IMPA | Medida e integração | 2/2025

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Lista 8

1. A function f is said to be in $L^1_{\text{loc}}(\mathbb{R}^n)$ if for every ball B, $f \cdot \chi_B \in L^1(\mathbb{R}^n)$. Show that the Lebesgue Differentiation Theorem still holds for $L^1_{\text{loc}}(\mathbb{R}^n)$.

2. Let φ be a Lebesgue measurable function in $\mathbb{R}^n$, that satisfies the following property: for every n-dimensional rectable Q, it holds that

$$\left| \int_Q \varphi(x) dx \right| \leqslant \frac{Mm(Q)}{1 + m(Q)},$$

for a constant M. Show that for every $f \in L^1(\mathbb{R}^n)$ it holds that

$$\lim_{k \to \infty} \int_{\mathbb{R}^n} \varphi(kx) f(x) dx = 0.$$

3. Let $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and consider the centered maximal operador

$$Mg(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |g(y) dy.$$

Let $n \geqslant 3$ and consider $f_\alpha(x) = |x|^{-\alpha}$, where $0 < \alpha < n$.

(a) Show that $Mf_\alpha(x) = C_\alpha f(x), \forall x \in \mathbb{R}^n$, where C_α is a constant.

(b) ** For each $0 < \alpha < n$, decide if $C_\alpha = 1$ or $C_\alpha > 1$ and prove it.

** - hard and optional.

4. Let E be a Lebesgue set in $\mathbb{R}$, the upper and lower limits of the quotients

$$\frac{m(E \cap (x - \delta, x + \delta))}{2\delta}$$

are called the upper and lower densities of E at X. If these aree qual, their common value $D_E(x)$ is the density of E at x. If $D_E(x) = 1$, x is a point of density of E. Prove that $D_E(x) = 1$ for almost all $x \in E$ and that $D_E(x) = 0$ for almost all $x \notin E$.

5. Let $A, B \subset \mathbb{R}$, put $A + B = \{a + b, a \in A, b \in B\}$. Suppose $m(A) > 0$, $m(B) > 0$ and prove that $A + B$ contains a segment. **Hint:** Use density points in A and B.