Contents

Chapter 1

Euclidean Spaces

1 Smooth Functions on a Euclidean Space

Problem 1.1.

Proof.

$$g(x) = \int_0^x t^{1/3} dt = \frac{3}{4} x^4/3$$

$$g'(x) = x^{1/3}$$

So, as seen before, g' is C^0 but not C^1. As such, g is C^1, but not C^2. And $h = \int_0^x g(t)dt$, which has $h' = g$, is C^2 but not C^3. ☐

Problem 1.2.

(a) *Proof.* Base case: $k = 0$, it is obviously true, with $p_0 = 1$. Suppose it's true for $k > 1$. Then

$$f^{(k)}(x) = p_{2k}(1/x)e^{-1/x}$$

$$\begin{aligned} f^{(k+1)}(x) &= (p_{2k}(1/x))' \cdot (e^{-1/x}) + (p_{2k}(1/x)) \cdot (e^{-1/x})' \\ &= (p_{2k})'(1/x)\frac{1}{x^2} \cdot (e^{-1/x}) + p_{2k}(1/x) \cdot e^{-1/x}\frac{1}{x^2} \\ &= e^{-1/x} \cdot \frac{(p_{2k})'(1/x) + p_{2k}(1/x)}{x^2} \end{aligned}$$

Now, $(p_{2k})'(1/x)/x^2$ is a polynomial on $(1/x)$ of degree $2k+1$ and $p_{2k}(1/x)/x^2$ is of degree $2k+2$, so $(p_{2k})'(1/x)/x^2 + p_{2k}(1/x)/x^2$ is a polynomial on $(1/x)$ of degree $2k+2$, proving the hypothesis. ☐

(b) *Proof.* These formula are certainly valid for any $x \neq 0$. For $x \to 0$, it suffices to notice that $e^{-1/x} \lll p_{2k}(1/x)$ for any k. So f^k is defined for all $\mathbb{R}$ and, (taking the limit) is 0 at 0 for any k. ☐

Problem 1.3.

(a) *Proof.* tan is C^∞ on $(-\pi/2, \pi/2)$ as, taking derivatives, on the denominators only cos appears and they never 0 on this interval. Its inverse, arctan has derivative $\frac{1}{1+x^2}$ which also is C^∞. ☐

(b) *Proof.* Consider

$$h(x) = \frac{x - (b+a)/2}{(b-a)/2}$$

$\square$

(c) *Proof.* Consider $h(x) = \exp(x) + a$ and $g(x) = b - \exp(x)$, then clearly h and g are diffeomorphisms, and we may compose the inverses to find that by the diffeomorphism $g \circ h^{-1}$, the intervals are diffeomorphic. $\square$

Problem 1.4.

Proof. Consider the smooth inverse

$$g : \mathbb{R}^n \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^n, \quad g(x_1, \dots, x_n) = (\arctan(x_1), \dots, \arctan(x_n))$$

$\square$

Problem 1.5.

(a) *Proof.* We parametrize the line between $(0,0,1)$ and (a,b,c) by t and solve for when $z = 0$.

$$l(t) = (0,0,1) + t \cdot ((a,b,c) - (0,0,1)) = (ta, tb, 1 + t(c-1))$$

$$l_3(t) = 0 \iff 1 + t(c-1) = 0 \iff t = \frac{1}{1-c}$$

yielding precisely

$$g(a,b,c) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right)$$

as $(a,b,c) \in S$, we know that $c = 1 - \sqrt{1 - a^2 - b^2}$.

For the inverse, we procede the same way, solving the line equation for when $|l(t) - (0,0,1)| = 1$. This time it is given by:

$$l(t) = (0,0,1) + t \cdot ((x,y,0) - (0,0,1))$$

So, for $|l(t) - (0,0,1)| = 1$ to happen, we must have:

$$t^2x^2 + t^2y^2 + t^2 = 1 \iff t^2(x^2 + y^2 + 1) = 1$$

yielding $t = \pm 1/\sqrt{x^2 + y^2 + 1}$, as we know our solution is in the lower hemisphere, we have $t = 1/\sqrt{x^2 + y^2 + 1}$. Substituting back on the line equation we find

$$(a,b,c) = \left(\frac{x}{\sqrt{x^2+y^2+1}}, \frac{y}{\sqrt{x^2+y^2+1}}, 1 - \frac{1}{\sqrt{x^2+y^2+1}}\right)$$

$\square$

(b) *Proof.* $h^{-1} = f^{-1} \circ g^{-1}$, g^{-1} was found in the previous item, and f^{-1} is simply the projection to the xy plane. So

$$h^{-1}(u,v) = \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}\right)$$

which is C^∞. h is a diffeomorphism. $\square$

(c) *Proof.* This is the most interesting item, but we do exactly the same thing looking at the S^n dimensional sphere in R^{n+1}. Consider the stereographic projection $g : S \to \mathbb{R}^n$ from $(0,0,1)$ given by:

$$g(x_1, x_2, \ldots, x_n, x_{n+1}) = \left(\frac{x_1}{1-c}, \ldots, \frac{x_n}{1-c}, 1 - \frac{1}{1-c} \right)$$

where $c = 1 - \left(\sum_1^n (x_i)^2 \right)^{1/2}$ Then, following the same construction as before, we find h and h^{-1} where:

$$h(x_1, x_2 \ldots, x_n) = \left(\frac{x_i}{1-c} \right)_{i=1}^n$$

where the expression on the right is a vector. Similarly, h^{-1} is defined as:

$$h^{-1}(x_1, x_2 \ldots, x_n) = \left(\frac{x_i}{\sqrt{1 + (x_1)^2 \cdots + (x_n)^2}} \right)_{i=1}^n$$

□

Problem 1.6.

