Packing Feedback Arc Sets in Reducible Flow Graphs

Han Xiao

Department of Mathematics, The University of Hong Kong,
Hong Kong, China
hxiao.math@connect.hku.hk

Abstract

We establish a min-max relation in arc-weighted reducible flow graphs in this paper. In particular, we prove that the maximum cardinality of feedback arc set packings equals the minimum total weight of cycles. We also present an $O(n^2m)$ algorithm for finding a maximum feedback arc set packing in reducible flow graphs. Our result relies heavily on Ramachandran's [10, 11] structural analysis of reducible flow graphs.

Keywords. Min-max relation, feedback arc set, packing, reducible flow graph, algorithm

1 Introduction

A reducible flow graph is a special flow graph which corresponds to the flowchart of computer programs via structured programming. As described in [6], the "divide-and-conquer" approach applies to reducible flow graphs, which makes the analysis easier on such graphs. Reducible flow graphs also occur frequently in practice. Statistics suggest that most flow graphs associated with computer programs are reducible flow graphs. Therefore, reducible flow graphs have attracted many research efforts over the past few decades. Various characterizations of reducible flow graphs can be found in [7]. The first efficient algorithm for recognizing reducible flow graphs was given by Hopcroft and Ullman [9]. Based on their work, Tarjan [13] derived a more efficient algorithm for testing flow graphs reducibility.

In this paper, by a cycle (*resp.* path) in a digraph we mean a directed one. A digraph is called *acyclic* if it contains no cycles, and is abbreviated a *DAG* (directed acyclic graph). Let $G = (V, A)$ be a digraph with a nonnegative integral function w defined on arcs in A. A set $F \subseteq A$ is a *feedback arc set* (*FAS*) for G if $G' = (V, A \backslash F)$ is acyclic. Let $H = (A, \mathcal{C})$ be a hypergraph whose vertex set is the arc set of G and edges (with repetition allowed) are the arc sets of cycles in G. A subset $\mathcal{C}'$ of edge set $\mathcal{C}$ in H is called a *cycle packing* if each arc $a \in A$ occurs at most $w(a)$ times in cycles of $\mathcal{C}'$. A *transversal* of a hypergraph is a subset of its vertex

set which intersects each edge of the hypergraph. Note that a transversal of $H = (A, \mathcal{C})$ is a feedback arc set of G. Let ν_w denote the maximum cardinality of cycle packings and τ_w denote the minimum weight of feedback arc sets. It is easy to prove that $\nu_w \leq \tau_w$. When equality $\nu_w = \tau_w$ holds for all nonnegative integral function w, the hypergraph H is said to satisfy the min-max relation and the graph G is said to be *cycle Mengerian* (*CM*). In [10], Ramachandran gave an $O(n^2 m \log(n^2/m))$ algorithm for finding a minimum weight feedback arc set in arc-weighted reducible graphs. She [11] also established a conjecture of Frank and Gyarfas [5] by showing that the minimum cardinality of feedback arc sets is equal to the maximum cardinality of cycle packings in unweighted reducible flow graphs. Her proof yielded an $O(\min\{mn^{5/3}, m^2\})$ algorithm for finding a maximum collection of arc disjoint cycles. Chen and Zang [3] pointed out that although Ramachandran's proof [11] was intended for unweighted reducible flow graphs, it could be adapted to handle the weighted case. Based on Ramachandran's work [10, 11], Chen and Zang proved that every reducible flow graph is CM. They also reduced the time complexity of finding a maximum cycle packing in weighted reducible flow graphs to $O(n^2 m \log(n^2/m))$ by avoiding using augmenting path method for finding the maximum flow as a subroutine in their algorithm.

A *clutter* is a hypergraph no edge of which is contained in another. Note that hypergraph $H = (A, \mathcal{C})$ is a clutter. The hypergraph $H^{\perp} = (A, \mathcal{F})$ is called the *blocker* of H, where vertex set of $H^{\perp}$ is the arc set of G and edges (with repetition allowed) are the collection of feedback arc sets. Analogous to the definition of cycle packing, we call a subset $\mathcal{F}'$ of edge set $\mathcal{F}$ in $H^{\perp}$ a *feedback arc set packing* if each arc a occurs at most $w(a)$ times in feedback arc sets of $\mathcal{F}'$. Note that a transversal of $H^{\perp}$ is a cycle in G. Let λ_w denote the maximum cardinality of feedback arc set packings and μ_w denote the minimum weight of cycles. Obviously, $\lambda_w \leq \mu_w$. When equality $\lambda_w = \mu_w$ holds for all nonnegative integral function w, the hypergraph $H^{\perp}$ is said to satisfy the min-max relation and the graph G is said to be *feedback arc set Mengerian* (*FASM*). It has been proved that hypergraph $H = (A, \mathcal{C})$ derived from reducible flow graph $G = (V, A)$ satisfies the min-max relation. Hence every reducible flow graph is CM. A natural question is whether the blocker $H^{\perp} = (A, \mathcal{F})$ of hypergraph H satisfies the min-max relation as well. The main contribution of this paper is twofold. First, we give an affirmative answer to this question, hence every reducible flow graph is FASM. Second, we present an $O(n^2 m)$ algorithm for finding a maximum feedback arc set packing in reducible flow graphs. Our result relies heavily on the structural analysis for reducible flow graphs of Ramachandran's work [10, 11].

