Contents

Chapter 1

Euclidean Spaces

1.1 Smooth Functions on a Euclidean Space

Problem 1.1.1.

Proof.

$$g(x) = \int_0^x t^{1/3} dt = \frac{3}{4} x^4/3$$
$$g'(x) = x^{1/3}$$

So, as seen before, g' is C^0 but not C^1. As such, g is C^1, but not C^2. And $h = \int_0^x g(t)dt$, which has $h' = g$, is C^2 but not C^3. □

Problem 1.1.2.

(a) *Proof.* Base case: $k = 0$, it is obviously true, with $p_0 = 1$. Suppose it's true for $k > 1$. Then

$$f^{(k)}(x) = p_{2k}(1/x)e^{-1/x}$$

$$\begin{aligned} f^{(k+1)}(x) &= (p_{2k}(1/x))' \cdot (e^{-1/x}) + (p_{2k}(1/x)) \cdot (e^{-1/x})' \\ &= (p_{2k})'(1/x)\frac{1}{x^2} \cdot (e^{-1/x}) + p_{2k}(1/x) \cdot e^{-1/x}\frac{1}{x^2} \\ &= e^{-1/x} \cdot \frac{(p_{2k})'(1/x) + p_{2k}(1/x)}{x^2} \end{aligned}$$

Now, $(p_{2k})'(1/x)/x^2$ is a polynomial on $(1/x)$ of degree $2k+1$ and $p_{2k}(1/x)/x^2$ is of degree $2k+2$, so $(p_{2k})'(1/x)/x^2 + p_{2k}(1/x)/x^2$ is a polynomial on $(1/x)$ of degree $2k+2$, proving the hypothesis. □

(b) *Proof.* These formula are certainly valid for any $x \neq 0$. For $x \to 0$, it suffices to notice that $e^{-1/x} \lll p_{2k}(1/x)$ for any k. So f^k is defined for all $\mathbb{R}$ and, (taking the limit) is 0 at 0 for any k. □

Problem 1.1.3.

(a) *Proof.* tan is C^∞ on $(-\pi/2, \pi/2)$ as, taking derivatives, on the denominators only cos appears and they never 0 on this interval. Its inverse, arctan has derivative $\frac{1}{1+x^2}$ which also is C^∞. □

(b) *Proof.* Consider

$$h(x) = \frac{x - (b+a)/2}{(b-a)/2}$$

$\square$

(c) *Proof.* Consider $h(x) = \exp(x) + a$ and $g(x) = b - \exp(x)$, then clearly h and g are diffeomorphisms, and we may compose the inverses to find that by the diffeomorphism $g \circ h^{-1}$, the intervals are diffeomorphic. $\square$

Problem 1.1.4.

Proof. Consider the smooth inverse

$$g : \mathbb{R}^n \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)^n, \quad g(x_1, \dots, x_n) = (\arctan(x_1), \dots, \arctan(x_n))$$

$\square$

Problem 1.1.5.

(a) *Proof.* We parametrize the line between $(0,0,1)$ and (a,b,c) by t and solve for when $z = 0$.

$$l(t) = (0,0,1) + t \cdot ((a,b,c) - (0,0,1)) = (ta, tb, 1 + t(c-1))$$

$$l_3(t) = 0 \iff 1 + t(c-1) = 0 \iff t = \frac{1}{1-c}$$

yielding precisely

$$g(a,b,c) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right)$$

as $(a,b,c) \in S$, we know that $c = 1 - \sqrt{1 - a^2 - b^2}$.

For the inverse, we procede the same way, solving the line equation for when $|l(t) - (0,0,1)| = 1$. This time it is given by:

$$l(t) = (0,0,1) + t \cdot ((x,y,0) - (0,0,1))$$

So, for $|l(t) - (0,0,1)| = 1$ to happen, we must have:

$$t^2x^2 + t^2y^2 + t^2 = 1 \iff t^2(x^2 + y^2 + 1) = 1$$

yielding $t = \pm 1/\sqrt{x^2 + y^2 + 1}$, as we know our solution is in the lower hemisphere, we have $t = 1/\sqrt{x^2 + y^2 + 1}$. Substituting back on the line equation we find

$$(a,b,c) = \left(\frac{x}{\sqrt{x^2+y^2+1}}, \frac{y}{\sqrt{x^2+y^2+1}}, 1 - \frac{1}{\sqrt{x^2+y^2+1}}\right)$$

