Strongly Kernel Mengerian Orientations of Line Multigraphs

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Abstract

Given a digraph D, let $\sigma(D)$ denote the linear system consisting of domination, independence and nonnegativity constraints, and $FK(D)$ denote the set of all solutions to $\sigma(D)$. We call D strongly kernel ideal if $FK(D')$ is integral for each induced subgraph D' of D and strongly kernel Mengerian if $\sigma(D')$ is TDI for each induced subgraph D' of D. We also call D good if it is clique-acyclic and each directed odd cycle has a chord.

In this paper, we prove that a digraph is strongly kernel ideal iff it is strongly kernel Mengerian iff it is good. Our result strengthens the theorem of Borodin *et al.* [3] on kernel perfect digraphs and generalizes the theorem of Király and Pap [7] on stable matching problem.

1 Introduction

An undirected graph is called *simple* if it contains neither loops nor parallel edges and is called a *multigraph* if it contains no loops but parallel edges are allowed. *Simple* digraphs and *multidigraphs* are defined analogously.

Let G be an undirected graph. The *line graph* of G is an undirected simple graph that represents the adjacency of edges of G. The *line multigraph* of G is an undirected multigraph that represents the adjacency of edges of G and if edges e and f have two common ends in G, their corresponding vertices are joined by two edges. In this paper, line graphs and line multigraphs of G are both denoted by $L(G)$ if no ambiguity occurs. Observe that the line graph and line multigraph of G are the same when G is simple. And for any two vertices in a line multigraph, there exist at most two parallel edges between them.

Let $D = (V, A)$ be a digraph. For $U \subseteq V$, We call U an *independent* set of D if no two vertices in U are connected by an arc, call U a *dominating* set of D if for each vertex $v \notin U$, there is an arc from v to U, and call U a *kernel* of D if it is both independent and dominating. A *clique* of D is a subset of V such that every two distinct vertices are connected by an arc. We call D *clique-acyclic* if for every clique of D the induced subdigraph of one-way arcs is acyclic, and call D *good* if it is clique-acyclic and each directed odd cycle has a chord or a pseudochord[1].

Theorem 1.1 (Borodin *et al.* [3])**.** *Let G be a line multigraph. The orientation D of G is kernel perfect if and only if it is good.*

A subset P of $\mathbb{R}^n$ is called a *polytope* if it is the convex hull of finitely many vectors in $\mathbb{R}^n$. A point x in P is called a *vertex* or an *extreme point* if there exist no distinct points y and z in P such that $x = \alpha y + (1-\alpha)z$ for $0 < \alpha < 1$. It is well known that P is the convex hull of its vertices, and that there exists a linear system $Ax \le b$ such that $P = \{x \in \mathbb{R}^n : Ax \le b\}$. We call P *$1/k$-integral* if its vertices are $1/k$-integral vectors, where $k \in \mathbb{N}$. By a theorem in linear programming, P is $1/k$-integral if and only if $\max\{c^T x : Ax \le b\}$ has an optimal $1/k$-integral solution, for every integral vector c for which the optimum is finite. If, instead, $\max\{c^T x : Ax \le b\}$ has a dual optimal $1/k$-integral solution, we call linear system $Ax \le b$ *totally dual $1/k$-integral* (TDI/k). It is easy to verify that $Ax \le b$ is TDI/k if and only if $Bx \le b$ is TDI, where $B = A/k$. Thus, from a theorem by Edmonds and Giles [6], we deduce that if $Ax \le b$ is TDI/k and b is integal, then $P = \{x \in \mathbb{R}^n : Ax \le b\}$ is $1/k$-integral.

Let $\sigma(D)$ denote the linear system consisting of the following inequalities:

$$x(v) + x(N_D^+(v)) \ge 1 \qquad \forall\ v\ \in V, \tag{1.1}$$

$$x(Q) \le 1 \qquad \forall\ Q \in \mathcal{Q}, \tag{1.2}$$

$$x(v) \ge 0 \qquad \forall\ v\ \in V, \tag{1.3}$$

where $x(U) = \sum_{u \in U} x(u)$ for any $U \subseteq V$, $N_D^+(v)$ denotes the set of all out-neighbors of vertex v, and $\mathcal{Q}$ denotes the set of all cliques of D. Observe that incidence vectors of kernels of D are precisely integral solutions $x \in \mathbb{Z}^A$ to $\sigma(D)$.

The *kernel polyotpe* of D, denoted by $K(D)$, is the convex hull of incidence vectors of all kernels of D. The *fractional kernel polytope* of D, denoted by $FK(D)$, is the set of all solutions $x \in \mathbb{R}^A$ to $\sigma(D)$. Clearly, $K(D) \subseteq FK(D)$.

