Contents

Chapter 1

Euclidean Spaces

1.1 Smooth Functions on a Euclidean Space

Problem 1.1.1.

Proof.

$$g(x) = \int_0^x t^{1/3} dt = \frac{3}{4} x^4/3$$
$$g'(x) = x^{1/3}$$

So, as seen before, g' is C^0 but not C^1. As such, g is C^1, but not C^2. And $h = \int_0^x g(t)dt$, which has $h' = g$, is C^2 but not C^3. □

Problem 1.1.2.

(a) *Proof.* Base case: $k = 0$, it is obviously true, with $p_0 = 1$. Suppose it's true for $k > 1$. Then

$$f^{(k)}(x) = p_{2k}(1/x)e^{-1/x}$$

$$\begin{aligned} f^{(k+1)}(x) &= (p_{2k}(1/x))' \cdot (e^{-1/x}) + (p_{2k}(1/x)) \cdot (e^{-1/x})' \\ &= (p_{2k})'(1/x)\frac{1}{x^2} \cdot (e^{-1/x}) + p_{2k}(1/x) \cdot e^{-1/x}\frac{1}{x^2} \\ &= e^{-1/x} \cdot \frac{(p_{2k})'(1/x) + p_{2k}(1/x)}{x^2} \end{aligned}$$

Now, $(p_{2k})'(1/x)/x^2$ is a polynomial on $(1/x)$ of degree $2k+1$ and $p_{2k}(1/x)/x^2$ is of degree $2k+2$, so $(p_{2k})'(1/x)/x^2 + p_{2k}(1/x)/x^2$ is a polynomial on $(1/x)$ of degree $2k+2$, proving the hypothesis. □

(b) *Proof.* These formula are certainly valid for any $x \neq 0$. For $x \to 0$, it suffices to notice that $e^{-1/x} \lll p_{2k}(1/x)$ for any k. So f^k is defined for all $\mathbb{R}$ and, (taking the limit) is 0 at 0 for any k. □

Problem 1.1.3.

(a) *Proof.* tan is C^∞ on $(-\pi/2, \pi/2)$ as, taking derivatives, on the denominators only cos appears and they never 0 on this interval. Its inverse, arctan has derivative $\frac{1}{1+x^2}$ which also is C^∞. □

(b) *Proof.* Consider

$$h(x) = \frac{x - (b+a)/2}{(b-a)/2}$$

$\square$

(c) *Proof.* Consider $h(x) = \exp(x)+a$ and $g(x) = b-\exp(x)$, then clearly h and g are diffeomorphisms, and we may compose the inverses to find that by the diffeomorphism $g \circ h^{-1}$, the intervals are diffeomorphic. $\square$

Problem 1.1.4.

Proof. Consider the smooth inverse

$$g : \mathbb{R}^n \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^n, \quad g(x_1, \dots, x_n) = (\arctan(x_1), \dots, \arctan(x_n))$$

$\square$

Problem 1.1.5.

(a) *Proof.* We parametrize the line between $(0,0,1)$ and (a,b,c) by t and solve for when $z = 0$.

$$l(t) = (0,0,1) + t \cdot ((a,b,c) - (0,0,1)) = (ta, tb, 1 + t(c-1))$$

$$l_3(t) = 0 \iff 1 + t(c-1) = 0 \iff t = \frac{1}{1-c}$$

yielding precisely

$$g(a,b,c) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right)$$

as $(a,b,c) \in S$, we know that $c = 1 - \sqrt{1 - a^2 - b^2}$.