Proof. We apply Taylor twice. As before, consider the function on the line $f(tx, ty))$. By the chain rule

$$D_t f(tx, ty) = \partial_x f(tx, ty)x + \partial_y f(tx, ty)y$$

So, integrating, we find

$$f(x, y) - f(0, 0) = D_t f(tx, ty)\Big]_0^1 = x \int_0^1 \partial_x f(tx, ty)dt + y \int_0^1 \partial_y f(tx, ty)dt$$

$$f(x, y) = f(0, 0) + x \int_0^1 \partial_x f(tx, ty)dt + y \int_0^1 \partial_y f(tx, ty)dt$$

Now we do the same for $\partial_x f(tx, ty)$ and $\partial_y f(tx, ty)$, we find:

$$\partial_x f(x, y) = \partial_x f(0, 0) + x \int_0^1 \partial_{xx} f(tx, ty)dt + y \int_0^1 \partial_{xy} f(tx, ty)dt$$

$$\partial_y f(x, y) = \partial_y f(0, 0) + x \int_0^1 \partial_{yx} f(tx, ty)dt + y \int_0^1 \partial_{yy} f(tx, ty)dt$$

Substituting in the $f(x, y)$ expansion:

$$\begin{aligned} f(x, y) = f(0, 0) &+ x \int_0^1 \left(\partial_x f(0, 0) + x \int_0^1 \partial_{xx} f(stx, sty)ds + y \int_0^1 \partial_{xy} f(stx, sty)ds \right) dt \\ &+ y \int_0^1 \left(\partial_y f(0, 0) + x \int_0^1 \partial_{yx} f(stx, sty)ds + y \int_0^1 \partial_{yy} f(stx, sty)ds \right) dt \\ = f(0, 0) &+ x\partial_x f(0, 0) + y\partial_y f(0, 0) + x^2 g_{11}(x, y) + xy g_{12}(x, y) + y^2 g_{22}(x, y) \end{aligned}$$

□

Problem 1.7.

Proof. $g(t,u)$ is 0 at $t=0$. And, by expanding f, we find, for $t \neq 0$:

$$g(t,u) = \frac{1}{t}\Big(f(0,0) + \partial_x f(0,0)t + \partial_y f(0,0)tu + t^2 g_{11}(t,tu) + t^2 u g_{12}(t,tu) + t^2u^2 g_{22}(t,tu)\Big)$$

Noticing $f(0,0) = \partial_x f(0,0) = \partial_y f(0,0) = 0$ we get:

$$g(t,u) = tg_{11}(t,tu) + tug_{12}(tu) + tu^2 g_{22}(t,tu)$$

Because $g(0,u) = 0$, this formula is valid for $t = 0$ as well, and this expression is C^∞. ☐

Problem 1.8. $f^{-1} = x^{1/3}$ which is not differentiable at 0. In complex analysis, as a consequence of Rouche's theorem, if $f'(z) = 0$, then $f(z+s) = f''(z)s^2 + \dots$, and it can be shown that for sufficiently small s, we have at least two solutions.

2 Tangent Vectors in Rn as Derivations

Problem 2.1.

Proof.

$$X = x\partial_x + y\partial_y$$

$$f(x,y,z) = x^2 + y^2 + z^2$$

Then, computing Xf is as simple as applying X to f at every point:

$$Xf = x\partial_x f + y\partial_y f = 2x^2 + 2y^2$$

☐

Problem 2.2.

Proof. We define all such operations point-wise on C_p^∞. For $f, g \in C_p^\infty$ and $\lambda \in \mathbb{R}$, for any $x \in U$:

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) = g(x) + f(x) = (g+f)(x)\\ (f\cdot g)(x) &= f(x)\cdot g(x) = g(x)\cdot f(x) = (g\cdot f)(x)\\ (\lambda f)(x) &= \lambda \cdot f(x)\end{aligned}$$

Such operations are closed in C_p^∞ as differentiability is a local property closed under these operations. ☐

Problem 2.3.

(a) *Proof.* Let D, D' be derivation at p. Then both D, D' are linear maps of the form $C_p^\infty \to \mathbb{R}$, that satisfy the Leibniz rule.

$$(D+D')(\lambda f + g) = D(\lambda f + g) + D'(\lambda f + g) = \lambda(D+D')f + (D+D')g$$

So $D + D'$ is linear. We also have:

$$(D+D')(fg) = D(fg) + D'(fg) = (Df)g + f(Dg) + (D'f)g + f(D'g) = (D+D')(f)g + f(D+D')(g)$$

As we wanted to show. ☐

(b) *Proof.* Certainly is a linear map and the c pops inside the Leibniz rule

$$cD(fg) = c((Df)g + f(Dg)) = (cDf)g + f(cDg)$$

$\square$

Problem 2.4.

Proof. Let $D_1, D_2 : A \to A$, then $D_1 \circ D_2 : A \to A$. And:

$$D_1 \circ D_2(ab) = D_1(a(D_2b)) + D_1((D_2a)b) = (D_1a)(D_2b) + a(D_1D_2b) + (D_1D_2a)b + (D_2a)(D_1b)$$

which certainly isn't necessairly equal to:

$$a(D_1(D_2b)) + (D_1(D_2a))b$$

Now let's consider $D_1 \circ D_2 - D_2 \circ D_1$, which is clearly a linear map.

$$\begin{aligned}(D_1 \circ D_2 - D_2 \circ D_1)(ab) &= (D_1a)(D_2b) + a(D_1D_2b) + (D_1D_2a)b + (D_2a)(D_1b) \\ &\quad - (D_2a)(D_1b) - a(D_2D_1b) - (D_2D_1a)b - (D_1a)(D_2b) \\ &= a[D_1D_2 - D_2D_1](b) + [D_1D_2 - D_2D_1](a)b\end{aligned}$$

As we wanted to show. $\square$

3 The Exterior Algebra of Multivectors

In some chapters, before the problem section there are some exercises within the text.

3.1 Within Text Exercises

Exercise 3.6.

Proof. We know looking at it that the inversions are $(2, 1), (3, 1), (4, 1), (5, 1)$. But we might have a clearer view writing it in matricial form:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$$

$\square$

Exercise 3.13.

Proof. Consider τSf.

$$\begin{aligned}\tau Sf &= \sum_{\sigma \in S_k} \tau(\sigma f) \\ &= \sum_{\sigma \in S_k} (\tau\sigma f) \\ &= \sum_{\sigma \in S_k} \sigma f\end{aligned}$$

As we have set equality $\{\tau\sigma, \sigma \in S_k\} = \{\sigma \in S_k\}$ $\square$

Exercise 3.15.