We call D *kernel perfect* if each of its induced subgraphs has a kernel, *strongly kernel ideal* if $FK(D')$ is integral for each induced subgraph D' of D, and *strongly kernel Mengerian* if $\sigma(D')$ is TDI for each induced subgraph D' of D.

As described in Egres Open [1], the polyhedral description of kernels remains open. Chen *et al.* [4] attained a polyhedral characterization of kernels by replacing clique constraints $x(Q) \le 1$ for $Q \in \mathcal{Q}$ with independence constraints $x(u) + x(v) \le 1$ for $(u, v) \in A$. In this paper we show

[1] A pseudochord is an arc (v_i, v_{i-1}) in a directed cycle $v_1 v_2 \ldots v_l$

that kernels in digraph D can be thoroughly characterized by polyhedral approaches if D is an orientation of some line multigraph.

Theorem 1.2. *Let G be a line multigraph, and let D be an orientation of G. Then the following statements are equivalent:*

(i) *D is good;*

(ii) *D is kernel perfect;*

(iii) *D is strongly kernel ideal;*

(iv) *D is strongly kernel Mengerian.*

The equivalence of (i) and (ii) was established by Borodin *et al.* [3] (Maffray [8] proved the case when G is perfect). Király and Pap [7] proved the theorem when G is the line graph of a bipartite graph, which will be given in next section. Our theorem strengthens the result of Borodin *et al.* and generalizes the result of Király and Pap to general line multigraphs.

2 Preliminaries

Let $G = (V, E)$ be an undirected graph. For each $v \in V$, let $\delta(v)$ be the set of all edges incident to v and let $\prec_v$ be a strict linear order on $\delta(v)$. We call $\prec_v$ the *preference* of v and say that v *prefers* e to f if $e \prec_v f$. Let $\prec$ be collection of all these $\prec_v$ for $v \in V$. We call $(G, \prec)$ a *preference system* and call $(G, \prec)$ *simple* if G is simple. Edge e is said to *dominate* edge f if they have a common end v such that $e \preceq_v f$. Let M be a matching of G. We call M *stable* if each edge of G is dominated by some edge in M.

Let $(G, \prec)$ be a preference system. For each $e \in E$, let $\varphi(e)$ denote the set of all edges of G that dominate e. Let $\pi(G, \prec)$ be the linear system consisting of the following inequalites:

$$x(\varphi(e)) \geq 1 \qquad \forall\, e \in E, \tag{2.1}$$

$$x(\delta(v)) \leq 1 \qquad \forall\, v \in V, \tag{2.2}$$

$$x(e) \geq 0 \qquad \forall\, e \in E. \tag{2.3}$$

As observed by Abeledo and Rothblum [2], incidence vectors of stable matchings of $(G, \prec)$ are precisely integral solutions $x \in \mathbb{Z}^E$ of $\pi(G, \prec)$.

The *stable matching polytope*, denoted by $SM(G, \prec)$, is the convex hull of incidence vectors of all stable matchings of $(G, \prec)$. The *fractional stable matching polytope*, denoted by $FSM(G, \prec)$, is the set of all solutions $x \in \mathbb{R}^E$ to $\pi(G, \prec)$. Clearly, $SM(G, \prec) \subseteq FSM(G, \prec)$.

Theorem 2.1 (Rothblum [9])**.** *Let $(G, \prec)$ be a simple preference system. If G is bipartite, then $SM(G, \prec) = FSM(G, \prec)$.*

Theorem 2.2 (Király and Pap [7])**.** *Let $(G, \prec)$ be a simple preference system. If G is bipartite, then $\pi(G, \prec)$ is totally dual integral.*

Let $C = v_1 v_2 \dots v_l$ be a cycle in G, we say that C has *cyclic preferences* in $(G, \prec)$ if $v_{i-1}v_i \prec_{v_i} v_i v_{i+1}$ for $i = 1, 2, \dots, l$ or $v_{i-1}v_i \succ_{v_i} v_i v_{i+1}$ for $i = 1, 2, \dots, l$, where indices are taken modulo l. For $x \in FSM(G, \prec)$, let $E_\alpha(x)$ denote the set of all edges with $x(e) = \alpha$ where $\alpha \in \mathbb{R}$ and $E_+(x)$ denote the set of all edges with $x(e) > 0$.

