Ostfalia
Hochschule für angewandte Wissenschaften

Introduction: 23.09.2024

1. Who is Dr. Wagner
2. Lecture „FEM mit Labor“
3. FEM-Application
4. Wind-Turbine Tower Benchmark
5. Lectur
6. FEM-Video YouTube

Technische Universität Braunschweig

PROMOTIONSURKUNDE

Die Fakultät für Maschinenbau
der Technischen Universität Braunschweig
verleiht unter der Präsidentin
Universitätsprofessorin Dr.-Ing. Anke Kaysser-Pyzalla
und unter dem Dekanat des Universitätsprofessors Dr.-Ing. Christoph Herrmann

Herrn
Dipl.-Ing. Heinz Norbert Ronald Wagner
geboren am 06.05.1986 in Magdeburg

den Grad eines
Doktor-Ingenieurs
(Dr.-Ing.)

nachdem in ordnungsgemäßem Promotionsverfahren durch die Dissertation

„Robust Design of Buckling Critical Thin-Walled Shell Structures“

sowie durch die mündliche Prüfung am 21. Juni 2018
die wissenschaftliche Befähigung erwiesen und dabei
das Gesamtprädikat „sehr gut bestanden“ erteilt wurde.

Braunschweig, 21. Juni 2018

TECHNISCHE UNIVERSITÄT CAROLO-WILHELMINA 45 BRAUNSCHWEIG

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Präsidentin
Technische Universität Braunschweig

Prof. Dr.-Ing. Christoph Herrmann
Dekan
Fakultät für Maschinenbau

OTTO-VON-GUERICKE-UNIVERSITÄT MAGDEBURG

FAKULTÄT FÜR MASCHINENBAU

DIPLOM

Herrn Ronald Wagner

geboren am 06. Mai 1986 in Magdeburg

wird nach bestandener Diplomprüfung im Studiengang

Maschinenbau

der akademische Grad

Diplomingenieur
(Dipl.-Ing.)

verliehen.

Magdeburg, 26. April 2013

Der Dekan

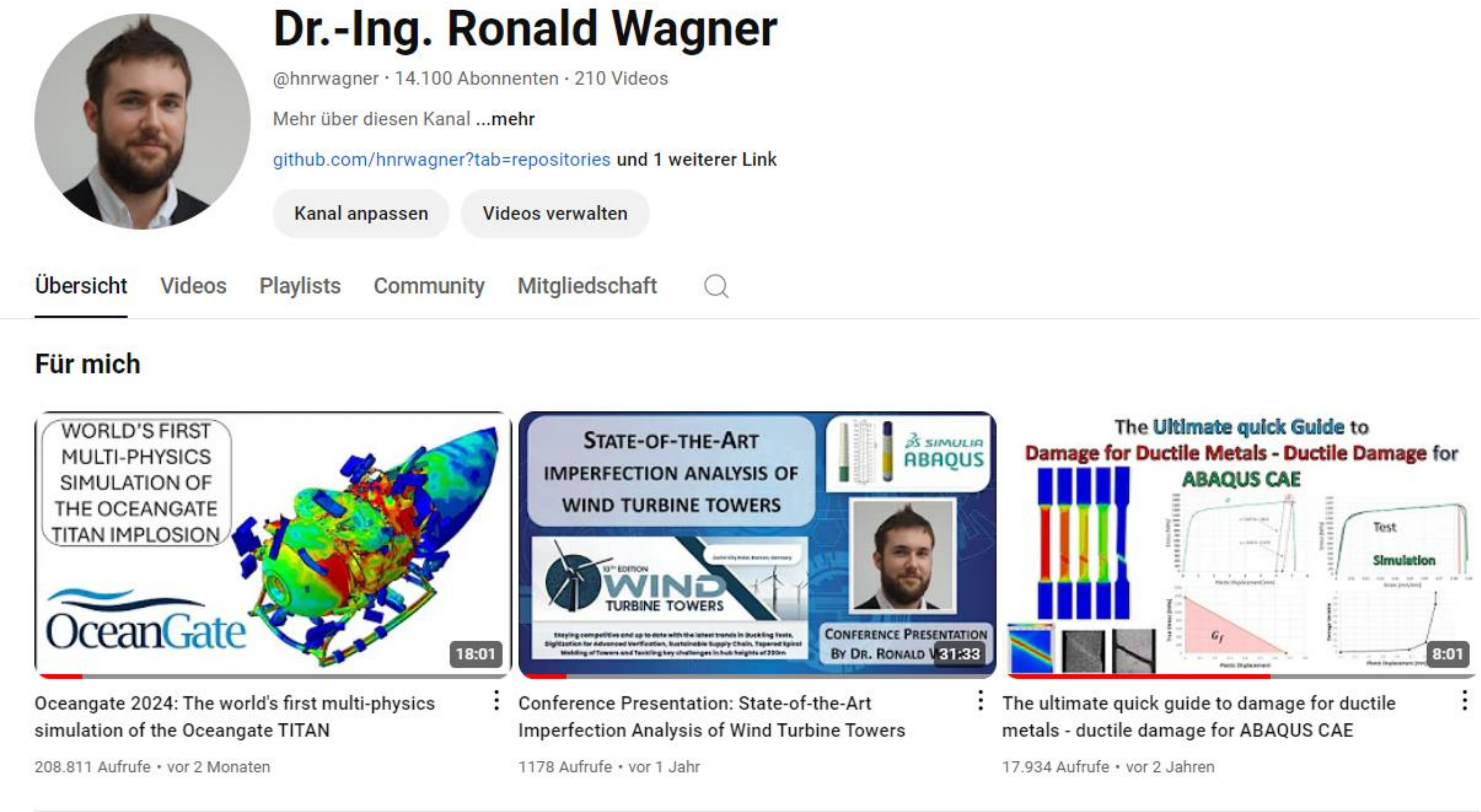
Prof. Dr.-Ing. K.-H. Grote

Otto-von-Guericke-Universität Magdeburg 11

Der Vorsitzende des Prüfungsausschusses

Prof. Dr. rer. nat. M. Scheffler

180804



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Dr.-Ing. Ronald Wagner

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A repository of ABAQUS FORTRAN subroutines
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USDFLD_FGM Public
ABAQUS Subroutine for functionally graded material
Fortran ☆ 10 Forks 3

CLT Public
This function calculates the ABD Composite Stiffness Matrix Components # Classical Lamination Theory (CLT)
Python ☆ 9 Forks 3

8 contributions in the last year

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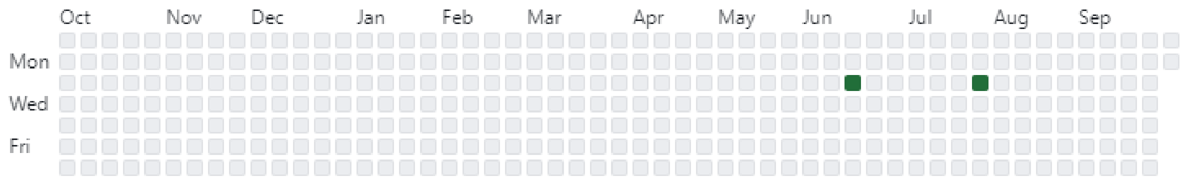

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2024
2023
2022
2021
2020

https://github.com/hnrwagner

Heinz Norbert Ronald Wagner

Siemens
Verified email at siemens.com
Shell buckling

FOLLOW

TITLE	CITED BY	YEAR
Robust design criterion for axially loaded cylindrical shells-Simulation and Validation Q1 + HNR Wagner, C Hühne, S Niemann, R Khakimova Thin-Walled Structures 115, 154-162	108	2017
Decision tree-based machine learning to optimize the laminate stacking of composite cylinders for maximum buckling load and minimum imperfection sensitivity Q1 + HNR Wagner, H Köke, S Dähne, S Niemann, C Hühne, R Khakimova Composite Structures 220, 45-63	89	2019
Robust knockdown factors for the design of axially loaded cylindrical and conical composite shells–development and validation Q1 + HNR Wagner, C Hühne, S Niemann Composite structures 173, 281-303	87	2017
Stability and vibrations of thin-walled composite structures H Abramovich, R Wagner Woodhead Publishing	86	2017
Robust knockdown factors for the design of cylindrical shells under axial compression: potentials, practical application and reliability analysis Q1 + HNR Wagner, C Hühne International Journal of Mechanical Sciences 135, 410-430	84	2018
Robust knockdown factors for the design of spherical shells under external pressure: Development and validation HNR Wagner, C Hühne, S Niemann International Journal of Mechanical Sciences 141, Pages 58-77	83	2018
Robust knockdown factors for the design of cylindrical shells under axial compression: Analysis and modeling of stiffened and unstiffened cylinders Q1 + HNR Wagner, C Hühne, S Niemann, K Tian, B Wang, P Hao Thin-Walled Structures 127, 629-645	72	2018

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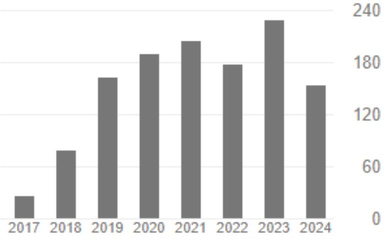

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Co-authors EDIT

- Christian Hühne — DLR & TU Braunschweig
- Jinyu Zhang (张金钰) — Xi'an Jiaotong University
- Eduardo M. Sosa — Research Associate Professor, ...
- Sascha Dähne — Deutsches Zentrum für Luft und ...

https://scholar.google.de/citations?user=a4sKEKsAAAAJ&hl=en

https://orcid.org/0000-0003-2749-1455 ORCID Connecting research and researchers

Veranstaltungsplan Wagner, Ronald

Woche: 23. - 29.09.2024

	Mo, 23.09.2024	Di, 24.09.2024	Mi, 25.09.2024	Do, 26.09.2024	Fr, 27.09.2024	
8:00	Mo, 23.09.2024 V FEM m. L. 6. Semester ABE, 6. Semester AGF 8:00 - 9:30 Uhr D-202					8:00
9:00						9:00
10:00	Mo, 23.09.2024 V FEM m. L. 6. Semester ABE, 6. Semester AGF 9:45 - 11:15 Uhr D-223					10:00
11:00						11:00
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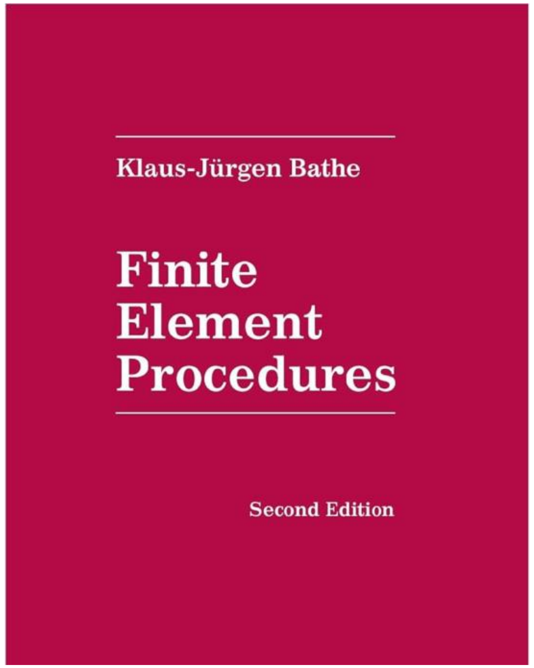
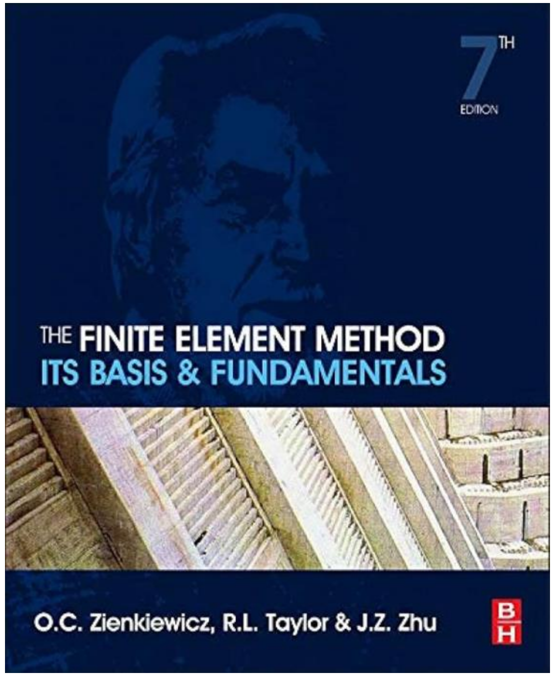
16:00						16:00
17:00						17:00
18:00						18:00
19:00						19:00