Proof. We write the expression containing the 6 permutations and their signs:

$$\begin{aligned} Af(v_1, v_2, v_3) = f(v_1, v_2, v_3) + (-1)f(v_1, v_3, v_2) + f(v_2, v_1, v_3) \\ + (-1)f(v_2, v_3, v_1) + (-1)f(v_3, v_1, v_2) + f(v_3, v_2, v_1) \end{aligned}$$

□

Exercise 3.17.

Proof. let f be k-linear, g l-linear and h m-linear. Then:

$$(f \otimes g) \otimes h = (f(v_1, \dots v_k) \cdot g(v_{k+1}, \dots v_{k+l})) \otimes h = f(v_1, \dots v_k) \cdot g(v_{k+1}, \dots v_{k+l}) \cdot h(v_{k+l+1}, \dots v_{k+l+m})$$

Similarly, as g is l-linear and h m-linear, $(g \otimes h)$ is $l + m$ linear, as such:

$$f \otimes (g \otimes h) = f(v_1, \dots v_k) \otimes (g \otimes h) = f(v_1, \dots v_k) \otimes (g(w_1, \dots w_{k+l}) \cdot h(w_{l+1}, \dots w_{l+m})$$

Resulting in what was expected:

$$f \otimes (g \otimes h) = f(v_1, \dots v_k) \cdot g(v_{k+1}, \dots v_{k+l}) \cdot h(v_{k+l+1}, \dots v_{k+l+m})$$

□

Exercise 3.20.

Proof.

$$f \wedge g(v_1, v_2, v_3, v_4) = \sum_{(2,2)-\text{shuffles } \sigma} f(v_{\sigma(1)}, v_{\sigma(2)}) g(v_{\sigma(3)}, v_{\sigma(4)})$$

As $\binom{4}{2} = 6$ we have the following big sum:

$$\begin{aligned} (f \wedge g)(v_1, v_2, v_3, v_4) = f(v_1, v_2) g(v_3, v_4) + (-1) f(v_1, v_3) g(v_2, v_4) \\ + f(v_1, v_4) g(v_2, v_3) + f(v_2, v_3) g(v_1, v_4) \\ + (-1) f(v_2, v_4) g(v_1, v_3) + f(v_3, v_4) g(v_1, v_2) \end{aligned}$$

□

Exercise 3.22.

Proof. It suffices to count the number of inversions, but this is simple, each of the first l elements of the permutation have k inversions with the last k elements, yielding lk inversions. As such, $\operatorname{sgn}(\tau) = (-1)^{kl}$. □

3.2 Problems

Problem 3.1.

Proof. We begin by remembering $\alpha_i : v \to v_i$, so we may write the tensor product

$$\alpha_i \otimes \alpha_j : (v, w) \to \alpha_i(v) \cdot \alpha_j(w)$$

So the transformation becomes:

$$f = \sum_{1 \leqslant i,j \leqslant n} g_{ij} \cdot \alpha_i \otimes \alpha_j$$

□

Problem 3.2.

(a) *Proof.* This is a simple consequence of the kernel image theorem. We know, from that result, the following identity:

$$\dim(V) = \dim(\ker(f)) + \dim(f(V))$$

As f is a linear, non-zero, and sends on $\mathbb{R}$, we know $f(V) = \mathbb{R}$, and $\dim(f(V)) = 1$. Substituting back, $\dim(\ker(f)) = n - 1$. $\square$

(b) *Proof.* If V has finite dimension, this is a consequence of the previous item. Being the kernel the same of dimension n-1 , by taking any vector v such that $f(v) \neq 0$, we may chose $c = g(v)/f(v)$. Now notice $\ker(cf - g)$ has dimension at least n, and as such is the whole space. Let's generalize using the first isomorphism theorem. Notice that

$$\frac{V}{\ker(f)} = \frac{V}{\ker(g)} \cong \mathbb{R}$$

This means that $\frac{V}{\ker(f)}$ and $\frac{V}{\ker(g)}$ are one-dimensional, with the same elements. Take v such that $\bar{v} \neq 0$, choose $c = g(v)/f(v)$, and as before notice that $\ker(cf - g) = V$. $\square$

Problem 3.3.

Proof. First of linear independence, consider I the set of multi-indices and set e_I as before. Suppose

$$\sum_{(i_1,\ldots,i_k)=I} c_I \alpha_{i_1} \otimes \alpha_{i_2} \cdots \otimes \alpha_{i_n} = 0$$

Now to uncover each c_I, apply the tranformation to e_I. If $J \neq I$, then there is a first index $j_k \neq i_k$. As such, when applying to e_I:

$$\alpha_{j_1} \otimes \cdots \alpha_{j_k} \cdots \otimes \alpha_{j_n}(e_I) = \alpha_{j_1}(e_{i_1}) \cdots \alpha_{j_k}(e_{i_k}) \cdots \alpha_{j_n}(e_{i_n}) = 0$$

As $\alpha_{j_k}(e_{i_k}) = 0$. And we find:

$$\left[\sum_{(i_1,\ldots,i_k)=I} c_I \cdot \alpha_{i_1} \otimes \alpha_{i_2} \cdots \otimes \alpha_{i_n} \right] (e_I) = c_I = 0$$

Now for span. Notice only by linearity that if two k-linear transformation coincide on the indices, then they coincide for every value. As such, given $f \in L_k(V)$, we define g by:

$$g = \sum_I f(e_I) \cdot \alpha_I$$

where here $\alpha_I = \alpha_{i_1} \otimes \alpha_{i_2} \cdots \otimes \alpha_{i_n}$. $\square$

Problem 3.4.

Proof. We say f is alternating if, for a permutation σ:

$$\sigma f = \operatorname{sgn}(\sigma) f$$

To show equivalence is to show that a function has the flipping property, iff it satisfies this permutation property. Clearly, permutation implies flipping, as doing any single 2-transposition changes the sign of the permutation. Now, for the other side, we record that the sign of a permutation is $(-1)^m$, where m is the number of inversions revelead when describing the permutation as 2-transpositions. Following this definition, we see that flipping is sufficient. $\square$

Problem 3.5.