$\square$

(b) *Proof.* $h^{-1} = f^{-1} \circ g^{-1}$, g^{-1} was found in the previous item, and f^{-1} is simply the projection to the xy plane. So

$$h^{-1}(u,v) = \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}\right)$$

which is C^∞. h is a diffeomorphism. $\square$

(c) *Proof.* This is the most interesting item, but we do exactly the same thing looking at the S^n dimensional sphere in R^{n+1}. Consider the stereographic projection $g : S \to \mathbb{R}^n$ from $(0,0,1)$ given by:

$$g(x_1, x_2, \ldots, x_n, x_{n+1}) = \left(\frac{x_1}{1-c}, \ldots, \frac{x_n}{1-c}, 1 - \frac{1}{1-c} \right)$$

where $c = 1 - \left(\sum_1^n (x_i)^2 \right)^{1/2}$ Then, following the same construction as before, we find h and h^{-1} where:

$$h(x_1, x_2 \ldots, x_n) = \left(\frac{x_i}{1-c} \right)_{i=1}^n$$

where the expression on the right is a vector. Similarly, h^{-1} is defined as:

$$h^{-1}(x_1, x_2 \ldots, x_n) = \left(\frac{x_i}{\sqrt{1 + (x_1)^2 \cdots + (x_n)^2}} \right)_{i=1}^n$$

□

Problem 1.1.6.

Proof. We apply Taylor twice. As before, consider the function on the line $f(tx, ty))$. By the chain rule

$$D_t f(tx, ty) = \partial_x f(tx, ty)x + \partial_y f(tx, ty)y$$

So, integrating, we find

$$f(x, y) - f(0, 0) = D_t f(tx, ty) \Big]_0^1 = x \int_0^1 \partial_x f(tx, ty)dt + y \int_0^1 \partial_y f(tx, ty)dt$$

$$f(x, y) = f(0, 0) + x \int_0^1 \partial_x f(tx, ty)dt + y \int_0^1 \partial_y f(tx, ty)dt$$

Now we do the same for $\partial_x f(tx, ty)$ and $\partial_y f(tx, ty)$, we find:

$$\partial_x f(x, y) = \partial_x f(0, 0) + x \int_0^1 \partial_{xx} f(tx, ty)dt + y \int_0^1 \partial_{xy} f(tx, ty)dt$$

$$\partial_y f(x, y) = \partial_y f(0, 0) + x \int_0^1 \partial_{yx} f(tx, ty)dt + y \int_0^1 \partial_{yy} f(tx, ty)dt$$

Substituting in the $f(x, y)$ expansion:

$$\begin{aligned} f(x, y) = f(0, 0) &+ x \int_0^1 \left(\partial_x f(0, 0) + x \int_0^1 \partial_{xx} f(stx, sty)ds + y \int_0^1 \partial_{xy} f(stx, sty)ds \right) dt \\ &+ y \int_0^1 \left(\partial_y f(0, 0) + x \int_0^1 \partial_{yx} f(stx, sty)ds + y \int_0^1 \partial_{yy} f(stx, sty)ds \right) dt \\ = f(0, 0) &+ x\partial_x f(0, 0) + y\partial_y f(0, 0) + x^2 g_{11}(x, y) + xyg_{12}(x, y) + y^2 g_{22}(x, y) \end{aligned}$$

□

Problem 1.1.7.

Proof. $g(t,u)$ is 0 at $t=0$. And, by expanding f, we find, for $t \neq 0$:

$$g(t,u) = \frac{1}{t}\Big(f(0,0) + \partial_x f(0,0)t + \partial_y f(0,0)tu + t^2 g_{11}(t,tu) + t^2 u g_{12}(t,tu) + t^2 u^2 g_{22}(t,tu)\Big)$$

Noticing $f(0,0) = \partial_x f(0,0) = \partial_y f(0,0) = 0$ we get:

$$g(t,u) = tg_{11}(t,tu) + tug_{12}(tu) + tu^2 g_{22}(t,tu)$$

Because $g(0,u)=0$, this formula is valid for $t=0$ as well, and this expression is C^∞. □

Problem 1.1.8. $f^{-1} = x^{1/3}$ which is not differentiable at 0. In complex analysis, as a consequence of Rouche's theorem, if $f'(z)=0$, then $f(z+s) = f''(z)s^2 + \dots$, and it can be shown that for sufficiently small s, we have at least two solutions.

1.2 Tangent Vectors in Rn as Derivations

Problem 1.2.1.

