IMPA | Medida e integração | 2/2025

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Lista 5

1. Let $\{f_n\}$ be a sequence of real nonnegative functions on $\mathbb{R}$, and consider the following four statements:

   (a) If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.

   (b) If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.

   (c) If each f_n is upper semicontinuous, then $\sum_{i=1}^{\infty} f_i$ is upper semicontinuous.

   (d) If each f_n is lower semicontinuous, then $\sum_{i=1}^{\infty} f_i$ is lower semicontinuous.

   Show that three of these are true and that one is false. What happens if the word "nonnegative" is omitted? Is the truth of the statements affected if $\mathbb{R}$ is replaced by a general topological space?

2. Construct a totally disconnected compact set $K \subset \mathbb{R}$ such that $m(K) > 0$. (Here K is required to have no connected subset consisting of more than one point.)

   If v is lower semicontinuous and $v \leqslant \chi_K$, show that actually $v \leqslant 0$. Hence χ_K cannot be approximated from below by lower semicontinuous functions, in the sense of the Vitali–Carathéodory theorem.

3. (Jensen's inequality) Let μ be a positive measure on a σ- algebra $\mathcal{M}$ in a set Ω, so that $\mu(\Omega) = 1$. If f is a real function in $L^2(\mu)$, if $a < f(x) < b$, where $-\infty \leqslant a < b \leqslant \infty$, for all $x \in \Omega$, and if φ is convex on (a, b), then show that

$$\varphi\left(\int_\Omega f(x)d\mu(x)\right) \leqslant \int_\Omega (\varphi(f(x))d\mu(x)$$

4. Denote $e(x) = e^{2\pi i x}$ and let $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$. We say that $(x_n)_n$ is equidistributed modulo 1 if, for all $m \neq 0$ it holds that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(mx_n) = 0.$$

   Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{f_n(x) : n = 1, 2, \ldots\}$ a sequence of measurable functions $f_n : X \to \mathbb{R}$.

Let $0 < p < \infty$ and assume that for each positive integer m we have

$$\sum_{N=1}^{\infty} N^{-p-1} \int_X \left| \sum_{n=1}^{N} e\big(m f_n(x)\big) \right|^p d\mu(x) < \infty.$$

Show that for μ-almost all points $x \in X$ the sequence $\{f_n(x) : n = 1, 2, \ldots\}$ is equidistributed in modulo 1.

5. Suppose $0 < p < 1$. If f and g are non-negative functions on $(x, \mathcal{M}, \mu)$, show that

$$\left(\int_X (f+g)^p d\mu \right)^{1/p} \geqslant \left(\int_X f^p d\mu \right)^{1/p} + \left(\int_X g^p d\mu \right)^{1/p}$$