Theorem 2.3 (Abeledo and Rothblum [2])**.** *Let $(G, \prec)$ be a simple preference system. Then $FSM(G, \prec)$ is $1/2$-integral. Moreover, for each $1/2$-integral point x in $FSM(G, \prec)$, $E_{1/2}(x)$ consists of vertex disjoint cycles with cyclic preferences.*

Theorem 2.4 (Chen *et al.* [5])**.** *Let $(G, \prec)$ be a simple preference system. Then $\pi(G, \prec)$ is totally dual $1/2$-integral. Moreover, $\pi(G, \prec)$ is totally dual integral if and only if $SM(G, \prec) = FSM(G, \prec)$.*

3 Reductions

To study kernels in multidigraphs, it suffices to work on its minimal spanning subdigraph that preserves all the connection relations among vertices. In the remainder of this paper, we assume that D is an orientation of a line multigraph $L(H)$ such that any two distinct vertices in H are joined by at most two edges and parallel edges in $L(H)$ are orientated oppositely. Hence D is a multidigraph such that any two vertices are joined by at most one arc in each direction.

Kernels are closely related to stable matchings. Let D be a clique-acyclic orientation of line multigraph $L(H)$. For any two adjacent edges e and f in H, define $e \prec_v f$ if (e, f) is an arc in D. Hence each clique-acyclic orientation D of line multigraph $L(H)$ is associated with a preference system $(H, \prec)$. Recall that $\sigma(D)$ denotes the linear system which defines $FK(D)$. Consequently, $\sigma(D)$ can be interpreted in terms of preference system $(H, \prec)$. The equivalence of constraints (1.3) and constraints (2.3) follows directly. Constraints (1.1) can be viewed as constraints (2.1) because of the one to one correspondence between dominating vertex set $\{v\} \cup N_D^+(v)$ for $v \in V(D)$ and stable edge set $\varphi(e)$ for $e \in E(H)$. Observe that cliques of D correspond to three types of edge sets in H:

- $\delta(v)$ for $v \in V(H)$,
- nontrivial subsets of $\delta(v)$ for $v \in V(H)$,
- complete subgraphs of H induced on three vertices (with parallel edges allowed).

Hence constraints (1.2) can be viewed as constraints (2.2) together with some extra constraints on $(H, \prec)$. Let $\mathcal{O}(H)$ denote the set of all complete subgraphs of H induced on three vertices. It follows that $\sigma(D)$ can be viewed as a linear system defined on preference system $(H, \prec)$ by the following inequalities:

$$
\begin{aligned}
x(\varphi(e)) &\geq 1 & & \forall\ e \in E(H), & (3.1)\\
x(\delta(v)) &\leq 1 & & \forall\ v \in V(H), & (3.2)\\
x(S) &\leq 1 & \emptyset \subset S \subset \delta(v), & \forall\ v \in V(H), & (3.3)\\
x(O) &\leq 1 & & \forall\ O \in \mathcal{O}(H), & (3.4)\\
x(e) &\geq 0 & & \forall\ e \in E(H). & (3.5)
\end{aligned}
$$

Notice that constraints (3.1), (3.2) and (3.5) constitute the Rothblum system $\pi(H, \prec)$ which defines $FSM(H, \prec)$. Constraints (3.3) are redundant with respect to $\pi(H, \prec)$ due to constraints (3.2). As we shall see, constraints (3.4) are also redundant with respect to $\pi(H, \prec)$ when D is good. Hence $FK(D)$ is essentially defined by Rothblum system $\pi(H, \prec)$, or equivalently $FK(D) = FSM(H, \prec)$, when D is good.

Lemma 3.1. *For parallel edges e and e' in H, there exists no edge f such that $e \prec_v f \prec_v e'$, where v is an endpoint of e and e'.*

Proof. Since D is clique-acyclic, by the construction of preference system $(H, \prec)$, the lemma follows directly. □

By Lemma 3.1, parallel edges play exactly the same role in preference system $(H, \prec)$. Hence we turn to study simple preference systems defined on its underlying simple graphs. Let $(\hat{H}, \hat{\prec})$ be a simple preference system, where $\hat{H}$ is a spanning subgraph of H obtained by keeping one edge between every pair of adjacent vertices and $\hat{\prec}$ is the restriction of $\prec$ on $\hat{H}$. Before proceeding, we introduce a technical lemma first.

Lemma 3.2. *Let*

$$Ax \leq b,\ x \geq 0 \tag{3.6}$$

and

$$\bar{A}\bar{x} \leq b,\ \bar{x} \geq 0 \tag{3.7}$$

be two linear systems, where $\bar{A}$ is obtained from A by duplicating some columns. If (3.6) is totally dual $1/k$-integral, then so is (3.7), where $k \in \mathbb{N}$.

Proof. It suffices to prove that the theorem holds for $\bar{A}$ with one duplicate column. Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{n+1}) \in \mathbb{R}^{n+1}$ and $\bar{A} = (\bar{a}_1, \ldots, \bar{a}_{n+1}) = (A, a_k)$, where a_k is the kth column of A. Let $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_{n+1}) \in \mathbb{Z}^{n+1}$ be an arbitrary integral vector such that $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \le b,\ \bar{x} \ge 0\}$ is finite.

