Strongly Kernel Mengerian Orientations of Line Multigraphs

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Abstract

Given a digraph D, let $\sigma(D)$ denote the linear system consisting of domination, independence and nonnegativity constraints, and $FK(D)$ denote the set of all solutions to $\sigma(D)$. We call D strongly kernel ideal if $FK(D')$ is integral for each induced subgraph D' of D and strongly kernel Mengerian if $\sigma(D')$ is TDI for each induced subgraph D' of D. We also call D good if it is clique-acyclic and each directed odd cycle has a chord.

In this paper, we prove that a digraph is strongly kernel ideal iff it is strongly kernel Mengerian iff it is good. Our result strengthens the theorem of Borodin *et al.* [3] on kernel perfect digraphs and generalizes the theorem of Király and Pap [7] on stable matching problem.

1 Introduction

An undirected graph is called *simple* if it contains neither loops nor parallel edges and is called a *multigraph* if it contains no loops but parallel edges are allowed. *Simple* digraphs and *multidigraphs* are defined analogously.

Let G be an undirected graph. The *line graph* of G is an undirected simple graph that represents the adjacency of edges of G. The *line multigraph* of G is an undirected multigraph that represents the adjacency of edges of G and if edges e and f have two common ends in G, their corresponding vertices are joined by two edges. In this paper, line graphs and line multigraphs of G are both denoted by $L(G)$ if no ambiguity occurs. Observe that the line graph and line multigraph of G are the same when G is simple. And for any two vertices in a line multigraph, there exist at most two parallel edges between them.

Let $D = (V, A)$ be a digraph. For $U \subseteq V$, We call U an *independent* set of D if no two vertices in U are connected by an arc, call U a *dominating* set of D if for each vertex $v \notin U$, there is an arc from v to U, and call U a *kernel* of D if it is both independent and dominating. A *clique* of D is a subset of V such that every two distinct vertices are connected by an arc. We call D *clique-acyclic* if for every clique of D the induced subdigraph of one-way arcs is acyclic, and call D *good* if it is clique-acyclic and each directed odd cycle has a chord or a pseudochord[1].

Theorem 1.1 (Borodin *et al.* [3]). *Let G be a line multigraph. The orientation D of G is kernel perfect if and only if it is good.*

A subset P of $\mathbb{R}^n$ is called a *polytope* if it is the convex hull of finitely many vectors in $\mathbb{R}^n$. A point x in P is called a *vertex* or an *extreme point* if there exist no distinct points y and z in P such that $x = \alpha y + (1 - \alpha)z$ for $0 < \alpha < 1$. It is well known that P is the convex hull of its vertices, and that there exists a linear system $Ax \leq b$ such that $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. We call P *$1/k$-integral* if its vertices are $1/k$-integral vectors, where $k \in \mathbb{N}$. By a theorem in linear programming, P is $1/k$-integral if and only if $\max\{c^T x : Ax \leq b\}$ has an optimal $1/k$-integral solution, for every integral vector c for which the optimum is finite. If, instead, $\max\{c^T x : Ax \leq b\}$ has a dual optimal $1/k$-integral solution, we call linear system $Ax \leq b$ *totally dual $1/k$-integral* (TDI/k). It is easy to verify that $Ax \leq b$ is TDI/k if and only if $Bx \leq b$ is TDI, where $B = A/k$. Thus, from a theorem by Edmonds and Giles [6], we deduce that if $Ax \leq b$ is TDI/k and b is integal, then $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is $1/k$-integral.

Let $\sigma(D)$ denote the linear system consisting of the following inequalities:

$$x(v) + x(N_D^+(v)) \geq 1 \qquad \forall\ v\ \in V, \tag{1.1}$$

$$x(Q) \leq 1 \qquad \forall\ Q \in \mathcal{Q}, \tag{1.2}$$

$$x(v) \geq 0 \qquad \forall\ v\ \in V, \tag{1.3}$$

where $x(U) = \sum_{u \in U} x(u)$ for any $U \subseteq V$, $N_D^+(v)$ denotes the set of all out-neighbors of vertex v, and $\mathcal{Q}$ denotes the set of all cliques of D. Observe that incidence vectors of kernels of D are precisely integral solutions $x \in \mathbb{Z}^A$ to $\sigma(D)$.

The *kernel polyotpe* of D, denoted by $K(D)$, is the convex hull of incidence vectors of all kernels of D. The *fractional kernel polytope* of D, denoted by $FK(D)$, is the set of all solutions $x \in \mathbb{R}^A$ to $\sigma(D)$. Clearly, $K(D) \subseteq FK(D)$.

We call D *kernel perfect* if each of its induced subgraphs has a kernel, *strongly kernel ideal* if $FK(D')$ is integral for each induced subgraph D' of D, and *strongly kernel Mengerian* if $\sigma(D')$ is TDI for each induced subgraph D' of D.