Proof. We may notice, using the wedge products as a basis for the co-vectors that if f is alternating, then $f(v_1, \ldots, v_k) = 0$ if there are $i \neq j$ with $v_i = v_j$ by looking at the decomposition. But that is actually harder.

Suppose f is n-alternating and let $v_1, \ldots v_n$ be n vectors with $v_i = v_j$ for $i < j$. Then after a permutation σ that makes i next to j, we have:

$$(\sigma f)(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = \text{sgn}(\sigma) f(v_1, \ldots, v_i, v_j, \ldots, v_n)$$

Being $v_i = v_j$

$$\text{sgn}(\sigma) f(v_1, \ldots, v_i, v_j, \ldots, v_n) = \text{sgn}(\sigma) f(v_1, \ldots, v_j, v_i, \ldots, v_n)$$

But, because f is alternating, we also have:

$$\text{sgn}(\sigma) f(v_1, \ldots, v_i, v_j, \ldots, v_n) = -\text{sgn}(\sigma) f(v_1, \ldots, v_j, v_i, \ldots, v_n)$$

So both are equal to 0, and as such $f(v_1, \ldots, v_n) = 0$.

Now suppose that whenever two vectors are equal, $f = 0$. Then, given any 2 positions $i < j$ we may write (to simplify, we suppose that they are the only ones):

$$0 = f(v_i + v_j, v_i + v_j) = f(v_i, v_i) + f(v_i, v_j) + f(v_j, v_i) + f(v_j, v_j) = f(v_i, v_j) + f(v_j, v_i)$$

So flipping a coordinate changes the sign. □

Problem 3.6.

Proof.

$$af \wedge bg = \frac{1}{k!l!} A(af \otimes bg) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn}(\sigma))(\sigma(af \otimes bg))$$

But we can pop out the constants from the tensor product. Yielding:

$$\frac{ab}{k!l!} \sum_{\sigma \in S_{k+l}} (\text{sgn}(\sigma))(\sigma(f \otimes g)) = ab(f \wedge g)$$

□

Problem 3.7.

Proof. I was thinking of using the relation for covectors α:

$$(\alpha^1 \wedge \alpha^2 \cdots \wedge \alpha^k)(v_1, v_2, \ldots, v_k) = \det[\alpha^i(v_j)]$$

But I don't know how to use it here. I guess we can expand stuff:

$$\beta^1 \wedge \beta^2 \cdots \wedge \beta^k = \bigwedge_{i=1}^{k} \sum_{j=1}^{k} a^i_j \gamma^j$$

Now, by linearity of the wedge product, we may separate each sum and write the following:

$$\bigwedge_{i=1}^{k} \sum_{j=1}^{k} a^i_j \gamma^j = \sum_{\substack{[i_1, i_2, \ldots, i_k] \\ \in [k]^k}} (a^1_{i_1} \gamma_{i_1} \wedge a^2_{i_2} \gamma_{i_2} \cdots \wedge a^k_{i_k} \gamma_{i_k})$$

Now, we know that, because the wedge product is alternating, we only care about permutations of $[k]$, because if we choose two γ_i's it will zero. As such, we are left with the following, (where the sign comes from the alternating property):

$$\sum_{\substack{[i_1,i_2,\ldots,i_k] \\ \in[k]^k}} (a^1_{i_1}\gamma_{i_1} \wedge a^2_{i_2}\gamma_{i_2} \cdots \wedge a^k_{i_k}\gamma_{i_k}) = \sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\cdot(a^1_{\sigma(1)}\gamma_1 \wedge a^2_{\sigma(2)}\gamma_2 \cdots \wedge a^k_{\sigma(k)}\gamma_k)$$

which, by the previous problem is equal to:

$$\sum_{\sigma\in S_k} \mathrm{sgn}(\sigma)\cdot a^1_{\sigma(1)}a^2_{\sigma(2)}\cdots a^k_{\sigma(k)}\cdot(\gamma_1\wedge\gamma_2\cdots\wedge\gamma_k) = (\det A)\,\gamma_1\wedge\gamma_2\cdots\wedge\gamma_k$$

□

Problem 3.8.

Proof. This is a collorary from the fact that given a basis B of V and their α^i duals, $\alpha^1\wedge\alpha^2\wedge\cdots\wedge\alpha^n$ is a basis for A_n. If ω is a n-covector, then it is of the form $c(\alpha^1\wedge\alpha^2\wedge\cdots\wedge\alpha^n)$, so if it is zero for B, then $c=0$.

Another way of seeing this is writing $\omega(v_1,v_2,\ldots,v_n) = \sum_{\sigma\in S_k} C_\sigma\omega(e_1,e_2,\ldots,e_n) = 0$. □

Problem 3.9.

Proof. This seems to have an easy and a hard direction. If the covectors are NOT linearly independent, then we may write, say $\alpha^k = \sum_{i=1}^{k-1} c_i\alpha^i$. As such

$$\alpha_1\wedge\alpha_2\cdots\wedge\alpha_k = \sum_{i=1}^{k-1} c_i\alpha_1\wedge\alpha_2\cdots\wedge\alpha_{k-1}\wedge\alpha_i = 0$$

Now, if they are linearly independent, let's use induction on the determinant formula.

$$\alpha^1\wedge\alpha^2\cdot\wedge\alpha^k(v_1,v_2\ldots,v_k) = \det[\alpha^i(v_j)]$$

Base case k = 1: $\alpha^1\neq 0$ Now, suppose it's valid for $k-1$ and Sps for sake of contradiction that $\alpha^1\wedge\alpha^2\cdot\wedge\alpha^k=0$, we may then consider, for a given fixed choice of $w_2,w_3,\ldots,w_k$ the linear transformation on v

$$x(v) = \alpha^1\wedge\alpha^2\cdots\wedge\alpha^k(v,w_2\ldots,w_k) = \sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)(\sigma\bigotimes_{i=1}^{k}\alpha^i)(v,w_2,\ldots,w_k) = 0$$

