On Kernel Mengerian Orientations of Line Multigraphs

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Abstract

We present a polyhedral description of kernels in orientations of line multigraphs. Given a digraph D, let $FK(D)$ denote the fractional kernel polytope defined on D, and let $\sigma(D)$ denote the linear system defining $FK(D)$. A digraph D is called kernel perfect if every induced subdigraph D' has a kernel, called kernel ideal if $FK(D')$ is integral for each induced subdigraph D', and called kernel Mengerian if $\sigma(D')$ is TDI for each induced subdigraph D'. We show that an orientation of line multigraph is kernel perfect iff it is kernel ideal iff it is kernel Mengerian. Our result strengthens the theorem of Borodin *et al.* [3] on kernel perfect digraphs and generalizes the theorem of Király and Pap [9] on stable matching problem.

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1 Introduction

A graph is called *simple* if it contains neither loops nor parallel edges, and is called a *multigraph* if it has parallel edges. A *simple* digraph is an orientation of simple graph. A *multi-digraph* is an orientation of multigraph.

Let G be a graph. The *line graph* of G, denoted by $L(G)$, is a graph such that: each vertex of $L(G)$ corresponds to an edge of G, and two vertices of $L(G)$ are adjacent if and only if they are incident as edges in G. We call $L(G)$ the *line multigraph* of G if any two vertices of $L(G)$ are connected by as many edges as the number of their common ends in G. We call G a *root* of $L(G)$.

Let $D = (V, A)$ be a digraph. For $U \subseteq V$, we call U an *independent* set of D if no two vertices in U are connected by an arc, call U a *dominating* set of D if for each vertex $v \notin U$, there is an arc from v to U, and call U a *kernel* of D if it is both independent and dominating. We call D *kernel perfect* if each of its induced subdigraphs has a kernel. A *clique* of D is a subset of V such that any two vertices are connected by an arc. We call D *clique-acyclic* if for each clique of D the induced subdigraph of one-way arc is acyclic, and call D *good* if it is clique-acyclic and every directed odd cycle has a (pseudo-)chord[1].

Theorem 1.1 (Borodin *et al.* [3])**.** *Let G be a line multigraph. The orientation D of G is kernel perfect if and only if it is good.*

A subset P of $\mathbb{R}^n$ is called a *polytope* if it is the convex hull of finitely many vectors in $\mathbb{R}^n$. A point x in P is called a *vertex* or an *extreme point* if there exist no distinct points y and z in P such that $x = \alpha y + (1 - \alpha)z$ for $0 < \alpha < 1$. It is well known that P is the convex hull of its vertices, and that there exists a linear system $Ax \leq b$ such that $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. We say P is *$1/k$-integral* if its vertices are $1/k$-integral vectors, where $k \in \mathbb{N}$. By a theorem in linear programming, P is $1/k$-integral if and only if $\max\{c^T x : Ax \leq b\}$ has an optimal $1/k$-integral solution for every integral vector c for which the optimum is finite. If, instead, $\max\{c^T x : Ax \leq b\}$ has a dual optimal $1/k$-integral solution, we say $Ax \leq b$ is *totally dual $1/k$-integral* (TDI/k). It is easy to verify that $Ax \leq b$ is TDI/k if and only if $Bx \leq b$ is TDI, where $B = A/k$ and $k \in \mathbb{N}$. Thus from a theorem of Edmonds and Giles [8], we deduce that if $Ax \leq b$ is TDI/k and b is integral, then $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is $1/k$-integral.

[1] A pseudo-chord is an arc (v_i, v_{i-1}) in a directed cycle $v_1 v_2 \dots v_l v_1$.