Let $\hat{c} \in \mathbb{Z}^n$ be defined by

$$\hat{c}_i = \begin{cases} \bar{c}_i & i \neq k, \\ \max\{\bar{c}_k, \bar{c}_{n+1}\} & i = k, \end{cases}$$

and $\hat{x} \in \mathbb{R}^n$ be defined by

$$\hat{x}_i = \begin{cases} \bar{x}_i & i \neq k, \\ \bar{x}_k + \bar{x}_{n+1} & i = k. \end{cases}$$

Clearly, $\max\{\hat{c}^T\hat{x} : A\hat{x} \le b,\ \hat{x} \ge 0\} \ge \max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \le b,\ \bar{x} \ge 0\}$. We claim that equality always holds in this inequality. Given an arbitrary optimal solution $\hat{x}^*$ to $\max\{\hat{c}^T\hat{x} : A\hat{x} \le b,\ \hat{x} \ge 0\}$, define $\bar{x}^*$ by $\bar{x}^*_i = \hat{x}^*_i$ for $i \neq k, n+1$, if $\bar{c}_k > \bar{c}_{n+1}$, $\bar{x}^*_k = \hat{x}^*_k$ and $\bar{x}^*_{n+1} = 0$, otherwise $\bar{x}^*_k = 0$ and $\bar{x}^*_{n+1} = \hat{x}^*_k$. It is easy to verify that $\bar{x}^*$ is a feasible solution to $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \le b,\ \bar{x} \ge 0\}$. Moreover, $\hat{c}^T\hat{x}^* = \bar{c}^T\bar{x}^*$ implies $\max\{\hat{c}^T\hat{x} : A\hat{x} \le b,\ \hat{x} \ge 0\} \le \max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \le b,\ \bar{x} \ge 0\}$. Hence equality follows.

Since $Ax \le b$ is TDI/k, there exists a dual optimal $1/k$-integral solution y^* to $\max\{\hat{c}^T\hat{x} : A\hat{x} \le b,\ \hat{x} \ge 0\}$ such that $(y^*)^T b = \max\{\hat{c}^T\hat{x} : A\hat{x} \le b,\ \hat{x} \ge 0\}$, $(y^*)^T A \ge \hat{c}^T$ and $y^* \ge 0$. It follows that

$$(y^*)^T\bar{A} = (y^*)^T(A, a_k) = ((y^*)^T A, (y^*)^T a_k) \ge (\hat{c}^T, \hat{c}_k) \ge \bar{c}^T,$$

implying that y^* is a feasible dual solution to $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \le b,\ \bar{x} \ge 0\}$. By the following inequalities

$$(y^*)^T b = \max\{\hat{c}^T\hat{x} : A\hat{x} \le b,\ \hat{x} \ge 0\} = \max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \le b,\ \bar{x} \ge 0\} \le (y^*)^T b,$$

y^* is also a dual optimal solution to $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \le b,\ \bar{x} \ge 0\}$, where the last inequality is from the weak duality theorem. Hence the lemma follows. □

Lemma 3.3. *If $\pi(\hat{H}, \hat{\prec})$ is totally dual $1/k$-integral, then so is $\pi(H, \prec)$, where $k \in \mathbb{N}$.*

Proof. By Lemma 3.1, columns corresponding to parallel edges in the left hand side matrix of $\pi(H, \prec)$ are identical. Hence the left hand side matrix of $\pi(H, \prec)$ can be obtained from that of $\pi(\hat{H}, \hat{\prec})$ by duplicating columns corresponding to parallels edges. Then the lemma follows from Lemma 3.2. □

By Theorem 2.4 and Lemma 3.3, we conclude that $\pi(H, \prec)$ is TDI/2. By the construction of $(H, \prec)$, when D is good, $(H, \prec)$ admits no odd cycles with cyclic preferences. It follows that constraints (3.4) are redundant with respect to $\pi(H, \prec)$ when D is good.

Lemma 3.4. *$FSM(H, \prec)$ is integral if and only if $FSM(\hat{H}, \hat{\prec})$ is integral.*

Proof. For simplicity, assume that $H = (V, E)$ and $\hat{H} = (V, \hat{E})$.

We prove the only if part by showing that $FSM(\hat{H}, \hat{\prec})$ is a projection of $FSM(H, \prec)$. Let $\hat{x}$ be a point in $FSM(\hat{H}, \hat{\prec})$ and define $x = (\hat{x}, 0) \in \mathbb{R}^{\hat{E}} \times \mathbb{R}^{E-\hat{E}}$. To show that $x \in FSM(H, \prec)$, it suffices to prove that $x(\varphi(e)) \geq 1$ for $e \in E - \hat{E}$. Let $e' \in \hat{E}$ be the edge parallel with $e \in E - \hat{E}$. By Lemma 3.1, $x(\varphi(e)) = x(\varphi(e')) = \hat{x}(\varphi(e')) \geq 1$. Hence $FSM(\hat{H}, \hat{\prec})$ is a projection of $FSM(H, \prec)$.