Proof.

$$X = x\partial_x + y\partial_y$$
$$f(x,y,z) = x^2 + y^2 + z^2$$

Then, computing Xf is as simple as applying X to f at every point:

$$Xf = x\partial_x f + y\partial_y f = 2x^2 + 2y^2$$

□

Problem 1.2.2.

Proof. We define all such operations point-wise on C_p^∞. For $f,g \in C_p^\infty$ and $\lambda \in \mathbb{R}$, for any $x \in U$:

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) = g(x) + f(x) = (g+f)(x)\\ (f\cdot g)(x) &= f(x)\cdot g(x) = g(x)\cdot f(x) = (g\cdot f)(x)\\ (\lambda f)(x) &= \lambda \cdot f(x)\end{aligned}$$

Such operations are closed in C_p^∞ as differentiability is a local property closed under these operations. □

Problem 1.2.3.

(a) *Proof.* Let D, D' be derivation at p. Then both D, D' are linear maps of the form $C_p^\infty \to \mathbb{R}$, that satisfy the Leibniz rule.

$$(D+D')(\lambda f + g) = D(\lambda f + g) + D'(\lambda f + g) = \lambda(D+D')f + (D+D')g$$

So $D+D'$ is linear. We also have:

$$(D+D')(fg) = D(fg) + D'(fg) = (Df)g + f(Dg) + (D'f)g + f(D'g) = (D+D')(f)g + f(D+D')(g)$$

As we wanted to show. □

(b) *Proof.* Certainly is a linear map and the c pops inside the Leibniz rule

$$cD(fg) = c((Df)g + f(Dg)) = (cDf)g + f(cDg)$$

$\square$

Problem 1.2.4.

Proof. Let $D_1, D_2 : A \to A$, then $D_1 \circ D_2 : A \to A$. And:

$$D_1 \circ D_2(ab) = D_1(a(D_2 b)) + D_1((D_2 a)b) = (D_1 a)(D_2 b) + a(D_1 D_2 b) + (D_1 D_2 a)b + (D_2 a)(D_1 b)$$

which certainly isn't necessairly equal to:

$$a(D_1(D_2 b)) + (D_1(D_2 a))b$$

Now let's consider $D_1 \circ D_2 - D_2 \circ D_1$, which is clearly a linear map.

$$\begin{aligned}(D_1 \circ D_2 - D_2 \circ D_1)(ab) &= (D_1 a)(D_2 b) + a(D_1 D_2 b) + (D_1 D_2 a)b + (D_2 a)(D_1 b) \\ &- (D_2 a)(D_1 b) - a(D_2 D_1 b) - (D_2 D_1 a)b - (D_1 a)(D_2 b) \\ &= a[D_1 D_2 - D_2 D_1](b) + [D_1 D_2 - D_2 D_1](a)b\end{aligned}$$

As we wanted to show. $\square$

1.3 The Exterior Algebra of Multivectors

In some chapters, before the problem section there are some exercises within the text.

1.3.1 Within Text Exercises

Exercise 3.6.

Proof. We know looking at it that the inversions are $(5,1), (5,2), (5,3), (5,4)$. But we might have a clearer view writing it in matricial form:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}$$

$\square$

Exercise 3.13.

Proof. Consider $\tau S f$.

$$\begin{aligned}\tau S f &= \sum_{\sigma \in S_k} \tau(\sigma f) \\ &= \sum_{\sigma \in S_k} (\tau \sigma f) \\ &= \sum_{\sigma \in S_k} \sigma f\end{aligned}$$

As we have set equality $\{\tau\sigma, \sigma \in S_k\} = \{\sigma \in S_k\}$ $\square$

Exercise 3.15.

Proof. We write the expression containing the 6 permutations and their signs:

$$\begin{aligned} Af(v_1, v_2, v_3) = & f(v_1, v_2, v_3) + (-1)f(v_1, v_3, v_2) + f(v_2, v_1, v_3) \\ & + (-1)f(v_2, v_3, v_1) + (-1)f(v_3, v_1, v_2) + f(v_3, v_2, v_1) \end{aligned}$$

□

Exercise 3.17.

