(1).a) Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. Given $\begin{bmatrix} x \\ y \end{bmatrix}_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix}$$
$$= (3-\lambda)(4-\lambda) - 2$$
$$= 12 - 4\lambda - 3\lambda + \lambda^2 - 2$$
$$= (\lambda - 2)(\lambda - 5)$$

Eigenvalues: $\lambda_1 = 2 \quad \lambda_2 = 5$

$\lambda_1:$ $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 2\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ $\lambda_2:$ $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 5\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$3c_1 + 2c_2 = 2c_1$ $3c_1 + 2c_2 = 5c_1$
$c_1 + 4c_2 = 2c_2$ $c_1 + 4c_2 = 5c_2$

Choose $c_1 = 2$ Choose $c_1 = 1$
So $c_2 = -1$ So $c_2 = 1$

$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is an associated. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an associated
eigenvector for λ_1 eigenvector for λ_2

General Solution: $\vec{V}_n = a_2 \cdot 5^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_1 \cdot 2^n \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Solve w/ $1 = a_2 5^0 + a_1 \cdot 2 \cdot 2^0$
Init. Values: $3 = a_2 5^0 - a_1 \cdot 2^0$

$1 = a_2 + 2a_1$ $3 = a_2 - a_1$
$1 - 2a_1 = a_2$ $3 = 1 - 2a_1 - a_1$
$1 - 2(-\frac{2}{3}) = a_2$ $-\frac{2}{3} = a_1$
$\frac{7}{3} = a_2$
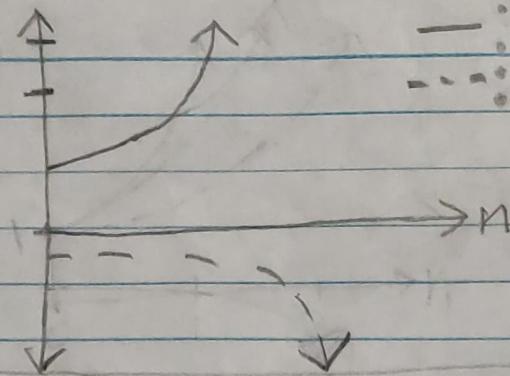
Particular Solution: $\vec{V}_n = \frac{7}{3} \cdot 5^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{2}{3} \cdot 2^n \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

a) Continued
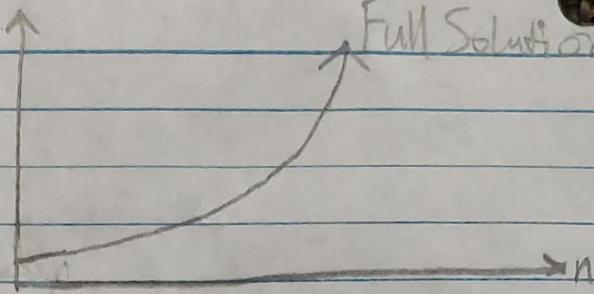
Sketch

Legend: $—: \frac{7}{3}5^n$, $---: -\frac{2}{3}2^n$

Full Solution

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b) $A = \begin{bmatrix} -1 & 3 \\ 0 & 1/3 \end{bmatrix}$ $\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 3 \\ 0 & \frac{1}{3}-\lambda \end{vmatrix}$

$\lambda = \dfrac{-\frac{2}{3} \pm \sqrt{(2/3)^2 + 4(1/3)}}{2}$ $= (-1-\lambda)(\frac{1}{3}-\lambda) - 0$

$= -\frac{1}{3} - \frac{1}{3}\lambda + \lambda + \lambda^2$

$\boxed{\lambda_1 = -1 \qquad \lambda_2 = 1/3}$

$\lambda_1:$ $\begin{bmatrix} -1 & 3 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = -\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ $\qquad$ $\lambda_2:$ $\begin{bmatrix} -1 & 3 \\ 0 & 1/3 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$

$-C_1 + 3C_2 = -C_1$ $\qquad\qquad\qquad\qquad$ $-C_1 + 3C_2 = 1/3\, C_1$

$\frac{1}{3}C_2 = -C_2$ $\qquad\qquad\qquad\qquad\quad$ $\frac{1}{3}C_2 = \frac{1}{3}C_2$

Choose $C_1 = 1$, $\quad C_2 = 0$ $\qquad\qquad$ Choose $C_2 = 1$, $C_1 = \frac{9}{4}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, associated eigenvector 1 $\qquad$ $\begin{bmatrix} 9/4 \\ 1 \end{bmatrix}$, associated eigenvector 2