The remainder of this paper is organized as follows. In section 2, we introduce some preliminary knowledge on reducible flow graphs. In section 3, we first prove that every reducible flow graph is FASM, then we present our algorithm and establish its correctness.

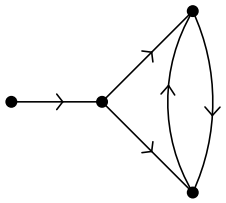


Figure 1: The obstruction F

2 Preliminaries

A *flow graph* is a digraph with a root such that the root reaches all other vertices via paths. Let $G = (V, A; r)$ be a flow graph with root r. A *DAG* of G is a maximal acyclic subgraph of G which remains a flow graph with root r. We call G *reducible* if the DAG of G is unique. A DAG of G obtained from a depth-first spanning tree rooted at r by adding a maximal subset of arc set A is called a *DFS DAG* of G. For $u, v \in V$, we say u *dominates* v if every $r - v$ path passes through u. Clearly, every vertex dominates itself. For a subgraph G' of G, a vertex v of G' is called an *entry vertex* if $v = r$ or there is an arc (u, v) of G with u outside G'. Observe that arcs in a reducible flow graph can be uniquely partitioned into the *forward* (or *DAG*) *arc* set and the *back arc* set.

Following are some useful characterizations of reducible flow graphs.

Theorem 2.1. *Let $G = (V, A; r)$ be a flow graph and let D be a DFS DAG of G. Then the following statements are equivalent:*

(a) *G is a reducible flow graph.*

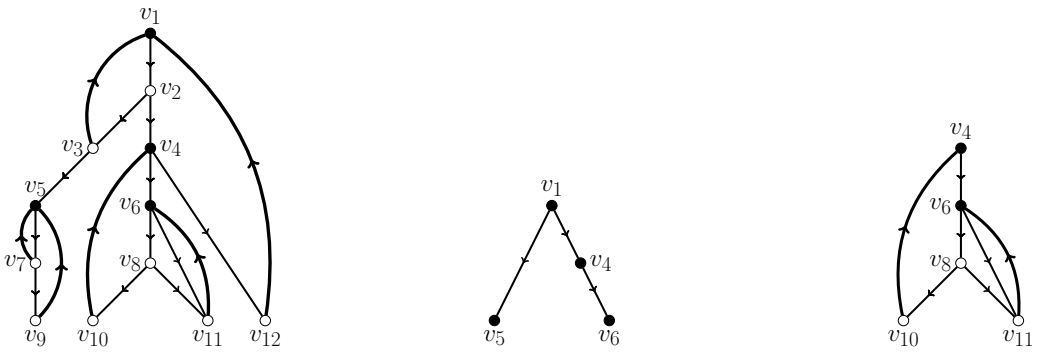
(b) *F is the unique obstruction for the transitive closure of G (see Figure 1).*

(c) *D is the unique DAG of G.*

(d) *The arc set A can be partitioned into A_1 and A_2 such that $(V, A_1; r)$ is a DAG of G and u dominates v in G for each arc (v, u) in A_2.*

(e) *Every cycle C in G contains exactly one back arc $a \in A \backslash A(D)$. Moreover, the head of back arc $a \in A \backslash A(D)$ is the entry vertex of C which dominates other vertices on C.*

Proof. The equivalence of $(a) - (d)$ can be found in [7] and [8]. The implication $(a) \Rightarrow (e)$ is proved in [12] and the converse direction $(e) \Rightarrow (a)$ is contained in [3]. □