Writing this sum separetely, depending where v is:

$$\begin{aligned} x(v) = &\sum_{\sigma\in S_k,\sigma(1)=1}\alpha^1(v)\cdot\mathrm{sgn}(\sigma)\cdot\alpha^2(w_{\sigma(2)})\cdots\alpha^k(w_{\sigma(k)}) \\ &+\sum_{\sigma\in S_k,\sigma(2)=1}\alpha^2(v)\cdot\mathrm{sgn}(\sigma)\cdot\alpha^1(w_{\sigma(1)})\cdots\alpha^k(w_{\sigma(k)}) \\ &\cdots \\ &+\sum_{\sigma\in S_k,\sigma(k)=1}\alpha^k(v)\cdot\mathrm{sgn}(\sigma)\cdot\alpha^1(w_{\sigma(1)})\cdots\alpha^{k-1}(w_{\sigma(k-1)}) \end{aligned}$$

If we show the term following $\alpha_1(v)$ is not 0, then we will have shown linear dependence. We can use induction for this! The terms that follows $\alpha_1(v)$ is actually $(\alpha^2(w_2)\wedge\alpha^3(w_3)\cdots\wedge\alpha^k(w_k))$, by the induction hypothesis, there is a choice of $w_2,w_3,\ldots,w_n$ with the expression non-zero. And we win! Reordering, we get that α^1 is a linear combination of α^j for $j>1$. □

Problem 3.10.

Proof. The converse is obvious. Sps $\alpha \wedge \gamma = 0$. Being $\alpha \neq 0$ and V finite dimensional (let's say n), we may use it to complete a basis: $(\alpha = \alpha^1), \alpha^2, \dots \alpha^n$. Then, we may, using the usual basis for the k-covectors $A_k(V)$, span γ by:

$$\gamma = \sum \gamma(e_I)\alpha_I$$

Meaning:

$$\alpha \wedge \gamma = \sum \gamma(e_I)\alpha^1 \wedge \alpha_I = 0$$

But the non-zero terms $\alpha^1 \wedge \alpha_I$ are L.I in $A_{k+1}(V)$, so that for each of the α_I that do not contain α^1, $\gamma(e_I) = 0$. As such γ contains only basis vectors that contain α^1 on the index. That is:

$$\gamma = \alpha \wedge \left(\sum_I \gamma(e_I)\alpha_{I/\{1\}} \right)$$

□

4 Differential forms on Rn

4.1 Within Text Exercises

Exercise 4.3.

Proof. As seen before, one basis is: $(dx^2 \wedge dx^3 \wedge dx^4)_p$, $(dx^1 \wedge dx^3 \wedge dx^4)_p$, $(dx^1 \wedge dx^2 \wedge dx^4)_p$ and $(dx^1 \wedge dx^2 \wedge dx^3)_p$. □

Exercise 4.4.

Proof. As seen before, the wedge product is defined point-wise, so at a point p we get:

$$(\omega \wedge \tau)_p(X_p, Y_p, Z_p) = (\omega_p \wedge \tau_p)(X_p, Y_p, Z_p)$$

Being ω a 2-form, ω_p is a 2-covector, similarly, τ_p is a 1-covector. By the covector formula (using shuffles) we get

$$[\omega_p \wedge \tau_p](v_1, v_2, v_3) = \Big(\omega_p(v_1, v_2)\tau_p(v_3) - \omega_p(v_1, v_3)\tau_p(v_2) + \omega_p(v_2, v_3)\tau_p(v_1)\Big)$$

So, as this is valid for all p, we may write it as

$$(\omega \wedge \tau)(X, Y, Z) = \Big(\omega(X, Y)\tau(Z) - \omega(X, Z)\tau(Y) + \omega(Y, Z)\tau(X)\Big)$$

□

Exercise 4.9.

Proof. We need to show that $d\omega = 0$. As $\deg(1/(x^2 + y^2)) = 0$, we have, by the anti-derivation rule:

$$d\omega = d\left(\frac{1}{x^2 + y^2}\right) \wedge (-ydx + xdy) + \left(\frac{1}{x^2 + y^2}\right) d(-ydx + xdy)$$

Developing the first half:

$$d\left(\frac{1}{x^2 + y^2}\right) = -\frac{2xdx + 2ydy}{(x^2 + y^2)^2}$$

So:

$$\left(-\frac{2xdx+2ydy}{(x^2+y^2)^2}\right)\wedge(-ydx+xdy)=\frac{-1}{(x^2+y^2)^2}\Bigg(-2xydx\wedge dx+2x^2dx\wedge dy$$
$$+-2y^2dy\wedge dx+2yxdy\wedge dx\Bigg)$$

Yielding

$$\frac{-1}{(x^2+y^2)^2}(2x^2+2y^2)dx\wedge dy=\frac{-2}{x^2+y^2}dx\wedge dy$$

Now we have to develop the other, easier part. Notice:

$$d(-ydx+xdy)=-(dy\wedge dx)+dx\wedge dy=2(dx\wedge dy)$$

So we get (joining both):

$$\frac{-2}{x^2+y^2}dx\wedge dy+\frac{2}{x^2+y^2}dx\wedge dy=0$$

□

4.2 Problems

Problem 4.1.

Proof.

$$\omega(X)=[zdx-dz](y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y})=zy\left(dx\frac{\partial}{\partial x}\right)+zx\left(dx\frac{\partial}{\partial y}\right)-y\left(dz\frac{\partial}{\partial x}\right)-x\left(dz\frac{\partial}{\partial y}\right)$$

Applying the covectors and cancelling out the zero terms we get:

$$\omega(X)_{(x,y,z)}=zy$$

□

Problem 4.2.