As described in Egres Open [1], the polyhedral description of kernels remains open. Chen *et al.* [4] attained a polyhedral characterization of kernels by replacing clique constraints $x(Q) \leq 1$ for $Q \in \mathcal{Q}$ with independence constraints $x(u) + x(v) \leq 1$ for $(u, v) \in A$. In this paper we show

[1]A pseudochord is an arc (v_i, v_{i-1}) in a directed cycle $v_1 v_2 \ldots v_l$

that kernels in digraph D can be thoroughly characterized by polyhedral approaches if D is an orientation of some line multigraph.

Theorem 1.2. *Let G be a line multigraph, and let D be an orientation of G. Then the following statements are equivalent:*

(i) *D is good;*

(ii) *D is kernel perfect;*

(iii) *D is strongly kernel ideal;*

(iv) *D is strongly kernel Mengerian.*

The equivalence of (i) and (ii) was established by Borodin *et al.* [3] (Maffray [8] proved the case when G is perfect). Király and Pap [7] proved the theorem when G is the line graph of a bipartite graph, which will be given in next section. Our theorem strengthens the result of Borodin *et al.* and generalizes the result of Király and Pap to general line multigraphs.

2 Preliminaries

Let $G = (V, E)$ be an undirected graph. For each $v \in V$, let $\delta(v)$ be the set of all edges incident to v and let $\prec_v$ be a strict linear order on $\delta(v)$. We call $\prec_v$ the *preference* of v and say that v *prefers* e to f if $e \prec_v f$. Let $\prec$ be collection of all these $\prec_v$ for $v \in V$. We call $(G, \prec)$ a *preference system* and call $(G, \prec)$ *simple* if G is simple. Edge e is said to *dominate* edge f if they have a common end v such that $e \preceq_v f$. Let M be a matching of G. We call M *stable* if each edge of G is dominated by some edge in M.

Let $(G, \prec)$ be a preference system. For each $e \in E$, let $\varphi(e)$ denote the set of all edges of G that dominate e. Let $\pi(G, \prec)$ be the linear system consisting of the following inequalites:

$$x(\varphi(e)) \geq 1 \qquad \forall\, e \in E, \tag{2.1}$$

$$x(\delta(v)) \leq 1 \qquad \forall\, v \in V, \tag{2.2}$$

$$x(e) \geq 0 \qquad \forall\, e \in E. \tag{2.3}$$

As observed by Abeledo and Rothblum [2], incidence vectors of stable matchings of $(G, \prec)$ are precisely integral solutions $x \in \mathbb{Z}^E$ of $\pi(G, \prec)$.

The *stable matching polytope*, denoted by $SM(G, \prec)$, is the convex hull of incidence vectors of all stable matchings of $(G, \prec)$. The *fractional stable matching polytope*, denoted by $FSM(G, \prec)$, is the set of all solutions $x \in \mathbb{R}^E$ to $\pi(G, \prec)$. Clearly, $SM(G, \prec) \subseteq FSM(G, \prec)$.

Theorem 2.1 (Rothblum [9])**.** *Let $(G, \prec)$ be a simple preference system. If G is bipartite, then $SM(G, \prec) = FSM(G, \prec)$.*

Theorem 2.2 (Király and Pap [7])**.** *Let* $(G, \prec)$ *be a simple preference system. If* G *is bipartite, then* $\pi(G, \prec)$ *is totally dual integral.*

Let $C = v_1 v_2 \dots v_l$ be a cycle in G, we say that C has *cyclic preferences* in $(G, \prec)$ if $v_{i-1}v_i \prec_{v_i} v_i v_{i+1}$ for $i = 1, 2, \dots, l$ or $v_{i-1}v_i \succ_{v_i} v_i v_{i+1}$ for $i = 1, 2, \dots, l$, where indices are taken modulo l. For $x \in FSM(G, \prec)$, let $E_\alpha(x)$ denote the set of all edges with $x(e) = \alpha$ where $\alpha \in \mathbb{R}$ and $E_+(x)$ denote the set of all edges with $x(e) > 0$.

Theorem 2.3 (Abeledo and Rothblum [2])**.** *Let* $(G, \prec)$ *be a simple preference system. Then* $FSM(G, \prec)$ *is* $1/2$*-integral. Moreover, for each* $1/2$*-integral point* x *in* $FSM(G, \prec)$*,* $E_{1/2}(x)$ *consists of vertex disjoint cycles with cyclic preferences.*

Theorem 2.4 (Chen *et al.* [5])**.** *Let* $(G, \prec)$ *be a simple preference system. Then* $\pi(G, \prec)$ *is totally dual* $1/2$*-integral. Moreover,* $\pi(G, \prec)$ *is totally dual integral if and only if* $SM(G, \prec) = FSM(G, \prec)$*.*

3 Reductions

To study kernels in multidigraphs, it suffices to work on its minimal spanning subdigraph that preserves all the connection relations among vertices. In the remainder of this paper, we assume that D is an orientation of a line multigraph $L(H)$ such that any two distinct vertices in H are joined by at most two edges and parallel edges in $L(H)$ are orientated oppositely. Hence D is a multidigraph such that any two vertices are joined by at most one arc in each direction.