Gen. Solution: $\vec{V_n} = a_1(-1)^n\begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2(1/3)^n\begin{bmatrix} 9/4 \\ 1 \end{bmatrix}$

Init. Con.: $\quad 2 = a_1(-1)^0 + a_2(1/3)^0 \cdot \frac{9}{4}$ $\qquad\qquad$ $3 = a_2$
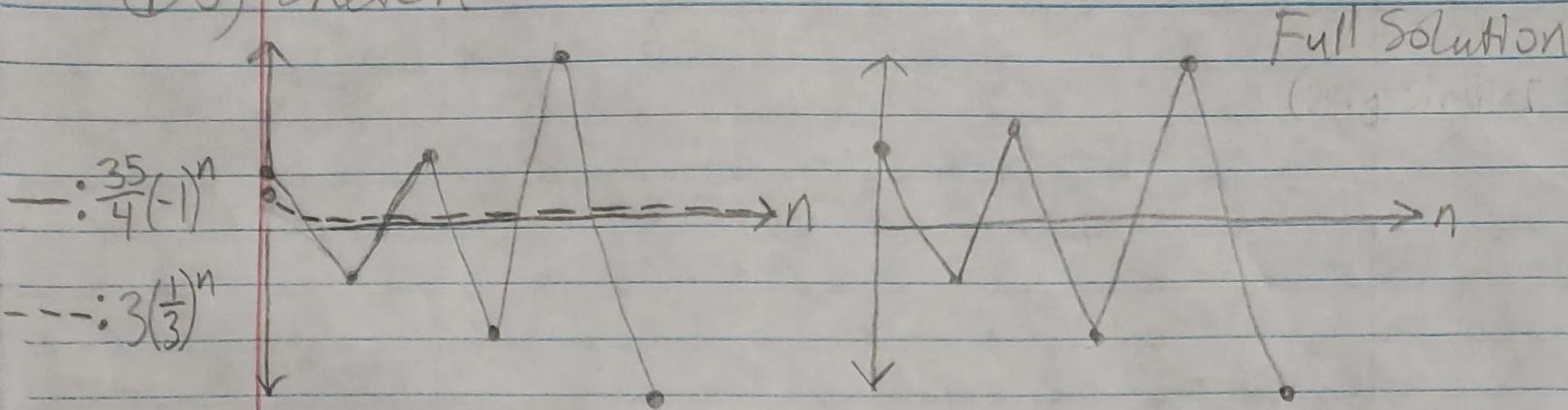
$\begin{bmatrix} x \\ y \end{bmatrix}_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\quad 3 = \quad 0 \quad + a_2(1/3)^0$ $\qquad\qquad\qquad$ $2 = a_1 + 3(\frac{9}{4})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \frac{35}{4} = a_1$

Particular Solution: $\vec{V_n} = \frac{35}{4}(-1)^n\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3(1/3)^n\begin{bmatrix} 9/4 \\ 1 \end{bmatrix}$

① b) Sketch

Full Solution

$-: \frac{35}{4}(-1)^n$

$---: 3(\frac{1}{3})^n$

② a) $\qquad a_{n+1} = a_n + K(M-a_n)(a_n-m) \qquad K>0$

Variables: a_{n+1} and a_n

Parameters: K, M, and m

The a_n term represents the population of the previous generation.

The $K(M-a_n)(a_n-m)$ represents the increase or decrease in the population from year n to year $n+1$.

For $a_n > M$, the second term (the "change" term) is negative. This fits the description in that populations above the carrying capacity will decrease.

For $a_n < m$, the second term is negative, meaning the population continues to decrease when below the minimum survival level as described.

② b). Fixed points:

$$a^* = a^* + 0.0001(5000 - a^*)(a^* - 100)$$
$$0 = (5000 - a^*)(a^* - 100)$$

$$\boxed{a^* = 5000, \ 100}$$

$$a_{n+1} = a_n + 0.0001(5000 - a_n)(a_n - 100)$$
$$= 1.51a_n - 0.0001a_n^2 - 50$$