(a) G (b) T_h (c) $G_s(V_{v_4})$

Figure 2: Reducible flow graph G and corresponding T_h, $G_s(V_{v_4})$

We follow some notations introduced by Hecht, Ullman [7, 8] and Ramachandran [10, 11]. Here V_h denotes the set of heads of all back arcs in G, and T_h denotes the *head dominator tree* of G which demonstrates the domination relation on $V_h \cup \{r\}$: the descendants of a vertex u in T_h are precisely the vertices in V_h dominated by u in G. For each $u \in V_h \cup \{r\}$, let (u_1, v_1), (u_2, v_2),...,(u_k, v_k) be all the back arcs in G whose heads are dominated by u. We use V_u to denote the *dominated back arc vertex set* of u which consists of all the vertices in V lying on some DAG path from u to u_i, for $1 \leq i \leq k$. Ramachandran [10, 11] proved that u dominates all vertices in V_u. We use $G_s(V_u)$ to denote the subgraph of G induced by V_u (see Figure 2).

The following properties on dominated back arc vertex sets were established by Chen and Zang [3].

Theorem 2.2. *Let $G = (V, A; r)$ be a reducible flow graph and let u, v be two vertices in $V_h \cup \{r\}$. Then the following statements hold:*

(*a*) *$G_s(V_u)$ is a reducible flow graph rooted at u.*

(*b*) *u is the unique entry vertex of $G_s(V_u)$ in G.*

(*c*) *If $G_s(V_u)$ and $G_s(V_v)$ have a common vertex, then either u dominates v or v dominates u.*

3 Maximum feedback arc set packings in reducible flow graphs

3.1 Min-max relation

Let $G = (V, A; r)$ be a reducible flow graph with a nonnegative integral weight $w(a)$ on each arc $a \in A$. We construct a network $N(r, t)$ from G as follows. Add a new vertex t. For $v \in V_h$, make a copy v', redirect back arcs from v to v' and connect v' to t by a *newly added arc* (v', t).

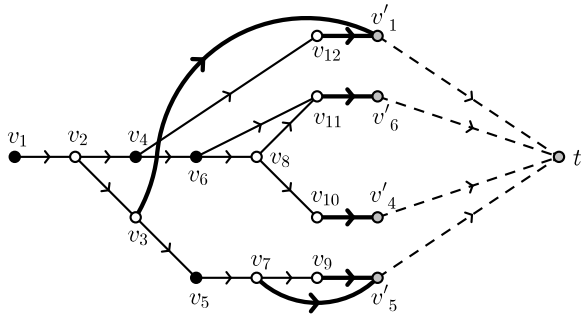


Figure 3: $N(v_1, t)$ constructed from $G = (V, A; v_1)$

We extend the weight function w on arcs in $G = (V, A; r)$ to a length function l on arcs in N: first, set $l(a) = w(a')$ for each arc $a \in A(N)$ that corresponds to arc $a' \in A$; then, for each newly added arc $a = (v', t)$ in $A(N)$, set $l(a) = d - d(v)$, where $d(v)$ is the distance from r to v in N and $d := \max_{v \in V_h} d(v)$ stands for the largest distance from r to every vertex in V_h. Obviously, $l(a) \geq 0$ for all $a \in A(N)$. In network $N(r, t)$, the set of all vertices at distance less than i from r is denoted by U_i and the set of all outgoing arcs of set U_i is denoted by $\partial^+(U_i)$.

Lemma 3.1. *For $v \in V_h$, the following statements hold:*

(*a*) *Every $r - v'$ path goes through v.*

(*b*) *There is a one to one correspondence between cycles whose head of the back arc is v in G and $v - t$ paths going through v' in $N(r, t)$.*

(*c*) *The length of the shortest $r - t$ paths through v' is $d + d(v, v')$.*

Proof. By Theorem 2.2, v is the unique entry vertex of $G_s(V_v)$. Then by the construction of $N(r, t)$, (*a*) and (*b*) follows directly.

Since any part of a shortest path is also a shortest path, so the length of the shortest $r - t$ paths going through v' is $d(v) + d(v, v') + d(v', t) = d(v) + d(v, v') + d - d(v) = d + d(v, v')$. □

Let K denote the minimum total weight of cycles in G. By Lemma 3.1, $d(t) = \min_{v \in V_h}\{d + d(v, v')\} = d + K$, implying that the total number of $r - t$ cuts $\partial^+(U_i)$ in N is $d + K$.