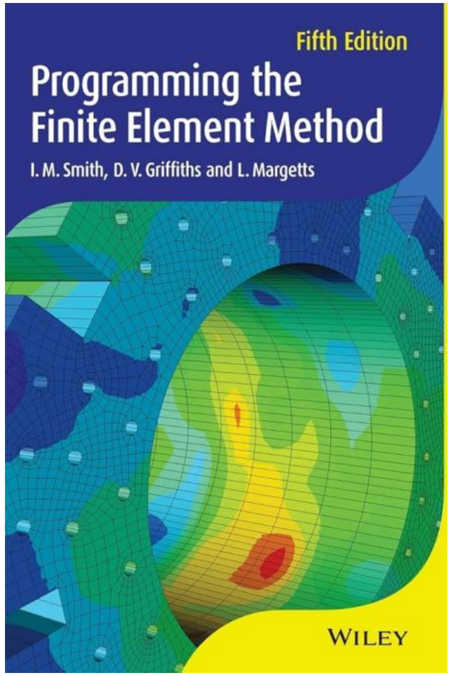
Summe Ist-Stunden: 4:00

Finite Elemente Methode (FEM)	
formale Angaben:	
Semester:	5
Häufigkeit:	jährlich
Art:	Wahlpflicht
Gesamtumfang:	4 SWS
ECTS-Punkte:	5
Workload gesamt:	150 h
davon in Präsenz:	60 h
davon Selbststudium:	90 h
erforderliche Vorkenntnisse (nur bei Wahlpflichtfächern):	Technische Mechanik I, II. und III
Verwendbarkeit:	Automotive Engineering, Automotive Engineering im Praxisverbund
Prüfungsform:	K 90 + EA
Modulverantwortlich:	Dr.-Ing. Ronald Wagner
Qualifikationsziele:	
Fachliche Kompetenz:	Studierende verstehen den direkten und den variationformulierten Zugangsweg zur FEM. Sie können die Problemkategorien statischer, dynamisch transienter und eigenwertproblembezogener physikalischer Fragestellungen richtig einordnen.
Methodische Kompetenz:	Studierende verwenden geeignete numerische Lösungsverfahren selbständig und korrekt. Studierende können strukturmechanische und andere physikalische Fragestellungen selbständig in einem FEM-System lösen. Sie können die selbst erzielten Ergebnisse richtig bewerten.
Sozialkompetenz:	Studierende organisieren sich selbständig in Arbeitsgruppen und können in unterschiedlichen Rollen (Teamplayer/Teamleader) miteinander kooperieren sowie Ergebnisse produzieren.
Persönliche Kompetenz:	Studierende können Ihre erzielten Ergebnisse kritisch bewerten und reflektieren anhand von selbst erarbeiteten, analytischen Referenzmodellen.
Lehrveranstaltungen:	
FEM	
Typ:	Vorlesung
Umfang:	2 SWS





Themen:	• Einleitung: Idee; Entwicklung, Software, typische Problemstellungen und techn. Anwendungen • Matrix-Steifigkeitsmethode Stabelement, Elementmatrizen, Assemblierung, Einbau Randbedingung, Lösung, h-Konvergenz • Ebene Stabwerke Kraft-Verschiebungsgesetz, ebene Elementmatrix, Koinzidenztransformation, Randbedingungsklassen Balkenelemente Bernoulli-Balkenelement, Formfunktionen, Energieintegral, Steifigkeitsmatrix; Anwendungen (z.B. Spaceframe-Leichtbau) • Variationsprinzip und FEM-Problemklassen [Kein Titel] Variationsprinzip, Minimum Gesamtpotential, Eigenwertprobleme der Dynamik und der Stabilität, transiente Berechnungen • Elementtechnologie: Simplex-Elemente, Ebene Scheiben- & Plattenelemente, höhere Ansatzfunktionen, Lagrange- /isopara- /isogeometrische Elemente, Locking, Hourglassing • Vernetzungsmethoden Linien- /Oberflächen-/ Volumen- /struktruierte- /unstrukturierte- /Delauny- /Octree- Vernetzung; automatische Netzadaption, lokale Verfeinerung, Übergänge, Netzqualität • Nichtlineare Problemstellungen Große Verformungen, Hyperelastizität, Plastizität, Kontakt, Schädigung; Anwendungen • Numerische Methoden Direkte und iterative Lösung lin. & nichtlinearer Gleichungssysteme, Eigenwertlöser, Bogenlängenverfahren, Zeitintegrationsverfahren.
Literatur:	• Betten, J:"Finite Elemente für Ingenieure 1 & 2". Berlin Heidelberg, 2003 • Rust, W.:"Nichtlineare Finite-Elemente-Berechnungen: Kontakt, Kinematik, Material". Springer, Berlin, Heidelberg, 2016 • Öchsner, A.; Öchsner, M.:"A first introduction to the finite element analysis program MSC Marc/Mentat". Springer, Berlin, Heidelberg 2018
Labor FEM	
Typ:	Labor
Umfang:	2 SWS
Themen:	Inhalte aus der Vorlesung werden in praktischen Übungen am Computer umgesetzt



International round-robin exercise in computational shell buckling

8-MW wind turbine tower benchmark

2nd May 2022

Imperial College London

SIEMENS Gamesa RENEWABLE ENERGY

Released: 2nd May 2022
Submit by: 1st October 2022

https://www.sciencedirect.com/science/article/pii/S135063072300078X

Engineering Failure Analysis 148 (2023) 107124

Contents lists available at ScienceDirect

Engineering Failure Analysis

journal homepage: www.elsevier.com/locate/engfailanal

8-MW wind turbine tower computational shell buckling benchmark. Part 1: An international 'round-robin' exercise

Adam J. Sadowski [a,*], Marc Seidel [b], Hussain Al-Lawati [c], Esmaeil Azizi [d], Hagen Balscheit [e], Manuela Böhm [f], Lei Chen [g], Ingmar van Dijk [h], Cornelia Doerich-Stavridis [i], Oluwole Kunle Fajuyitan [j], Achilleas Filippidis [a], Astrid Winther Fischer [l], Claas Fischer [m], Simos Gerasimidis [n], Hassan Karampour [o], Lijithan Kathirkamanathan [a], Frithjof Marten [q], Yasuko Mihara [r], Shashank Mishra [s], Volodymyr Sakharov [t], Amela Shahini [u], Saravanan Subramanian [v], Cem Topkaya [w], Heinz Norbert Ronald Wagner [x], Jianze Wang [y], Jie Wang [z], Kshitij Kumar Yadav [aa], Xiang Yun [ab], Pan Zhang [ac]

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[m] TÜV Nord EnSys GmbH, Hamburg, Germany
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[y] Department of Civil Engineering, Sichuan University, China
[z] Department of Architecture and Civil Engineering, University of Bath, UK
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[ab] Department of Civil and Structural Engineering, University of Sheffield, UK
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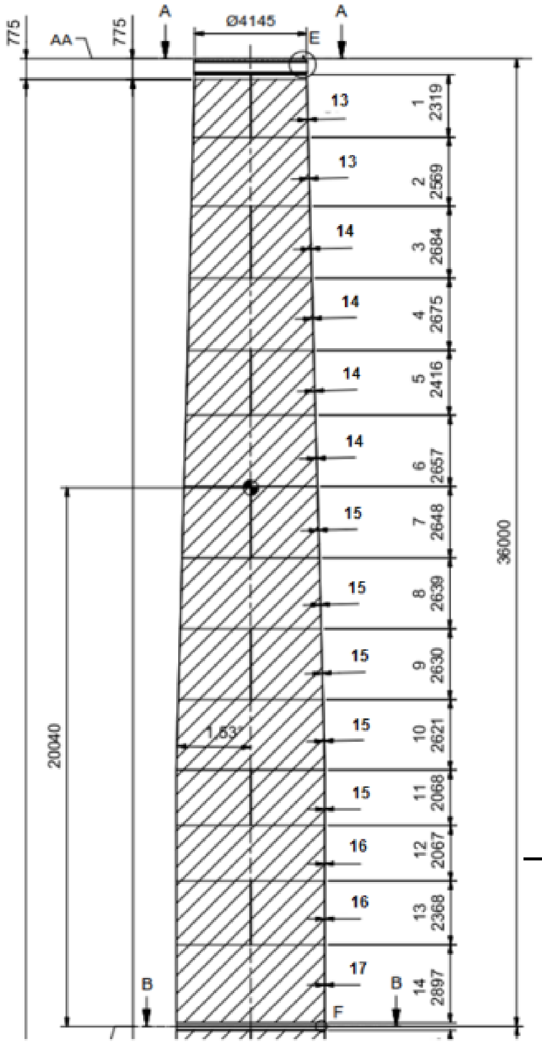
Tower geometry

- Consider only the tower segment shown
- Loads are applied through top flange (see next slide)
- Bottom flange provides restraint
- Material: S355J0 steel (E = 210 GPa, f_y = 345 MPa, ρ = 7850 kg/m^3)
- 'Excellent' (but not perfect) construction quality
- Top diameter: 4145 mm
- Bottom diameter: 5500 mm

Note: Assume that the diameters are those of the shell midsurface.