Stability: $f'(a) = -0.0002a_n + 1.51$

$$f'(5000) = -0.0002(5000) + 1.51$$
$$= 0.51$$

$$f'(100) = -0.0002(100) + 1.51$$
$$= 1.49$$

$$\boxed{\begin{array}{l} a = 5000 \text{ is stable because } |f'(5000)| < 1 \\ a = 100 \text{ is unstable because } |f'(100)| > 1 \end{array}}$$
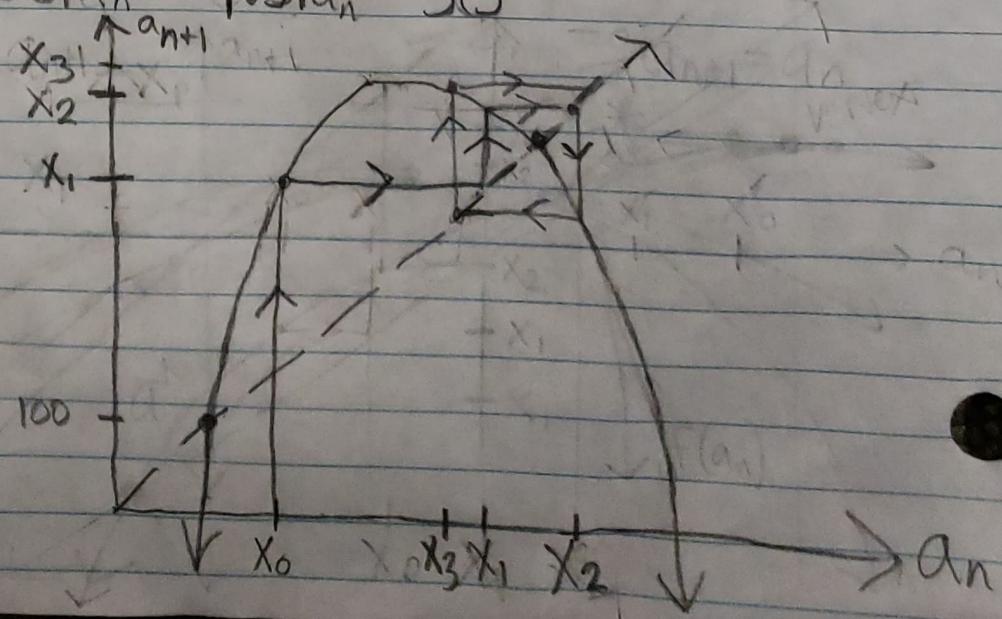
c). $a_{n+1} = a_n + 0.0001(5000 - a_n)(a_n - 100)$
$a_{n+1} = -0.0001a_n^2 + 1.51a_n - 50$
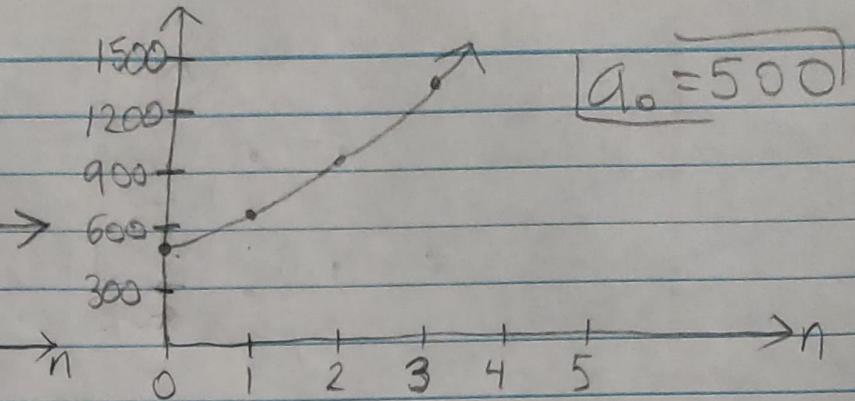
As expected from b),
$a^* = 5000$ is stable
while $a^* = 100$
is unstable.

2. d)

$a_0 = 100$

$a_0 = 500$

$a_0 = 100,000$

$a_0 = 10$

e). As seen in the last two sketch graphs, too small or too big of initial conditions cause the model to immediately plummet to negative values.
We can solve for when $a_{n+1} < 0$ to see what initial populations lead to this.

$$a_{n+1} = 0 = -0.0001 a_n^2 + 1.51 a_n - 50$$

$$a_n = \frac{-1.51 \pm \sqrt{1.51^2 - 4(0.0001)(50)}}{-0.0002}$$

$$a_n \approx 33, \ 15067$$

The graph of a_{n+1} vs. a_n is a concave-down parabola, so for all $a_0 < 33$ or $a_0 > 15067$, the model predicts that the next value, a_1, will immediately be negative. It might not make sense for a huge population to suddenly vanish.

(4) Juveniles: Iterating through annual harvests ranging from 1 to 10 juveniles reveals that at most 5 can be harvested without driving the population to extinction.

Mature / Reproductive Adults: By the same process as above, at most 3 reproductive adults can be harvested annually while still avoiding extinction.