For $v \in V_h$, let C_v denote the collection of all cycles in G whose head of the back arc is v. With a slight abuse of notation, we call $\partial^+(U_i)$ is a *FAS* of C_v if it cuts off all $v - v'$ paths in N, since it yields a FAS of C_v naturally. We call $\partial^+(U_i)$ a *quasi-FAS* of C_v if deleting $\partial^+(U_i)$ breaks all $v - t$ paths in N. Let $\mathcal{C}$ be the collection of C_v for $v \in S \subseteq V_h$. We call $\partial^+(U_i)$ a *quasi-FAS* of $\mathcal{C}$ if $\partial^+(U_i)$ is a quasi-FAS of C_v for $C_v \in \mathcal{C}$, and we call a quasi-FAS $\partial^+(U_i)$ of $\mathcal{C}$

a *quasi-FAS* of G if $\mathcal{C}$ is the collection of C_v for $v \in V_h$. By Lemma 3.1, it is easy to see that $\partial^+(U_i)$ is a *FAS* of C_v if $d(v) < i \leq \min\{d(t), d(v')\}$, a quasi-FAS of C_v if $d(v) < i \leq d(t)$, a quasi-FAS of $\mathcal{C}$ if $\max_{v:C_v \in \mathcal{C}} d(v) < i \leq d(t)$ and a quasi-FAS of G if $d < i \leq d(t)$. Furthermore, for $v \in V_h$, the number of quasi-FASs of C_v is $d(t) - d(v)$ and the number of FASs of C_v is $\min\{d(t), d(v')\} - d(v)$; the number of quasi-FASs of $\mathcal{C}$ is $d(t) - \max_{v:C_v \in \mathcal{C}} d(v)$. In particular, the number of quasi-FASs of G is $d(t) - d = K$.

As we shall see, starting with $\mathcal{C} = \{C_v : v \in V_h\}$, we always arrive at an FAS of G by recursively choosing some quasi-FAS $\partial^+(U_i)$ of current $\mathcal{C}$ and deleting C_v that are broken by $\partial^+(U_i)$ from $\mathcal{C}$.

To construct a FAS, one need to break all cycles. One straightforward idea is to choose a cut of C_v in N for $v \in V_h$. Let $\mathcal{C}$ be the collection of C_v for $v \in V_h$.

Whenever there are quasi cut for $\mathcal{C}$ we always arrive at a FAS.

When $\mathcal{C}$ is not empty, we recursively choose a quasi-cut of current $\mathcal{C}$

Without loss of generality, assume there are newly added arcs in the initial quasi-FAS $\partial^+(U_i)$ and. As we shall see, the first available[1] quasi-cut of $\mathcal{C}$ is always available cut of some $C_v \in \mathcal{C}$ throughout our construction process. When $\mathcal{C}$ is not empty, we recursively choose the first available quasi-cut of current $\mathcal{C}$ as a supplement of the initial quasi-FAS $\partial^+(U_i)$ and delete every C_v from $\mathcal{C}$ that is broken by this quasi-cut. We call an execution of choosing the first available quasi-cut of current $\mathcal{C}$ and deleting the broken cycle classes from $\mathcal{C}$ an *iteration*. As long as we have available quasi cuts breaking some $C_v \in \mathcal{C}$ in each iteration before $\mathcal{C}$ becomes empty, the cardinality of $\mathcal{C}$ decreases by at least one in each iteration and we will finally arrive at a FAS of G. When $\mathcal{C}$ is empty, take the union of the initial quasi-FAS and all the chosen quasi-cuts, delete all newly added arcs from its union and replace each arc by its corresponding arc in G. We call the process of constructing a valid FAS a *stage*.

Observe that the initial quasi FAS of each stage is actually the first available quasi cut of $\mathcal{C}$, where $\mathcal{C}$ is the collection of C_v for every $v \in V_h$. Therefore our construction process in each stage has a slightly different interpretation. Instead of starting with a quasi FAS, we start with an empty set F and initialize $\mathcal{C}$ by the collection of C_v for every $v \in V_h$. When $\mathcal{C}$ is not empty, we recursively choose the first available quasi cut $\partial^+(U_i)$ of current $\mathcal{C}$, merge $\partial^+(U_i)$ into F, and update $\mathcal{C}$ by deleting every C_v that is broken by $\partial^+(U_i)$. When $\mathcal{C}$ becomes empty, the resulting F breaks all C_v for every $v \in V_h$. Deleting all newly added arcs in F and replacing each arc in F by its corresponding arc in G give rise to a valid FAS of G.

Let $\mathcal{C}$ be the collection of C_v for $v \in V_h$. To construct a FAS, we start with an empty set

[1]'First available' here refers to the $r - t$ cut $\partial^+(U_i)$ with the least index i among all unused cuts.

