On Kernel Mengerian Orientations of Line Multigraphs

Han Xiao *

Department of Mathematics,
The University of Hong Kong,
Hong Kong, China

Abstract

We present a polyhedral description of kernels in orientations of line multigraphs. Given a digraph D, let $FK(D)$ denote the fractional kernel polytope defined on D, and let $\sigma(D)$ denote the linear system defining $FK(D)$. A digraph D is called kernel perfect if every induced subdigraph D' has a kernel, called kernel ideal if $FK(D')$ is integral for each induced subdigraph D', and called kernel Mengerian if $\sigma(D')$ is TDI for each induced subdigraph D'. We show that an orientation of a line multigraph is kernel perfect iff it is kernel ideal iff it is kernel Mengerian. Our result strengthens the theorem of Borodin *et al.* [3] on kernel perfect digraphs and generalizes the theorem of Király and Pap [7] on stable matching problem.

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*hxiao.math@connect.hku.hk

1 Introduction

An undirected graph is called *simple* if it contains neither loops nor parallel edges and is called a *multigraph* if parallel edges are allowed. *Simple* digraphs and *directed multigraphs* are defined analogously.

Let G be an undirected graph. The *line graph* of G, denoted by $L(G)$, is an undirected graph that represents the adjacency of edges of G: each vertex of $L(G)$ corresponds to an edge of G, and two vertices of $L(G)$ are adjacent if and only if they are incident as edges in G. We call $L(G)$ the *line multigraph* of G if any two vertices of $L(G)$ are connected by as many edges as the number of their common ends in G. We call G a *root* of $L(G)$.

Let $D = (V, A)$ be a digraph. For $U \subseteq V$, we call U an *independent* set of D if no two vertices in U are connected by an arc, call U a *dominating* set of D if for each vertex $v \notin U$, there is an arc from v to U, and call U a *kernel* of D if it is both independent and dominating. We call D *kernel perfect* if each of its induced subdigraphs has a kernel. A *clique* of D is a subset of V such that any two vertices are connected by an arc. We call D *clique-acyclic* if for each clique of D the induced subdigraph of one-way arcs is acyclic, and call D *good* if it is clique-acyclic and every directed odd cycle has a chord or a pseudo-chord[1].

Theorem 1.1 (Borodin *et al.* [3])**.** *Let G be a line multigraph. The orientation D of G is kernel perfect if and only if it is good.*

A subset P of $\mathbb{R}^n$ is called a *polytope* if it is the convex hull of finitely many vectors in $\mathbb{R}^n$. A point x in P is called a *vertex* or an *extreme point* if there exist no distinct points y and z in P such that $x = \alpha y + (1-\alpha)z$ for $0 < \alpha < 1$. It is well known that P is the convex hull of its vertices, and that there exists a linear system $Ax \le b$ such that $P = \{x \in \mathbb{R}^n : Ax \le b\}$. We say P is $1/k$-*integral* if its vertices are $1/k$-integral vectors, where $k \in \mathbb{N}$. By a theorem in linear programming, P is $1/k$-integral if and only if $\max\{c^T x : Ax \le b\}$ has an optimal $1/k$-integral solution for every integral vector c for which the optimum is finite. If, instead, $\max\{c^T x : Ax \le b\}$ has a dual optimal $1/k$-integral solution, we say $Ax \le b$ is *totally dual $1/k$-integral* (TDI/k). It is easy to verify that $Ax \le b$ is TDI/k if and only if $Bx \le b$ is TDI, where $B = A/k$ and $k \in \mathbb{N}$. Thus from a theorem of Edmonds and Giles [6], we deduce that if $Ax \le b$ is TDI/k and b is integral, then $P = \{x \in \mathbb{R}^n : Ax \le b\}$ is $1/k$-integral.

[1] A pseudo-chord is an arc (v_i, v_{i-1}) in a directed cycle $v_1 v_2 \ldots v_l$.

Let $\sigma(D)$ denote the linear system consisting of the following inequalities:

$$x(v) + x(N_D^+(v)) \geq 1 \qquad \forall\ v\ \in V, \tag{1.1}$$

$$x(Q) \leq 1 \qquad \forall\ Q \in \mathcal{Q}, \tag{1.2}$$

$$x(v) \geq 0 \qquad \forall\ v\ \in V, \tag{1.3}$$

where $x(U) = \sum_{u \in U} x(u)$ for any $U \subseteq V$, $N_D^+(v)$ denotes the set of all out-neighbors of vertex v, and $\mathcal{Q}$ denotes the set of all cliques of D. Observe that incidence vectors of kernels of D are precisely integral solutions $x \in \mathbb{Z}^A$ to $\sigma(D)$. The *kernel polytope* of D, denoted by $K(D)$, is the convex hull of incidence vectors of all kernels of D. The *fractional kernel polytope* of D, denoted by $FK(D)$, is the set of all solutions $x \in \mathbb{R}^A$ to $\sigma(D)$. Clearly, $K(D) \subseteq FK(D)$. We call D *kernel ideal* if $FK(D')$ is integral for each induced subdigraph D', and *kernel Mengerian* if $\sigma(D')$ is TDI for each induced subdigraph D'.

