Lista 9

1. Prove Young's inequality for $1 < p, q, r < \infty$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function with $f' \in L^1(-\infty, \infty)$. Prove that we have

$$\int_{-\infty}^{\infty} |f(x+h) - f(x)| dx \leqslant c|h|,$$

   where c is a constant that does not depend on h.

3. Let $f : [0, 1] \to \mathbb{R}$ be an absolutely continuous function such that $f' \in L^2([0, 1])$. Show that for any $\varepsilon > 0$, there exists an $\delta > 0$ such that if $0 \leqslant y < x \leqslant 1$ be such that $|y - x| < \delta$, then

$$\frac{|f(x) - f(y)|^2}{|x - y|} < \varepsilon.$$

4. Let $f_n : \mathbb{R} \to [0, \infty)$ be non-decreasing functions for each $n \in \mathbb{N}$. Suppose that for every $x \in \mathbb{R}$ we have

$$f(x) = \sum_{n=1}^{\infty} f_n(x) < \infty.$$

   Prove that

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

   almost everywhere.

5. Let $(X_i, \mathcal{M}_i, \mu_i)$, $i = 1, 2$ be σ-finite measure spaces. Let $f : X_1 \times X_2 \to [0, \infty)$ be a measurable function in the product space. Prove that for $1 \leqslant p < \infty$

$$\left(\int \left(\int f(x_1, x_2) d\mu_2 \right)^p d\mu_1 \right)^{1/p} \leqslant \int \left(\int f(x_1, x_2)^p d\mu_1 \right)^{1/p} d\mu_2$$

6. Let A, B be measurable sets of positive finite measure in $\mathbb{R}$. Using convolution of functions, prove that

$$A + B = \{x + y; x \in A, y \in B\}$$

   contains a segment.