F. When $\mathcal{C}$ is not empty, we recursively choose a quasi-cut $\partial^+(U_i)$ of current $\mathcal{C}$, merge $\partial^+(U_i)$ into F, and update $\mathcal{C}$ by deleting every C_v that is broken by $\partial^+(U_i)$. We call an execution of choosing the first available quasi-cut of current $\mathcal{C}$ and deleting the broken cycle classes from $\mathcal{C}$ an *iteration*. When $\mathcal{C}$ becomes empty, the resulting F breaks all C_v for every $v \in V_h$. Deleting all newly added arcs in F and replacing each arc in F by its corresponding arc in G give rise to a FAS of G.

Now the problem boils down to showing that in the construction of each stage, the first available quasi cut of current $\mathcal{C}$ always breaks some $C_v \in \mathcal{C}$ before $\mathcal{C}$ becomes empty. Notice that in the worst case, we need a proper cut for every $C_v \in \mathcal{C}$. Therefore if we can show that we always have enough available proper cuts for every $C_v \in \mathcal{C}$ throughout the construction process, we are done. The following lemma guarantees that we always have enough available proper cuts of C_v for every $v \in V_h$ and hence iterations always end in each stage.

Lemma 3.2. *Let $\mathcal{C}$ be the collection of C_v for every $v \in V_h$. At each stage, if we initially start with $\mathcal{C}$, recursively choose the first available quasi cut of $\mathcal{C}$ in $N(r, t; l)$ and delete every cycle class C_v that is broken by this quasi cut from $\mathcal{C}$, then we always end up with an empty set $\mathcal{C}$. Moreover, at the end of stage k, the number of available proper cuts of C_v for each $v \in V_h$ is at least $K - k$.*

Our proof of Lemma 3.2 is based on the following technical lemma.

Lemma 3.3. *In each iteration of a stage, let $\mathcal{C}' \subseteq \mathcal{C}$ be the collection of all cycle classes in $\mathcal{C}$ such that the first available quasi cut $\partial^+(U_i)$ of current $\mathcal{C}$ is also the first available proper cut for each cycle class in $\mathcal{C}'$. If $\mathcal{C}'$ is not empty, then there exists a cycle class $C_u \in \mathcal{C}'$ such that the number of available proper cuts of C_u is decreased by precisely one in this stage.*

Proof. Assume to the contrary that the number of available proper cuts of each cycle class in $\mathcal{C}'$ is decreased by more than one in this stage. Since after this iteration, all cycle classes in $\mathcal{C}'$ will be deleted from $\mathcal{C}$. Therefore no proper cuts of each cycle class in $\mathcal{C}'$ will be chosen in the succeeding iterations of this stage. It follows that all other proper cuts of each cycle class in $\mathcal{C}'$ are chosen before this iteration. However, in this case, $\partial^+(U_i)$ is no longer the first available quasi cut of $\mathcal{C}$ as all cycle classes in $\mathcal{C}'$ are deleted from $\mathcal{C}$ before this iteration and the first available quasi cut of $\mathcal{C}$ will not depend on them, a contradiction. □

Notice that each $r - t$ cut $\partial^+(U_i)$ of N can only be used once in constructing the collection of FAS. Therefore, at the end of each stage, we go through every $r - t$ cut $\partial^+(U_i)$ in N and decrease the value of $l(a)$ by one for each arc $a \in \partial^+(U_i)$ if $\partial^+(U_i)$ is a chosen quasi cut at this

stage. As a result, the constantly updated length function l and the latest distance $d(v)$ for $v \in V(N)$ provide some useful information in every repetition of stages: for each $v \in V_h$, the difference $\min\{d(t), d(v')\} - d(v)$ is exactly the number of currently available proper cuts of C_v. Besides, the first available quasi cut of current $\mathcal{C}$ in each iteration of a stage is exactly the first cut of C_v with the largest $d(v)$ in $\mathcal{C}$.

Now we are ready to present a proof of our key lemma.

Proof of Lemma 3.2. We apply induction on the number of stages.

Observe that the initial state, viewed as stage 0, automatically satisfies the induction hypothesis. Therefore, the proof of basis and the proof of induction step are essentially the same. So we proceed to the induction step. Suppose the statement in Lemma 3.2 holds for stage $k \geq 0$. We aim to establish the statement for stage $k+1$.