As described in Egres Open [1], the polyhedral description of kernels remains open. Chen *et al.* [4] attained a polyhedral characterization of kernels by replacing clique constraints $x(Q) \leq 1$ for $Q \in \mathcal{Q}$ with independence constraints $x(u) + x(v) \leq 1$ for $(u, v) \in A$. In this paper we show that kernels in digraph D can be fully characterized by polyhedral approaches if D is an orientation of line multigraphs.

Theorem 1.2. *Let D be an orientation of a line multigraph. Then the following statements are equivalent:*

(i) *D is good;*

(ii) *D is kernel perfect;*

(iii) *D is kernel ideal;*

(iv) *D is kernel Mengerian.*

The equivalence of (i) and (ii) was established by Borodin *et al.* [3] (Maffray [8] proved the case when D is perfect). Király and Pap [7] proved Theorem 1.2 for the case when the root of D is bipartite. Our result strengthens the theorem of Borodin *et al.* [3] and generalizes the theorem of Király and Pap [7] to line multigraphs.

2 Preliminaries

Let $G = (V, E)$ be an undirected graph. For $v \in V$, let $\delta(v)$ denote the set of all edges incident to v and $\prec_v$ be a strict linear order on $\delta(v)$. We call $\prec_v$ the *preference* of v, and for edges e and

f incident to v we say v *prefers* e to f or e *dominates* f if $e \preceq_v f$. Let $\prec$ be the set of all $\prec_v$ for $v \in V$. We call $(G, \prec)$ a *preference system* and say $(G, \prec)$ is *simple* if G is simple. For $e \in E$, let $\varphi(e)$ denote the set of all edges that dominate e in $(G, \prec)$. Let M be a matching of G. We say M is *stable* in $(G, \prec)$ if every edge of G is dominated by some edge in M.

Let $\pi(G, \prec)$ denote the linear system consisting of the following inequalities:

$$x(\varphi(e)) \geq 1 \qquad \forall\ e \in E, \tag{2.1}$$

$$x(\delta(v)) \leq 1 \qquad \forall\ v \in V, \tag{2.2}$$

$$x(e) \geq 0 \qquad \forall\ e \in E. \tag{2.3}$$

As observed by Abeledo and Rothblum [2], incidence vectors of stable matchings of $(G, \prec)$ are precisely integral solutions $x \in \mathbb{Z}^E$ to $\pi(G, \prec)$. The *stable matching polytope*, denoted by $SM(G, \prec)$, is the convex hull of incidence vectors of all stable matchings of $(G, \prec)$. The *fractional stable matching polytope*, denoted by $FSM(G, \prec)$, is the set of all solutions $x \in \mathbb{R}^E$ to $\pi(G, \prec)$. Clearly, $SM(G, \prec) \subseteq FSM(G, \prec)$.

Theorem 2.1 (Rothblum [9])**.** *Let $(G, \prec)$ be a simple preference system. If G is bipartite, then $SM(G, \prec) = FSM(G, \prec)$.*

Theorem 2.2 (Király and Pap [7])**.** *Let $(G, \prec)$ be a simple preference system. If G is bipartite, then $\pi(G, \prec)$ is totally dual integral.*

Let $C = v_1 v_2 \dots v_l$ be a cycle in G, we say that C has *cyclic preferences* in $(G, \prec)$ if $v_{i-1}v_i \prec_{v_i} v_i v_{i+1}$ for $i = 1, 2, \dots, l$ or $v_{i-1}v_i \succ_{v_i} v_i v_{i+1}$ for $i = 1, 2, \dots, l$, where indices are taken modulo l. For $x \in FSM(G, \prec)$, let $E_\alpha(x)$ denote the set of all edges with $x(e) = \alpha$ where $\alpha \in \mathbb{R}$ and $E_+(x)$ denote the set of all edges with $x(e) > 0$.