Proof. For this problem it is helpful to remember Prop 3.27. Which states: if $\alpha^1,\alpha^2,\dots\alpha^k$ are covectors, then

$$[\alpha^1\wedge\alpha^2\wedge\cdots\wedge\alpha^k](v_1,v_2,\dots,v_k)=\det[\alpha^i(v_j)]$$

ω then becomes:

$$\omega_p=p^3(dx^1\wedge dx^2)$$

To check this is true, we can write out:

$$[dx^1\wedge dx^2](a,b)=dx^1(a)dx^2(b)-dx^1(b)dx^2(a)=a^1b^2-a^2b^1$$

As we wanted to show. □

Problem 4.3.

Proof. We consider $x(r,\theta), y(r,\theta) \in \Omega_0(\mathbb{R}^2)$ as 0-degree covectors. We may then apply d to them yielding:

$$dx_p = d(r\cos(\theta)) = \left(\frac{\partial}{\partial r} x dr\right) + \left(\frac{\partial}{\partial \theta} x d\theta\right) = \cos(\theta)dr - r\sin(\theta)d\theta$$

similarly we find:

$$dy_p = \sin(\theta)dr + r\cos(\theta)d\theta$$

We may then calculate $(dx \wedge dy)_p$ as (ignoring the subscript)

$$dx \wedge dy = \cos(\theta)\sin(\theta)dr \wedge dr + r(\cos(\theta))^2 dr \wedge d\theta - r(\sin(\theta))^2 d\theta \wedge dr - r^2\sin(\theta)\cos(\theta)d\theta \wedge d\theta$$

Cancelling what we can and flipping the wedge product we find:

$$dx \wedge dy = r dr \wedge d\theta$$

□

Problem 4.4.

Proof. Doing exactly the same thing as the previous problem, we consider x, y, z as 0-covectors and apply the differential.

$$\begin{aligned}
dx &= \sin(\phi)\cos(\theta)d\rho + \rho\cos(\phi)\cos(\theta)d\phi - \rho\sin(\phi)\sin(\theta)d\theta \\
dy &= \sin(\phi)\sin(\theta)d\rho + \rho\cos(\phi)\sin(\theta)d\phi + \rho\sin(\phi)\cos(\theta)d\theta \\
dz &= \cos(\theta)d\rho - \rho\sin(\theta)d\theta
\end{aligned}$$

Now $dx \wedge dy \wedge dz$ is considerably more difficult to calculate, expanding each term linearly and taking only the non repeating values we may reduce it to:

$$\begin{aligned}
dx \wedge dy \wedge dz = {} & \sin(\phi)\cos(\theta)d\rho \wedge \rho\cos(\phi)\sin(\theta)d\phi \wedge (-\rho\sin(\theta)d\theta) \\
& + \rho\cos(\phi)\cos(\theta)d\phi \wedge \sin(\phi)\sin(\theta)d\rho \wedge (-\rho\sin(\theta)d\theta) \\
& + \rho\cos(\phi)\cos(\theta)d\phi \wedge \rho\sin(\phi)\cos(\theta)d\theta \wedge \cos(\theta)d\rho \\
& + (-\rho\sin(\phi)\sin(\theta)d\theta) \wedge \rho\cos(\phi)\sin(\theta)d\phi \wedge \cos(\theta)d\rho
\end{aligned}$$

Now, we may rearrange all products, and pull the functions on front yielding:

$$\begin{aligned}
dx \wedge dy \wedge dz = {} & -\rho^2\cos(\phi)\sin(\phi)\cos(\theta)(\sin(\theta))^2(d\rho \wedge d\phi \wedge d\theta) \\
& + \rho^2\cos(\phi)\sin(\phi)\cos(\theta)(\sin(\theta))^2(d\rho \wedge d\phi \wedge d\theta) \\
& + \rho^2\cos(\phi)\sin(\phi)(\cos(\theta))^3(d\rho \wedge d\phi \wedge d\theta) \\
& + \rho^2\cos(\phi)\sin(\phi)\cos(\theta)(\sin(\theta))^2(d\rho \wedge d\phi \wedge d\theta)
\end{aligned}$$

Canceling the first two and using $\cos^3 + \cos\sin^2 = \cos$ on the latter, we find:

$$dx \wedge dy \wedge dz = \rho^2\cos(\phi)\sin(\phi)\cos(\theta)(d\rho \wedge d\phi \wedge d\theta)$$

□

Problem 4.5.

Proof. We ignore products with repeating co-vectors and alternate positions changing signs.

$$\alpha \wedge \beta = a_1 b_1 + a_2 b_2 + a_3 b_3 (dx^1 \wedge dx^2 \wedge dx^3)$$

□

Problem 4.6.

Proof. Let's uncover $\alpha \wedge \beta$ and then transform it to a vector following the correspondence.

$$\begin{aligned}
\alpha \wedge \beta &= a_1 b_2 \alpha^1 \wedge \alpha^2 + a_1 b_3 \alpha^1 \wedge \alpha^3 \\
&\quad + a_2 b_1 \alpha^2 \wedge \alpha^1 + a_2 b_3 \alpha^2 \wedge \alpha^3 \\
&\quad + a_3 b_1 \alpha^3 \wedge \alpha^1 + a_3 b_2 \alpha^3 \wedge \alpha^2 \\
&= (a_2 b_3 - a_3 b_2) \alpha^2 \wedge \alpha^3 + (a_1 b_3 - a_3 b_1) \alpha^1 \wedge \alpha^3 + (a_1 b_2 - a_2 b_1) \alpha^1 \wedge \alpha^2 \\
&= (a_2 b_3 - a_3 b_2) \alpha^2 \wedge \alpha^3 - (a_1 b_3 - a_3 b_1) \alpha^3 \wedge \alpha^1 + (a_1 b_2 - a_2 b_1) \alpha^1 \wedge \alpha^2
\end{aligned}$$

So that:

$$v_{\alpha \wedge \beta} = \langle (a_2 b_3 - a_3 b_2), -(a_1 b_3 - a_3 b_1), (a_1 b_2 - a_2 b_1) \rangle = v_\alpha \times v_\beta$$

□

Problem 4.7.