Proof. let f be k-linear, g l-linear and h m-linear. Then:

$$(f \otimes g) \otimes h = (f(v_1, \dots v_k) \cdot g(v_{k+1}, \dots v_{k+l})) \otimes h = f(v_1, \dots v_k) \cdot g(v_{k+1}, \dots v_{k+l}) \cdot h(v_{k+l+1}, \dots v_{k+l+m})$$

Similarly, as g is l-linear and h m-linear, $(g \otimes h)$ is $l+m$ linear, as such:

$$f \otimes (g \otimes h) = f(v_1, \dots v_k) \otimes (g \otimes h) = f(v_1, \dots v_k) \otimes (g(w_1, \dots w_{k+l}) \cdot h(w_{l+1}, \dots w_{l+m})$$

Resulting in what was expected:

$$f \otimes (g \otimes h) = f(v_1, \dots v_k) \cdot g(v_{k+1}, \dots v_{k+l}) \cdot h(v_{k+l+1}, \dots v_{k+l+m})$$

□

Exercise 3.20.

Proof.

$$f \wedge g(v_1, v_2, v_3, v_4) = \sum_{(2,2)-\text{shuffles } \sigma} f(v_{\sigma(1)}, v_{\sigma(2)}) g(v_{\sigma(3)}, v_{\sigma(4)})$$

As $\binom{4}{2} = 6$ we have the following big sum:

$$\begin{aligned} (f \wedge g)(v_1, v_2, v_3, v_4) = & f(v_1, v_2)g(v_3, v_4) + (-1)f(v_1, v_3)g(v_2, v_4) \\ & + f(v_1, v_4)g(v_2, v_3) + f(v_2, v_3)g(v_1, v_4) \\ & + (-1)f(v_2, v_4)g(v_1, v_3) + f(v_3, v_4)g(v_1, v_2) \end{aligned}$$

□

Exercise 3.22.

Proof. It suffices to count the number of inversions, but this is simple, each of the first l elements of the permutation have k inversions with the last k elements, yielding lk inversions. As such, $\text{sgn}(\tau) = (-1)^{kl}$. □

1.3.2 Problems

Problem 1.3.1.

Proof. We begin by remembering $\alpha_i : v \to v_i$, so we may write the tensor product

$$\alpha_i \otimes \alpha_j : (v, w) \to \alpha_i(v) \cdot \alpha_j(w)$$

So the transformation becomes:

$$f = \sum_{1 \leqslant i,j \leqslant n} g_{ij} \cdot \alpha_i \otimes \alpha_j$$

□

Problem 1.3.2.

(a) *Proof.* This is a simple consequence of the kernel image theorem. We know, from that result, the following identity:

$$\dim(V) = \dim(\ker(f)) + \dim(f(V))$$

As f is a linear, non-zero, and sends on $\mathbb{R}$, we know $f(V) = \mathbb{R}$, and $\dim(f(V)) = 1$. Substituting back, $\dim(\ker(f)) = n - 1$. □

(b) *Proof.* If V has finite dimension, this is a consequence of the previous item. Being the kernel the same of dimension n-1 , by taking any vector v such that $f(v) \neq 0$, we may chose $c = g(v)/f(v)$. Now notice $\ker(cf - g)$ has dimension at least n, and as such is the whole space. Let's generalize using the first isomorphism theorem. Notice that

$$\frac{V}{\ker(f)} = \frac{V}{\ker(g)} \cong \mathbb{R}$$

This means that $\frac{V}{\ker(f)}$ and $\frac{V}{\ker(g)}$ are one-dimensional, with the same elements. Take v such that $\bar{v} \neq 0$, choose $c = g(v)/f(v)$, and as before notice that $\ker(cf - g) = V$. □

Problem 1.3.3.

Proof. First of linear independence, consider I the set of multi-indices and set e_I as before. Suppose

$$\sum_{(i_1,\dots,i_k)=I} c_I \alpha_{i_1} \otimes \alpha_{i_2} \cdots \otimes \alpha_{i_n} = 0$$

Now to uncover each c_I, apply the tranformation to e_I. If $J \neq I$, then there is a first index $j_k \neq i_k$. As such, when applying to e_I:

$$\alpha_{j_1} \otimes \cdots \alpha_{j_k} \cdots \otimes \alpha_{j_n}(e_I) = \alpha_{j_1}(e_{i_1}) \cdots \alpha_{j_k}(e_{i_k}) \cdots \alpha_{j_n}(e_{i_n}) = 0$$

As $\alpha_{j_k}(e_{i_k}) = 0$. And we find:

$$\left[\sum_{(i_1,\dots,i_k)=I} c_I \cdot \alpha_{i_1} \otimes \alpha_{i_2} \cdots \otimes \alpha_{i_n} \right](e_I) = c_I = 0$$

Now for span. Notice only by linearity that if two k-linear transformation coincide on the indices, then they coincide for every value. As such, given $f \in L_k(V)$, we define g by:

$$g = \sum_I f(e_I) \cdot \alpha_I$$

where here $\alpha_I = \alpha_{i_1} \otimes \alpha_{i_2} \cdots \otimes \alpha_{i_n}$. □

Problem 1.3.4.