Theorem 2.3 (Abeledo and Rothblum [2])**.** *Let $(G, \prec)$ be a simple preference system. Then $FSM(G, \prec)$ is $1/2$-integral. Moreover, for each $1/2$-integral point x in $FSM(G, \prec)$, $E_{1/2}(x)$ consists of vertex disjoint cycles with cyclic preferences.*

Theorem 2.4 (Chen *et al.* [5])**.** *Let $(G, \prec)$ be a simple preference system. Then $\pi(G, \prec)$ is totally dual $1/2$-integral. Moreover, $\pi(G, \prec)$ is totally dual integral if and only if $SM(G, \prec) = FSM(G, \prec)$.*

3 Reductions

To study kernels in directed multigraphs, it suffices to work on its minimal spanning subdigraph that preserves all the connection relations of vertices. In the remainder of this section, assume that D is an orientation of line multigraph $L(H)$ such that any two vertices in H are joined by at most two edges and parallel edges in $L(H)$ are orientated oppositely. Hence D is a directed multigraph such that any two distinct vertices are joined by at most one arc in each direction.

Kernels are closely related to stable matchings. When D is clique-acyclic, let $e \prec_v f$ if (e, f) is an arc in D for any two incident edges e and f with common end v in H. Hence each clique-acyclic orientation D of line multigraph $L(H)$ is associated with a preference system $(H, \prec)$. Recall that $\sigma(D)$ denotes the linear system which defines $FK(D)$. Consequently, $\sigma(D)$ can be viewed as a linear system defined on preference system $(H, \prec)$. The equivalence of constraints (1.3) and constraints (2.3) follows directly. Constraints (1.1) can be viewed as constraints (2.1) because of the one to one correspondence between dominating vertex set $\{v\} \cup N_D^+(v)$ for $v \in V(D)$ and stable edge set $\varphi(e)$ for $e \in E(H)$. Observe that cliques of D correspond to three types of edge sets in H:

- $\delta(v)$ for $v \in V(H)$,
- nontrivial subsets of $\delta(v)$ for $v \in V(H)$,
- complete subgraphs of H induced on three vertices (with parallel edges allowed).

Hence constraints (1.2) can be viewed as constraints (2.2) together with some extra constraints on $(H, \prec)$. Let $\mathcal{O}(H)$ denote the set of all complete subgraphs of H induced on three vertices. Then $\sigma(D)$ can be reformulated in terms of preference system $(H, \prec)$:

$$x(\varphi(e)) \geq 1 \qquad \forall\, e \in E(H), \tag{3.1}$$

$$x(\delta(v)) \leq 1 \qquad \forall\, v \in V(H), \tag{3.2}$$

$$x(S) \leq 1 \qquad \emptyset \subset S \subset \delta(v), \quad \forall\, v \in V(H), \tag{3.3}$$

$$x(O) \leq 1 \qquad \forall\, O \in \mathcal{O}(H), \tag{3.4}$$

$$x(e) \geq 0 \qquad \forall\, e \in E(H). \tag{3.5}$$

Notice that constraints (3.1), (3.2) and (3.5) constitute the Rothblum system $\pi(H, \prec)$ which defines $FSM(H, \prec)$. Constraints (3.3) are redundant with respect to $\pi(H, \prec)$ due to constraints (3.2). As we shall see, constraints (3.4) are also redundant with respect to $\pi(H, \prec)$ when D is good. Hence $FK(D)$ is essentially defined by Rothblum system $\pi(H, \prec)$, or equivalently, that $FK(D) = FSM(H, \prec)$, when D is good.

Lemma 3.1. *For parallel edges e and e' in H, there exists no edge f such that $e \prec_v f \prec_v e'$, where v is a common end of e and e'.*

Proof. Since D is clique-acyclic, by the construction of preference system $(H, \prec)$, the lemma follows directly. $\square$

By Lemma 3.1, parallel edges play the same role in preference system $(H, \prec)$. Hence we turn to study underlying simple preference systems of $(H, \prec)$. Let $(\hat{H}, \hat{\prec})$ be a simple preference system, where $\hat{H}$ is a spanning subgraph of H obtained by keeping one edge between every pair of adjacent vertices and $\hat{\prec}$ is the restriction of $\prec$ on $\hat{H}$. Before proceeding, we introduce a technical lemma.

Lemma 3.2. *Let*

$$Ax \leq b,\ x \geq 0 \tag{3.6}$$

and

$$\bar{A}\bar{x} \leq b,\ \bar{x} \geq 0 \tag{3.7}$$

be two linear systems, where $\bar{A}$ is obtained from A by duplicating some columns. If (3.6) is totally dual $1/k$-integral, then so is (3.7), where $k \in \mathbb{N}$.