Kernels are closely related to stable matchings. Let D be a clique-acyclic orientation of line multigraph $L(H)$. For any two adjacent edges e and f in H, define $e \prec_v f$ if (e, f) is an arc in D. Hence each clique-acyclic orientation D of line multigraph $L(H)$ is associated with a preference system $(H, \prec)$. Recall that $\sigma(D)$ denotes the linear system which defines $FK(D)$. Consequently, $\sigma(D)$ can be interpreted in terms of preference system $(H, \prec)$. The equivalence of constraints (1.3) and constraints (2.3) follows directly. Constraints (1.1) can be viewed as constraints (2.1) because of the one to one correspondence between dominating vertex set $\{v\} \cup N_D^+(v)$ for $v \in V(D)$ and stable edge set $\varphi(e)$ for $e \in E(H)$. Observe that cliques of D correspond to three types of edge sets in H:

- $\delta(v)$ for $v \in V(H)$,
- nontrivial subsets of $\delta(v)$ for $v \in V(H)$,
- complete subgraphs of H induced on three vertices (with parallel edges allowed).

Hence constraints (1.2) can be viewed as constraints (2.2) together with some extra constraints on $(H, \prec)$. Let $\mathcal{O}(H)$ denote the set of all complete subgraphs of H induced on three vertices. It follows that $\sigma(D)$ can be viewed as a linear system defined on preference system $(H, \prec)$ by the following inequalities:

$$
\begin{aligned}
x(\varphi(e)) &\geq 1 & & \forall\ e \in E(H), & (3.1)\\
x(\delta(v)) &\leq 1 & & \forall\ v \in V(H), & (3.2)\\
x(S) &\leq 1 & \emptyset \subset S \subset \delta(v), & \forall\ v \in V(H), & (3.3)\\
x(O) &\leq 1 & & \forall\ O \in \mathcal{O}(H), & (3.4)\\
x(e) &\geq 0 & & \forall\ e \in E(H). & (3.5)
\end{aligned}
$$

Notice that constraints (3.1), (3.2) and (3.5) constitute the Rothblum system $\pi(H, \prec)$ which defines $FSM(H, \prec)$. Constraints (3.3) are redundant with respect to $\pi(H, \prec)$ due to constraints (3.2). As we shall see, constraints (3.4) are also redundant with respect to $\pi(H, \prec)$ when D is good. Hence $FK(D)$ is essentially defined by Rothblum system $\pi(H, \prec)$, or equivalently $FK(D) = FSM(H, \prec)$, when D is good.

Lemma 3.1. *For parallel edges e and e' in H, there exists no edge f such that $e \prec_v f \prec_v e'$, where v is an endpoint of e and e'.*

Proof. Since D is clique-acyclic, by the construction of preference system $(H, \prec)$, the lemma follows directly. □

By Lemma 3.1, parallel edges play exactly the same role in preference system $(H, \prec)$. Hence we turn to study simple preference systems defined on its underlying simple graphs. Let $(\hat{H}, \hat{\prec})$ be a simple preference system, where $\hat{H}$ is a spanning subgraph of H obtained by keeping one edge between every pair of adjacent vertices and $\hat{\prec}$ is the restriction of $\prec$ on $\hat{H}$. Before proceeding, we introduce a technical lemma first.

Lemma 3.2. *Let*

$$Ax \leq b,\ x \geq 0 \tag{3.6}$$

and

$$\bar{A}\bar{x} \leq b,\ \bar{x} \geq 0 \tag{3.7}$$

be two linear systems, where $\bar{A}$ is obtained from A by duplicating some columns. If (3.6) is totally dual $1/k$-integral, then so is (3.7), where $k \in \mathbb{N}$.

Proof. It suffices to prove that the theorem holds for $\bar{A}$ with one duplicate column. Let $\bar{x} = (\bar{x}_1, \dots, \bar{x}_{n+1}) \in \mathbb{R}^{n+1}$ and $\bar{A} = (\bar{a}_1, \dots, \bar{a}_{n+1}) = (A, a_k)$, where a_k is the kth column of A. Let $\bar{c} = (\bar{c}_1, \dots, \bar{c}_{n+1}) \in \mathbb{Z}^{n+1}$ be an arbitrary integral vector such that $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$ is finite.