Proof. We say f is alternating if, for a permutation σ:

$$\sigma f = \operatorname{sgn}(\sigma) f$$

To show equivalence is to show that a function has the flipping property, iff it satisfies this permutation property. Clearly, permutation implies flipping, as doing any single 2-transposition changes the sign of the permutation. Now, for the other side, we record that the sign of a permutation is $(-1)^m$, where m is the number of inversions revelead when describing the permutation as 2-transpositions. Following this definition, we see that flipping is sufficient. □

Problem 1.3.5.

Proof. We may notice, using the wedge products as a basis for the co-vectors that if f is alternating, then $f(v_1, \ldots, v_k) = 0$ if there are $i \neq j$ with $v_i = v_j$ by looking at the decomposition. But that is actually harder.

Suppose f is n-alternating and let $v_1, \ldots v_n$ be n vectors with $v_i = v_j$ for $i < j$. Then after a permutation σ that makes i next to j, we have:

$$(\sigma f)(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_n) = \operatorname{sgn}(\sigma) f(v_1, \ldots, v_i, v_j, \ldots, v_n)$$

Being $v_i = v_j$

$$\operatorname{sgn}(\sigma) f(v_1, \ldots, v_i, v_j, \ldots, v_n) = \operatorname{sgn}(\sigma) f(v_1, \ldots, v_j, v_i, \ldots, v_n)$$

But, because f is alternating, we also have:

$$\operatorname{sgn}(\sigma) f(v_1, \ldots, v_i, v_j, \ldots, v_n) = -\operatorname{sgn}(\sigma) f(v_1, \ldots, v_j, v_i, \ldots, v_n)$$

So both are equal to 0, and as such $f(v_1, \ldots, v_n) = 0$.

Now suppose that whenever two vectors are equal, $f = 0$. Then, given any 2 positions $i < j$ we may write (to simplify, we suppose that they are the only ones):

$$0 = f(v_i + v_j, v_i + v_j) = f(v_i, v_i) + f(v_i, v_j) + f(v_j, v_i) + f(v_j, v_j) = f(v_i, v_j) + f(v_j, v_i)$$

So flipping a coordinate changes the sign. □

Problem 1.3.6.

Proof.

$$af \wedge bg = \frac{1}{k!l!} A(af \otimes bg) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}(\sigma))(\sigma(af \otimes bg))$$

But we can pop out the constants from the tensor product. Yielding:

$$\frac{ab}{k!l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn}(\sigma))(\sigma(f \otimes g)) = ab(f \wedge g)$$

□

Problem 1.3.7.

Proof. I was thinking of using the relation for covectors α:

$$(\alpha^1 \wedge \alpha^2 \cdots \wedge \alpha^k)(v_1, v_2, \ldots, v_k) = \det[\alpha^i(v_j)]$$

But I don't know how to use it here. I guess we can expand stuff:

$$\beta^1 \wedge \beta^2 \cdots \wedge \beta^k = \bigwedge_{i=1}^{k} \sum_{j=1}^{k} a^i_j \gamma^j$$

Now, by linearity of the wedge product, we may separate each sum and write the following:

$$\bigwedge_{i=1}^{k} \sum_{j=1}^{k} a^i_j \gamma^j = \sum_{\substack{[i_1, i_2, \ldots, i_k] \\ \in [k]^k}} (a^1_{i_1} \gamma_{i_1} \wedge a^2_{i_2} \gamma_{i_2} \cdots \wedge a^k_{i_k} \gamma_{i_k})$$

Now, we know that, because the wedge product is alternating, we only care about permutations of $[k]$, because if we choose two γ_i's it will zero. As such, we are left with the following, (where the sign comes from the alternating property):

$$\sum_{\substack{[i_1,i_2,\ldots,i_k] \\ \in [k]^k}} (a^1_{i_1}\gamma_{i_1} \wedge a^2_{i_2}\gamma_{i_2} \cdots \wedge a^k_{i_k}\gamma_{i_k}) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot (a^1_{\sigma(1)}\gamma_1 \wedge a^2_{\sigma(2)}\gamma_2 \cdots \wedge a^k_{\sigma(k)}\gamma_k)$$

which, by the previous problem is equal to:

$$\sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot a^1_{\sigma(1)} a^2_{\sigma(2)} \cdots a^k_{\sigma(k)} \cdot (\gamma_1 \wedge \gamma_2 \cdots \wedge \gamma_k) = (\det A)\, \gamma_1 \wedge \gamma_2 \cdots \wedge \gamma_k$$

□

Problem 1.3.8.