Proof. It suffices to prove that the theorem holds for $\bar{A}$ with one duplicate column. Let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{n+1}) \in \mathbb{R}^{n+1}$ and $\bar{A} = (\bar{a}_1, \ldots, \bar{a}_{n+1}) = (A, a_k)$, where a_k is the kth column of A. Let $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_{n+1}) \in \mathbb{Z}^{n+1}$ be an integral vector such that $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$ is finite.

Define $\hat{c} \in \mathbb{Z}^n$ by

$$\hat{c}_i := \begin{cases} \bar{c}_i & i \neq k, \\ \max\{\bar{c}_k, \bar{c}_{n+1}\} & i = k, \end{cases}$$

and define $\hat{x} \in \mathbb{R}^n$ by

$$\hat{x}_i := \begin{cases} \bar{x}_i & i \neq k, \\ \bar{x}_k + \bar{x}_{n+1} & i = k. \end{cases}$$

Clearly, $\max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\} \geq \max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$. We claim that equality always holds in this inequality. Given an optimal solution $\hat{x}^*$ to $\max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\}$, define $\bar{x}^*$ by $\bar{x}^*_i := \hat{x}^*_i$ for $i \neq k, n+1$; if $\bar{c}_k > \bar{c}_{n+1}$, $\bar{x}^*_k := \hat{x}^*_k$ and $\bar{x}^*_{n+1} := 0$, otherwise $\bar{x}^*_k := 0$ and $\bar{x}^*_{n+1} := \hat{x}^*_k$. It is easy to verify that $\bar{x}^*$ is a feasible solution to $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$. Moreover, $\hat{c}^T\hat{x}^* = \bar{c}^T\bar{x}^*$ implies $\max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\} \leq \max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$. Hence equality follows.

Since $Ax \leq b$ is TDI/k, there exists a dual optimal $1/k$-integral solution y^* to $\max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\}$ such that $(y^*)^Tb = \max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\}$, $(y^*)^TA \geq \hat{c}^T$ and $y^* \geq 0$. It follows that

$$(y^*)^T\bar{A} = (y^*)^T(A, a_k) = ((y^*)^TA, (y^*)^Ta_k) \geq (\hat{c}^T, \hat{c}_k) \geq \bar{c}^T,$$

implying that y^* is a feasible dual solution to $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$. By the following inequalities

$$(y^*)^T b = \max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\} = \max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\} \leq (y^*)^T b,$$

y^* is also a dual optimal solution to $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$, where the last inequality is from the weak duality theorem. Hence the lemma follows. □

Lemma 3.3. *If $\pi(\hat{H}, \hat{\prec})$ is totally dual $1/k$-integral, then so is $\pi(H, \prec)$, where $k \in \mathbb{N}$.*

Proof. By Lemma 3.1, columns corresponding to parallel edges in the left hand side matrix of $\pi(H, \prec)$ are identical. Hence the left hand side matrix of $\pi(H, \prec)$ can be obtained from that of $\pi(\hat{H}, \hat{\prec})$ by duplicating columns corresponding to parallels edges. Then the lemma follows from Lemma 3.2. □

By Theorem 2.4 and Lemma 3.3, we deduce that $\pi(H, \prec)$ is TDI/2. By the construction of $(H, \prec)$, when D is good, $(H, \prec)$ admits no odd cycles with cyclic preferences. Hence constraints (3.4) are redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$ when D is good.

Lemma 3.4. *$FSM(H, \prec)$ is integral if and only if $FSM(\hat{H}, \hat{\prec})$ is integral.*

Proof. For simplicity, assume that $H = (V, E)$ and $\hat{H} = (V, \hat{E})$.

We prove the "only if" part by showing that $FSM(\hat{H}, \hat{\prec})$ is a projection of $FSM(H, \prec)$. Let $\hat{x}$ be a point in $FSM(\hat{H}, \hat{\prec})$ and define $x := (\hat{x}, 0) \in \mathbb{R}^{\hat{E}} \times \mathbb{R}^{E-\hat{E}}$. To show that $x \in FSM(H, \prec)$, it suffices to prove that $x(\varphi(e)) \geq 1$ for $e \in E - \hat{E}$. Let $e' \in \hat{E}$ be the edge parallel with $e \in E - \hat{E}$. By Lemma 3.1, $x(\varphi(e)) = x(\varphi(e')) = \hat{x}(\varphi(e')) \geq 1$. Hence $FSM(\hat{H}, \hat{\prec})$ is a projection of $FSM(H, \prec)$.