Let $\hat{c} \in \mathbb{Z}^n$ be defined by

$$\hat{c}_i = \begin{cases} \bar{c}_i & i \neq k, \\ \max\{\bar{c}_k, \bar{c}_{n+1}\} & i = k, \end{cases}$$

and $\hat{x} \in \mathbb{R}^n$ be defined by

$$\hat{x}_i = \begin{cases} \bar{x}_i & i \neq k, \\ \bar{x}_k + \bar{x}_{n+1} & i = k. \end{cases}$$

Clearly, $\max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\} \geq \max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$. We claim that equality always holds in this inequality. Given an arbitrary optimal solution $\hat{x}^*$ to $\max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\}$, define $\bar{x}^*$ by $\bar{x}^*_i = \hat{x}^*_i$ for $i \neq k, n+1$, if $\bar{c}_k > \bar{c}_{n+1}$, $\bar{x}^*_k = \hat{x}^*_k$ and $\bar{x}^*_{n+1} = 0$, otherwise $\bar{x}^*_k = 0$ and $\bar{x}^*_{n+1} = \hat{x}^*_k$. It is easy to verify that $\bar{x}^*$ is a feasible solution to $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$. Moreover, $\hat{c}^T\hat{x}^* = \bar{c}^T\bar{x}^*$ implies $\max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\} \leq \max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$. Hence equality follows.

Since $Ax \leq b$ is TDI/k, there exists a dual optimal $1/k$-integral solution y^* to $\max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\}$ such that $(y^*)^Tb = \max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\}$, $(y^*)^TA \geq \hat{c}^T$ and $y^* \geq 0$. It follows that

$$(y^*)^T\bar{A} = (y^*)^T(A, a_k) = ((y^*)^TA, (y^*)^Ta_k) \geq (\hat{c}^T, \hat{c}_k) \geq \bar{c}^T,$$

implying that y^* is a feasible dual solution to $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$. By the following inequalities

$$(y^*)^Tb = \max\{\hat{c}^T\hat{x} : A\hat{x} \leq b,\ \hat{x} \geq 0\} = \max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\} \leq (y^*)^Tb,$$

y^* is also a dual optimal solution to $\max\{\bar{c}^T\bar{x} : \bar{A}\bar{x} \leq b,\ \bar{x} \geq 0\}$, where the last inequality is from the weak duality theorem. Hence the lemma follows. □

Lemma 3.3. *If $\pi(\hat{H}, \hat{\prec})$ is totally dual $1/k$-integral, then so is $\pi(H, \prec)$, where $k \in \mathbb{N}$.*

Proof. By Lemma 3.1, columns corresponding to parallel edges in the left hand side matrix of $\pi(H, \prec)$ are identical. Hence the left hand side matrix of $\pi(H, \prec)$ can be obtained from that of $\pi(\hat{H}, \hat{\prec})$ by duplicating columns corresponding to parallels edges. Then the lemma follows from Lemma 3.2. □

By Theorem 2.4 and Lemma 3.3, we conclude that $\pi(H, \prec)$ is TDI/2. By the construction of $(H, \prec)$, when D is good, $(H, \prec)$ admits no odd cycles with cyclic preferences. It follows that constraints (3.4) are redundant with respect to $\pi(H, \prec)$ when D is good.

Lemma 3.4. *$FSM(H, \prec)$ is integral if and only if $FSM(\hat{H}, \hat{\prec})$ is integral.*

Proof. For simplicity, assume that $H = (V, E)$ and $\hat{H} = (V, \hat{E})$.

We prove the only if part by showing that $FSM(\hat{H}, \hat{\prec})$ is a projection of $FSM(H, \prec)$. Let $\hat{x}$ be a point in $FSM(\hat{H}, \hat{\prec})$ and define $x = (\hat{x}, 0) \in \mathbb{R}^{\hat{E}} \times \mathbb{R}^{E-\hat{E}}$. To show that $x \in FSM(H, \prec)$, it suffices to prove that $x(\varphi(e)) \geq 1$ for $e \in E - \hat{E}$. Let $e' \in \hat{E}$ be the edge parallel with $e \in E - \hat{E}$. By Lemma 3.1, $x(\varphi(e)) = x(\varphi(e')) = \hat{x}(\varphi(e')) \geq 1$. Hence $FSM(\hat{H}, \hat{\prec})$ is a projection of $FSM(H, \prec)$.