We initially start with $\mathcal{C}$ which consists of C_v for every $v \in V_h$. By induction hypothesis, at the beginning of stage $k+1$, the number of available proper cut of C_v for every $v \in V_h$ is at least $K - k > 0$. Therefore, every C_v for $v \in V_h$ has available proper cuts. Let C_u be an arbitrary cycle class in $\mathcal{C}$. Let $\partial^+(U_{i_\tau}), \ldots, \partial^+(U_{i_1})$ with $i_\tau < \cdots < i_1$ be all the proper cuts of C_u used in stage $k+1$. Assume these $r - t$ cuts are chosen as the first available quasi cut of a series of nested subsets $\mathcal{C}^{(\tau)} \subset \cdots \subset \mathcal{C}^{(1)} := \mathcal{C}$ respectively. If $\tau = 1$, the number of available proper cuts of C_u decreases by one. By induction hypothesis, the number of available proper cuts of C_u is at least $K - k - 1$ at the end of stage $k+1$. So assume $\tau \geq 2$. Let $u_j \in V_h$ be a vertex with the largest $d(v)$ among all $C_v \in \mathcal{C}^{(j)}$ for $j = \tau, \ldots, 1$. Since every C_v for $v \in V_h$ has available proper cuts, the first available quasi cut $\partial^+(U_{i_j})$ of $\mathcal{C}^{(j)}$ is essentially the first available proper cut of $C_{u_j} \in \mathcal{C}^{(j)}$. Without loss of generality, assume C_{u_j} is a cycle class in $\mathcal{C}^{(j)}$ whose number of available proper cuts decreases by one in this stage. By Lemma 3.3, such C_{u_j} exists. It follows that $d(u) \leq d(u_\tau) < d(u'_\tau) \leq \cdots \leq d(u_1) < \min\{d(t), d(u')\}$. Obviously, C_u is not the cycle class with the least number available proper cuts among all vertices in V_h, since $\min\{d(t), d(u')\} - d(u) > d(u'_\tau) - d(u_\tau)$. Moreover, by Lemma 3.3, the number of available proper cuts of C_{u_τ} decreases by one in this stage. Then inequality $\min\{d(t), d(u')\} - d(u) > d(u'_\tau) - d(u_\tau)$ together with the induction hypothesis implies that the number of available proper cuts of C_u decreases by $\tau \geq 2$, but is no less than $K - k - 1$ at the end of stage $k+1$.

Therefore, we always have available proper cuts of C_v for every $v \in V_h$ in each iteration of a stage. So the first available quasi cut of current $\mathcal{C}$ is also the first available proper cut of some $C_v \in \mathcal{C}$. As a result, the cardinality of $\mathcal{C}$ decreases in each iteration. We finally arrive at an empty set $\mathcal{C}$ and hence iterations end in each stage. □

Recall that there are K quasi FASs in N, and we only use one quasi FAS in a stage. As

a result, there will be K stages throughout the construction process. Moreover, Lemma 3.2 guarantees each stage returns a valid FAS of G. Therefore we can attain an FAS packing of cardinality K. Since K is also the minimum total weight of cycles in G, the following theorem follows directly.

Theorem 3.4. *Every reducible flow graph is FASM.*

3.2 Algorithm and complexity

The constructive proof given above naturally yields an algorithm for finding a maximum FAS packing in a reducible flow graph G. However, regardless of other operations, we still need to repeat stage K times before we arrive at a maximum FAS packing, which makes our algorithm pseudo-polynomial. Fortunately, consecutive stages might return the same valid FAS because those chosen proper quasi cuts in each stage might be the same. So a natural idea is to merge these consecutive stages returning the same valid FAS into one new stage. Suppose we have merged the first $k-1$ stages returning the same valid FAS into $\kappa-1$ new STAGEs respectively and updated the length function l accordingly. Now we are at stage k, which cannot be merged into STAGE $\kappa-1$. We start our construction with $\mathcal{C}$ which initially consists of C_v for every $v \in V_h$. Then we recursively choose the first available quasi cut of current $\mathcal{C}$ in N, delete every cycle class C_v from $\mathcal{C}$ that is broken by the chosen quasi cut, and update d by $\max_{C_v \in \mathcal{C}} d(v)$. After $j \geq 0$ iterations, we arrive at a nonempty set $\mathcal{C}$, and $\partial^+(U_{d+1})$ is the first available quasi cut of current $\mathcal{C}$. Observe that the following $\min_{(u,w)\in\partial^+(U_{d+1})}\{d(w)-d\}$ quasi cuts of current $\mathcal{C}$ including $\partial^+(U_{d+1})$ are actually the same. If current $\mathcal{C}$ occurs in succeeding stages, we might use the same $r-t$ cut again. In each iteration of stage k, we update α (initialized with a value large enough) by the smaller value of current α and $\min_{(u,w)\in\partial^+(U_{d+1})}\{d(w)-d\}$. When $\mathcal{C}$ becomes empty, quasi cuts chosen in the succeeding α stages are actually the same. It follows that the succeeding α consecutive stages including stage k give rise to the same valid FAS of G. Therefore, we merge these α stages including stage k into new STAGE κ. Besides, observe that arcs might be chosen repeatedly in more than one iterations of a stage. Let $\mathcal{T}$ be the collection of all the chosen quasi cuts of a stage. At the end of STAGE κ, instead of updating the length function l separately, we go through every $r-t$ cut T in $\mathcal{T}$ and decrease $l(a)$ by α for each arc $a \in T$.