	H	t	D_{top}	D_{bot}
101 flange	600	50	4145.0	4145.0
102 section	2319	13	4145.0	4257.9
103	2569	13	4257.9	4393.4
104	2684	14	4393.4	4535.1
105	2675	14	4535.1	4676.2
106	2416	14	4676.2	4803.7
107	2657	14	4803.7	4943.9
108	2648	15	4943.9	5083.7
109	2639	15	5083.7	5222.9
110	2630	15	5222.9	5361.7
111	2621	15	5361.7	5500.0
112	2068	15	5500.0	5500.0
113	2067	16	5500.0	5500.0
114	2368	16	5500.0	5500.0
115	2897	17	5500.0	5500.0
116 flange	142	17	5500.0	5500.0

Two Load Cases (LCs)

LC1 – no torsion

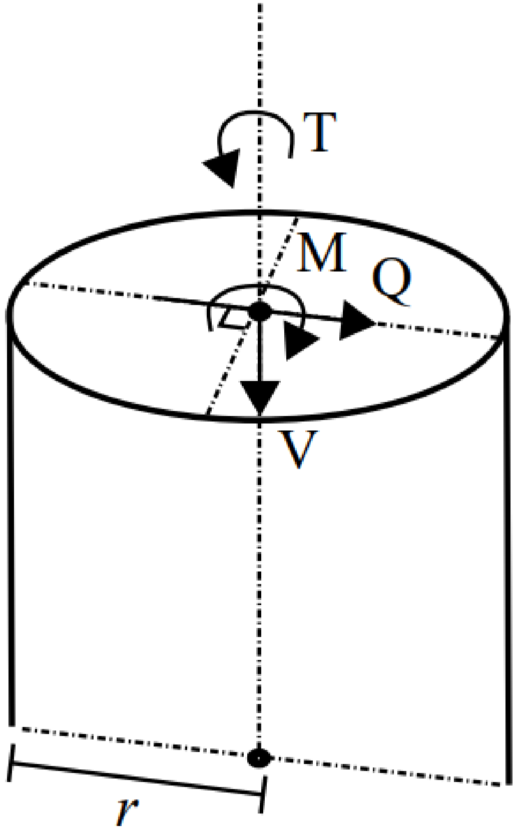
- Shear force: Q = 1.76 MN
- Bending moment in direction of Q: M = 33 MNm
- Vertical force: V = 4 MN
- Self-weight

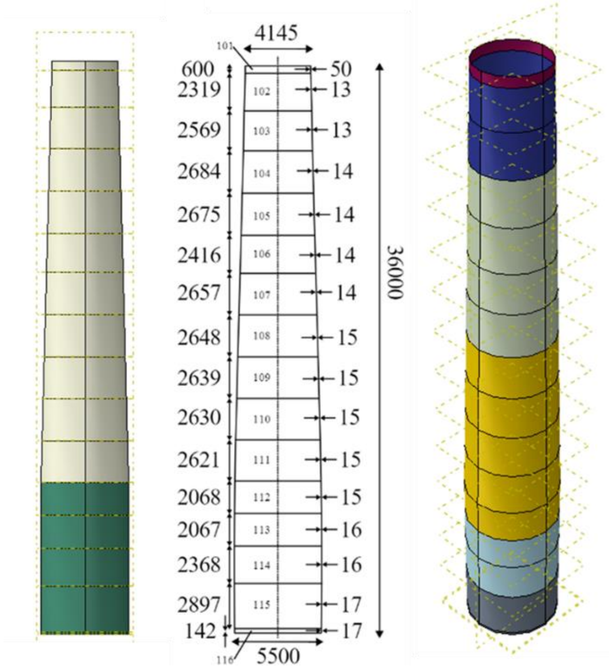
LC2 – with torsion

- Shear force: Q = 1.6 MN
- Bending moment in direction of Q: M = 30 MNm
- Vertical force: V = 4 MN
- Self-weight
- Torque moment about centroidal axis: T = 22 MNm

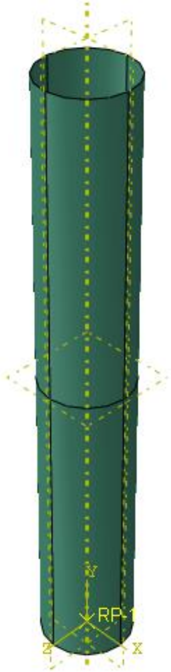
No additional partial factors need be applied (loads are design loads according to IEC 61400-1, with partial safety factors already included)

Loads acting at centroid through top of top flange





FEM Complex Model



Simplified Model:

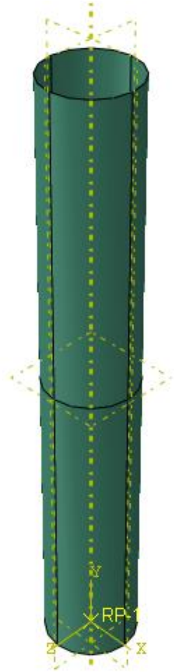
- constant cross-section
- Radius **R** = 2750 mm
- Length **L** = 36000 mm
- Wall Thickness **t** = 17 mm
- Elasticity Modulus **E** = 210 000 MPa
- Yield Stres **Y** = 345 MPa

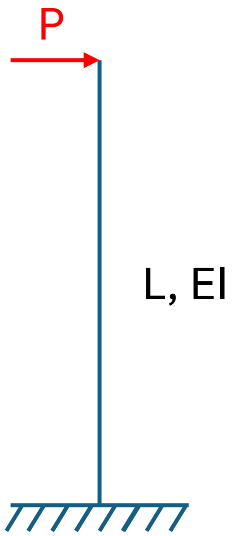
Direct Method

Variational Method

Galerkin Method of Weighted Residuals

Finite Element Method (FEM)





Direct Method

Problem Definition

Boundary Conditions

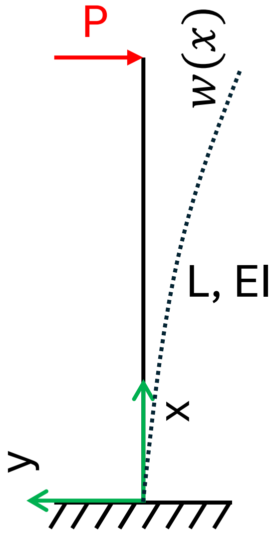
We are solving for the deflection $w(x)$ of a **clamped beam** under a **point load** P at the **free end**.

The governing equation is the **Euler-Bernoulli beam equation**:

$$EIw(x),xxxx = 0 \quad \text{for} \quad 0 \leq x \leq L$$

Where:

- EI is the flexural rigidity,
- $w(x)$ is the deflection of the beam as a function of x,
- x is the position along the length of the beam,
- L is the total length of the beam.



We also have the following **boundary conditions**:

1. **At the clamped end** $x = 0$:
   - $w(0) = 0$ (zero deflection),
   - $w(x),x\big|_{x=0} = 0$ (zero slope).
2. **At the free end** $x = L$:
   - $w(x),xx\big|_{x=L} = 0$ (zero bending moment),
   - $EIw(x),xxx\big|_{x=L} = P$ (shear force equals the applied point load).

Direct Method

Integrate Equation

We will integrate this equation step by step and apply the boundary conditions to solve for the constants of integration.

Step 1: First Integration

Integrating the equation once with respect to x:

$$EIw(x),xxx = C_1$$

Step 2: Second Integration

Integrating again:

$$EIw(x),xx = C_1x + C_2$$

Step 3: Third Integration

Integrating again:

$$EIw(x),x = \frac{C_1}{2}x^2 + C_2x + C_3$$

Step 4: Fourth Integration

One more integration gives the deflection:

$$EIw(x) = \frac{C_1}{6}x^3 + \frac{C_2}{2}x^2 + C_3x + C_4$$

Direct Method

Apply Boundary Conditions

Now, apply the boundary conditions at the clamped end ($x = 0$) and the free end ($x = L$).

Boundary Condition 1: $w(0) = 0$ **(deflection at the clamped end)**

$$w(0) = \frac{C_1}{6} \cdot 0^3 + \frac{C_2}{2} \cdot 0^2 + C_3 \cdot 0 + C_4 = 0$$

$$C_4 = 0$$

Boundary Condition 2: $w(x), x|_{x=0} = 0$ **(slope at the clamped end)**

$$w(x), x|_{x=0} = \frac{C_1}{2} \cdot 0^2 + C_2 \cdot 0 + C_3 = 0$$

$$C_3 = 0$$

Now the equation for the deflection simplifies to:

$$EIw(x) = \frac{C_1}{6} x^3 + \frac{C_2}{2} x^2$$

Boundary Condition 3: $w(x), xx|_{x=L} = 0$ **(bending moment at the free end)**

At $x = L$:

$$w(x), xx|_{x=L} = C_1 L + C_2 = 0$$

$$C_2 = -C_1 L$$

The equation now becomes:

$$EIw(x) = \frac{C_1}{6} x^3 - \frac{C_1 L}{2} x^2$$

Boundary Condition 4: $w(x), xxx|_{x=L} = \frac{P}{EI}$ **(shear force due to the point load at the free end)**

At $x = L$:

$$EIw(x), xxx|_{x=L} = C_1 = -P$$

$$C_1 = -P$$

Now substitute $C_1 = -P$ into the deflection equation:

$$EIw(x) = \frac{-P}{6} x^3 + \frac{PL}{2} x^2$$

Direct Method

Final deflection equation

The final deflection equation $w(x)$ is:

$$w(x) = \frac{P}{6EI}\left(-x^3 + 3Lx^2\right)$$

This is the deflection of the cantilever beam under a point load P at the free end. The deflection is maximum at $x = L$ and zero at the clamped end $x = 0$.

Maximum Deflection at the Free End $x = L$:

At $x = L$, the deflection is:

$$w(L) = \frac{P}{6EI}\left(-L^3 + 3L \cdot L^2\right) = \frac{PL^3}{3EI}$$

So, the maximum deflection at the free end of the beam is:

