\log\sigma_{t-1}^2$ of the measurement equation in (4) and the GARCH equation in (5) together. To describe extreme cases, we allow an explosive regime in the lower regime, while the stationarity condition is satisfied in the upper regime of $\log\sigma_t^2$ by

$$|\beta_1^{(1)} + \alpha_1^{(1)}\psi| < b_3, \\ |\beta_1^{(2)} + \alpha_1^{(2)}\psi| < 1, \tag{6}$$

where $b_3 = 1 + \epsilon$, $0 < \epsilon < 1$, is a specified constant that is greater than one. In other words, we relax a common constraint in the lower regime and make inferences through Bayesian MCMC methods to estimate model parameters. The Bayesian estimation enables efficiency and flexibly handles hysteresis R-GARCH models.

3 | BAYESIAN INFERENCE

The posterior distributions enable us to obtain the Bayesian estimation of model parameters via combining a likelihood function and prior distributions. The MCMC methods are effective and can solve complicated high-dimensional models. One can easily achieve any specified quantile of 1-day-ahead returns from the predictive distribution via this MCMC sampling method. The MCMC sampling approach has a practical advantage, in addition to computational convenience, in that volatility, VaR, and ES forecasting are obtainable.

Suppose that $\boldsymbol{\theta}$ denotes the parameters for the realized hysteretic GARCH model; $\boldsymbol{\theta} = (\boldsymbol{\phi}_1', \boldsymbol{\phi}_2', \boldsymbol{\alpha}_1', \boldsymbol{\alpha}_2', \boldsymbol{\xi}', \sigma_u^2, \boldsymbol{\omega})'$ with $\boldsymbol{\phi}_i = (\phi_0^{(i)}, \phi_1^{(i)})'$, $\boldsymbol{\alpha}_i = (\alpha_0^{(i)}, \alpha_1^{(i)}, \beta_1^{(i)})'$, $\boldsymbol{\xi} = (\xi, \psi, \tau_1, \tau_2)'$, and $\boldsymbol{\omega} = (c_L, c_U)'$. We use Bayesian inference to estimate all unknown parameters and forecast VaR (ES) simultaneously through MCMC methods. Let $\boldsymbol{r} = (r_1, r_2, ..., r_n)'$ and $\boldsymbol{x} = (x_1, x_2, ..., x_n)'$. When $\{\zeta_t\}$ follows a standard normal distribution, the joint log-likelihood function is

$$\begin{aligned} \log \mathcal{L}(\boldsymbol{r}, \boldsymbol{x}|\boldsymbol{\theta}) &= \sum_{t=2}^{n} \{f_1(r_t|r_1, x_1, ..., r_{t-1}, x_{t-1}) \\ &\quad + f_2(x_t|r_1, x_1, ..., r_{t-1}, x_{t-1}, r_t)\}, \\ &= con + \sum_{t=2}^{n} \left[-\frac{1}{2}\log \sigma_t^2 - \frac{(r_t - \phi_0^{(1)} - \phi_1^{(1)} r_{t-1})^2}{2\sigma_t^2} \right] \mathrm{I}(S_t = 1) \\ &\quad + \sum_{t=2}^{n} \left[-\frac{1}{2}\log \sigma_t^2 - \frac{(r_t - \phi_0^{(2)} - \phi_1^{(2)} r_{t-1})^2}{2\sigma_t^2} \right] \mathrm{I}(S_t = 2) \\ &\quad + \sum_{t=2}^{n} \left\{ -\frac{1}{2}\log \sigma_u^2 - \frac{u_t^2}{2\sigma_u^2} \right\}, \end{aligned} \tag{7}$$

where con is a constant and $u_t = \log x_t - [\xi + \psi \log \sigma_t^2 + \tau_1 \zeta_t + \tau_2(\zeta_t^2 - 1)]$.

Most financial time series exhibit leptokurtic, asymmetric, and volatility clustering. The empirical distributions of asset returns often have a heavy tail and therefore may also be skewed. We thus employ the skew Student's t distribution of Hansen (1994) for all R-GARCH-type models. The probability density function of skew Student's t distribution by Hansen (1994) denoted by $St(\eta, \nu)$ is

$$p(\zeta_t|\nu, \eta) = \begin{cases} bc\left[1 + \frac{1}{\nu-2}\left(\frac{b\zeta_t + a}{1-\eta}\right)^2\right]^{-(\nu+1)/2}, & \text{if } \zeta_t < -\frac{a}{b}, \\ bc\left[1 + \frac{1}{\nu-2}\left(\frac{b\zeta_t + a}{1+\eta}\right)^2\right]^{-(\nu+1)/2}, & \text{if } \zeta_t \geq -\frac{a}{b}, \end{cases}$$

where degrees of freedom ν and skewness parameter η follows $2 < \nu < \infty$ and $-1 < \eta < 1$, respectively. We set the constants respectively as

$$a = 4\eta c\left(\frac{\nu - 2}{\nu - 1}\right), b^2 = 1 + 3\eta^2 - a^2, c = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi(\nu - 2)}\Gamma(\nu/2)}.$$

We divide $\boldsymbol{\theta}$ into the following groups: (i) $\boldsymbol{\phi}_j$, $j = 1, 2$ under the return equation; (ii) $\boldsymbol{\alpha}_j$, $j = 1, 2$ under the volatility equation; (iii) $\boldsymbol{\xi}$ under the measurement equation; (iv) $\boldsymbol{\omega}$ is the hysteresis variable vector; (v) σ_u^2; (vi) ν is degrees of freedom; and (vii) η is a skewness parameter.

3.1 | Prior distributions

The prior for $\boldsymbol{\phi}_j$ in the mean equation is

$$\boldsymbol{\phi_j} \sim N(\boldsymbol{\phi}_0, \Sigma)\mathrm{I}(A_{1j}), \tag{8}$$

where A_{1j} needs to satisfy condition $|\phi_1^{(j)}| < 1$. This is a truncated Gaussian prior with mean $\boldsymbol{\phi}_0$ and variance Σ. To specify the prior for $\boldsymbol{\alpha}_j$, we adopt the prior constraint in Equation (6) to allow $\log \sigma_t^2$ to be nonstationary in the low regime. We specify the joint prior $p(\boldsymbol{\alpha}_j, \boldsymbol{\xi}) = p(\boldsymbol{\alpha}_j|\boldsymbol{\xi})p(\boldsymbol{\xi})$, where $p(\boldsymbol{\alpha}_j|\boldsymbol{\xi}) \propto \mathrm{I}(A_{2j})$. Moreover, A_{2j} satisfies the limitation in (6), and $p(\boldsymbol{\xi}) \propto \mathrm{I}$. We follow the idea of Chen et al. (2011) to employ a data-dependent prior on c_L and c_U.

$$c_L \sim \mathrm{Unif}(a_1, b_1) \text{and} c_U|c_L \sim \mathrm{Unif}(a_2, b_2), \tag{9}$$

where a_1 and b_1, respectively, are the $100h$th and $100(1-2h)$th percentiles of the observations. Furthermore, we let a_2 be $c_L + b^*$ and b_2 be the $100(1-h)$th percentiles of the observations, where b^* is a selected number that ensures $c_L + b^* < c_U$ to guarantee there are at least $100h\%$ observations in each regime. We denote the degrees of freedom ν as $\nu^* = \nu^{-1}$ and assign a uniform prior for ν^*, $\nu^* \sim$ Unif(0, 0.25) so that $\nu > 4$. In order to make a flat prior, we denote the degrees of skew parameter η as $\eta \in (-1,1)$. Lastly, we set an inverse Gamma prior for σ_u^2.

3.2 | Posterior distributions

We explain the Bayesian estimation for the hysteresis R-GARCH model. The conditional posterior distribution for each group is proportional to the product of the likelihood function and prior probability. Let $\boldsymbol{\theta}_l$ be a parameter group, and $p(\boldsymbol{\theta}_l|\boldsymbol{r},\boldsymbol{x},\boldsymbol{\theta}_{\setminus l})$ denotes a conditional posterior distribution given $\boldsymbol{r}$, $\boldsymbol{x}$, and $\boldsymbol{\theta}_{\setminus l}$, where $\boldsymbol{\theta}_{\setminus l}$ is the vector of all parameters without $\boldsymbol{\theta}_l$, which we express as

$$p(\boldsymbol{\theta}_l|\boldsymbol{r},\boldsymbol{x},\boldsymbol{\theta}_{\setminus l}) \propto \mathcal{L}(\boldsymbol{r},\boldsymbol{x}|\boldsymbol{\theta}) \times p(\boldsymbol{\theta}_l|\boldsymbol{\theta}_{\setminus l}),$$

where $p(\boldsymbol{\theta}_l)$ is a prior density.

