IMPA Medida e integração 2/2025

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# Lista 7

1. (Ergoroff's Theorem) Let $\mu(X) < \infty$, $(f_n)$ a sequence of complex measurable functions which converges pointwise at every point of $X$, and $\varepsilon > 0$. Prove that there is a measure set $E \subset X$, with $\mu(X - E) < \varepsilon$ such that $(f_n)$ converges uniformly on $E$.

2. Suppose that $\mu$ is a measure on $X$, $\mu(X) < \infty$, $f_n \in L^1(\mu)$, $f_n(x)$ converges to $f(x)$ a.e, and there exists $p > 1$ and $C < \infty$ such that $\int_X |f_n|^p d\mu < C$, for all $n$. Prove that

$$\lim_{n\to\infty} \int_X |f - f_n| d\mu = 0.$$

3. Suppose $X$ consists of two points $a$ and $b$; define $\mu(\{a\}) = 1$, $\mu(\{b\}) = \mu(X) = \infty$. Is it true, for this $\mu$, that $L^\infty$ is the dual space of $L^1(\mu)$?

4. (Lebesgue-Stiltjes Integral) Let $F : [a, b] \to \mathbb{R}$ be an increasing function.

   (a) Prove that $F$ has a coutable number of discontinuities.

   (b) If $x_0$ is a discontinuity point, prove that

   $$F(x_0^+) = \lim_{\substack{x > x_0 \\ x \to x_0}} F(x)$$

   is well-defined (i.e the limit exists).

   (c) By possibly modifying $F$ in its discontinuity points $x_0$, assume that $F(x_0) = F(x_0^+)$.

   Prove that there is a unique measure $\mu$ on Borel sets of $\mathbb{R}$ such that $\mu((a, b]) = F(b) - F(a)$. We write, for $f \in L^1(\mu)$

   $$\int_a^b f(x) d\mu(x) = \int_a^b f(x) dF(x).$$

   (d) Prove that if $F$ is continuously differentiable function then

   $$\int_a^b f(x) dF(x) = \int_a^b f(x) F'(x) dx,$$

   for all $f$ that is $L^1(\mu)$.

(e) Let $(x_n)$ be a sequence in $[a, b]$, $(\alpha_n)$ a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} \alpha_n < \infty.$$

Let

$$j_n(x) = \begin{cases} 0 & \text{if } x < x_n \\ 1 & \text{otherwise} \end{cases}$$

and define the function

$$F(x) = \sum_{n=1}^{\infty} \alpha_n j_n(x).$$

For $f \in L^1(dF)$, find

$$\int_a^b f(x) dF(x).$$