$$w_{\max} = \frac{PL^3}{3EI}$$

In Python Code:

```
P/(6*EI)*(-x**3+3*L*x**2)
```

```
P*L**3/(3*EI)
```

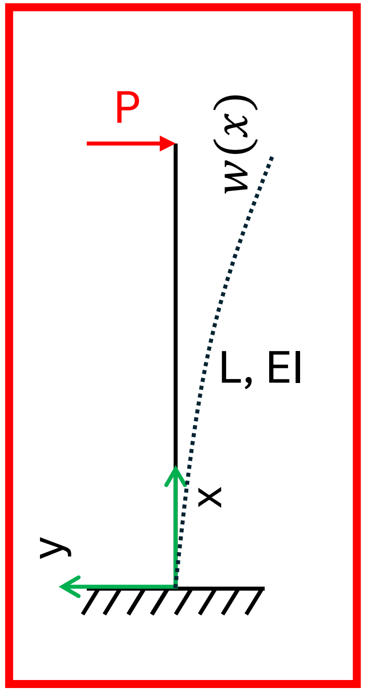
Direct Method

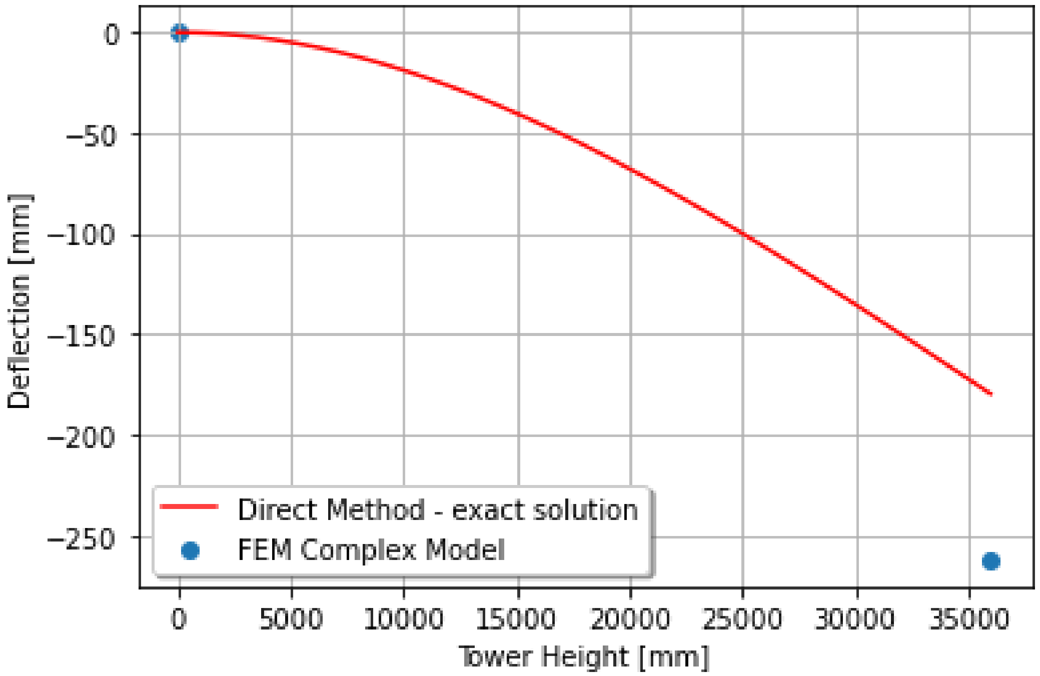
Simplified Model:

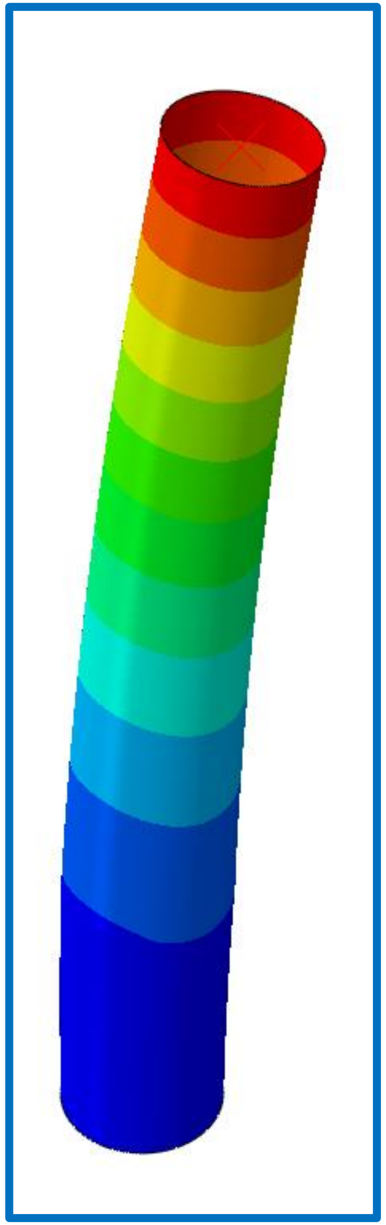
- **CONSTANT CROSS-SECTION**
- Radius **R** = 2750 mm
- Length **L** = 36000 mm
- Wall Thickness **t** = 17 mm
- Elasticity Modulus **E** = 210 000 MPa
- Yield Stres **Y** = 345 Mpa

- → higher stiffness than complex model
- → lower deflection (-32 %) (179 mm **vs.** 262 mm)

FEM Complex Model







Variational Method (Ritz Method)

Problem Definition

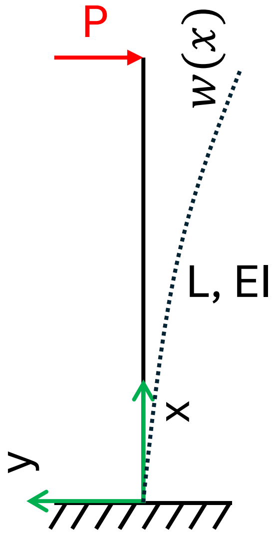
We are solving for the deflection $w(x)$ of a **clamped beam** under a **point load** P at the **free end**.

The governing equation is the **Euler-Bernoulli beam equation:**

$$EIw(x),xxxx = 0 \quad \text{for} \quad 0 \le x \le L$$

Where:

- EI is the flexural rigidity,
- $w(x)$ is the deflection of the beam as a function of x,
- x is the position along the length of the beam,
- L is the total length of the beam.



Boundary Conditions

We also have the following **boundary conditions:**

1. **At the clamped end** $x = 0$:
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   - $w(x),x\big|_{x=0} = 0$ (zero slope).
2. **At the free end** $x = L$:
   - $w(x),xx\big|_{x=L} = 0$ (zero bending moment),
   - $EIw(x),xxx\big|_{x=L} = P$ (shear force equals the applied point load).

Variational Method (Ritz Method)

1. **Define the Trial Function:**
   - The first step in the Ritz method is to define a trial (approximate) function, $\tilde{w}(x)$, that represents the unknown solution (such as deflection in the case of a beam).
   - The trial function must:
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$$U = \frac{1}{2}\int_0^L EI\left(\tilde{w}(x), xx\right)^2 dx$$
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   - Set the derivatives of the total potential energy Π with respect to the unknown coefficients equal to zero to obtain a system of algebraic equations:
$$\frac{\partial \Pi}{\partial c_1} = 0, \quad \frac{\partial \Pi}{\partial c_2} = 0, \quad \ldots$$
   - Solving these equations yields the values of the unknown coefficients.
6. **Obtain the Approximate Solution:**
   - Once the coefficients of the trial function are determined, the trial function $\tilde{w}(x)$ becomes the approximate solution to the problem.
   - The more terms in the trial function, the closer the approximation becomes to the exact solution.

Walter Ritz (1878 – 1909).

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(Ritz Method)

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Step 1: Define the Trial Function

We use the following **quadratic trial function** that satisfies the boundary conditions at $x = 0$ (clamped end):

$$\tilde{w}(x) = c_1x^2$$

This trial function automatically satisfies:

- $\tilde{w}(0) = 0$ (deflection is zero at $x = 0$),
- $\tilde{w}(x), x|_{x=0} = 0$ (slope is zero at $x = 0$).

The free end boundary conditions will be handled by minimizing the total potential energy.

Variational Method (Ritz Method)

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The total potential energy Π of the system consists of two parts:

1. **Strain energy** U **due to bending.**
2. **Potential energy of the external load** V**.**

Variational Method (Ritz Method)

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Strain Energy U

The strain energy U is given by:

$$U = \frac{1}{2}\int_0^L EI\left(\tilde{w}(x), xx\right)^2 dx$$

First, calculate the second derivative of the trial function $\tilde{w}(x)$:

$$\tilde{w}(x), xx = 2c_1$$

Thus, the strain energy becomes:

$$U = \frac{1}{2}\int_0^L EI\left(2c_1\right)^2 dx$$

$$U = 2c_1^2 EI\int_0^L dx = 2c_1^2 EIL$$

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Potential Energy of the External Load V

The potential energy of the point load P applied at the free end $x = L$ is:

$$V = -P\tilde{w}(L)$$

Substitute $\tilde{w}(L) = c_1L^2$:

$$V = -Pc_1L^2$$

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The total potential energy Π is the sum of the strain energy and the potential energy of the external load:

$$\Pi = U + V$$

Substitute the expressions for U and V:

$$\Pi = 2c_1^2EIL - Pc_1L^2$$

To find the coefficient c_1, minimize the total potential energy with respect to c_1:

$$\frac{d\Pi}{dc_1} = 0$$

Differentiate Π with respect to c_1:

$$\frac{d\Pi}{dc_1} = 4c_1EIL - PL^2 = 0$$

Solve for c_1:

$$4c_1EIL = PL^2$$

$$c_1 = \frac{PL}{4EI}$$

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The final trial function for the deflection $\tilde{w}(x)$ is:

$$\tilde{w}(x) = \frac{PL}{4EI}x^2$$

At $x = L$, the deflection is:

$$\tilde{w}(L) = \frac{PL}{4EI}L^2 = \frac{PL^3}{4EI}$$

In Python Code:

```
P*L/(4*EI)*x**2
```

```
P*L**3/(4*EI)
```

Variational Method (Ritz Method)

Direct Method

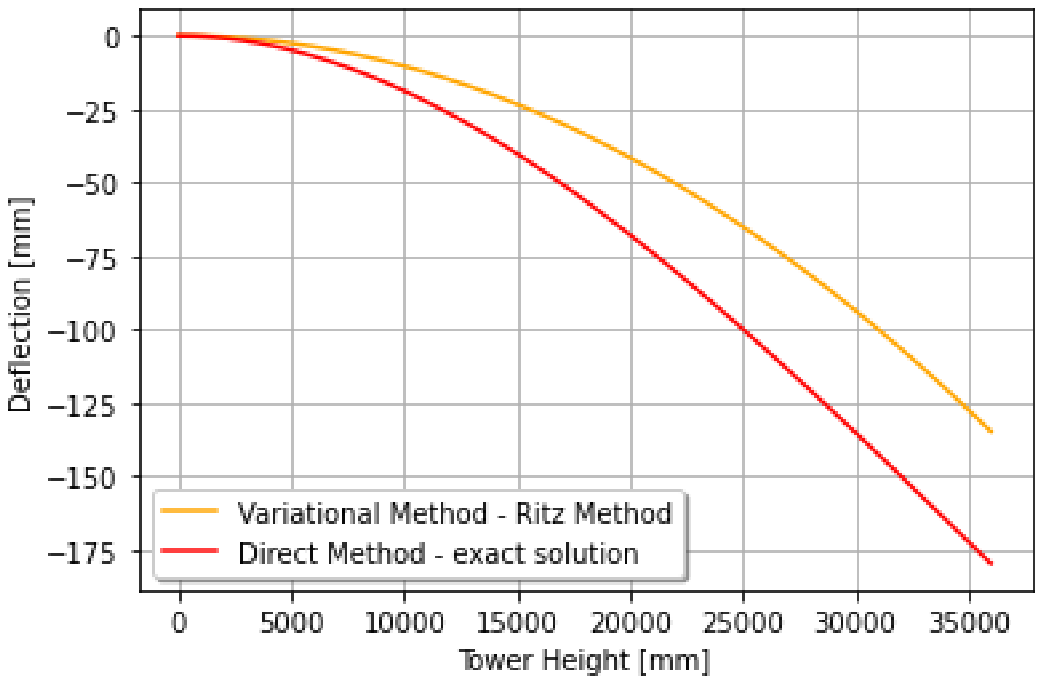
Ritz Method:

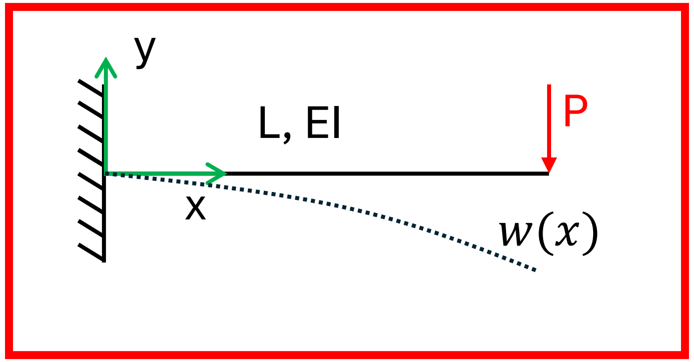
- **THE TRIAL FUNCTION FOR RITZ METHOD WAS QUADRATIC BUT THE EXACT SOLUTION WAS CUBIC**

→ lower deflection (-25 %)
(134 mm **vs.** 179 mm)

$$\tilde{w}(x) = \frac{PL}{4EI} x^{2}$$

$$w(x) = \frac{P}{6EI}\left(-x^{3} + 3Lx^{2}\right)$$





Galerkin Method of Weighted Residuals

Problem Definition

Boundary Conditions

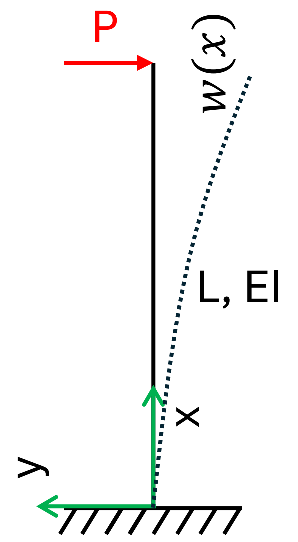
We are solving for the deflection $w(x)$ of a **clamped beam** under a **point load** P at the **free end**.

The governing equation is the **Euler-Bernoulli beam equation**:

$$EIw(x), xxxx = 0 \quad \text{for} \quad 0 \leq x \leq L$$

Where:

- EI is the flexural rigidity,
- $w(x)$ is the deflection of the beam as a function of x,
- x is the position along the length of the beam,
- L is the total length of the beam.



We also have the following **boundary conditions**:

1. **At the clamped end** $x = 0$:
   - $w(0) = 0$ (zero deflection),
   - $w(x), x\big|_{x=0} = 0$ (zero slope).
2. **At the free end** $x = L$:
   - $w(x), xx\big|_{x=L} = 0$ (zero bending moment),
   - $EIw(x), xxx\big|_{x=L} = P$ (shear force equals the applied point load).

Galerkin Method of Weighted Residuals

Summary

1. **Trial Function**: The trial function $\tilde{w}(x)$ is approximated as a linear combination of basis functions $\varphi_i(x)$.

$$\tilde{w}(x) = \sum_{i=1}^{N} c_i \varphi_i(x)$$

2. **Weight Function**: The weight function $\varphi_j(x)$ is the **derivative** of the trial function with respect to the unknown coefficient c_j, which simplifies to $\varphi_j(x)$ (the same as the basis function).

$$\varphi_j(x) = \frac{\partial \tilde{w}(x)}{\partial c_j}$$

3. **Residual**: The residual $R(x)$ is computed by substituting the trial function into the differential equation.
4. **Orthogonality Condition**: The residual is made orthogonal to the weight functions by solving:

$$\int_\Omega R(x)\varphi_j(x)\,dx = 0$$

5. **Solve for Coefficients**: This leads to a system of equations for the unknown coefficients, which is solved to find $c_1, c_2, \ldots, c_N$.
6. **Construct Approximate Solution**: The final approximate solution $\tilde{w}(x)$ is obtained by substituting the coefficients back into the trial function.

Boris Grigorievich Galerkin (1871 – 1945).

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6. **Construct Approximate Solution:** The final approximate solution $\tilde{w}(x)$ is obtained by substituting the coefficients back into the trial function.

We approximate the deflection $w(x)$ using a **trial function** (also called a basis function) that satisfies the boundary conditions at $x = 0$. A suitable **4th-order polynomial** can be used as the trial function:

$$w(x) \approx \tilde{w}(x) = c_1 x^2 + c_2 x^3 + c_3 x^4$$

Where c_1, c_2, and c_3 are unknown coefficients that we will determine.

This trial function satisfies the boundary conditions at the **clamped end** ($x = 0$):

- $w(0) = 0$,
- $w(x),x\big|_{x=0} = 0$.

Now, we calculate the necessary derivatives of $\tilde{w}(x)$ to apply the boundary conditions:

1. **First derivative** $w(x),x$ (slope):

$$\tilde{w}(x),x = 2c_1 x + 3c_2 x^2 + 4c_3 x^3$$

2. **Second derivative** $w(x),xx$ (curvature):

$$\tilde{w}(x),xx = 2c_1 + 6c_2 x + 12c_3 x^2$$

3. **Third derivative** $w(x),xxx$ (related to shear force):

$$\tilde{w}(x),xxx = 6c_2 + 24c_3 x$$

4. **Fourth derivative** $w(x),xxxx$ (governing the beam equation):

$$\tilde{w}(x),xxxx = 24c_3$$

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Now we apply the boundary conditions at the **free end** ($x = L$):

Moment-free condition $w(x), xx\big|_{x=L} = 0$

Substitute $x = L$ into the second derivative:

$$w(x), xx\big|_{x=L} = 2c_1 + 6c_2L + 12c_3L^2 = 0$$

This is our **first equation**:

$$2c_1 + 6c_2L + 12c_3L^2 = 0$$

Shear force condition $EIw(x), xxx\big|_{x=L} = P$

Substitute $x = L$ into the third derivative:

$$w(x), xxx\big|_{x=L} = 6c_2 + 24c_3L$$

Using the boundary condition $EIw(x), xxx\big|_{x=L} = P$, we get:

$$EI(6c_2 + 24c_3L) = P$$

This is our **second equation**:

$$6c_2 + 24c_3L = \frac{P}{EI}$$

Solve for c_2 in terms of c_3

From equation (2):

$$6c_2 + 24c_3L = \frac{P}{EI}$$

Solve for c_2:

$$c_2 = \frac{P}{6EI} - 4Lc_3$$

Substitute c_2 into equation (1)

Substitute $c_2 = \frac{P}{6EI} - 4Lc_3$ into equation (1):

$$2c_1 + 6\left(\frac{P}{6EI} - 4Lc_3\right)L + 12c_3L^2 = 0$$

Simplifying:

$$2c_1 + P\frac{L}{EI} - 24L^2c_3 + 12L^2c_3 = 0$$

$$2c_1 + P\frac{L}{EI} - 12L^2c_3 = 0$$

Solve for c_1:

$$2c_1 = -P\frac{L}{EI} + 12L^2c_3$$

$$c_1 = -\frac{PL}{2EI} + 6L^2c_3$$

Now we have c_1 and c_2 in terms of c_3.

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$$c_1 = -\frac{PL}{2EI} + 6L^2 c_3 \qquad c_2 = \frac{P}{6EI} - 4Lc_3$$

The trial function $\tilde{w}(x)$ is:

$$\tilde{w}(x) = c_1 x^2 + c_2 x^3 + c_3 x^4$$

Now, substitute the expressions for c_1 and c_2:

$$\tilde{w}(x) = \left(-\frac{PL}{2EI} + 6L^2 c_3\right) x^2 + \left(\frac{P}{6EI} - 4Lc_3\right) x^3 + c_3 x^4$$

Galerkin Method of Weighted Residuals

Summary

1. **Trial Function**: The trial function $\tilde{w}(x)$ is approximated as a linear combination of basis functions $\varphi_i(x)$.

$$\tilde{w}(x) = \sum_{i=1}^{N} c_i \varphi_i(x)$$

2. **Weight Function**: The weight function $\varphi_j(x)$ is the **derivative** of the trial function with respect to the unknown coefficient c_j, which simplifies to $\varphi_j(x)$ (the same as the basis function).

$$\varphi_j(x) = \frac{\partial \tilde{w}(x)}{\partial c_j}$$

3. **Residual**: The residual $R(x)$ is computed by substituting the trial function into the differential equation.
4. **Orthogonality Condition**: The residual is made orthogonal to the weight functions by solving:

$$\int_\Omega R(x)\varphi_j(x)\,dx = 0$$

5. **Solve for Coefficients**: This leads to a system of equations for the unknown coefficients, which is solved to find $c_1, c_2, \ldots, c_N$.
6. **Construct Approximate Solution**: The final approximate solution $\tilde{w}(x)$ is obtained by substituting the coefficients back into the trial function.

Now, derive the trial function $\varphi_j(x)$ with respect to c_3, which gives the **weight function**:

$$\varphi_j(x) = \frac{\partial \tilde{w}(x)}{\partial c_j} = \frac{\partial \tilde{w}(x)}{\partial c_3} = \frac{\partial}{\partial c_3}\left(\left(-\frac{PL}{2EI} + 6L^2 c_3\right)x^2 + \left(\frac{P}{6EI} - 4Lc_3\right)x^3 + c_3 x^4\right)$$

Differentiating term by term:

1. $\frac{\partial}{\partial c_3}\left(\left(-\frac{PL}{2EI} + 6L^2c_3\right)x^2\right) = 6L^2x^2,$
2. $\frac{\partial}{\partial c_3}\left(\left(\frac{P}{6EI} - 4Lc_3\right)x^3\right) = -4Lx^3,$
3. $\frac{\partial}{\partial c_3}\left(c_3x^4\right) = x^4.$

Thus, the **weight function** $\varphi_j(x)$ is:

$$\varphi_j(x) = \frac{\partial \tilde{w}(x)}{\partial c_j} = \frac{\partial \tilde{w}(x)}{\partial c_3} = 6L^2x^2 - 4Lx^3 + x^4$$

Galerkin Method of Weighted Residuals

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$$\int_\Omega R(x)\varphi_j(x)\,dx = 0$$

5. **Solve for Coefficients**: This leads to a system of equations for the unknown coefficients, which is solved to find $c_1, c_2, \ldots, c_N$.
6. **Construct Approximate Solution**: The final approximate solution $\tilde{w}(x)$ is obtained by substituting the coefficients back into the trial function.