Regarding the posterior distributions of the parameter groups $(\boldsymbol{\phi}_i,\boldsymbol{\omega},\sigma_u^2,\nu,\eta)$, their closed forms are unknown. We then use a random walk Metropolis–Hastings (MH) algorithm to draw the MCMC estimates; see Metropolis et al. (1953) and Hastings (1970) for these parameter groups. Chen and So (2006) combine the random walk MH algorithm and the independent kernel MH algorithm to accelerate convergence and reach optimal mixing. The random walk Metropolis algorithm brings out high posterior correlations and slow convergence for the parameters $\boldsymbol{\beta}$ and $\boldsymbol{\xi}$ iterates. We thus use the adaptive MCMC sampling method of Chen and So (2006).

Regarding the MCMC diagnostic, we use trace plots to assess mixing and use autocorrelation function (ACF) plots to ensure convergence of MCMC iterates. Gelman et al. (1996) prove that the good features of convergence are the acceptance rate being between 25% and 50%. We utilize the MCMC algorithm to generate 1-day-ahead volatility and predicted 1-day-ahead return. We obtain VaR and ES forecasts via the predictive distribution. The detailed procedure appears in the next subsection.

3.3 | Bayesian forecasting of VaR and ES

VaR allows one to confirm that a financial establishment can still operate after a catastrophic event. It is a popular measure in risk management, which denotes the probability of the worst potential loss over a given time period for a given probability. Mathematically, the 1-day-ahead forecast of VaR $_{n+1}$ satisfies:

$$Pr(r_{n+1} < \mathrm{VaR}_{n+1}|F_n) = \alpha,$$

where α is the probability level. We forecast and collect the 1-day-ahead return, r_{n+1}, which enable us to forecast VaR and ES built on the designed MCMC algorithm.

1. Collect $\mu_{n+1}^{[j]}|\boldsymbol{\theta}^{[j]}$ and $\sigma_{n+1}^{[j]}|\boldsymbol{\theta}^{[j]}$ dependent upon the in-sample information F_n, where $\boldsymbol{\theta}^{[j]}$ is the jth MCMC iterate.
2. Draw $\zeta_{n+1} \sim St\left(\eta^{[j]},\nu^{[j]}\right)$.
3. Compute $r_{n+1}^{[j]} = \mu_{n+1}^{[j]} + \sigma_{n+1}^{[j]}\zeta_{n+1}$ and go to Step 1.

We forecast VaR$_{n+1}$ relating to the αth conditional quantile of r_{n+1}, which is the one-step-ahead forecast of the αth quantile of r_{n+1} given on the information F_n.

ES is recognized to be a desirable risk function in terms of its fundamental properties. ES is a coherent risk measure and exhibits sub-additivity, which allows one to easily adjust a financial portfolio. We compute ES for the expected value of loss given that we find r_t to be less than VaR$_t$ during the out-of-sample period. We obtain ES forecasting by the following adaptive MCMC scheme.

1. Generate one-step-ahead return $r_{n+1}^{[j]}|F_n$, $\boldsymbol{\theta}^{[j]}$, where $\boldsymbol{\theta}^{[j]}$ is the jth MCMC iterate.
2. Forecast VaR$_{n+1}$ from the α-percentile of $\{r_{n+1}^{[j]}, j = M+1,\ldots,N\}$.
3. Forecast ES$_{n+1}$ regarding the following equation:

$$\widehat{\mathrm{ES}}_{n+1} = \frac{\sum_{j=M+1}^{N} r_{n+1}^{[j]} \times \mathrm{I}\left(r_{n+1}^{[j]} < \mathrm{VaR}_{n+1}\right)}{\sum_{j=M+1}^{N} \mathrm{I}\left(r_{n+1}^{[j]} < \mathrm{VaR}_{n+1}\right)},$$

where M and N are the numbers of samples for the burn-in period and the total MCMC period, respectively. We then estimate the 1-day-ahead VaR and ES forecasting by a rolling window approach.

3.3.1 | Evaluation of VaR

Our research aims to evaluate the volatility forecasts and quantile forecasts. We use VRate for an evaluation of VaR. VRate is a simple appraisal of VaR performance and runs as follows:

$$\mathrm{VRate} = \frac{1}{m}\sum_{t=n+1}^{n+m} \mathrm{I}(r_t < \mathrm{VaR}_t),$$

where m is forecast period size and n is the in-sample size.

3.3.2 | Backtesting methods

We utilize three common backtesting methods for assessing the accuracy of VaR estimation: the UC test of Kupiec (1995), the CC test of Christoffersen (1998), and the DQ test of Engle and Manganelli (2004). The UC test evaluates whether the true loss exceeds VaR. The CC test evaluates whether VaR estimates have accurate coverage in each time, which combines the likelihood ratio test and the UC test. The DQ test is more powerful than the traditional VaR test, because the traditional VaR test cannot control for the dependence of violations.

3.3.3 | Evaluation of ES

We employ the measures proposed by Embrechts et al. (2005) who use two measures to evaluate ES. $V_1(\alpha)$ takes the VaR estimates, and $V_2(\alpha)$ is a penalty term that corrects the method and strongly depends on the VaR estimates; see Takahashi et al. (2016). Let $q(\alpha)$ be the α-quantile of $\delta_t(\alpha)$. Here, $k(\alpha)$ and $\pi(\alpha)$ are the time points when there is a violation and when $\delta_t(\alpha) < q(\alpha)$ happens, respectively. The measure is

$$\begin{aligned} \delta_t(\alpha) &= r_t - \mathrm{ES}_t(\alpha), \\ V_1(\alpha) &= \frac{1}{T_1}\sum_{t \in k(\alpha)} \delta_t(\alpha), V_2(\alpha) = \frac{1}{T_2}\sum_{t \in \pi(\alpha)} \delta_t(\alpha), \\ V(\alpha) &= \frac{|V_1(\alpha)| + |V_2(\alpha)|}{2}, \end{aligned}$$

where T_1 and T_2 are the number of time points in $k(\alpha)$ and $\pi(\alpha)$, respectively. Alternatively, Chen et al. (2012) use the ES rate, which is similar to VRate for VaR forecasts.

3.3.4 | Evaluation of volatility

According to Patton (2011), we use two loss functions: one is MSE, and the other is QLIKE. They enable us to evaluate volatility forecasts to measure the performance of the target models. We download the realized kernel series from the Oxford-Man Institute and treat it as the volatility proxy. Empirical data in Patton and Sheppard (2009) and Hansen (2005) prove that MSE and QLIKE have a better ability to find good volatility prediction models. We define $\widehat{\sigma}_t^2$ and $\widetilde{\sigma}_t^2$ as the volatility forecast and the volatility proxy, respectively.

$$\begin{aligned} \mathrm{MSE} &= \frac{1}{m}\sum_{t=n+1}^{n+m} \frac{\left(\widetilde{\sigma}_t^2 - \hat{\sigma}_t^2\right)^2}{2}, \\ \mathrm{QLIKE} &= \frac{1}{m}\sum_{t=n+1}^{n+m} \left[\frac{\widetilde{\sigma}_t^2}{\hat{\sigma}_t^2} - \ln\frac{\widetilde{\sigma}_t^2}{\hat{\sigma}_t^2} - 1\right], \end{aligned}$$

where m is the sample size in the out-of-sample period.

4 | SIMULATION STUDY

We use a simulation study to justify the performance of the proposed Bayesian approach with the parameter estimates and tail risk measures from the R-HGARCH models. We consider three scenarios in this simulation study: (i) different variant hysteresis zones, (ii) highest volatility and explosive volatility persistence in Regime 1, and (iii) using a misspecified model. For the simulation study, we assess whether parameter estimation, VaR, and ES forecasting via the Bayesian MCMC methods are approximate to the true values.

We set the hyper-parameter $h = 0.15$, which relates to the range of hysteretic variable $\boldsymbol{\omega}$ in Equation (9), as well as consider $\epsilon = 0.5$ ($b_3 = 1 + \epsilon$) to allow the magnitude of explosive volatility in Regime 1. The initial values for the parameters are $\boldsymbol{\phi}_j = (0,0)'$, $\boldsymbol{\alpha}_j = (0.1, 0.1, 0.1)'$, and $\boldsymbol{\xi}_i = (0.1, 0.5, 0.1, -0.1)'$, $j = 1,2$. We choose the initial value of the degrees of freedom $\nu = 200$, because we start with a Gaussian error and set up the skew parameter $\eta = 0$ as a symmetrical distribution.