We prove the "if" part by showing that each vertex of $FSM(H, \prec)$ can be obtained from some vertex of $FSM(\hat{H}, \hat{\prec})$ by adding some zero entries. Let x be a vertex of $FSM(H, \prec)$. We first show that $x(e) \cdot x(e') = 0$ for parallel edges e and e'. Assume to the contrary that $x(e) \cdot x(e') \neq 0$, then $x(e) = x(e') = 1/2$ follows. Let y and z be duplicates of x, and further set $y(e) := z(e') := 1$ and $y(e') := z(e) := 0$. It follows that $x = (y + z)/2$. Clearly, $y, z \in FSM(H, \prec)$, contradicting to the fact that x is a vertex. Now define $\hat{x} \in \mathbb{R}^{\hat{E}}$ by, for $e \in \hat{E}$, $\hat{x}(e) := x(e) + x(e')$, where $e' \in E - \hat{E}$ is parallel with e. By Lemma 3.1, it is easy to verify that $\hat{x} \in FSM(\hat{H}, \hat{\prec})$. We claim that $\hat{x}$ is a vertex of $FSM(\hat{H}, \hat{\prec})$. Assume to the contrary that there exist $\hat{x}_1, \hat{x}_2 \in FSM(\hat{H}, \hat{\prec})$ such that $\hat{x} = \alpha\hat{x}_1 + (1 - \alpha)\hat{x}_2$, where $0 < \alpha < 1$. For $i = 1, 2$, we extend $\hat{x}_i \in \mathbb{R}^{\hat{E}}$ to $x_i \in \mathbb{R}^E$ by, for $e \in \hat{E}$ without parallel edges in H, $x_i(e) := \hat{x}_i(e)$; for $e \in \hat{E}$ and its parallel edge $e' \in E - \hat{E}$, if $x(e) > x(e')$, $x_i(e) := \hat{x}_i(e)$ and $x_i(e') := 0$, otherwise $x_i(e) := 0$ and $x_i(e') := \hat{x}_i(e)$. By

Lemma 3.1, it is easy to see that $x_1, x_2 \in FSM(H, \prec)$. Since $x(e) \cdot x(e') = 0$ for parallel edges e and e', $x = \alpha x_1 + (1-\alpha)x_2$ follows, a contradiction. Hence $\hat{x}$ is vertex of $FSM(\hat{H}, \hat{\prec})$. Therefore each vertex x of $FSM(H, \prec)$ is associated with a vertex $\hat{x}$ of $FSM(\hat{H}, \hat{\prec})$, and x can be obtained from $\hat{x}$ by, for $e \in \hat{E}$ without parallel edges in H, $x(e) := \hat{x}(e)$; for $e \in \hat{E}$ and its parallel edge $e' \in E - \hat{E}$, if $x(e) > x(e')$, $x(e) := \hat{x}(e)$ and $x(e') := 0$, otherwise $x(e) := 0$ and $x(e') := \hat{x}(e)$. □

We end this section with a summary. When D is clique-acyclic, it is associated with a preference system $(H, \prec)$ and a simple preference system $(\hat{H}, \hat{\prec})$, where $\hat{H}$ is a simple spanning subgraph of H maximizing the edge set and $\hat{\prec}$ is the restriction of $\prec$ on $\hat{H}$. Hence constraints (3.3) and (3.4) are redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$ and $FK(D) = FSM(H, \prec)$ follows. To show $FK(D)$ is integral, by Lemma 3.4 it suffices to show that $FSM(\hat{H}, \hat{\prec})$ is integral. To show $\sigma(D)$ is TDI, by Lemma 3.3 it suffices to show $\pi(\hat{H}, \hat{\prec})$ is TDI. Moreover, when D is good both $(H, \prec)$ and $(\hat{H}, \hat{\prec})$ admit no odd cycles with cyclic preferences.

4 Proofs

Before presenting our proof of the main theorem, we exhibit some properties of simple preference systems admitting no odd cycles with cyclic preferences.

Lemma 4.1. *Let $(G, \prec)$ be a simple preference system. If $(G, \prec)$ admits no odd cycles with cyclic preferences, then $SM(G, \prec) = FSM(G, \prec)$.*

By Theorem 2.4, integrality of $FSM(G, \prec)$ is equivalent to total dual integrality of $\pi(G, \prec)$, where $(G, \prec)$ is a simple preference system. A corollary follows directly.