We prove the if part by showing that each vertex of $FSM(H, \prec)$ can be obtained from some vertex of $FSM(\hat{H}, \hat{\prec})$ by adding some zero entries. Let x be a vertex of $FSM(H, \prec)$. Define $\hat{x} \in \mathbb{R}^{\hat{E}}$ by, for $e \in \hat{E}$, $\hat{x}(e) = x(e) + x(e')$, where $e' \in E - \hat{E}$ is parallel with e. By Lemma 3.1, it is easy to verify that $\hat{x} \in FSM(\hat{H}, \hat{\prec})$. We claim that $\hat{x}$ is a vertex of $FSM(\hat{H}, \hat{\prec})$. Assume to the contrary that there exist $\hat{x}_1, \hat{x}_2 \in FSM(\hat{H}, \hat{\prec})$ such that $\hat{x} = \alpha\hat{x}_1 + (1-\alpha)\hat{x}_2$, where $0 < \alpha < 1$. For $i = 1, 2$, we extend $\hat{x}_i \in \mathbb{R}^{\hat{E}}$ to $x_i \in \mathbb{R}^E$ by, for $e \in \hat{E}$ without parallel edges in H, $x_i(e) = \hat{x}_i(e)$; for $e \in \hat{E}$ and its parallel edge $e' \in E - \hat{E}$, if $x(e) > x(e')$, $x_i(e) = \hat{x}_i(e)$ and $x_i(e') = 0$, otherwise $x_i(e) = 0$ and $x_i(e') = \hat{x}_i(e)$. By Lemma 3.1, it is easy to see that $x_1, x_2 \in FSM(H, \prec)$. Furthermore, $x(e) \cdot x(e') = 0$ for parallel edges e and e'. Assume to the contrary that $x(e) \cdot x(e') \neq 0$, it follows that $x(e) = x(e') = 1/2$. Let y and z are duplicates of x and set $y(e) = z(e') = 1$ and $y(e') = z(e) = 0$. It follows that $x = (y + z)/2$. Clearly, $y, z \in FSM(H, \prec)$, contradicting to the assumption that x is a vertex. Hence $x = \alpha x_1 + (1-\alpha)x_2$ follows, a contradiction again. Therefore, $\hat{x}$ is vertex of $FSM(\hat{H}, \hat{\prec})$. It follows that each vertex x of $FSM(H, \prec)$ is associated with a vertex $\hat{x}$ of $FSM(\hat{H}, \hat{\prec})$ and x can be obtained from $\hat{x}$ by, for $e \in \hat{E}$ without parallel edges in H, $x(e) = \hat{x}(e)$; for $e \in \hat{E}$ and its parallel edge $e' \in E - \hat{E}$, if $x(e) > x(e')$, $x(e) = \hat{x}(e)$ and $x(e') = 0$, otherwise $x(e) = 0$ and $x(e') = \hat{x}(e)$. □

We end this section with a summary. When D is clique-acyclic, it is associated with a preference system $(H, \prec)$ and a simple preference system $(\hat{H}, \hat{\prec})$, where $\hat{H}$ is a simple spanning subgraph of H maximizing the edge set and $\hat{\prec}$ is the restriction of $\prec$ on $\hat{H}$. Hence constraints (3.3) and (3.4) are redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$ and $FK(D) = FSM(H, \prec)$. To show $FK(D)$ is integral, by Lemma 3.4 it suffices to show that $FSM(\hat{H}, \hat{\prec})$ is integral. To show $\sigma(D)$ is TDI, by Lemma 3.3 it suffices to show $\pi(\hat{H}, \hat{\prec})$ is TDI. Moreover, when D is good, both $(H, \prec)$ and $(\hat{H}, \hat{\prec})$ admit no odd cycles with cyclic preferences.

4 Proofs

Before presenting our proof to the main theorem, we explore some properties of simple preference systems admitting no odd cycles with cyclic preferences.

Lemma 4.1. *Let $(G, \prec)$ be a simple preference system. If $(G, \prec)$ admits no odd cycles with cyclic preferences, then $SM(G, \prec) = FSM(G, \prec)$.*

As observed by Chen *et al.* [5], integrality of $FSM(G, \prec)$ is equivalent to total dual integrality of $\pi(G, \prec)$. A corollary follows directly.