In the MCMC sampling procedure, we set the total iterations $N = 20{,}000$ and abandon the first iterations $M = 8000$ as the burn-in period. In order to reduce the autocorrelation of the MCMC samples, we only retain every fourth sampling of the chain after the burn-in period. Given $\boldsymbol{\theta}^{[j]}$, the jth iteration of the MCMC sampling at α level, we compute $\mathrm{VaR}_{n+1}^{[j]}$ by

$$\mathrm{VaR}_{n+1}^{[j]} = \mu_{n+1}^{[j]} + \sigma_{n+1}^{[j]} St_\alpha^{-1}\left(\eta^{[j]}, \nu^{[j]}\right), \tag{10}$$

where St_α^{-1} is the α quantile of the skewed Student's t distribution. We are able to obtain the ES forecast via Equation (10) as follows:

$$\mathrm{ES}_{n+1}^{[j]} = \mu_{n+1}^{[j]} + \sigma_{n+1}^{[j]} E\left[r_{n+1} | r_{n+1} < \mathrm{VaR}_{n+1}^{[j]}\right], \tag{11}$$

where r_{n+1} is obtainable in the simulation study. The mean absolute percentage error (MAPE) is one of the most broadly used measures of forecast accuracy, due to its advantages of scale-independency. Therefore, to assess the forecasting performance we adopt the modified version of the MAPE in Chen et al. (2021) to assess the performance of VaR and ES.

$$\text{MAPE} = \frac{1}{N-M}\sum_{j=M+1}^{N}|\text{PE}_{n+1}^{[j]}|, \quad \text{PE}_{n+1}^{[j]} = \frac{q_{n+1}-\hat{q}_{n+1}^{[j]}}{q_{n+1}},$$

where $\hat{q}_{n+1}^{[j]}$ is a conditional quantile forecast given $\boldsymbol{\theta}^{[j]}$ at α level. The q_{n+1} stands for either VaR_{n+1} or ES_{n+1}. MAPE could produce infinite or undefined values for zero or close-to-zero actual values, which is not applicable in this study.

Tables 1–4 provide Bayesian parameter estimates via averages of posterior means, medians, standard deviations, and 95% credible intervals (Low CI and Up CI), and the prediction performance of tail risk measures—VaR and ES forecasts. To check the performance of the proposed method under the different hysteretic zones, we consider two setups of threshold parameter bounds for the hysteresis regime, which we illustrate in Tables 1 and 2. We observe that both averages of posterior means and medians are close to the true parameters and that 95% credible intervals cover the true values. The estimates of (c_L, c_U) are $(-0.2103, 0.2741)$ and $(-0.4003, 0.1208)$ in the two setups when the true values are $(-0.20, 0.27)$ and $(-0.40, 0.12)$, respectively. We set an explosive regime in the lower regime; that is, $\beta_1^{(1)} + \alpha_1^{(1)}\psi > 1$. The results in Table 3 confirm that the posterior estimates are not significantly far away from their true values.

We further consider an alternative scenario; that is, data are actually generated from a realized two-regime threshold GARCH model. We fit the proposed R-HGARCH model to estimate the unknown parameters' underlying process generated by the R-TGARCH model. In this case, we set only one threshold variable $c = -0.2$ for the true model. The results in Table 4 present that the estimates are close to the true value, except the threshold value. This indicates robust properties of the estimators

TABLE 1 Simulation results for the realized hysteretic GARCH model with $(c_L, c_U) = (-0.20, 0.27)$ in Equations (3)–(5) via 100 replications

Parameter	True	Mean	Median	SD	Low CI	Up CI	MAPE
$\phi_0^{(1)}$	−0.02	−0.0257	−0.0271	0.0235	−0.0668	0.0234	
$\phi_1^{(1)}$	0.02	0.0177	0.0176	0.0340	−0.0487	0.0847	
$\phi_0^{(2)}$	0.02	0.0243	0.0243	0.0331	−0.0409	0.0893	
$\phi_1^{(2)}$	0.00	−0.0004	−0.0004	0.0390	−0.0769	0.0762	
$\alpha_0^{(1)}$	0.15	0.1521	0.1518	0.0176	0.1185	0.1881	
$\alpha_1^{(1)}$	0.30	0.3049	0.3042	0.0242	0.2595	0.3542	
$\beta_1^{(1)}$	0.68	0.6737	0.6741	0.0265	0.6205	0.7242	
$\alpha_0^{(2)}$	−0.07	−0.0711	−0.0714	0.0126	−0.0952	−0.0458	
$\alpha_1^{(2)}$	0.13	0.1363	0.1357	0.0217	0.0954	0.1804	
$\beta_1^{(2)}$	0.80	0.7932	0.7937	0.0257	0.7412	0.8418	
ξ	−0.20	−0.1986	−0.1986	0.0361	−0.2688	−0.1278	
ψ	1.00	0.9935	0.9921	0.0425	0.9135	1.0806	
τ_1	−0.03	−0.0301	−0.0301	0.0132	−0.0560	−0.0045	
τ_2	0.15	0.1490	0.1488	0.0089	0.1319	0.1672	
σ_u^2	0.28	0.2811	0.2810	0.0089	0.2641	0.2992	
c_L	−0.20	−0.2103	−0.2095	0.0224	−0.2532	−0.1694	
c_U	0.27	0.2741	0.2734	0.0208	0.2368	0.3163	
ν	7.00	7.8323	7.6334	1.3757	5.7550	11.0759	
η	−0.15	−0.1479	−0.1481	0.0315	−0.2093	−0.0856	
1%VaR		−2.7210	−2.7168	0.1219	−2.9691	−2.4941	3.69%
5%VaR		−1.6845	−1.6836	0.0659	−1.8159	−1.5583	3.25%
1%ES		−3.4425	−3.4320	0.1950	−3.8496	−3.0893	4.60%
5%ES		−2.3417	−2.3388	0.1013	−2.5472	−2.1519	3.56%

Parameter	True	Mean	Median	SD	Low CI	Up CI	MAPE
$\phi_0^{(1)}$	−0.02	−0.0241	−0.0252	0.0215	−0.0614	0.0202	
$\phi_1^{(1)}$	0.02	0.0194	0.0192	0.0352	−0.0496	0.0885	
$\phi_0^{(2)}$	0.02	0.0194	0.0194	0.0223	−0.0241	0.0633	
$\phi_1^{(2)}$	0.00	−0.0010	−0.0009	0.0335	−0.0671	0.0647	
$\alpha_0^{(1)}$	0.15	0.1491	0.1487	0.0218	0.1071	0.1926	
$\alpha_1^{(1)}$	0.30	0.3044	0.3038	0.0259	0.2555	0.3571	
$\beta_1^{(1)}$	0.68	0.6722	0.6727	0.0293	0.6132	0.7287	
$\alpha_0^{(2)}$	−0.07	−0.0711	−0.0713	0.0128	−0.0958	−0.0451	
$\alpha_1^{(2)}$	0.13	0.1325	0.1319	0.0197	0.0954	0.1728	
$\beta_1^{(2)}$	0.80	0.7945	0.7951	0.0227	0.7481	0.8375	
ξ	−0.20	−0.1979	−0.1981	0.0419	−0.2799	−0.1160	
ψ	1.00	1.0008	0.9992	0.0454	0.9156	1.0944	
τ_1	−0.03	−0.0300	−0.0299	0.0132	−0.0558	−0.0041	
τ_2	0.15	0.1479	0.1477	0.0089	0.1312	0.1659	
σ_u^2	0.28	0.2819	0.2817	0.0089	0.2649	0.2999	
c_L	−0.40	−0.4003	−0.4014	0.0211	−0.4381	−0.3600	
c_U	0.12	0.1208	0.1217	0.0266	0.0706	0.1706	
ν	7.00	7.7771	7.5843	1.3549	5.7010	10.9613	
η	−0.15	−0.1506	−0.1506	0.0315	−0.2119	−0.0885	
1%VaR		−2.4022	−2.3990	0.1084	−2.6241	−2.1989	3.67%
5%VaR		−1.4831	−1.4829	0.0583	−1.5983	−1.3700	3.33%
1%ES		−3.0424	−3.0330	0.1742	−3.4102	−2.7271	4.60%
5%ES		−2.0659	−2.0637	0.0900	−2.2492	−1.8960	3.56%

TABLE 2 Simulation results for the realized hysteretic GARCH model with $(c_L, c_U) = (-0.40, 0.12)$ in Equations (3)–(5) via 100 replications

under model misspecification. Nevertheless, the estimates of (c_L, c_U) still cover the true threshold value, −0.2. In summary, the results of the proposed Bayesian approach are applicable and reliable, even though the model is misspecified.

We then calculate VaR and ES at the $\alpha = 1\%, 5\%$ levels using the proposed model and obtain MAPE to assess the performance of forecasting risk measures. The averages of MAPE measurements in Tables 1 and 2 show excellent prediction accuracy for VaR and ES because all averages of MAPE measurements are less than 5%. However, for the case of an explosive regime in Regime 1 in Table 3, the ES MAPE at the 1% level is 5.31%, which shows that ES at $\alpha = 1\%$ suffers a little bit of loss in accuracy. The results of misspecification case in Table 4 present similar findings, whereby MAPE of ES at the 1% level is 5.75%. Overall, the results of the MAPE measurements being lower or close to 5% indicate consistency and robustness of the proposed Bayesian methods.