We prove the if part by showing that each vertex of $FSM(H, \prec)$ can be obtained from some vertex of $FSM(\hat{H}, \hat{\prec})$ by adding some zero entries. Let x be a vertex of $FSM(H, \prec)$. Define $\hat{x} \in \mathbb{R}^{\hat{E}}$ by, for $e \in \hat{E}$, $\hat{x}(e) = x(e) + x(e')$, where $e' \in E - \hat{E}$ is parallel with e. By Lemma 3.1, it is easy to verify that $\hat{x} \in FSM(\hat{H}, \hat{\prec})$. We claim that $\hat{x}$ is a vertex of $FSM(\hat{H}, \hat{\prec})$. Assume to the contrary that there exist $\hat{x}_1, \hat{x}_2 \in FSM(\hat{H}, \hat{\prec})$ such that $\hat{x} = \alpha\hat{x}_1 + (1-\alpha)\hat{x}_2$, where $0 < \alpha < 1$. For $i = 1, 2$, we extend $\hat{x}_i \in \mathbb{R}^{\hat{E}}$ to $x_i \in \mathbb{R}^E$ by, for $e \in \hat{E}$ without parallel edges in H, $x_i(e) = \hat{x}_i(e)$; for $e \in \hat{E}$ and its parallel edge $e' \in E - \hat{E}$, if $x(e) > x(e')$, $x_i(e) = \hat{x}_i(e)$ and $x_i(e') = 0$, otherwise $x_i(e) = 0$ and $x_i(e') = \hat{x}_i(e)$. By Lemma 3.1, it is easy to see that $x_1, x_2 \in FSM(H, \prec)$. Furthermore, $x(e) \cdot x(e') = 0$ for parallel edges e and e'. Assume to the contrary that $x(e) \cdot x(e') \neq 0$, it follows that $x(e) = x(e') = 1/2$. Let y and z are duplicates of x and set $y(e) = z(e') = 1$ and $y(e') = z(e) = 0$. It follows that $x = (y+z)/2$. Clearly, $y, z \in FSM(H, \prec)$, contradicting to the assumption that x is a vertex. Hence $x = \alpha x_1 + (1-\alpha)x_2$ follows, a contradiction again. Therefore, $\hat{x}$ is vertex of $FSM(\hat{H}, \hat{\prec})$. It follows that each vertex x of $FSM(H, \prec)$ is associated with a vertex $\hat{x}$ of $FSM(\hat{H}, \hat{\prec})$ and x can be obtained from $\hat{x}$ by, for $e \in \hat{E}$ without parallel edges in H, $x(e) = \hat{x}(e)$; for $e \in \hat{E}$ and its parallel edge $e' \in E - \hat{E}$, if $x(e) > x(e')$, $x(e) = \hat{x}(e)$ and $x(e') = 0$, otherwise $x(e) = 0$ and $x(e') = \hat{x}(e)$. $\square$

We end this section with a summary. When D is clique-acyclic, it is associated with a preference system $(H, \prec)$ and a simple preference system $(\hat{H}, \hat{\prec})$, where $\hat{H}$ is a simple spanning subgraph of H maximizing the edge set and $\hat{\prec}$ is the restriction of $\prec$ on $\hat{H}$. Hence constraints (3.3) and (3.4) are redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$ and $FK(D) = FSM(H, \prec)$. To show $FK(D)$ is integral, by Lemma 3.4 it suffices to show that $FSM(\hat{H}, \hat{\prec})$ is integral. To show $\sigma(D)$ is TDI, by Lemma 3.3 it suffices to show $\pi(\hat{H}, \hat{\prec})$ is TDI. Moreover, when D is good, both $(H, \prec)$ and $(\hat{H}, \hat{\prec})$ admit no odd cycles with cyclic preferences.

4 Proofs

Before presenting our proof to the main theorem, we explore some properties of simple preference systems admitting no odd cycles with cyclic preferences.

Lemma 4.1. *Let $(G, \prec)$ be a simple preference system. If $(G, \prec)$ admits no odd cycles with cyclic preferences, then $SM(G, \prec) = FSM(G, \prec)$.*

As observed by Chen *et al.* [5], integrality of $FSM(G, \prec)$ is equivalent to total dual integrality of $\pi(G, \prec)$. A corollary follows directly.

Corollary 4.2. *Let $(G, \prec)$ be a simple preference system. If $(G, \prec)$ admits no odd cycles with cyclic preferences, then $\pi(G, \prec)$ is totally dual integral.*

Proof of Lemma 4.1

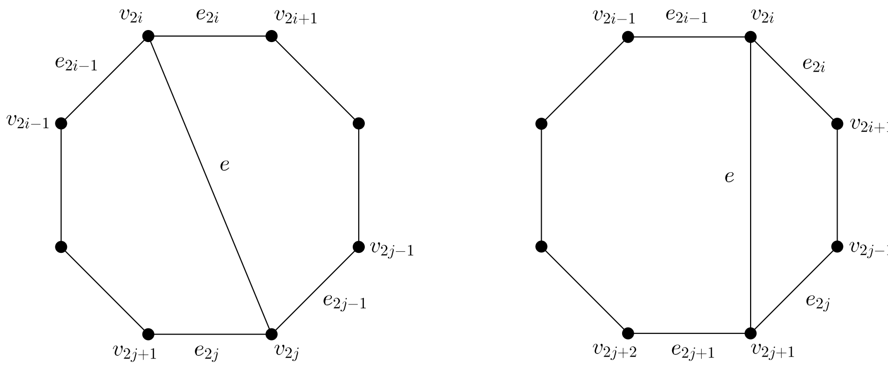
Let $(G, \prec)$ be a simple preference system admitting no odd cycles with cyclic preferences. By Theorem 2.3, $FSM(G, \prec)$ is 1/2-integral. Let x be a 1/2-integral point in $FSM(G, \prec)$. Consider $E_{1/2}(x)$, which consists of vertex disjoint even cycles $C_1, C_2, \ldots, C_r$ with cyclic preferences. For $i = 1, 2, \ldots, r$, label vertices and edges of $C_i \in E_{1/2}(x)$ such that $C_i = v_1^i v_2^i \ldots v_l^i$ and $e_k^i \prec_{v_{k+1}^i} e_{k+1}^i$ for $k = 1, 2, \ldots, l$, where $e_k^i = v_k^i v_{k+1}^i$ and indices are taken modulo l. We remark that the parity of vertices and edges refers to the parity of their indices. Define $z \in \mathbb{R}^{E(G)}$ by