Corollary 4.2. *Let $(G, \prec)$ be a simple preference system. If $(G, \prec)$ admits no odd cycles with cyclic preferences, then $\pi(G, \prec)$ is totally dual integral.*

Proof of Lemma 4.1. Let $(G, \prec)$ be a simple preference system admitting no odd cycles with cyclic preferences. By Theorem 2.3, $FSM(G, \prec)$ is 1/2-integral. Let x be a 1/2-integral point in $FSM(G, \prec)$. Consider $E_{1/2}(x)$, which consists of even cycles $C_1, C_2, \dots, C_r$ with cyclic preferences. For $i = 1, 2, \dots, r$, label vertices and edges of $C_i \in E_{1/2}(x)$ such that $C_i = v_1^i v_2^i \dots v_l^i$ and $e_k^i \prec_{v_{k+1}^i} e_{k+1}^i$ for $k = 1, 2, \dots, l$, where $e_k^i = v_k^i v_{k+1}^i$ and indices are taken modulo l. We remark that the parity of vertices and edges refers to the parity of their indices. Define $z \in \mathbb{R}^{E(G)}$ by

$$z(e) = \begin{cases} 1 & e \text{ is an even edge in some } C \in E_{1/2}(x), \\ -1 & e \text{ is an odd edge in some } C \in E_{1/2}(x), \\ 0 & \text{otherwise.} \end{cases}$$

We are going to exclude x from vertices of $FSM(G, \prec)$ by adding perturbation ϵz for small ϵ to x and showing that $x \pm \epsilon z \in FSM(G, \prec)$. Tight constraints in (2.1)-(2.3) under perturbation ϵz play a key role here. Observe that tight constraints in (2.2) and (2.3) are invariant under perturbation ϵz. It remains to show that perturbation ϵz does not affect tight constraints in (2.1) either. Let e be an edge with $x(\varphi(e)) = 1$. Clearly, $|\varphi(e) \cap E_+(x)| \in \{1, 2\}$. When $|\varphi(e) \cap E_+(x)| = 1$, $x(e) = 1$ follows, which is trivial. When $|\varphi(e) \cap E_+(x)| = 2$, the following claim guarantees that corresponding tight constraints are also invariant under perturbation ϵz.

Claim: The parity of dominating edges of e doest not agree.

To prove this claim, we distinguish four cases.

Case 1. Edge e is an edge from some $C \in E_{1/2}(x)$. This case is trivial since C admits cyclic preferences.

Case 2. Edge e is a chord in some $C \in E_{1/2}(x)$. We first show that endpoints of e do not have the same parity in C. Assume to the contrary that $e = v_{2i} v_{2j}$. If $e_{2i} \prec e$, it follows that $e_{2i-1} \prec e$.

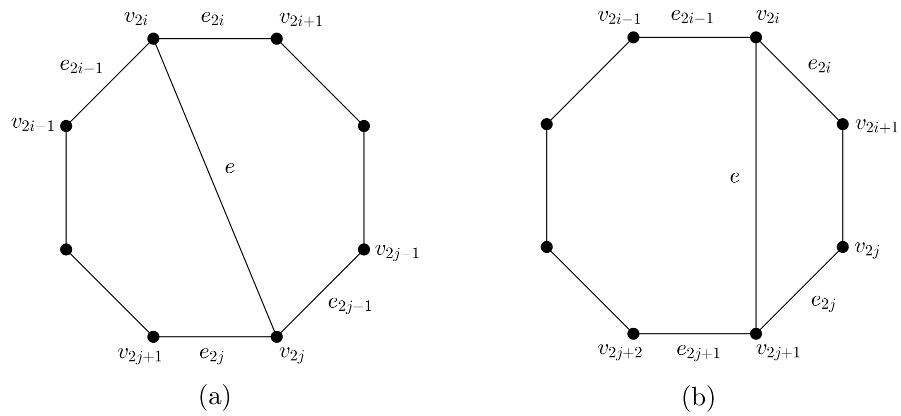

Figure 1: Case 2

Since $x(\varphi(e)) = 1$, $e \prec e_{2j-1}$ and $e \prec e_{2j}$ follow. However, $v_{2i}ev_{2j}e_{2j}v_{2j+1}\dots v_{2i-1}e_{2i-1}v_{2i}$ consistute an odd cycle with cyclic preferences, a contradiction. Hence $e \prec e_{2i}$ follows.

Similarly, if $e_{2j} \prec e$, it follow that $e_{2j-1} \prec e$. Eqaulity $x(\varphi(e)) = 1$ implies that $e \prec e_{2i}$ and $e \prec e_{2i-1}$. However, $v_{2i}e_{2i}v_{2i+1}\dots v_{2j-1}e_{2j-1}v_{2j}ev_{2i}$ constitute an odd cycle with cyclic preferences, a contradiction. Hence $e \prec e_{2j}$ follows.