Corollary 4.2. *Let $(G, \prec)$ be a simple preference system. If $(G, \prec)$ admits no odd cycles with cyclic preferences, then $\pi(G, \prec)$ is totally dual integral.*

Proof of Lemma 4.1. By Theorem 2.3, $FSM(G, \prec)$ is 1/2-integral as $(G, \prec)$ is a simple preference system. Let x be a 1/2-integral point in $FSM(G, \prec)$. Since $(G, \prec)$ admits no odd cycles with cyclic preferences, $E_{1/2}(x)$ consists of even cycles $C_1, C_2, \ldots, C_r$ with cyclic preferences. For $i = 1, 2, \ldots, r$, label vertices and edges of $C_i \in E_{1/2}(x)$ such that $C_i = v_1^i v_2^i \ldots v_l^i$ and $e_k^i \prec_{v_{k+1}^i} e_{k+1}^i$ for $k = 1, 2, \ldots, l$, where $e_k^i = v_k^i v_{k+1}^i$ and indices are taken modulo l. We remark that the parity of vertices and edges refers to the parity of their indices. Define $z \in \mathbb{R}^{E(G)}$ by

$$z(e) := \begin{cases} 1 & e \text{ is an even edge in some } C \in E_{1/2}(x), \\ -1 & e \text{ is an odd edge in some } C \in E_{1/2}(x), \\ 0 & \text{otherwise.} \end{cases}$$

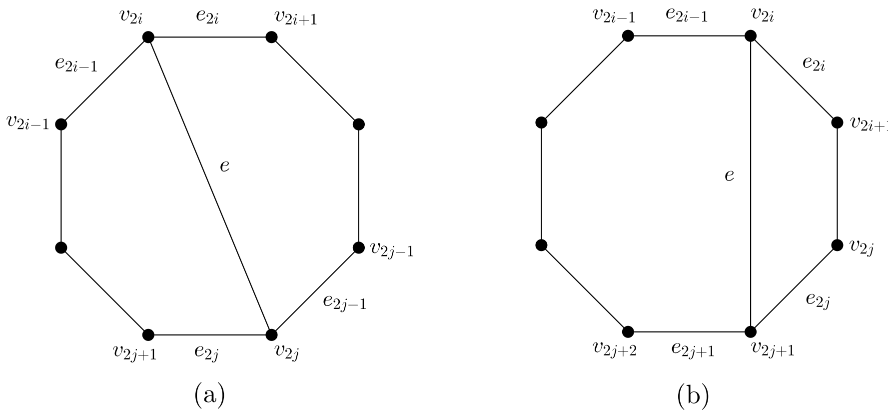

Figure 1: Case 2

We are going to exclude x from vertices of $FSM(G, \prec)$ by adding perturbation ϵz for small ϵ to x and showing that $x \pm \epsilon z \in FSM(G, \prec)$. Tight constraints in (2.1)-(2.3) under perturbation ϵz play a key role here. Observe that tight constraints in (2.2) and (2.3) are invariant under perturbation ϵz. It remains to show that perturbation ϵz does not affect tight constraints in (2.1) either. Let e be an edge with $x(\varphi(e)) = 1$. Clearly, $|\varphi(e) \cap E_+(x)| \in \{1, 2\}$. When $|\varphi(e) \cap E_+(x)| = 1$, $x(e) = 1$ follows, which is trivial. When $|\varphi(e) \cap E_+(x)| = 2$, we claim that the parity of dominating edges in $E_{1/2}(x)$ of e does not agree (relabeling vertices and edges in $E_{1/2}(x)$ if necessary). Hence corresponding tight constraints in (2.1) are also invariant under perturbation ϵz. To justify this claim, we distinguish four cases.

Case 1. Edge e is an edge from some $C \in E_{1/2}(x)$. This case is trivial since C admits cyclic preferences.

Case 2. Edge e is a chord in some $C \in E_{1/2}(x)$. We first show that endpoints of e have different parity in C. We prove it by contradiction. Without loss of generality, let $e = v_{2i}v_{2j}$.

If $e_{2i} \prec e$, then $e_{2i-1} \prec e$. Since $x(\varphi(e)) = 1$, it follows that $e \prec e_{2j-1}$ and $e \prec e_{2j}$. However, $v_{2i}ev_{2j}e_{2j}v_{2j+1} \dots v_{2i-1}e_{2i-1}v_{2i}$ constitute an odd cycle with cyclic preferences, a contradiction. Hence $e \prec e_{2i}$.

Similarly, if $e_{2j} \prec e$, then $e_{2j-1} \prec e$. Equality $x(\varphi(e)) = 1$ implies that $e \prec e_{2i}$ and $e \prec e_{2i-1}$. However, $v_{2i}e_{2i}v_{2i+1} \dots v_{2j-1}e_{2j-1}v_{2j}ev_{2i}$ constitute an odd cycle with cyclic preferences, a contradiction. Hence $e \prec e_{2j}$.

