Lista 2

1. Suppose $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, \ldots$, and $f_1 \geqslant f_2 \geqslant \ldots \geqslant 0$, $f_n(x) \to f(x)$ for every $x \in X$, $f_1 \in L^1(\mu)$. Prove that then
$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$
and show that this conclusion does not follow if $f \in L^1(\mu)$ is omitted.

2. Prove that there exists $f_n : X \to [0, \infty]$ sequence of measurable functions such that
$$\int_X (\liminf_{n\to\infty} f_n) d\mu < \liminf_{n\to\infty} \int_X f_n d\mu,$$
strictly. That is, Fatou's lemma can yield a strict inequality.

3. Suppose μ is a measure on X, $f : X \to [0, \infty]$ is measurable and $\int_X f d\mu = c$, with $0 < c < \infty$. Let α be a constant. Then
$$\lim_{n\to\infty} \int_X n \log(1 + (f/n)^\alpha) d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1 \\ c & \text{if } \alpha = 1 \\ 0 & \text{if } 1 < \alpha < \infty. \end{cases}$$
Hint: Use different theorems of convergence when $\alpha \geqslant 1$ and when $0 < \alpha < 1$.

4. Suppose $\mu(X) < \infty$, and (f_n) is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that
$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$
Show that the hypothesis $\mu(X) < \infty$ cannot be omitted.

5. Suppose $f \in L^1(\mu)$. Prove that for every $\varepsilon > 0$, there exists an $\delta > 0$ such that, if $\mu(E) < \delta$, then
$$\int_E |f| d\mu < \varepsilon.$$

6. Let (E_k) be a sequence of measurable sets in X, such that

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty.$$

   Then almost all $x \in X$ lie in at most finitely many sets of E_k.

7. Let f and g be complex measurable functions. Prove that $\{x : f(x) = g(x)\}$ is measurable.