$$z(e) = \begin{cases} 1 & e \text{ is an even edge in some } C \in E_{1/2}(x), \\ -1 & e \text{ is an odd edge in some } C \in E_{1/2}(x), \\ 0 & \text{otherwise.} \end{cases}$$

We are going to exclude x from vertices of $FSM(G, \prec)$ by adding perturbation ϵz for small ϵ to x and showing that $x \pm \epsilon z \in FSM(G, \prec)$. Tight constraints in (2.1)-(2.3) under perturbation ϵz play a key role in our proof. Observe that tight constraints in (2.2) and (2.3) are invariant under perturbation ϵz. It remains to show that perturbation ϵz does not affect tight constraints in (2.1) either. Clearly, $|\varphi(e) \cap E_+(x)| \in \{1, 2\}$ for $e \in E(G)$. When $|\varphi(e) \cap E_+(x)| = 1$, it is trivial. When $|\varphi(e) \cap E_+(x)| = 2$, the following lemma guarantees that corresponding tight constraints are also invariant under perturbation ϵz.

Lemma 4.3. *Let x be a 1/2-integral point in $FSM(G, \prec)$ and let e be an edge in $E(G)$ such that $x(\varphi(e)) = 1$ and $|\varphi(e) \cap E_+(x)| = 2$. Then the parity of dominating edges of e does not agree.*

Then edge e falls into the following four categories:



(a) Case 1　　(b) Case 2

Figure 1: Chord $e \in C$

(*i*) *e is an edge from some* $C \in E_{1/2}(x)$*;*

(*ii*) *e is a chord in some* $C \in E_{1/2}(x)$*;*

(*iii*) *e is a hanging edge of some* $C \in E_{1/2}(x)$ *and dominated by two edges from* C*;*

(*iv*) *e is a connecting edge bewteen* C_i *and* C_j *and dominated by one edge from* C_i *and one edge from* C_j *respectively, where* $C_i, C_j \in E_{1/2}(x)$*.*

Proof of Lemma 4.3 (*i*)-(*iii*). We distinguish four cases.

The lemma holds trivially for edges in categories (*i*) and (*iii*), since edges in (*i*) and (*iii*) are dominated by consecutive edges from some $C \in E_{1/2}(x)$.

To prove the lemma holds for category (*ii*), we distinguish two cases by the parity of endpoints of chord e in C. We first show that e cannot be a chord in any C such that endpoints of e have the same parity in C. Assume to the contrary that chord e have the same parity in some $C \in E_{1/2}(x)$. Without loss of generality, let $e = v_{2i}v_{2j}$.

If $e_{2i} \prec e$, it follows that $e_{2i-1} \prec e$. Since $x(\varphi(e)) = 1$, $e \prec e_{2j-1}$ and $e \prec e_{2j}$ follow. However, $v_{2i}ev_{2j}e_{2j}v_{2j+1}\ldots v_{2i-1}e_{2i-1}v_{2i}$ form an odd cycle with cyclic preferences, a contradiction. Hence, $e \prec e_{2i}$.

Similarly, if $e_{2j} \prec e$, it follow that $e_{2j-1} \prec e$. Eqaulity $x(\varphi(e)) = 1$ implies that $e \prec e_{2i}$ and $e \prec e_{2i-1}$. However, $v_{2i}e_{2i}v_{2i+1}\ldots v_{2j-1}e_{2j-1}v_{2j}ev_{2i}$ form an odd cycle with cyclic preferences, a contradiction. Hence, $e \prec e_{2j}$.

Now $e \prec e_{2i}$ and $e \prec e_{2j}$, it follows that $e_{2i-1} \prec e$ and $e_{2j-1} \prec e$ since $x(\varphi(e)) = 1$. But in this case the two odd cycles with cyclic preferences mentioned above occur at the same time.

Therefore, e cannot be a chord whose endpoints have the same parity in any C. It remains to show that in the latter case, dominating edges of chord e have different parity. Without loss

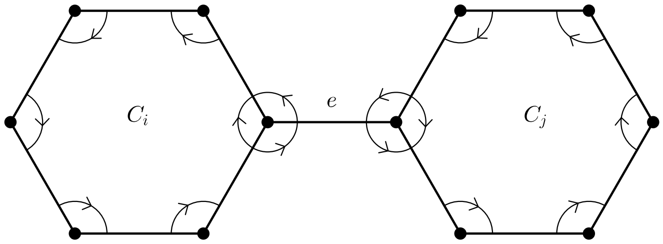


Figure 2: $e \in F_i \cap F_j$

of generality, let $e = v_{2i}v_{2j+1}$.