The residual R(x) is the difference between the approximate and actual forces:

$$R(x) = EI\tilde{w}(x), xxxx - f(x)$$

For a homogeneous case (no external force):

$$R(x) = EI\tilde{w}(x), xxxx$$

$$\tilde{w}(x), xxxx = 24c_3$$

Galerkin Method of Weighted Residuals

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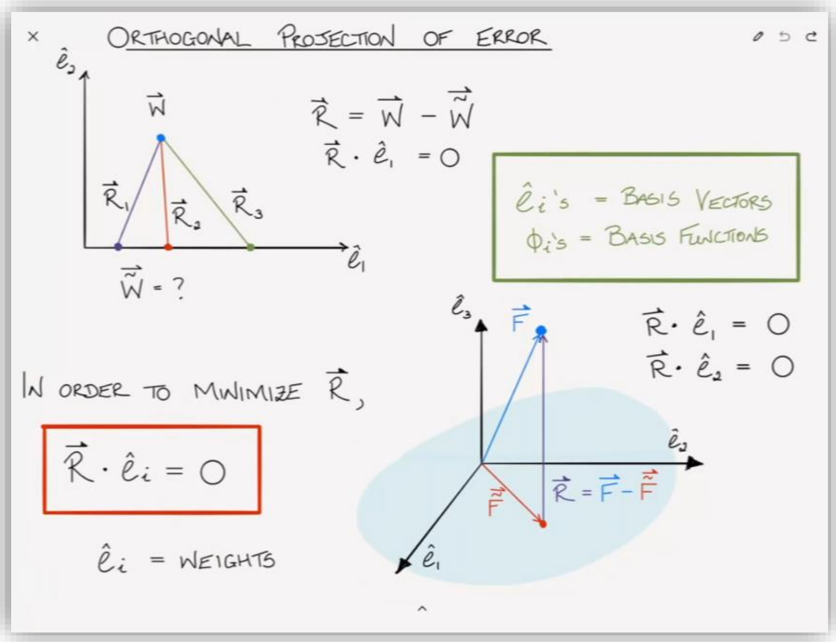
6. **Construct Approximate Solution**: The final approximate solution $\tilde{w}(x)$ is obtained by substituting the coefficients back into the trial function.

Substitute $R(x) = EI \cdot 24c_3$ and the weight function:

$$EI \cdot 24c_3 \int_0^L \left(6L^2x^2 - 4Lx^3 + x^4\right)\, dx = 0$$

$$c_3 = 0$$

To fulfill this equation $c_3 = 0$ because EI and L $\neq 0$



Galerkin Method of Weighted Residuals

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$$c_1 = -\frac{PL}{2EI} + 6L^2 c_3 \qquad c_2 = \frac{P}{6EI} - 4Lc_3$$

$$c_3 = 0$$

Solve for c_2:

$$c_2 = \frac{P}{6EI}$$

Solve for c_1:

$$c_1 = -\frac{PL}{2EI}$$

Galerkin Method of Weighted Residuals

Summary

1. **Trial Function**: The trial function $\tilde{w}(x)$ is approximated as a linear combination of basis functions $\varphi_i(x)$.

$$\tilde{w}(x) = \sum_{i=1}^{N} c_i \varphi_i(x)$$

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5. **Solve for Coefficients**: This leads to a system of equations for the unknown coefficients, which is solved to find $c_1, c_2, \ldots, c_N$.
6. **Construct Approximate Solution**: The final approximate solution $\tilde{w}(x)$ is obtained by substituting the coefficients back into the trial function.

Now that we have c_1, c_2, and c_3, we substitute these values into the trial function:

$$w(x) = c_1 x^2 + c_2 x^3 + c_3 x^4$$

Substitute the values:

$$w(x) = \left(-\frac{PL}{2EI}\right)x^2 + \left(\frac{P}{6EI}\right)x^3$$

Thus, the **approximate solution** for the deflection $w(x)$ is:

$$w(x) = \frac{P}{6EI}\left(x^3 - 3Lx^2\right)$$

Finally, to find the maximum deflection at $x = L$:

$$w(L) = \frac{P}{6EI}\left(L^3 - 3LL^2\right) = -\frac{PL^3}{3EI}$$

This is the maximum deflection at the free end.

Galerkin Method of Weighted Residuals

Direct Method

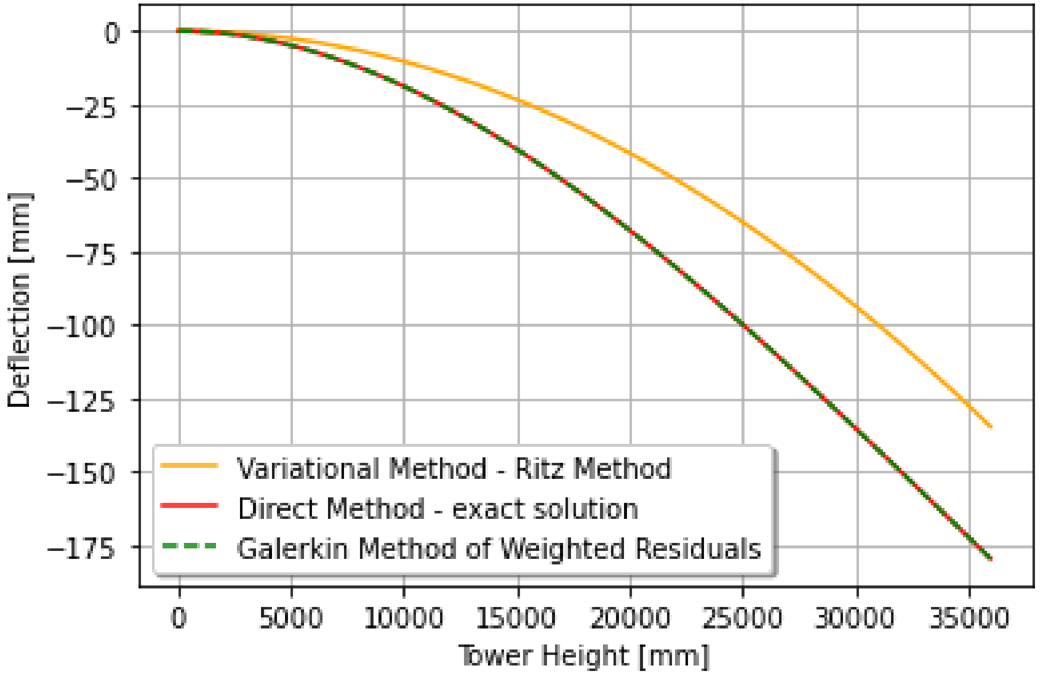
Galerkin Method:

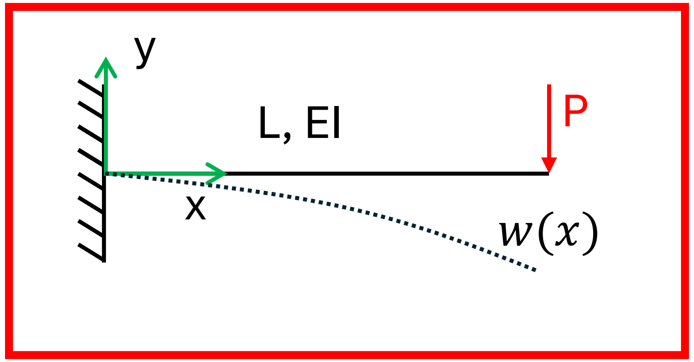
- **THE TRIAL FUNCTION FOR GALERKIN METHOD WAS 4TH ORDER AND THE EXACT SOLUTION WAS CUBIC**

→ Galerkin leads to exact solution

$$w(x) = \frac{P}{6EI}\left(-x^3 + 3Lx^2\right)$$

$$w(x) = \frac{P}{6EI}\left(-x^3 + 3Lx^2\right)$$





Finite Element Method (FEM)

Problem Definition

Boundary Conditions

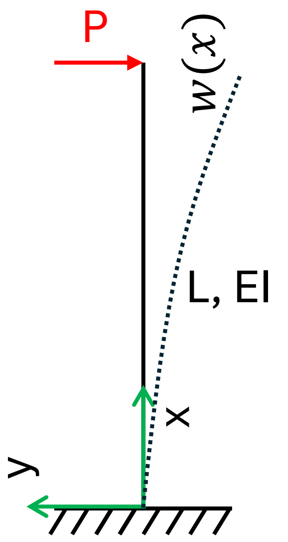
We are solving for the deflection $w(x)$ of a **clamped beam** under a **point load** P at the **free end**.

The governing equation is the **Euler-Bernoulli beam equation**:

$$EIw(x),xxxx = 0 \quad \text{for} \quad 0 \leq x \leq L$$

Where:

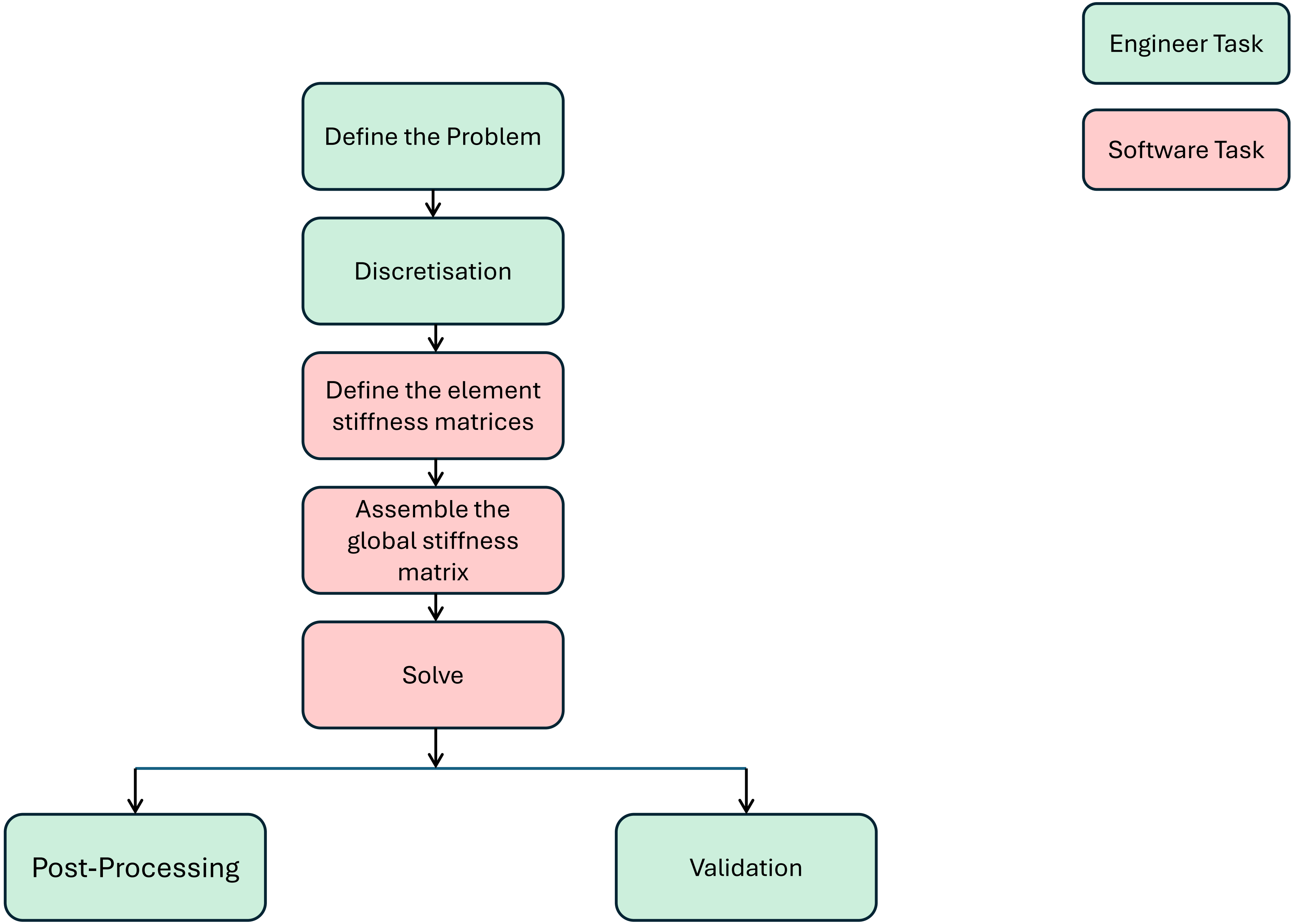
- EI is the flexural rigidity,
- $w(x)$ is the deflection of the beam as a function of x,
- x is the position along the length of the beam,
- L is the total length of the beam.