The following lemma guarantees that, by making the biggest jump on stages in exchange for efficiency, our construction ends in $O(n)$ STAGEs, where n is number of vertices in G.

Lemma 3.5. *By merging consecutive stages returning the same valid FAS into a new STAGE, the construction ends in $O(n)$ STAGEs, where n is the number of vertices in G.*

Proof. At the beginning of our algorithm, we may decompose all vertices of $V(N)$ into at most $|V(N)|$ subsets of $V(N)$ according to the distance $d(v)$ with respect to initial length function l. At each STAGE, there exists a chosen quasi cut S satisfying equality $\alpha = \min_{(u,w)\in S} d(w) - \max_{(u,w)\in S} d(u)$. At the end of this STAGE, the length of each arc in quasi cut S decreases by α. It follows that the vertex set of distance $\min_{(u,w)\in S} d(w)$ and the vertex set of distance $\max_{(u,w)\in S} d(u)$ will merge into one vertex set of the same distance. Therefore, the number of decompositions decreases by at least one in each STAGE. After $O(|V(N)|)$ STAGEs, all vertices in $V(N)$ will belong to the same distance class, implying the termination of our algorithm. Also notice that $|V(N)| \leq 2n + 1$, where n is the number of vertices in G. Hence the construction ends in $O(n)$ STAGEs. □

Now we are ready to present our algorithm.

In our algorithm, we need to repeatedly find all the $d(v)$ with respect to latest length function l in N. The following lemma ensures that this task can always be finished efficiently due to the special structure of N, an acyclic digraph.

Lemma 3.6. *All the single source shortest distances in an acyclic digraph $D = (V, A)$ can be found in $O(n + m)$, where $n = |V|$ and $m = |A|$.*

Proof. Let $l : A \to \mathbb{R}$ be the length function defined on arcs of D and $r \in V$ be the source given. Denote the distance of v from r by $d(v)$ for every $v \in V$. Initially, set $d(r) = 0$ and $d(v) = +\infty$ for $v \in V\backslash\{r\}$, which takes $O(n)$ time. Then topologically sort all vertices and denote the ordering by π. This process can be finished in $O(m + n)$ time [1]. Visit every vertex v in topological order and update $d(v)$ by $\min_{u:\pi(u)<\pi(v)}\{d(u) + l(u, v)\}$. This step returns the global shortest distance $d(v)$ because vertices behind v in a topological order cannot be on any $r-v$ paths. Throughout the updating process, each arc is only visited once which requires $O(m)$ time. Hence the time complexity of finding all the shortest distances from r is the same as the time complexity of topological sort, which is $O(m + n)$. □

Theorem 3.7. *In an arc-weighted reducible flow graph $G = (V, A; r)$, a maximum FAS packing $\mathcal{F}$ can be found in $O(n^2m)$ time, where $n = |V|$ and $m = |A|$.*

Proof. By the construction of network $N(r, t; l)$, $|V(N)| \leq 2n + 1$ and $|A(N)| \leq m + n$. It follows that $O(|V(N)|) = O(n)$. Since G is connected, $O(|A(N)|) = O(m)$. Notice that the construction of network N involves finding the shortest distance $d(v)$ for $v \in V_h$. By Lemma 3.6, we can find all the distance $d(v)$ from r in $O(m + n) = O(m)$ time. So the construction of network N requires $O(m)$ time. It is easy to see that all operations outside STAGEs take $O(m)$

Algorithm 1: FAS packing algorithm

Input : An arc-weighted reducible flow graph $G = (V, A; r)$
Output: A maximum FAS packing $\mathcal{F}$ of G