5 | EMPIRICAL EXAMPLES

We evaluate the performance of the proposed model using daily returns and realized measures for four stock markets: Japan, South Korea, the United Kingdom, and the United States. For comparisons of the performance of tail forecasting, we also consider two competing models, R-GARCH and realized two-regime threshold GARCH, which have already gained popularity in the literature. We download datasets from Oxford-Man Institute of Quantitative Finance, which include Nikkei 225 in Japan, KOSPI in South Korea, FTSE 100 in the United Kingdom, and DJIA in the United States as the empirical data. We employ the daily closing prices to

TABLE 3 Simulation results for the realized hysteretic GARCH model with explosive volatility in Regime 1 ($\beta_1^{(1)} + \alpha_1^{(1)}\psi) > 1$ in Equations (3)–(5) via 100 replications

Parameter	True	Mean	Median	SD	Low CI	Up CI	MAPE
$\phi_0^{(1)}$	−0.02	−0.0284	−0.0294	0.0235	−0.0694	0.0197	
$\phi_1^{(1)}$	0.02	0.0098	0.0098	0.0361	−0.0607	0.0807	
$\phi_0^{(2)}$	0.02	0.0198	0.0198	0.0230	−0.0253	0.0648	
$\phi_1^{(2)}$	0.00	−0.0006	−0.0006	0.0321	−0.0635	0.0626	
$\alpha_0^{(1)}$	0.20	0.1943	0.1938	0.0176	0.1616	0.2306	
$\alpha_1^{(1)}$	0.35	0.3572	0.3568	0.0238	0.3121	0.4052	
$\beta_1^{(1)}$	0.70	0.6831	0.6835	0.0237	0.6354	0.7286	
$\alpha_0^{(2)}$	−0.07	−0.0716	−0.0719	0.0115	−0.0935	−0.0482	
$\alpha_1^{(2)}$	0.13	0.1365	0.1361	0.0192	0.1003	0.1755	
$\beta_1^{(2)}$	0.80	0.7963	0.7968	0.0201	0.7555	0.8344	
ξ	−0.20	−0.1859	−0.1851	0.0375	−0.2606	−0.1140	
ψ	1.00	0.9957	0.9951	0.0259	0.9456	1.0486	
τ_1	−0.03	−0.0312	−0.0311	0.0129	−0.0568	−0.0058	
τ_2	0.15	0.1449	0.1447	0.0090	0.1278	0.1631	
σ_u^2	0.28	0.2788	0.2786	0.0089	0.2620	0.2967	
c_L	−0.40	−0.3988	−0.3990	0.0155	−0.4275	−0.3683	
c_U	0.12	0.1245	0.1240	0.0186	0.0920	0.1610	
ν	7.00	8.3709	8.1124	1.6500	5.9820	12.2517	
η	−0.15	−0.1505	−0.1507	0.0315	−0.2122	−0.0885	
1%VaR		−3.1347	−3.1326	0.1480	−3.4351	−2.8617	4.01%
5%VaR		−1.9542	−1.9555	0.0799	−2.1110	−1.8025	3.41%
1%ES		−3.9403	−3.9304	0.2321	−4.4262	−3.5212	5.31%
5%ES		−2.7008	−2.6997	0.1230	−2.9491	−2.4733	3.83%

compute daily return r_t, $r_t = (\ln P_t - \ln P_{t-1}) \times 100$, where P_t is the index price at time t from the target market. We use realized kernel (x_t) for a realized measurement of volatility on day t. Using high-frequency data can improve the accuracy of the model and balance the impact of market microstructure noise.

The period of the data spans from January 4, 2010, to December 31, 2020. We separate the datasets into two parts: an in-sample period from January 4, 2010, to December 31, 2016, and a 4-year out-of-sample period from the first trading day of January 2017 to the last trading day of December 2020. The out-of-sample period includes the COVID-19 pandemic period. The out-of-sample data contain 969, 978, 1010, and 996 observations for Nikkei 225, KOSPI, FTSE 100, and DJIA, respectively. We employ a rolling window approach to produce 1-day-ahead forecasting of σ_{n+1}^2, VaR, and ES for the proposed models.

Figures 1 and 2 show time plots of returns and logarithms of realized kernel ($\log x_t$) for the entire time span. The return series and $\log x_t$ show volatility clustering, in which large changes tend to be followed by further large changes, of either sign, and small changes tend to be followed by small changes. We observe that the COVID-19 outbreak in 2020 has resulted in unprecedented volatility in these financial markets. Moreover, we find that the return and $\log x_t$ of Nikkei 225 fluctuated the most in 2011 when the earthquake and tsunami happened on March 11 and influenced the stock market with a great shock.

Table 5 provides descriptive statistics for the full sample and the out-of-sample periods of the return and $\log x_t$ of the four financial markets. The average returns for the four markets are all close to 0, while the average $\log x_t$ is near −1.0. We observe that returns are negatively skewed and those of $\log x_t$ are positively

TABLE 4 Simulation results for the realized hysteretic GARCH model via 100 replications: The actual data generating process is from the realized threshold GARCH model

Parameter	True	Mean	Median	SD	Low CI	Up CI	MAPE
$\phi_0^{(1)}$	−0.02	−0.0215	−0.0225	0.0215	−0.0591	0.0227	
$\phi_1^{(1)}$	0.02	0.0196	0.0194	0.0350	−0.0489	0.0887	
$\phi_0^{(2)}$	0.02	0.0185	0.0184	0.0259	−0.0326	0.0695	
$\phi_1^{(2)}$	0.00	0.0008	0.0010	0.0373	−0.0730	0.0736	
$\alpha_0^{(1)}$	0.15	0.1188	0.1185	0.0223	0.0754	0.1635	
$\alpha_1^{(1)}$	0.30	0.3048	0.3041	0.0261	0.2559	0.3584	
$\beta_1^{(1)}$	0.68	0.6714	0.6719	0.0301	0.6109	0.7291	
$\alpha_0^{(2)}$	−0.07	−0.0608	−0.0608	0.0145	−0.0892	−0.0324	
$\alpha_1^{(2)}$	0.13	0.1452	0.1446	0.0215	0.1051	0.1892	
$\beta_1^{(2)}$	0.80	0.7852	0.7858	0.0257	0.7333	0.8338	
ξ	−0.20	−0.1925	−0.1932	0.0437	−0.2761	−0.1049	
ψ	1.00	0.9977	0.9960	0.0488	0.9061	1.0988	
τ_1	−0.03	−0.0301	−0.0301	0.0134	−0.0564	−0.0039	
τ_2	0.15	0.1483	0.1480	0.0092	0.1310	0.1666	
σ_u^2	0.28	0.2847	0.2845	0.0090	0.2675	0.3029	
c	−0.20	-	-	-	-	-	
c_L	-	−0.2860	−0.2840	0.0374	−0.3626	−0.2224	
c_U	-	0.1261	0.1235	0.0379	0.0616	0.2021	
ν	7.00	8.0967	7.8684	1.5062	5.8671	11.6330	
η	−0.15	−0.1517	−0.1518	0.0316	−0.2132	−0.0894	
1%VaR		−2.2252	−2.2213	0.1039	−2.4382	−2.0339	4.81%
5%VaR		−1.3783	−1.3773	0.0564	−1.4903	−1.2713	4.09%
1%ES		−2.8107	−2.8014	0.1646	−3.1582	−2.5159	5.75%
5%ES		−1.9148	−1.9117	0.0865	−2.0913	−1.7546	4.67%

skewed. The high excess kurtosis of each returns series reveals that the time series contains many extreme values. The Ljung–Box test shows that all squared returns and $\log x_t$ have autocorrelation. We choose the same setups for the hyper-parameter and initial values as in the simulation study.

Tables 6–10 provide the parameter estimates of the R-GARCH model, the R-TGARCH model, and the R-HGARCH model for these four stock markets. We provide the averages of posterior means and 95% credible intervals for the three models. Our findings are summarized as follows.