Now $e \prec e_{2i}$ and $e \prec e_{2j}$, it follows that $e_{2i-1} \prec e$ and $e_{2j-1} \prec e$ since $x(\varphi(e)) = 1$. But in this case two odd cycles with cyclic preferences mentioned above occur at the same time. Therefore, endpoints of e have different parity in C. Hence let $e = v_{2i}v_{2j+1}$. If $e_{2i} \prec e$ (*resp.* $e_{2j+1} \prec e$), it follows that $e_{2i-1} \prec e$ (*resp.* $e_{2j} \prec e$). Then e is dominated by two consecutive

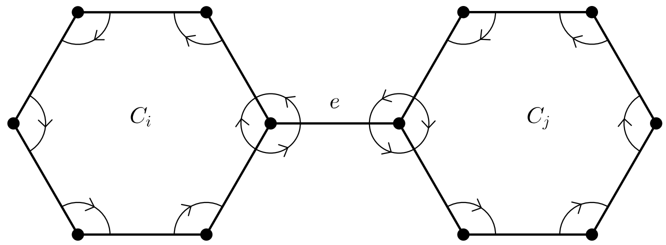

Figure 2: Case 4

edges from C, which is trivial. So assume that $e \prec e_{2i}$ and $e \prec e_{2j+1}$. Since $x(\varphi(e)) = 1$, it follows that $e_{2i+1} \prec e$ and $e_{2j} \prec e$. Therefore e is dominated by two edges with different parity.

Case 3. Edge e is a hanging edge of some $C \in E_{1/2}(x)$ and dominated by two edges from C. This case is trivial.

Case 4. Edge e is a connecting edge between C_i and C_j and dominated by one edge from C_i and one edge from C_j respectively, where $C_i, C_j \in E_{1/2}(x)$. For $k = 1, 2, \ldots, r$, let F_k be the subset of edges in this case and incident to C_k. Then $\cup_{i=1}^{i=r} F_i \cup C_i$ induces a subgraph of G. It suffices to work on a component of the induced subgraph. We apply induction on the number α of cycles from $E_{1/2}(x)$ in a component.

When $\alpha = 1$, it is trivial. Hence assume the claim holds for components with $\alpha \geq 1$ cycles from $E_{1/2}(x)$. We consider a component with $\alpha + 1$ cycles $C_1, \ldots, C_\alpha, C_{\alpha+1}$ from $E_{1/2}(x)$. Without loss of generality, assume that deleting $C_{\alpha+1}$ yields a new component with α cycles. By induction hypothesis, the claim holds for the resulting component. It remains to check edges in $F_{\alpha+1}$. If there exists an edge in $F_{\alpha+1}$ violating the claim, relabel vertices and edges in $C_{\alpha+1}$. After at most one relabeling, all edges in $F_{\alpha+1}$ satisfy the claim. We prove it by contradiction. Let $f_1, f_2 \in F_{\alpha+1}$ be edges such that f_1 satisfies the claim but f_2 violates the claim. For $i = 1, 2$, let $f_i = u_i w_i$, where u_i is the endpoint in the resulting component and w_i is the endpoint in $C_{\alpha+1}$. By assumption, u_1 and w_1 have different parity and u_2 and w_2 have the same parity. Analogous to the definition of cycles with cyclic preferences, we call path $P = v_1 v_2 \ldots v_l$ a *$v_1 v_l$-path with linear preferences* if $v_i v_{i+1} \prec_{v_{i+1}} v_{i+1} v_{i+2}$ for $i = 1, 2, \ldots, l-2$. Clearly, for any two vertices in the same component, there exists a path with linear preferences between them. Hence there exist a $u_1 u_2$-path P_α and a $w_2 w_1$-path $P_{\alpha+1}$, both of which admit linear preferences. Moreover, $u_1 P_\alpha u_2 f_2 w_2 P_{\alpha+1} w_1 f_1 u_1$ constitute a cycle with cyclic preferences. We justify this cycle is odd by showing that the $u_1 u_2$-path P_α is even (*resp.* odd) if u_1 and u_2 have the same (*resp.* different) parity.