For the inverse, we procede the same way, solving the line equation for when $|l(t) - (0,0,1)| = 1$. This time it is given by:

$$l(t) = (0,0,1) + t \cdot ((x,y,0) - (0,0,1))$$

So, for $|l(t) - (0,0,1)| = 1$ to happen, we must have:

$$t^2x^2 + t^2y^2 + t^2 = 1 \iff t^2(x^2 + y^2 + 1) = 1$$

yielding $t = \pm 1/\sqrt{x^2+y^2+1}$, as we know our solution is in the lower hemisphere, we have $t = 1/\sqrt{x^2+y^2+1}$. Substituting back on the line equation we find

$$(a,b,c) = \left(\frac{x}{\sqrt{x^2+y^2+1}}, \frac{y}{\sqrt{x^2+y^2+1}}, 1 - \frac{1}{\sqrt{x^2+y^2+1}}\right)$$

$\square$

(b) *Proof.* $h^{-1} = f^{-1} \circ g^{-1}$, g^{-1} was found in the previous item, and f^{-1} is simply the projection to the xy plane. So

$$h^{-1}(u,v) = \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}\right)$$

which is C^∞. h is a diffeomorphism. $\square$

(c) *Proof.* This is the most interesting item, but we do exactly the same thing looking at the S^n dimensional sphere in R^{n+1}. Consider the stereographic projection $g : S \to \mathbb{R}^n$ from $(0,0,1)$ given by:

$$g(x_1, x_2, \ldots, x_n, x_{n+1}) = \left(\frac{x_1}{1-c}, \ldots, \frac{x_n}{1-c}, 1 - \frac{1}{1-c}\right)$$

where $c = 1 - \left(\sum_1^n (x_i)^2\right)^{1/2}$ Then, following the same construction as before, we find h and h^{-1} where:

$$h(x_1, x_2 \ldots, x_n) = \left(\frac{x_i}{1-c}\right)_{i=1}^n$$

where the expression on the right is a vector. Similarly, h^{-1} is defined as:

$$h^{-1}(x_1, x_2 \ldots, x_n) = \left(\frac{x_i}{\sqrt{1 + (x_1)^2 \cdots + (x_n)^2}}\right)_{i=1}^n$$

□

Problem 1.1.6.

Proof. We apply Taylor twice. As before, consider the function on the line $f(tx, ty))$. By the chain rule

$$D_t f(tx, ty) = \partial_x f(tx, ty)x + \partial_y f(tx, ty)y$$

So, integrating, we find

$$f(x, y) - f(0, 0) = D_t f(tx, ty)\Big]_0^1 = x \int_0^1 \partial_x f(tx, ty)dt + y \int_0^1 \partial_y f(tx, ty)dt$$

$$f(x, y) = f(0, 0) + x \int_0^1 \partial_x f(tx, ty)dt + y \int_0^1 \partial_y f(tx, ty)dt$$

Now we do the same for $\partial_x f(tx, ty)$ and $\partial_y f(tx, ty)$, we find:

$$\partial_x f(x, y) = \partial_x f(0, 0) + x \int_0^1 \partial_{xx} f(tx, ty)dt + y \int_0^1 \partial_{xy} f(tx, ty)dt$$

$$\partial_y f(x, y) = \partial_y f(0, 0) + x \int_0^1 \partial_{yx} f(tx, ty)dt + y \int_0^1 \partial_{yy} f(tx, ty)dt$$

Substituting in the $f(x, y)$ expansion:

$$\begin{aligned} f(x, y) = f(0, 0) &+ x \int_0^1 \left(\partial_x f(0, 0) + x \int_0^1 \partial_{xx} f(stx, sty)ds + y \int_0^1 \partial_{xy} f(stx, sty)ds\right) dt \\ &+ y \int_0^1 \left(\partial_y f(0, 0) + x \int_0^1 \partial_{yx} f(stx, sty)ds + y \int_0^1 \partial_{yy} f(stx, sty)ds\right) dt \\ = f(0, 0) &+ x\partial_x f(0, 0) + y\partial_y f(0, 0) + x^2 g_{11}(x, y) + xyg_{12}(x, y) + y^2 g_{22}(x, y) \end{aligned}$$

□

Problem 1.1.7.