1. To describe an extreme case, we allow an explosive regime in the first regime. We observe the following inequality:
$$|\beta_1^{(2)}+\alpha_1^{(2)}\psi|<|\beta_1^{(1)}+\alpha_1^{(1)}\psi|.$$
The first regime has high volatility and explosive persistence.
2. The lagged AR(1) effect helps judge the mean reversion and market efficiency in finance. The coefficient of AR(1) usually is not significant. There are some exceptions. The estimate of $\phi_1^{(2)}$ in the UK stock market is significant regarding the R-TGARCH model, while the estimates of $\phi_1^{(1)}$ in Japan market, $\phi_1^{(2)}$ in the US and UK markets are significant built on the R-HGARCH model.
3. Regarding threshold values for the R-HGARCH model, c_L is always a negative estimate, in which 95% credible intervals do not include zero for all markets. The estimate of $c_U \approx 0$ for the two Asian stock

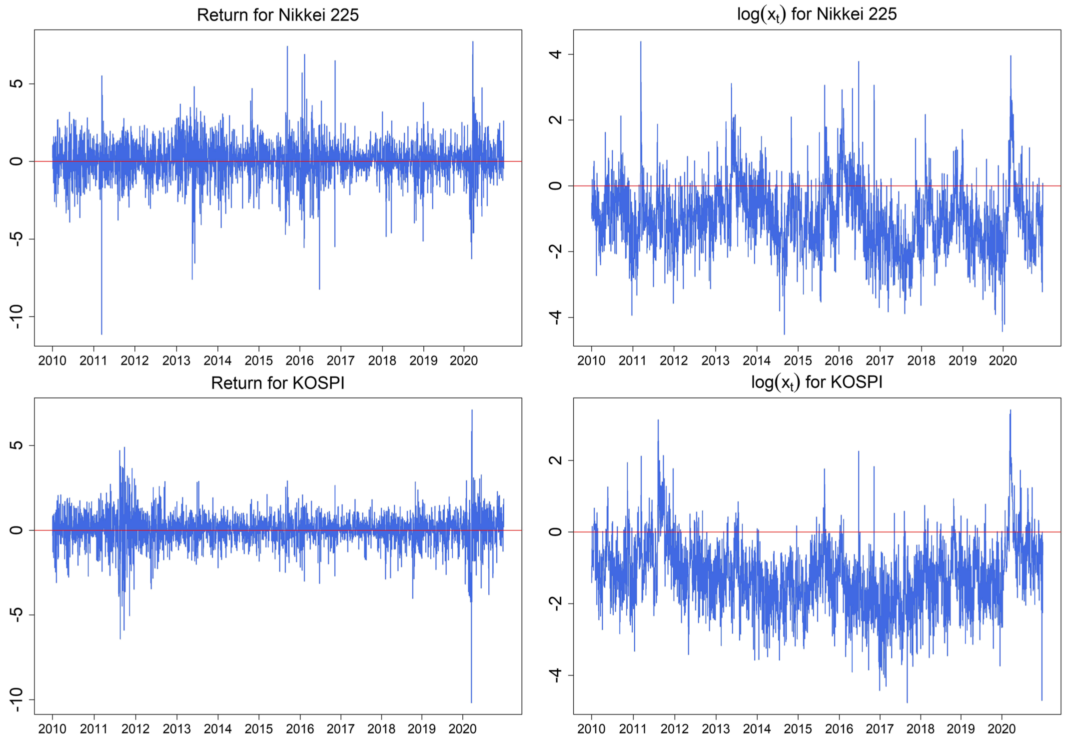

FIGURE 1 Time plots of daily returns and $\log x_t$ for Nikkei 225 and KOSPI

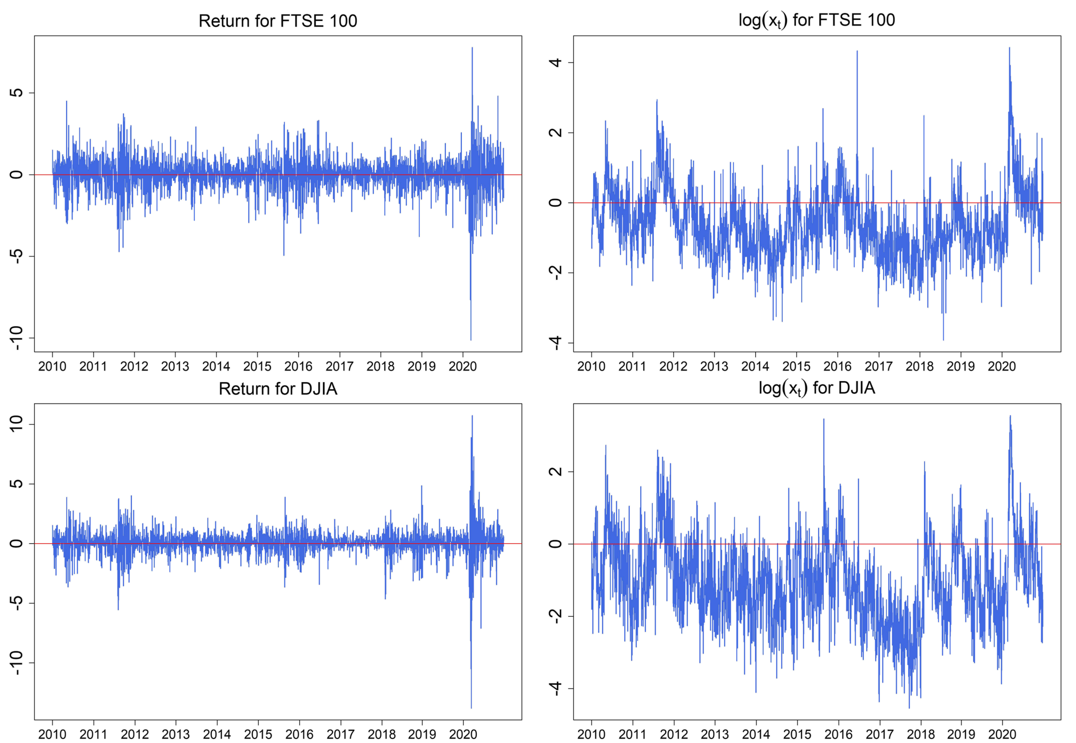

FIGURE 2 Time plots of daily returns and $\log x_t$ for FTSE 100 and DJIA

TABLE 5 Descriptive statistics of daily returns and $\log x_t$ for the full sample (F) and the out-of-sample (I) period (January 3, 2017–December 30, 2020)

Market	Variable	Mean	SD	Min	Max	Skewness	Excess Kurtosis	$Q(5)^{(2)}$ p-value	$Q^2(5)^{(3)}$ p-value
Nikkei: F									
	r_t	0.0356	1.3421	−11.1534	7.7314	−0.4485	5.4246	0.2117	<0.0001
	$\log x_t$	−0.9637	1.0680	−4.5205	4.3894	0.4768	1.1182	<0.0001	
Nikkei: I									
	r_t	0.0373	1.1735	−6.2736	7.7314	−0.1200	5.9529	0.1001	<0.0001
	$\log x_t$	−1.2552	1.0766	−4.4289	3.9601	0.6850	1.4806	<0.0001	
KOSPI: F									
	r_t	0.0197	1.0166	−10.1786	7.1106	−0.5522	8.4506	0.0354	<0.0001
	$\log x_t$	−1.2043	0.9774	−4.7654	3.4170	0.3022	1.0571	<0.0001	
KOSPI: I									
	r_t	0.0355	1.0657	−10.1786	7.1106	−0.7633	14.1170	<0.0001	<0.0001
	$\log x_t$	−1.2435	1.0620	−4.7654	3.4170	0.2800	1.0121	<0.0001	
FTSE: F									
	r_t	0.0064	1.0343	−10.1365	7.7823	−0.5572	7.4725	0.4405	<0.0001
	$\log x_t$	−0.5975	0.9516	−3.9248	4.4319	0.6724	1.3979	<0.0001	
FTSE: I									
	r_t	−0.0099	1.1100	−10.1365	7.7823	−0.9972	13.0584	0.3859	<0.0001
	$\log x_t$	−0.7502	1.0362	−3.9248	4.4319	1.0771	2.3554	<0.0001	
DJIA: F									
	r_t	0.0390	1.0946	−13.8069	10.7538	−0.9790	22.1139	<0.0001	<0.0001
	$\log x_t$	−1.0526	1.1652	−4.5459	3.5616	0.3417	0.4360	<0.0001	
DJIA: I									
	r_t	0.0438	1.3678	−13.8069	10.7538	−1.1949	23.3630	<0.0001	<0.0001
	$\log x_t$	−1.2407	1.3186	−4.5459	3.5616	0.5458	0.4992	<0.0001	

Notes: (1) All series fails the normality assumption by the Jarque–Bera normality test. (2) $Q(5)$ standard for the Ljung–Box test for the target series. (3) $Q^2(5)$ standards for the Ljung–Box test for squared target series.

markets, while the estimate of c_U is significantly positive for the UK and the US stock markets.

4. The parameters (ξ,ψ) are an unbiased estimator of true volatility if $(\xi,\psi)=(0,1)$. The posterior means of ξ and ψ are within (−1.325, −0.312) and (0.893, 1.060) for the R-HGARCH model in the four markets.
5. The estimates for degrees of freedom, ν, are within (5.465, 11.681) for all scenarios. As expected, the innovation distributions are leptokurtic.
6. The negative estimates of skewness parameter, η, are within (−0.157, −0.089) for all structures. We conclude that the skew standardized t distribution is appropriate.

We provide the ACF plots and trace plots of MCMC estimates for each parameter of R-GARCH model, R-TGARCH model, and R-HGARCH model for the US market in the supplementary material. We observe that the lag ACF plots die down quickly and that the trace plots are steady, indicating that the iterations completely reach convergence from these plots.

5.1 | Evaluation of VaR and ES

We use VRate and backtesting methods to evaluate the performance of one-step-ahead forecasts for the three risk models on the four financial stock markets. We employ three backtesting methods, the UC, CC, and DQ tests, to compare the three risk models. We summarize our findings in Table 11 as follows.