Proof. Notice, firstly, as D_1 and D_2 are super derivations of degree m_1 and m_2, then, for any k, given $A_k(V)$, $D_1(A_k(V)) \subseteq A_{k+m_1}(V)$ and, as such, $D_2(D_1(A_k(V))) \subseteq A_{k+m_1+m_2}(V)$. Similarly, $D_1(D_2(A_k(V))) \subseteq A_{k+m_1+m_2}(V)$. We need to check the antiderivation property. Calculating $[D_1, D_2]$ on $a \in A_k(V)$ and $b \in A_l(V)$: Let's expand it by parts, first internaly (using the property inside the arguments):

$$D_1((D_2 a)b + (-1)^{km_2} a(D_2 b)) - (-1)^{m_1 m_2} D_2((D_1 a)b + (-1)^{km_1} a(D_1 b))$$

Then on the outer layer:

$$\begin{aligned}
[D_1, D_2](a, b) &= [D_1 \circ D_2](a)b + (-1)^{(k+m_2)m_1} (D_2 a)(D_1 b) \\
&\quad + (-1)^{km_2} ((D_1 a)(D_2 b) + (-1)^{km_1} a [D_1 \circ D_2](b)) \\
&\quad - (-1)^{m_1 m_2} ([D_2 \circ D_1](a)b + (-1)^{(k+m_1)m_2} (D_1 a)(D_2 b)) \\
&\quad - (-1)^{m_1 m_2} ((-1)^{km_1} ((D_2 a)(D_1 b) + (-1)^{km_2} a [D_2 \circ D_1](b)))
\end{aligned}$$

Which is absolutely terrible to read, but we may organize it a bit better, cancelling $(-1)^{(k+m_2)m_1}$ with $-(-1)^{m_1 m_2}((-1)^{km_1}$; And $(-1)^{km_2}$ with $-(-1)^{m_1 m_2}(-1)^{(k+m_1)m_2}$. Yielding

$$[D_1 \circ D_2](a)b + (-1)^{k(m_1+m_2)} a [D_1 \circ D_2](b) - (-1)^{m_1 m_2} \Big([D_2 \circ D_1](a)b + (-1)^{k(m_1+m_2)} a [D_2 \circ D_1](b) \Big)$$

Which is finally equal to:

$$[D_1, D_2](a)b - (-1)^{k(m_1+m_2)} a [D_1 \circ D_2](b)$$

□

Chapter 2

Manifolds

Chapter 3

Appendices

A Point-Set Topology

A.1 Exercises

Exercise A.5.

Proof. We know, that for every α, $I_\alpha \subset \sum_\beta I_\beta$, so we find that

$$Z\left(\sum_\alpha I_\alpha\right) \subset Z(I_\alpha)$$

As this is valid for every α,

$$Z\left(\sum_\alpha I_\alpha\right) \subset \bigcap_\alpha Z(I_\alpha)$$

The other side is true, for p is a zero of every element of the basis, then it is a common zero for all elements in the ideal.

For the second part, notice, by looking at generators that $Z(I) \cup Z(J) \subset Z(IJ)$. Now consider $p \in Z(IJ)$, then we know that p is a common zero of all pairs $f_i g_j$, with $f_i \in I$ and $g_j \in J$. Now suppose that there are $f \in I, g \in J$ s.t $f(p) \neq 0$ and $g(p) \neq 0$. As we are in a domain, $f(p)g(p) \neq 0$, as such $p \notin Z(IJ)$. □

Exercise A.33.

Proof. Let F_1, F_2 be two closed sets, by proposition A.32, they are compact. For each element $p \in F_2$, apply proposition A.31 finding open sets U_p, V_p s.t $F_1 \subset U_p$ and $p \in V_p$. Now, as F_2 is compact, take a finite cover

$$F_2 \subset \bigcup_\alpha V_\alpha = B$$

and a finite intersection:

$$F_1 \subset \bigcap_\alpha U_\alpha = A$$

Notice $A \cap B = \varnothing$, as:

$$A \cap B = \bigcup_\alpha (V_\alpha \cap A) \subset \bigcup_\alpha (V_\alpha \cap U_\alpha) = \varnothing$$

□

Exercise A.37.

Proof. Given a covering of the union, take a finite subcover for each of the finite compacts. There are only finitely many open sets chosen. $\square$

Exercise A.53.

(a) *Proof.* $\overline{A} \cup \overline{B}$ is a closed set that contains $A \cup B$, so $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. But also, as $A \subset A \cup B$ and $B \subset A \cup B$, we have $\overline{A} \subset \overline{A \cup B}$ and $\overline{B} \subset \overline{A \cup B}$. So $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. $\square$

(b) *Proof.* Just as before, $\overline{A} \cap \overline{B}$ is a closed set that contains $A \cap B$. $\square$

A.2 Problems

Problem A.1.

Proof. This is almost tautological. If $(u, v) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$, then $u \in U_1$ with $v \in V_1$ and $u \in U_2$ and $v \in V_2$, so $(u, v) \in U_1 \times V_1 \cap U_2 \times V_2$. Similarly, if $(u, v) \in U_1 \times V_1 \cap U_2 \times V_2$, then $u \in U_1 \cap U_2$ and $v \in V_1 \cap V_2$, so $(u, v) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$. $\square$

Problem A.2.

Proof.

$$(U_1 \cap U_2) \cap (V_1 \cup V_2) = (U_1 \cap U_2 \cap V_1) \cup (U_1 \cap U_2 \cap V_2) = \varnothing$$

$\square$

Problem A.3.

(a) *Proof.* Each F_i is of the form $(U_i)^c$. As such, we find

$$\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} (U_i)^c = \left(\bigcap_{i=1}^{n} U_i \right)^c$$

But, $\bigcap_{i=1}^{n} U_i$ is open, so our finite union is closed. $\square$

(b) *Proof.* By exactly the same expression:

$$\bigcap_{i=1}^{n} F_i = \bigcap_{i=1}^{n} (U_i)^c = \left(\bigcup_{i=1}^{n} U_i \right)^c$$

$\square$

Problem A.4.