Let $\sigma(D)$ denote the linear system consisting of the following inequalities:

$$x(v) + x(N^+(v)) \geq 1 \qquad \forall\ v\ \in V, \tag{1.1}$$

$$x(Q) \leq 1 \qquad \forall\ Q \in \mathcal{Q}, \tag{1.2}$$

$$x(v) \geq 0 \qquad \forall\ v\ \in V, \tag{1.3}$$

where $x(U) = \sum_{u \in U} x(u)$ for any $U \subseteq V$, $N^+(v)$ denotes the set of all out-neighbors of vertex v, and $\mathcal{Q}$ denotes the set of all cliques of D. Observe that incidence vectors of kernels of D are precisely integral solutions $x \in \mathbb{Z}^A$ to $\sigma(D)$. The *kernel polytope* of D, denoted by $K(D)$, is the convex hull of incidence vectors of all kernels of D. The *fractional kernel polytope* of D, denoted by $FK(D)$, is the set of all solutions $x \in \mathbb{R}^A$ to $\sigma(D)$. Clearly, $K(D) \subseteq FK(D)$. We call D *kernel ideal* if $FK(D')$ is integral for each induced subdigraph D', and *kernel Mengerian* if $\sigma(D')$ is TDI for each induced subdigraph D'.

As described in Egres Open [1], the polyhedral description of kernels remains open. Chen *et al.* [5] attained a polyhedral characterization of kernels by replacing clique constraints $x(Q) \leq 1$ for $Q \in \mathcal{Q}$ with independence constraints $x(u) + x(v) \leq 1$ for $(u, v) \in A$. In this paper we show that kernels in orientations of line multigraph can be characterized polyhedrally.

Theorem 1.2. *Let D be an orientation of a line multigraph. Then the following statements are equivalent:*

(i) *D is good;*

(ii) *D is kernel perfect;*

(iii) *D is kernel ideal;*

(iv) *D is kernel Mengerian.*

The equivalence of (i) and (ii) was established by Borodin *et al.* [3] (Maffray [10] proved the case when D is perfect). Király and Pap [9] proved Theorem 1.2 for the case when the root of D is bipartite. Our result strengthens the theorem of Borodin *et al.* [3] and generalizes the theorem of Király and Pap [9] to line multigraphs.

2 Preliminaries

Kernels are closely related to stable matchings. Before proceeding, we introduce the notation of stable matching and some results that will be used later. Let $G = (V, E)$ be a graph. For $v \in V$, let $\delta(v)$ denote the set of edges incident to v and $\prec_v$ be a strict linear order on $\delta(v)$. We call $\prec_v$

the *preference* of v, and for edges e and f incident to v we say v *prefers* e to f or e *dominates* f if $e \prec_v f$. Let $\prec$ be the set of linear order $\prec_v$ for $v \in V$. We call the pair $(G, \prec)$ *preference system*, and call $(G, \prec)$ *simple* if G is simple. For $e \in E$, let $\varphi(e)$ denote the set consisting of e itself and edges that dominate e in $(G, \prec)$, and let $\varphi_v(e)$ denote the set of edges that dominate e at vertex v in $(G, \prec)$. Given a matching M in G, we call M *stable* in $(G, \prec)$ if every edge of G is either in M or is dominated by some edge in M.

Let $\pi(G, \prec)$ denote the linear system consisting of the following linear inequalities:

$$x(\varphi(e)) \geq 1 \qquad \forall\ e \in E, \tag{2.1}$$

$$x(\delta(v)) \leq 1 \qquad \forall\ v \in V, \tag{2.2}$$

$$x(e) \geq 0 \qquad \forall\ e \in E. \tag{2.3}$$

As observed by Abeledo and Rothblum [2], incidence vectors of stable matchings of $(G, \prec)$ are precisely integral solutions $x \in \mathbb{Z}^E$ to $\pi(G, \prec)$. The *stable matching polytope*, denoted by $SM(G, \prec)$, is the convex hull of incidence vectors of all stable matchings of $(G, \prec)$. The *fractional stable matching polytope*, denoted by $FSM(G, \prec)$, is the set of all solutions $x \in \mathbb{R}^E$ to $\pi(G, \prec)$. Clearly, $SM(G, \prec) \subseteq FSM(G, \prec)$.