TABLE 6 Posterior mean, 2.5th percentile (Low CI), and 97.5th percentile (Up CI) of the unknown parameters for the realized GARCH model in Equation (1)

	Nikkei 225			KOSPI		
Parameter	**Mean**	**Low CI**	**Up CI**	**Mean**	**Low CI**	**Up CI**
ϕ_0	0.0054	−0.0195	0.0285	0.0131	−0.0163	0.0413
ϕ_1	−0.0194	−0.0441	0.0051	−0.0228	−0.0555	0.0099
α_0	0.3933	0.3346	0.4627	0.1716	0.1326	0.2185
α_1	0.2541	0.2151	0.2985	0.1855	0.1535	0.2232
β_1	0.6511	0.5913	0.7031	0.8017	0.7665	0.8333
ξ	−1.3792	−1.4895	−1.2792	−0.9710	−1.0689	−0.8778
ψ	1.0636	0.9383	1.1870	0.9405	0.8666	1.0259
τ_1	−0.1123	−0.1488	−0.0769	−0.1551	−0.1930	−0.1179
τ_2	0.1621	0.1414	0.1828	0.1445	0.1202	0.1659
σ_u^2	0.5666	0.5300	0.6057	0.5089	0.4756	0.5452
ν	8.1498	5.9633	11.5395	5.4646	4.3206	7.1953
η	−0.1263	−0.1910	−0.0597	−0.1214	−0.1787	−0.0613
	FTSE 100			DJIA		
ϕ_0	−0.0030	−0.0151	0.0139	0.0071	−0.0047	0.0179
ϕ_1	0.0017	−0.0402	0.0418	−0.0042	−0.0182	0.0083
α_0	0.0595	0.0358	0.0830	0.3216	0.2665	0.3813
α_1	0.2800	0.2396	0.3173	0.4442	0.4009	0.4922
β_1	0.6627	0.6176	0.7050	0.5442	0.4948	0.5889
ξ	−0.2488	−0.3149	−0.1843	−0.7678	−0.8608	−0.6846
ψ	1.0569	0.9856	1.1404	0.9091	0.8517	0.9656
τ_1	−0.0760	−0.1035	−0.0498	−0.1551	−0.1908	−0.1206
τ_2	0.1669	0.1499	0.1858	0.1351	0.1167	0.1598
σ_u^2	0.2941	0.2758	0.3138	0.4504	0.4223	0.4819
ν	11.6807	7.3562	20.1272	5.8502	4.5790	7.7318
η	−0.1277	−0.1895	−0.0623	−0.1569	−0.2196	−0.0943

1. The R-HGARCH model outperforms the other two models in terms of VRate at the 1% level for the four markets. The same winner does not always cover all cases at the 5% level.
2. The DQ test indicates whether the series of violations is i.i.d. from a Bernoulli distribution for the proposed model. The R-HGARCH model has the least rejection counts among the three models for the four markets when we vary the lag of the DQ test from lag 1 up to 3. The DQ_3 test rejects all violation series at the 1% level for the UK stock market.
3. All R-GARCH-type models are appropriate for the Japan stock market because the three backtests support the hypothesis. The nominal VRate of the R-HGARCH model is the closest to the 1% level among the three models for the Japan and South Korean stock markets.

We illustrate 1-day-ahead VaR predictions and returns under three risk models for the two non-Asia markets in Figures 3 and 4. VaR predictions for the two Asia markets are available in the Supporting Information. There are two violations on March 9 and 12, 2020, for FTSE under the R-HGARCH model. FTSE 100 experienced its two greatest drops (see Figure 3), when it suffered from both the COVID-19 pandemic and uncertainty over Brexit. Figure 4 shows that the R-HGARCH model has few numbers of violations at the 1% level. The R-HGARCH model fails to capture two consecutive violations on February 2 and 5, 2018, while the others fail to capture three consecutive violations starting on February 1, 2018, for the DJIA.

To evaluate ES, we employ the measures from Embrechts et al. (2005) and select the smallest value to

TABLE 7 Posterior mean, 2.5th percentile (Low CI), and 97.5th percentile (Up CI) of the unknown parameters for the realized threshold GARCH model in Equation (2)

	Nikkei 225			KOSPI		
Parameter	Mean	Low CI	Up CI	Mean	Low CI	Up CI
$\phi_0^{(1)}$	0.0045	−0.0157	0.0248	−0.0081	−0.0353	0.0183
$\phi_1^{(1)}$	−0.0121	−0.0307	0.0074	−0.0219	−0.0566	0.0115
$\phi_0^{(2)}$	0.0008	−0.0184	0.0196	0.0009	−0.0267	0.0276
$\phi_1^{(2)}$	−0.0041	−0.0234	0.0147	0.0047	−0.0279	0.0369
$\alpha_0^{(1)}$	0.4636	0.4026	0.5298	0.2559	0.1996	0.3123
$\alpha_1^{(1)}$	0.2347	0.1906	0.2811	0.2038	0.1540	0.2567
$\beta_1^{(1)}$	0.7276	0.6696	0.7804	0.7921	0.7333	0.8447
$\alpha_0^{(2)}$	0.2312	0.1817	0.2841	0.0801	0.0452	0.1254
$\alpha_1^{(2)}$	0.1917	0.1603	0.2234	0.1387	0.1082	0.1749
$\beta_1^{(2)}$	0.6937	0.6499	0.7335	0.8166	0.7758	0.8505
ξ	−1.3437	−1.4312	−1.2680	−0.9537	−1.0460	−0.8696
ψ	1.0829	0.9877	1.1884	0.9924	0.8723	1.1205
τ_1	−0.1176	−0.1460	−0.0858	−0.1542	−0.1921	−0.1219
τ_2	0.1590	0.1404	0.1771	0.1464	0.1231	0.1662
γ	−0.5360	−0.5884	−0.4953	−0.3038	−0.3542	−0.1653
σ_u^2	0.5281	0.5007	0.5577	0.5039	0.4711	0.5384
ν	7.0998	5.4647	9.2899	5.4964	4.2895	7.4005
η	−0.1019	−0.1511	−0.0535	−0.1245	−0.1849	−0.0642

TABLE 8 Posterior mean, 2.5th percentile (Low CI), and 97.5th percentile (Up CI) of the unknown parameters for the realized threshold GARCH model in Equation (2)

	FTSE 100			DJIA		
Parameter	Mean	Low CI	Up CI	Mean	Low CI	Up CI
$\phi_0^{(1)}$	−0.0130	−0.0526	0.0290	0.0059	−0.0101	0.0215
$\phi_1^{(1)}$	0.0106	−0.0488	0.0702	−0.0043	−0.0259	0.0163
$\phi_0^{(2)}$	0.0145	−0.0306	0.0572	0.0130	−0.0060	0.0322
$\phi_1^{(2)}$	−0.0650	−0.1218	−0.0058	−0.0031	−0.0231	0.0171
$\alpha_0^{(1)}$	0.1933	0.1665	0.2211	0.3624	0.3167	0.4105
$\alpha_1^{(1)}$	0.2481	0.2139	0.2851	0.3837	0.3485	0.4206
$\beta_1^{(1)}$	0.7376	0.7013	0.7754	0.6224	0.5826	0.6589
$\alpha_0^{(2)}$	−0.0719	−0.0938	−0.0487	−0.0997	−0.1461	−0.0537
$\alpha_1^{(2)}$	0.1375	0.1065	0.1710	0.1975	0.1521	0.2423
$\beta_1^{(2)}$	0.7958	0.7579	0.8301	0.7473	0.6958	0.7983
ξ	−0.2966	−0.3618	−0.2368	−0.6117	−0.6791	−0.5464
ψ	1.0436	0.9807	1.1100	0.9455	0.9021	0.9896
τ_1	−0.0834	−0.1073	−0.0602	−0.1219	−0.1497	−0.0941
τ_2	0.1575	0.1400	0.1746	0.1503	0.1328	0.1681
γ	−0.1406	−0.1738	−0.1261	0.1775	0.1085	0.2257
σ_u^2	0.2857	0.2713	0.3007	0.4433	0.4202	0.4671
ν	10.5941	7.4274	15.6851	6.6283	5.2402	8.4830
η	−0.1544	−0.2057	−0.1012	−0.1495	−0.1985	−0.1008

TABLE 9 Posterior mean, 2.5th percentile (Low CI), and 97.5th percentile (Up CI) of the unknown parameters for the realized hysteretic GARCH model: Asia stock markets