Now $e \prec e_{2i}$ and $e \prec e_{2j}$, it follows that $e_{2i-1} \prec e$ and $e_{2j-1} \prec e$ since $x(\varphi(e)) = 1$. But in this case two odd cycles with cyclic preferences mentioned above occur at the same time. Therefore, endpoints of e have different parity in C. Without loss of generality, let $e = v_{2i}v_{2j+1}$. If $e_{2i} \prec e$ or $e_{2j+1} \prec e$, it follows that $e_{2i-1} \prec e$ or $e_{2j} \prec e$ respectively. Then e is dominated by two consecutive edges from C, which is trivial. So assume that $e \prec e_{2i}$ and $e \prec e_{2j+1}$. Since $x(\varphi(e)) = 1$, it follows that $e_{2i+1} \prec e$ and $e_{2j} \prec e$. Hence e is dominated by two nonadjacent edges with different parity.

Case 3. Edge e is a hanging edge of some $C \in E_{1/2}(x)$ and dominated by two edges from C. This case is trivial.

Case 4. Edge e is a connecting edge bewteen C_i and C_j and dominated by one edge from C_i and one edge from C_j respectively, where $C_i, C_j \in E_{1/2}(x)$. For $k = 1, 2, \dots, r$, let F_k be the set of edges incident to C_k in case 4. It suffices to work on a component of the induced subgraph $\cup_{i=1}^{i=r} F_i \cup C_i$. We apply induction on the number α of cycles from $E_{1/2}(x)$ in a component.

When $\alpha = 1$, it is trivial. Hence assume the claim holds for components of with $\alpha \geq 1$ cycles from $E_{1/2}(x)$. We consider a component with $\alpha + 1$ cycles $C_1, \dots, C_\alpha, C_{\alpha+1}$ from $E_{1/2}(x)$. Without loss of generality, assume that deleting $C_{\alpha+1}$ yields a new component with α cycles. By induction hypothesis, the claim holds for the resulting component. It remains to check edges in $F_{\alpha+1}$. If there exists an edge in $F_{\alpha+1}$ violating the claim, we relabel vertices and edges in $C_{\alpha+1}$.

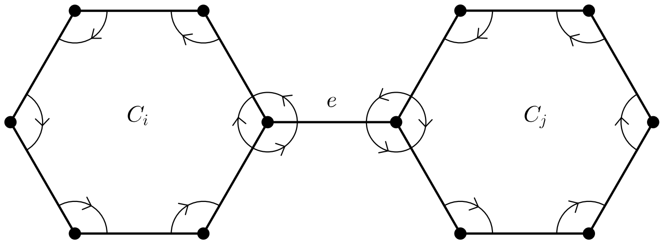

Figure 2: Case 4

After at most one relabeling, all edges in $F_{\alpha+1}$ satisfy the claim. We prove it by contradiction. Let $f_1, f_2 \in F_{\alpha+1}$ be edges such that f_1 satisfies the claim but f_2 violates the claim. For $i = 1, 2$, let $f_i = u_i w_i$, where u_i is the endpoint in the resulting component and w_i is the endpoint in $C_{\alpha+1}$. By assumption, u_1 and w_1 have different parity and u_2 and w_2 have the same parity.

Analogous to the definition of cycles with cyclic preferences, we call $P = v_1 v_2 \dots v_l$ a $v_1 v_l$-*path with linear preferences* if $v_i v_{i+1} \prec_{v_{i+1}} v_{i+1} v_{i+2}$ for $i = 1, 2, \dots, l-2$. We show that for any two vertices in the resulting component, there exists a path with linear preferences between them. Moreover, if they have the same parity, the path is of even length; if they have different parity, the path is of odd length.