Theorem 2.1 (Rothblum [11]). *Let $(G, \prec)$ be a simple preference system. If G is bipartite, then $SM(G, \prec) = FSM(G, \prec)$.*

Theorem 2.2 (Király and Pap [9]). *Let $(G, \prec)$ be a simple preference system. If G is bipartite, then $\pi(G, \prec)$ is totally dual integral.*

Given a cycle $C = v_1 v_2 \ldots v_l v_1$ in G, we call C of *cyclic preferences* in $(G, \prec)$ if $v_{i-1}v_i \prec_{v_i} v_i v_{i+1}$ for $i = 1, 2, \ldots, l$ or $v_{i-1}v_i \succ_{v_i} v_i v_{i+1}$ for $i = 1, 2, \ldots, l$, where indices are taken modulo l. For $x \in FSM(G, \prec)$, let $E_\alpha(x)$ denote the set of all edges with $x(e) = \alpha$ where $\alpha \in \mathbb{R}$ and $E_+(x)$ denote the set of all edges with $x(e) > 0$.

Theorem 2.3 (Abeledo and Rothblum [2]). *Let $(G, \prec)$ be a simple preference system. Then $FSM(G, \prec)$ is $1/2$-integral. Moreover, for each $1/2$-integral point x in $FSM(G, \prec)$, $E_{1/2}(x)$ consists of vertex disjoint cycles with cyclic preferences.*

Theorem 2.4 (Chen *et al.* [6]). *Let $(G, \prec)$ be a simple preference system. Then $\pi(G, \prec)$ is totally dual $1/2$-integral. Moreover, $\pi(G, \prec)$ is totally dual integral if and only if $SM(G, \prec) = FSM(G, \prec)$.*

3 Reductions

Given a clique-acyclic orientation D of line multigraph $L(H)$, let $e \prec_v f$ if (f, e) is an arc in D for any two incident edges e and f with common end v in H. Hence D is associated with a preference system $(H, \prec)$. Recall that $\sigma(D)$ denotes the linear system which defines $FK(D)$. Consequently, $\sigma(D)$ can be viewed as a linear system defined on preference system $(H, \prec)$. The equivalence of constraints (1.3) and constraints (2.3) follows directly. Constraints (1.1) can be viewed as constraints (2.1) because of the one to one correspondence between dominating vertex set $\{v\} \cup N_D^+(v)$ for $v \in V(D)$ and stable edge set $\varphi(e)$ for $e \in E(H)$. Observe that cliques of D correspond to three types of edge set in H:

- $\delta(v)$ for $v \in V(H)$,
- nontrivial subsets of $\delta(v)$ for $v \in V(H)$,
- complete subgraphs of H induced on three vertices,

and all three types allow parallel edges. Hence constraints (1.2) can be viewed as constraints (2.2) together with some extra constraints on $(H, \prec)$. Let $\mathcal{O}(H)$ denote the set of all complete subgraphs of H induced on three vertices. Then $\sigma(D)$ can be reformulated in terms of preference system $(H, \prec)$:

$$x(\varphi(e)) \geq 1 \qquad \forall\, e \in E(H), \tag{3.1}$$

$$x(\delta(v)) \leq 1 \qquad \forall\, v \in V(H), \tag{3.2}$$

$$x(S) \leq 1 \qquad \emptyset \subset S \subset \delta(v), \quad \forall\, v \in V(H), \tag{3.3}$$

$$x(O) \leq 1 \qquad \forall\, O \in \mathcal{O}(H), \tag{3.4}$$

$$x(e) \geq 0 \qquad \forall\, e \in E(H). \tag{3.5}$$

Notice that constraints (3.1), (3.2) and (3.5) form the Rothblum system $\pi(H, \prec)$ which defines $FSM(H, \prec)$. Constraints (3.3) are redundant with respect to $\pi(H, \prec)$ due to constraints (3.2). As we shall see, constraints (3.4) are also redundant with respect to $\pi(H, \prec)$. Hence $FK(D)$ is essentially defined by Rothblum system $\pi(H, \prec)$, or equivalently that $FK(D) = FSM(H, \prec)$.

Observe that H is a multigraph. To bridge the gap between simple preference system and $(H, \prec)$, we exploit the gadget introduced by Cechlárová and Fleiner [4]. We define a simple preference system $(H', \prec')$ from $(H, \prec)$ by substituting each parallel edge e with endpoints u and v in H by a 6-cycle with two hanging edges as in Figure 1 such that uu_0 (*resp.* vv_0) has the same order with uv in $\prec_u$ (*resp.* $\prec_v$). Observe that the construction preserves the parity of all cycles with cyclic preferences in H.