Parameter	Nikkei 225			KOSPI		
	Mean	Low CI	Up CI	Mean	Low CI	Up CI
$\phi_0^{(1)}$	−0.0421	−0.0826	0.0134	−0.0234	−0.0543	0.0173
$\phi_1^{(1)}$	−0.1366	−0.1988	−0.0727	−0.0685	−0.1388	0.0040
$\phi_0^{(2)}$	0.0158	−0.0380	0.0686	0.0087	−0.0386	0.0550
$\phi_1^{(2)}$	−0.0289	−0.0859	0.0295	−0.0143	−0.0832	0.0634
$\alpha_0^{(1)}$	0.4300	0.3734	0.4906	0.2567	0.1835	0.3446
$\alpha_1^{(1)}$	0.2413	0.2002	0.2843	0.1840	0.1260	0.2498
$\beta_1^{(1)}$	0.7190	0.6597	0.7661	0.8752	0.7935	0.9514
$\alpha_0^{(2)}$	0.2462	0.1963	0.2997	0.0133	−0.0486	0.0757
$\alpha_1^{(2)}$	0.2046	0.1714	0.2439	0.1143	0.0769	0.1559
$\beta_1^{(2)}$	0.6885	0.6442	0.7316	0.7998	0.7419	0.8492
ξ	−1.3254	−1.3924	−1.2571	−0.9657	−1.0775	−0.8276
ψ	1.0600	0.9817	1.1599	0.8931	0.7569	1.0458
τ_1	−0.1124	−0.1424	−0.0839	−0.1455	−0.1811	−0.1111
τ_2	0.1541	0.1374	0.1706	0.1357	0.1136	0.1577
c_L	−0.5600	−0.6081	−0.4955	−0.4336	−0.6483	−0.1303
c_U	0.0294	−0.0213	0.0888	0.1092	−0.1387	0.3613
σ_u^2	0.5298	0.5024	0.5588	0.4878	0.4551	0.5213
ν	7.0294	5.5525	9.0620	5.5958	4.3054	7.5880
η	−0.0931	−0.1456	−0.0428	−0.0893	−0.1512	−0.0242

TABLE 10 Posterior mean, 2.5th percentile (Low CI), and 97.5th percentile (Up CI) of the unknown parameters for the realized hysteretic GARCH model in Equations (3)–(5)

Parameter	FTSE 100			DJIA		
	Mean	Low CI	Up CI	Mean	Low CI	Up CI
$\phi_0^{(1)}$	−0.0220	−0.0541	0.0207	0.0052	−0.0467	0.0558
$\phi_1^{(1)}$	0.0059	−0.0553	0.0694	−0.0327	−0.1095	0.0445
$\phi_0^{(2)}$	0.0260	−0.0148	0.0661	0.0743	0.0405	0.1075
$\phi_1^{(2)}$	−0.0846	−0.1428	−0.0290	−0.0846	−0.1414	−0.0282
$\alpha_0^{(1)}$	0.1882	0.1602	0.2162	0.3732	0.3285	0.4193
$\alpha_1^{(1)}$	0.2533	0.2151	0.2930	0.3843	0.3404	0.4305
$\beta_1^{(1)}$	0.7634	0.7187	0.8060	0.6438	0.5911	0.6944
$\alpha_0^{(2)}$	−0.0523	−0.0734	−0.0267	−0.0372	−0.0761	0.0045
$\alpha_1^{(2)}$	0.1451	0.1117	0.1810	0.2120	0.1751	0.2513
$\beta_1^{(2)}$	0.7877	0.7475	0.8264	0.7036	0.6616	0.7431
ξ	−0.3117	−0.3788	−0.2515	−0.6065	−0.6835	−0.5350
ψ	1.0004	0.9432	1.0616	0.9495	0.9021	0.9945
τ_1	−0.0818	−0.1060	−0.0577	−0.1111	−0.1407	−0.0822
τ_2	0.1603	0.1423	0.1765	0.1487	0.1322	0.1674
c_L	−0.3244	−0.3485	−0.2956	−0.1854	−0.2062	−0.1659
c_U	0.0822	0.0552	0.1170	0.1155	0.1035	0.1337
σ_u^2	0.2884	0.2736	0.3044	0.4460	0.4237	0.4701
ν	10.4783	7.4689	15.6929	6.5835	5.1676	8.5239
η	−0.1568	−0.2069	−0.1055	−0.1363	−0.1881	−0.0847

TABLE 11 Violation rates and backtests at the 1% and 5% levels

Markets	Model	VioNo	VioRate	UC	CC	DQ_1	DQ_2	DQ_3
Nikkei 225	$\alpha = 1\%$							
	R-GARCH	11	1.1352%	0.6789	0.8088	0.9468	0.9727	0.9854
	R-TGARCH	8	0.8256%	0.5737	0.7986	0.9261	0.9722	0.9897
	R-HGARCH	9	0.9288%	0.8216	0.8959	0.9455	0.9797	0.9922
	$\alpha = 5\%$							
	R-GARCH	44	4.5408%	0.5055	0.8012	0.9284	0.6600	0.7114
	R-TGARCH	40	4.1280%	0.1996	0.3737	0.5714	0.5842	0.7182
	R-HGARCH	42	4.3344%	0.1499	0.2010	0.6539	0.6828	0.7074
KOSPI	$\alpha = 1\%$							
	R-GARCH	13	1.3292%	0.3245	0.5165	0.4280	0.5322	0.6163
	R-TGARCH	15	1.5337%	0.1198	0.2361	0.3614	0.4629	0.5476
	R-HGARCH	8	1.1247%	0.7007	0.8194	0.8925	0.9552	0.9818
	$\alpha = 5\%$							
	R-GARCH	60	6.1350%	0.1152	0.2849	0.4346	0.3154	0.1159
	R-TGARCH	65	6.6462%	0.0242	0.0744	0.1171	0.1454	0.0441
	R-HGARCH	62	6.3395%	0.0645	0.1554	0.1807	0.1859	0.1031
FTSE 100	$\alpha = 1\%$							
	R-GARCH	19	1.8812%	0.0121	0.0286	0.0094	0.0139	0.0001
	R-TGARCH	19	1.8812%	0.0121	0.0299	0.0020	0.0027	0.0000
	R-HGARCH	7	0.6931%	0.2994	0.5558	0.0543	0.0924	0.0026
	$\alpha = 5\%$							
	R-GARCH	57	5.6436%	0.3574	0.0945	0.0870	0.0235	0.0082
	R-TGARCH	59	5.8416%	0.2314	0.4661	0.4335	0.2363	0.1792
	R-HGARCH	59	5.8416%	0.2314	0.3442	0.3500	0.1972	0.1357
DJIA	$\alpha = 1\%$							
	R-GARCH	20	2.0080%	0.0049	0.0004	<0.0001	<0.0001	<0.0001
	R-TGARCH	19	1.9076%	0.0105	0.0055	<0.0001	<0.0001	<0.0001
	R-HGARCH	11	1.1044%	0.7446	0.2580	0.0150	0.0252	0.0430
	$\alpha = 5\%$							
	R-GARCH	59	5.9237%	0.1932	0.0351	0.0015	0.0029	0.0065
	R-TGARCH	58	5.8233%	0.2448	0.0903	0.0017	0.0016	0.0038
	R-HGARCH	59	5.9237%	0.1932	0.0878	0.0010	0.0004	0.0008

Note: The DQ_i test stands for DQ test with ith lag.

decide which is the best model. Table 12 presents that the evaluation of ES is at the 1% level and that the R-TGARCH model performs superior for Nikkei 225 and DJIA. The R-HGARCH model has the highest ranking performance for KOSPI and FTSE 100 when the evaluation of ES is at the 1% significance level.

To evaluate volatility forecasting, we convert RK to the estimate of close-to-close volatility based on Hansen (2005) and take it as the volatility proxy. We provide root MSE (RMSE) and QLIKE for four target series in Table 13. The R-HGARCH model works splendidly in the US and UK stock markets in terms of the lowest RMSE values. The R-TGARCH model performs well in the two Asia markets under the RMSE and QLIKE criteria.

