IMPA Medida e integração 2/2025

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Lista 3

1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that at each point, f is either right continuous or left continuous (or both). Is f necessarily Borel measurable?

2. If $f : [a, b] \to \mathbb{R}$ is a continuous function, show that its Riemann integral is equal to its Lebesgue integral.

3. Let X be a set and $\mu^\star$ an outer measure in X. Consider the measure space $(X, \mathcal{M}_{\mu^\star}, \mu^\star)$ and $(f_n)_{n=1}^\infty$ a sequence of real-valued functions defined almost everywhere in X. Suppose that $(\varepsilon_n)_{n=1}^\infty$ is a sequence of non-negative real numbers such that
$$\sum_{n=1}^\infty \varepsilon_n < \infty$$
$$\sum_{n=1}^\infty \mu^\star(\{x : |f_{n+1}(x) - f_n(x)| \geqslant \varepsilon_n\}) < \infty.$$
Show that $\lim_{n\to\infty} f_n$ is defined (as a real-valued function) almost everywhere.

4. If f is a Lebesgue measurable complex function on $\mathbb{R}$, prove that there is a Borel function g on $\mathbb{R}$ such that $f = g$ almost everywhere.

5. Construct a sequence of continuous functions f_n on $[0, 1]$ such that $0 \leqslant f_n \leqslant 1$, such that
$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$$
but such that the sequence $(f_n(x))$ converges for no $x \in [0, 1]$.

6. Show that there is a function $f : \mathbb{R}^2 \to \{0, 1\}$ such that
   (a) the Lebesgue integral $\int_{\mathbb{R}} f(x, t)dx$ is defined and equal to 1 for every $t \neq 0$;
   (b) $\liminf_{t\to 0} f(x, t)$ is not Lebesgue measurable.