If $e_{2i} \prec e$ or $e_{2j+1} \prec e$, it follows that $e_{2i-1} \prec e$ or $e_{2j} \prec e$ respectively. Then e is dominated by two consecutive edges from C, which is trivial. So assume that $e \prec e_{2i}$ and $e \prec e_{2j+1}$. Since $x(\varphi(e)) = 1$, it follows that $e_{2i+1} \prec e$ and $e_{2j} \prec e$. Therefore, e is dominated by two nonadjacent edges with different parity. $\square$

To prove the remaining part of Lemma 4.3, we need some preparations. Let F_i be the set of hanging edges of $C_i \in E_{1/2}(x)$ in category (iv) of Lemma 4.3. Clearly, if $e \in F_i$, then there exists F_j $(j \neq i)$ such that $e \in F_j$. Analogous to the definition of cycles with cyclic preferences, we call $P = v_1v_2 \dots v_l$ a *v_1v_l-path with linear preferences* if $v_iv_{i+1} \prec_{v_{i+1}} v_{i+1}v_{i+2}$ for $i = 1, 2, \dots, l-2$. We have the following characterization for any two vertices from the same component of the induced subgraph $\cup_{i=1}^{i=r}(F_i \cup C_i)$ of G.

Lemma 4.4. *Suppose Lemma 4.3 is true. For any two vertices in the same component of induced subgraph $\cup_{i=1}^{i=r}(F_i \cup C_i)$, there exists a path with linear preferences between them. Moreover, if they have the same parity, the path is of even length; if they have different parity, the path is of odd length.*

Proof. When vertices are from the same cycle $C \in E_{1/2}(x)$, the lemma holds trivially. Hence take $v^s \in C_s$ and $v^t \in C_t$ where $s \neq t$. The existence of paths with linear preferences is trivial. Without loss of generality, consider a v^sv^t-path with linear preferences. We prove this lemma by induction on the number τ of cycles from $E_{1/2}(x)$ involved in the v^sv^t-path. Clearly, $\tau \geq 2$.

When $\tau = 2$. Take $v^s_+v^t_- \in F_s \cap F_t$. Let P_s be the path with linear preferences from v^s to v^s_+ in C_s and P_t be the path with linear preferences from v^t_- to v^t in C_t. Then $P = v^sP_sv^s_+v^t_-P_tv^t$ is a v^sv^t-path with linear preference. By Lemma 4.3, v^s_+ and v^t_- have different parity since $v^s_+v^t_- \in F_s \cap F_t$. If v^s and v^t have the same parity, then lengths of P_s and P_t have different parity, implying that P is of even length; if v^s and v^t have different parities, then lengths of P_s and P_t have the same parity, implying that P is of odd length.

Now assume that lemma is true for all $\tau \geq 2$. Let $C_{k_1}, \ldots, C_{k_\tau}, C_{k_{\tau+1}}$ be cycles from $E_{1/2}(x)$ involved along the $v^s v^t$-path with linear preferences. Take $v_+^{k_\tau} v_-^{k_{\tau+1}} \in F_{k_\tau} \cap F_{k_{\tau+1}}$. Let P_{s,k_τ} denote the $v^s v_+^{k_\tau}$-path and $P_{k_\tau,t}$ denote the $v_+^{k_\tau} v^t$-path, both of which have linear preferences. Clearly, $P = v^s P_{s,k_\tau} v_+^{k_\tau} P_{k_\tau,t} v^t$ is a $v^s v^t$-path with linear preferences. Since P_{s,k_τ} is a path involving only τ cycles and $P_{k_\tau,t}$ is a path involving only two cycles, their lengths both depend on the parity of their endpoints. It is easy to see that P is of even length when v^s and v^t have the same parity and P is of odd length when v^s and v^t have different parity. □

Proof of Lemma 4.3 (iv). It suffices to prove that the lemma holds for each component in the induced subgraph $\cup_{i=1}^{i=r} F_i \cup C_i$. We apply induction on the number α of cycles from $E_{1/2}(x)$ in a component.

When $\alpha = 1$, it is trivial.

Assume that Lemma 4.3 (*iv*) is true for components with $\alpha \geq 1$ cycles from $E_{1/2}(x)$. We consider a component with $\alpha + 1$ cycles $C_1, C_2, \ldots, C_\alpha, C_{\alpha+1}$ from $E_{1/2}(x)$. We further require that $C_{\alpha+1}$ is not a cut vertex if we view these cycles as vertices in this component. Hence deleting $C_{\alpha+1}$ yields a new component with α cycles, and Lemma 4.3 applies to the resulting new component. It remains to show that edges in $F_{\alpha+1}$ satisfy Lemma 4.3. If there exists edge $e \in F_{\alpha+1}$ violating Lemma 4.3, we relabel vertices and edges in $C_{\alpha+1}$. We will show that after at most one relabeling, all edges in $F_{\alpha+1}$ satisfy Lemma 4.3. We prove it by contradiction. Let $f_1, f_2 \in F_{\alpha+1}$ be edges such that f_1 satisfies Lemma 4.3 but f_2 violates Lemma 4.3. For $i = 1, 2$, let $f_i = u_i w_i$, where u_i is a vertex in the resulting new component and w_i is a vertex in $C_{\alpha+1}$. By assumption, u_1 and w_1 have different parity and u_2 and w_2 have the same parity.