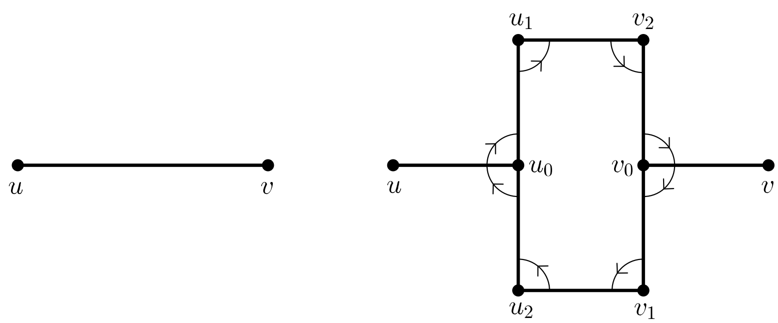

Figure 1: The gadget introduced for parallel edges

Lemma 3.1. *$FSM(H, \prec)$ is a projection of $FSM(H', \prec')$.*

Proof. Take $x \in FSM(H, \prec)$. For each parallel edge e with endpoints u and v in H, we define the value of edges in the gagdet corresponding to e as follows:

1. Set $x'_{uu_0} = x'_{vv_0} := x_e$;
2. Set $x'_{u_0u_1} = x'_{v_0v_2} := 1 - x_e - x(\varphi_u(e))$;
3. Set $x'_{u_0u_2} = x'_{v_0v_1} := x(\varphi_u(e))$;
4. Set $x'_{u_1v_2} := x_e + x(\varphi_u(e))$;
5. Set $x'_{u_2v_1} := 1 - x(\varphi_u(e))$.

Next for each edge f without parallel edges in H, set $x'_f := x_f$. It is easy to check that $x' \in FSM(H', \prec')$. Hence $FSM(H, \prec)$ is a projection of $FSM(H', \prec')$. $\square$

By Theorem 2.3 and Lemma 3.1, $FSM(H, \prec)$ is 1/2-integral since $FSM(H', \prec')$ is 1/2-integral. Hence constraints (3.4) only work for vertices in $FSM(H, \prec)$ with $x(O) = 3/2$, where $O \in \mathcal{O}(H)$. However by Theorem 2.3 again, each O with $x(O) = 3/2$ admits a 3-cycle with cyclic preferences which arises from a directed 3-cycle from D, contradicting to the assumption that D is clique-acyclic. Therefore, constraints (3.4) are unbinding for all vertices of $FSM(H, \prec)$.

Lemma 3.2. *If $\pi(H', \prec')$ is totally dual integral, then so is $\pi(H, \prec)$.*

Proof. We show that $\pi(H, \prec)$ can be obtained from $\pi(H', \prec')$ after a series of Fourier-Motzkin eliminations. It suffices to demonstrate one elimination process from a gadget to an edge. Given a gadget arising from edge e as in Figure 1, eliminate u_1v_2 from $\pi(H', \prec')$ first. Then all constraints involving $x_{u_1v_2}$ are replaced by equality $x_{u_0u_1} = x_{v_0v_2}$. Similarly, eliminating u_2v_1