Proof. For $p \in]a, a[^n$, $|p| < \sqrt{n}a$. And if $p \in B(0, \sqrt{n}a)$, then for each coordinate, $|p^i| < \sqrt{n}a$, as such $p \in]\sqrt{n}a, \sqrt{n}a[^n$. Now, to prove the cubes form a basis, we use the definition, for every open set U and $p \in U$, choose the ball centered in p contained in U and the open cube centered in p contained in the ball. $\square$

Problem A.5.

Proof. A^c is open, B^c is open, so $A^c \times Y$ and $X \times B^c$ are open. So $A \times Y$ and $X \times B$ are closed, and we have $A \times Y \cap X \times B = A \times B$ closed. $\square$

Problem A.6.

Proof. If S is Haussdorf, consider $(a, b) \in S \times S - \Delta$. There are disjoint open sets A, B with $a \in A$ and $b \in B$, then $A \times B \subset S \times S - \Delta$ is a open neighborhood of (a, b), as (a, b) were arbitrary, Δ is closed. Now, if Δ is closed, $S \times S - \Delta$ is open, and if $a \neq b \in S$, $(a, b) \in S \times S - \Delta$ is contained in some open neighborhood $A \times B$ (because these sets form a basis for $S \times S$). Take, as before A and B. $\square$

Problem A.7.

Proof. Consider an open set $A = \bigcup_\alpha U_\alpha \times V_\alpha$. We find that $\pi(A) = \bigcup_\alpha U_\alpha$ which is open. $\square$

Problem A.8.

Proof. If f is continuous, then the preimage of every open set is open. In particular, given p, the preimage of the open ball $B(f(p), \varepsilon)$ is an open set of A, which contains p. As $p \in f^{-1}(B(f(p), \varepsilon)$ open, there is $\delta > 0$ s.t. $B(p, \delta) \subset f^{-1}(B(f(p), \varepsilon)$ as we wanted to show.

For the converse, sps f satisfies the ε-δ criterion. Consider V open set of R^n, if $V = \varnothing$, then we have nothing to prove, as the preimages of V is the empty set, which is open in A. Now let $f(p) \in V$, because V is open, there is $r > 0$ s.t. $B(f(p), r) \subset V$. By the criterion, there is $\delta > 0$ s.t. $f(B(p, \delta)) \subset B(f(p), r)$, that is, $B(p, \delta) \subset f^{-1}(V)$, which means $f^{-1}(V)$ is open - as p was arbitrary. $\square$

Problem A.9.

Proof. For any function $f : X \to Y$ and set $A \subset Y$, $f^{-1}(A^c) = (f^{-1}(A))^c$. As such, if f is continuous and $F = U^c$ is a closed set. $f^{-1}(F) = f^{-1}(U^c) = (f^{-1}(U))^c$ which is closed. Absolutely the same proof is valid for proving the assertion from the closed to the open sets. $\square$

Problem A.10.

Proof. We know that the projections are continuous, as such, if f is continuous, so are their compositions. If both projections are continuous, then by noticicng that $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$ which is a finite intersection of open sets, we check that f is continuous. (To generalize to any open set in the product topology, separate it into basis elements) $\square$

Problem A.11.

Proof. Sps $f \times g$ is continuous, then given an open set $U \subset X'$, consider $(f \times g)^{-1}(U \times Y') = f^{-1}(U) \times Y$ which is open by the continuity of $f \times g$. By problem A.7, $f^{-1}(U)$ is open. The same argument is valid for g.

If f, g are continuous, and $\bigcup_\alpha U'_\alpha \times V'_\alpha = A$ is an open set of $X' \times Y'$, then $(f \times g)^{-1}(A) = f^{-1}(\bigcup_\alpha U'_\alpha) \times g^{-1}(\bigcup_\alpha V'_\alpha)$ is open as well. $\square$

Problem A.12.

Proof. The inverse function f^{-1} has the closed pre-images property (because f is closed). So by a previous problem f is continuous. $\square$

Problem A.13.

Proof. Given any covering of the topological space, consider the countable set of basis elements contained in the cover, for each such set, choose one set of the covering which contains it. $\square$

Problem A.14.

Proof. Take a covering of such union, take the union of the finite coverings for each compact set. □

Problem A.15.

Proof. Notice that this is precisely the definion of connectedness for the subspace when noticing $A = (A \cap U) \cup (A \cap V)$. □

Problem A.16.

Proof. If C is a connected component and $p \in C$, by the local connected property, there is a connected neighborhood V_p, as $C \cup V_p$ is connected $C \cup V_p = C$, as such, $V_p \subset C$. So C is open, as p was arbitrary. □

Problem A.17.

Proof. Clearly, if $U \cap A \neq \varnothing$ then $U \cap \overline{A} \neq \varnothing$. Now suppose there is $p \in U \cap \overline{A}$, then as U is open, there is an open neighborhood $V_p \subset U$. But, as $p \in \overline{A}$, every such neighborhood intersects A, so $V_p \cap A \neq \varnothing$. □

Problem A.18.

Proof. Clearly a countable basis for the whole set will contain a countable basis for every neighborhood. □

Problem A.19.

Proof. Sps, for the sake of contradiction that there were two limits, x and y. As the space is $Hausdorff$, there is a separation $U \cap V = \varnothing$ with $x \in U, y \notin U$ and $x \notin V, y \in V$. By the property of the limit, eventually the sequence is forever in U, never in V, contradicting y being a limit point. □

Problem A.20.

Proof. $cl_S(A) \times Y$ is a closed set of $S \times Y$ that contains $A \times Y$, as such, $cl_{S\times Y}(A \times Y) \subset cl_S(A) \times Y$.

Sps $(a, y) \in cl_S(A) \times Y$, then if $a \in A$, there's nothing to show. Then, suppose $a \in ac(A)$. we must show $(a, y) \in ac(A \times Y)$. Take any open neighborhood of (a, y) from the basis of the product topology, then $(a, y) \in U \times V$, as such, $U \cap A \neq \varnothing$, and of course $U \times V \cap A \times Y \neq \varnothing$. □

Problem A.21.

(a) *Proof.* Follows directly from proposition A.48 and $S = cl(A)$ □

(b) *Proof.* From A.18, we saw that any two non-empty open sets intersect. So every neighborhood intersects with A. □