IMPA | Medida e integração | 2/2025

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Lista 4

1. Let $\lambda^\star$ be the Lebesgue outer measure in $\mathbb{R}$. Show that if $\lambda^\star(A) < \infty$ and
$$\lambda^\star(A) = \sup\{\lambda^\star(K), K \subset A \text{ compact}\}$$
then, for every $E \subset \mathbb{R}$ it holds that
$$\lambda^\star(E) = \lambda^\star(E \cap A) + \lambda^\star(E \cap A^c).$$

2. Let μ be a measure as in the Riesz Representation Theorem. Assume, moreover, that for every Borel set A, it holds that
$$\mu(A) = \sup\{\mu(K), K \subset A\},$$
and that X is compact and $\mu(X) = 1$. Prove that there is a compact set $K \subset X$ such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K.

3. Find the class of all compact sets of $\mathbb{R}$ that are the support of a continuous function.

4. Define the distance between two points (x_1, y_1), (x_2, y_2) in $\mathbb{R}^2$ to be
$$\begin{array}{rl} |y_1 - y_2| & \text{if } x_1 = x_2 \\ 1 + |y_1 - y_2| & \text{if } x_1 \neq x_2. \end{array}$$
Show that this is a metric, and that the resulting metric space if locally compact. Moreover, show that for $f \in C_c(\mathbb{R}^2)$ (with respect to this metric), there are finitely many points $x_1, \ldots, x_n$ for which $f(x, y) \neq 0$ for at least one y. In this case, define
$$\Lambda(f) = \sum_{j=1}^{n} \int_{\mathbb{R}} f(x_j, y)dy.$$
Consider the measure associated with this Λ given by the Riesz Representation Theorem. Prove that if E is the x-axis then $\mu(E) = \infty$ and $\mu(K) = 0$ for every compact $K \subset E$.

5. (Constructing the Cantor set) Let $C_0 = [0, 1]$. For $n \geqslant 1$, obtain C_n from C_{n-1} by removing the open middle third of every closed interval in C_{n-1}. Let
$$C = \bigcap_{n=0}^{\infty} C_n$$
be the *Cantor set*.

(a) Describe C_1, C_2 explicitly. Show that every C_n is a finite union of closed intervals and that C is closed.

(b) Prove that C is *nowhere dense* in $[0, 1]$, i.e., the interior of C is empty.

(c) Prove that the Lebesgue measure of C is zero.

(d) Show that C is uncountable.

(e) Give the ternary expansion characterization: prove that

$$x \in C \iff x \text{ has a base-3 expansion using only digits 0 and 2.}$$