yields equality $x_{v_0v_1} = x_{u_0u_2}$. Next eliminating uu_0 gives $x(\delta(u)\backslash\{uu_0\}) \leq x_{u_0u_1} + x_{u_0u_2}$ and $x_{u_0u_2} \leq x(\varphi_u(uv))$. After eliminating u_0u_1 and u_0u_2, we arrive at $x(\delta(u)\backslash\{uu_0\}) \leq x_{v_0v_1} + x_{v_0v_2}$ and $x_{v_0v_1} \leq x(\varphi_u(uv))$. In the end, canceling v_0v_1 and v_0v_2 gives $x_{vv_0} + x(\varphi_v(vv_0) + x(\varphi_u(uu_0)) \geq 1$ and $x_{vv_0} + x(\delta(u)\backslash\{uu_0\}) \leq 1$. Besides, $x_{vv_0} + x(\delta(v)\backslash\{vv_0\}) \leq 1$ is unchanged. Notice that $x_{vv_0} + x(\varphi_v(vv_0) + x(\varphi_u(uu_0)) \geq 1$ can be viewed as $x(\varphi(e)) \geq 1$, $x_{vv_0} + x(\delta(u)\backslash\{uu_0\}) \leq 1$ can be viewed as $x(\delta(u)) \leq 1$, and $x_{vv_0} + x(\delta(v)\backslash\{vv_0\}) \leq 1$ can be viewed as $x(\delta(v)) \leq 1$. Hence we reduce the linear system involving the gadget to a linear system only related to its corresponding edge e. Performing Fourier-Motzkin eliminations in such an order for each gadget in H' leads to a linear system defined on $(H, \prec)$, which is precisely the same with $\pi(H, \prec)$ (renaming variables and removing redundant constraints if necessary). As proved by Cook [7], total dual integrality is preserved under Fourier-Motzkin elimination of a variable if it occurs in each constraint with coefficient 0 or ± 1. Hence the lemma follows. $\square$

4 Proofs

Observe that when D is good, both $(H, \prec)$ and $(H', \prec')$ admit no odd cycles with cyclic preferences. Hence we exhibit some properties of simple preference systems admitting no odd cycles with cyclic preferences first.

Lemma 4.1. *Let $(G, \prec)$ be a simple preference system. If $(G, \prec)$ admits no odd cycles with cyclic preferences, then $SM(G, \prec) = FSM(G, \prec)$.*

By Theorem 2.4, integrality of $FSM(G, \prec)$ is equivalent to total dual integrality of $\pi(G, \prec)$, where $(G, \prec)$ is a simple preference system. A corollary follows directly.

Corollary 4.2. *Let $(G, \prec)$ be a simple preference system. If $(G, \prec)$ admits no odd cycles with cyclic preferences, then $\pi(G, \prec)$ is totally dual integral.*

Proof of Lemma 4.1. By Theorem 2.3, $FSM(G, \prec)$ is 1/2-integral as $(G, \prec)$ is a simple preference system. Let x be a 1/2-integral point in $FSM(G, \prec)$. Since $(G, \prec)$ admits no odd cycles with cyclic preferences, $E_{1/2}(x)$ consists of even cycles $C_1, C_2, \ldots, C_r$ with cyclic preferences. For $i = 1, 2, \ldots, r$, label vertices and edges of $C_i \in E_{1/2}(x)$ such that $C_i = v_1^i v_2^i \ldots v_l^i$ and $e_k^i \prec_{v_{k+1}^i} e_{k+1}^i$ for $k = 1, 2, \ldots, l$, where $e_k^i = v_k^i v_{k+1}^i$ and indices are taken modulo l. We remark that the parity of vertices and edges refers to the parity of their indices. Define $z \in \mathbb{R}^{E(G)}$ by

$$z(e) := \begin{cases} 1 & e \text{ is an even edge in some } C \in E_{1/2}(x), \\ -1 & e \text{ is an odd edge in some } C \in E_{1/2}(x), \\ 0 & \text{otherwise.} \end{cases}$$