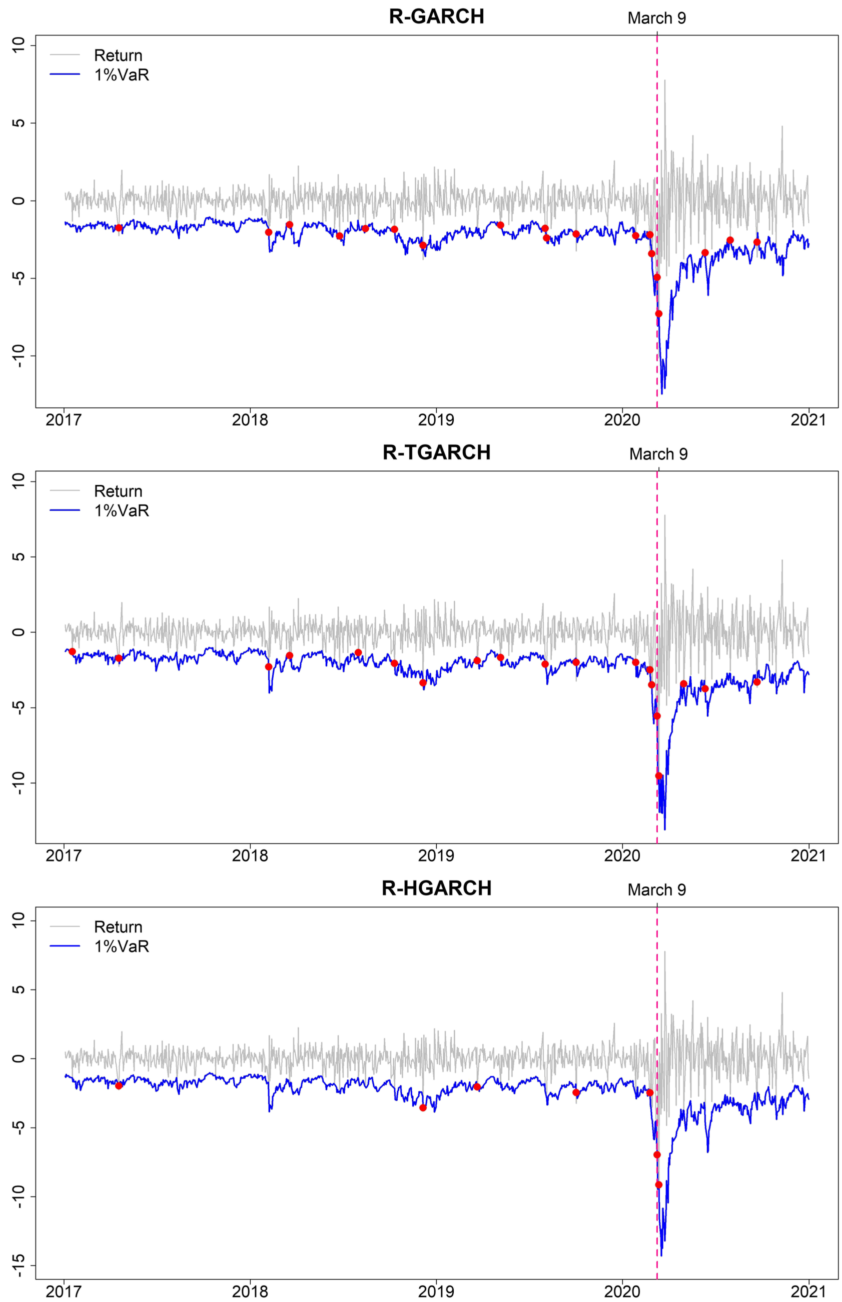

FIGURE 3 One-day-ahead VaR predictions using the three realized GARCH-type models: UK stock market

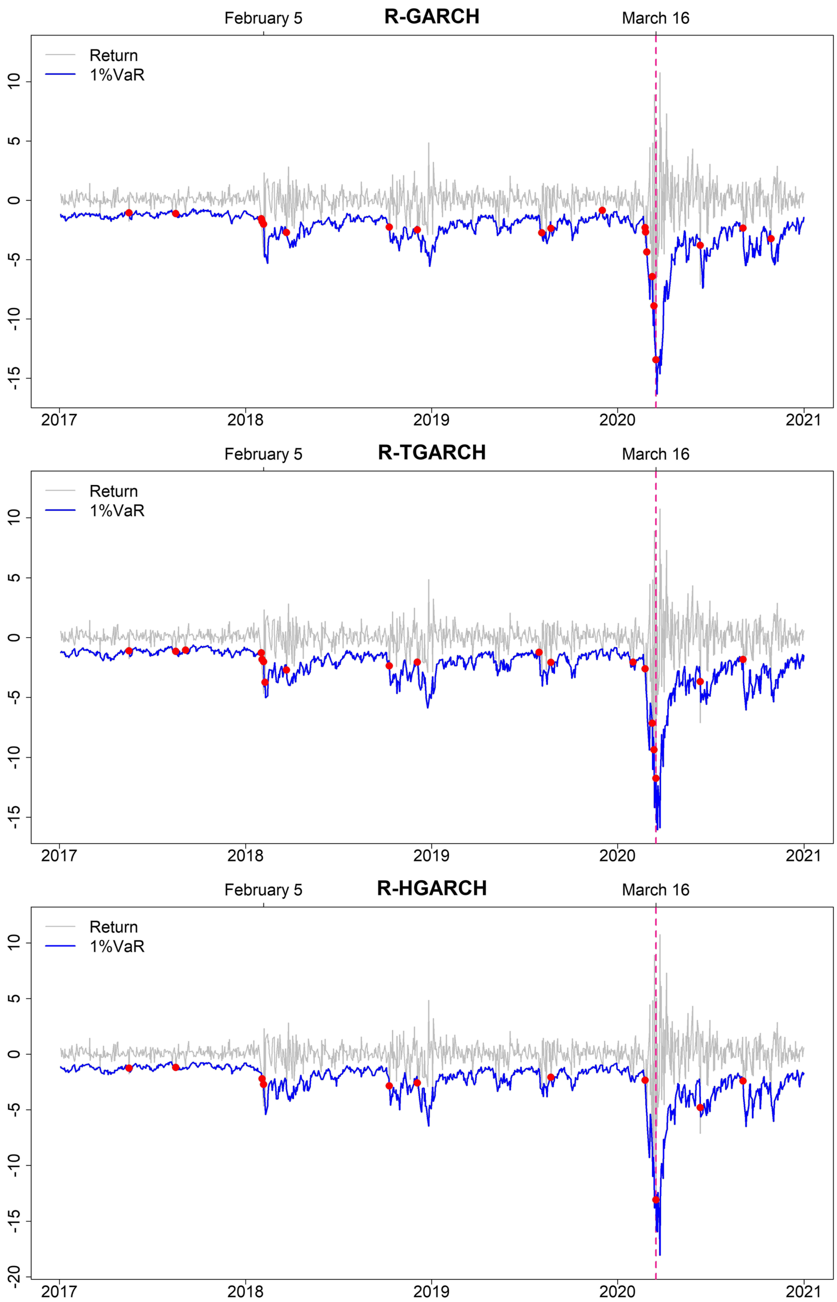

FIGURE 4 One-day-ahead VaR predictions using the three realized GARCH-type models: US stock market

TABLE 12 Evaluation of ES by the Embrechts et al. (2005) method

Markets	α	R-GARCH	R-TGARCH	R-HGARCH
Nikkei 225	1%	0.3522	$\boxed{0.2860}$	0.3241
	5%	0.0853	0.1225	0.0736
KOSPI	1%	0.1614	0.2401	$\boxed{0.0921}$
	5%	0.0875	0.0892	0.0776
FTSE 100	1%	0.2109	$\boxed{0.1222}$	0.1462
	5%	0.2832	0.1817	0.0719
DJIA	1%	0.3546	0.3476	$\boxed{0.1591}$
	5%	0.3366	0.3505	0.2127

Note: A box indicates the best model at the 1% level.

TABLE 13 RMSE and QLIKE

Market	Loss function	R-GARCH	R-TGARCH	R-HGARCH
Nikkei 225	RMSE	2.7489	2.6523	2.7014
	QLIKE	0.3886	0.3716	0.4017
KOSPI	RMSE	1.8105	1.6482	1.6731
	QLIKE	0.3387	0.3274	0.3296
FTSE 100	RMSE	2.5182	2.4530	2.3915
	QLIKE	0.2566	0.2339	0.2336
DJIA	RMSE	2.8145	2.7900	2.7638
	QLIKE	0.3787	0.3770	0.3797

6 | CONCLUSIONS

This research proposes a new model, R-HGARCH, by combining daily returns and realized kernel in the three-regime nonlinear model. The proposed nonlinear model presents explosive persistence and high volatility in Regime 1 in order to capture extreme cases. Through the MCMC method in the Bayesian estimation scheme, we estimate the model parameters and forecast tail risk efficiently. We incorporate the skew Student's t distribution into the risk models for estimation and forecasting the quantiles of returns and volatility via sampling in the predictive distribution. We employ four stock markets in the empirical examples with a 4-year out-of-sample period and also consider the R-GARCH and R-TGARCH models for comparing their quantile forecasts.

The results indicate the R-HGARCH model outperforms among the realized models at the 1% level in terms of VRates and backtests. There is no clear winner based on the performance in ES. When the evaluation of ES is at the 1% significance level, the R-HGARCH model has the best performance for KOSPI and DJIA and the realized two-regime threshold GARCH model performs the best for Nikkei 225 and FTSE 100. The R-HGARCH model performs well in the US and UK stock markets based on RMSE, while the R-TGARCH model performs well in the two Asia stock markets. We only focus on realized single, two-regime, and hysteretic GARCH models for comparison. Other R-GARCH-type models are available in the literature, including the realized exponential GARCH model (Hansen & Huang, 2016) and the realized heterogeneous autoregressive GARCH model (Huang et al., 2016). Chen et al. (2021) propose Bayesian estimation and quantile forecasting for these two R-GARCH models. We may include these two models in the future.

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DATA AVAILABILITY STATEMENT

The dataset underlying the results described in this paper can be found in Realized library, Oxford-Man Institute of Quantitative Finance. https://realized.oxford-man.ox.ac.uk/data/download.

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SUPPORTING INFORMATION

Additional supporting information can be found online in the Supporting Information section at the end of this article.

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