Proof. $g(t,u)$ is 0 at $t=0$. And, by expanding f, we find, for $t \neq 0$:

$$g(t,u) = \frac{1}{t}\Big(f(0,0) + \partial_x f(0,0)t + \partial_y f(0,0)tu + t^2 g_{11}(t,tu) + t^2 u g_{12}(t,tu) + t^2u^2 g_{22}(t,tu)\Big)$$

Noticing $f(0,0) = \partial_x f(0,0) = \partial_y f(0,0) = 0$ we get:

$$g(t,u) = tg_{11}(t,tu) + tug_{12}(tu) + tu^2 g_{22}(t,tu)$$

Because $g(0,u) = 0$, this formula is valid for $t = 0$ as well, and this expression is C^∞. □

Problem 1.1.8. $f^{-1} = x^{1/3}$ which is not differentiable at 0. In complex analysis, as a consequence of Rouche's theorem, if $f'(z) = 0$, then $f(z+s) = f''(z)s^2 + \dots$, and it can be shown that for sufficiently small s, we have at least two solutions.

1.2 Tangent Vectors in Rn as Derivations

Problem 1.2.1.

Proof.

$$X = x\partial_x + y\partial_y$$
$$f(x,y,z) = x^2 + y^2 + z^2$$

Then, computing Xf is as simple as applying X to f at every point:

$$Xf = x\partial_x f + y\partial_y f = 2x^2 + 2y^2$$

□

Problem 1.2.2.

Proof. We define all such operations point-wise on C_p^∞. For $f, g \in C_p^\infty$ and $\lambda \in \mathbb{R}$, for any $x \in U$:

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) = g(x) + f(x) = (g+f)(x)\\ (f\cdot g)(x) &= f(x)\cdot g(x) = g(x)\cdot f(x) = (g\cdot f)(x)\\ (\lambda f)(x) &= \lambda \cdot f(x)\end{aligned}$$

Such operations are closed in C_p^∞ as differentiability is a local property closed under these operations. □

Problem 1.2.3.

(a) *Proof.* Let D, D' be derivation at p. Then both D, D' are linear maps of the form $C_p^\infty \to \mathbb{R}$, that satisfy the Leibniz rule.

$$(D+D')(\lambda f + g) = D(\lambda f + g) + D'(\lambda f + g) = \lambda(D+D')f + (D+D')g$$

So $D + D'$ is linear. We also have:

$$(D+D')(fg) = D(fg) + D'(fg) = (Df)g + f(Dg) + (D'f)g + f(D'g) = (D+D')(f)g + f(D+D')(g)$$

As we wanted to show. □

(b) *Proof.* Certainly is a linear map and the c pops inside the Leibniz rule

$$cD(fg) = c((Df)g + f(Dg)) = (cDf)g + f(cDg)$$

$\square$

Problem 1.2.4.

Proof. Let $D_1, D_2 : A \to A$, then $D_1 \circ D_2 : A \to A$. And:

$$D_1 \circ D_2(ab) = D_1(a(D_2b)) + D_1((D_2a)b) = (D_1a)(D_2b) + a(D_1D_2b) + (D_1D_2a)b + (D_2a)(D_1b)$$

which certainly isn't necessairly equal to:

$$a(D_1(D_2b)) + (D_1(D_2a))b$$

Now let's consider $D_1 \circ D_2 - D_2 \circ D_1$, which is clearly a linear map.

$$\begin{aligned}(D_1 \circ D_2 - D_2 \circ D_1)(ab) &= (D_1a)(D_2b) + a(D_1D_2b) + (D_1D_2a)b + (D_2a)(D_1b) \\ &\quad - (D_2a)(D_1b) - a(D_2D_1b) - (D_2D_1a)b - (D_1a)(D_2b) \\ &= a[D_1D_2 - D_2D_1](b) + [D_1D_2 - D_2D_1](a)b\end{aligned}$$

As we wanted to show. $\square$