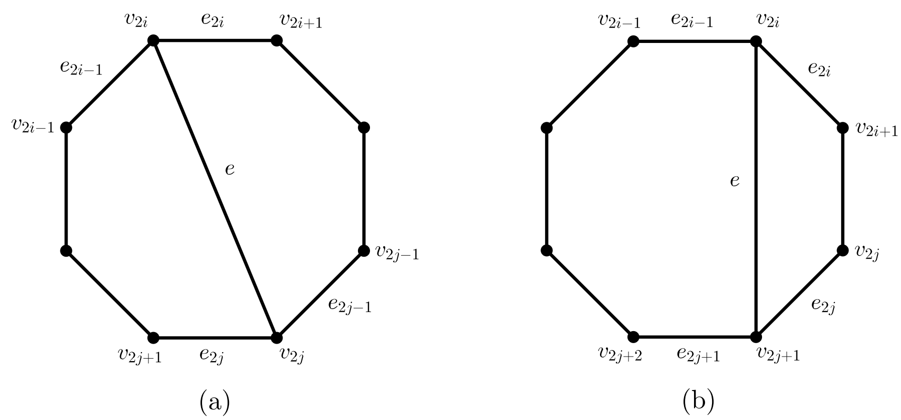

Figure 2: Case 2

We are going to exclude x from vertices of $FSM(G, \prec)$ by adding perturbation ϵz for small ϵ to x and showing that $x \pm \epsilon z \in FSM(G, \prec)$. Tight constraints in (2.1)-(2.3) under perturbation ϵz play a key role here. Observe that tight constraints in (2.2) and (2.3) are invariant under perturbation ϵz. It remains to show that perturbation ϵz does not affect tight constraints in (2.1) either. Let e be an edge with $x(\varphi(e)) = 1$. Clearly, $|\varphi(e) \cap E_+(x)| \in \{1, 2\}$. When $|\varphi(e) \cap E_+(x)| = 1$, $x(e) = 1$ follows, which is trivial. When $|\varphi(e) \cap E_+(x)| = 2$, we claim that the parity of dominating edges in $E_{1/2}(x)$ of e does not agree (relabeling vertices and edges in $E_{1/2}(x)$ if necessary). Hence corresponding tight constraints in (2.1) are also invariant under perturbation ϵz. To justify this claim, we distinguish four cases.

Case 1. Edge e is an edge from some $C \in E_{1/2}(x)$. This case is trivial since C admits cyclic preferences.

Case 2. Edge e is a chord in some $C \in E_{1/2}(x)$. We first show that endpoints of e have different parity in C. We prove it by contradiction. Without loss of generality, let $e = v_{2i}v_{2j}$.

If $e_{2i} \prec e$, then $e_{2i-1} \prec e$. Since $x(\varphi(e)) = 1$, it follows that $e \prec e_{2j-1}$ and $e \prec e_{2j}$. However, $v_{2i}ev_{2j}e_{2j}v_{2j+1} \ldots v_{2i-1}e_{2i-1}v_{2i}$ form an odd cycle with cyclic preferences, a contradiction. Hence $e \prec e_{2i}$.

Similarly, if $e_{2j} \prec e$, then $e_{2j-1} \prec e$. Equality $x(\varphi(e)) = 1$ implies that $e \prec e_{2i}$ and $e \prec e_{2i-1}$. However, $v_{2i}e_{2i}v_{2i+1} \ldots v_{2j-1}e_{2j-1}v_{2j}ev_{2i}$ form an odd cycle with cyclic preferences, a contradiction. Hence $e \prec e_{2j}$.

Now $e \prec e_{2i}$ and $e \prec e_{2j}$, it follows that $e_{2i-1} \prec e$ and $e_{2j-1} \prec e$ since $x(\varphi(e)) = 1$. But in this case two odd cycles with cyclic preferences mentioned above occur at the same time. Therefore, endpoints of e have different parity in C. Hence let $e = v_{2i}v_{2j+1}$. If $e_{2i} \prec e$ (*resp.* $e_{2j+1} \prec e$), it follows that $e_{2i-1} \prec e$ (*resp.* $e_{2j} \prec e$). Then e is dominated by two consecutive

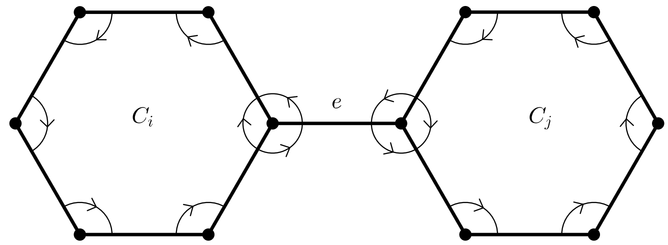

Figure 3: Case 4

edges from C, which is trivial. So assume that $e \prec e_{2i}$ and $e \prec e_{2j+1}$. Since $x(\varphi(e)) = 1$, it follows that $e_{2i+1} \prec e$ and $e_{2j} \prec e$. Therefore e is dominated by two edges with different parity.