We also have the following **boundary conditions**:

1. **At the clamped end** $x = 0$:
   - $w(0) = 0$ (zero deflection),
   - $w(x), x\big|_{x=0} = 0$ (zero slope).
2. **At the free end** $x = L$:
   - $w(x), xx\big|_{x=L} = 0$ (zero bending moment),
   - $EIw(x), xxx\big|_{x=L} = P$ (shear force equals the applied point load).

Finite Element Method (FEM)



Finite Element Method (FEM)

Define the Problem

↓

Discretisation

↓

Define the element stiffness matrices

↓

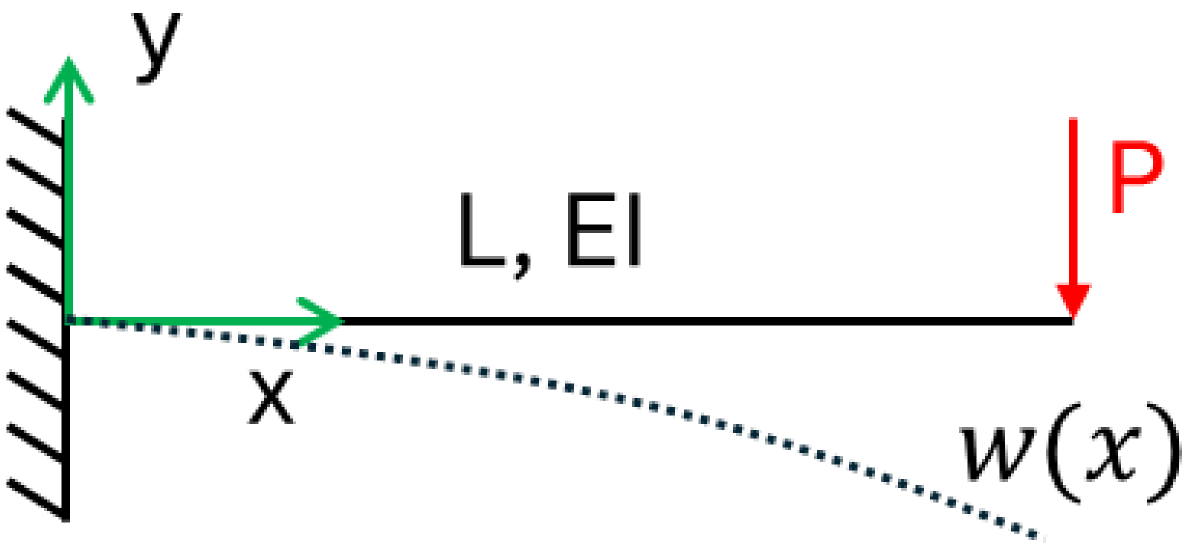
Assemble the global stiffness matrix

↓

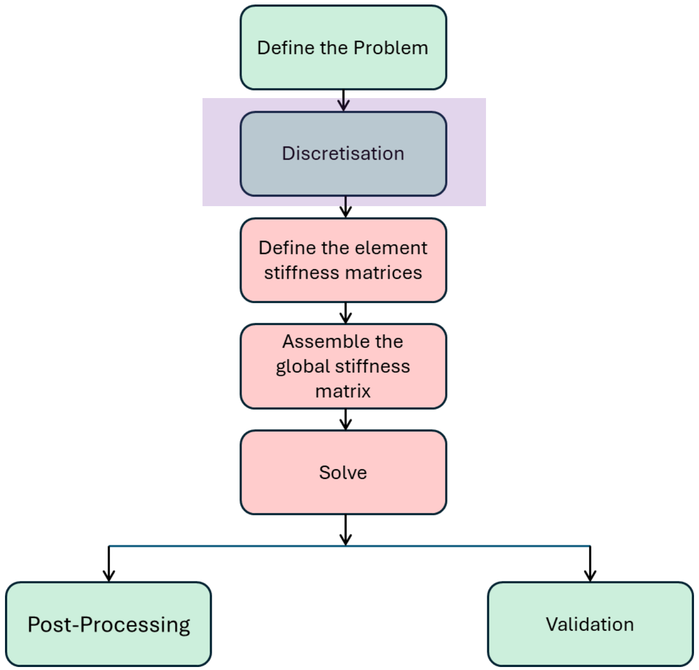
Solve

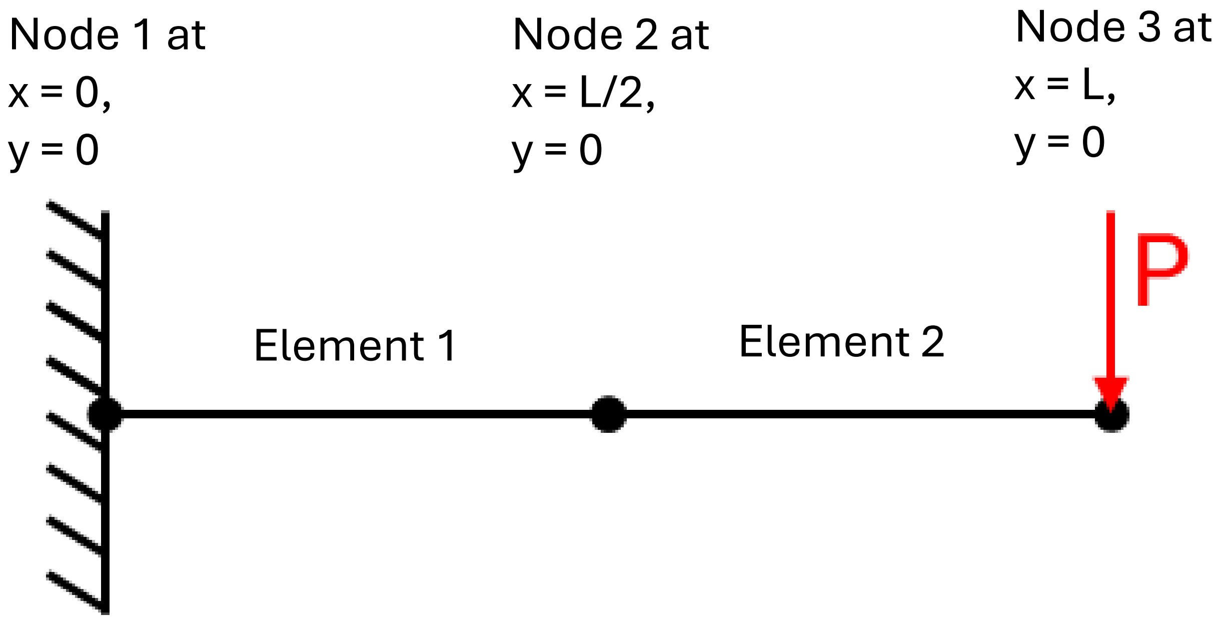
↓

Post-Processing | Validation



Finite Element Method (FEM)





We divide the beam of length L into **two equal elements** of length $L/2$.

- **Element 1**: From $x = 0$ to $x = L/2$,
- **Element 2**: From $x = L/2$ to $x = L$.

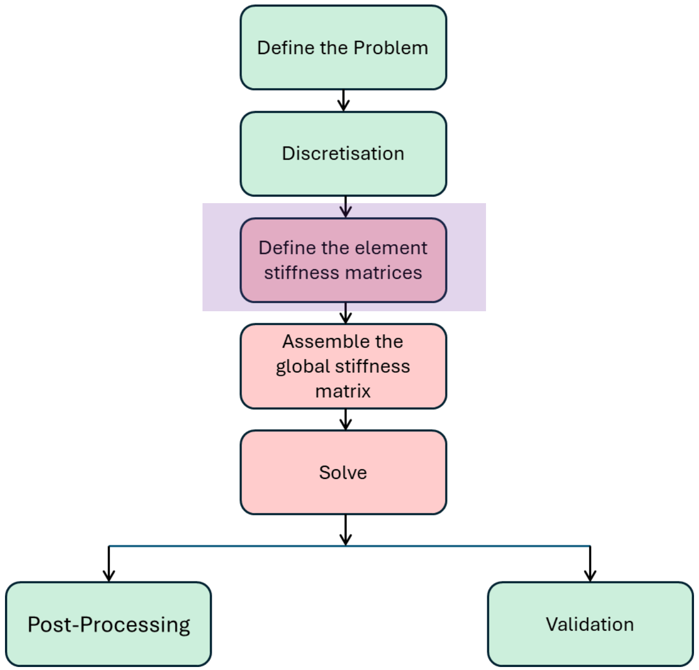
Each element has **two nodes**:

- **Element 1** has nodes 1 and 2.
- **Element 2** has nodes 2 and 3.

The nodal displacements and rotations at each node are denoted as:

- w_1, θ_1 for node 1,
- w_2, θ_2 for node 2,
- w_3, θ_3 for node 3.

Finite Element Method (FEM)



The **local stiffness matrix** $\mathbf{K}_e$ for a beam element is derived from the **beam bending energy**. For a beam element, the local stiffness matrix in terms of the degrees of freedom (DOFs) $w_i, \theta_i, w_j, \theta_j$ is given by:

$$\mathbf{K}_e = \frac{EI}{L_e^3} \begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix}$$

Where:

- EI is the flexural rigidity of the beam,
- L_e is the length of the beam element.

Force Vector

The force vector $\mathbf{F}_{\text{local}}$, corresponding to these displacements, is:

$$\mathbf{F}_{\text{local}} = \begin{bmatrix} F_{w1} \\ M_1 \\ F_{w2} \\ M_2 \end{bmatrix}$$

Where:

- F_{w1}: Shear force at node 1 (related to w_1).
- M_1: Moment at node 1 (related to θ_1).
- F_{w2}: Shear force at node 2 (related to w_2).
- M_2: Moment at node 2 (related to θ_2).