When vertices are from the same cycle C_k where $k \in \{1, 2, \dots, \alpha\}$, it is trivial. Hence take $v^s \in C_s$ and $v^t \in C_t$ where $s, t \in \{1, 2, \dots, \alpha\}$ and $s \neq t$. The existence of paths with linear preferences is trivial. Without loss of generality, consider a $v^s v^t$-path with linear preferences. We apply induction on the number τ of cycles from $E_{1/2}(x)$ involved in the $v^s v^t$-path. Clearly, $\tau \geq 2$. When $\tau = 2$. Take $v^s_+ v^t_- \in F_s \cap F_t$. Let P_s be a path with linear preferences from v^s to v^s_+ in C_s and P_t be a path with linear preferences from v^t_- to v^t in C_t. It follows that $v^s P_s v^s_+ v^t_- P_t v^t$ constitute a $v^s v^t$-path P with linear preference. By assumption, v^s_+ and v^t_- have different parity since $v^s_+ v^t_- \in F_s \cap F_t$. If v^s and v^t have the same parity, then lengths of P_s and P_t have different parity, implying that P is of even length; if v^s and v^t have different parities, then lengths of P_s and P_t have the same parity, implying that P is of odd length. Now assume $\tau \geq 2$. Let $C_{k_1}, \dots, C_{k_\tau}, C_{k_{\tau+1}}$ be cycles from $E_{1/2}(x)$ involved along a $v^s v^t$-path with linear preferences. Take $v^{k_\tau}_+ v^{k_{\tau+1}}_- \in F_{k_\tau} \cap F_{k_{\tau+1}}$. Let P_{s,k_τ} denote a $v^s v^{k_\tau}_+$-path and $P_{k_\tau,t}$ denote a $v^{k_\tau}_+ v^t$-path, both of which have linear preferences. Clearly, $P = v^s P_{s,k_\tau} v^{k_\tau}_+ P_{k_\tau,t} v^t$ is a $v^s v^t$-path with linear preferences. Since P_{s,k_τ} is a path involving only τ cycles and $P_{k_\tau,t}$ is a path involving only two cycles, their lengths both depend on the parity of their endpoints. It follows that P is of even length when v^s and v^t have the same parity and P is of odd length when v^s and v^t have different parity.

Hence there exists a $u_1 u_2$-path P_α with linear preferences and a $w_2 w_1$-path $P_{\alpha+1}$ with linear preferences. Now $u_1 P_\alpha u_2 f_2 w_2 P_{\alpha+1} w_1 f_1 u_1$ constitute a cycle with cyclic preferences. We show that this cycle is odd, which leads to a contradiction.

If u_1 and u_2 have the same parity, then w_1 and w_2 have different parity, implying that P_α is of even length and $P_{\alpha+1}$ is of odd length; if u_1 and u_2 have different parity, then w_1 and w_2 have the same parity, implying that P_α is of odd length and $P_{\alpha+1}$ is of even length. Either case yields an odd cycle with cyclic preferences.

Therefore 1/2-integral points are not vertices of $FSM(G, \prec)$ as they can be perturbed by ϵz for small ϵ without leaving $FSM(G, \prec)$. By Theorem 2.3, $SM(G, \prec) = FSM(G, \prec)$ follows. $\square$

Now we are ready to present a proof to our main theorem.

Proof of Theorem 1.2. It suffices to show the equivalence of (i), (iii) and (iv). Without loss of generality, assume that D is an orientation of line multigraph $L(H)$ such that any two vertices in H are joined by at most two edges and parallel edges in $L(H)$ are orientated oppositely. When D is good, D is associated with a preference system $(H, \prec)$ and a simple preference system $(\hat{H}, \hat{\prec})$, both of which admit no odd cycles with cyclic preferences, where $\hat{H}$ is a simple spanning subgraph of H maximizing the edge set and $\hat{\prec}$ is the restriction of $\prec$ on $\hat{H}$. Hence $\sigma(D)$ can be viewed as a linear system defined on preference system $(H, \prec)$ and consisting of constraints (3.1)-(3.5), where constraints (3.1), (3.2) and (3.5) constitute the Rothblum system $\pi(H, \prec)$.

By Lemma 4.1, $FSM(\hat{H}, \hat{\prec})$ is integral. Integrality of $FSM(H, \prec)$ follows from Lemma 3.4. Hence constraints (3.3) and (3.4) are both redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$. Therefore $FK(D) = FSM(H, \prec)$, implying that $FK(D)$ is integral. Similar arguments apply to any induced subgraphs of D. Hence $(i) \implies (iii)$.

By Theorem 2.4, integrality of $FSM(\hat{H}, \hat{\prec})$ implies total dual integrality of $\pi(\hat{H}, \hat{\prec})$. Total dual integrality of $\pi(H, \prec)$ follows from Lemma 3.3. Since $\pi(H, \prec)$ is part of $\sigma(D)$ and the other constraints (3.3)-(3.4) are redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$, total dual integrality of $\sigma(D)$ follows. Similar arguments apply to any induced subgraphs of D. Hence $(iii) \implies (iv)$.

By a theorem of Edmonds and Giles [6], $(iv) \implies (iii)$ follows directly.

To see $(iii) \implies (i)$, we prove it by contradiction. Observe that strong kernel idealness of D implies the existence of kernels for any induced subgraphs of D. Let D be a digraph such that D is strongly kernel ideal but not good. Then there exists either a clique containing directed cycles or a directed odd cycle without chords nor pseudochords in D. We show that neither case is possible. If D has a clique containing directed cycles, we consider the subgraph induced on this clique. There is no kernel for this induced subgraph, a contradiction. If D contains a directed odd cycle without chords nor pseudochords, we restrict ourselves to the subgraph induced on this directed odd cycle. There is no kernel for this induced subgraph either, a contradiction. □

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