Case 3. Edge e is a hanging edge of some $C \in E_{1/2}(x)$ and dominated by two edges from C. This case is trivial.

Case 4. Edge e is a connecting edge between C_i and C_j and dominated by one edge from C_i and one edge from C_j respectively, where $C_i, C_j \in E_{1/2}(x)$. For $k = 1, 2, \ldots, r$, let F_k be the subset of edges in this case and incident to C_k. Then $\cup_{i=1}^{i=r} F_i \cup C_i$ induces a subgraph of G. It suffices to work on a component of the induced subgraph. We apply induction on the number α of cycles from $E_{1/2}(x)$ in a component.

When $\alpha = 1$, it is trivial. Hence assume the claim holds for components with $\alpha \geq 1$ cycles from $E_{1/2}(x)$. We consider a component with $\alpha + 1$ cycles $C_1, \ldots, C_\alpha, C_{\alpha+1}$ from $E_{1/2}(x)$. Without loss of generality, assume that deleting $C_{\alpha+1}$ yields a new component with α cycles. By induction hypothesis, the claim holds for the resulting component. It remains to check edges in $F_{\alpha+1}$. If there exists an edge in $F_{\alpha+1}$ violating the claim, relabel vertices and edges in $C_{\alpha+1}$. After at most one relabeling, all edges in $F_{\alpha+1}$ satisfy the claim. We prove it by contradiction. Let $f_1, f_2 \in F_{\alpha+1}$ be edges such that f_1 satisfies the claim but f_2 violates the claim. For $i = 1, 2$, let $f_i = u_i w_i$, where u_i is the endpoint in the resulting component and w_i is the endpoint in $C_{\alpha+1}$. By assumption, u_1 and w_1 have different parity and u_2 and w_2 have the same parity. Analogous to the definition of cycles with cyclic preferences, we call path $P = v_1 v_2 \ldots v_l$ a *$v_1 v_l$-path with linear preferences* if $v_i v_{i+1} \prec_{v_{i+1}} v_{i+1} v_{i+2}$ for $i = 1, 2, \ldots, l-2$. Clearly, for any two vertices in the same component, there exists a path with linear preferences between them. Hence there exist a $u_1 u_2$-path P_α and a $w_2 w_1$-path $P_{\alpha+1}$, both of which admit linear preferences. Moreover, $u_1 P_\alpha u_2 f_2 w_2 P_{\alpha+1} w_1 f_1 u_1$ form a cycle with cyclic preferences. We justify this cycle is odd by showing that the $u_1 u_2$-path P_α is even (*resp.* odd) if u_1 and u_2 have the same (*resp.* different) parity.

If u_1 and u_2 belong to the same cycle from $E_{1/2}(x)$, it is trivial. Hence assume $u_1 \in C_s$ and $u_2 \in C_t$, where $s, t \in \{1, 2, \ldots, \alpha\}$ and $s \neq t$. We apply induction on the number τ of cycles from $E_{1/2}(x)$ involved in P_α. Clearly, $\tau \geq 2$. When $\tau = 2$. Take $v^s v^t \in F_s \cap F_t$ on P_α. Let P_s be the part of P_α from u_1 to v^s in C_s and P_t be the part of P_α from v^t to u_2 in C_t. It follows that $u_1 P_s v^s v^t P_t u_2$ form P_α. By primary induction hypothesis, v^s and v^t have different parity since $v^s v^t \in F_s \cap F_t$. If u_1 and u_2 have the same parity, then P_s and P_t have different parity, implying that P_α is even; if u_1 and u_2 have different parity, then P_s and P_t have the same parity, implying that P_α is odd. Now assume $\tau \geq 2$. Let $C_{k_1}, \ldots, C_{k_\tau}, C_{k_{\tau+1}}$ be cycles from $E_{1/2}(x)$ involved along P_α. Take $v^{k_\tau} v^{k_{\tau+1}} \in F_{k_\tau} \cap F_{k_{\tau+1}}$ on P_α. Let P_{s,k_τ} denote the part of P_α from u_1 to v^{k_τ} and $P_{k_\tau,t}$ denote the part of P_α from v^{k_τ} to u_2. Clearly, $P_\alpha = u_1 P_{s,k_\tau} v^{k_\tau} P_{k_\tau,t} u_2$. Since P_{s,k_τ} involves τ cycles and $P_{k_\tau,t}$ involves two cycles, both length depend on the parity of endpoints. It follows that P_α is even when u_1 and u_2 have the same parity, and P_α is odd when u_1 and u_2 have different parity.