If u_1 and u_2 belong to the same cycle from $E_{1/2}(x)$, it is trivial. Hence assume $u_1 \in C_s$ and $u_2 \in C_t$, where $s, t \in \{1, 2, \ldots, \alpha\}$ and $s \neq t$. We apply induction on the number τ of cycles from $E_{1/2}(x)$ involved in P_α. Clearly, $\tau \geq 2$. When $\tau = 2$. Take $v^s v^t \in F_s \cap F_t$ on P_α. Let P_s be the part of P_α from u_1 to v^s in C_s and P_t be the part of P_α from v^t to u_2 in C_t. It follows that $u_1 P_s v^s v^t P_t u_2$ constitute P_α. By primary induction hypothesis, v^s and v^t have different parity since $v^s v^t \in F_s \cap F_t$. If u_1 and u_2 have the same parity, then P_s and P_t have different parity, implying that P_α is even; if u_1 and u_2 have different parity, then P_s and P_t have the same parity, implying that P_α is odd. Now assume $\tau \geq 2$. Let $C_{k_1}, \ldots, C_{k_\tau}, C_{k_{\tau+1}}$ be cycles from $E_{1/2}(x)$ involved along P_α. Take $v^{k_\tau} v^{k_{\tau+1}} \in F_{k_\tau} \cap F_{k_{\tau+1}}$ on P_α. Let P_{s,k_τ} denote the part of P_α from u_1 to v^{k_τ} and $P_{k_\tau,t}$ denote the part of P_α from v^{k_τ} to u_2. Clearly, $P_\alpha = u_1 P_{s,k_\tau} v^{k_\tau} P_{k_\tau,t} u_2$. Since P_{s,k_τ} involves τ cycles and $P_{k_\tau,t}$ involves two cycles, both length depend on the parity of endpoints. It follows that P_α is even when u_1 and u_2 have the same parity, and P_α is odd when u_1 and u_2 have different parity.

Hence when u_1 and u_2 have the same parity, w_1 and w_2 have different parity, implying that P_α is even and $P_{\alpha+1}$ is odd; when u_1 and u_2 have different parity, w_1 and w_2 have the same parity, implying that P_α is odd and $P_{\alpha+1}$ is even. Either case yields an odd cycle with cyclic preferences, a contradiction.

Therefore 1/2-integral points are not vertices of $FSM(G, \prec)$ as they can be perturbed by ϵz for small ϵ without leaving $FSM(G, \prec)$. By Theorem 2.3, $SM(G, \prec) = FSM(G, \prec)$ follows. $\square$

Now we are ready to present a proof of our main theorem.

Proof of Theorem 1.2. It suffices to show the equivalence of (i), (iii) and (iv). Without loss of generality, let D be an orientation of line multigraph $L(H)$ such that any two vertices in H are joined by at most two edges and parallel edges in $L(H)$ are orientated oppositely. When D is good, D is associated with a preference system $(H, \prec)$ and a simple preference system $(\hat{H}, \hat{\prec})$, both of which admit no odd cycles with cyclic preferences, where $\hat{H}$ is a simple spanning subgraph of H maximizing the edge set and $\hat{\prec}$ is the restriction of $\prec$ on $\hat{H}$. Hence $\sigma(D)$ can be viewed as a linear system defined on preference system $(H, \prec)$ and consisting of constraints (3.1)-(3.5), where constraints (3.1), (3.2) and (3.5) constitute the Rothblum system $\pi(H, \prec)$.

By Lemma 4.1, $FSM(\hat{H}, \hat{\prec})$ is integral. Integrality of $FSM(H, \prec)$ follows from Lemma 3.4. Hence constraints (3.3) and (3.4) are both redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$. Therefore $FK(D) = FSM(H, \prec)$, implying that $FK(D)$ is integral. Similar arguments apply to any induced subdigraphs of D. Hence $(i) \implies (iii)$.

By Corollary 4.2, $\pi(\hat{H}, \hat{\prec})$ is TDI. Total dual integrality of $\pi(H, \prec)$ follows from Lemma 3.3. Since $\pi(H, \prec)$ is part of $\sigma(D)$ and the other constraints (3.3)-(3.4) are redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$, total dual integrality of $\sigma(D)$ follows. Similar arguments apply to any induced subdigraphs of D. Hence $(iii) \implies (iv)$.

By a theorem of Edmonds and Giles [6], implication $(iv) \implies (iii)$ follows directly.

To prove implication $(iii) \implies (i)$, we assume the contrary. Observe that strong kernel idealness of D implies the existence of kernels for any induced subdigraphs of D. Let D be a digraph such that D is kernel ideal but not good. Then there exists either a clique containing directed cycles or a directed odd cycle without chords nor pseudo-chords in D. We show that neither case is possible. If D has a clique containing directed cycles, we consider the subdigraph induced on this clique. There is no kernel for this induced subdigraph, a contradiction. If D contains a directed odd cycle without chords nor pseudo-chords, we restrict ourselves to the subdigraph induced on this directed odd cycle. There is no kernel for this induced subdigraph either, a contradiction. $\square$

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