Proof. This is a collorary from the fact that given a basis B of V and their α^i duals, $\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^n$ is a basis for A_n. If ω is a n-covector, then it is of the form $c(\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^n)$, so if it is zero for B, then $c = 0$.

Another way of seeing this is writing $\omega(v_1, v_2, \ldots, v_n) = \sum_{\sigma \in S_k} C_\sigma \omega(e_1, e_2, \ldots, e_n) = 0$. □

Problem 1.3.9.

Proof. This seems to have an easy and a hard direction. If the covectors are NOT linearly independent, then we may write, say $\alpha^k = \sum_{i=1}^{k-1} c_i \alpha^i$. As such

$$\alpha_1 \wedge \alpha_2 \cdots \wedge \alpha_k = \sum_{i=1}^{k-1} c_i \alpha_1 \wedge \alpha_2 \cdots \wedge \alpha_{k-1} \wedge \alpha_i = 0$$

Now, if they are linearly independent, let's use induction on the determinant formula.

$$\alpha^1 \wedge \alpha^2 \cdot \wedge \alpha^k(v_1, v_2 \ldots, v_k) = \det[\alpha^i(v_j)]$$

Base case k = 1: $\alpha^1 \neq 0$ Now, suppose it's valid for $k-1$ and Sps for sake of contradiction that $\alpha^1 \wedge \alpha^2 \cdot \wedge \alpha^k = 0$, we may then consider, for a given fixed choice of $w_2, w_3, \ldots, w_k$ the linear transformation on v

$$x(v) = \alpha^1 \wedge \alpha^2 \cdots \wedge \alpha^k(v, w_2 \ldots, w_k) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)(\sigma \bigotimes_{i=1}^{k} \alpha^i)(v, w_2, \ldots, w_k) = 0$$

Writing this sum separetely, depending where v is:

$$\begin{aligned} x(v) = &\sum_{\sigma \in S_k, \sigma(1)=1} \alpha^1(v) \cdot \operatorname{sgn}(\sigma) \cdot \alpha^2(w_{\sigma(2)}) \cdots \alpha^k(w_{\sigma(k)}) \\ &+ \sum_{\sigma \in S_k, \sigma(2)=1} \alpha^2(v) \cdot \operatorname{sgn}(\sigma) \cdot \alpha^1(w_{\sigma(1)}) \cdots \alpha^k(w_{\sigma(k)}) \\ &\cdots \\ &+ \sum_{\sigma \in S_k, \sigma(k)=1} \alpha^k(v) \cdot \operatorname{sgn}(\sigma) \cdot \alpha^1(w_{\sigma(1)}) \cdots \alpha^{k-1}(w_{\sigma(k-1)}) \end{aligned}$$

If we show the term following $\alpha_1(v)$ is not 0, then we will have shown linear dependence. We can use induction for this! The terms that follows $\alpha_1(v)$ is actually $(\alpha^2(w_2) \wedge \alpha^3(w_3) \cdots \wedge \alpha^k(w_k))$, by the induction hypothesis, there is a choice of $w_2, w_3, \ldots, w_n$ with the expression non-zero. And we win! Reordering, we get that α^1 is a linear combination of α^j for $j > 1$. □

Problem 1.3.10.

Proof. The converse is obvious. Sps $\alpha \wedge \gamma = 0$. Being $\alpha \neq 0$ and V finite dimensional (let's say n), we may use it to complete a basis:

$$(\alpha = \alpha^1), \alpha^2, \dots \alpha^n$$

. Then, we may, using the usual basis for the k-covectors $A_k(V)$, span γ by:

$$\gamma = \sum \gamma(e_I)\alpha_I$$

Meaning:

$$\alpha \wedge \gamma = \sum \gamma(e_I)\alpha^1 \wedge \alpha_I = 0$$

But the non-zero terms $\alpha^1 \wedge \alpha_I$ form a basis for $A_{k+1}(V)$, that for each of the α_I that do not contain α^1, $\gamma(e_I) = 0$. As such γ contains only basis vectors that contain α^1 on the index. That is:

$$\gamma = \alpha \wedge \left(\sum_I \gamma(e_I)\alpha_{I/\{1\}} \right)$$

$\square$

1.4 Differential forms on Rn