Hence when u_1 and u_2 have the same parity, w_1 and w_2 have different parity, implying that P_α is even and $P_{\alpha+1}$ is odd; when u_1 and u_2 have different parity, w_1 and w_2 have the same parity, implying that P_α is odd and $P_{\alpha+1}$ is even. Either case yields an odd cycle with cyclic preferences, a contradiction.

Therefore 1/2-integral points are not vertices of $FSM(G, \prec)$ as they can be perturbed by ϵz for small ϵ without leaving $FSM(G, \prec)$. By Theorem 2.3, $SM(G, \prec) = FSM(G, \prec)$ follows. □

Now we are ready to present a proof of our main theorem.

Proof of Theorem 1.2. It suffices to show the equivalence of (i), (iii) and (iv). Let D be a good orientation of line multigraph $L(H)$. Construct preference system $(H, \prec)$ from D, and construct simple preference system $(H', \prec')$ from $(H, \prec)$ by substituting each parallel edge with a gadget as in Figure 1. By the construction, $(H', \prec')$ admits no odd cycles with cyclic preferences. Now $\sigma(D)$ can be viewed as a linear system defined on preference system $(H, \prec)$ and consisting of constraints (3.1)-(3.5). Observe that constraints (3.1), (3.2) and (3.5) form the Rothblum system $\pi(H, \prec)$, and constraints (3.3)-(3.4) are redundant with respect to $\pi(H, \prec)$. Hence $FK(D) = FSM(H, \prec)$ follows.

By Lemma 4.1, $FSM(H', \prec')$ is integral. Integrality of $FSM(H, \prec)$ follows from Lemma 3.1, implying $FK(D)$ is integral. Similar arguments apply to any induced subdigraphs of D. Hence $(i) \implies (iii)$.

By Corollary 4.2, $\pi(H', \prec')$ is TDI. Total dual integrality of $\pi(H, \prec)$ follows from Lemma 3.2. Since $\pi(H, \prec)$ is part of $\sigma(D)$ and the other constraints (3.3)-(3.4) are redundant in $\sigma(D)$ with respect to $\pi(H, \prec)$, total dual integrality of $\sigma(D)$ follows. Similar arguments apply to any induced subdigraphs of D. Hence $(iii) \implies (iv)$.

By a theorem of Edmonds and Giles [8], implication $(iv) \implies (iii)$ follows directly.

To prove implication $(iii) \implies (i)$, we assume the contrary. Observe that D being kernel ideal implies the existence of kernels for any induced subdigraphs of D. Let D be a digraph such that D is kernel ideal but not good. Then there either exists a clique containing directed cycles or exists a directed odd cycle without (pseudo-)chords. We show that neither case is possible. If D has a clique containing directed cycles, we consider the subdigraph induced on this clique. There is no kernel for this induced subdigraph, a contradiction. If D contains a directed odd cycle without (pseudo-)chords, we restrict ourselves to the subdigraph induced on this directed odd cycle. There is no kernel for this induced subdigraph either, a contradiction. $\square$

5 Discussions

Superorientations seems to break the connection between kernels and stable matchings. And being kernel perfect is not sufficient for being kernel ideal in superorientations, not to mention being kernel mengerian. We may consider the digraph $D = (V, A)$, where $V = \{1, 2, 3, 4\}$ and $A = \{(1, 2), (2, 1), (3, 2), (4, 1), (4, 3)\}$. This digraph can be view as a superorientation of a 4-cycle, which is a line graph. However, $FK(D)$ has three vertices, among which there exists half-integral one.

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