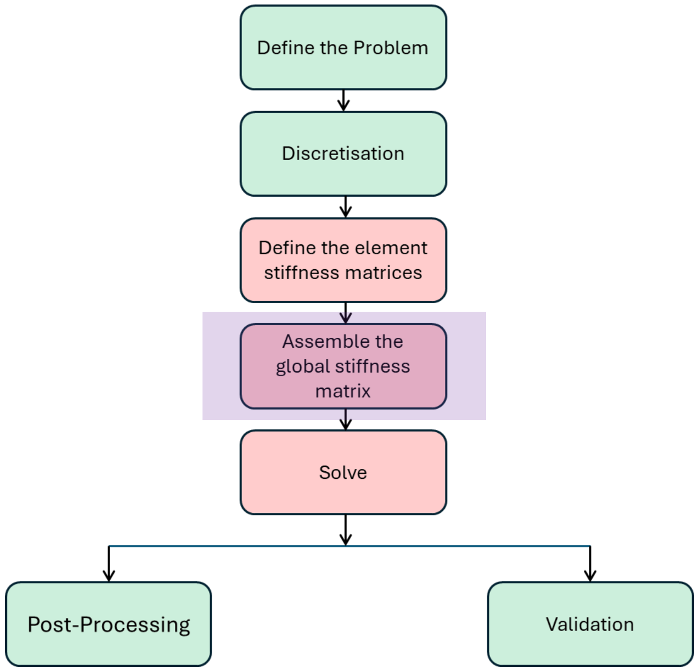
The displacement vector for this simplified beam element becomes:

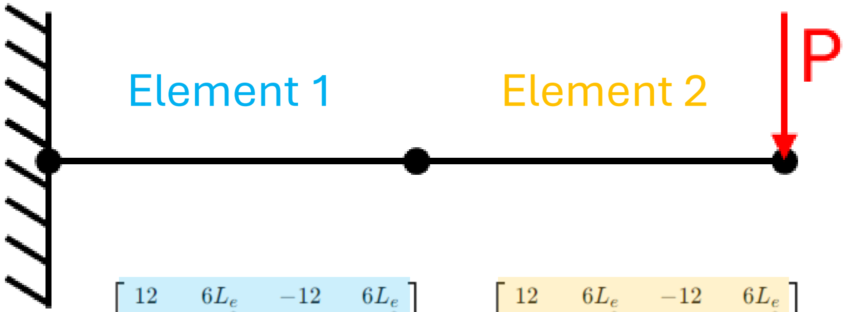
$$\mathbf{d}_{\text{local}} = \begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{bmatrix}$$

Where:

- w_1: Transverse deflection (perpendicular displacement) at node 1.
- θ_1: Rotation (slope) at node 1.
- w_2: Transverse deflection at node 2.
- θ_2: Rotation at node 2.

Finite Element Method (FEM)





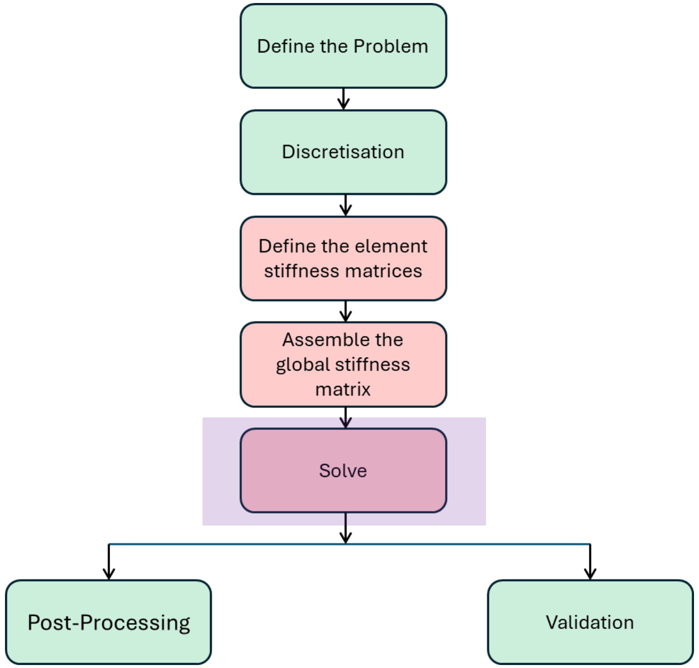
$$\frac{EI}{L_e^3}\begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix} \qquad \frac{EI}{L_e^3}\begin{bmatrix} 12 & 6L_e & -12 & 6L_e \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 \\ -12 & -6L_e & 12 & -6L_e \\ 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix}$$

To form the global stiffness matrix, we combine the two local stiffness matrices. The key idea is that the degrees of freedom at **node 2** are shared between the two elements, so we must add the contributions from both elements at this node.

The global stiffness matrix $\mathbf{K}_{\text{global}}$ will be a 6×6 matrix, as there are three nodes, each with two degrees of freedom. The global stiffness matrix looks like this:

$$\mathbf{K}_{\text{global}} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & -6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ -6L & 2L^2 & 0 & 8L^2 & 6L & 2L^2 \\ 0 & 0 & -12 & 6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Finite Element Method (FEM)



$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ -6L & 2L^2 & 0 & 8L^2 & 6L & 2L^2 \\ 0 & 0 & -12 & 6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{bmatrix}$$

Since node 1 is clamped ($w_1 = 0$ and $\theta_1 = 0$), we remove the first two rows and first two columns from the matrix, leaving the following **reduced global stiffness matrix**:

$$\mathbf{K}_{\text{global}}^{\text{reduced}} = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & -12 & 6L \\ 0 & 8L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

This is now a 4×4 matrix, corresponding to the degrees of freedom for w_2, θ_2, w_3, and θ_3.

The force vector $\mathbf{F}_{\text{reduced}}$ will now be:

$$\mathbf{F}_{\text{reduced}} = \begin{bmatrix} 0 \\ 0 \\ P \\ 0 \end{bmatrix}$$

The displacement vector is also reduced by removing the entries corresponding to the clamped degrees of freedom (w_1 and θ_1):

$$\mathbf{d}_{\text{reduced}} = \begin{bmatrix} w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{bmatrix}$$

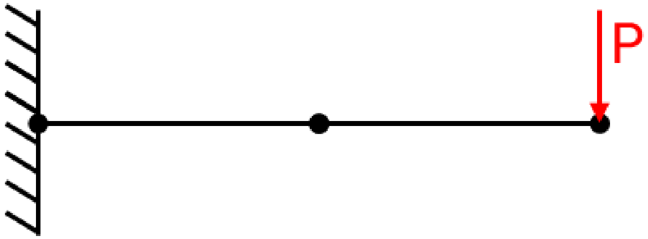
Finite Element Method (FEM)

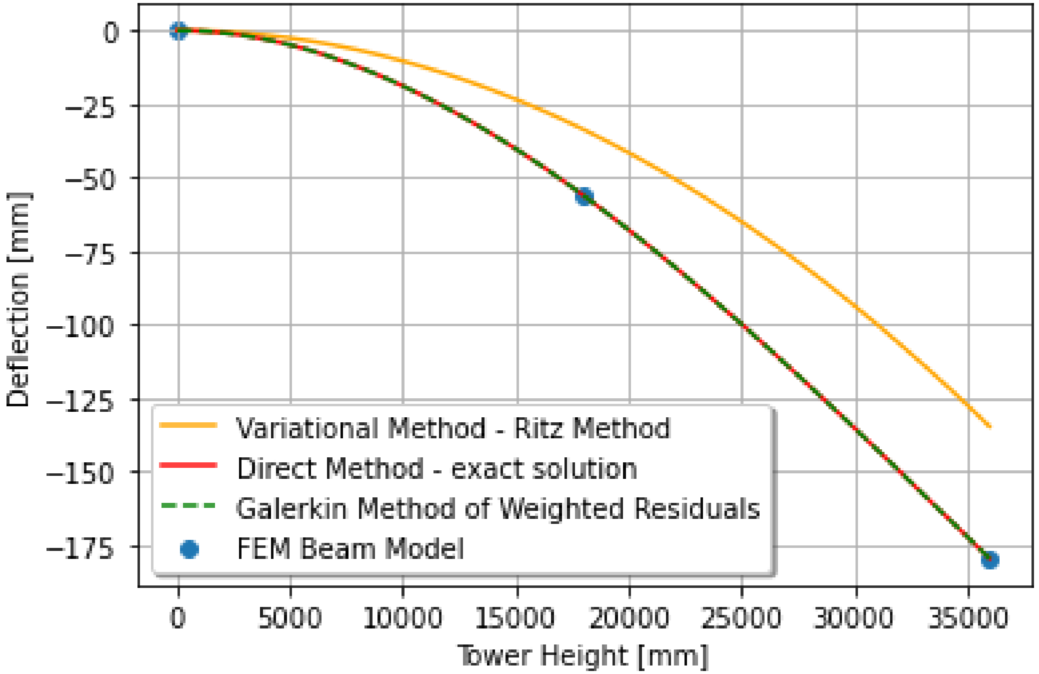
Direct Method

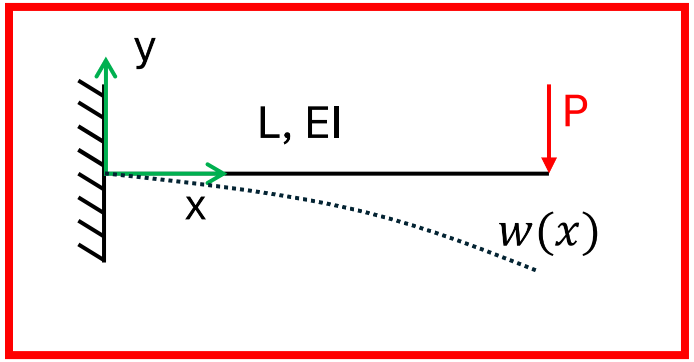
FEM:

- **FEM NODE DISPLACEMENTS ARE SIMILAR TO THE SOLUTION OF THE DIRECT METHOD FOR THIS EXAMPLE**

$$w(x) = \frac{P}{6EI}\left(-x^3 + 3Lx^2\right)$$







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Where:

- EI is the flexural rigidity of the beam,
- L_e is the length of the beam element.

In Python Code:

```
# Element length
L_e = L / 2

# Local stiffness matrix for a beam element
def beam_element_stiffness(E, I, L_e):
    return (E * I / L_e**3) * np.array([
        [12, 6*L_e, -12, 6*L_e],
        [6*L_e, 4*L_e**2, -6*L_e, 2*L_e**2],
        [-12, -6*L_e, 12, -6*L_e],
        [6*L_e, 2*L_e**2, -6*L_e, 4*L_e**2]
    ])
```

To form the global stiffness matrix, we combine the two local stiffness matrices. The key idea is that the degrees of freedom at **node 2** are shared between the two elements, so we must add the contributions from both elements at this node.

The global stiffness matrix $\mathbf{K}_{\text{global}}$ will be a 6×6 matrix, as there are three nodes, each with two degrees of freedom. The global stiffness matrix looks like this:

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In Python Code:

```
# Assemble the global stiffness matrix
K_local = beam_element_stiffness(E, I, L_e)

# Global stiffness matrix for two elements
K_global = np.zeros((6, 6))

# Assemble the stiffness matrix
K_global[:4, :4] += K_local  # Element 1
contribution
K_global[2:, 2:] += K_local  # Element 2
contribution
```

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & -6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ -6L & 2L^2 & 0 & 8L^2 & 6L & 2L^2 \\ 0 & 0 & -12 & 6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \end{bmatrix}$$

Since node 1 is clamped ($w_1 = 0$ and $\theta_1 = 0$), we remove the first two rows and first two columns from the matrix, leaving the following **reduced global stiffness matrix**:

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This is now a 4×4 matrix, corresponding to the degrees of freedom for w_2, θ_2, w_3, and θ_3.

In Python Code:

```
# Apply boundary conditions (clamped at node 1, remove rows and columns for w1 and θ1)

K_reduced = K_global[2:, 2:]
```