By Lemma 4.4, there exists a path P_α with linear preferences between u_1 and u_2. Without loss of generality, let P_α be a $u_1 u_2$-path with linear preferences. There also exists a path $P_{\alpha+1}$ with linear preferences from w_2 to w_1. Now $u_1 P_\alpha u_2 f_2 w_2 P_{\alpha+1} w_1 f_1 u_1$ form a cycle $\hat{C}$ with cyclic preferences. We will show that $\hat{C}$ is an odd cycle with cyclic prefereces.

If u_1 and u_2 have the same parity, then w_1 and w_2 have different parity. It follow that P_α is of even length and $P_{\alpha+1}$ is of odd length. Hence $\hat{C}$ is an odd cycle with cyclic preferences.

If u_1 and u_2 have different parity, then w_1 and w_2 have the same parity. It follows that P_α is of odd length and $P_{\alpha+1}$ is of even length. Hence $\hat{C}$ is an odd cycle with cyclic preferences again.

Either case leads to an odd cycle with cyclic preferences, a contradiction. □

Proof of Lemma 4.1. By Lemma 4.3, each 1/2-integral point $x \in FSM(G, \prec)$ can be perturbed by ϵz for small ϵ without leaving $FSM(G, \prec)$. Hence 1/2-integral points cannot be vertices of $FSM(G, \prec)$. By Theorem 2.3, $SM(G, \prec) = FSM(G, \prec)$ follows. □

Now we are ready to present a proof to our main theorem.

Proof of Theorem 1.2. It suffices to show the equivalence among (i), (iii) and (iv). Without loss of generality, assume that D is an orientation of line multigraph $L(H)$ such that any two vertices in H are joined by at most two edges and parallel edges in $L(H)$ are orientated oppositely. When D is clique-acyclic, D is associated with a preference system $(H, \prec)$. Hence $\sigma(D)$ can be viewed as a linear system defined on preference system $(H, \prec)$ and consisting of constraints (3.1)-(3.5) where constraints (3.1), (3.2) and (3.5) form the Rothblum system $\pi(H, \prec)$.

Assume D is good. Then the preference system $(H, \prec)$ constructed from D admits no odd cycles with cyclic preferences and so is $(\hat{H}, \hat{\prec})$, where $\hat{H}$ is a spanning subgraph of H obtained by keeping one edge between every pair of adjacent vertices in H and $\hat{\prec}$ is the restriction of $\prec$ on $\hat{H}$. By Lemma 4.1, $FSM(\hat{H}, \hat{\prec})$ is integral. Integrality of $FSM(H, \prec)$ follows from Lemma 3.4. Hence constraints (3.3) and (3.4) are both redundant in $\sigma(D)$ which defines $FK(D)$. Therefore $FK(D) = FSM(H, \prec)$, implying that $FK(D)$ is integral. Similar arguments apply to any induced subgraphs of D. Hence $(i) \implies (iii)$. Further notice that, by Theorem 2.4 integrality of $FSM(\hat{H}, \hat{\prec})$ implies total dual integrality of $\pi(\hat{H}, \hat{\prec})$. Then total dual integrality of $\pi(H, \prec)$ follows from Lemma 3.3. Since $\pi(H, \prec)$ is part of $\sigma(D)$ and the other constraints (3.3), (3.4) are redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$, total dual integrality of $\sigma(D)$ follows. Similar arguments apply to any induced subgraphs of D. Therefore, $(i) \implies (iv)$.

To see $(iii) \implies (i)$, we prove it by contradiction. Observe that strong kernel idealness of D implies the existence of kernels for any induced subgraphs of D. Let D be a digraph such that D is strongly kernel ideal but not good. Then there exists either a clique containing directed cycles or a directed odd cycle without chords and pseudochords in D. We show that neither case is possible. If D has a clique containing directed cycles, we consider the subgraph induced on this clique. There is no kernel for this induced subgraph, a contradiction. If D contains a directed odd cycle without chords and pseudochords, we restrict ourselves to the subgraph induced on this directed odd cycle. There is no kernel for this induced subgraph either, a contradiction.

To see $(iv) \implies (i)$, notice that $(iv) \implies (iii)$ is a directly result from a theorem by Edmonds and Giles [6]. Hence $(iv) \implies (i)$ follows from $(iii) \implies (i)$.

□

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