Construct $N(r, t; l)$ from $G = (V, A; r)$;
Find $d(v)$ for every $v \in V(N)$ with respect to l in N;
$d \leftarrow \max_{v \in V_h} d(v)$;
$\mathcal{F} \leftarrow \emptyset$;
STAGE: **while** $d < d(t)$ **do**
  $\mathcal{T} \leftarrow \emptyset$;
  $F \leftarrow \emptyset$;
  $\alpha \leftarrow d(t)$;
  `Augment` $(\partial^+(U_{d+1}))$;
  Delete all newly added arcs from F;
  Replace remaining arcs in F by corresponding arcs in G;
  $\mathcal{F} \leftarrow \mathcal{F} \cup \{(F, \alpha)\}$;
  `Update` (l);
  Find $d(v)$ for every $v \in V(N)$ with respect to current l in N;
  $d \leftarrow \max_{v \in V_h} d(v)$;
Return $\mathcal{F}$;

procedure `Augment`(S)
  $\mathcal{T} \leftarrow \mathcal{T} \cup \{S\}$;
  $F \leftarrow F \cup S$;
  $\alpha \leftarrow \min\{\alpha, \min_{(u,w) \in S}\{d(w) - d\}\}$;
  if $\exists (v', t) \in S$ **then**
    $d \leftarrow \max_{(v',t) \in S} d(v)$;
    `Augment` $(\partial^+(U_{d+1}))$;
procedure `Update`(l)
  for $T \in \mathcal{T}$ **do**
    for $a \in T$ **do**
      $l(a) \leftarrow l(a) - \alpha$;

time. By Lemma 3.5, our algorithm ends in $O(n)$ STAGEs. It remains to show that the time complexity of each STAGE is $O(nm)$. Notice that each STAGE contains a recursive function $Augment(S)$ and a function $Update(l)$. Each invocation of function $Augment(S)$ in a STAGE decreases the number of newly added arcs in S by at least one. So $Augment(S)$ in a STAGE ends in $O(n)$ invocations. Function $Augment(S)$ itself involves finding $\min_{(u,w) \in S}\{d(w) - d\}$ and $\max_{(v',t) \in S} d(v)$, both of which can be done in $O(n)$ time. So the complexity of function $Augment(S)$ in one STAGE is $O(n^2)$. As to function $Update(l)$, it requires $O(nm)$ time in one STAGE. Hence the time complexity of each STAGE is $O(nm)$. The theorem follows. □

Acknowledgments

The author would like to thank Prof. Wenan Zang for his invaluable suggestions.

References

[1] Ahuja, R.K., Magnanti, T.L., Orlin, J.B.: Network Flows - Theory, Algorithms, and Applications. Prentice Hall Inc., Englewood Cliffs, NJ (1993)

[2] Chen, X., Ding, G., Hu, X., Zang, W.: A min-max relation on packing feedback vertex sets. Math. Oper. Res. **31**(4), 777–788 (2006)

[3] Chen, X., Zang, W.: An efficient algorithm for finding maximum cycle packings in reducible flow graphs. Algorithmica **44**(3), 195–211 (2006)

[4] Ding, G., Zang, W.: Packing cycles in graphs. J. Combin. Theory Ser. B **86**(2), 381–407 (2002)

[5] Frank, A., Gyarfas, A.: Directed graphs and computer programs. In: Problemes Combinatoires et Theorie des Graphes, 260, pp. 157–158 (1976)

[6] Hecht, M.S.: Flow analysis of computer programs. North-Holland, New York (1977)

[7] Hecht, M.S., Ullman, J.D.: Flow graph reducibility. SIAM J. Comput. **1**(2), 188–202 (1972)

[8] Hecht, M.S., Ullman, J.D.: Characterizations of reducible flow graphs. J. Assoc. Comput. Mach. **21**(3), 367–375 (1974)

[9] Hopcroft, J.E., Ullman, J.D.: An $n \log n$ algorithm for detecting reducible graphs. In: Proc. 6th Annual Princeton Conference on Information Sciences and Systems, pp. 119–122. Princeton, NJ (1972)

[10] Ramachandran, V.: Finding a minimum feedback arc set in reducible flow graphs. J. Algorithms **9**(3), 299–313 (1988)

[11] Ramachandran, V.: A minimax arc theorem for reducible flow graphs. SIAM J. Discrete Math. **3**(4), 554–560 (1990)

[12] Shamir, A.: A linear time algorithm for finding minimum cutsets in reducible graphs. SIAM J. Comput. **8**(4), 645–655 (1979)

[13] Tarjan, R.E.: Testing flow graph reducibility. J. Comput. System Sci. **9**(3), 355–365 (1974)